General δ-shell interactions for the two-dimensional Dirac operator: self-adjointness and approximation

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Abstract. In this work we consider the two-dimensional Dirac operator with general local singular interactions supported on a closed curve. A systematic study of the interaction is performed by decomposing it into a linear combination of four elementary interactions: electrostatic, Lorentz scalar, magnetic, and a fourth one which can be absorbed by using unitary transformations. We address the self-adjointness and the spectral description of the underlying Dirac operator. In the non-critical case, we do so by providing a boundary triple, and in the critical purely magnetic case, by exploiting the phenomenon of confinement and super-symmetry. Moreover, we justify our model by showing that Dirac operators with singular interactions are limits in the strong resolvent sense of Dirac operators with regular potentials.

Contents

1. Introduction

In the present paper we study the two-dimensional Dirac operator with a singular interaction supported on a closed curve. Our main motivation is to treat the most general local

interactions. Besides electrostatic δ-shell interactions and the Lorentz scalar δ-shell interactions we include into the analysis the magnetic δ-shell interactions, which correspond to the magnetic field supported on a curve. The main two questions addressed in the present paper are self-adjointness of the underlying Dirac operator and its approximation by Dirac operators with regular potentials.

Recall that the Dirac operator was firstly introduced in relativistic quantum mechanics to describe the dynamics of spin-$\frac{1}{2}$ particles (see, e.g., the monograph [68]), and it was later associated to the evolution of quasi-particles in new materials, such as the graphene (see, e.g., [28]). Dirac operators with singular interactions supported on sets of lower dimensions serve as idealized models for Dirac operators with more realistic (regular) potentials.

Hamiltonians with interactions supported on sets of zero Lebesgue measure have been studied intensively in the mathematical physics. Initially, the case of Schrödinger operators with singular interactions was investigated, see, e.g., [2, 15, 33]. In recent years the focus partially shifted to the Dirac operators with singular interactions. While for Schrödinger operators quadratic forms are a convenient tool to define the underlying Hamiltonian [22], in the Dirac setting more subtle techniques are necessary due to the lack of semi-boundedness. The case of one-dimensional Dirac operators with point-interactions is well understood [2,20,24,39,55]. Three-dimensional Dirac operators with singular interactions supported on surfaces are considered in, e.g., [4–6, 8, 9, 12, 13, 30]. Finally, the two-dimensional case without the magnetic interaction has recently been analysed in [14, 57]. The interest to include the magnetic δ-shell interaction stems from applications in modern physics [37, 50, 56]. The closely related model of magnetic links in three dimensions has been recently considered in [58–60].

The approximation of Dirac operators with singular interactions by Dirac operators with regular interactions provides a justification of the idealized model under consideration. In the one-dimensional setting the analysis is performed in [42, 43, 67, 69], whereas a generalization to three dimensions has recently appeared in [48, 49]. In the present manuscript, we modify to our setting some techniques that worked efficiently in the one-dimensional case.

Recall that the action of the two-dimensional free Dirac operator $\mathcal{D}_0$ is given by the differential expression

$$(1.1) \qquad D_0 := -i\sigma \cdot \nabla + m\sigma_3 = -i(\sigma_1\partial_1 + \sigma_2\partial_2) + m\sigma_3,$$

where $m \in \mathbb{R}$ is the mass and σ_1, σ_2 and σ_3 are the Pauli matrices

$$(1.2) \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is self-adjoint on dom $\mathcal{D}_0 := H^1(\mathbb{R}^2; \mathbb{C}^2) \subset L^2(\mathbb{R}^2; \mathbb{C}^2)$ and essentially self-adjoint on $C_c^\infty(\mathbb{R}^2; \mathbb{C}^2)$. The spectrum of $\mathcal{D}_0$ is purely absolutely continuous and

$$\sigma(\mathcal{D}_0) = \sigma_{ac}(\mathcal{D}_0) = (-\infty, -|m|] \cup [|m|, +\infty).$$

The interaction under consideration will be supported on the boundary $\Sigma := \partial\Omega$ of a C^∞-smooth bounded simply connected open set $\Omega \subset \mathbb{R}^2$. The curve Σ splits the Euclidean space into a disjoint union, that is, $\mathbb{R}^2 = \Omega_+ \cup \Sigma \cup \Omega_-$, where $\Omega_+ := \Omega$ and

$\Omega_- := \mathbb{R}^2 \setminus \overline{\Omega}_+$. We will call the curve Σ a *shell*. Let us denote the outer unit normal to Ω_+ and the unit vector tangent to the boundary Σ in $x \in \Sigma$ by $\mathbf{n} \equiv (n_1, n_2) = \mathbf{n}(x)$ and $\mathbf{t} \equiv (t_1, t_2) = \mathbf{t}(x)$, respectively. For definiteness, we put $t_1 = -n_2$ and $t_2 = n_1$. For any $\mathbb{C}^2$-valued function f defined on $\mathbb{R}^2$, we set $f_\pm = f \upharpoonright \Omega_\pm$. When it is defined in a suitable sense, we denote $\mathcal{T}_\pm^D f_\pm$ the Dirichlet trace of $f_\pm$ at Σ, and we define the distribution $\delta_\Sigma f$ by

$$\langle \delta_\Sigma f, \varphi \rangle := \int_\Sigma \frac{1}{2}(\mathcal{T}_+^D f_+ + \mathcal{T}_-^D f_-) \cdot \varphi \, ds \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^2; \mathbb{C}^2),$$

where ds means integration with respect to the arc-length of Σ.

 We are interested in the Dirac operator in $L^2(\mathbb{R}^2; \mathbb{C}^2)$ given by the formal expression

$$(1.3) \qquad -i\sigma \cdot (\nabla + i(\lambda \mathbf{t} + \omega \mathbf{n})\delta_\Sigma) + (\eta \mathbb{I}_2 + \tau\sigma_3)\delta_\Sigma + m\sigma_3$$
$$= D_0 + \big(\eta \mathbb{I}_2 + \tau\sigma_3 + \lambda(\sigma \cdot \mathbf{t}) + \omega(\sigma \cdot \mathbf{n})\big)\delta_\Sigma,$$

where $\eta, \tau, \lambda, \omega$ are smooth real-valued functions, and where we used the notation

$$(1.4) \qquad\qquad\qquad \sigma \cdot A := \sigma_1 A_1 + \sigma_2 A_2 \quad \text{for } A \in \mathbb{C}^2.$$

For any given $x \in \Sigma$, the matrices $\mathbb{I}_2, \sigma_3, \sigma \cdot \mathbf{t}(x)$, and $\sigma \cdot \mathbf{n}(x)$ constitute a basis of the Hermitian 2×2 matrices, so at every point there is the most general Hermitian matrix as a coefficient of δ_Σ. The *electrostatic δ-shell interaction* $\eta \mathbb{I}_2 \delta_\Sigma$ and the *Lorentz scalar δ-shell interaction* $\tau\sigma_3 \delta_\Sigma$ describe a distribution of charges and masses on the curve Σ, respectively. The novelty in our treatment is the *magnetic δ-shell interaction* $\lambda(\sigma \cdot \mathbf{t})\delta_\Sigma$, which describes a magnetic field supported on Σ. We remark that the vector potential associated with the latter interaction is given by $\mathbf{A}_\Sigma = \lambda(t_1\delta_\Sigma, t_2\delta_\Sigma)$, and we will show in Appendix B that the underlying magnetic field is given by $B_\Sigma = \lambda \partial_\mathbf{n} \delta_\Sigma$, where $\partial_\mathbf{n} \delta_\Sigma$ stands for the double layer distribution. Finally, we prove that, under some restrictions on the parameters $\eta, \tau, \lambda, \omega$, the interaction term $\omega(\sigma \cdot \mathbf{n})\delta_\Sigma$ can be gauged away in the spirit of [47,54]; see Theorem 2.1 for details. In particular, this term can be always gauged away when all parameters are constant. Due to this observation, we may focus on the case when $\omega = 0$ and the other parameters are smooth real-valued functions.

 The self-adjoint operator $\mathcal{D}_{\eta,\tau,\lambda}$ associated with the formal expression (1.3) with $\omega = 0$ is constructed in Section 6 rigorously as a self-adjoint extension of the symmetric operator with infinite deficiency indices acting in the Hilbert space $L^2(\mathbb{R}^2; \mathbb{C}^2)$, and it is given by

$$Sf = (-i\sigma \cdot \nabla + m\sigma_3)f, \quad \text{dom } S = H_0^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2).$$

To this end, we build an ordinary boundary triple for S^*, which is a modification of the boundary triple constructed in [14]. This modification is necessary to treat the magnetic δ-shell interaction. In this construction the operator $\mathcal{D}_{\eta,\tau,\lambda}$ acts as D_0 on $\Omega_\pm$ and it is subject to the local boundary conditions on Σ, which involves the parameters η, τ, λ, the tangential vector $\mathbf{t}$, and the normal vector $\mathbf{n}$. The construction boils the question of self-adjointness of $\mathcal{D}_{\eta,\tau,\lambda}$ down to self-adjointness of a certain first-order pseudo-differential operator on Σ. The latter is shown by conventional techniques under the condition

$$(1.5) \qquad\qquad \left(\frac{\eta^2 - \tau^2 - \lambda^2}{4} - 1\right)^2 - \lambda^2 \neq 0 \quad \text{everywhere on } \Sigma.$$

The case when the above expression on the left-hand side vanishes is called critical and needs special treatment. In the present paper we cover in Section 7 the special sub-case of purely magnetic critical shell interaction ($\eta = \tau = 0$ and $\lambda = \pm 2$), in which case $\mathcal{D}_{0,0,\pm 2}$ is defined as a self-adjoint operator by a different and more direct method. It is also remarkable that the condition $\eta^2 - \tau^2 - \lambda^2 = -4$ is necessary and sufficient for the confinement to take place, where by confinement we understand that the operator $\mathcal{D}_{\eta,\tau,\lambda}$ can be decomposed into the orthogonal sum with respect to the decomposition $L^2(\mathbb{R}^2; \mathbb{C}^2) = L^2(\Omega_+; \mathbb{C}^2) \oplus L^2(\Omega_-; \mathbb{C}^2)$; cf. Section 2.3 for details. In particular, the choice $\eta = \tau = 0$ and $\lambda = \pm 2$ gives rise to *zig-zag* boundary conditions.

Finally, we will find in Section 8 approximations by regular potentials in the strong resolvent sense for the Dirac operator with δ-shell potentials in the non-critical and non-confining case, i.e., when $\eta^2 - \tau^2 - \lambda^2 \neq -4$ everywhere on Σ and (1.5) holds true. Approximations of δ-shell interactions by more realistic regular potentials provide a justification for the idealized models, which is necessary to understand the physical nature of the rather abstract δ-shell interactions. Mathematical consequences of the strong resolvent convergence comprise the strong convergence of the corresponding unitary propagators and the principle of non-expansion of the spectrum of the limit operator; see Section VIII.7 of [63] for details. Finally, the strong resolvent convergence is necessary to the norm resolvent convergence. Proving it would be a stronger result and our results are a step towards it. In this paper, we explicitly construct regular symmetric potentials $\mathbb{V}_{\eta,\tau,\lambda;\varepsilon} \in L^\infty(\mathbb{R}^2; \mathbb{C}^{2\times 2})$ supported on a tubular ε-neighbourhood of Σ and such that

$$\mathbb{V}_{\eta,\tau,\lambda;\varepsilon} \xrightarrow{\varepsilon \to 0} (\eta \mathbb{I}_2 + \tau \sigma_3 + \lambda (\sigma \cdot \mathbf{t})) \delta_\Sigma \quad \text{in the sense of distributions,}$$

and we investigate the strong resolvent limit of $\mathcal{D}_0 + \mathbb{V}_{\eta,\tau,\lambda;\varepsilon}$ as $\varepsilon \to 0$. It turns out that $\mathcal{D}_0 + \mathbb{V}_{\eta,\tau,\lambda;\varepsilon} \xrightarrow{\varepsilon \to 0} \mathcal{D}_{\hat{\eta},\hat{\tau},\hat{\lambda}}$ for appropriate $\hat{\eta}, \hat{\tau}, \hat{\lambda} \in C^\infty(\Sigma; \mathbb{R})$ that are in general different from the starting η, τ, λ, but are expressed explicitly in terms of them. This phenomenon was observed firstly in the one-dimensional case [67] and then in the three-dimensional setting [49] – we say in this situation that a *renormalization of the coupling constants* occurs.

We finish this introduction by pointing out that when concluding the preparation of this manuscript, we learnt that the three-dimensional analogue of the magnetic δ-shell interaction introduced here was being considered simultaneously and independently in [18]; the reader may see Section 9 for more details.

Organization of the paper

In Section 2 we formulate and discuss all the main results of the present paper. Section 3 contains preliminary material that is used throughout the paper. In Section 4 we obtain spectral relations for the point spectrum of the Dirac operator with δ-shell interactions under special transforms of the interaction strengths and, moreover, we show how the fourth interaction $\omega(\sigma \cdot \mathbf{n}) \delta_\Sigma$ can be eliminated by a properly constructed unitary transform. Further, in Section 5 we provide a condition on the interaction strengths, which gives the confinement. In Section 6 we analyse the non-critical case, prove the self-adjointness of the underlying Dirac operator and obtain its basic spectral properties. The self-adjointness and spectral properties of the Dirac operator with purely magnetic critical

interaction are investigated in Section 7. In Section 8 we construct strong resolvent approximations of Dirac operators with δ-shell interactions by sequences of Dirac operators with suitably scaled regular potentials. Possible generalization for higher dimensions is briefly discussed in Section 9.

The paper is complemented by two appendices. In Appendix A we focus on exponentials of 2×2 matrices of a special structure. Finally, in Appendix B we compute the magnetic field associated with the magnetic δ-shell interaction.

2. Main results

We briefly discuss here the main results of this paper, referring to the various sections below for more detailed results. For an open set $\Omega \subset \mathbb{R}^2$, we define

$$(2.1) \qquad H(\sigma, \Omega) := \{ f \in L^2(\Omega; \mathbb{C}^2) \mid \sigma \cdot \nabla f \in L^2(\Omega; \mathbb{C}^2) \}.$$

For any

$$f = f_+ \oplus f_- \in H(\sigma, \Omega_+) \oplus H(\sigma, \Omega_-) \subset L^2(\Omega_+; \mathbb{C}^2) \oplus L^2(\Omega_-; \mathbb{C}^2) \equiv L^2(\mathbb{R}^2; \mathbb{C}^2),$$

it was shown in [16] that $f_\pm$ admit Dirichlet traces $\mathcal{T}_\pm^D f_\pm$ in $H^{-1/2}(\Sigma; \mathbb{C}^2)$, see Section 3.5 for details.

Given $\eta, \tau, \lambda, \omega \in C^\infty(\Sigma; \mathbb{R})$, we define

$$(2.2) \quad \mathrm{dom}(\mathcal{D}_{\eta,\tau,\lambda,\omega}) := \big\{ f = f_+ \oplus f_- \in H(\sigma, \Omega_+) \oplus H(\sigma, \Omega_-) :$$
$$i(\sigma \cdot \mathbf{n})(\mathcal{T}_-^D f_- - \mathcal{T}_+^D f_+) = \tfrac{1}{2}(\eta \mathbb{I}_2 + \tau \sigma_3 + \lambda(\sigma \cdot \mathbf{t}) + \omega(\sigma \cdot \mathbf{n}))(\mathcal{T}_-^D f_- + \mathcal{T}_+^D f_+) \big\}$$

and

$$(2.3) \qquad \mathcal{D}_{\eta,\tau,\lambda,\omega} f := D_0 f_+ \oplus D_0 f_- \quad \text{for all } f \in \mathrm{dom}(\mathcal{D}_{\eta,\tau,\lambda,\omega}).$$

In (2.2) the condition on $\mathcal{T}_\pm^D f_\pm$ is understood in $H^{-1/2}(\Sigma; \mathbb{C}^2)$. By means of an integration by parts, it can be seen (see, e.g., [52]) that $\mathcal{D}_{\eta,\tau,\lambda,\omega}$ is the operator representing the formal differential expression (1.3).

Since most of this article focuses on the case $\omega = 0$, due to the results presented in Section 4, for the sake of brevity, we also set $\mathcal{D}_{\eta,\tau,\lambda} := \mathcal{D}_{\eta,\tau,\lambda,0}$, i.e.,

$$(2.4) \quad \mathrm{dom}(\mathcal{D}_{\eta,\tau,\lambda}) := \big\{ f = f_+ \oplus f_- \in H(\sigma, \Omega_+) \oplus H(\sigma, \Omega_-) :$$
$$i(\sigma \cdot \mathbf{n})(\mathcal{T}_-^D f_- - \mathcal{T}_+^D f_+) = \tfrac{1}{2}(\eta \mathbb{I}_2 + \tau \sigma_3 + \lambda(\sigma \cdot \mathbf{t}))(\mathcal{T}_-^D f_- + \mathcal{T}_+^D f_+) \big\}$$

and

$$(2.5) \qquad \mathcal{D}_{\eta,\tau,\lambda} f := D_0 f_+ \oplus D_0 f_- \quad \text{for all } f \in \mathrm{dom}(\mathcal{D}_{\eta,\tau,\lambda}).$$

Finally, we denote

$$(2.6) \qquad d := \eta^2 - \tau^2 - \lambda^2 \in C^\infty(\Sigma; \mathbb{R}).$$

2.1. Reduction to $\omega = 0$

In Section 4, we will prove the following result.

Theorem 2.1. *Given* $\omega \in \mathbb{R}$ *and* η, τ, $\lambda \in C^\infty(\Sigma; \mathbb{R})$ *such that* $d := \eta^2 - \tau^2 - \lambda^2$ *is a constant function on* Σ, *let* X *be a solution to*

$$(2.7) \qquad dX^2 - 4 + (4 + \omega^2 - d)X = 0$$

and

$$(2.8) \qquad z := \frac{dX^2 + 4}{X(4 + d - \omega^2 + 4\omega i)}.$$

Then $X \in \mathbb{R} \setminus \{0\}$, $z \in \mathbb{C}$ *satisfies* $|z| = 1$, *and* $\mathcal{D}_{\eta,\tau,\lambda,\omega} = U_z \mathcal{D}_{X\eta,X\tau,X\lambda,0} U_{\bar{z}}$, *where*

$$U_z \varphi := \chi_{\Omega_+} \varphi + z \chi_{\Omega_-} \varphi \quad (\text{for all } \varphi \in L^2(\mathbb{R}^2; \mathbb{C}^2))$$

is unitary in $L^2(\mathbb{R}^2; \mathbb{C}^2)$.

Roughly speaking, Theorem 2.1 implies that a spectral study for $\mathcal{D}_{\eta,\tau,\lambda} = \mathcal{D}_{\eta,\tau,\lambda,0}$ suffices to treat the general case $\mathcal{D}_{\eta,\tau,\lambda,\omega}$, hence the formal term $\omega(\sigma \cdot \mathbf{n})\delta_\Sigma$ in the δ-shell interaction is indeed superfluous. In a classical (absolutely continuous) framework, one would say that this term can be gauged away. This is reminiscent of a similar effect for magnetic potentials in the Coulomb gauge, see Remark 1.5 of [26], and [21, 27, 35, 36]. In Section 4 we show that the unitary transform U_z can always be taken different from the identity except for the case $(d, \omega) = (-4, 0)$, which corresponds to confining δ-shell interactions, see Section 5. In particular, $\mathcal{D}_{\eta,\tau,\lambda,\omega}$ never yields confinement if $(d, \omega) \neq (-4, 0)$.

At the end of Section 4 we find some isospectral transformations as a byproduct of our result, and we describe the charge conjugation properties of the operator $\mathcal{D}_{\eta,\tau,\lambda}$.

2.2. The non-critical case

We say that we are in the *non-critical* case when (1.5) holds true, i.e., everywhere on Σ,

$$(2.9) \qquad \mathfrak{C}(\eta, \tau, \lambda)(x) := \left(\frac{d}{4} - 1\right)^2 - \lambda^2 = \left(\frac{\eta^2 - \tau^2 - \lambda^2}{4} - 1\right)^2 - \lambda^2 \neq 0.$$

In the following theorem we gather the properties of $\mathcal{D}_{\eta,\tau,\lambda}$ in the case when (2.9) holds true. We point out that the non-magnetic case ($\lambda = 0$) has been already treated in Theorem 1.1 of [14] for constant $\eta, \tau \in \mathbb{R}$.

Theorem 2.2. *Let* η, τ, $\lambda \in C^\infty(\Sigma; \mathbb{R})$ *and let* $\mathcal{D}_{\eta,\tau,\lambda}$ *be defined as in* (2.4) *and* (2.5). *Moreover, let either* $d(x) \neq 0$ *for all* $x \in \Sigma$, *or let* η, τ, λ *be constant and such that* $d = 0$. *If* (2.9) *holds true, then* $\mathcal{D}_{\eta,\tau,\lambda}$ *is self-adjoint in* $L^2(\mathbb{R}^2; \mathbb{C}^2)$ *with* dom $\mathcal{D}_{\eta,\tau,\lambda} \subset H^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$. *The essential spectrum of* $\mathcal{D}_{\eta,\tau,\lambda}$ *is*

$$\sigma_{\text{ess}}(\mathcal{D}_{\eta,\tau,\lambda}) = (-\infty, -|m|] \cup [|m|, +\infty),$$

and its discrete spectrum is finite.

The proof of Theorem 2.2 mimics the strategy of the proof of Theorem 1.1 of [14], taking into account the necessary modifications to treat the additional interaction $\lambda(\sigma \cdot \mathbf{t})\delta_\Sigma$. In Section 6, where we also show a Krein-type resolvent formula, we provide an abstract version of the Birman–Schwinger principle and obtain the spectral properties of $\mathcal{D}_{\eta,\tau,\lambda}$.

2.3. Confining δ-shell interactions

If $d = -4$ everywhere on Σ, the phenomenon of *confinement* arises. Physically, this means that a particle initially located in $\Omega_\pm$ can not escape this region during the quantum evolution associated with $\mathcal{D}_{\eta,\tau,\lambda}$. Such phenomenon was firstly noticed in [30] in the three-dimensional setting. We describe the corresponding Hamiltonian in the following theorem, which will be proved in Section 5.

Theorem 2.3. *Let $\eta, \tau, \lambda \in C^\infty(\Sigma; \mathbb{R})$ and let $\mathcal{D}_{\eta,\tau,\lambda}$ be defined as in (2.4) and (2.5). If $d = -4$ everywhere on Σ, then $\mathcal{D}_{\eta,\tau,\lambda}$ decouples into the direct sum*

$$\mathcal{D}_{\eta,\tau,\lambda} = \mathcal{D}^+_{\eta,\tau,\lambda} \oplus \mathcal{D}^-_{\eta,\tau,\lambda},$$

with

$$(2.10) \quad \begin{cases} \mathrm{dom}\,\mathcal{D}^\pm_{\eta,\tau,\lambda} := \big\{ f_\pm \in H(\sigma, \Omega_\pm) : \\ \qquad\qquad [\pm i(\sigma \cdot \mathbf{n}) + \tfrac{1}{2}(\eta\mathbb{I}_2 + \tau\sigma_3 + \lambda\,(\sigma \cdot \mathbf{t}))]\mathcal{T}^D_\pm f_\pm = 0\}, \\ \mathcal{D}^\pm_{\eta,\tau,\lambda} f_\pm := D_0 f_\pm \quad \textit{for all } f_\pm \in \mathrm{dom}\,\mathcal{D}^\pm_{\eta,\tau,\lambda}. \end{cases}$$

Remark 2.4. We expect Theorem 2.3 to hold under weaker regularity assumptions, since its proof follows from purely algebraic considerations, but we are not investigating this in the present manuscript to keep the exposition homogeneous. Recall that we assume that $\eta, \tau, \lambda \in C^\infty(\Sigma; \mathbb{R})$ and $\Sigma = \partial\Omega$ of class C^∞ in order to exploit the theory of pseudo-differential operators, but we expect that these assumptions can be weakened. However, in the case that Σ has only Lipschitz regularity and in the case that η, τ, λ are not regular, different phenomena are expected, see [25, 45, 57], see also [13, 62], where coefficients with lower regularity are considered in the three-dimensional case.

Let us look closer at $\mathcal{D}^+_{\eta,\tau,\lambda}$, the Dirac operator on Ω_+. If $\eta = 0$, $d = -4$ implies that there exists $\theta \in C^\infty(\Sigma; \mathbb{R})$ such that $\sin\theta = -\lambda/2$ and $\cos\theta = \tau/2$. Since $i(\sigma \cdot \mathbf{n})\sigma_3 = \sigma \cdot \mathbf{t}$ and $(\sigma \cdot \mathbf{n})^2 = \mathbb{I}_2$ (see (3.12), (3.13), and (3.14) below), we may rewrite the condition for f_+ in (2.10) as

$$(2.11) \qquad\qquad [\mathbb{I}_2 - \cos\theta(\sigma \cdot \mathbf{t}) - \sin\theta\,\sigma_3]\mathcal{T}^D_+ f_+ = 0.$$

These are the *quantum dot* boundary conditions (see [16, 17, 57] and references therein). In particular, we have the *infinite mass* boundary conditions when $(\eta, \tau, \lambda) = (0, \pm 2, 0)$, and the *zig-zag* boundary conditions when $(\eta, \tau, \lambda) = (0, 0, \pm 2)$. By means of confinement with the electric and Lorentz scalar δ-shell interactions only, it is possible to realise the quantum dot boundary conditions (2.11) for $\theta \in [0, 2\pi) \setminus \{\pi/2, 3\pi/2\}$, that is, all the possible ones except the zig-zag boundary condition, see Remark 4.2 of [14]. Considering also the magnetic δ-shell interaction allows us to describe every Dirac operator on a domain with quantum dot boundary conditions as a Dirac operator with a δ-shell interaction.

For any choice of parameters η, τ, λ such that, everywhere on Σ, $d = -4$ and $(\eta, \tau, \lambda) \neq (0, 0, \pm 2)$, the operator $\mathcal{D}_{\eta,\tau,\lambda}$ is already described in Theorem 2.2. The cases $(\eta, \tau, \lambda) = (0, 0, \pm 2)$ are *critical*, because we have $\mathfrak{C}(\eta, \tau, \lambda) = 0$. They are discussed in the following subsection.

2.4. The critical case

The analogous critical case in the three-dimensional setting was firstly described in [11] and [53] and later in the two-dimensional setting in [14]. In Theorem 1.2 of [14], the case $\mathfrak{C}(\eta, \tau, \lambda) = \lambda = 0$, $\eta, \tau \in \mathbb{R}$ is described, namely, the self-adjointness is proved and the spectral properties are analysed. We complement this result by analysing the case $\mathfrak{C}(\eta, \tau, \lambda) = \eta = \tau = 0$, i.e., we prove the self-adjointness and give a detailed description of the spectrum for purely magnetic critical potentials $\pm 2(\sigma \cdot \mathbf{t}) \delta_\Sigma$.

Theorem 2.5. *The operator $\mathcal{D}_{0,0,\lambda}$ with $\lambda \in \{-2, 2\}$ is self-adjoint. The restriction $\mathcal{D}_{0,0,\lambda} \upharpoonright H^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$ is essentially self-adjoint. We have that $\operatorname{dom} \mathcal{D}_{0,0,\lambda} \not\subset H^s(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$ for any $s > 0$. Finally, the spectrum of $\mathcal{D}_{0,0,\lambda}$ is characterised as follows:*

(i) $\sigma(\mathcal{D}_{0,0,\lambda}) = (-\infty, -|m|] \cup [|m|, \infty)$,

(ii) $\pm m$ *are eigenvalues of infinite multiplicity,*

(iii) *there is a sequence of (embedded) eigenvalues $\{\pm \sqrt{m^2 + \lambda_k}\}_{k \geq 1}$, where λ_k are the eigenvalues of the Dirichlet Laplacian on Ω_+ enumerated in non-decreasing order and counted with multiplicities.*

The proof of Theorem 2.5 is provided in Section 7. We underline that we do not exploit the strategies in [11, 14, 53] but we take advantage of the phenomenon of confinement and decomposition in Theorem 2.3. To show our result we adapt the analysis of the massless Dirac operator on a domain with the zig-zag boundary conditions given in [65] to the case with a mass. This allows us to give a more detailed description. Note that the presence of embedded eigenvalues was already observed in the non-critical confining case in the three-dimensional setting in Theorem 3.7 of [5], Proposition 3.3 of [12].

2.5. Approximation of δ-shell interactions by regular potentials

In the present paper, we find approximations by regular potentials in the strong resolvent sense for the Dirac operator with δ-shell potentials in the non-critical and non-confining case, that is, for $\mathcal{D}_{\eta,\tau,\lambda}$ when $\mathfrak{C}(\eta, \tau, \lambda) \neq 0$ and $d \neq -4$ everywhere on Σ. To this end, for $\eta, \tau, \lambda \in C^\infty(\Sigma; \mathbb{R})$, we construct regular symmetric potentials $\mathbb{V}_{\eta,\tau,\lambda;\varepsilon} \in L^\infty(\mathbb{R}^2; \mathbb{C}^{2 \times 2})$ supported on an ε-neighbourhood of Σ and such that

$$\mathbb{V}_{\eta,\tau,\lambda;\varepsilon} \xrightarrow{\varepsilon \to 0} (\eta \mathbb{I}_2 + \tau \sigma_3 + \lambda(\sigma \cdot \mathbf{t})) \delta_\Sigma \quad \text{in the sense of distributions,}$$

and we investigate the strong resolvent limit of $\mathcal{D}_0 + \mathbb{V}_{\eta,\tau,\lambda;\varepsilon}$ as $\varepsilon \to 0$. It turns out that $\mathcal{D}_0 + \mathbb{V}_{\eta,\tau,\lambda;\varepsilon} \to \mathcal{D}_{\hat\eta,\hat\tau,\hat\lambda}$ as $\varepsilon \to 0$ for appropriate $\hat\eta, \hat\tau, \hat\lambda \in C^\infty(\Sigma; \mathbb{R})$ that are in general different from the starting η, τ, λ.

In the three-dimensional setting [49], the proof of the strong resolvent convergence is an adaptation to the relativistic scenario of the approach used in [7] for the case of

Schrödinger operators with δ-shell interactions. In [7], the co-dimension of the shell is strictly smaller than the order of the differential operator (the Laplacian). As a consequence, the singularities of the kernels of the boundary integral operators used in [7] are weak enough to be controlled uniformly along the approximation procedure. This, in particular, leads to the convergence in the norm resolvent sense in the case of the Schrödinger operator. However, in the case of the Dirac operator, the co-dimension of the shell is exactly the same as the order of the differential operator. This has an important effect on the nature of the corresponding boundary integral operators, which now are singular integral operators instead of compact. Due to this new obstruction with respect to the Schrödinger case, the approach used in [7] was adapted in [49] to the Dirac case to show the convergence in the strong resolvent sense assuming uniform smallness of the approximating potentials. Nevertheless, the question of strong resolvent convergence can also be addressed by more direct methods, which do not require any smallness assumption on the approximating potentials, such as by proving the convergence in the strong graph limit sense and then applying Theorem VIII.26 of [63], which says that, in the self-adjoint setting, the strong graph convergence and the strong resolvent convergence are equivalent.

Originally, this approach was used in the one-dimensional setting [42, 43]. In this way, one can find approximating potentials for any type of δ-potential. The norm resolvent convergence of approximations is harder to tackle. However, since in the one-dimensional case one can perform very explicit calculations with the resolvents of the approximations and the resolvents of their limit operators, it can be proved as well [67, 69]. In the present work, we will modify the ideas of [42] to get the sequence of approximating potentials for general linear combinations of δ-shell interactions, not only for the purely electrostatic or purely Lorentz scalar δ-shell interactions as in [49]. It will converge in the strong resolvent sense, without any smallness assumption on the approximating potentials. We expect that a similar approach can be applied in the three-dimensional case as well.

Definition of $\mathbb{V}_{\eta,\tau,\lambda;\varepsilon}$. In order to describe the approximating potentials $\mathbb{V}_{\eta,\tau,\lambda;\varepsilon}$ explicitly, we will introduce an additional notation, referring to Section 3.2 for details. For $\beta > 0$, $\Sigma_\beta := \{x \in \mathbb{R}^2 \mid \mathrm{dist}(x, \Sigma) < \beta\}$ is the tubular neighbourhood of Σ of width β. If $\beta > 0$ is sufficiently small, Σ_β is parametrized as

$$\Sigma_\beta = \{x_\Sigma + p\,\mathbf{n}(x_\Sigma) \mid x_\Sigma \in \Sigma,\ p \in (-\beta, \beta)\}.$$

Furthermore, let

$$h \in L^\infty(\mathbb{R}; \mathbb{R}), \quad \text{with supp}\, h \subset (-1, 1) \text{ and } \int_{-1}^{1} h(t)\, dt = 1.$$

The function h will determine the transverse profile of $\mathbb{V}_{\eta,\tau,\lambda;\varepsilon}$. For $0 < \varepsilon < \beta$, let

$$h_\varepsilon(p) := \frac{1}{\varepsilon} h\left(\frac{p}{\varepsilon}\right) \quad \text{for all } p \in \mathbb{R}.$$

We have supp $h_\varepsilon \subset (-\varepsilon, \varepsilon)$ and $\lim_{\varepsilon \to 0} h_\varepsilon = \delta_0$ in the sense of the distributions $\mathcal{D}'(\mathbb{R})$, where δ_0 is the Dirac δ-function supported at the origin. For $\eta, \tau, \lambda \in C^\infty(\Sigma; \mathbb{R})$, let

$$(2.12) \qquad B_{\eta,\tau,\lambda} \in C^\infty(\Sigma; \mathbb{C}^{2\times 2}), \quad B_{\eta,\tau,\lambda}(x_\Sigma) := (\eta \mathbb{I}_2 + \tau \sigma_3 + \lambda(\sigma \cdot \mathbf{t}))(x_\Sigma).$$

The function $B_{\eta,\tau,\lambda}$ will encode the matrix structure of the approximating potentials; for all $x_\Sigma \in \Sigma$, the matrix $B_{\eta,\tau,\lambda}(x_\Sigma)$ is symmetric. Finally, for any $\varepsilon \in (0, \beta)$, we define the symmetric approximating potentials $\mathbb{V}_{\eta,\tau,\lambda;\varepsilon} \in L^\infty(\mathbb{R}^2; \mathbb{C}^{2\times 2})$ as follows:

$$(2.13) \qquad \mathbb{V}_{\eta,\tau,\lambda;\varepsilon}(x) := \begin{cases} B_{\eta,\tau,\lambda}(x_\Sigma)h_\varepsilon(p) & \text{if } x = x_\Sigma + p\,\mathbf{n}(x_\Sigma) \in \Sigma_\beta, \\ 0 & \text{if } x \in \mathbb{R}^2 \setminus \Sigma_\beta. \end{cases}$$

It is easy to see that $\lim_{\varepsilon\to 0} \mathbb{V}_{\eta,\tau,\lambda;\varepsilon} = B_{\eta,\tau,\lambda}\delta_\Sigma$ in $\mathcal{D}'(\mathbb{R}^2; \mathbb{C}^{2\times 2})$.

For $0 < \varepsilon < \beta$, we define the family of Dirac operators $\{\mathcal{E}_{\eta,\tau,\lambda;\varepsilon}\}_\varepsilon$ as follows:

$$(2.14) \qquad \begin{aligned} \operatorname{dom}\mathcal{E}_{\eta,\tau,\lambda;\varepsilon} &:= \operatorname{dom}\mathcal{D}_0 = H^1(\mathbb{R}^2; \mathbb{C}^2), \\ \mathcal{E}_{\eta,\tau,\lambda;\varepsilon}\psi &:= \mathcal{D}_0\psi + \mathbb{V}_{\eta,\tau,\lambda;\varepsilon}\psi \quad \text{for all } \psi \in \operatorname{dom}\mathcal{E}_{\eta,\tau,\lambda;\varepsilon}. \end{aligned}$$

Since $\mathbb{V}_{\eta,\tau,\lambda;\varepsilon}$ are bounded and symmetric, the operators $\mathcal{E}_{\eta,\tau,\lambda;\varepsilon}$ are self-adjoint.

We can now state the main result of this subsection.

Theorem 2.6. *Let either $\eta, \tau, \lambda \in C^\infty(\Sigma; \mathbb{R})$ be such that $d(x) \neq k^2\pi^2$, for all $k \in \mathbb{N}_0$ and for all $x \in \Sigma$, or $\eta, \tau, \lambda \in \mathbb{R}$ be such that $d = (2k_0)^2\pi^2$ for some $k_0 \in \mathbb{N}_0$. Let $\hat{\eta}, \hat{\tau}, \hat{\lambda} \in C^\infty(\Sigma; \mathbb{R})$ be defined as follows:*

$$(2.15) \qquad (\hat{\eta}, \hat{\tau}, \hat{\lambda}) = \frac{\tan(\sqrt{d}/2)}{\sqrt{d}/2}(\eta, \tau, \lambda) \qquad \text{if } d > 0,$$

$$(2.16) \qquad (\hat{\eta}, \hat{\tau}, \hat{\lambda}) = (\eta, \tau, \lambda) \qquad\qquad \text{if } d = 0,$$

$$(2.17) \qquad (\hat{\eta}, \hat{\tau}, \hat{\lambda}) = \frac{\tanh(\sqrt{-d}/2)}{\sqrt{-d}/2}(\eta, \tau, \lambda) \quad \text{if } d < 0.$$

Let $\mathcal{E}_{\eta,\tau,\lambda;\varepsilon}$ be defined as in (2.14) and let $\mathcal{D}_{\hat{\eta},\hat{\tau},\hat{\lambda}}$ be defined as in (2.4) and (2.5). If $\mathfrak{C}(\hat{\eta}, \hat{\tau}, \hat{\lambda})(x) \neq 0$ for all $x \in \Sigma$, then

$$\mathcal{E}_{\eta,\tau,\lambda;\varepsilon} \xrightarrow{\varepsilon\to 0} \mathcal{D}_{\hat{\eta},\hat{\tau},\hat{\lambda}} \quad \text{in the strong resolvent sense.}$$

The proof of Theorem 2.6 is in Section 8.

Remark 2.7. In the case $d = 0$, the phenomenon of renormalization of the coupling constants does not occur. This was already observed in the one-dimensional setting in [69].

Remark 2.8. Thanks to Theorem 2.6, for all $\eta \in \mathbb{R} \setminus \{(2k + 1)\pi, (1/2 + k)\pi \mid k \in \mathbb{Z}\}$, $\tau \in \mathbb{R}$, we have $\mathcal{E}_{\eta,0,0;\varepsilon} \to \mathcal{D}_{\hat{\eta},0,0}$ and $\mathcal{E}_{0,\tau,0;\varepsilon} \to \mathcal{D}_{0,\hat{\tau},0}$ as $\varepsilon \to 0$, with $\hat{\eta} = 2\tan(\eta/2)$ and $\hat{\tau} = 2\tanh(\tau/2)$. Exactly the same renormalization of the coupling constant appeared in the one-dimensional [67] and three-dimensional [49] cases. The renormalization of the coupling constant for a general one-dimensional relativistic point interaction was investigated in [43] and later in [69], where exactly the same formulae for renormalization as in Theorem 2.6 were discovered. In Theorem 2.6, the relations (2.15), (2.16), and (2.17) between η, τ, λ and $\hat{\eta}, \hat{\tau}, \hat{\lambda}$ descend from the necessity to satisfy condition (8.4), instrumental in the proof.

Remark 2.9. From (2.15)–(2.17),

$$\hat{d} := \hat{\eta}^2 - \hat{\tau}^2 - \hat{\lambda}^2 = \begin{cases} 4\tan^2(\sqrt{d}/2) & \text{if } d > 0, \\ 0 & \text{if } d = 0, \\ -4\tanh^2(\sqrt{-d}/2) & \text{if } d < 0. \end{cases}$$

Since in all the cases $\hat{d} > -4$, Theorem 2.6 does not provide strong convergence to a Dirac operator with a δ-shell causing confinement (see Theorem 2.3). However, we recover the case $\hat{d} = -4$ in the limit $d \to -\infty$. This suggests that it should be possible to get the confining cases by means of an approximation procedure in which we choose the coefficients $\eta = \eta_\varepsilon$, $\tau = \tau_\varepsilon$ and $\lambda = \lambda_\varepsilon$ dependent on the parameter ε so that the associated parameter $\hat{d} = \hat{d}_\varepsilon$ satisfies $\hat{d}_\varepsilon > -4$ and $\hat{d}_\varepsilon \to -4$ uniformly in the limit $\varepsilon \to 0$. Finally, the fact that $\hat{d} > -4$ is not a limitation, since for $\hat{\eta}, \hat{\tau}, \hat{\lambda} \in C^\infty(\Sigma; \mathbb{R})$ such that $\hat{d} < -4$, $\mathcal{D}_{\hat{\eta}, \hat{\tau}, \hat{\lambda}}$ is unitarily equivalent to $\mathcal{D}_{\tilde{\eta}, \tilde{\tau}, \tilde{\lambda}}$, for $\tilde{\eta}, \tilde{\tau}, \tilde{\lambda} \in C^\infty(\Sigma; \mathbb{R})$ such that $\tilde{d} := \tilde{\eta}^2 - \tilde{\tau}^2 - \tilde{\lambda}^2 > -4$, see Section 4.2.

As a consequence of Theorem 2.6, we get the second result of this section.

Corollary 2.10. *Let* $\hat{\eta}, \hat{\tau}, \hat{\lambda} \in C^\infty(\Sigma; \mathbb{R})$ *be such that* $\hat{d} := \hat{\eta}^2 - \hat{\tau}^2 - \hat{\lambda}^2 > -4$ *and* $\mathbb{C}(\hat{\eta}, \hat{\tau}, \hat{\lambda}) \neq 0$ *everywhere on* Σ. *Let* $\eta, \tau, \lambda \in C^\infty(\Sigma; \mathbb{R})$ *be defined as follows:*

- *If* $\hat{d} > 0$*, then*

$$(\eta, \tau, \lambda) = \frac{\arctan\sqrt{\hat{d}}/2 + k\pi}{\sqrt{\hat{d}}/2}(\hat{\eta}, \hat{\tau}, \hat{\lambda}) \quad \text{for } k \in \mathbb{Z}.$$

- *If* $\hat{d} = 0$*, and* $\hat{\eta}, \hat{\tau}$ *and* $\hat{\lambda}$ *are constant, then*

$$(\eta, \tau, \lambda) = (\hat{\eta}, \hat{\tau}, \hat{\lambda}).$$

In particular, if $\hat{\eta} = \hat{\tau} = \hat{\lambda} = 0$*, then* $(\eta, \tau, \lambda) = (\hat{\eta}, \hat{\tau}, \hat{\lambda})$ *or* $\eta, \tau, \lambda \in \mathbb{R}$ *are such that* $\eta^2 - \tau^2 - \lambda^2 = (2k\pi)^2$*, for* $k \in \mathbb{N}$.

- *If* $-4 < \hat{d} < 0$*, then*

$$(\eta, \tau, \lambda) = \frac{\operatorname{arctanh}\sqrt{-\hat{d}}/2}{\sqrt{-\hat{d}}/2}(\hat{\eta}, \hat{\tau}, \hat{\lambda}).$$

Then

$$\mathcal{E}_{\eta, \tau, \lambda; \varepsilon} \xrightarrow{\varepsilon \to 0} \mathcal{D}_{\hat{\eta}, \hat{\tau}, \hat{\lambda}} \quad \text{in the strong resolvent sense.}$$

The proof of Corollary 2.10 is also given in Section 8.

Remark 2.11. In the case $\hat{d} \geq 0$, the correspondence $(\hat{\eta}, \hat{\tau}, \hat{\lambda}) \mapsto (\eta, \tau, \lambda)$ is not one-to-one. One can choose the coupling constants in the approximating potentials arbitrarily large and still end up with the same limit operator. From the physical perspective, we suppose that this surprising behaviour is possible due to the *Klein effect* (also called the Klein paradox). Usually, the Klein effect is related to the scattering on the electrostatic barrier

when, speaking vaguely, the transmission coefficient does not depend on the height of the barrier monotonously, see [31] for an overview. Clearly, this effect occurs for the pure electrostatic interaction, for which $\hat{d} > 0$. On the other hand, one can push $\hat{d}$ below zero, and thus eliminate the Klein effect, by switching sufficiently strong Lorentz scalar/magnetic fields on.

We conclude the presentation of our results by underlining that, in the case $\hat{\eta} = \hat{\tau} = 0$ and $\hat{\lambda} \in \mathbb{R} \setminus \{\pm 2\}$, it is possible to give a simple direct proof of Corollary 2.10, constructing an alternative sequence of approximations without making use of "parallel coordinates", see Section 8.1.

3. Preliminaries

In order to prove our results we need to introduce a number of mathematical objects and related results. First, we discuss in Sections 3.1 and 3.2 planar curves and their tubular neighbourhoods. Then in Section 3.3 we provide some identities related to Pauli matrices. Further, we give in Section 3.4 basic ideas on the Sobolev spaces and pseudo-differential operators on Σ. Then we recall in Section 3.5 the concept of the trace operator. After that, we outline in Section 3.6 the approach of boundary triples to the extensions theory of symmetric operators. Finally, we recall some properties of the free Dirac operator in Section 3.7, and define several associated auxiliary integral operators on Σ in Section 3.8. In this preliminary section, we partially follow the presentation in [14], which gives the theoretical background and the technical instruments for our analysis. We refer to it and the references therein for the proofs of the results in this section and for further details.

3.1. Tangent, normal and curvature of Σ

We gather here some elementary facts on curves, in order to fix the notations. Details can be found, e.g., in [1].

We recall that $\Omega \subset \mathbb{R}^2$ is a bounded open simply connected set with C^∞ boundary $\Sigma := \partial\Omega$. Set $\ell := |\Sigma|$ and let $\gamma : \mathbb{R}/\ell\mathbb{Z} \to \Sigma \subset \mathbb{R}^2$ be a smooth arc-length parametrization of Σ with positive orientation. Let

$$(3.1) \qquad \mathbf{t}_\gamma : \mathbb{R}/\ell\mathbb{Z} \to \mathbb{R}^2, \quad \mathbf{t}_\gamma(s) = \dot{\gamma}(s),$$

$$(3.2) \qquad \mathbf{n}_\gamma : \mathbb{R}/\ell\mathbb{Z} \to \mathbb{R}^2, \quad \mathbf{n}_\gamma(s) = (\dot{\gamma}_2(s), -\dot{\gamma}_1(s)),$$

where the dot stands for the derivative with respect to the arc-length s. It is clear that $\{\mathbf{n}_\gamma(s), \mathbf{t}_\gamma(s)\}$ is a positively oriented basis of $\mathbb{R}^2$ for any $s \in \mathbb{R}/\ell\mathbb{Z}$. Moreover, by the Frenet–Serret formulas, there exists function κ_γ, called the signed curvature, such that

$$(3.3) \qquad \dot{\mathbf{t}}_\gamma = -\kappa_\gamma \mathbf{n}_\gamma, \quad \dot{\mathbf{n}}_\gamma = \kappa_\gamma \mathbf{t}_\gamma.$$

Therefore, we have

$$(3.4) \qquad |\kappa_\gamma(s)| = \|\dot{\mathbf{t}}_\gamma(s)\| = \|\ddot{\gamma}(s)\|.$$

We set $\mathbf{t} := \mathbf{t}_\gamma \circ \gamma^{-1}, \mathbf{n} := \mathbf{n}_\gamma \circ \gamma^{-1} \colon \Sigma \to \mathbb{R}^2$, and $\kappa := \kappa_\gamma \circ \gamma^{-1} \colon \Sigma \to \mathbb{R}$. The functions $\mathbf{t}, \mathbf{n}, \kappa$ are independent of the particular choice of the positively oriented arc-length parametrization γ; $\mathbf{n}$ is the unit normal vector field along Σ which points outwards of Ω_+, $\mathbf{t}$ is the unit tangent vector field along Σ, counter-clockwise oriented, and κ is the signed curvature of Σ. We remark that we choose the definition of the curvature κ so that it is non-negative for convex domains Ω.

3.2. Tubular neighbourhoods of Σ

Below we recall some elementary properties of tubular neighbourhoods of planar curves. Details can be found in Chapter 1 of [34], and [7, 46], see also Sections 1.6 and 2.2 of [1]. For $\beta > 0$,

$$\Sigma_\beta := \{x \in \mathbb{R}^2 \mid \operatorname{dist}(x, \Sigma) < \beta\}$$

is the tubular neighbourhood of Σ of width β. Let us introduce the following mapping:

$$\mathcal{L}_\gamma : \mathbb{R}/\ell\mathbb{Z} \times (-\beta, \beta) \to \mathbb{R}^2, \quad \mathcal{L}_\gamma(s, p) = \gamma(s) + p\,\mathbf{n}_\gamma(s).$$

The following theorem shows that, for all β small enough, the map $\mathcal{L}_\gamma$ is a smooth parametrization of Σ_β.

Theorem 3.1 (Theorem 2.2.5 of [1]). *There exists $\beta_0 \in (0, (\max |\kappa_\gamma|)^{-1})$ such that, for all $\beta \in (0, \beta_0)$, $\mathcal{L}_\gamma$ is a bijection of $\mathbb{R}/\ell\mathbb{Z} \times (-\beta, \beta)$ onto Σ_β.*

In the following we will always assume that $0 < \beta < \beta_0$, for $\beta_0 > 0$ given by Theorem 3.1. Thanks to the second formula in (3.3), we get

$$(3.5) \qquad \nabla\mathcal{L}_\gamma(s, p) = \big(\partial_s \mathcal{L}_\gamma(s, p) \quad \partial_p \mathcal{L}_\gamma(s, p)\big) = \big((1 + p\kappa_\gamma(s))\mathbf{t}_\gamma(s) \quad \mathbf{n}_\gamma(s)\big),$$

where $\partial_s \mathcal{L}_\gamma$, $\partial_p \mathcal{L}_\Gamma$, $\mathbf{t}_\gamma$, and $\mathbf{n}_\gamma$ should be understood as column vectors. Thanks to (3.2), we obtain

$$(3.6) \quad \det(\nabla\mathcal{L}_\gamma)(s, p) = \det \begin{pmatrix} (1 + p\kappa_\gamma(s))t_1(\gamma(s)) & +t_2(\gamma(s)) \\ (1 + p\kappa_\gamma(s))t_2(\gamma(s)) & -t_1(\gamma(s)) \end{pmatrix} = -(1 + p\kappa_\gamma(s)).$$

We remark that $\det(\nabla\mathcal{L}_\gamma)(s, p) < 0$ for all $(s, p) \in \mathbb{R}/\ell\mathbb{Z} \times (-\beta, \beta)$, since $|p\kappa_\gamma(s)| < \beta\beta_0^{-1} < 1$. Thus, we have

$$(3.7) \quad \int_{\Sigma_\beta} f(x)\,dx = \int_0^\ell \int_{-\beta}^\beta f(\gamma(s) + p\,\mathbf{n}_\gamma(s))(1 + p\kappa_\gamma(s))\,dp\,ds \quad \text{for all } f \in L^1(\Sigma_\beta).$$

Next, we define

$$(3.8) \qquad \mathcal{P}_\gamma := (\mathcal{L}_\gamma^{-1})_1 : \Sigma_\beta \to \mathbb{R}/\ell\mathbb{Z}, \quad \mathcal{P}_\gamma(\gamma(s) + p\,\mathbf{n}_\gamma(s)) = s,$$

$$(3.9) \qquad \mathcal{P}_\perp := (\mathcal{L}_\gamma^{-1})_2 : \Sigma_\beta \to (-\beta, \beta), \quad \mathcal{P}_\perp(\gamma(s) + p\,\mathbf{n}_\gamma(s)) = p,$$

Thanks to (3.5), (3.6), and the inverse function theorem, for all $x = \gamma(s) + p\,\mathbf{n}_\gamma(s) \in \Sigma_\beta$, we have that

$$(3.10) \qquad \nabla\mathcal{P}_\gamma(x) = \frac{1}{1 + p\kappa_\gamma(s)}\,\mathbf{t}_\gamma(s), \quad \nabla\mathcal{P}_\perp(x) = \mathbf{n}_\gamma(s).$$

Finally, it is also convenient to define

$$\mathcal{L} : \Sigma \times (-\beta, \beta) \to \mathbb{R}^2, \quad \mathcal{L}(x_\Sigma, p) = \mathcal{L}_\gamma(\gamma^{-1}(x_\Sigma), p) = x_\Sigma + p\,\mathbf{n}(x_\Sigma) \in \Sigma_\beta,$$

which is a bijection of $\Sigma \times (-\beta, \beta)$ onto Σ_β, by Theorem 3.1, and

$$(3.11) \qquad \mathcal{P}_\Sigma := \mathcal{L}_1^{-1} : \Sigma_\beta \to \Sigma, \quad \mathcal{P}_\Sigma(x_\Sigma + p\,\mathbf{n}(x_\Sigma)) = x_\Sigma.$$

3.3. Pauli matrices

By an explicit calculation, one can verify that

$$(3.12) \qquad \sigma_3^2 = (\sigma \cdot \mathbf{n})^2 = (\sigma \cdot \mathbf{t})^2 = \mathbb{I}_2,$$

$$(3.13) \qquad i(\sigma \cdot \mathbf{n})\sigma_3 = \sigma \cdot \mathbf{t},$$

$$(3.14) \qquad (\sigma \cdot \mathbf{n})(\sigma \cdot \mathbf{t}) = i\sigma_3,$$

where in (3.13) and (3.14) we have used the fact that $(t_1, t_2) = (-n_2, n_1)$.

3.4. Sobolev spaces and pseudo-differential calculus on Σ

We denote by $\mathbb{T}$ the torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$; the space of the periodic smooth functions on the torus $\mathbb{T}$ and the space of periodic distributions on the torus $\mathbb{T}$ will be denoted by $\mathcal{D}(\mathbb{T}) = C^\infty(\mathbb{T})$ and $\mathcal{D}(\mathbb{T})'$, respectively. For $f \in \mathcal{D}(\mathbb{T})'$, we define its Fourier coefficients using the duality pairing $\langle \cdot, \cdot \rangle_{\mathcal{D}(\mathbb{T})', \mathcal{D}(\mathbb{T})}$ as follows:

$$\widehat{f}(n) := \langle f, e_{-n} \rangle_{\mathcal{D}(\mathbb{T})', \mathcal{D}(\mathbb{T})}, \quad e_n : t \in \mathbb{T} \mapsto e^{i2\pi nt}.$$

For $s \in \mathbb{R}$, the Sobolev space of order s on $\mathbb{T}$ is defined as

$$H^s(\mathbb{T}) := \Big\{ f \in \mathcal{D}(\mathbb{T})' \ \Big| \ \sum_{n=-\infty}^{+\infty} (1 + |n|)^{2s} |\widehat{f}(n)|^2 < +\infty \Big\}.$$

A linear operator H on $C^\infty(\mathbb{T})$ is a periodic pseudo-differential operator on $\mathbb{T}$ if there exists $h : \mathbb{T} \times \mathbb{Z} \to \mathbb{C}$ such that:

(i)   for all $n \in \mathbb{Z}$, $h(\cdot, n) \in C^\infty(\mathbb{T})$,

(ii) H acts as $Hf = \sum_{n \in \mathbb{Z}} h(\cdot, n) \widehat{f}(n) e_n$,

(iii)   there exists $\alpha \in \mathbb{R}$ such that for all $p, q \in \mathbb{N}_0$, there exists $c_{p,q} > 0$ such that

$$(3.15) \qquad \Big| \Big(\frac{d^p}{dt^p} (\omega^q h) \Big)(t, n) \Big| \leq c_{p,q} (1 + |n|)^{\alpha - q},$$

where the operator ω is defined by $(\omega h)(t, n) := h(t, n+1) - h(t, n)$ for all $(t, n) \in \mathbb{T} \times \mathbb{Z}$.

The number α is called the order of the pseudo-differential operator H. The set of all pseudo-differential operators of order α on $\mathbb{T}$ is denoted Ψ^α, and we define

$$\Psi^{-\infty} := \bigcap_{\alpha \in \mathbb{R}} \Psi^\alpha.$$

Recall that $\ell = |\Sigma|$ and that $\gamma \colon \mathbb{R}/\ell\mathbb{Z} \to \Sigma$ is a smooth arc-length parametrization of Σ. We define the map $U^* \colon \mathcal{D}(\mathbb{T}) \to \mathcal{D}(\Sigma)$ as

$$(U^* g)(x) := \ell^{-1} g(\ell^{-1} \gamma^{-1}(x)), \quad x \in \Sigma,$$

where we have set $\mathcal{D}(\Sigma) := C^\infty(\Sigma)$, and the map $U \colon \mathcal{D}(\Sigma)' \to \mathcal{D}(\mathbb{T})'$ as

(3.16)
$$\langle Uf, g \rangle_{\mathcal{D}(\mathbb{T})', \mathcal{D}(\mathbb{T})} := \langle f, U^* g \rangle_{\mathcal{D}(\Sigma)', \mathcal{D}(\Sigma)}.$$

The Sobolev space of order $s \in \mathbb{R}$ on Σ is defined as

$$H^s(\Sigma) := \{ f \in \mathcal{D}(\Sigma)' \mid Uf \in H^s(\mathbb{T}) \}.$$

For all $s \geq 0$, $H^{-s}(\Sigma) = (H^s(\Sigma))'$ and the duality pairing $\langle \phi, \psi \rangle_{H^{-s}, H^s}$ is defined for all $\phi \in H^{-s}(\Sigma)$, $\psi \in H^s(\Sigma)$.

A linear operator H on $C^\infty(\Sigma)$ is a periodic pseudo-differential operator on Σ of order $\alpha \in \mathbb{R}$ if $H_0 := UHU^{-1} \in \Psi^\alpha$. The set of pseudo-differential operators on Σ of order α is denoted Ψ_Σ^α and we set

$$\Psi_\Sigma^{-\infty} := \bigcap_{\alpha \in \mathbb{R}} \Psi_\Sigma^\alpha.$$

In the next proposition we gather some useful properties of the pseudo-differential operators on Σ (for proofs, see Sections 5.8 and 5.9 of [64]).

Proposition 3.2. *Let $\alpha, \beta \in \mathbb{R}$, $A \in \Psi_\Sigma^\alpha$ and $B \in \Psi_\Sigma^\beta$.*

(i) *For all $s \in \mathbb{R}$, A extends uniquely to a bounded linear operator, denoted by the same letter, from $H^s(\Sigma)$ to $H^{s-\alpha}(\Sigma)$.*

(ii) *We have*

$$A + B \in \Psi_\Sigma^{\max(\alpha, \beta)}, \quad AB \in \Psi_\Sigma^{\alpha + \beta}, \quad [A, B] \in \Psi_\Sigma^{\alpha + \beta - 1}.$$

3.4.1. The operator Λ^α. We describe an example of pseudo-differential operator that is useful for our purposes. For $\alpha \in \mathbb{R}$, consider the operator Λ^α on $C^\infty(\Sigma)$:

(3.17) $\quad \Lambda^\alpha := U^{-1} L^\alpha U, \quad \text{with } L^\alpha u(x) = \sum_{n \in \mathbb{Z}} (1 + |n|)^{\alpha/2} \, \hat{u}(n) \, e_n(t), \quad u \in \mathcal{D}(\mathbb{T}).$

Thanks to (3.15) and (3.17), one can show that $\Lambda^\alpha \in \Psi_\Sigma^{\alpha/2}$. Due to Proposition 3.2 (i), Λ^α extends uniquely to a bounded linear operator from $H^s(\Sigma)$ to $H^{s-\alpha/2}(\Sigma)$, for any $s \in \mathbb{R}$; and such extension is in fact an isomorphism, by the definition of $H^s(\Sigma)$. Of course, Λ^α can also be seen as an unbounded operator on $H^s(\Sigma)$, for all $s \in \mathbb{R}$.

In particular, the operator $\Lambda := \Lambda^1$ is used repeatedly in the paper, and its action on vector-valued functions is understood component-wise. It is useful to remark that for all $\phi, \psi \in L^2(\Sigma)$, we have $\Lambda\phi \in H^{-1/2}(\Sigma)$, $\Lambda^{-1}\psi \in H^{1/2}(\Sigma)$, and

(3.18)
$$\langle \Lambda\phi, \Lambda^{-1}\psi \rangle_{H^{-1/2}, H^{1/2}} = \langle \phi, \psi \rangle_{L^2}.$$

3.5. Trace operators

For any open set $U \subset \mathbb{R}^2$, recall the definition of $H(\sigma, U)$ in (2.1). Thanks to Lemma 2.2 of [16], $H(\sigma, U) \subset L^2(U; \mathbb{C}^2) \cap H^1_{\text{loc}}(U)$, which is a Hilbert space, endowed with the norm

$$\|f\|^2_{H(\sigma, U)} = \|f\|^2_{L^2(U; \mathbb{C}^2)} + \|-i\sigma \cdot \nabla f\|^2_{L^2(U; \mathbb{C}^2)}.$$

We recall that $\Omega \subset \mathbb{R}^2$ is a bounded open simply connected set with C^∞ boundary $\Sigma := \partial\Omega$, and $\mathbb{R}^2 = \Omega_+ \cup \Sigma \cup \Omega_-$, where $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^2 \setminus \overline{\Omega_+}$. Thanks to Lemmata 2.3 and 2.4 of [16] (see also Lemmata 15 and 18 of [3]), the Dirichlet trace operators

$$\mathcal{T}^D_{\pm,0} : H^1(\Omega_\pm; \mathbb{C}^2) \to H^{1/2}(\Sigma; \mathbb{C}^2)$$

extend uniquely to the bounded linear operators

$$\mathcal{T}^D_\pm : H(\sigma, \Omega_\pm) \to H^{-1/2}(\Sigma; \mathbb{C}^2).$$

and the following holds.

Proposition 3.3. *For $f \in H(\sigma, \Omega_\pm)$, $\mathcal{T}^D_\pm f \in H^{1/2}(\Sigma; \mathbb{C}^2)$ if and only if $f \in H^1(\Omega_\pm; \mathbb{C}^2)$.*

3.6. Theory of the boundary triples

In this section we review the theory of the boundary triples, referring to [14, 23, 29, 61] and to the monographs [10, 66] for details.

We start with the definition of a boundary triple for a symmetric operator.

Definition 3.4. Let A be a closed densely defined symmetric operator in a Hilbert space $\mathcal{H}$. Moreover, let $\mathcal{G}$ be another Hilbert space and $\Gamma_0, \Gamma_1 : \text{dom } A^* \to \mathcal{G}$ be linear maps. The triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a *boundary triple* for A^* if and only if

(i) for all $f, g \in \text{dom } A^*$, we have

$$\langle A^* f, g \rangle_\mathcal{H} - \langle f, A^* g \rangle_\mathcal{H} = \langle \Gamma_1 f, \Gamma_0 g \rangle_\mathcal{G} - \langle \Gamma_0 f, \Gamma_1 g \rangle_\mathcal{G},$$

(ii) the map $f \in \text{dom } A^* \mapsto (\Gamma_0 f, \Gamma_1 f) \in \mathcal{G} \times \mathcal{G}$ is surjective.

Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triple for the adjoint A^* of a densely defined closed symmetric operator A on a Hilbert space $\mathcal{H}$. Then $B := A^* \upharpoonright \ker \Gamma_0$ is self-adjoint, and for any $z \in \rho(B)$, one has the direct sum decomposition

$$\text{dom } A^* = \text{dom } B \dotplus \ker(A^* - z) = \ker \Gamma_0 \dotplus \ker(A^* - z).$$

In particular, $\Gamma_0 \upharpoonright \ker(A^* - z)$ is bijective. We define the γ-field G_z and the Weyl function M_z associated to the triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$:

(3.19) $$G_z : z \in \rho(B) \mapsto (\Gamma_0 \upharpoonright \ker(A^* - z))^{-1} : \mathcal{G} \to \mathcal{H},$$

(3.20) $$M_z : z \in \rho(B) \mapsto \Gamma_1 G_z : \mathcal{G} \to \mathcal{G}.$$

For $z \in \rho(B)$, the operators G_z and M_z are bounded, and $z \mapsto G_z$ and $z \mapsto M_z$ are holomorphic on $\rho(B)$. Furthermore, the adjoints of G_z and M_z are given by

$$G_z^* = \Gamma_1 (B - \bar{z})^{-1} \quad \text{and} \quad M_z^* = M_{\bar{z}}.$$

For A a closed densely defined symmetric operator in a Hilbert space $\mathcal{H}$, the knowledge of a boundary triple for the operator A^* allows to move the study of its self-adjoint restrictions and their spectral properties to the (sometimes) easier setting of the Hilbert space $\mathcal{G}$. This is shown in the next proposition, for which we need to introduce some notation. Let $\mathcal{G}_\Pi$ be a closed subspace of $\mathcal{G}$, viewed as a Hilbert space when endowed with the induced inner product. Denote the projection and the canonical embedding as

$$\Pi : \mathcal{G} \to \mathcal{G}_\Pi \quad \text{and} \quad \Pi^* : \mathcal{G}_\Pi \to \mathcal{G},$$

respectively. Let Θ be a linear operator in $\mathcal{G}_\Pi$. We define the operator $B_{\Pi,\Theta} := A^* \upharpoonright \operatorname{dom} B_{\Pi,\Theta}$, where

(3.21) $\quad \operatorname{dom} B_{\Pi,\Theta} := \big\{ f \in \operatorname{dom} A^* \mid (\mathbb{I} - \Pi^*\Pi)\Gamma_0 f = 0,$
$$\Pi\Gamma_0 f \in \operatorname{dom}\Theta, \ \Pi\Gamma_1 f = \Theta\Pi\Gamma_0 f \big\}.$$

Theorem 3.5 (Theorem 2.12 of [14]). *The operator $B_{\Pi,\Theta}$ is* (*essentially*) *self-adjoint in $\mathcal{H}$ if and only if Θ is* (*essentially*) *self-adjoint in $\mathcal{G}$. Furthermore, if Θ is self-adjoint and $z \in \rho(B)$, then the following assertions hold*:

(i) $\quad z \in \sigma(B_{\Pi,\Theta})$ *if and only if* $0 \in \sigma(\Theta - \Pi M_z \Pi^*)$,

(ii) $\quad z \in \sigma_\mathrm{p}(B_{\Pi,\Theta})$ *if and only if* $0 \in \sigma_\mathrm{p}(\Theta - \Pi M_z \Pi^*)$, *in which case we have that* $\ker(B_{\Pi,\Theta} - z) = G_z \Pi^* \ker(\Theta - \Pi M_z \Pi^*)$,

(iii) *for all $z \in \rho(B_{\Pi,\Theta}) \cap \rho(B)$, one has*

$$(B_{\Pi,\Theta} - z)^{-1} = (B - z)^{-1} + G_z \Pi^* (\Theta - \Pi M_z \Pi^*)^{-1} \Pi G_{\bar{z}}^*.$$

3.7. The free Dirac operator

Recall that the free Dirac operator $\mathcal{D}_0$ is defined as follows:

$$\mathcal{D}_0 f := D_0 f, \quad \operatorname{dom}\mathcal{D}_0 := H^1(\mathbb{R}^2; \mathbb{C}^2).$$

For any $z \in \rho(\mathcal{D}_0) = \mathbb{C} \setminus ((-\infty, -|m|] \cup [|m|, +\infty))$, we have

$$(\mathcal{D}_0 - z)^{-1} f(x) = \int_{\mathbb{R}^2} \phi_z(x - y) f(y)\, dy, \quad f \in L^2(\mathbb{R}^2; \mathbb{C}^2),$$

where the Green function ϕ_z is given for $x \neq 0$ by

(3.22) $\quad \phi_z(x) := \dfrac{1}{2\pi} K_0\big(\sqrt{m^2 - z^2}|x|\big)\big(m\sigma_3 + z\mathbb{I}_2\big)$
$$+ i\Big(\sigma \cdot \frac{x}{|x|}\Big)\frac{\sqrt{m^2 - z^2}}{2\pi} K_1\big(\sqrt{m^2 - z^2}|x|\big),$$

the functions K_j are the modified Bessel functions of the second kind of order j, and we are taking the principal square root function, i.e., for $z \in \mathbb{C} \setminus (-\infty, 0]$, $\operatorname{Re}\sqrt{z} > 0$.

We denote by S the restriction of $\mathcal{D}_0$ to the functions vanishing at Σ, i.e.,

(3.23) $\quad Sf = (-i\sigma \cdot \nabla + m\sigma_3)f, \quad \operatorname{dom} S = H_0^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2).$

It is easy to see that S^* is the maximal realization of D_0 in $\mathbb{R}^2 \setminus \Sigma$, i.e., for $f = f_+ \oplus f_- \in L^2(\mathbb{R}^2; \mathbb{C}^2) \equiv L^2(\Omega_+; \mathbb{C}^2) \oplus L^2(\Omega_-; \mathbb{C}^2)$,

$$
(3.24) \quad
\begin{aligned}
\operatorname{dom} S^* &= \{ f = f_+ \oplus f_- \in L^2(\Omega_+; \mathbb{C}^2) \oplus L^2(\Omega_-; \mathbb{C}^2) \mid f_\pm \in H(\sigma, \Omega_\pm) \}, \\
S^* f &= (-i\sigma \cdot \nabla + m\sigma_3) f_+ \oplus (-i\sigma \cdot \nabla + m\sigma_3) f_-.
\end{aligned}
$$

We finally recall some properties of the essential spectrum of any self-adjoint extension of S.

Proposition 3.6 (Propositions 3.8 and 3.9 of [14]). *Let A be a self-adjoint extension of S. Then the following hold*:

(i) $(-\infty, -|m|] \cup [|m|, +\infty) \subset \sigma_{\mathrm{ess}}(A)$,

(ii) *if* $\operatorname{dom} A \subset H^s(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$ *for some $s > 0$, then the spectrum of A in $(-|m|, |m|)$ is purely discrete and finite.*

3.8. Auxiliary integral operators

We introduce now several integral operators related to the Green function ϕ_z.

Let us denote the Dirichlet trace operator in $H^1(\mathbb{R}^2; \mathbb{C}^2)$ on Σ by $\mathcal{T}^D \colon H^1(\mathbb{R}^2; \mathbb{C}^2) \to H^{1/2}(\Sigma; \mathbb{C}^2)$. It is well known that $\mathcal{T}^D$ is bounded, surjective, and $\ker \mathcal{T}^D = H^1_0(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$, see Theorems 3.37 and 3.40 of [51]. For $z \in \rho(\mathcal{D}_0)$, we define

$$
(3.25) \qquad \Phi'_z := \mathcal{T}^D (\mathcal{D}_0 - \bar{z})^{-1} \colon L^2(\mathbb{R}^2; \mathbb{C}^2) \to H^{1/2}(\Sigma; \mathbb{C}^2)
$$

and its anti-dual

$$
(3.26) \qquad \Phi_z := (\mathcal{T}^D (\mathcal{D}_0 - \bar{z})^{-1})' \colon H^{-1/2}(\Sigma; \mathbb{C}^2) \to L^2(\mathbb{R}^2; \mathbb{C}^2),
$$

where the potential operator Φ_z is a bounded bijective operator from $H^{-1/2}(\Sigma; \mathbb{C}^2)$ onto $\ker(S^* - z)$. Moreover, for $\varphi \in L^2(\Sigma; \mathbb{C}^2)$, one has the integral representation

$$
\Phi_z \varphi(x) = \int_\Sigma \phi_z(x - y) \varphi(y) \, ds(y) \quad \text{for a.e. } x \in \mathbb{R}^2 \setminus \Sigma.
$$

We denote by C_Σ the *Cauchy transform* on Σ. To define it, we identify $\mathbb{R}^2 \sim \mathbb{C}$: writing $\mathbb{R}^2 \ni x = (x_1, x_2) \sim x_1 + i x_2 =: \xi \in \mathbb{C}$ and $\mathbb{R}^2 \ni y = (y_1, y_2) \sim y_1 + i y_2 =: \zeta \in \mathbb{C}$. Then we set

$$
(3.27) \qquad C_\Sigma u(\xi) := \frac{i}{\pi} \, \mathrm{p.v.} \int_\Sigma \frac{u(\zeta)}{\xi - \zeta} \, d\zeta \quad \text{for all } u \in C^\infty(\Sigma), \, \xi \in \Sigma,
$$

where the complex line integral is understood as its principal value. Furthermore, let C'_Σ be the operator which satisfies $(C_\Sigma u, v)_{L^2(\Sigma)} = (u, C'_\Sigma v)_{L^2(\Sigma)}$ for all $u, v \in C^\infty(\Sigma)$. The periodic pseudo-differential operators C_Σ, C'_Σ belong to Ψ^0_Σ and give rise to bounded operators in $H^s(\Sigma)$, for all $s \in \mathbb{R}$. Moreover,

$$
(3.28) \qquad C'_\Sigma C_\Sigma - \mathbb{I}, \; C_\Sigma C'_\Sigma - \mathbb{I}, \; C_\Sigma - C'_\Sigma \in \Psi^{-\infty}_\Sigma
$$

(see Proposition 2.9 of [14]) and $C^2_\Sigma = C'^2_\Sigma = \mathbb{I}$ (see Lemma 4.2.3 of [64]).

For $z \in \rho(\mathcal{D}_0)$, we define the boundary integral operator

$$(3.29) \qquad \mathcal{C}_z \, \varphi(x) := \text{p.v.} \int_{\Sigma} \phi_z(x - y) \varphi(y) \, ds(y)$$

$$\text{for a.e. } x \in \Sigma \text{ and for all } \varphi \in C^{\infty}(\Sigma; \mathbb{C}^2).$$

The pseudo-differential operator $\mathcal{C}_z$ belongs to Ψ_{Σ}^0, and, in particular, it gives rise to a bounded operator in $H^s(\Sigma; \mathbb{C}^2)$, for any $s \in \mathbb{R}$. Its realization in $L^2(\Sigma; \mathbb{C}^2)$ satisfies $(\mathcal{C}_z)^* = \mathcal{C}_{\bar{z}}$. Furthermore, for the tangent vector field $\mathbf{t} = (t_1, t_2)$ along Σ, we denote

$$(3.30) \qquad\qquad\qquad\qquad T = t_1 + i t_2.$$

Then one has

$$(3.31) \qquad \mathcal{C}_z = \frac{1}{2} \begin{pmatrix} 0 & C_{\Sigma} \bar{T} \\ T C_{\Sigma}' & 0 \end{pmatrix} + \frac{\ell}{4\pi} \begin{pmatrix} (z+m)\mathbb{I} & 0 \\ 0 & (z-m)\mathbb{I} \end{pmatrix} \Lambda^{-2} + \hat{\Psi},$$

where ℓ is the length of Σ and $\hat{\Psi} \in \Psi_{\Sigma}^{-2}$, see Proposition 3.4 of [14].

The operators Φ_z and $\mathcal{C}_z$ are related to each other by the following relation, analogous in this context to the Plemelj–Sokhotskii formula (see Proposition 3.5 of [14]):

$$\mathcal{T}_{\pm}^D \Phi_z \varphi = \mp \frac{i}{2} (\sigma \cdot \mathbf{n}) \varphi + \mathcal{C}_z \varphi \quad \text{for all } \varphi \in H^{-1/2}(\Sigma; \mathbb{C}^2).$$

4. Unitary equivalences

4.1. Reduction to $\omega = 0$

Recall that the operator $\mathcal{D}_{\eta, \tau, \lambda, \omega}$ is defined as in (2.2), (2.3). The purpose of this section is to show that, in many cases, $\mathcal{D}_{\eta, \tau, \lambda, \omega}$ is unitarily equivalent to $\mathcal{D}_{\tilde{\eta}, \tilde{\tau}, \tilde{\lambda}, 0}$ for certain $\tilde{\eta}$, and $\tilde{\tau}, \tilde{\lambda} \colon \Sigma \to \mathbb{R}$ defined in terms of η, τ, λ, and ω. This unitary equivalence is based on the following simple transformation. Given $z \in \mathbb{C}$ such that $|z| = 1$, let

$$U_z \colon L^2(\mathbb{R}^2; \mathbb{C}^2) \to L^2(\mathbb{R}^2; \mathbb{C}^2), \quad U_z \varphi = \chi_{\Omega_+} \varphi + z \chi_{\Omega_-} \varphi \quad \text{for all } \varphi \in L^2(\mathbb{R}^2; \mathbb{C}^2).$$

It is clear that $(U_z)^* = U_{\bar{z}}$ and, since $z\bar{z} = |z|^2 = 1$, that $(U_z)^* U_z = U_z (U_z)^* = \mathbb{I}_2$. Hence, U_z is a unitary operator in $L^2(\mathbb{R}^2; \mathbb{C}^2)$. With this at hand, we can introduce the operator

$$\text{dom}(\mathcal{D}_{\eta, \tau, \lambda, \omega}^z) := U_{\bar{z}}(\text{dom}(\mathcal{D}_{\eta, \tau, \lambda, \omega})),$$
$$\mathcal{D}_{\eta, \tau, \lambda, \omega}^z f := U_{\bar{z}} \mathcal{D}_{\eta, \tau, \lambda, \omega} U_z f \quad \text{for all } f \in \text{dom}(\mathcal{D}_{\eta, \tau, \lambda, \omega}^z),$$

which is unitarily equivalent to $\mathcal{D}_{\eta, \tau, \lambda, \omega}$ by construction.

Before addressing the proof of Theorem 2.1, let us make some observations on the values of X and z, which were introduced in (2.7) and (2.8), respectively, depending on d and ω.

Case $d = 0$. In this situation, (2.7) and (2.8) rewrite as

$$X = \frac{4}{4 + \omega^2} \quad \text{and} \quad z = \frac{4 + \omega^2}{4 - \omega^2 + 4\omega i}.$$

Therefore, we clearly have that $X \in \mathbb{R} \setminus \{0\}$ and $z \in \mathbb{C}$ are constant. Also, to check that $|z| = 1$ is straightforward.

Case $d \neq 0$ and $\omega = 0$. In this situation, (2.7) and (2.8) rewrite as

$$dX^2 - 4 + (4 - d)X = 0 \quad \text{and} \quad z = \frac{dX^2 + 4}{X(4 + d)}.$$

Since the number of solutions of the first equation strongly depends on the values of d, we must distinguish two cases. On one hand, if $d \neq -4$, then we get

$$X = \frac{1}{2d}\left(d - 4 \pm \sqrt{(d - 4)^2 + 16d}\right) = \frac{1}{2d}(d - 4 \pm |d + 4|)$$

and, thus, the solutions to (2.7) are $X = 1$ and $X = -4/d$. For $X = 1$, we get $z = 1$, and for $X = -4/d$, we get $z = -1$. On the other hand, if $d = -4$, then (2.7) rewrites as $(X - 1)^2 = 0$, hence the unique solution is $X = 1$. In this case, z formally corresponds to

$$z = \frac{dX^2 + 4}{X(4 + d)} = \frac{d + 4}{4 + d} = 1.$$

Case $d \neq 0$ and $\omega \neq 0$. From (2.7), we get

$$X = \frac{1}{2d}\left(d - \omega^2 - 4 \pm \sqrt{(d - \omega^2 - 4)^2 + 16d}\right).$$

A simple computation shows that

$$(d - \omega^2 - 4)^2 + 16d = (d - \omega^2 + 4)^2 + 16\omega^2 \geq 16\omega^2 > 0$$

and, therefore, X can chosen as either one of the two real, nonzero, and different solutions to (2.7). In this setting, it is clear that $z \in \mathbb{C}$ is constant because $X, \omega \neq 0$. Let us now show that $|z| = 1$. We have

$$\begin{aligned}|z|^2 &= \frac{(dX^2 + 4)^2}{X^2((4 + d - \omega^2)^2 + 16\omega^2)} = \frac{(dX^2 + 4)^2}{X^2((-4 + d - \omega^2)^2 + 16d)}\\ &= \frac{(dX^2 + 4)^2}{((d - \omega^2 - 4)X)^2 + 16dX^2} = \frac{(dX^2 + 4)^2}{(dX^2 - 4)^2 + 16dX^2},\end{aligned}$$

where we used (2.7) in the last equality. Therefore,

$$|z|^2 = \frac{(dX^2 + 4)^2}{(dX^2 - 4)^2 + 16dX^2} = \frac{(dX^2 + 4)^2}{(dX^2 + 4)^2} = 1,$$

as desired.

We have checked that we always have $X \in \mathbb{R} \setminus \{0\}$, and $z \in \mathbb{C}$ always satisfies $|z| = 1$. This shows the first statement of Theorem 2.1. The rest of the proof of Theorem 2.1 strongly relies on the following result, which requires some notation. We set

$$(4.1) \qquad M_{\eta,\tau,\lambda,\omega}^{\pm} := \pm i(\sigma \cdot \mathbf{n}) + \frac{1}{2}(\eta \mathbb{I}_2 + \tau \sigma_3 + \lambda(\sigma \cdot \mathbf{t}) + \omega(\sigma \cdot \mathbf{n})).$$

Note that the boundary condition in (2.2) can be expressed as

$$M_{\eta,\tau,\lambda,\omega}^{+} \mathcal{T}_+^D f_+ = -M_{\eta,\tau,\lambda,\omega}^{-} \mathcal{T}_-^D f_-.$$

Lemma 4.1. *Given η, τ, λ, $\omega \in \mathbb{R}$, let X and z be as in Theorem 2.1. If $(d, \omega) \neq (-4, 0)$, then, for every $x \in \Sigma$, $M_{\eta,\tau,\lambda,\omega}^{\pm}$ and $M_{X\eta,X\tau,X\lambda,0}^{\pm}$ are invertible matrices. Moreover,*

$$(4.2) \qquad (M_{\eta,\tau,\lambda,\omega}^{+})^{-1} M_{\eta,\tau,\lambda,\omega}^{-} (M_{X\eta,X\tau,X\lambda,0}^{-})^{-1} M_{X\eta,X\tau,X\lambda,0}^{+} = \bar{z}\, \mathbb{I}_2.$$

Proof. We introduce the auxiliary matrix

$$\widetilde{M}_{\eta,\tau,\lambda,\omega}^{\pm} := \mp i(\sigma \cdot \mathbf{n}) + \frac{1}{2}(\eta \mathbb{I}_2 - \tau \sigma_3 - \lambda(\sigma \cdot \mathbf{t}) - \omega(\sigma \cdot \mathbf{n})).$$

Thanks to (3.12) and (3.13),

$$M_{\eta,\tau,\lambda,\omega}^{\pm} \widetilde{M}_{\eta,\tau,\lambda,\omega}^{\pm} = \left(\pm i(\sigma \cdot \mathbf{n}) + \frac{1}{2}(\eta \mathbb{I}_2 + \tau \sigma_3 + \lambda(\sigma \cdot \mathbf{t}) + \omega(\sigma \cdot \mathbf{n})) \right)$$

$$\times \left(\mp i(\sigma \cdot \mathbf{n}) + \frac{1}{2}(\eta \mathbb{I}_2 - \tau \sigma_3 - \lambda(\sigma \cdot \mathbf{t}) - \omega(\sigma \cdot \mathbf{n})) \right)$$

$$= \left(1 + \frac{1}{4}(\eta^2 - \tau^2 - \lambda^2 - \omega^2) \mp \omega i \right) \mathbb{I}_2 = \frac{1}{4}(4 + d - \omega^2 \mp 4\omega i)\mathbb{I}_2.$$

Since $(d, \omega) \neq (-4, 0)$, we see that $4 + d - \omega^2 \mp 4\omega i \neq 0$. Therefore, $M_{\eta,\tau,\lambda,\omega}^{\pm}$ is invertible whenever $(d, \omega) \neq (-4, 0)$, and its inverse is given by

$$(M_{\eta,\tau,\lambda,\omega}^{\pm})^{-1} = \frac{4}{4 + d - \omega^2 \mp 4\omega i} \, \widetilde{M}_{\eta,\tau,\lambda,\omega}^{\pm}$$

$$= \frac{4}{4 + d - \omega^2 \mp 4\omega i} \left(\mp i(\sigma \cdot \mathbf{n}) + \frac{1}{2}(\eta \mathbb{I}_2 - \tau \sigma_3 - \lambda(\sigma \cdot \mathbf{t}) - \omega(\sigma \cdot \mathbf{n})) \right).$$

A similar computation can be carried out to find $(M_{X\eta,X\tau,X\lambda,0}^{\pm})^{-1}$. In this case, one gets

$$(4.3) \qquad (M_{X\eta,X\tau,X\lambda,0}^{\pm})^{-1} = \frac{4}{4 + dX^2} \left(\mp i(\sigma \cdot \mathbf{n}) + \frac{X}{2}(\eta \mathbb{I}_2 - \tau \sigma_3 - \lambda(\sigma \cdot \mathbf{t})) \right).$$

Using (2.7) together with the assumption that $(d, \omega) \neq (-4, 0)$ and the fact that $X \neq 0$, we have

$$(dX^2 + 4)^2 = (dX^2 - 4)^2 + 16dX^2 = ((d - \omega^2 - 4)X)^2 + 16dX^2$$

$$= X^2((d - \omega^2 - 4)^2 + 16d) = X^2((d - \omega^2 + 4)^2 + 16\omega^2) > 0.$$

Hence, the right-hand side of (4.3) is well defined. This shows that $M_{X\eta,X\tau,X\lambda,0}^{\pm}$ is invertible whenever $(d, \omega) \neq (-4, 0)$.

Let us address the proof of (4.2). The first step is to compute $(M_{\eta,\tau,\lambda,\omega}^{\pm})^{-1}M_{\eta,\tau,\lambda,\omega}^{\mp}$. We have

$$
(M_{\eta,\tau,\lambda,\omega}^{\pm})^{-1}M_{\eta,\tau,\lambda,\omega}^{\mp}
$$
$$
= \frac{4}{4 + d - \omega^2 \mp 4\omega i}\Big(\mp i(\sigma \cdot \mathbf{n}) + \frac{1}{2}(\eta\mathbb{I}_2 - \tau\sigma_3 - \lambda(\sigma \cdot \mathbf{t}) - \omega(\sigma \cdot \mathbf{n}))\Big)
$$
$$
\times \Big(\mp i(\sigma \cdot \mathbf{n}) + \frac{1}{2}(\eta\mathbb{I}_2 + \tau\sigma_3 + \lambda(\sigma \cdot \mathbf{t}) + \omega(\sigma \cdot \mathbf{n}))\Big)
$$
$$
= \frac{4}{4 + d - \omega^2 \mp 4\omega i}\Big(\frac{-4 + d - \omega^2}{4}\mathbb{I}_2 \pm (\lambda\sigma_3 - \eta i(\sigma \cdot \mathbf{n}) - \tau(\sigma \cdot \mathbf{t}))\Big)
$$

and, similarly,

$$
(M_{X\eta,X\tau,X\lambda,0}^{\pm})^{-1}M_{X\eta,X\tau,X\lambda,0}^{\mp}
$$
$$
= \frac{4}{4 + dX^2}\Big(\frac{-4 + dX^2}{4}\mathbb{I}_2 \pm X(\lambda\sigma_3 - \eta i(\sigma \cdot \mathbf{n}) - \tau(\sigma \cdot \mathbf{t}))\Big).
$$

From these calculations, we obtain

(4.4) $\quad (M_{\eta,\tau,\lambda,\omega}^{+})^{-1}M_{\eta,\tau,\lambda,\omega}^{-}(M_{X\eta,X\tau,X\lambda,0}^{-})^{-1}M_{X\eta,X\tau,X\lambda,0}^{+}$

$$
= \frac{4}{4 + d - \omega^2 - 4\omega i}\Big(\frac{-4 + d - \omega^2}{4}\mathbb{I}_2 + \lambda\sigma_3 - \eta i(\sigma \cdot \mathbf{n}) - \tau(\sigma \cdot \mathbf{t})\Big)
$$
$$
\times \frac{4}{4 + dX^2}\Big(\frac{-4 + dX^2}{4}\mathbb{I}_2 - X\lambda\sigma_3 + X\eta i(\sigma \cdot \mathbf{n}) + X\tau(\sigma \cdot \mathbf{t})\Big).
$$

Given $a, \tilde{a} \in \mathbb{R}$, a computation shows that

(4.5) $\quad \big(a\mathbb{I}_2 + \lambda\sigma_3 - \eta i(\sigma \cdot \mathbf{n}) - \tau(\sigma \cdot \mathbf{t})\big)\big(\tilde{a}\mathbb{I}_2 - X\lambda\sigma_3 + X\eta i(\sigma \cdot \mathbf{n}) + X\tau(\sigma \cdot \mathbf{t})\big)$
$$
= (a\tilde{a} + dX)\mathbb{I}_2 + (\tilde{a} - aX)\big(\lambda\sigma_3 - \eta i(\sigma \cdot \mathbf{n}) - \tau(\sigma \cdot \mathbf{t})\big).
$$

By taking $a = (-4 + d - \omega^2)/4$ and $\tilde{a} = (-4 + dX^2)/4$, and using (2.7), we see that

(4.6) $\qquad a\tilde{a} + dX = \dfrac{(-4 + d - \omega^2)X(-4 + dX^2)}{16X} + dX$

$$
= \frac{(dX^2 - 4)^2}{16X} + dX = \frac{(dX^2 + 4)^2}{16X}
$$

and

(4.7) $\qquad \tilde{a} - aX = \dfrac{1}{4}\big(dX^2 - 4 + (4 - d + \omega^2)X\big) = 0.$

Plugging (4.6) and (4.7) in (4.5), and combining then with (4.4), we conclude that

$$
(M_{\eta,\tau,\lambda,\omega}^{+})^{-1}M_{\eta,\tau,\lambda,\omega}^{-}(M_{X\eta,X\tau,X\lambda,0}^{-})^{-1}M_{X\eta,X\tau,X\lambda,0}^{+}
$$
$$
= \frac{4}{4 + d - \omega^2 - 4\omega i}\frac{4}{4 + dX^2}\frac{(dX^2 + 4)^2}{16X}\mathbb{I}_2 = \frac{dX^2 + 4}{X(4 + d - \omega^2 - 4\omega i)}\mathbb{I}_2 = \bar{z}\,\mathbb{I}_2,
$$

where we used (2.8) in the last equality above. Therefore, (4.2) holds and the lemma follows. ∎

Proof of Theorem 2.1. We have already shown that $X \in \mathbb{R} \setminus \{0\}$ and $z \in \mathbb{C}$, with $|z| = 1$, see the comments above. It only remains to prove that

$$(4.8) \qquad \mathcal{D}^z_{\eta,\tau,\lambda,\omega} = \mathcal{D}_{X\eta,X\tau,X\lambda,0}.$$

Once this is shown, we would get that $\mathcal{D}_{\eta,\tau,\lambda,\omega}$ and $\mathcal{D}_{X\eta,X\tau,X\lambda,0}$ are unitarily equivalent through the unitary operator U_z, because $\mathcal{D}^z_{\eta,\tau,\lambda,\omega} = U_{\bar{z}} \mathcal{D}_{\eta,\tau,\lambda,\omega} U_z$ by definition.

Note that (4.8) is obvious if $(d,\omega) = (-4,0)$ because then $X = 1$, $z = 1$, and $U_z = \mathbb{I}_2$. From now on we assume that $(d,\omega) \neq (-4,0)$. Then Lemma 4.1 yields

$$(4.9) \qquad (M^-_{\eta,\tau,\lambda,\omega})^{-1} M^+_{\eta,\tau,\lambda,\omega} = z (M^-_{X\eta,X\tau,X\lambda,0})^{-1} M^+_{X\eta,X\tau,X\lambda,0}.$$

Recalling now (2.2) and (4.1), we have

$$\mathrm{dom}(\mathcal{D}_{\eta,\tau,\lambda,\omega}) = \{f = f_+ \oplus f_- \in H(\sigma,\Omega_+) \oplus H(\sigma,\Omega_-) :$$
$$M^+_{\eta,\tau,\lambda,\omega} \mathcal{T}^D_+ f_+ = -M^-_{\eta,\tau,\lambda,\omega} \mathcal{T}^D_- f_-\}.$$

Therefore, using (4.9) and that $z\bar{z} = |z|^2 = 1$, we deduce that

$$\mathrm{dom}(\mathcal{D}^z_{\eta,\tau,\lambda,\omega}) = U_{\bar{z}}(\mathrm{dom}(\mathcal{D}_{\eta,\tau,\lambda,\omega}))$$
$$= \{U_{\bar{z}} f = f_+ \oplus \bar{z} f_- \in H(\sigma,\Omega_+) \oplus H(\sigma,\Omega_-) :$$
$$M^+_{\eta,\tau,\lambda,\omega} \mathcal{T}^D_+ f_+ = -M^-_{\eta,\tau,\lambda,\omega} \mathcal{T}^D_- f_-\}$$
$$= \{U_{\bar{z}} f = f_+ \oplus \bar{z} f_- \in H(\sigma,\Omega_+) \oplus H(\sigma,\Omega_-) :$$
$$(M^-_{\eta,\tau,\lambda,\omega})^{-1} M^+_{\eta,\tau,\lambda,\omega} \mathcal{T}^D_+ f_+ = -\mathcal{T}^D_- f_-\}$$
$$= \{U_{\bar{z}} f = f_+ \oplus \bar{z} f_- \in H(\sigma,\Omega_+) \oplus H(\sigma,\Omega_-) :$$
$$z (M^-_{X\eta,X\tau,X\lambda,0})^{-1} M^+_{X\eta,X\tau,X\lambda,0} \mathcal{T}^D_+ f_+ = -\mathcal{T}^D_- f_-\}$$
$$= \{U_{\bar{z}} f = f_+ \oplus \bar{z} f_- \in H(\sigma,\Omega_+) \oplus H(\sigma,\Omega_-) :$$
$$M^+_{X\eta,X\tau,X\lambda,0} \mathcal{T}^D_+ f_+ = -M^-_{X\eta,X\tau,X\lambda,0} \mathcal{T}^D_- (\bar{z} f_-)\}$$
$$= \{g = g_+ \oplus g_- \in H(\sigma,\Omega_+) \oplus H(\sigma,\Omega_-) :$$
$$M^+_{X\eta,X\tau,X\lambda,0} \mathcal{T}^D_+ g_+ = -M^-_{X\eta,X\tau,X\lambda,0} \mathcal{T}^D_- g_-\}$$
$$= \mathrm{dom}(\mathcal{D}_{X\eta,X\tau,X\lambda,0}).$$

Now, let $f \in \mathrm{dom}(\mathcal{D}^z_{\eta,\tau,\lambda,\omega}) = \mathrm{dom}(\mathcal{D}_{X\eta,X\tau,X\lambda,0})$. Then $f = U_{\bar{z}} \varphi$ for some $\varphi \in \mathrm{dom}(\mathcal{D}_{\eta,\tau,\lambda,\omega})$. Recall that, although η, τ, and λ may be non-constant, we assume that d and ω are constant along Σ, which implies that z is also constant in $\overline{\Omega}_-$. This yields $\bar{z} D_0 \varphi_- = D_0(\bar{z} \varphi_-)$ in Ω_-. Hence,

$$(4.10) \quad \mathcal{D}^z_{\eta,\tau,\lambda,\omega} f = U_{\bar{z}} \mathcal{D}_{\eta,\tau,\lambda,\omega} U_z f = U_{\bar{z}} \mathcal{D}_{\eta,\tau,\lambda,\omega} U_z U_{\bar{z}} \varphi = U_{\bar{z}} \mathcal{D}_{\eta,\tau,\lambda,\omega} \varphi$$
$$= U_{\bar{z}}(D_0 \varphi_+ \oplus D_0 \varphi_-) = D_0 \varphi_+ \oplus \bar{z} D_0 \varphi_- = D_0 \varphi_+ \oplus D_0(\bar{z} \varphi_-)$$
$$= D_0(U_{\bar{z}} \varphi)_+ \oplus D_0(U_{\bar{z}} \varphi)_- = D_0 f_+ \oplus D_0 f_- = \mathcal{D}_{X\eta,X\tau,X\lambda,0} f.$$

That is, $\mathcal{D}^z_{\eta,\tau,\lambda,\omega} f = \mathcal{D}_{X\eta,X\tau,X\lambda,0} f$ for all $f \in \mathrm{dom}(\mathcal{D}^z_{\eta,\tau,\lambda,\omega}) = \mathrm{dom}(\mathcal{D}_{X\eta,X\tau,X\lambda,0})$. Therefore, (4.8) holds and Theorem 2.1 follows. ∎

4.2. Spectral relations

From the proof of Theorem 2.1 we realize that if $\omega = 0$, we can also allow d to be variable in Σ in the conclusion of Theorem 2.1, as long as $d(x) \notin \{0, -4\}$ for all $x \in \Sigma$. This is because (4.10) holds whenever z is constant in Ω_-, and for $\omega = 0$ and $d \neq 0, -4$ we can always take $z = -1$, as we explained below the statement of Theorem 2.1. Thus, we can take $X = -4/d$, which yields the isospectral transformation of parameters

$$(\eta, \tau, \lambda) \mapsto (X\eta, X\tau, X\lambda) = -\frac{4}{d}(\eta, \tau, \lambda)$$

for all $\eta, \tau, \lambda \in C^\infty(\Sigma; \mathbb{R})$ such that $\eta^2(x) - \tau^2(x) - \lambda^2(x) = d(x) \notin \{0, -4\}$ for all $x \in \Sigma$, with no more restrictions on d in Σ. We underline that this correspondence maps the set $\{(\eta, \tau, \lambda) \in \mathbb{R}^3 \mid \eta^2 - \tau^2 - \lambda^2 < -4\}$ onto $\{(\eta, \tau, \lambda) \in \mathbb{R}^3 \mid -4 < \eta^2 - \tau^2 - \lambda^2 < 0\}$ and $\{(\eta, \tau, \lambda) \in \mathbb{R}^3 \mid \eta^2 - \tau^2 - \lambda^2 > 0\}$ onto itself.

Next, we apply the observation above on $\mathcal{D}_{\eta,\tau,\lambda,0} = \mathcal{D}_{\eta,\tau,\lambda}$ with constant parameters. Moreover, we investigate the spectral relation between $\mathcal{D}_{\eta,\tau,\lambda}$ and its charge conjugation.

Proposition 4.2. *Let $\eta, \tau, \lambda \in \mathbb{R}$, and $\mathcal{D}_{\eta,\tau,\lambda}$ be defined as in* (2.4), (2.5). *The following hold*:

 (i) *if $d \neq 0$, then $z \in \sigma_p(\mathcal{D}_{\eta,\tau,\lambda})$ if and only if $z \in \sigma_p(\mathcal{D}_{-4\eta/d, -4\tau/d, -4\lambda/d})$,*

 (ii) $z \in \sigma_p(\mathcal{D}_{\eta,\tau,\lambda})$ *if and only if $-z \in \sigma_p(\mathcal{D}_{-\eta,\tau,-\lambda})$.*

Remark 4.3. The previous proposition reduces to known results if $\lambda = 0$. In the three-dimensional setting, the equivalent of Proposition 4.2 was observed for the first time in Theorem 3.3 and Theorem 3.6 of [5] for the purely electric interaction and in Theorem 2.3 of [41] for the Lorentz scalar interaction. For the proof of (ii) we adapt the strategy of the proof of Proposition 4.2 (iii) of [14]; the proof of (i) descends from the previous arguments, or it can be obtained by adapting the proof of Proposition 4.2 (i) of [14].

Proof. (i) The case $d = -4$ is obvious. The case $d \neq -4$ follows from Theorem 2.1 with $\omega = 0$ by simply noting that one can take $X = -4/d$ and $z = -1$.

   (ii). Let C be the antilinear charge conjugation operator

$$C : L^2(\mathbb{R}^2; \mathbb{C}^2) \to L^2(\mathbb{R}^2; \mathbb{C}^2), \quad Cf = \sigma_1 \bar{f}, \quad f \in L^2(\mathbb{R}^2; \mathbb{C}^2).$$

The operator C is an involution, i.e., $C^2 f = f$ for all $f \in L^2(\mathbb{R}^2; \mathbb{C}^2)$. The result follows if we show that

$$(4.11) \qquad\qquad CD_{\eta,\tau,\lambda} = -D_{-\eta,\tau,-\lambda}C.$$

Taking the complex conjugate of the condition in the definition of $\mathrm{dom}\, \mathcal{D}_{\eta,\tau,\lambda}$, we see that $f \in \mathrm{dom}\, D_{\eta,\tau,\lambda}$ if and only if

$$-i(\bar{\sigma} \cdot \mathbf{n})(\mathcal{T}_-^D \bar{f}_- - \mathcal{T}_+^D \bar{f}_+) = \frac{1}{2}(\eta \mathbb{I}_2 + \tau\sigma_3 + \lambda(\bar{\sigma} \cdot \mathbf{t}))(\mathcal{T}_-^D \bar{f}_- + \mathcal{T}_+^D \bar{f}_+),$$

where we denoted $\bar{\sigma} := (\bar{\sigma}_1, \bar{\sigma}_2)$ and $\bar{\sigma}_j$ is the matrix that has the conjugate entries of the matrix σ_j, $j = 1, 2$. Since $\bar{\sigma} = (\sigma_1, -\sigma_2)$, multiplying the last equation by σ_1, we get

$$i(\sigma \cdot \mathbf{n})(\mathcal{T}_-^D(\sigma_1 \bar{f}_-) - \mathcal{T}_+^D(\sigma_1 \bar{f}_+)) = \frac{1}{2}(-\eta \mathbb{I}_2 + \tau\sigma_3 - \lambda(\sigma \cdot \mathbf{t}))(\mathcal{T}_-^D(\sigma_1 \bar{f}_-) + \mathcal{T}_+^D(\sigma_1 \bar{f}_+)),$$

i.e., $Cf \in \text{dom } D_{-\eta,\tau,-\lambda}$. We have showed that $\text{dom}(CD_{\eta,\tau,\lambda}) = \text{dom}(D_{-\eta,\tau,-\lambda}C)$. With an explicit computation, one sees that $(-i\sigma \cdot \nabla + m\sigma_3)Cf = -C(-i\sigma \cdot \nabla + m\sigma_3)f$. Thus, we get (4.11). ∎

We finally mention that the three-dimensional analogue of $\mathcal{D}_{\eta,0,0,\omega}$ was investigated in [47], where the same transformation of the coefficients $(\eta, \omega) \rightarrow (\tilde{\eta}, 0)$ by means of X and z was discovered. Since here we also admit $\tau, \lambda \neq 0$, and, with a restriction, also non-constant coefficients, Theorem 2.1 can be understood as a generalization of [47] to the two-dimensional scenario for more general δ-shell interactions.

5. Confinement

In the following lemma, we describe the properties of confinement and transmission induced by the boundary condition in (2.4).

Lemma 5.1. *Let* $\eta, \tau, \lambda \in C^\infty(\Sigma; \mathbb{R})$ *and let* $\mathcal{D}_{\eta,\tau,\lambda}$ *be defined as in* (2.4) *and* (2.5). *Then the following hold*:

(i) *If* $d \neq -4$ *everywhere on* Σ, *there exists an invertible matrix function* $R_{\eta,\tau,\lambda}$ *(explicitly defined in* (5.6) *below) such that every* $f = f_+ \oplus f_- \in H(\sigma, \Omega_+) \oplus H(\sigma, \Omega_-)$ *belongs to* $\text{dom } \mathcal{D}_{\eta,\tau,\lambda}$ *if and only if*

$$\text{(5.1)} \qquad \mathcal{T}_+^D f_+ = R_{\eta,\tau,\lambda}\mathcal{T}_-^D f_-.$$

(ii) *If* $d = -4$ *everywhere on* Σ, *every* $f = f_+ \oplus f_- \in H(\sigma, \Omega_+) \oplus H(\sigma, \Omega_-)$ *belongs to* $\text{dom } \mathcal{D}_{\eta,\tau,\lambda}$ *if and only if*

$$\text{(5.2)} \qquad \left[i(\sigma \cdot \mathbf{n}) \pm \frac{1}{2}(\eta\mathbb{I}_2 + \tau\sigma_3 + \lambda(\sigma \cdot \mathbf{t}))\right]\mathcal{T}_\pm^D f_\pm = 0.$$

Proof. Let $f = f_+ \oplus f_- \in H(\sigma, \Omega_+) \oplus H(\sigma, \Omega_-)$. From (2.4), $f \in \text{dom } \mathcal{D}_{\eta,\tau,\lambda}$ if and only if

$$\text{(5.3)} \qquad \left(i(\sigma \cdot \mathbf{n}) + \frac{1}{2}(\eta\mathbb{I}_2 + \tau\sigma_3 + \lambda(\sigma \cdot \mathbf{t}))\right)\mathcal{T}_+^D f_+$$
$$= \left(i(\sigma \cdot \mathbf{n}) - \frac{1}{2}(\eta\mathbb{I}_2 + \tau\sigma_3 + \lambda(\sigma \cdot \mathbf{t}))\right)\mathcal{T}_-^D f_-.$$

Thanks to (3.12), this is equivalent to

$$\text{(5.4)} \qquad (\mathbb{I}_2 + M)\mathcal{T}_+^D f_+ = (\mathbb{I}_2 - M)\mathcal{T}_-^D f_-,$$

with

$$M := -\frac{i}{2}(\sigma \cdot \mathbf{n})(\eta\mathbb{I}_2 + \tau\sigma_3 + \lambda(\sigma \cdot \mathbf{t})).$$

Due to (3.12), (3.13), and (3.14), we have

$$\text{(5.5)} \qquad M = -\frac{\eta}{2}i(\sigma \cdot \mathbf{n}) - \frac{\tau}{2}\sigma \cdot \mathbf{t} + \frac{\lambda}{2}\sigma_3,$$
$$M^2 = -\frac{d}{4}\mathbb{I}_2, \quad (\mathbb{I}_2 + M)(\mathbb{I}_2 - M) = \frac{4 + d}{4}\mathbb{I}_2.$$

When $d \neq -4$, the matrix $\mathbb{I}_2 + M$ is invertible, by (5.5), and we get (i) by setting

$$(5.6) \quad R_{\eta,\tau,\lambda} := (\mathbb{I}_2 + M)^{-1}(\mathbb{I}_2 - M) = \frac{4}{4+d}(\mathbb{I}_2 - M)^2 = \frac{4}{4+d}\Big(\frac{4-d}{4}\mathbb{I}_2 - 2M\Big)$$

$$= \frac{4}{4+d}\Big(\frac{4-d}{4}\mathbb{I}_2 + i\eta(\sigma \cdot \mathbf{n}) + \tau(\sigma \cdot \mathbf{t}) - \lambda\sigma_3\Big).$$

If $d = -4$, then $M^2 = \mathbb{I}_2$. Multiplying (5.4) by $\mathbb{I}_2 \pm M$, we get

$$0 = (\mathbb{I}_2 \pm M)^2 \mathcal{T}_\pm^D f_\pm = 2(\mathbb{I}_2 \pm M)\mathcal{T}_\pm^D f_\pm.$$

Multiplying the previous equation by $\frac{i}{2}(\sigma \cdot \mathbf{n})$ and using (3.12) we arrive at (5.2). Vice versa, (5.2) implies (5.3) trivially. ∎

If $d = \eta^2 - \tau^2 - \lambda^2 \neq -4$ everywhere on Σ, Lemma 5.1 (i) states that the values of f_+ and f_- along Σ are related via the matrix $R_{\eta,\tau,\lambda}$. The presence of the δ-shell implies a transmission condition for the functions in the domain of $\mathcal{D}_{\eta,\tau,\lambda}$ across the surface Σ.

If $d = \eta^2 - \tau^2 - \lambda^2 = -4$ everywhere on Σ, Lemma 5.1 (ii) implies Theorem 2.3.

6. The non-critical case

In this section we prove Theorem 2.2 stated in Section 2. The operator $\mathcal{D}_{\eta,\tau,\lambda}$ is defined again as in (2.4) and (2.5), where in general the condition in (2.4) is understood in the sense of $H^{-1/2}(\Sigma)$. We show the self-adjointness and some further properties of the operator $\mathcal{D}_{\eta,\tau,\lambda}$. The strategy of the proofs mainly follows [14], but we need to modify the boundary triple that we use in order to include the magnetic interaction.

It will be convenient to introduce some extra notation. Recall the definition $T := t_1 + i t_2 \in \mathbb{C}$ from (3.30), where $\mathbf{t} = (t_1, t_2)$ is the tangent vector to Σ at the point $x \in \Sigma$. We define the following matrix-valued functions on Σ:

$$(6.1) \qquad V := \begin{pmatrix} 1 & 0 \\ 0 & \overline{T} \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} \eta + \tau & \lambda \\ \lambda & \eta - \tau \end{pmatrix}.$$

For all $x \in \Sigma$, the matrix $V(x)$ is unitary and

$$(6.2) \qquad \eta\mathbb{I}_2 + \tau\sigma_3 + \lambda(\sigma \cdot \mathbf{t}) = V^*BV \quad \text{in } \Sigma.$$

Finally, we have $\det B = \eta^2 - \tau^2 - \lambda^2 = d \in C^\infty(\Sigma; \mathbb{R})$, cf. (2.6).

In the following proposition we adapt to our setting the boundary triple constructed in Proposition 3.6 of [14] for the operator S^*, defined in (3.24). In the formulation of the proposition we extend the operators Λ and $\mathcal{C}_\zeta$ defined in (3.17) and (3.29), respectively, onto two-component functions applying the respective mappings component-wise.

Proposition 6.1. *Let $\zeta \in \rho(\mathcal{D}_0)$ and let $\Gamma_0, \Gamma_1 : \operatorname{dom} S^* \to L^2(\Sigma; \mathbb{C}^2)$ be defined by*

$$(6.3) \quad \begin{aligned} \Gamma_0 f &= i\Lambda^{-1}V(\sigma \cdot \mathbf{n})\big(\mathcal{T}_+^D f_+ - \mathcal{T}_-^D f_-\big), \\ \Gamma_1 f &= \frac{1}{2}\Lambda V\big((\mathcal{T}_+^D f_+ + \mathcal{T}_-^D f_-) - (\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})V^*\Lambda\Gamma_0 f\big), \end{aligned}$$

where $f = f_+ \oplus f_- \in \text{dom } S^*$. Then $\{L^2(\Sigma; \mathbb{C}^2), \Gamma_0, \Gamma_1\}$ is a boundary triple for S^* such that $\mathcal{D}_0 = S^* \upharpoonright \ker \Gamma_0$. Moreover, the corresponding γ-field is

(6.4) $$G_z : z \in \rho(\mathcal{D}_0) \mapsto \Phi_z V^* \Lambda$$

and the Weyl function is

(6.5) $$M_z : z \in \rho(\mathcal{D}_0) \mapsto \Lambda V \left(\mathcal{C}_z - \frac{1}{2} (\mathcal{C}_\xi + \mathcal{C}_{\bar{\xi}}) \right) V^* \Lambda.$$

Proof. In Proposition 3.6 of [14], it is proved that $\{L^2(\Sigma; \mathbb{C}^2), \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is a boundary triple for S^*, with

$$\widetilde{\Gamma}_0 f = i \Lambda^{-1} (\sigma \cdot \mathbf{n}) (\mathcal{T}_+^D f_+ - \mathcal{T}_-^D f_-),$$

$$\widetilde{\Gamma}_1 f = \frac{1}{2} \Lambda \left((\mathcal{T}_+^D f_+ + \mathcal{T}_-^D f_-) - (\mathcal{C}_\xi + \mathcal{C}_{\bar{\xi}}) \Lambda \widetilde{\Gamma}_0 f \right), \quad \text{where } f = f_+ \oplus f_- \in \text{dom } S^*.$$

Moreover, $\mathcal{D}_0 = S^* \upharpoonright \ker \widetilde{\Gamma}_0$ and the γ-field $\widetilde{G}_z$ and the Weyl function $\widetilde{M}_z$ associated to the boundary triple $\{L^2(\Sigma; \mathbb{C}^2), \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ are defined by

(6.6)
$$\widetilde{G}_z : z \in \rho(D_0) \mapsto (\widetilde{\Gamma}_0 \upharpoonright \ker(A^* - z))^{-1} = \Phi_z \Lambda,$$
$$\widetilde{M}_z : z \in \rho(D_0) \mapsto \widetilde{\Gamma}_1 \widetilde{G}_z = \Lambda \left(\mathcal{C}_z - \frac{1}{2} (\mathcal{C}_\xi + \mathcal{C}_{\bar{\xi}}) \right) \Lambda.$$

We define Γ_0, Γ_1 as in (6.3). The key observation of our proof is that

(6.7) $$\Gamma_0 = \Lambda^{-1} V \Lambda \, \widetilde{\Gamma}_0 \quad \text{and} \quad \Gamma_1 = \Lambda V \Lambda^{-1} \widetilde{\Gamma}_1,$$

and $\Lambda^{-1} V \Lambda$, $\Lambda V \Lambda^{-1}$ are bounded and boundedly invertible on $L^2(\Sigma; \mathbb{C}^2)$, since $\Lambda : H^s(\Sigma) \to H^{s-1/2}(\Sigma)$ is an isomorphism for all $s \in \mathbb{R}$ and $V \in C^\infty(\Sigma; \mathbb{C}^2)$ is pointwise unitary. Consequently, the map $f \in \text{dom } S^* \mapsto (\Gamma_0 f, \Gamma_1 f) \in L^2(\Sigma; \mathbb{C}^2) \times L^2(\Sigma; \mathbb{C}^2)$ is surjective, i.e., the condition (ii) in Definition 3.4 of the boundary triple is fulfilled. In order to verify the condition (i) of the definition, we observe that, for all $\phi, \psi \in \text{dom } S^*$,

(6.8) $\langle \Gamma_0 \phi, \Gamma_1 \psi \rangle_{L^2} = \langle \Lambda \Gamma_0 \phi, \Lambda^{-1} \Gamma_1 \psi \rangle_{H^{-1/2}, H^{1/2}} = \langle V \Lambda \widetilde{\Gamma}_0 \phi, V \Lambda^{-1} \widetilde{\Gamma}_1 \psi \rangle_{H^{-1/2}, H^{1/2}},$

due to (3.18) and (6.7). The last term in the previous equation equals

(6.9) $\langle \Lambda \widetilde{\Gamma}_0 \phi, V^* V \Lambda^{-1} \widetilde{\Gamma}_1 \psi \rangle_{H^{-1/2}, H^{1/2}} = \langle \Lambda \widetilde{\Gamma}_0 \phi, \Lambda^{-1} \widetilde{\Gamma}_1 \psi \rangle_{H^{-1/2}, H^{1/2}},$

because $V^* V = \mathbb{I}_2$. Combining (6.8) and (6.9), and using (3.18) again, we get

(6.10) $\langle \Gamma_0 \phi, \Gamma_1 \psi \rangle_{L^2} = \langle \Lambda \widetilde{\Gamma}_0 \phi, \Lambda^{-1} \widetilde{\Gamma}_1 \psi \rangle_{H^{-1/2}, H^{1/2}} = \langle \widetilde{\Gamma}_0 \phi, \widetilde{\Gamma}_1 \psi \rangle_{L^2}.$

This yields (i) in Definition 3.4. Therefore, we conclude that $\{L^2(\Sigma; \mathbb{C}^2), \Gamma_0, \Gamma_1\}$ is a boundary triple for S^*. Since $\ker \Gamma_0 = \ker \widetilde{\Gamma}_0$, it is true that $\mathcal{D}_0 = S^* \upharpoonright \ker \Gamma_0$. From (3.19), we get that, for all $z \in \rho(\mathcal{D}_0)$,

$G_z := (\Gamma_0 \upharpoonright \ker(A^* - z))^{-1} = (\widetilde{\Gamma}_0 \upharpoonright \ker(A^* - z))^{-1} \Lambda^{-1} V^* \Lambda = \widetilde{G}_z \Lambda^{-1} V^* \Lambda = \Phi_z V^* \Lambda,$

i.e., (6.4). Plugging this result together with (6.7) into (3.20), we obtain

$$M_z := \Gamma_1 G_z = \Lambda V \Lambda^{-1} \widetilde{\Gamma}_1 \widetilde{G}_z \Lambda^{-1} V^* \Lambda = \Lambda V \Lambda^{-1} \widetilde{M}_z \Lambda^{-1} V^* \Lambda$$
$$= \Lambda V \Big(\mathcal{C}_z - \frac{1}{2}(\mathcal{C}_\xi + \mathcal{C}_{\bar{\xi}}) \Big) V^* \Lambda,$$

for all $z \in \rho(\mathcal{D}_0)$. This is just (6.5). ∎

The following lemma is a regularity result concerning the boundary triple defined in Proposition 6.1.

Lemma 6.2. *Let $f \in \operatorname{dom} S^*$. Then $f \in H^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$ if and only if $\Gamma_0 f \in H^1(\Sigma; \mathbb{C}^2)$.*

Proof. The proof is analogous to Lemma 3.7 of [14], reasoning as in the proof of Proposition 6.1. ∎

The following proposition is a modification of Proposition 4.3 of [14] taking into account the magnetic δ-shell interaction.

Proposition 6.3. *Let $\eta, \tau, \lambda \in C^\infty(\Sigma; \mathbb{R})$, and let B be defined as in (6.1). Let the operator $\mathcal{D}_{\eta,\tau,\lambda}$ be defined as in (2.4) and (2.5). Then the following hold:*

(i) *Assume $d(x) \neq 0$ for all $x \in \Sigma$. Let $\theta \in \Psi^1_\Sigma$ be given by*

(6.11) $$\theta := -\Lambda \Big[B^{-1} + \frac{1}{2} V(\mathcal{C}_\xi + \mathcal{C}_{\bar{\xi}})V^* \Big] \Lambda,$$

and let Θ be its maximal realization, i.e.,

$$\Theta\varphi := \theta\varphi, \quad \operatorname{dom} \Theta := \{\varphi \in L^2(\Sigma; \mathbb{C}^2) \mid \theta\varphi \in L^2(\Sigma; \mathbb{C}^2)\}.$$

Then

(6.12) $$\operatorname{dom} \mathcal{D}_{\eta,\tau,\lambda} = \{f \in \operatorname{dom} S^* \mid \Gamma_0 f \in \operatorname{dom} \Theta, \ \Gamma_1 f = \Theta\Gamma_0 f\}.$$

(ii) *Assume $\eta, \tau, \lambda \in \mathbb{R}$ and $\eta = \pm\sqrt{\tau^2 + \lambda^2} \neq 0$. Then there exist*

$$\Pi_\pm : L^2(\Sigma; \mathbb{C}^2) \to L^2(\Sigma) \quad and \quad \Pi_\pm^* : L^2(\Sigma) \to L^2(\Sigma; \mathbb{C}^2)$$

such that $\Pi_\pm^* \Pi_\pm$ are orthogonal projectors and, defining $\theta_\pm \in \Psi^1_\Sigma$ by

$$\theta_\pm := -\Lambda \Big[\frac{1}{2\eta} \mathbb{I} + \Pi_\pm \frac{1}{2} V(\mathcal{C}_\xi + \mathcal{C}_{\bar{\xi}})V^* \Pi_\pm^* \Big] \Lambda,$$

and letting $\Theta_\pm$ be its maximal realization, i.e.,

$$\Theta_\pm\varphi := \theta_\pm\varphi, \quad \operatorname{dom} \Theta_\pm := \{\varphi \in L^2(\Sigma) \mid \theta_\pm\varphi \in L^2(\Sigma)\},$$

we have

(6.13) $$\operatorname{dom} \mathcal{D}_{\eta,\tau,\lambda} = \{f \in \operatorname{dom} S^* \mid \Pi_\pm\Gamma_0 f \in \operatorname{dom} \Theta_\pm,$$
$$\Pi_\pm\Gamma_1 f = \Theta_\pm\Pi_\pm\Gamma_0 f, \ (\mathbb{I} - \Pi_\pm^*\Pi_\pm)\Gamma_0 f = 0\}.$$

Remark 6.4. In the case that $\eta = \pm\sqrt{\tau^2 + \lambda^2} = 0$, $\mathcal{D}_{\eta,\tau,\lambda}$ is in fact the free Dirac operator $\mathcal{D}_0$, defined in Section 3.7.

Proof. From (6.3), we see that

$$i(\sigma \cdot \mathbf{n})(\mathcal{T}^D_+ f_+ - \mathcal{T}^D_- f_-) = V^*\Lambda\Gamma_0 f,$$

$$\frac{1}{2}(\mathcal{T}^D_+ f_+ + \mathcal{T}^D_- f_-) = V^*\Lambda^{-1}\Gamma_1 f + \frac{1}{2}(\mathcal{C}_\xi + \mathcal{C}_{\bar{\xi}})V^*\Lambda\Gamma_0 f,$$

so the transmission condition in (2.4) rewrites as follows:

$$(6.14) \quad -V^*\Lambda\Gamma_0 f = (\eta\mathbb{I}_2 + \tau\sigma_3 + \lambda(\sigma \cdot \mathbf{t}))\left(V^*\Lambda^{-1}\Gamma_1 f + \frac{1}{2}(\mathcal{C}_\xi + \mathcal{C}_{\bar{\xi}})V^*\Lambda\Gamma_0 f\right).$$

Multiplying the last equation by V and then using (6.2) together with the identity $V^*V = VV^* = \mathbb{I}_2$, we get

$$(6.15) \quad B\Lambda^{-1}\Gamma_1 f = -\left(\mathbb{I}_2 + \frac{1}{2}B\, V(\mathcal{C}_\xi + \mathcal{C}_{\bar{\xi}})V^*\right)\Lambda\Gamma_0 f.$$

We prove now (i). In the case that $d(x) \neq 0$, the matrix $B = B(x)$ is invertible for all $x \in \Sigma$. Thanks to (6.15), we obtain the representation in (6.12).

Next, we pass to the proof of (ii). Let

$$\Xi_+, \Xi_- : L^2(\Sigma; \mathbb{C}^2) \to L^2(\Sigma), \quad \Xi_+\begin{pmatrix}\varphi_1\\\varphi_2\end{pmatrix} := \varphi_1, \quad \Xi_-\begin{pmatrix}\varphi_1\\\varphi_2\end{pmatrix} := \varphi_2,$$

and let $\mathbb{U}$ be the unitary matrix such that

$$(6.16) \quad B = \mathbb{U}^*\begin{pmatrix}\eta + \sqrt{\tau^2 + \lambda^2} & 0\\ 0 & \eta - \sqrt{\tau^2 + \lambda^2}\end{pmatrix}\mathbb{U}.$$

Finally, let $\Pi_\pm := \Xi_\pm\mathbb{U}$. One sees immediately that $\Pi^*_\pm\Pi_\pm$ are orthogonal projectors in $L^2(\Sigma; \mathbb{C}^2)$, and that $\Pi^*_\pm\Pi_\pm(L^2(\Sigma; \mathbb{C}^2)) \subset L^2(\Sigma; \mathbb{C}^2)$ is isometrically isomorphic to $\Pi_\pm(L^2(\Sigma; \mathbb{C}^2)) \subset L^2(\Sigma)$.

We give only a proof in the case $\eta = \sqrt{\tau^2 + \lambda^2}$, the other case being analogous. From (6.15), we infer that

$$(6.17) \quad -\Lambda\Gamma_0 f = \mathbb{U}^*\begin{pmatrix}2\eta & 0\\ 0 & 0\end{pmatrix}\mathbb{U}\left(\Lambda^{-1}\Gamma_1 f + \frac{1}{2}V(\mathcal{C}_\xi + \mathcal{C}_{\bar{\xi}})V^*\Lambda\Gamma_0 f\right)$$

$$= 2\eta\,\Pi^*_+\Pi_+\left(\Lambda^{-1}\Gamma_1 f + \frac{1}{2}V(\mathcal{C}_\xi + \mathcal{C}_{\bar{\xi}})V^*\Lambda\Gamma_0 f\right).$$

We will show that this equation is equivalent to (6.13). Multiplying (6.17) by $\mathbb{I} - \Pi^*_+\Pi_+$, we get

$$(6.18) \quad (\mathbb{I} - \Pi^*_+\Pi_+)\Lambda\Gamma_0 f = 0.$$

Since $\Pi_+\Pi^*_+ = \mathbb{I}$, multiplying (6.17) by Π_+, we get

$$\Pi_+\left[\frac{1}{2\eta}\mathbb{I} + \frac{1}{2}V(\mathcal{C}_\xi + \mathcal{C}_{\bar{\xi}})V^*\right]\Lambda\Gamma_0 f = -\Pi_+\Lambda^{-1}\Gamma_1 f.$$

Thanks to (6.18), we have

$$(6.19) \qquad \left[\frac{1}{2\eta}\,\mathbb{I} + \Pi_+\frac{1}{2}\,V(\mathcal{C}_\xi + \mathcal{C}_{\bar\xi})V^*\Pi_+^*\right]\Pi_+\Lambda\Gamma_0 f = -\Pi_+\Lambda^{-1}\Gamma_1 f.$$

Since $\eta, \tau, \lambda \in \mathbb{R}$, Λ commutes with Π_+ and Π_+^*. Therefore, taking the bijectivity of Λ into account, (6.18) and (6.19) yield

$$(\mathbb{I} - \Pi_+^*\Pi_+)\Gamma_0 f = 0,$$
$$\Pi_+\Gamma_1 f = -\Lambda\left[\frac{1}{2\eta}\mathbb{I} + \Pi_+\frac{1}{2}\,V(\mathcal{C}_\xi + \mathcal{C}_{\bar\xi})V^*\Pi_+^*\right]\Lambda\,\Pi_+\Gamma_0 f,$$

that is, we get the conditions in (6.13). ∎

In the *non-critical* case, $\mathfrak{C}(\eta, \tau, \lambda) \neq 0$ everywhere on Σ, with $\mathfrak{C}(\eta, \tau, \lambda)$ defined as in (2.9), we can show the self-adjointness of $\mathcal{D}_{\eta,\tau,\lambda}$ using Theorem 3.5 and Proposition 6.3, together with the self-adjointness of the operator Θ. The proof is an adaptation of the proof of Lemma 4.5 of [14] that takes the new interaction into account.

Lemma 6.5. *Let $\eta, \tau, \lambda \in C^\infty(\Sigma; \mathbb{R})$ be such that $\mathfrak{C}(\eta, \tau, \lambda) \neq 0$ for all $x \in \Sigma$. Then the following hold*:

(i) *if $d(x) \neq 0$ for all $x \in \Sigma$, then we have that $\mathrm{dom}\,\Theta = H^1(\Sigma; \mathbb{C}^2)$ and Θ is self-adjoint in $L^2(\Sigma; \mathbb{C}^2)$,*

(ii) *if $\eta, \tau, \lambda \in \mathbb{R}$ and $\eta = \pm\sqrt{\tau^2 + \lambda^2} \neq 0$, then $\mathrm{dom}\,\Theta_\pm = H^1(\Sigma)$ and $\Theta_\pm$ is self-adjoint in $L^2(\Sigma)$.*

Proof. Since the multiplication by V is bounded in $L^2(\Sigma; \mathbb{C}^2)$, from (3.31), we get

$$(6.20) \qquad V(\mathcal{C}_\xi + \mathcal{C}_{\bar\xi})V^* = V\begin{pmatrix} 0 & C_\Sigma\bar{T} \\ TC_\Sigma' & 0 \end{pmatrix}V^* + \hat\Psi_1 = \begin{pmatrix} 0 & C_\Sigma \\ C_\Sigma' & 0 \end{pmatrix} + \hat\Psi_1,$$

with $\hat\Psi_1 \in \Psi_\Sigma^{-1}$.

We start the proof of (i) by putting $\Theta_1 := \Theta \restriction H^1(\Sigma; \mathbb{C}^2)$. Since $\theta \in \Psi_\Sigma^1$, the operator Θ_1 is well defined as an operator in $L^2(\Sigma; \mathbb{C}^2)$. To show (i), we prove that $\Theta_1 = \Theta$ and Θ_1 is self-adjoint in $L^2(\Sigma; \mathbb{C}^2)$. Since $(V(\mathcal{C}_\xi + \mathcal{C}_{\bar\xi})V^*)^* = V(\mathcal{C}_{\bar\xi} + \mathcal{C}_\xi)V^*$ and Λ is self-adjoint as an operator in $L^2(\Sigma)$, Θ_1 is symmetric. Moreover, since $(\Theta \restriction C^\infty) \subset \Theta_1$, we have $\Theta_1 \subset \Theta_1^* \subset (\Theta \restriction C^\infty)^*$. We conclude that $\Theta_1 \subset \Theta$ since Θ is the maximal realization of θ, and so $(\Theta \restriction C^\infty)^* = \Theta$.

Hence, to show that $\Theta_1 = \Theta$, it is now sufficient to prove that $\Theta \subset \Theta_1$, that is to say $\mathrm{dom}\,\Theta \subset \mathrm{dom}\,\Theta_1 = H^1(\Sigma; \mathbb{C}^2)$. We fix $\varphi \in \mathrm{dom}\,\Theta$. Thanks to Proposition 6.3 (i) and (6.20), we have

$$\theta\varphi = -\Lambda P\Lambda\varphi + \hat\Psi_2\,\varphi,$$

with $\hat\Psi_2 \in \Psi_\Sigma^0$ and

$$(6.21) \qquad P = \frac{1}{d}\begin{pmatrix} \eta - \tau & -\lambda \\ -\lambda & \eta + \tau \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & C_\Sigma \\ C_\Sigma' & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{2(\eta-\tau)}{d} & C_\Sigma - \frac{2\lambda}{d} \\ C_\Sigma' - \frac{2\lambda}{d} & \frac{2(\eta+\tau)}{d} \end{pmatrix}.$$

We have $\Lambda P \Lambda \varphi \in L^2(\Sigma; \mathbb{C}^2)$ and $P \Lambda \varphi \in H^{1/2}(\Sigma; \mathbb{C}^2)$ because $\Lambda \colon H^{1/2}(\Sigma; \mathbb{C}^2) \to L^2(\Sigma; \mathbb{C}^2)$ is an isomorphism. Since $C_\Sigma, C'_\Sigma \in \Psi^0_\Sigma$, these pseudo-differential operators give rise to bounded operators in $H^{1/2}(\Sigma; \mathbb{C}^2)$, and this implies that

$$
\frac{1}{2} \begin{pmatrix} \frac{2(\eta+\tau)}{d} & -C_\Sigma - \frac{2\lambda}{d} \\ -C'_\Sigma - \frac{2\lambda}{d} & \frac{2(\eta-\tau)}{d} \end{pmatrix} P \Lambda \varphi
$$

$$
= \frac{1}{d^2} \begin{pmatrix} d + 2\lambda^2 - \frac{d^2}{4} C_\Sigma C'_\Sigma + \frac{\lambda d}{2}(C_\Sigma - C'_\Sigma) & -2(\eta+\tau)\lambda \\ -2(\eta-\tau)\lambda & d + 2\lambda^2 - \frac{d^2}{4} C'_\Sigma C_\Sigma + \frac{\lambda d}{2}(C'_\Sigma - C_\Sigma) \end{pmatrix} \Lambda \varphi
$$

belongs to $H^{1/2}(\Sigma; \mathbb{C}^2)$. Using (3.28), we conclude that

$$
M \Lambda \varphi := \frac{1}{d^2} \begin{pmatrix} \eta^2 - \tau^2 + \lambda^2 - \frac{d^2}{4} & -2(\eta+\tau)\lambda \\ -2(\eta-\tau)\lambda & \eta^2 - \tau^2 + \lambda^2 - \frac{d^2}{4} \end{pmatrix} \Lambda \varphi \in H^{1/2}(\Sigma; \mathbb{C}^2).
$$

Note that

$$
(6.22) \qquad \det M = \frac{1}{d^4}\left[\left(\eta^2 - \tau^2 + \lambda^2 - \frac{d^2}{4}\right)^2 - 4(\eta^2 - \tau^2)\lambda^2\right] = \frac{1}{d^2}\,\mathfrak{C}(\eta, \tau, \lambda)
$$

and, by our hypothesis, M is invertible. Therefore, we get $\Lambda \varphi \in H^{1/2}(\Sigma; \mathbb{C}^2)$ and $\varphi \in H^1(\Sigma; \mathbb{C}^2)$, because $\Lambda \colon H^1(\Sigma; \mathbb{C}^2) \to H^{1/2}(\Sigma; \mathbb{C}^2)$ is an isomorphism. This completes the proof of the case (i).

Now we pass to the proof of (ii). Arguing as in the proof of (i), let $\Theta_{\pm,1} := \Theta_\pm \restriction H^1(\Sigma)$. It is true that $\Theta_{\pm,1} \subset \Theta^*_{\pm,1} \subset \Theta_\pm$, and so we can conclude the proof if we show that $\operatorname{dom} \Theta_\pm \subset \operatorname{dom} \Theta_{\pm,1} = H^1(\Sigma)$. By Proposition 6.3 (ii) and (6.20), we have

$$
\Theta_\pm \varphi = -\Lambda\left[\frac{1}{2\eta}\mathbb{I} + \frac{1}{2}\Pi_\pm \begin{pmatrix} 0 & C_\Sigma \\ C'_\Sigma & 0 \end{pmatrix} \Pi^*_\pm\right] \Lambda \varphi + \widehat{\Psi} \varphi \quad \text{for all } \varphi \in \operatorname{dom} \Theta_\pm,
$$

with some symmetric operator $\widehat{\Psi} \in \Psi^0_\Sigma$. Since $\Lambda \colon H^{1/2}(\Sigma) \to L^2(\Sigma)$ is an isomorphism, the last equation implies that

$$
(6.23) \qquad \left[\frac{1}{2\eta}\mathbb{I} + \frac{1}{2}\Pi_\pm \begin{pmatrix} 0 & C_\Sigma \\ C'_\Sigma & 0 \end{pmatrix} \Pi^*_\pm\right] \Lambda \varphi \in H^{1/2}(\Sigma).
$$

For the unitary matrix $\mathbb{U}$ in (6.16), we may choose

$$
\mathbb{U} = \begin{cases} \mathbb{I}_2 & \text{if } \lambda = 0, \\ ((\tau + \sqrt{\tau^2 + \lambda^2})^2 + \lambda^2)^{-1/2} \begin{pmatrix} \tau + \sqrt{\tau^2 + \lambda^2} & \lambda \\ -\lambda & \tau + \sqrt{\tau^2 + \lambda^2} \end{pmatrix} & \text{if } \lambda \neq 0. \end{cases}
$$

When $\lambda = 0$ our choice of $\mathbb{U}$ gives $\Pi_\pm = \Xi_\pm$, and from (6.23), we get

$$
(6.24) \qquad \left[\frac{1}{2\eta}\mathbb{I} + \frac{1}{2}\Pi_\pm \begin{pmatrix} 0 & C_\Sigma \\ C'_\Sigma & 0 \end{pmatrix} \Pi^*_\pm\right] \Lambda \varphi = \frac{1}{2\eta}\Lambda \varphi \in H^{1/2}(\Sigma).
$$

Since $\Lambda\colon H^1(\Sigma) \to H^{1/2}(\Sigma)$ is an isomorphism, we get $\varphi \in H^1(\Sigma)$. If $\lambda \neq 0$, then (6.23) yields

$$\left[\frac{1}{2\eta}\,\mathbb{I} + \frac{1}{2}\,\Pi_\pm \begin{pmatrix} 0 & C_\Sigma \\ C'_\Sigma & 0 \end{pmatrix} \Pi_\pm^* \right]\Lambda\varphi = \left[\frac{1}{2\eta}\,\mathbb{I} \pm \frac{\lambda}{4\eta}(C_\Sigma + C'_\Sigma) \right]\Lambda\varphi \in H^{1/2}(\Sigma).$$

Since $C_\Sigma, C_\Sigma \in \Psi_\Sigma^0$, we get

$$\left(\mathbb{I} \mp \frac{\lambda}{2}(C_\Sigma + C'_\Sigma)\right)\left(\mathbb{I} \pm \frac{\lambda}{2}(C_\Sigma + C'_\Sigma)\right)\Lambda\varphi = \left[\mathbb{I} - \frac{\lambda^2}{4}(C_\Sigma + C'_\Sigma)^2\right]\Lambda\varphi \in H^{1/2}(\Sigma).$$

Taking (3.28) into account, we finally obtain

$$(\mathbb{I} - \lambda^2)\Lambda\varphi \in H^{1/2}(\Sigma).$$

We conclude that $\Lambda\varphi \in H^{1/2}(\Sigma)$, since for $d = 0$, the condition $\mathfrak{C}(\eta, \tau, \lambda) \neq 0$ forces $\lambda^2 \neq 1$. Since $\Lambda\colon H^1(\Sigma) \to H^{1/2}(\Sigma)$ is an isomorphism, we get $\varphi \in H^1(\Sigma)$ also in this case. ∎

We are now ready to show the self-adjointness of $\mathcal{D}_{\eta,\tau,\lambda}$ in the non-critical case.

Theorem 6.6. *Let $\eta, \tau, \lambda \in C^\infty(\Sigma; \mathbb{R})$ be such that $\mathfrak{C}(\eta, \tau, \lambda)(x) \neq 0$. Moreover, let either $d(x) \neq 0$ for all $x \in \Sigma$, or let η, τ, λ be constant such that $d = 0$. Then $\mathcal{D}_{\eta,\tau,\lambda}$, defined by (2.4) and (2.5), is self-adjoint in $L^2(\mathbb{R}^2; \mathbb{C}^2)$ with domain $\operatorname{dom}\mathcal{D}_{\eta,\tau,\lambda} \subset H^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$. Moreover, for all $z \in \rho(\mathcal{D}_{\eta,\tau,\lambda}) \cap \rho(\mathcal{D}_0)$, the operator $\mathbb{I}_2 + (\eta\mathbb{I}_2 + \tau\sigma_3 + \lambda(\sigma \cdot \mathbf{t}))\mathcal{C}_z$ is bounded and boundedly invertible in $H^{1/2}(\Sigma; \mathbb{C}^2)$ and*

$$
\begin{aligned}
(6.25) \quad & (\mathcal{D}_{\eta,\tau,\lambda} - z)^{-1} \\
& = (\mathcal{D}_0 - z)^{-1} - \Phi_z\big(\mathbb{I}_2 + (\eta\mathbb{I}_2 + \tau\sigma_3 + \lambda(\sigma \cdot \mathbf{t}))\mathcal{C}_z\big)^{-1}(\eta\mathbb{I}_2 + \tau\sigma_3 + \lambda(\sigma \cdot \mathbf{t}))\Phi'_{\bar{z}}.
\end{aligned}
$$

Proof. The proof is analogous to the proof of Theorem 4.6 of [14], when we have set the right framework.

The self-adjointness of $\mathcal{D}_{\eta,\tau,\lambda}$ follows from the self-adjointness of Θ and $\Theta_\pm$ in $L^2(\Sigma; \mathbb{C}^2)$ and $L^2(\Sigma)$, respectively, thanks to Theorem 3.5. Moreover, Lemma 6.2 implies that $\operatorname{dom}\mathcal{D}_{\eta,\tau,\lambda} \subset H^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$, since, by Lemma 6.5, $\operatorname{dom}\Theta = H^1(\Sigma; \mathbb{C}^2)$ and $\operatorname{dom}\Theta_\pm = H^1(\Sigma)$.

We show (6.25) in the case $d(x) \neq 0$ for all $x \in \Sigma$. By Theorem 3.5 (iii), $\Theta - M_z$ is boundedly invertible in $L^2(\Sigma; \mathbb{C}^2)$, for $z \in \rho(\mathcal{D}_{\eta,\tau,\lambda}) \cap \rho(\mathcal{D}_0)$, and

$$(\mathcal{D}_{\eta,\tau,\lambda} - z)^{-1} = (\mathcal{D}_0 - z)^{-1} + G_z(\Theta - M_z)^{-1}G_{\bar{z}}^*.$$

From the definition of M_z (6.5), we get

$$(6.26) \quad \Theta - M_z = -\Lambda V(\eta\mathbb{I}_2 + \tau\sigma_3 + \lambda(\sigma \cdot \mathbf{t}))^{-1}(\mathbb{I}_2 + (\eta\mathbb{I}_2 + \tau\sigma_3 + \lambda(\sigma \cdot \mathbf{t}))\mathcal{C}_z)V^*\Lambda.$$

The operator $\Theta - M_z$ is bijective in $L^2(\Sigma; \mathbb{C}^2)$ when defined on $\operatorname{dom}\Theta = H^1(\Sigma; \mathbb{C}^2)$, and $(\mathbb{I}_2 + (\eta\mathbb{I}_2 + \tau\sigma_3 + \lambda(\sigma \cdot \mathbf{t}))\mathcal{C}_z)$ is well defined and bounded in $H^{1/2}(\Sigma; \mathbb{C}^2)$. Recalling the definition of G_z from (6.4), we get (6.25). The case $d = 0$ is analogous and will be omitted. ∎

In the next proposition we gather some basic results on the spectrum of $\mathcal{D}_{\eta,\tau,\lambda}$.

Proposition 6.7. *Let $\eta, \tau, \lambda \in C^\infty(\Sigma; \mathbb{R})$ be such that $\mathfrak{C}(\eta, \tau, \lambda)(x) \neq 0$ everywhere on Σ. Moreover, let either $d(x) \neq 0$ for all $x \in \Sigma$, or let η, τ and λ be constant such that $d = 0$. Let $\mathcal{D}_{\eta,\tau,\lambda}$ be defined as in (2.4) and (2.5). Then the following hold:*

(i) *We have $\sigma_{\text{ess}}(\mathcal{D}_{\eta,\tau,\lambda}) = (-\infty, -|m|] \cup [|m|, +\infty)$; if, in particular, $m = 0$, then $\sigma(\mathcal{D}_{\eta,\tau,\lambda}) = \sigma_{\text{ess}}(\mathcal{D}_{\eta,\tau,\lambda}) = \mathbb{R}$.*

(ii) *If $m \neq 0$, then $\mathcal{D}_{\eta,\tau,\lambda}$ has at most finitely many eigenvalues in $(-|m|, |m|)$.*

(iii) *Assume $m \neq 0$. Then $z \in (-|m|, |m|)$ is a discrete eigenvalue of $\mathcal{D}_{\eta,\tau,\lambda}$ if and only if there exists $\varphi \in H^{1/2}(\Sigma; \mathbb{C}^2)$ such that $(\mathbb{I} + (\eta \mathbb{I} + \tau\sigma_3 + \lambda(\sigma \cdot \mathbf{t}))\mathcal{C}_z)\varphi = 0$.*

Proof. The proof is analogous to the proof of Theorem 4.7 of [14]. Firstly, let us show (i) and (ii). Thanks to Proposition 3.6 (i), $(-\infty, -|m|] \cup [|m|, +\infty) \subset \sigma_{\text{ess}}(\mathcal{D}_{\eta,\tau,\lambda})$. Moreover, since $\text{dom}\,\mathcal{D}_{\eta,\tau,\lambda} \subset H^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$, by Theorem 6.6, Proposition 3.6 (ii) implies that the spectrum of $\mathcal{D}_{\eta,\tau,\lambda}$ in $(-|m|, |m|)$ is discrete and finite.

We show (iii) only in the case $d(x) \neq 0$ for all $x \in \Sigma$, the case $d = 0$ being similar. By Theorem 3.5 (i) combined with (6.26), $z \in \rho(\mathcal{D}_0)$ is an eigenvalue of $\mathcal{D}_{\eta,\tau,\lambda}$ if and only if there exists $\psi \in \text{dom}\,\Theta = H^1(\Sigma; \mathbb{C}^2)$ such that

$$-\Lambda V(\eta \mathbb{I}_2 + \tau\sigma_3 + \lambda(\sigma \cdot \mathbf{t}))^{-1}(\mathbb{I}_2 + (\eta \mathbb{I}_2 + \tau\sigma_3 + \lambda(\sigma \cdot \mathbf{t}))\mathcal{C}_z)V^*\Lambda\psi = 0,$$

i.e., if and only if $\varphi = V^*\Lambda\psi \in H^{1/2}(\Sigma; \mathbb{C}^2)$ satisfies

$$(\mathbb{I}_2 + (\eta \mathbb{I}_2 + \tau\sigma_3 + \lambda(\sigma \cdot \mathbf{t}))\mathcal{C}_z)\varphi = 0. \qquad \blacksquare$$

7. The purely magnetic critical interaction

In this section we give the proof of Theorem 2.5. We consider the case $\lambda = 2$ only, since the case $\lambda = -2$ can be treated analogously. Recall that, by Theorem 2.3, the operator $\mathcal{D}_{0,0,2}$ can be decomposed into the orthogonal sum

$$\mathcal{D}_{0,0,2} = \mathcal{D}_{0,0,2}^+ \oplus \mathcal{D}_{0,0,2}^- =: \mathcal{D}^+ \oplus \mathcal{D}^-,$$

with

$$\text{dom}\,\mathcal{D}^\pm := \{f_\pm \in H(\sigma, \Omega_\pm) \mid [\pm i(\sigma \cdot \mathbf{n}) + (\sigma \cdot \mathbf{t}))]\mathcal{T}_\pm^D f_\pm = 0\},$$
$$\mathcal{D}^\pm f_\pm := \mathcal{D}_0 f_\pm \quad \text{for all } f_\pm \in \text{dom}\,\mathcal{D}^\pm.$$

Using that $n_1 = t_2$, $n_2 = -t_1$, we find

$$i(\sigma \cdot \mathbf{n}) + (\sigma \cdot \mathbf{t}) = \begin{pmatrix} 0 & 0 \\ 2T & 0 \end{pmatrix}, \quad -i(\sigma \cdot \mathbf{n}) + (\sigma \cdot \mathbf{t}) = \begin{pmatrix} 0 & 2\overline{T} \\ 0 & 0 \end{pmatrix}.$$

Hence, $\mathcal{D}^\pm$ have the following representations:

$$\mathcal{D}^+ f = \mathcal{D}_0 f, \quad \text{dom}\,\mathcal{D}^+ = \{f = (f_1, f_2)^\top \in H(\sigma, \Omega_+) \mid \mathcal{T}_+^D f_1 = 0\},$$
$$\mathcal{D}^- f = \mathcal{D}_0 f, \quad \text{dom}\,\mathcal{D}^- = \{f = (f_1, f_2)^\top \in H(\sigma, \Omega_-) \mid \mathcal{T}_-^D f_2 = 0\}.$$

Using the Cauchy–Riemann differential expressions

$$\partial_z := \frac{1}{2}(\partial_1 - i\,\partial_2) \quad \text{and} \quad \partial_{\bar{z}} := \frac{1}{2}(\partial_1 + i\,\partial_2),$$

we can represent $\mathcal{D}^\pm$ as follows:

$$\mathcal{D}^\pm f = \begin{pmatrix} m & -2i\,\partial_z \\ -2i\,\partial_{\bar{z}} & -m \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

$$\operatorname{dom}\mathcal{D}^+ = \{ f = (f_1, f_2)^\top \mid f_1, f_2, \partial_z f_2, \partial_{\bar{z}} f_1 \in L^2(\Omega_+),\ \mathcal{T}_+^D f_1 = 0 \},$$

$$\operatorname{dom}\mathcal{D}^- = \{ f = (f_1, f_2)^\top \mid f_1, f_2, \partial_z f_2, \partial_{\bar{z}} f_1 \in L^2(\Omega_-),\ \mathcal{T}_-^D f_2 = 0 \}.$$

In view of Lemma 18 of [3] (see also Lemma 3.1 of [14]), the domains of $\mathcal{D}^\pm$ can altern-atively be given by

$$\operatorname{dom}\mathcal{D}^+ = \{ f = (f_1, f_2)^\top \mid f_2, \partial_z f_2 \in L^2(\Omega_+),\ f_1 \in H_0^1(\Omega_+) \},$$

$$\operatorname{dom}\mathcal{D}^- = \{ f = (f_1, f_2)^\top \mid f_1, \partial_{\bar{z}} f_1 \in L^2(\Omega_-),\ f_2 \in H_0^1(\Omega_-) \}.$$

The operators $\mathcal{D}^\pm$ can be viewed as bounded symmetric perturbations of the respective massless Dirac operators. Since, by Proposition 1 of [65], the unperturbed massless Dirac operators are self-adjoint, we conclude that $\mathcal{D}^\pm$ are self-adjoint as well.

Next, we will show the symmetry of the spectrum of $\mathcal{D}^\pm$. For $\nu \in \mathbb{C}$ such that $\nu^2 - m^2 \neq 0$, we introduce the matrix

$$T_\nu = \begin{pmatrix} \sqrt{\frac{m+\nu}{m-\nu}} & 0 \\ 0 & \sqrt{\frac{m-\nu}{m+\nu}} \end{pmatrix}.$$

Clearly, T_ν is invertible and $T_\nu^{-1} = T_{-\nu}$. Moreover, for any $\nu \in \mathbb{C} \setminus \{-m, m\}$, we have $T_\nu(\operatorname{dom}\mathcal{D}^\pm) = \operatorname{dom}\mathcal{D}^\pm$ and

$$(7.1) \qquad\qquad (\mathcal{D}^\pm + \nu)T_{-\nu} = T_\nu(\mathcal{D}^\pm - \nu).$$

Let $\nu \in \sigma(\mathcal{D}^\pm) \setminus \{-m, m\}$. Then there exists a sequence $(\psi_n)_n$ in $\operatorname{dom}\mathcal{D}^\pm$ such that

$$\lim_{n\to\infty} \frac{\|(\mathcal{D}^\pm - \nu)\psi_n\|}{\|\psi_n\|} = 0.$$

Let $\phi_n := T_\nu^{-1}\psi_n = T_{-\nu}\psi_n$. Then we get

$$\frac{\|(\mathcal{D}^\pm + \nu)\phi_n\|}{\|\phi_n\|} = \frac{\|(\mathcal{D}^\pm + \nu)T_{-\nu}\psi_n\|}{\|T_{-\nu}\psi_n\|} = \frac{\|T_\nu(\mathcal{D}^\pm - \nu)\psi_n\|}{\|T_\nu^{-1}\psi_n\|}$$

$$\leq \|T_\nu\|^2 \frac{\|(\mathcal{D}^\pm - \nu)\psi_n\|}{\|\psi_n\|} \to 0 \quad \text{as } n \to \infty.$$

Hence, we conclude that $-\nu \in \sigma(\mathcal{D}^\pm)$. Moreover, if $\nu \neq \pm m$ is an eigenvalue of $\mathcal{D}^\pm$, then, in view of identity (7.1), $-\nu$ is also an eigenvalue of $\mathcal{D}^\pm$.

Now we perform the spectral analysis of $\mathcal{D}^+$. First of all, we notice the inclusion

$$\ker(\mathcal{D}^+ + m) \supset \left\{ \begin{pmatrix} 0 \\ f_2 \end{pmatrix} \ \middle| \ f_2 \in L^2(\Omega_+), \ \partial_z f_2 = 0 \right\}.$$

Indeed,

$$\mathcal{D}^+ \begin{pmatrix} 0 \\ f_2 \end{pmatrix} = \begin{pmatrix} -2i\,\partial_z f_2 \\ -mf_2 \end{pmatrix} = -m \begin{pmatrix} 0 \\ f_2 \end{pmatrix}.$$

Since the space of square-integrable anti-holomorphic functions on Ω_+ is infinite-dimensional, $-m$ is an eigenvalue of infinite multiplicity in the spectrum of $\mathcal{D}^+$. In particular, $\operatorname{dom}\mathcal{D}^+ \not\subset H^s(\Omega_+; \mathbb{C}^2)$ for any $s > 0$, as otherwise the spectrum of $\mathcal{D}^+$ would be purely discrete, due to the compactness of embedding of the Sobolev spaces $H^s(\Omega_+; \mathbb{C}^2)$, $s > 0$, into $L^2(\Omega_+; \mathbb{C}^2)$. Thus, $\operatorname{dom}\mathcal{D}_{0,0,2} \not\subset H^s(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$ for any $s > 0$.

Consider the auxiliary operators

$$(7.2) \qquad \begin{aligned} A_+\psi &= -2i\,\partial_{\bar{z}}\psi, \quad \operatorname{dom}A_+ = H_0^1(\Omega_+), \\ A_-\psi &= -2i\,\partial_z\psi, \quad \operatorname{dom}A_- = H_0^1(\Omega_-). \end{aligned}$$

The adjoints of $A_\pm$ are characterised in the spirit of Proposition 1 of [65] as

$$\begin{aligned} A_+^*\psi &= -2i\,\partial_z\psi, \quad \operatorname{dom}A_+^* = \{\psi \in L^2(\Omega_+) \mid \partial_z\psi \in L^2(\Omega_+)\}, \\ A_-^*\psi &= -2i\,\partial_{\bar{z}}\psi, \quad \operatorname{dom}A_-^* = \{\psi \in L^2(\Omega_-) \mid \partial_{\bar{z}}\psi \in L^2(\Omega_-)\}. \end{aligned}$$

The quadratic form for $(\mathcal{D}^+)^2$ is given by

$$\mathfrak{h}^+[f] := \|\mathcal{D}^+ f\|_{L^2(\Omega_+;\mathbb{C}^2)}^2, \quad \operatorname{dom}\mathfrak{h}^+ := \operatorname{dom}\mathcal{D}^+.$$

Next, we compute

$$\begin{aligned} \mathfrak{h}^+[f] = \|\mathcal{D}^+ f\|_{L^2(\Omega_+;\mathbb{C}^2)}^2 &= \|mf_1 - 2i\,\partial_z f_2\|_{L^2(\Omega_+)}^2 + \|-mf_2 - 2i\,\partial_{\bar{z}} f_1\|_{L^2(\Omega_+)}^2 \\ &= 4\|\partial_z f_2\|_{L^2(\Omega_+)}^2 + 4\|\partial_{\bar{z}} f_1\|_{L^2(\Omega_+)}^2 \\ &\quad + 4m\,\operatorname{Re}[(f_2, i\,\partial_{\bar{z}} f_1)_{L^2(\Omega_+)} - (f_1, i\,\partial_z f_2)_{L^2(\Omega_+)}] \\ &\quad + m^2\|f_1\|_{L^2(\Omega_+)}^2 + m^2\|f_2\|_{L^2(\Omega_+)}^2. \end{aligned}$$

Integrating by parts, we find with the aid of $\operatorname{dom}\mathfrak{h}^+ = \operatorname{dom}\mathcal{D}^+$ that

$$\begin{aligned} \operatorname{Re}[(f_2, &\, i\,\partial_{\bar{z}} f_1)_{L^2(\Omega_+)} - (f_1, i\,\partial_z f_2)_{L^2(\Omega_+)}] \\ &= \operatorname{Re}[(i\,\partial_z f_2, f_1)_{L^2(\Omega_+)} - (f_1, i\,\partial_z f_2)_{L^2(\Omega_+)}] = 0. \end{aligned}$$

Thus, the expression for $\mathfrak{h}^+$ simplifies as

$$\begin{aligned} \mathfrak{h}^+[f] &= 4\|\partial_z f_2\|_{L^2(\Omega_+)}^2 + 4\|\partial_{\bar{z}} f_1\|_{L^2(\Omega_+)}^2 + m^2\|f_1\|_{L^2(\Omega_+)}^2 + m^2\|f_2\|_{L^2(\Omega_+)}^2 \\ &= \|A_+^* f_2\|_{L^2(\Omega_+)}^2 + \|A_+ f_1\|_{L^2(\Omega_+)}^2 + m^2\|f_1\|_{L^2(\Omega_+)}^2 + m^2\|f_2\|_{L^2(\Omega_+)}^2. \end{aligned}$$

The domain of $\mathfrak{h}^+$ can be written as

$$\operatorname{dom}\mathfrak{h}^+ = \{f = (f_1, f_2)^\top \mid f_1 \in \operatorname{dom}A_+, \ f_2 \in \operatorname{dom}A_+^*\} = \operatorname{dom}A_+ \oplus \operatorname{dom}A_+^*.$$

Therefore, we end up with the orthogonal decomposition

$$(\mathcal{D}^+)^2 = (A_+^* A_+ + m^2) \oplus (A_+ A_+^* + m^2),$$

from which we deduce, using Propositions 2 and 3 of [65], that

$$\sigma((\mathcal{D}^+)^2) \setminus \{m^2\} = \{m^2 + \mu \mid \mu \in \sigma(-\Delta_{\mathrm{D}}^{\Omega_+})\},$$

where $-\Delta_{\mathrm{D}}^{\Omega_+}$ is the Dirichlet Laplacian on Ω_+. Using the symmetry of the spectrum shown above, we obtain that

$$\sigma(\mathcal{D}^+) \setminus \{-|m|, |m|\} = \{\pm\sqrt{m^2 + \mu} \mid \mu \in \sigma(-\Delta_{\mathrm{D}}^{\Omega_+})\}.$$

Hence, (iii) of Theorem 2.5 follows. Moreover, we observe that $\mathcal{D}^+ \upharpoonright H^1(\Omega_+; \mathbb{C}^2)$ is essentially self-adjoint provided that $H_0^1(\Omega_+) \oplus H^1(\Omega_+)$ is a core for $\mathcal{D}^+$. The latter follows from the density of $H^1(\Omega_+)$ in dom A_+^*; cf. Lemma 14 of [3].

Now we perform the spectral analysis of $\mathcal{D}^-$. As in the analysis of $\mathcal{D}^+$, we notice the inclusion

$$\ker(\mathcal{D}^- - m) \supset \left\{ \begin{pmatrix} f_1 \\ 0 \end{pmatrix} \,\middle|\, f_1 \in L^2(\Omega_-), \, \partial_{\bar{z}} f_1 = 0 \right\}.$$

Indeed,

$$\mathcal{D}^- \begin{pmatrix} f_1 \\ 0 \end{pmatrix} = \begin{pmatrix} m f_1 \\ -2i \, \partial_{\bar{z}} f_1 \end{pmatrix} = m \begin{pmatrix} f_1 \\ 0 \end{pmatrix}.$$

We observe that the space of square-integrable holomorphic functions on Ω_- is also infinite-dimensional. Indeed, for an arbitrary $z_0 \in \Omega_+$, the family of linear independent functions $\{(z - z_0)^{-k}\}_{k \geq 2}$ is square-integrable and holomorphic in Ω_-. Hence, m is an eigenvalue of infinite multiplicity in the spectrum of $\mathcal{D}^-$ and thus, combining with the fact that $-m$ is an eigenvalue of infinite multiplicity in the spectrum of $\mathcal{D}_+$ shown above, the claim of (ii) of Theorem 2.5 follows. Next we consider the quadratic form

$$\mathfrak{h}^-[f] := \|\mathcal{D}^- f\|_{L^2(\Omega_-; \mathbb{C}^2)}^2, \quad \text{dom } \mathfrak{h}^- := \text{dom } \mathcal{D}^-$$

for $(\mathcal{D}^-)^2$. Repeating the same type of computation as we did for $\mathcal{D}^+$, we get

$$\begin{aligned}
\mathfrak{h}^-[f] := \|\mathcal{D}^- f\|_{L^2(\Omega_-; \mathbb{C}^2)}^2 &= \|m f_1 - 2i \, \partial_z f_2\|_{L^2(\Omega_-)}^2 + \|-m f_2 - 2i \, \partial_{\bar{z}} f_1\|_{L^2(\Omega_-)}^2 \\
&= 4\|\partial_z f_2\|_{L^2(\Omega_-)}^2 + 4\|\partial_{\bar{z}} f_1\|_{L^2(\Omega_-)}^2 + m^2\|f_1\|_{L^2(\Omega_-)}^2 + m^2\|f_2\|_{L^2(\Omega_-)}^2 \\
&= \|A_- f_2\|_{L^2(\Omega_-)}^2 + \|A_-^* f_1\|_{L^2(\Omega_-)}^2 + m^2\|f_1\|_{L^2(\Omega_-)}^2 + m^2\|f_2\|_{L^2(\Omega_-)}^2.
\end{aligned}$$

The domain of $\mathfrak{h}^-$ can be written as

$$\text{dom } \mathfrak{h}^- = \text{dom } A_-^* \oplus \text{dom } A_-.$$

Hence, we have the orthogonal decomposition

$$(\mathcal{D}^-)^2 = (A_- A_-^* + m^2) \oplus (A_-^* A_- + m^2),$$

which implies, in view of Propositions 2 and 3 of [65], that

$$\sigma((\mathcal{D}^-)^2) \setminus \{m^2\} = \{m^2 + \mu \mid \mu \in \sigma(-\Delta_{\mathrm{D}}^{\Omega_-}) \setminus \{0\}\},$$

where $-\Delta_D^{\Omega_-}$ is the Dirichlet Laplacian on Ω_-. Taking that $\sigma(-\Delta_D^{\Omega_-}) = [0, \infty)$ into account, we get $\sigma((\mathcal{D}^-)^2) = [m^2, +\infty)$. In view of the symmetry of the spectrum of $\mathcal{D}^-$ shown above, we necessarily get that $\sigma(\mathcal{D}^-) = (-\infty, -|m|] \cup [|m|, +\infty)$. Hence, (i) of Theorem 2.5 is shown. The essential self-adjointness of $\mathcal{D}^- \upharpoonright H^1(\Omega_-; \mathbb{C}^2)$ follows analogously to that of $\mathcal{D}^+ \upharpoonright H^1(\Omega_+; \mathbb{C}^2)$.

The essential self-adjointness of $\mathcal{D}_{0,0,2} \upharpoonright H^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$ follows from the essential self-adjointness of $\mathcal{D}^\pm \upharpoonright H^1(\Omega_\pm; \mathbb{C}^2)$. Thus, the proof is concluded.

Remark 7.1. The above spectral analysis of $\mathcal{D}^\pm$ is reminiscent of the spectral analysis of three-dimensional Dirac operators with zig-zag boundary conditions on general open sets performed in [40], which has appeared while the present paper was under preparation.

8. Approximation of δ-shell interactions by regular potentials

In this section we prove Theorem 2.6 on approximation of the Dirac operator with δ-shell interaction by a sequence of Dirac operators with regular scaled potentials.

Proof of Theorem 2.6. For all $0 < \varepsilon < \beta$, we define the self-adjoint operators $\mathcal{E}_{\eta,\tau,\lambda;\varepsilon}$ according to (2.14) and we define the operators $\mathcal{D}_{\hat{\eta},\hat{\tau},\hat{\lambda}}$ according to (2.4), (2.5). Since $\mathfrak{C}(\hat{\eta}, \hat{\tau}, \hat{\lambda}) \neq 0$ everywhere on Σ, Theorem 2.2 tells us that these operators are self-adjoint and dom $\mathcal{D}_{\hat{\eta},\hat{\tau},\hat{\lambda}} \subset H^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$. Thanks to Theorem VIII.26 of [63], since the limiting operators and the limit operator are self-adjoint, the family $\{\mathcal{E}_{\eta,\tau,\lambda;\varepsilon}\}_{\varepsilon \in (0,\beta)}$ converges in the strong resolvent sense to $\mathcal{D}_{\hat{\eta},\hat{\tau},\hat{\lambda}}$ as $\varepsilon \to 0$ if and only if it converges in the strong graph limit sense. The latter means that, for all $\psi \in \text{dom } \mathcal{D}_{\hat{\eta},\hat{\tau},\hat{\lambda}}$, there exists a family of vectors $\{\psi_\varepsilon\}_{\varepsilon \in (0,\beta)} \subset \text{dom } \mathcal{E}_{\eta,\tau,\lambda;\varepsilon} = H^1(\mathbb{R}^2; \mathbb{C}^2)$ such that

$$(8.1) \qquad \lim_{\varepsilon \to 0} \psi_\varepsilon = \psi \quad \text{and} \quad \lim_{\varepsilon \to 0} \mathcal{E}_{\eta,\tau,\lambda;\varepsilon} \psi_\varepsilon = \mathcal{D}_{\hat{\eta},\hat{\tau},\hat{\lambda}} \psi \quad \text{in } L^2(\mathbb{R}^2; \mathbb{C}^2).$$

Without loss of generality we can assume $m = 0$, because, by its very definition, the strong graph convergence (8.1) is stable with respect to bounded symmetric perturbations.

Let $\psi \equiv \psi_+ \oplus \psi_- \in \text{dom } \mathcal{D}_{\hat{\eta},\hat{\tau},\hat{\lambda}}$. From (2.15), (2.16) and (2.17), we observe that $\hat{d} := \hat{\eta}^2 - \hat{\tau}^2 - \hat{\lambda}^2 > -4$. Therefore, by Lemma 5.1 (i),

$$(8.2) \qquad \mathcal{T}_+^D \psi_+ = R_{\hat{\eta},\hat{\tau},\hat{\lambda}} \mathcal{T}_-^D \psi_-,$$

where

$$(8.3) \qquad R_{\hat{\eta},\hat{\tau},\hat{\lambda}}(x_\Sigma) = \frac{4}{4+\hat{d}} \Big(\frac{4-\hat{d}}{4} \mathbb{I}_2 + i\hat{\eta}(\sigma \cdot \mathbf{n}) + \hat{\tau}(\sigma \cdot \mathbf{t}) - \hat{\lambda}\sigma_3 \Big)(x_\Sigma).$$

Clearly, $R_{\hat{\eta},\hat{\tau},\hat{\lambda}} \in C^\infty(\Sigma; \mathbb{C}^{2 \times 2})$. Moreover, by Proposition 3.3 combined with the fact that dom $\mathcal{D}_{\hat{\eta},\hat{\tau},\hat{\lambda}} \subset H^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$, see Theorem 6.6, $\mathcal{T}_\pm^D \psi_\pm \in H^{1/2}(\Sigma; \mathbb{C}^2)$.

Recall that $\mathcal{E}_{\eta,\tau,\lambda;\varepsilon} = \mathcal{D}_0 + \mathbb{V}_{\eta,\tau,\lambda;\varepsilon}$, where, for all x in Σ_ε, i.e., in the ε-tubular neighbourhood of Σ,

$$\mathbb{V}_{\eta,\tau,\lambda;\varepsilon} = B_{\eta,\tau,\lambda}(x_\Sigma)h_\varepsilon(p),$$

where $B_{\eta,\tau,\lambda}$ was introduced in (2.12), and $\mathbb{V}_{\eta,\tau,\lambda;\varepsilon} = 0$ everywhere else, see (2.13). According to Lemma A.1,

$$(8.4) \qquad \exp[i(\sigma \cdot \mathbf{n}(x_\Sigma))B_{\eta,\tau,\lambda}(x_\Sigma)] = R_{\hat{\eta},\hat{\tau},\hat{\lambda}}(x_\Sigma) \quad \text{for all } x_\Sigma \in \Sigma.$$

Since, by definition, $\int_{-\varepsilon}^{\varepsilon} h_\varepsilon(t)\, dt = 1$, we can rewrite (8.2) as

$$(8.5) \qquad \exp\Big[-i\Big(\int_{-\varepsilon}^{0} h_\varepsilon(t)\, dt\Big)(\sigma \cdot \mathbf{n})B_{\eta,\tau,\lambda}\Big]\mathcal{T}_+^D \psi_+$$

$$= \exp\Big[i\Big(\int_{0}^{\varepsilon} h_\varepsilon(t)\, dt\Big)(\sigma \cdot \mathbf{n})B_{\eta,\tau,\lambda}\Big]\mathcal{T}_-^D \psi_-.$$

Let us now construct the family $\{\psi_\varepsilon\}_{\varepsilon \in (0,\beta)}$. For all $\varepsilon \in (0,\beta)$, we put

$$H_\varepsilon : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad H_\varepsilon(p) := \begin{cases} \int_p^\varepsilon h_\varepsilon(t)\, dt, & 0 < p < \varepsilon, \\ -\int_{-\varepsilon}^p h_\varepsilon(t)\, dt, & -\varepsilon < p < 0, \\ 0, & |p| \geq \varepsilon. \end{cases}$$

Note that $\operatorname{supp} H_\varepsilon \subset (-\varepsilon, \varepsilon)$ and $H_\varepsilon \in L^\infty(\mathbb{R})$. Since $\|H_\varepsilon\|_{L^\infty(\mathbb{R})} \leq \|h\|_{L^1(\mathbb{R})}$, $\{H_\varepsilon\}_\varepsilon$ is bounded uniformly in ε. For all $\varepsilon \in (0,\beta)$, the restrictions of H_ε to $\mathbb{R}_\pm$ are uniformly continuous, so finite limits at $p = 0$ exist, and differentiable a.e. with derivative being bounded, since $h_\varepsilon \in L^\infty(\mathbb{R};\mathbb{R})$. Furthermore, H_ε has a jump at the origin of size $\int_{-\varepsilon}^\varepsilon h_\varepsilon(t)\, dt = 1$. Next, we set

$$\mathbb{U}_\varepsilon : \mathbb{R}^2 \setminus \Sigma \to \mathbb{C}^{2\times 2},$$

$$(8.6) \qquad \mathbb{U}_\varepsilon(x) := \begin{cases} \exp[i(\sigma \cdot \mathbf{n})B_{\eta,\tau,\lambda}(\mathcal{P}_\Sigma(x))\, H_\varepsilon(\mathcal{P}_\perp(x))], & x \in \Sigma_\varepsilon \setminus \Sigma, \\ \mathbb{I}_2, & x \in \mathbb{R}^2 \setminus \Sigma_\varepsilon, \end{cases}$$

where the mappings $\mathcal{P}_\Sigma$ and $\mathcal{P}_\perp$ are defined as in (3.11) and (3.9), respectively. The matrix functions $\mathbb{U}_\varepsilon$ are bounded, uniformly in ε, and uniformly continuous in $\Omega_\pm$, with a jump discontinuity across Σ, i.e., for all $x_\Sigma \in \Sigma$, we have

$$(8.7) \qquad \mathbb{U}_\varepsilon(x_\Sigma^+) := \lim_{\substack{y \to x_\Sigma \\ y \in \Omega_+}} \mathbb{U}_\varepsilon(y) = \exp\Big[-i\Big(\int_{-\varepsilon}^0 h_\varepsilon(t)\, dt\Big)(\sigma \cdot \mathbf{n}(x_\Sigma))B_{\eta,\tau,\lambda}(x_\Sigma)\Big]$$

and

$$(8.8) \qquad \mathbb{U}_\varepsilon(x_\Sigma^-) := \lim_{\substack{y \to x_\Sigma \\ y \in \Omega_-}} \mathbb{U}_\varepsilon(y) = \exp\Big[i\Big(\int_0^\varepsilon h_\varepsilon(t)\, dt\Big)(\sigma \cdot \mathbf{n}(x_\Sigma))B_{\eta,\tau,\lambda}(x_\Sigma)\Big].$$

Finally, we put

$$(8.9) \qquad \psi_\varepsilon = \psi_{\varepsilon,+} \oplus \psi_{\varepsilon,-} := \mathbb{U}_\varepsilon \psi \in L^2(\mathbb{R}^2;\mathbb{C}^2).$$

It is immediate to see, by the dominated convergence theorem, that

$$(8.10) \qquad \psi_\varepsilon \xrightarrow{\varepsilon \to 0} \psi \quad \text{in } L^2(\mathbb{R}^2;\mathbb{C}^2),$$

since $\psi_\varepsilon - \psi = (\mathbb{U}_\varepsilon - \mathbb{I})\psi$, $\mathbb{U}_\varepsilon \in L^\infty(\mathbb{R}^2;\mathbb{C}^{2\times 2})$ with a uniform bound in $\varepsilon \in (0,\beta)$, $\operatorname{supp}(\mathbb{U}_\varepsilon - \mathbb{I}) \subset \Sigma_\varepsilon$, and $|\Sigma_\varepsilon| \to 0$ as $\varepsilon \to 0$.

We show now that $\psi_\varepsilon \in \operatorname{dom} \mathcal{E}_{\eta,\tau,\lambda;\varepsilon} = H^1(\mathbb{R}^2;\mathbb{C}^2)$ for all $\varepsilon \in (0,\beta)$. To do so, we verify that $\psi_{\varepsilon,\pm} \in H^1(\Omega_\pm;\mathbb{C}^2)$ and that $\mathcal{T}_+^D \psi_{\varepsilon,+} = \mathcal{T}_-^D \psi_{\varepsilon,-} \in H^{1/2}(\Sigma;\mathbb{C}^2)$. Let $\gamma : \mathbb{R}/\ell\mathbb{Z} \to \Sigma \subset \mathbb{R}^2$ be, as always, a smooth arc-length parametrization of Σ with positive orientation and

$$A \in C^\infty(\mathbb{R}/\ell\mathbb{Z};\mathbb{C}^{2\times 2}), \quad A(s) := i(\sigma \cdot \mathbf{n}(\gamma(s)))B_{\eta,\tau,\lambda}(\gamma(s)).$$

Thus, we may write (8.6) as

$$(8.11) \qquad \mathbb{U}_\varepsilon(x) = \begin{cases} \exp[A(\mathcal{P}_\gamma(x))H_\varepsilon(\mathcal{P}_\perp(x))], & x \in \Sigma_\varepsilon \setminus \Sigma, \\ \mathbb{I}_2, & x \in \mathbb{R}^2 \setminus \Sigma_\varepsilon, \end{cases}$$

where $\mathcal{P}_\gamma$ is defined as in (3.8). For $j = 1, 2$, $\operatorname{supp} \partial_j \mathbb{U}_\varepsilon \subset \Sigma_\varepsilon$ and, thanks to the Wilcox formula, cf. equation (4.1) of [71], for $x \in \Sigma_\varepsilon \setminus \Sigma$, we have

$$\partial_j \mathbb{U}_\varepsilon(x) = \int_0^1 \left[e^{zA(\mathcal{P}_\gamma(x))H_\varepsilon(\mathcal{P}_\perp(x))} \partial_j [A(\mathcal{P}_\gamma(x))H_\varepsilon(\mathcal{P}_\perp(x))] \right.$$
$$\left. \times e^{(1-z)A(\mathcal{P}_\gamma(x))H_\varepsilon(\mathcal{P}_\perp(x))} \right] dz.$$

Recall that we have set $s = \mathcal{P}_\gamma(x)$ and $p = \mathcal{P}_\perp(x)$. Using (3.10), we obtain

$$\partial_j [A(\mathcal{P}_\gamma(x))H_\varepsilon(\mathcal{P}_\perp(x))] = \partial_s A(s) \frac{(\mathbf{t}_\gamma(s))_j}{1 + p\kappa_\gamma(s)} H_\varepsilon(p) - A(s) h_\varepsilon(p)(\mathbf{n}_\gamma(s))_j.$$

Therefore, we arrive at

$$(8.12) \quad \partial_j \mathbb{U}_\varepsilon(x) = -A(s) h_\varepsilon(p)(\mathbf{n}_\gamma(s))_j \mathbb{U}_\varepsilon(x)$$
$$+ H_\varepsilon(p) \int_0^1 e^{zA(s)H_\varepsilon(p)} \partial_s A(s) \frac{(\mathbf{t}_\gamma(s))_j}{1 + p\kappa_\gamma(s)} \cdot e^{(1-z)A(s)H_\varepsilon(p)} dz$$
$$= -i(\sigma \cdot \mathbf{n}(x)) \mathbb{V}_{\eta,\tau,\lambda;\varepsilon}(x)(\mathbf{n}_\gamma(s))_j \mathbb{U}_\varepsilon(x) + R_{j;\varepsilon}(x),$$

where

$$R_{j;\varepsilon}(x) := H_\varepsilon(p) \int_0^1 e^{zA(s)H_\varepsilon(p)} \partial_s A(s) \frac{(\mathbf{t}_\gamma(s))_j}{1 + p\kappa_\gamma(s)} \cdot e^{(1-z)A(s)H_\varepsilon(p)} dz.$$

The matrix-valued functions $R_{j;\varepsilon}$ are bounded, uniformly in $\varepsilon \in (0, \beta)$, and $\operatorname{supp} R_{j;\varepsilon} \subset \Sigma_\varepsilon$. We observe that

$$(8.13) \qquad\qquad \mathbb{U}_\varepsilon, \partial_1 \mathbb{U}_\varepsilon, \partial_2 \mathbb{U}_\varepsilon \in L^\infty(\Omega_\pm; \mathbb{C}^{2\times 2}).$$

Since $\psi_\pm \in H^1(\Omega_\pm; \mathbb{C}^2)$, we conclude that $\psi_{\varepsilon,\pm} = \mathbb{U}_\varepsilon \psi_\pm \in H^1(\Omega_\pm; \mathbb{C}^2)$.

Thanks to Proposition 3.3, $\mathcal{T}_+^D \psi_{\varepsilon,\pm} \in H^{1/2}(\Sigma; \mathbb{C}^2)$ and, thanks to Chapter 4 of [32], for a.e. $x_\Sigma \in \Sigma$,

$$\mathcal{T}_\pm^D \psi_{\varepsilon,\pm}(x_\Sigma) = \lim_{r\to 0} \frac{1}{|B_r(x_\Sigma)|} \int_{\Omega_\pm \cap B_r(x_\Sigma)} \psi_\varepsilon(y)\, dy$$
$$= \lim_{r\to 0} \frac{1}{|B_r(x_\Sigma)|} \int_{\Omega_\pm \cap B_r(x_\Sigma)} \mathbb{U}_\varepsilon(y)\, \psi(y)\, dy;$$

similarly, we have

$$\mathbb{U}_\varepsilon(x_\Sigma^\pm) \mathcal{T}_\pm^D \psi_\pm(x_\Sigma) = \lim_{r\to 0} \frac{1}{|B_r(x_\Sigma)|} \int_{\Omega_\pm \cap B_r(x_\Sigma)} \mathbb{U}_\varepsilon(x_\Sigma^\pm)\, \psi(y)\, dy.$$

Since $\mathbb{U}_\varepsilon$ is continuous on $\overline{\Omega}_+$ and $\overline{\Omega}_-$, respectively, we get

$$\mathcal{T}_\pm^D \psi_{\varepsilon,\pm}(x_\Sigma) = \mathbb{U}_\varepsilon(x_\Sigma^\pm)\, \mathcal{T}_\pm^D \psi_\pm(x_\Sigma).$$

Taking (8.5), (8.7) and (8.8) into account, this yields $\mathcal{T}_+^D \psi_{\varepsilon,+} = \mathcal{T}_-^D \psi_{\varepsilon,-} \in H^{1/2}(\Sigma; \mathbb{C}^2)$, and so we conclude that $\psi_\varepsilon \in H^1(\mathbb{R}^2; \mathbb{C}^2)$ for all $\varepsilon \in (0, \beta)$.

To finish the proof, it remains to show that $\lim_{\varepsilon\to 0}\mathcal{E}_{\eta,\tau,\lambda;\varepsilon}\psi_\varepsilon = \mathcal{D}_{\eta,\tau,\lambda}\psi$ in $L^2(\mathbb{R}^2;\mathbb{C}^2)$. Since $\mathcal{E}_{\eta,\tau,\lambda;\varepsilon}\psi_\varepsilon - \mathcal{D}_{\eta,\tau,\lambda}\psi \in L^2(\mathbb{R}^2;\mathbb{C}^2)$, it suffices to show that $\lim_{\varepsilon\to 0}(\mathcal{E}_{\eta,\tau,\lambda;\varepsilon}\psi_\varepsilon - \mathcal{D}_{\eta,\tau,\lambda}\psi)_\pm = 0$ in $L^2(\Omega_\pm;\mathbb{C}^2)$. With an explicit computation, we have that

$$(8.14)\quad (\mathcal{E}_{\eta,\tau,\lambda;\varepsilon}\psi_\varepsilon - \mathcal{D}_{\eta,\tau,\lambda}\psi)_\pm = -i\sigma\cdot\nabla(\mathbb{U}_\varepsilon\psi_\pm) + \mathbb{V}_{\eta,\tau,\lambda;\varepsilon}\psi_{\varepsilon,\pm} + i\sigma\cdot\nabla\psi_\pm$$

$$= -i\sum_{j=1}^2 \sigma_j[(\partial_j\mathbb{U}_\varepsilon)\psi_\pm + (\mathbb{U}_\varepsilon - \mathbb{I}_2)\partial_j\psi_\pm] + \mathbb{V}_{\eta,\tau,\lambda;\varepsilon}\psi_{\varepsilon,\pm}.$$

Applying (8.12) together with (3.12), we get

$$(8.15)\qquad -i\sum_{j=1}^2 \sigma_j(\partial_j\mathbb{U}_\varepsilon)\psi_\pm = -i\sum_{j=1}^2 \sigma_j[-i(\sigma\cdot\mathbf{n})\mathbb{V}_{\eta,\tau,\lambda;\varepsilon}\,n_j\,\mathbb{U}_\varepsilon\psi_\pm + R_{j;\varepsilon}\psi_\pm]$$

$$= -(\sigma\cdot\mathbf{n})(\sigma\cdot\mathbf{n})\mathbb{V}_{\eta,\tau,\lambda;\varepsilon}\mathbb{U}_\varepsilon\psi_\pm - i\sum_{j=1}^2 \sigma_j R_{j;\varepsilon}\psi_\pm$$

$$= -\mathbb{V}_{\eta,\tau,\lambda;\varepsilon}\psi_{\varepsilon,\pm} + Q_\varepsilon\psi_\pm,$$

where $Q_\varepsilon \in L^\infty(\mathbb{R}^2;\mathbb{C}^{2\times 2})$, with the L^∞-norm uniformly bounded in $\varepsilon \in (0,\beta)$, and $\mathrm{supp}\,Q_\varepsilon \subset \Sigma_\varepsilon$. According to (8.14) and (8.15), we get

$$(8.16)\qquad (\mathcal{E}_{\eta,\tau,\lambda;\varepsilon}\psi_\varepsilon - \mathcal{D}_{\eta,\tau,\lambda}\psi)_\pm$$

$$= -i\sum_{j=1}^2 \sigma_j[(\mathbb{U}_\varepsilon - \mathbb{I}_2)\partial_j\psi_\pm] + Q_\varepsilon\psi_\pm \xrightarrow{\varepsilon\to 0} 0 \quad\text{in } L^2(\Omega_\pm;\mathbb{C}^2),$$

by the dominated convergence, since $\psi_\pm \in H^1(\Omega_\pm;\mathbb{C}^2)$, $\mathbb{U}_\varepsilon - \mathbb{I}_2$ and Q_ε are uniformly bounded in $\varepsilon \in (0,\beta)$ and supported on Σ_ε, and $\lim_{\varepsilon\to 0}|\Sigma_\varepsilon| = 0$.

Putting (8.16) and (8.10) together, we obtain (8.1). ∎

The problem of finding regular approximations for $\mathcal{D}_{\eta,\tau,\lambda,\omega}$ with $\eta,\tau,\lambda,\omega \in \mathbb{R}$ reduces to the problem of finding the approximations when $\omega = 0$. Indeed, according to Theorem 2.1, there exist $X \in \mathbb{R} \setminus \{0\}$ and a unitary operator U_z, where $z \in \mathbb{C}$, with $|z| = 1$, is a parameter that may be calculated in terms of η,τ,λ,ω, such that $U_{\bar z}\mathcal{D}_{\eta,\tau,\lambda,\omega}U_z = \mathcal{D}_{X\eta,X\tau,X\lambda}$. We will assume that $X^2 d > -4$, because if $X^2 d < -4$, then, employing Theorem 2.1 again, we can sandwich $\mathcal{D}_{X\eta,X\tau,X\lambda}$ by another unitary transform to get $\mathcal{D}_{\tilde\eta,\tilde\tau,\tilde\lambda}$ such that $\tilde d = \tilde\eta^2 - \tilde\tau^2 - \tilde\lambda^2 > -4$. Now, using Corollary 2.10, we find a family of approximating operators $\mathcal{E}_{\eta',\tau',\lambda';\varepsilon}$ such that $\mathcal{E}_{\eta',\tau',\lambda';\varepsilon} \to \mathcal{D}_{X\eta,X\tau,X\lambda}$ in the strong resolvent sense as $\varepsilon \to 0$. If, for $a \in \mathbb{R}$, we define the unitary multiplication operator

$$W_{a;\varepsilon} := \begin{cases} \mathbb{I}_2 & \text{in } \Omega_+ \setminus \Sigma_\varepsilon, \\ \exp\left[ia\int_{-\varepsilon}^{\mathcal{P}_\perp(\cdot)} h_\varepsilon(t)\,dt\right]\mathbb{I}_2 & \text{in } \Sigma_\varepsilon, \\ e^{ia}\mathbb{I}_2 & \text{in } \Omega_- \setminus \Sigma_\varepsilon. \end{cases}$$

then, with the help of (3.10), we get

$$W_{a;\varepsilon}^*\mathcal{E}_{\eta,\tau,\lambda;\varepsilon}W_{a;\varepsilon} = \mathcal{E}_{\eta,\tau,\lambda;\varepsilon} + a(\sigma\cdot\mathbf{n})\chi_{\Sigma_\varepsilon}h_\varepsilon,$$

where $\chi_{\Sigma_\varepsilon}$ is the indicator function of Σ_ε. Note that $\lim_{\varepsilon\to 0} W_{-\arg z;\varepsilon} = U_{\bar{z}}$ in the strong operator topology. Recalling that $U_z^{-1} = U_{\bar{z}} = U_z^*$, we conclude that

$$\mathcal{E}_{\eta',\tau',\lambda';\varepsilon} - \arg z (\sigma \cdot \mathbf{n}) \chi_{\Sigma_\varepsilon} h_\varepsilon$$
$$= W_{-\arg z;\varepsilon}^* \mathcal{E}_{\eta',\tau',\lambda';\varepsilon} W_{-\arg z;\varepsilon} \to U_z \mathcal{D}_{X\eta, X\tau, X\lambda} U_{\bar{z}} = \mathcal{D}_{\eta,\tau,\lambda,\omega},$$

in the strong resolvent sense as $\varepsilon \to 0$.

Proof of Corollary 2.10. The proof is immediate from Theorem 2.6 and Lemma A.2. ∎

8.1. Alternative approximations for purely magnetic interaction

If $\hat{\eta} = \hat{\tau} = 0$, and $\hat{\lambda} \in \mathbb{R} \setminus \{\pm 2\}$, then $R_{0,0,\hat{\lambda}} = \text{diag}((2-\hat{\lambda})/(2+\hat{\lambda}), (2+\hat{\lambda})/(2-\hat{\lambda}))$ is constant along Σ. This makes it possible to construct an alternative sequence of approximations in a remarkably direct way without employing "parallel coordinates" (s, p). The strategy will be to apply the method of [42, 43] that works for any type of one-dimensional δ-interaction. We will restrict ourselves to the case $\hat{\lambda} \in (-2, 2)$, the remaining cases, including their approximations, may be recovered using unitary equivalences, cf. Remark 2.9. Let us start with introducing a bounded operator $\mathcal{W} := \chi_{\Omega_+}\mathbb{I}_2 + \chi_{\Omega_-} R_{0,0,\hat{\lambda}}$ in $L^2(\mathbb{R}^2; \mathbb{C}^2)$.

Lemma 8.1. *We have $\mathcal{D}_{0,0,\hat{\lambda}} = \mathcal{W}(-i\sigma \cdot \nabla)\mathcal{W}$, where the operator at the right-hand side is defined on $\{\psi \in L^2(\mathbb{R}^2; \mathbb{C}^2) \mid \mathcal{W}\psi \in H^1(\mathbb{R}^2; \mathbb{C}^2)\}$.*

Proof. Since $-i\sigma \cdot \nabla$ is self-adjoint on $H^1(\mathbb{R}^2; \mathbb{C}^2)$, $\mathcal{W}^* = \mathcal{W}$, and $\mathcal{W}$ together with $\mathcal{W}^{-1}$ are bounded, $\mathcal{W}(-i\sigma \cdot \nabla)\mathcal{W}$ is also self-adjoint. Therefore, it is sufficient to show that $\mathcal{D}_{0,0,\hat{\lambda}} \subset \mathcal{W}(-i\sigma \cdot \nabla)\mathcal{W}$. Take $\psi \equiv \psi_+ \oplus \psi_- \in \text{dom}(\mathcal{D}_{0,0,\hat{\lambda}}) \subset H^1(\Omega_+; \mathbb{C}^2) \oplus H^1(\Omega_-; \mathbb{C}^2)$. Then $\mathcal{T}_+^D(\mathcal{W}\psi)_+ = \mathcal{T}_+^D \psi_+$ and, by (8.2), $\mathcal{T}_-^D(\mathcal{W}\psi)_- = R_{0,0,\hat{\lambda}}\mathcal{T}_-^D \psi_- = \mathcal{T}_+^D \psi_+$. Hence, $\mathcal{W}\psi \in H^1(\mathbb{R}^2; \mathbb{C}^2)$, i.e., $\psi \in \text{dom}(\mathcal{W}(-i\sigma \cdot \nabla)\mathcal{W})$. Finally, using the fact that for $j = 1, 2$, $R_{0,0,\hat{\lambda}} \sigma_j R_{0,0,\hat{\lambda}} = \sigma_j$, we get

$$\mathcal{W}\sigma \cdot \nabla \mathcal{W}\psi = \chi_{\Omega_+}\sigma \cdot \nabla\psi_+ + \chi_{\Omega_-} R_{0,0,\hat{\lambda}}\sigma R_{0,0,\hat{\lambda}} \cdot \nabla\psi_-$$
$$= \sigma \cdot \nabla\psi_+ \oplus \sigma \cdot \nabla\psi_- = i\mathcal{D}_{0,0,\hat{\lambda}}\psi. \qquad \blacksquare$$

Next, let $(g_\varepsilon)_{\varepsilon>0}$ be the standard two-dimensional mollifiers, i.e.,

$$g_\varepsilon(x) := \frac{1}{\varepsilon^2} g\left(\frac{x}{\varepsilon}\right), \quad \text{with } g \in C^\infty(\mathbb{R}^2; [0, +\infty)),$$

such that

$$\text{supp}(g) \subset \overline{B(0,1)} \quad \text{and} \quad \int_{B(0,1)} g = 1.$$

Note that we may write $\mathcal{W} = \exp(-\lambda\chi_{\Omega_-}\sigma_3)$ with $\lambda := 2\,\text{arctanh}\,\hat{\lambda}/2$, because $R_{0,0,\hat{\lambda}} = \exp(-\lambda\sigma_3)$. This suggests to introduce $\mathcal{W}_\varepsilon := \exp(-\lambda\chi_{\Omega_-}^\varepsilon\sigma_3)$, where $\chi_{\Omega_-}^\varepsilon := g_\varepsilon * \chi_{\Omega_-}$. Then we have the following result.

Proposition 8.2. *Let $\hat{\lambda} \in (-2, 2)$ be constant and $\lambda = 2\,\text{arctanh}\,\hat{\lambda}/2$. Then*

$$(8.17) \qquad \mathcal{D}_0 + \lambda(\sigma_2, -\sigma_1) \cdot \nabla\chi_{\Omega_-}^\varepsilon \xrightarrow{\varepsilon\to 0} \mathcal{D}_{0,0,\hat{\lambda}}$$

in the strong resolvent sense.

Proof. First, using Lemma 8.1, one shows that the self-adjoint operator $\mathcal{D}_\lambda^\varepsilon := \mathcal{W}_\varepsilon(-i\sigma \cdot \nabla)\mathcal{W}_\varepsilon$ defined on $\{\psi \in L^2(\mathbb{R}^2; \mathbb{C}^2) \mid \mathcal{W}_\varepsilon \psi \in H^1(\mathbb{R}^2; \mathbb{C}^2)\}$ converges to $\mathcal{D}_{0,0,\hat{\lambda}}$ in the strong graph limit sense as $\varepsilon \to 0$; for details, see the proof of Theorem 2 of [42]. This implies also the strong resolvent convergence. Since $\mathrm{dom}(\mathcal{D}_\lambda^\varepsilon) = \mathcal{W}_\varepsilon^{-1} H^1(\mathbb{R}^2; \mathbb{C}^2)$ and both $\mathcal{W}_\varepsilon$ and $\mathcal{W}_\varepsilon^{-1}$, viewed as matrix-valued functions, are smooth and bounded (including their derivatives), $\mathrm{dom}\,\mathcal{D}_\lambda^\varepsilon = H^1(\mathbb{R}^2; \mathbb{C}^2)$. Next, for any $\psi \in \mathrm{dom}(\mathcal{D}_\lambda^\varepsilon)$, we have

$$
\begin{aligned}
\mathcal{D}_\lambda^\varepsilon \psi = \mathcal{W}_\varepsilon(-i\sigma \cdot \nabla)\mathcal{W}_\varepsilon \psi &= -i\,\mathcal{W}_\varepsilon \sigma \mathcal{W}_\varepsilon \cdot \nabla \psi + i\lambda\,\mathcal{W}_\varepsilon \sigma \mathcal{W}_\varepsilon \cdot \nabla \chi_{\Omega_-}^\varepsilon \sigma_3 \psi \\
&= -i\sigma \cdot \nabla \psi + i\lambda\sigma \cdot \nabla \chi_{\Omega_-}^\varepsilon \sigma_3 \psi,
\end{aligned}
$$

where we used the observation that $\mathcal{W}_\varepsilon \sigma \mathcal{W}_\varepsilon = \sigma$ in the last equality. Therefore, $\mathcal{D}_\lambda^\varepsilon = \mathcal{D}_0 + \lambda(\sigma_2, -\sigma_1) \cdot \nabla \chi_{\Omega_-}^\varepsilon$. ∎

Note that in the sense of distributions,

$$
\lim_{\varepsilon \to 0} \nabla \chi_{\Omega_-}^\varepsilon = \lim_{\varepsilon \to 0} g_\varepsilon * \nabla \chi_{\Omega_-} = \delta_0 * \nabla \chi_{\Omega_-} = \nabla \chi_{\Omega_-} = \mathbf{n}\,\delta_\Sigma,
$$

so the distributional limit as $\varepsilon \to 0$ of the potential in (8.17) is $\lambda(\sigma \cdot \mathbf{t})$, and not $\hat{\lambda}(\sigma \cdot \mathbf{t})$.

9. Final remark: higher dimensions

We conclude the paper with a discussion on a possible generalization of our results to higher dimensional cases.

It is possible to define an analogue of the magnetic interaction in higher dimensions. This is not immediate, because the tangent unit vector is not uniquely defined. However, since $\sigma \cdot \mathbf{t} = i(\sigma \cdot \mathbf{n})\sigma_3$, see (3.13), this issue may be overcome. We rewrite the formal expression for (1.3) as follows:

$$
D_{\eta,\tau,\lambda,\omega} = D_0 + \big(\eta \mathbb{I}_2 + \tau\sigma_3 + \lambda i(\sigma \cdot \mathbf{n})\sigma_3 + \omega(\sigma \cdot \mathbf{n})\big)\delta_\Sigma.
$$

Put $N := 2^{\lfloor \frac{n+1}{2} \rfloor}$, where $\lfloor \cdot \rfloor$ denotes the integer part of a real number. It is well known (see, e.g., [38, 44]) that there exist Hermitian matrices $\alpha_1, \ldots, \alpha_n, \alpha_{n+1} \in \mathbb{C}^{N \times N}$ that satisfy the anticommutation relations

$$
\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{j,k}\,\mathbb{I}_N, \quad 1 \le j, k \le n+1,
$$

where $\delta_{j,k}$ stands for the Kronecker delta. The Dirac differential expression with a δ-shell interaction in $\mathbb{R}^n$ acts on functions $\psi: \mathbb{R}^n \to \mathbb{C}^N$ as follows:

$$
D^{[n]}_{\eta,\tau,\lambda,\omega} := -i\alpha \cdot \nabla + \big(\eta \mathbb{I}_N + \tau\alpha_{n+1} + \lambda i(\alpha \cdot \mathbf{n})\alpha_{n+1} + \omega(\alpha \cdot \mathbf{n})\big)\delta_\Sigma,
$$

where $\alpha \cdot \nabla := \sum_{j=1}^n \alpha_j\,\partial_j$. In particular, we define the Dirac differential expression in $\mathbb{R}^3$ with a δ-shell interaction as follows: denoting $\beta = \alpha_4$,

$$
D^{[3]}_{\eta,\tau,\lambda,\omega} = -i\alpha \cdot \nabla + m\beta + \big(\eta \mathbb{I}_4 + \tau\beta + \lambda i(\alpha \cdot \mathbf{n})\beta + \omega(\alpha \cdot \mathbf{n})\big)\delta_\Sigma.
$$

Adopting the terminology used for the potentials, we will call the interaction $\lambda i(\alpha \cdot \mathbf{n})\beta\,\delta_\Sigma$ the *anomalous-magnetic* δ-shell interaction.

We point out that when we were finishing this work, we learnt that the three dimensional case was being considered in [18]. In there, the author introduces the δ-shell interaction corresponding to the differential expression $D_0^{[3]} + (\xi\gamma_5 + i\lambda(\alpha \cdot \mathbf{n})\beta)\delta_\Sigma$, for $\xi, \lambda \in \mathbb{R}$ and $\gamma_5 := -i\alpha_1\alpha_2\alpha_3$. Using the strategy developed in [4], that is, based on fundamental solutions, in Section 6 of [18] the author shows some results which, in the case $\xi = \eta = \tau = 0$, agree with our Theorems 2.2, 2.3 and (the statements about self-adjointness in) 2.5. It is worth mentioning that his approach also works on surfaces Σ with low regularity. In this direction, see also [62], where general local interactions are considered, although no explicit reference to the anomalous magnetic potential is made.

A. Lemmata on exponential matrices

Let $\mathbf{t} = (t_1, t_2)$ be a unit vector and let $\mathbf{n} := (t_2, -t_1)$. Using the shorthand notation (1.4), for $\eta, \tau, \lambda \in \mathbb{R}$, let us consider the Hermitian matrix

$$(A.1) \qquad B_{\eta,\tau,\lambda} := \eta\mathbb{I}_2 + \tau\sigma_3 + \lambda(\sigma \cdot \mathbf{t}).$$

For $\hat{\eta}, \hat{\tau}, \hat{\lambda} \in \mathbb{R}$ such that $\hat{d} := \hat{\eta}^2 - \hat{\tau}^2 - \hat{\lambda}^2 \neq -4$, put

$$(A.2) \qquad R_{\hat{\eta},\hat{\tau},\hat{\lambda}} := \frac{4}{4+\hat{d}}\left(\frac{4-\hat{d}}{4}\mathbb{I}_2 + i\hat{\eta}(\sigma \cdot \mathbf{n}) + \hat{\tau}(\sigma \cdot \mathbf{t}) - \hat{\lambda}\sigma_3\right).$$

In this appendix, we address the following questions:

(i) Given $\eta, \tau, \lambda \in \mathbb{R}$, is it possible to find $\hat{\eta}, \hat{\tau}, \hat{\lambda} \in \mathbb{R}$ such that $\hat{d} \neq -4$ and

$$\exp[i(\sigma \cdot \mathbf{n})B_{\eta,\tau,\lambda}] = R_{\hat{\eta},\hat{\tau},\hat{\lambda}}\,?$$

(ii) Is this correspondence bijective?

Similar questions were already considered in the appendix of [69].

The following lemma gives an answer to question (i).

Lemma A.1. *Let $\eta, \tau, \lambda \in \mathbb{R}$, $d := \eta^2 - \tau^2 - \lambda^2$, and let $B_{\eta,\tau,\lambda}$ and $R_{\hat{\eta},\hat{\tau},\hat{\lambda}}$ be given by (A.1) and (A.2), respectively. Let $\hat{\eta}, \hat{\tau}, \hat{\lambda} \in \mathbb{R}$ be such that $\hat{d} = \hat{\eta}^2 - \hat{\tau}^2 - \hat{\lambda}^2 \neq -4$. Then $R_{\hat{\eta},\hat{\tau},\hat{\lambda}} = \exp[i(\sigma \cdot \mathbf{n})B_{\eta,\tau,\lambda}]$ if and only if one of the following holds:*

$$(A.3) \quad \begin{cases} d > 0,\ d \neq (2k+1)^2\pi^2 \text{ for all } k \in \mathbb{N}_0 \text{ and } (\hat{\eta}, \hat{\tau}, \hat{\lambda}) = \dfrac{\tan(\sqrt{d}/2)}{\sqrt{d}/2}(\eta, \tau, \lambda), \\[2ex] d = 0 \text{ and } (\hat{\eta}, \hat{\tau}, \hat{\lambda}) = (\eta, \tau, \lambda), \\[2ex] d < 0 \text{ and } (\hat{\eta}, \hat{\tau}, \hat{\lambda}) = \dfrac{\tanh(\sqrt{-d}/2)}{\sqrt{-d}/2}(\eta, \tau, \lambda). \end{cases}$$

If $d = (2k_0+1)^2\pi^2$, for $k_0 \in \mathbb{N}_0$, then there are no $\hat{\eta}, \hat{\tau}, \hat{\lambda} \in \mathbb{R}$ such that $R_{\hat{\eta},\hat{\tau},\hat{\lambda}} = \exp[i(\sigma \cdot \mathbf{n})B_{\eta,\tau,\lambda}]$.

Proof. Denoting $T = t_1 + it_2$ for $\mathbf{t} = (t_1, t_2)$, we get

$$(A.4) \qquad i(\sigma \cdot \mathbf{n})B_{\eta,\tau,\lambda} = \begin{pmatrix} -\lambda & (\tau-\eta)\overline{T} \\ (\tau+\eta)T & \lambda \end{pmatrix}.$$

The exponential of a general 2×2 matrix A is

(A.5) $$\exp[A] = \exp\left[\frac{\operatorname{Tr} A}{2}\right]\left(\cos \nu \, \mathbb{I}_2 + \frac{\sin \nu}{\nu}\left(A - \frac{\operatorname{Tr} A}{2} \, \mathbb{I}_2\right)\right),$$

with

$$\nu = \sqrt{\det A - \left(\frac{\operatorname{Tr} A}{2}\right)^2} \in \mathbb{C},$$

considering the principal branch of the square root, and where $(\sin \nu)/\nu$ is intended to be equal to 1 when $\nu = 0$, see, e.g., [19] and the appendix of [69]. Plugging (A.4) into (A.5), we get

$$\exp[i \, (\sigma \cdot \mathbf{n}) B_{\eta,\tau,\lambda}] = \cos \sqrt{d} \, \mathbb{I}_2 + \frac{\sin \sqrt{d}}{\sqrt{d}} \, i \, (\sigma \cdot \mathbf{n}) B_{\eta,\tau,\lambda}$$

$$= \cos \sqrt{d} \, \mathbb{I}_2 + \frac{\sin \sqrt{d}}{\sqrt{d}} \, (i \, (\sigma \cdot \mathbf{n})\eta + \tau(\sigma \cdot \mathbf{t}) - \lambda\sigma_3).$$

Since the matrices $\{\mathbb{I}_2, \sigma_3, \sigma \cdot \mathbf{t}, \sigma \cdot \mathbf{n}\}$ are a basis of the Hermitian 2×2 matrices, we have $R_{\hat{\eta},\hat{\tau},\hat{\lambda}} = \exp[i \, (\sigma \cdot \mathbf{n}) B_{\eta,\tau,\lambda}]$ if and only if the coefficients with respect to this basis are equal, i.e.,

(A.6) $$\cos \sqrt{d} = \frac{4 - \hat{d}}{4 + \hat{d}},$$

(A.7) $$\frac{\sin \sqrt{d}}{\sqrt{d}}(\eta, \tau, \lambda) = \frac{4}{4 + \hat{d}} \, (\hat{\eta}, \hat{\tau}, \hat{\lambda}).$$

For $d = (2k_0 + 1)^2\pi^2$, with $k_0 \in \mathbb{N}_0$, (A.6) has no solution $\hat{d} \in \mathbb{R} \setminus \{-4\}$. Consequently, there are no $\hat{\eta}, \hat{\tau}, \hat{\lambda} \in \mathbb{R}$ such that $R_{\hat{\eta},\hat{\tau},\hat{\lambda}} = \exp[i \, (\sigma \cdot \mathbf{n}) B_{\eta,\tau,\lambda}]$. We consider now $d \in \mathbb{R}$ such that $d \neq (2k + 1)^2\pi^2$, for all $k \in \mathbb{N}_0$. Dividing (A.7) by $1 + \cos \sqrt{d}$ and using (A.6), we get

$$\frac{\sin \sqrt{d}}{1 + \cos \sqrt{d}} \frac{1}{\sqrt{d}/2}(\eta, \tau, \lambda) = (\hat{\eta}, \hat{\tau}, \hat{\lambda}).$$

We conclude the proof by applying the elementary identity

$$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta} \quad \text{for all } \theta \in \mathbb{C} \setminus \{(2k + 1)\pi \mid k \in \mathbb{Z}\},$$

and by recalling that, for all $d < 0$, we have

$$\frac{\tan \sqrt{d}/2}{\sqrt{d}/2} = \frac{\tanh \sqrt{-d}/2}{\sqrt{-d}/2}. \qquad \blacksquare$$

By Lemma A.1, we have that the function $d := \eta^2 - \tau^2 - \lambda^2 \mapsto \hat{d} = \hat{\eta}^2 - \hat{\tau}^2 - \hat{\lambda}^2$ maps $d \in [0, +\infty)$ to $\hat{d} \in [0, +\infty)$ and $d \in (-\infty, 0)$ to $\hat{d} \in (-4, 0)$. Consequently, (ii) has a negative answer: the correspondence between (η, τ, λ) and $(\hat{\eta}, \hat{\tau}, \hat{\lambda})$ is not surjective since one can not find $(\eta, \tau, \lambda) \in \mathbb{R}^3$ such that $R_{\hat{\eta},\hat{\tau},\hat{\lambda}} = \exp[i \, (\sigma \cdot \mathbf{n}) B_{\eta,\tau,\lambda}]$ when $\hat{\eta}, \hat{\tau}, \hat{\lambda}$ are such that $\hat{d} < -4$. Moreover, the correspondence is not injective when $d \geq 0$, as the following lemma shows.

Lemma A.2. *Let $\hat{\eta}, \hat{\tau}, \hat{\lambda} \in \mathbb{R}$ be such that $\hat{d} := \hat{\eta}^2 - \hat{\tau}^2 - \hat{\lambda}^2 > -4$, and let $R_{\hat{\eta}, \hat{\tau}, \hat{\lambda}}$ be as in* (A.2). *Let $\eta, \tau, \lambda \in \mathbb{R}$, $d := \eta^2 - \tau^2 - \lambda^2$ and let $B_{\eta, \tau, \lambda}$ be as in* (A.1). *Then $R_{\hat{\eta}, \hat{\tau}, \hat{\lambda}} = \exp[i\,(\sigma \cdot \mathbf{n})\,B_{\eta, \tau, \lambda}]$ or equivalently* (A.3) *holds if and only if one of the following holds*:

(A.8) • $\hat{d} > 0$ *and* $(\eta, \tau, \lambda) = \dfrac{\arctan \sqrt{\hat{d}}/2 + k\pi}{\sqrt{\hat{d}}/2} \, (\hat{\eta}, \hat{\tau}, \hat{\lambda})$ *for $k \in \mathbb{Z}$,*

• $\hat{d} = 0$ *and* $(\eta, \tau, \lambda) = (\hat{\eta}, \hat{\tau}, \hat{\lambda})$; *for $\hat{d} = \hat{\eta} = \hat{\tau} = \hat{\lambda} = 0$, also any $\eta, \tau, \lambda \in \mathbb{R}$ such that $d = (2k_0\pi)^2$ for $k_0 \in \mathbb{N}$, are admissible,*

• $-4 < \hat{d} < 0$ *and* $(\eta, \tau, \lambda) = \dfrac{\operatorname{arctanh} \sqrt{-\hat{d}}/2}{\sqrt{-\hat{d}}/2} \, (\hat{\eta}, \hat{\tau}, \hat{\lambda}).$

Proof. If $\hat{d} \geq 0$, then, by (A.3), we have $d \geq 0$ and $\hat{d} = 4\tan^2(\sqrt{d}/2)$, that gives

(A.9) $$\tan \frac{\sqrt{d}}{2} = \frac{\sqrt{\hat{d}}}{2}, \quad \frac{\sqrt{d}}{2} = \arctan \frac{\sqrt{\hat{d}}}{2} + k\pi, \quad \text{for } k \in \mathbb{Z}.$$

If $\sqrt{d}/2 = k_0\pi$, for some $k_0 \in \mathbb{N}$, then (A.3) is true if and only if $\hat{\eta} = \hat{\tau} = \hat{\lambda} = 0$. If $\sqrt{d}/2 = 0$, then (A.3) gives $(\hat{\eta}, \hat{\tau}, \hat{\lambda}) = (\eta, \tau, \lambda)$. If $\sqrt{d}/2 \neq k\pi$, for all $k \in \mathbb{N}_0$, we can divide by $\tan \sqrt{d}/2$ in (A.3). Using (A.9), this yields (A.8). The proof of the case $-4 < \hat{d} < 0$ is analogous and even simpler, so it will be omitted. ∎

B. Magnetic field

Throughout this section λ is assumed to be a real constant. The term $\lambda(\sigma \cdot \mathbf{t})\delta_\Sigma$ in (1.3) corresponds to the singular vector potential $\mathbf{A}_\Sigma := \lambda(t_1\delta_\Sigma, t_2\delta_\Sigma)$ supported on Σ. Note that this is just a formal expression – in fact, this term is reflected in the transmission condition across Σ. We will introduce the magnetic field by the formula $B_\Sigma = \partial_1 A_2 - \partial_2 A_1 = \lambda(\partial_1(t_2\delta_\Sigma) - \partial_2(t_1\delta_\Sigma))$, i.e., exactly in the same manner as in a regular case. Here $t_i\delta_\Sigma$ is the simple layer and the derivatives are understood in the sense of distributions. Since $(n_1, n_2) = (t_2, -t_1)$, we obtain

$$\langle B_\Sigma, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^2), \mathcal{D}(\mathbb{R}^2)} = -\lambda\big(\langle n_1\delta_\Sigma, \partial_1\varphi \rangle_{\mathcal{D}'(\mathbb{R}^2), \mathcal{D}(\mathbb{R}^2)} + \langle n_2\delta_\Sigma, \partial_2\varphi \rangle_{\mathcal{D}'(\mathbb{R}^2), \mathcal{D}(\mathbb{R}^2)}\big)$$
$$= -\lambda \int_\Sigma \mathbf{n} \cdot \nabla\varphi \, d\sigma = \langle \lambda \partial_\mathbf{n}\delta_\Sigma, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^2), \mathcal{D}(\mathbb{R}^2)},$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$, where $\partial_\mathbf{n}\delta_\Sigma$ stands for the double layer distribution, cf. [70].

Alternatively, thanks to the divergence theorem, we may write

$$\langle B_\Sigma, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^2), \mathcal{D}(\mathbb{R}^2)} = -\lambda \int_\Omega \Delta\varphi \, dx = \langle -\lambda\Delta(\chi_\Omega), \varphi \rangle_{\mathcal{D}'(\mathbb{R}^2), \mathcal{D}(\mathbb{R}^2)}.$$

If, for $\varepsilon \in (0, \beta)$, we define the vector potential

(B.1) $\quad \mathbf{A}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2, \quad \mathbf{A}_\varepsilon(x) := \begin{cases} \lambda h_\varepsilon(p)\mathbf{t}(x_\Sigma) & \text{for } x = x_\Sigma + p\mathbf{n}(x_\Sigma) \in \Sigma_\varepsilon, \\ 0 & \text{for } x \in \mathbb{R}^2 \setminus \Sigma_\varepsilon, \end{cases}$

then the corresponding magnetic field B_ε reads

$$(B.2) \qquad B_\varepsilon(x) = \partial_1 \mathbf{A}_{\varepsilon,2}(x) - \partial_2 \mathbf{A}_{\varepsilon,1}(x)$$

$$= \begin{cases} \frac{\lambda h_\varepsilon(p)\kappa(s)}{1+p\kappa(s)} + \lambda h_\varepsilon'(p) & \text{for all } x = \gamma(s) + p\mathbf{n}_\gamma(s) \in \Sigma_\varepsilon, \\ 0 & \text{for } x \in \mathbb{R}^2 \setminus \Sigma_\varepsilon, \end{cases}$$

and we have the following result.

Proposition B.1. *Let A_ε and B_ε be defined as in (B.1) and (B.2), respectively. Then*

(i) $\mathbf{A}_\varepsilon \xrightarrow{\varepsilon \to 0} \lambda \mathbf{t}\delta_\Sigma = \mathbf{A}_\Sigma$ *in the sense of distributions,*

(ii) $B_\varepsilon \xrightarrow{\varepsilon \to 0} \lambda \partial_\mathbf{n}\delta_\Sigma = B_\Sigma$ *in the sense of distributions.*

Proof. Let γ be an arc-length parametrization of Σ, as in Section 3.1. In order to prove (i), let $\varphi \in \mathcal{D}(\mathbb{R}^2)$. Thanks to (3.7),

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} A_\varepsilon(x)\varphi(x)\,dx = \lim_{\varepsilon \to 0} \int_{\Sigma_\varepsilon} A_\varepsilon(x)\varphi(x)\,dx$$

$$= \lim_{\varepsilon \to 0} \int_0^\ell \int_{-\varepsilon}^\varepsilon \lambda h_\varepsilon(p)\mathbf{t}_\gamma(s)\varphi(\gamma(s)+p\mathbf{n}_\gamma(s))(1+p\kappa_\gamma(s))\,dp\,ds$$

$$= \lim_{\varepsilon \to 0} \int_0^\ell \int_{-1}^1 \lambda h(q)\mathbf{t}_\gamma(s)\varphi(\gamma(s)+\varepsilon q\mathbf{n}_\gamma(s))(1+\varepsilon q\kappa_\gamma(s))\,dq\,ds$$

$$= \int_0^\ell \lambda \mathbf{t}_\gamma(s)\varphi(\gamma(s))\,ds = \int_\Sigma \lambda \mathbf{t}(x_\Sigma)\varphi(x_\Sigma)\,dx_\Sigma = \langle \lambda \mathbf{t}\delta_\Sigma, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^2),\mathcal{D}(\mathbb{R}^2)}.$$

The second assertion follows from (i) combined with the continuity of distributional derivatives with respect to the convergence on $\mathcal{D}'(\mathbb{R}^2)$.   ∎

Note that the two-dimensional Dirac operator with the magnetic field associated with the vector potential $\mathbf{A}_\varepsilon$ is just $\mathcal{E}_{0,0,\lambda;\varepsilon}$. By Theorem 2.6, $\mathcal{E}_{0,0,\lambda;\varepsilon}$ converges to $\mathcal{D}_{0,0,\hat{\lambda}}$ in the strong resolvent sense, where $\hat{\lambda}$ is always (except for the trivial case $\lambda = 0$) different from λ. On the other hand, we have just shown that the formal limit of $\mathcal{E}_{0,0,\lambda;\varepsilon}$ is $\mathcal{D}_{0,0,\lambda}$.

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