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An asymptotic formula for the number of n-dimensional representations of SU(3)

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Abstract. We prove an asymptotic formula for the number of n-dimensional representations of the group SU(3). Main tools for the proof are Wright's circle method and the saddle point method.

1. Introduction and statement of results

The special unitary group SU(2) has (up to equivalence) one irreducible representation V_k of each dimension $k \in \mathbb{N}$. Each n-dimensional representation $\bigoplus_{k=1}^{\infty} r_k V_k$ corresponds to a unique partition

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_r, \qquad 1 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_r \le n, \tag{1.1}$$

such that r_k denotes the number of k's in (1.1). As a result, the number of n-dimensional representations equals $p(n)$, the number of integer partitions of n. The partition function has no elementary closed formula, nor does it satisfy any finite order recurrence. However, with $p(0) := 1$, its generating function has the following classical product expansion:

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \tag{1.2}$$

In [7], Hardy and Ramanujan showed the asymptotic formula

$$p(n) \sim \frac{1}{4\sqrt{3}n} \exp\left(\pi \sqrt{\frac{2n}{3}}\right), \quad \text{as } n \to \infty. \tag{1.3}$$

For their proof, they introduced the so-called Hardy–Ramanujan circle method, that uses modular type transformations to obtain a divergent asymptotic expansion whose truncations approximate $p(n)$ up to small errors. A later refinement by Rademacher [10] provided a convergent series

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right)$$

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for $p(n)$, where the $A_k(n)$ are Kloosterman sums and $I_{\frac{3}{2}}$ is the usual modified Bessel function of the first kind.

It is natural to ask whether the above correspondence between the number of representations of a group and a partition function can be generalized. The next obvious case is the unitary group SU(3), whose irreducible representations are a family of representations $W_{j,k}$ indexed by pairs of positive integers. Note that $\dim(W_{j,k}) = \frac{jk(j+k)}{2}$ (see Chapter 5 of [6]). Like in the case of SU(2), a general n-dimensional representation decomposes into a sum of these $W_{j,k}$, again each with some multiplicity. So analogously to (1.2), it is easy to see that the numbers $r(n)$ of n-dimensional representations, again with $r(0) := 1$, have the generating function

$$G(q) := \sum_{n=0}^{\infty} r(n)\, q^n = \prod_{j,k\geq 1} \frac{1}{1-q^{\frac{jk(j+k)}{2}}}$$

$$= 1 + q + q^2 + 3q^3 + 3q^4 + 3q^5 + 8q^6 + 8q^7 + \cdots . \tag{1.4}$$

In [11], Romik proved the following analogon of formula (1.3) for the sequence $r(n)$:

$$r(n) \sim \frac{C_0}{n^{\frac{3}{5}}} \exp\left(A_1 n^{\frac{2}{5}} - A_2 n^{\frac{3}{10}} - A_3 n^{\frac{1}{5}} - A_4 n^{\frac{1}{10}}\right), \quad \text{as } n \to \infty, \tag{1.5}$$

for $A_1, A_2, A_3, A_4, C_0 \in \mathbb{R}$, that are defined in the notation section. Romik asked for lower order terms in the asymptotic expansion of $r(n)$. We answer his question in the following theorem.

Theorem 1.1. *Let $L \in \mathbb{N}_0$. We have, as $n \to \infty$,*

$$r(n) = \frac{1}{n^{\frac{3}{5}}} \left(\sum_{j=0}^{L} \frac{C_j}{n^{\frac{j}{10}}} + O_L\left(n^{-\frac{L}{10}-\frac{3}{80}}\right)\right) \exp\left(A_1 n^{\frac{2}{5}} - A_2 n^{\frac{3}{10}} - A_3 n^{\frac{1}{5}} - A_4 n^{\frac{1}{10}}\right),$$

where the constants C_j do not depend on L and n and can all be calculated explicitly.

Remark. In (7.14) and (7.15), we compute some of these constants explicitly.

In order to prove (1.5) and Theorem 1.1, one has to carefully study the related Dirichlet series

$$\omega(s) := \sum_{j,k\geq 1} \frac{1}{\left(2\dim(W_{j,k})\right)^s} = \sum_{k,j\geq 1} \frac{1}{k^s j^s (k+j)^s}, \tag{1.6}$$

that converges absolutely for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{2}{3}$. It is natural to ask whether ω can be continued to a meromorphic function on the entire complex plane. This was answered in the positive by Matsumoto [9]. Romik [11] then studied more closely the properties of ω beyond its abscissa of convergence. Although $\omega(s)$ does not seem to possess a simple functional equation, it turns out that it has "trivial zeros" at $s \in -\mathbb{N}$. The key identity behind this is

$$\zeta(6n+2) = \frac{2(4n+1)!}{(6n+1)(2n)!^2} \sum_{k=1}^{n} \frac{\binom{2n}{2k-1}}{\binom{6n}{2n+2k-1}} \zeta(2n+2k)\,\zeta(4n-2k+2), \tag{1.7}$$

for $n \in \mathbb{N}$, where as usual, the *Riemann ζ-function* is

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{for } \mathrm{Re}(s) > 1.$$

Romik [11] proved that (1.7) is equivalent to the fact that $\omega(s)$ has zeros at $s \in -\mathbb{N}$. Both the trivial zeros and the distribution of poles of ω are of importance in the proof of Theorem 1.1. A key tool for the proof of the main theorem is to give uniform estimates for the function ω on vertical lines.

The paper is organized as follows. After recalling some preliminaries in Section 2, we prove in Section 3 an asymptotic expansion of the function $G(q)$ near $q = 1$ with an explicit formula for the occurring error term. In Section 4, we give a uniform upper bound for this error which is below used to show that it is negligible. Also, we prepare the error analysis for the error tails of the major arc integral when using the saddle point method. In Section 5, we give a power series expansion for the saddle points and show that they depend on the parameter n, but stay bounded as n increases (in fact, they even converge to 1). This saddle point function is then used in Section 6 to investigate the asymptotic expansion of $G(q)$ near to $q = 1$ in further detail. Finally, in Section 7 we use Wright's circle method and the saddle point method to prove Theorem 1.1. While the major arc integral can be evaluated using the preliminary sections, we use a lemma by Romik [11] to deal with the minor arcs. In Section 8, we finally discuss some open problems and related questions.

Notation

We now provide some frequently used notation. For $\beta \in \mathbb{R}$, we denote by $\{\beta\} := \beta - \lfloor \beta \rfloor$ the *fractional part* of β. As usual, $\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$. For $0 < \delta \leq \frac{\pi}{2}$, we define the cone

$$\mathcal{C}_\delta := \left\{ z \in -i\mathbb{H} : |\mathrm{Arg}(z)| \leq \frac{\pi}{2} - \delta \right\},$$

where Arg is the principal branch of the complex argument. Moreover, for $\delta \in \mathbb{R}$, we let

$$\mathbb{C}_\delta := \left\{ w \in \mathbb{C} : \mathrm{Im}(w) \leq 1 - \delta \right\}.$$

We denote for $r > 0$, $B_r(z) := \{w \in \mathbb{C} : |w - z| < r\}$. For $z_1, z_2 \in \mathbb{C}$, we define

$$[z_1, z_2] := \left\{ z = z_1 + \lambda(z_2 - z_1) : 0 \leq \lambda \leq 1 \right\}.$$

We frequently make use of the notation $f(x_1, \ldots, x_k) \ll g(x_1, \ldots, x_k)$ for $f, g \colon D \to \mathbb{C}$, where $D \subset \mathbb{C}$, which is equivalent to $|f(x_1, \ldots, x_k)| \leq C_{f,g} |g(x_1, \ldots, x_k)|$ for all $(x_1, \ldots, x_n) \in D$ and for a constant $C_{f,g} > 0$. In the case that $C_{f,g}$ depends on additional parameters $a, b, c, \ldots$, we write $f(x_1, \ldots, x_k) \ll_{a,b,c,\ldots} g(x_1, \ldots, x_k)$. We equivalently write $f(x_1, \ldots, x_k) = O(g(x_1, \ldots, x_k))$ and $f(x_1, \ldots, x_k) = O_{a,b,c,\ldots}(g(x_1, \ldots, x_k))$, respectively. We also use the notation $\ll_I$ if the implied constant depends on some interval I.

Throughout, we use the principal branch of the complex logarithm, denoted by Log, and the principal branches of the power functions, i.e., for real s, $z \mapsto z^s$ is real-valued for $z > 0$ and holomorphic in $\mathbb{C} \setminus (-\infty, 0]$. The branch being used in this definition matches the one associated with Taylor expansions in the binomial theorem,

$$(1+w)^s = \sum_{n=0}^{\infty} \binom{s}{n} w^n, \quad |w| < 1.$$

We also define the following constants, where $\Gamma(s)$ denotes the Gamma function:

$$\begin{aligned}
&X := \left(\tfrac{1}{9}\Gamma\left(\tfrac{1}{3}\right)^2 \zeta\left(\tfrac{5}{3}\right)\right)^{\frac{3}{10}} = 1.17117\ldots,\\
&Y := -\sqrt{\pi}\,\zeta\left(\tfrac{1}{2}\right)\zeta\left(\tfrac{3}{2}\right) = 6.76190\ldots,\\
&A_1 := 5X^2 = 6.85826\ldots, \quad A_2 := \frac{Y}{X} = 5.7736\ldots,\\
&A_3 := \frac{3Y^2}{80X^4} = 0.91134\ldots, \quad A_4 := \frac{11Y^3}{3200X^7} = 0.35163\ldots,\\
&C_0 := \frac{2\sqrt{3\pi}}{\sqrt{5}} X^{\frac{1}{3}} \exp\left(-\frac{Y^4}{2560X^{10}}\right) = 2.44629\ldots.
\end{aligned} \tag{1.8}$$

2. Preliminaries

In this section, we recall and prove results required for this paper. The following properties of the Gamma function are well known; the proof uses [1, 13].

Theorem 2.1. (1) *The Gamma function has a meromorphic continuation to $\mathbb{C}$, with simple poles only at $s \in -\mathbb{N}_0$, and satisfies the functional equation*

$$\Gamma(s+1) = s\Gamma(s).$$

(2) *For $s \in \mathbb{C}$ such that $-\pi + \delta \leq \mathrm{Arg}(s) \leq \pi - \delta$, with $\delta > 0$, we have the Stirling approximation*

$$\Gamma(s+1) \sim \sqrt{2\pi s} \cdot s^s e^{-s}, \quad \text{as } |s| \to \infty.$$

(3) *Writing $s = \sigma + it$ and $\sigma \in I$ for a compact interval $I \subset [\frac{1}{2}, \infty)$, we uniformly have, for $t \in \mathbb{R}$,*

$$\max\left\{1, |t|^{\sigma - \frac{1}{2}}\right\} e^{-\frac{\pi|t|}{2}} \ll_I |\Gamma(\sigma + it)| \ll_I \max\left\{1, |t|^{\sigma - \frac{1}{2}}\right\} e^{-\frac{\pi|t|}{2}}.$$

The estimate also holds for compact intervals $I \subset \mathbb{R}$, if we assume $|t| \geq 1$.

(4) *For all $s \in \mathbb{C}$, we have the following identity between meromorphic functions*:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

(5) *For $m \in \mathbb{N}_0$, we have*

$$\Gamma\left(m + \frac{1}{2}\right) = \frac{(2m)!}{4^m m!}\sqrt{\pi} \quad \text{and} \quad \Gamma\left(-m - \frac{1}{2}\right) = \frac{(-4)^{m+1}(m+1)!}{\big(2(m+1)\big)!}\sqrt{\pi}.$$

(6) *We have, uniformly for $x, y \geq 1$,*

$$\Gamma(x)\Gamma(y) \ll \Gamma(x+y).$$

Proof. Parts (1), (2), (4), and (5) are well known. For part (3), we use (see p. 21 of [1])

$$\left|\Gamma(\sigma+it)\right| = \sqrt{2\pi}\,|t|^{\sigma-\frac{1}{2}}\,e^{-\frac{\pi|t|}{2}}\left(1+O_I\left(|t|^{-1}\right)\right).$$

Part (6) requires the following well-known identity for the Beta function (see p. 35 of [1]):

$$\frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \quad \text{for } x, y > 0. \qquad \blacksquare$$

Define, for $x > 0$ and $s \in \mathbb{C}$, the *incomplete Gamma function*

$$\Gamma(s;x) := \int_x^\infty t^{s-1}e^{-t}\,dt$$

which satisfies the following important properties.

Theorem 2.2. (1) *The function $s \mapsto \Gamma(s;x)$ defines an entire function. One has the following asymptotic behavior for fixed $s \in \mathbb{C}$:*

$$\Gamma(s;x) \sim x^{s-1}e^{-x}, \quad \text{as } x \to \infty.$$

(2) *For $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we have the inequality*

$$\Gamma(n;x) \le n!\,\max\left\{1, x^{n-1}\right\}e^{-x}.$$

Proof. For part (1), see p. 98 of [12]. Part (2) uses that

$$\Gamma(n;x) = (n-1)!\,e^{-x}\sum_{k=0}^{n-1}\frac{x^k}{k!} \le (n-1)!\,e^{-x}\,n\max\left\{1, x^{n-1}\right\} = n!\max\left\{1, x^{n-1}\right\}e^{-x}. \quad \blacksquare$$

We also consider, for $s, z \in \mathbb{C}$ with $s \notin -\mathbb{N}$, the *generalized binomial coefficient*, defined by

$$\binom{s}{z} := \frac{\Gamma(s+1)}{\Gamma(z+1)\,\Gamma(s-z+1)}.$$

Lemma 2.3. *The following assertions hold true.*

(1) *We have, uniformly for $\beta > 0$ and $T \ge 0$,*

$$\left|\prod_{j=1}^{\lfloor\beta\rfloor}(\{\beta\}+j+iT)\right| \ll \Gamma(\beta+1)\max\left\{1, T^{\lfloor\beta\rfloor}\right\}.$$

(2) *We have, uniformly for $k \in \mathbb{N}_0$, $\alpha \in \mathbb{N}+\frac{1}{4}$, and $T \ge 0$,*

$$\left|\binom{k-\alpha-1+iT}{k}\right| = \left|\frac{\Gamma(k-\alpha+iT)}{\Gamma(-\alpha+iT)\,k!}\right| \ll \left|\binom{\alpha}{k}\right|\max\left\{1, T^k\right\}.$$

(3) *Let $0 < x < 1$. There exist $m, M > 0$ only dependent on x, such that, for all $k \in \mathbb{N}$,*

$$m k^{x-1} \le \binom{-x}{2k} \le M k^{x-1}.$$

Proof. (1) We assume that $\beta \geq 1$, as in the case $0 < \beta < 1$ the result is trivially true. We note that

$$\prod_{j=1}^{\lfloor\beta\rfloor}(\{\beta\}+j+iT)=\prod_{j=1}^{\lfloor\beta\rfloor}(\beta-\lfloor\beta\rfloor+j+iT)=\prod_{j=0}^{\lfloor\beta\rfloor-1}(\beta-j+iT)$$
$$=\prod_{j=0}^{\lfloor\beta\rfloor-1}(\beta-j)\prod_{j=0}^{\lfloor\beta\rfloor-1}\left(1+\frac{iT}{\beta-j}\right).$$

Since for $m := \lfloor\beta\rfloor - j \in \mathbb{N}$ (for $0 \leq j \leq \lfloor\beta\rfloor - 1$) and $T \geq 0$ one has

$$\left|1+\frac{iT}{m}\right| \leq \max\{1,T\}\left|1+\frac{i}{m}\right|,$$

it follows, using $\left|1+\frac{i}{\beta-j}\right| \leq \left|1+\frac{i}{\lfloor\beta\rfloor-j}\right|$, that

$$\prod_{j=0}^{\lfloor\beta\rfloor-1}\left|\left(1+\frac{iT}{\beta-j}\right)\right| \leq \max\left\{1,T^{\lfloor\beta\rfloor}\right\}\prod_{j=1}^{\lfloor\beta\rfloor}\left|\left(1+\frac{i}{j}\right)\right|.$$

Note that

$$\prod_{j=1}^{m}\left|\left(1+\frac{i}{j}\right)\right| = \sqrt{\prod_{j=1}^{m}\left(1+\frac{1}{j^2}\right)} \leq \sqrt{\prod_{j=1}^{\infty}\left(1+\frac{1}{j^2}\right)} < \infty.$$

Thus there exists some constant $C > 0$, independent of β and T, such that

$$\left|\prod_{j=1}^{\lfloor\beta\rfloor}(\{\beta\}+j+iT)\right| \leq C\prod_{j=0}^{\lfloor\beta\rfloor-1}(\beta-j)\max\left\{1,T^{\lfloor\beta\rfloor}\right\}. \tag{2.1}$$

By using Theorem 2.1 (1), one inductively sees that

$$\Gamma(\beta+1)=\Gamma(\{\beta\}+1)\prod_{j=0}^{\lfloor\beta\rfloor-1}(\beta-j),$$

and therefore

$$\prod_{j=0}^{\lfloor\beta\rfloor-1}(\beta-j) \leq \frac{\Gamma(\beta+1)}{\min_{1\leq x\leq 2}\Gamma(x)} \ll \Gamma(\beta+1). \tag{2.2}$$

Combining (2.1) and (2.2), we conclude the claim.

(2) We first assume that $0 \leq T \leq 1$. Then we have, by Theorem 2.1 (1),

$$\left|\binom{-\alpha+k-1+iT}{k}\right| = \binom{\alpha}{k}\prod_{m=0}^{k-1}\left|\left(1-\frac{iT}{\alpha-m}\right)\right|.$$

Every factor of the form $\left|1 - \frac{iT}{\ell+\frac{1}{4}}\right|$ equals 1 for $\ell \in \mathbb{Z}$ (note that $\alpha - m \in \mathbb{Z} + \frac{1}{4}$ by assumption, and thus non-zero). Hence, if we add an infinite number of such factors, the result is at least as big. Since we assume $0 \le T \le 1$, we obtain

$$\left|\binom{-\alpha + k - 1 + iT}{k}\right| \le \left|\binom{\alpha}{k}\right| \prod_{m=-\infty}^{\infty} \left|1 - \frac{i}{m + \frac{1}{4}}\right| \ll \left|\binom{\alpha}{k}\right|.$$

For $T > 1$, we find that

$$\frac{1}{k!}\prod_{j=0}^{k-1}(j - \alpha + iT)$$
$$= \frac{1}{k!}\prod_{j=0}^{k-1}(\alpha - j)T^k \prod_{m=0}^{k-1}\left(\frac{i}{\alpha - m} - \frac{1}{T}\right) = \binom{\alpha}{k} T^k \prod_{m=0}^{k-1}\left(\frac{i}{\alpha - m} - \frac{1}{T}\right).$$

Since $T > 1$, every factor in the product has absolute value smaller than $\left|1 - \frac{i}{\alpha - m}\right|$, hence

$$\left|\binom{-\alpha + k - 1 + iT}{k}\right| \le \left|\binom{\alpha}{k}\right| T^k \left|\prod_{m=0}^{k-1}\left(1 - \frac{i}{\alpha - m}\right)\right| \ll \left|\binom{\alpha}{k}\right| T^k$$

by the same arguments as above.

(3) Using Theorem 2.1 (4), one has

$$\binom{-x}{2k} = \frac{\Gamma(1-x)\,\Gamma(2k+x)\sin(\pi x)}{\pi\Gamma(2k+1)}.$$

With Theorem 2.1 (2), we obtain, as $k \to \infty$,

$$\frac{\Gamma(1-x)\,\Gamma(2k+x)\sin(\pi x)}{\pi\,\Gamma(2k+1)} \sim 2^{x-1}\,\frac{\Gamma(1-x)\sin(\pi x)}{\pi}\left(k + \frac{x-1}{2}\right)^{x-1}.$$

Since $1 - x > 0$, we have

$$k^{x-1} \sim (k+y)^{x-1}, \quad \text{as } k \to \infty,$$

for all $y \in \mathbb{R}$, so there exist constants $c_1, c_2 > 0$ such that

$$c_1\,k^{x-1} < \left(k + \frac{x-1}{2}\right)^{x-1} < c_2\,k^{x-1}$$

for all $k \in \mathbb{N}$ (note that $\frac{1-x}{2} < \frac{1}{2}$). Thus,

$$2^{x-1}\,k^{x-1}\,\frac{c_1\Gamma(1-x)\sin(\pi x)}{\pi} < \binom{-x}{2k} < 2^{x-1}\,k^{x-1}\,\frac{c_2\,\Gamma(1-x)\sin(\pi x)}{\pi}.$$

The result follows by setting

$$m := \frac{c_1\,2^{x-1}\Gamma(1-x)\sin(\pi x)}{\pi} \quad \text{and} \quad M := \frac{c_2\,2^{x-1}\Gamma(1-x)\sin(\pi x)}{\pi},$$

where we note that $\Gamma(1-x)\sin(\pi x) > 0$ for all $0 < x < 1$. ∎

We also need some analytic properties of the Riemann ζ-function, see [2, 3, 13]; more precisely, we refer to p. 129 of [3] for part (2).

Theorem 2.4. (1) *The ζ-function has a meromorphic continuation to $\mathbb{C}$ with only a simple pole at $s = 1$ with residue 1. For $s \in \mathbb{C}$, we have (as identity between meromorphic functions)*

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s).$$

(2) *Fix $c > 0$ and $\sigma_0 \in \mathbb{R}$. Then we have, uniformly for $|t| \geq 1$ and $\sigma \geq \sigma_0$,*

$$\zeta(\sigma + it) \ll_c |t|^{\mu(\sigma)+c},$$

where

$$\mu(\sigma) := \begin{cases} 0 & \text{if } \sigma \geq 1, \\ \frac{1}{2}(1-\sigma) & \text{if } 0 \leq \sigma < 1, \\ \frac{1}{2} - \sigma & \text{if } \sigma < 0. \end{cases}$$

In particular, we obtain, uniformly in $T \geq 0$,

$$\zeta\left(\frac{1}{2} + iT\right) \ll \max\left\{1, T^{\frac{1}{2}}\right\}.$$

The reader should note that all of these bounds are trivial, but can be improved by much more detailed investigations of the zeta function in the critical strip. However, the improved bounds do not improve our results.

To use the saddle point method, we also require the following approximation.

Lemma 2.5. *Let λ_n be an increasing unbounded sequence of positive real numbers, let $B > 0$, and let P be a polynomial of degree $m \in \mathbb{N}_0$. Then we have*

$$\int_{-\lambda_n}^{\lambda_n} P(x)\, e^{-Bx^2} dx = \int_{-\infty}^{\infty} P(x)\, e^{-Bx^2} dx + O_{B,P}\left(\lambda_n^{m-1}\, e^{-B\lambda_n^2}\right).$$

Proof. We use the triangle inequality for integrals and the substitution $t = Bx^2$ to get

$$\left| \int_{\lambda_n}^{\infty} P(x)\, e^{-Bx^2}\, dx \right| \leq \frac{1}{2\sqrt{B}} \int_{B\lambda_n^2}^{\infty} \left| P\left(\sqrt{\frac{t}{B}}\right) \right| \frac{e^{-t}}{\sqrt{t}}\, dt$$
$$\ll_{P,B} \int_{B\lambda_n^2}^{\infty} t^{\frac{m-1}{2}}\, e^{-t}\, dt = \Gamma\left(\frac{m+1}{2}\, ;\, B\lambda_n^2\right).$$

Using the asymptotic behavior in Theorem 2.2 (1), we obtain

$$\Gamma\left(\frac{m+1}{2}\, ;\, B\lambda_n^2\right) \ll_B \lambda_n^{m-1}\, e^{-B\lambda_n^2}.$$

The same argument works for the range $-\infty < x \leq -\lambda_n$, from which we easily deduce the lemma. ∎

3. An asymptotic expansion of G

The key to understanding the asymptotic behavior of the sequence $r(n)$ is to study the function $z \mapsto G(e^{-z})$ (see (1.4)) near $z = 0$. For this, we require the following properties of ω; note that the meromorphic continuation of ω was first obtained by Matsumoto [9].

Theorem 3.1 (Romik [11], Theorems 1.2 and 1.3). *We have the following.*

(1) *The series* (1.6) *converges for* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > \frac{2}{3}$, *and defines a holomorphic function in that region.*

(2) *The function* ω *can be analytically continued to a holomorphic function on* $\mathbb{C} \setminus (\{\frac{2}{3}\} \cup (\frac{1}{2} - \mathbb{N}_0))$.

(3) *The function* ω *has a simple pole at* $s = \frac{2}{3}$ *with residue*

$$\mathrm{Res}_{s=\frac{2}{3}}\, \omega(s) = \frac{\Gamma\left(\frac{1}{3}\right)^3}{2\sqrt{3}\,\pi}.$$

For each $m \in \mathbb{N}_0$, *it has a simple pole at* $s = \frac{1}{2} - m$ *with residue*

$$\mathrm{Res}_{s=\frac{1}{2}-m}\, \omega(s) = \frac{(-1)^m}{16^m}\binom{2m}{m}\zeta\left(\frac{1}{2} - 3m\right).$$

(4) *Let* $I \subset \mathbb{R}$ *be a compact interval. For all* $\sigma \in I$, $\omega(\sigma + it)$ *grows at most polynomially as* $|t| \to \infty$, *where the polynomial only depends on* I.

(5) *We have* $\omega(-n) = 0$ *for all* $n \in \mathbb{N}$.

We define, for $\eta \in \mathbb{R} \setminus \{-\frac{2}{3}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$ and $z \in -i\mathbb{H}$,

$$E(\eta; z) := \frac{1}{2\pi i}\int_{-\eta-i\infty}^{-\eta+i\infty} J(s; z)\, ds, \quad \text{where} \quad J(s; z) := \left(\frac{2}{z}\right)^s \Gamma(s)\,\zeta(s+1)\,\omega(s).$$

The next proposition provides some basic facts about the function $E(\eta; z)$. As usual, Log denotes the principal branch of the complex logarithm. For the proof, we shall use Theorems 2.1, 2.4 and 3.1, as well as results from [11].

Proposition 3.2.

(1) *Let* $\eta \in \mathbb{R} \setminus \{-\frac{2}{3}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}, \ldots\}$ *and* $0 < \delta \leq \frac{\pi}{2}$. *As* $z \to 0$ *in* $\mathcal{C}_\delta$,

$$E(\eta; z) = O_{\eta,\delta}\left(|z|^\eta\right).$$

(2) *Let* $\eta \in \mathbb{R}^+ \setminus \frac{1}{2}(2\mathbb{N}_0 + 1)$. *Then we have the functional equation*

$$E(\eta; z) = E(\eta + 1; z) + \mathrm{Res}_{s=-w}\, J(s; z),$$

where $w \in \mathbb{N}_0 + \frac{1}{2}$ *is unique with the property* $\eta < w < \eta + 1$. *If* $m + \frac{1}{2} < \eta_1, \eta_2 < m + \frac{3}{2}$ *for some* $m \in \mathbb{N}_0$, *then we have* $E(\eta_1; z) = E(\eta_2; z)$ *for all* $z \in -i\mathbb{H}$.

(3) *For all* $\eta \in \mathbb{R} \setminus \{-\frac{2}{3}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}, \ldots\}$, *the function* $z \mapsto E(\eta; z)$ *is holomorphic on* $-i\mathbb{H}$. *For all* $\eta < -\frac{2}{3}$ *and* $z \in -i\mathbb{H}$, *we have*

$$\mathrm{Log}\left(G\left(e^{-z}\right)\right) = E(\eta; z).$$

Proof. (1) With Theorems 2.1 (1), 2.4 (1) and 3.1 (2), we conclude that $s \mapsto J(s;z)$ is meromorphic. Since the only poles of $\zeta(s+1)$ and of $\Gamma(s)$ are at 0 and in $-\mathbb{N}_0$, respectively, by Theorem 3.1 (2) the only poles of $J(s;z)$ in $-i\mathbb{H}$ are at $\frac{1}{2}$ and $\frac{2}{3}$. To determine the poles of $s \mapsto J(s;z)$ in the left half-plane, we first note that $\left(\frac{2}{z}\right)^s \zeta(s+1)$ has no poles in that region. Potential poles of $\Gamma(s)$ at negative integers are cancelled by the trivial zeros of $\omega(s)$ in $-\mathbb{N}$, see Theorem 3.1 (5). As a result, the only poles of $s \mapsto J(s;z)$ are at $\frac{2}{3}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{3}{2}, \ldots$, and in particular, for $\eta \in \mathbb{R} \setminus \{-\frac{2}{3}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}, \ldots\}$ there are no poles of the integrand on the vertical integration curve in $E(\eta;z)$. With $|\mathrm{Arg}(z)| \leq \frac{\pi}{2} - \delta$ (since $z \in \mathcal{C}_\delta$), we obtain for $\eta \in \mathbb{R} \setminus \{\frac{2}{3}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{3}{2}, \ldots\}$,

$$\left| \int_{-\eta - i\infty}^{-\eta + i\infty} J(s;z)\, ds \right| \leq \left| \frac{z}{2} \right|^{\eta} \int_{-\infty}^{\infty} \left| \Gamma(-\eta + iv)\, \zeta(1 - \eta + iv)\, \omega(-\eta + iv) \right| e^{\left(\frac{\pi}{2} - \delta\right)|v|}\, dv.$$

Thus using Theorem 2.4 (2), Theorem 2.1 (3), and Theorem 3.1 (4), we may conclude (1).

(2) As the proof follows that of Romik, we only sketch the main ideas. According to Theorems 2.1 (3), 2.4 (2) and 3.1 (4), the function $s \mapsto J(s;z) = J(\sigma + it;z)$ has exponential decay for $|t| \to \infty$ and σ bounded for all $z \in -i\mathbb{H}$. We conclude with the residue theorem that $E(\eta_1;z) - E(\eta_2;z)$ with $\eta_2 > \eta_1 > 0$ only depends on the poles of $s \mapsto J(s;z)$ satisfying $-\eta_2 < -w < -\eta_1$, and can be expressed as a finite sum of residues. By Theorem 3.1 (2), we see that the set of residues is empty in the case $m + \frac{1}{2} < \eta_1, \eta_2 < m + \frac{3}{2}$, which proves that $E(\eta_1;z) - E(\eta_2;z) = 0$ in this case. If $\eta_2 = \eta_1 + 1$, then there is a unique pole $-w$ of $s \mapsto J(s;z)$ satisfying $-\eta - 1 < -w < -\eta$, again due to Theorem 3.1 (2), and one obtains $E(\eta;z) - E(\eta+1;z) = \mathrm{Res}_{s=-w} J(s;z)$.

(3) Again we only give a sketch of the proof. Using the exponential decay of $J(s;z)$ as in part (2), one sees that the integral $E(\eta;z)$ converges uniformly on compact subsets on the right half-plane. By Weierstrass' theorem, one concludes holomorphicity, and as $E(\eta;t) = \log(G(e^{-t}))$ for $\eta < -\frac{2}{3}$ was proved for $t \in \mathbb{R}^+$ on p. 33 of [11], the claim follows via analytic continuation. ∎

The next theorem gives the asymptotic expansion of $\mathrm{Log}\left(G(e^{-z})\right)$.

Theorem 3.3. *Let $\eta > \frac{1}{2}$ with $\eta \notin \frac{1}{2}(2\mathbb{N}+1)$. Then, for $z \in -i\mathbb{H}$, we have*

$$\mathrm{Log}\big(G(e^{-z})\big) = \frac{2^{\frac{2}{3}}\, 3 X^{\frac{10}{3}}}{z^{\frac{2}{3}}} - \frac{\sqrt{2}\, Y}{z^{\frac{1}{2}}} - \frac{1}{3} \mathrm{Log}(z) + \frac{1}{3} \log\left(16\pi^3\right) + z^{\frac{1}{2}} \sum_{0 \leq m < \eta - \frac{1}{2}} \nu_m z^m + E(\eta;z),$$

where the constants X and Y were defined in (1.8), and the coefficients ν_m are given by

$$\nu_m = \frac{\sqrt{2\pi}}{(16\pi)^3 \left(8^5 \pi^4\right)^m} \frac{1}{m+1} \binom{2m}{m} \frac{(6m+6)!}{(3m+3)!}\, \zeta\left(m + \frac{1}{2}\right) \zeta\left(3m + \frac{7}{2}\right).$$

Proof. By Proposition 3.2 (3), we have for $\eta < -\frac{2}{3}$ and $z \in -i\mathbb{H}$,

$$\mathrm{Log}\big(G(e^{-z})\big) = E(\eta;z).$$

The first coefficients in the asymptotic expansion of $E(\eta; z)$ can be obtained explicitly using the residue theorem. It was shown in equation (70) of [11] that, for $t \in \mathbb{R}^+$,

$$\operatorname{Log}\big(G(e^{-t})\big) = \frac{2^{\frac{2}{3}}\, 3X^{\frac{10}{3}}}{t^{\frac{2}{3}}} - \frac{\sqrt{2}\,Y}{t^{\frac{1}{2}}} - \frac{1}{3}\log(t) + \log\Big(2^{\frac{4}{3}}\pi\Big) + \nu t^{\frac{1}{2}} + E(1; t), \tag{3.1}$$

where the term $\nu t^{\frac{1}{2}}$ is the residue of $J(s; t)$ at $s = -\frac{1}{2}$, see also p. 34 of [11]. By Proposition 3.2(3), both sides of (3.1) continue to holomorphic functions on the right half-plane. As a result, by the identity theorem, equation (3.1) holds also for $t = z \in -i\mathbb{H}$. Let $a \in \mathbb{N}_0$. By using Proposition 3.2(2) inductively, one obtains with (3.1) for all $z \in -i\mathbb{H}$,

$$\begin{aligned}\operatorname{Log}\big(G(e^{-z})\big) = {} & \frac{2^{\frac{2}{3}}\, 3X^{\frac{10}{3}}}{z^{\frac{2}{3}}} - \frac{\sqrt{2}\,Y}{z^{\frac{1}{2}}} - \frac{1}{3}\operatorname{Log}(z) + \log\Big(2^{\frac{4}{3}}\pi\Big) \\ & + \sum_{m=0}^{a} \operatorname{Res}_{s=-m-\frac{1}{2}} J(s; z) + E(a+1; z).\end{aligned}$$

Let $\eta > \frac{1}{2}$ and $\eta \notin \frac{1}{2}(2\mathbb{N}+1)$. Since we clearly have $a + \frac{1}{2} < \eta < a + \frac{3}{2}$ for some $a \in \mathbb{N}_0$, by Proposition 3.2(2) we have $E(\eta; z) = E(a+1; z)$, which proves that

$$\begin{aligned}\operatorname{Log}\big(G(e^{-z})\big) = {} & \frac{2^{\frac{2}{3}}\, 3X^{\frac{10}{3}}}{z^{\frac{2}{3}}} - \frac{\sqrt{2}\,Y}{z^{\frac{1}{2}}} - \frac{1}{3}\operatorname{Log}(z) + \log\Big(2^{\frac{4}{3}}\pi\Big) \\ & + \sum_{m=0}^{a} \operatorname{Res}_{s=-m-\frac{1}{2}} J(s; z) + E(\eta; z).\end{aligned}$$

Finally, using Theorem 3.1(3), Theorem 2.1(5), and Theorem 2.4(1), one proves the equality

$$\operatorname{Res}_{s=-m-\frac{1}{2}} J(s; z) = \nu_m\, z^{m+\frac{1}{2}}$$

for all $m \in \mathbb{N}_0$, which concludes the proof of the theorem. ■

4. Error bounds

To give an asymptotic formula for $r(n)$, we use Wright's circle method and the saddle point method. For this we need a number of technical results. In the following, we always assume $\alpha \in \mathbb{N} + \frac{1}{4}$. To find a satisfactory estimate for $E(\alpha; z)$, we define, with $k_n := \lfloor n^{\frac{2}{55}} \rfloor$ and $z \in -i\mathbb{H}$,

$$\mathcal{E}_n(z) := E\left(k_n + \frac{1}{4}; \frac{2X^2 z}{n^{\frac{3}{5}}}\right). \tag{4.1}$$

Note that the choice of k_n guarantees that $k_n \to \infty$ on the one hand (so the contributions of the poles of $\omega(s)$ are included into the main term), but does not grow too fast in the sense that the error term $\mathcal{E}_n$ is bounded well enough. The choice $\frac{2}{55}$ is not unique, but helps to achieve a good bound, which becomes finally clear in the proof of the following result.

Theorem 4.1. *For $z \in \mathcal{C}_{\frac{\pi}{4}}$ we have, for some absolute constant $B > 0$,*

$$\mathcal{E}_n(z) \ll \left(\frac{B}{n^{\frac{1}{5}}}\right)^{n^{\frac{2}{55}}} |z|^{k_n+\frac{1}{4}}.$$

To prove Theorem 4.1, we need some auxiliary lemmas. The first proposition provides a uniform bound for ζ on vertical lines. Using Theorem 2.4 (1), Theorem 2.1 (1), (3), and Lemma 2.3 (1), we obtain the following.

Proposition 4.2. *We have, uniformly in $\beta \geq \frac{1}{4}$ and $T \geq 0$,*

$$\left|\zeta(-\beta + iT)\right| \ll (2\pi)^{-\beta}\,\Gamma(\beta+1)\max\left\{1, T^{\beta+\frac{1}{2}}\right\}.$$

The following statement was proved on pp. 20–23 of [11].

Proposition 4.3. *Let $M \in \mathbb{N}$. We have, for $s \in \mathbb{C}$ with $\frac{3}{4} - \frac{M}{2} < \mathrm{Re}(s) < M + \frac{1}{2}$,*

$$\omega(s) = \frac{\Gamma(2s-1)\,\Gamma(1-s)\,\zeta(3s-1)}{\Gamma(s)} + \frac{1}{\Gamma(s)}\sum_{k=0}^{M-1}(-1)^k\frac{\Gamma(s+k)}{k!}\,\zeta(2s+k)\,\zeta(s-k)$$
$$+ \frac{1}{2\pi i\,\Gamma(s)}\int_{M-\frac{1}{2}-i\infty}^{M-\frac{1}{2}+i\infty}\Gamma(s+z)\,\Gamma(-z)\,\zeta(2s+z)\,\zeta(s-z)\,dz.$$

The next proposition gives a uniform estimate for the function ω.

Proposition 4.4. *For some $C > 0$, we have, uniformly in $\alpha \in \mathbb{N} + \frac{1}{4}$ and $T \geq 0$,*

$$\left|\omega(-\alpha + iT)\right| \ll C^{\alpha}\,\Gamma(3\alpha+3)\,\Gamma\left(3\alpha + \frac{21}{4}\right)\max\left\{1, T^{5\alpha+\frac{11}{2}}\right\}.$$

We split the technical proof of Proposition 4.4 in several lemmas. The following lemma considers the first summand in Proposition 4.3 and follows by a direct calculation using Theorem 2.1 (1), (3), (4) and Proposition 4.2.

Lemma 4.5. *We have, uniformly in $\alpha \in \mathbb{N} + \frac{1}{4}$ and $T \geq 0$,*

$$\left|\frac{\Gamma(-2\alpha-1+2iT)\,\Gamma(\alpha+1-iT)\,\zeta(-3\alpha-1+3iT)}{\Gamma(-\alpha+iT)}\right|$$
$$\ll \left(\frac{9}{8\pi^3}\right)^{\alpha}\Gamma(3\alpha+2)\max\left\{1, T^{3\alpha+\frac{3}{2}}\right\}e^{-\frac{\pi T}{2}}.$$

Proof. Using Theorem 2.1 (4), with $s = -\alpha + iT$, we obtain

$$\left|\frac{\Gamma(-2\alpha-1+2iT)\,\Gamma(\alpha+1-iT)\,\zeta(-3\alpha-1+3iT)}{\Gamma(-\alpha+iT)}\right|$$
$$= \frac{1}{\pi}\left|\Gamma(-2\alpha-1+2iT)\,\Gamma(\alpha+1-iT)^2\sin(-\pi\alpha+\pi iT)\,\zeta(-3\alpha-1+3iT)\right|. \tag{4.2}$$

We inductively use Theorem 2.1 (1), to obtain

$$\Gamma(-2\alpha-1+2iT)=\frac{\Gamma\left(\frac{1}{2}+2iT\right)}{\prod_{j=0}^{2\lfloor\alpha\rfloor+1}(-2\alpha-1+j+2iT)},$$

$$\Gamma(\alpha+1-iT)^2=\prod_{j=0}^{\lfloor\alpha\rfloor-1}(\alpha-j-iT)^2\,\Gamma\left(\frac{5}{4}-iT\right)^2.$$

With this and (4.2), we see that

$$\begin{aligned}&\left|\frac{\Gamma(-2\alpha-1+2iT)\,\Gamma(\alpha+1-iT)\,\zeta(-3\alpha-1+3iT)}{\Gamma(-\alpha+iT)}\right|\\ &=\frac{1}{\pi}\left|\Gamma\left(\frac{1}{2}+2iT\right)\Gamma\left(\frac{5}{4}-iT\right)^2\sin\left(-\frac{\pi}{4}+\pi iT\right)\right|\\ &\quad\times\left|\frac{\prod_{j=0}^{\lfloor\alpha\rfloor-1}(\alpha-j-iT)^2}{\prod_{j=0}^{2\lfloor\alpha\rfloor+1}(-2\alpha-1+j+2iT)}\,\zeta(-3\alpha-1+3iT)\right|.\end{aligned}\tag{4.3}$$

We use Theorem 2.1 (3) and (4.3), to find, uniformly in $\alpha\in\mathbb{N}+\frac{1}{4}$ and $T\geq 0$,

$$\begin{aligned}&\left|\frac{\Gamma(-2\alpha-1+2iT)\,\Gamma(\alpha+1-iT)\,\zeta(-3\alpha-1+3iT)}{\Gamma(-\alpha+iT)}\right|\\ &\ll e^{-\frac{\pi T}{2}}\left|\frac{\prod_{j=0}^{\lfloor\alpha\rfloor-1}(\alpha-j-iT)^2}{\prod_{j=0}^{\lfloor\alpha\rfloor-1}(2\alpha+1-2j-2iT)\,(2\alpha-2j-2iT)}\right.\\ &\qquad\left.\times\frac{\zeta(-3\alpha-1+3iT)}{\left(2\alpha-2\lfloor\alpha\rfloor-2iT\right)\left(2\alpha+1-2\lfloor\alpha\rfloor-2iT\right)}\right|.\end{aligned}\tag{4.4}$$

With Proposition 4.2, we obtain that the last expression in (4.4) equals

$$\begin{aligned}&e^{-\frac{\pi T}{2}}\left|\prod_{j=0}^{\lfloor\alpha\rfloor-1}\frac{\alpha-j-iT}{2\left(2(\alpha-j)+1-2iT\right)}\right|\left|\frac{\zeta(-3\alpha-1+3iT)}{\left(2\alpha-2\lfloor\alpha\rfloor-2iT\right)\left(2\alpha+1-2\lfloor\alpha\rfloor-2iT\right)}\right|\\ &\ll e^{-\frac{\pi T}{2}}\,3^{-\alpha}\,(2\pi)^{-3\alpha}\,\Gamma(3\alpha+2)\max\left\{1,(3T)^{3\alpha+\frac{3}{2}}\right\}.\end{aligned}$$

■

In the next lemma, we give an estimate for the second summand in Proposition 4.3; the proof uses Proposition 4.2, Theorem 2.1 (6), Lemma 2.3 (2), and Theorem 2.4 (2).

Lemma 4.6. *We have, uniformly in $\alpha\in\mathbb{N}+\frac{1}{4}$ and $T\geq 0$,*

$$\begin{aligned}&\left|\sum_{k=0}^{2\lfloor\alpha\rfloor+2}(-1)^k\frac{\Gamma(-\alpha+k+iT)}{\Gamma(-\alpha+iT)k!}\,\zeta(-2\alpha+k+2iT)\,\zeta(-\alpha-k+iT)\right|\\ &\qquad\ll(2\pi^3)^{-\alpha}\,\Gamma\left(3\alpha+\frac{5}{2}\right)\max\left\{1,T^{5\alpha+72}\right\}.\end{aligned}$$

Proof. By Theorem 2.1 (6) we obtain, for all $0 \leq k \leq 2\lfloor \alpha \rfloor$,

$$\Gamma(2\alpha - k + 1)\Gamma(\alpha + k + 1) \ll \Gamma(3\alpha + 2). \tag{4.5}$$

Proposition 4.2 and equation (4.5) give, for all $0 \leq k \leq 2\lfloor \alpha \rfloor$,

$$\left|\zeta(k - 2\alpha + 2iT)\,\zeta(-\alpha - k + iT)\right| \ll (2\pi)^{-3\alpha}\, 2^{2\alpha - k}\, \Gamma(3\alpha + 2) \max\left\{1, T^{3\alpha+1}\right\}. \tag{4.6}$$

Using the standard triangle inequality, (4.6), Proposition 4.2, Lemma 2.3 (2), and Theorem 2.4 (2), we bound, uniformly in $\alpha \in \mathbb{N} + \frac{1}{4}$ and $T \geq 0$,

$$\begin{aligned}
&\left|\sum_{k=0}^{2\lfloor\alpha\rfloor+2} (-1)^k \frac{\Gamma(k - \alpha + iT)}{\Gamma(-\alpha + iT)k!}\, \zeta(k - 2\alpha + 2iT)\,\zeta(-\alpha - k + iT)\right| \\
&\qquad \ll \sum_{k=0}^{2\lfloor\alpha\rfloor} \left|\binom{\alpha}{k}\right| \max\left\{1, T^k\right\} (2\pi)^{-3\alpha}\, 2^{2\alpha-k}\, \Gamma(3\alpha + 2) \max\left\{1, T^{3\alpha+1}\right\} \\
&\qquad\quad + \left|\binom{\alpha}{2\lfloor\alpha\rfloor + 1} (2\pi)^{-3\alpha}\, \Gamma\left(3\alpha + \frac{3}{2}\right) \max\left\{1, T^{5\alpha+2}\right\}\right| \\
&\qquad\quad + \left|\binom{\alpha}{2\lfloor\alpha\rfloor + 2} (2\pi)^{-3\alpha}\, \Gamma\left(3\alpha + \frac{5}{2}\right) \max\left\{1, T^{5\alpha+\frac{7}{2}}\right\}\right|,
\end{aligned}$$

where we treat the summands for $k \in \{2\lfloor\alpha\rfloor + 1, 2\lfloor\alpha\rfloor + 2\}$ separately. Noting that

$$\left|\binom{\alpha}{2\lfloor\alpha\rfloor + 2}\right| = \left|\frac{\alpha - 2\lfloor\alpha\rfloor - 1}{2\lfloor\alpha\rfloor + 2}\, \frac{\Gamma(\alpha + 1)}{\Gamma(2\lfloor\alpha\rfloor + 2)\Gamma(\alpha - 2\lfloor\alpha\rfloor)}\right| \leq \left|\binom{\alpha}{2\lfloor\alpha\rfloor + 1}\right|$$

and taking maximal factors, we can bound the above sum by

$$\begin{aligned}
&\left(2\pi^3\right)^{-\alpha} \Gamma(3\alpha + 2) \max\left\{1, T^{3\alpha+1}\right\} \sum_{k=0}^{2\lfloor\alpha\rfloor} \binom{\alpha}{k} 2^{-k} \max\left\{1, T^k\right\} \\
&\qquad + \left|\binom{\alpha}{2\lfloor\alpha\rfloor + 1} (2\pi)^{-3\alpha}\, \Gamma\left(3\alpha + \frac{5}{2}\right) \max\left\{1, T^{5\alpha+\frac{7}{2}}\right\}\right|
\end{aligned}$$

$$\ll \left(2\pi^3\right)^{-\alpha} \Gamma(3\alpha + 2) \max\left\{1, T^{5\alpha+1}\right\} + (2\pi)^{-3\alpha}\, \Gamma\left(3\alpha + \frac{5}{2}\right) \max\left\{1, T^{5\alpha+\frac{7}{2}}\right\}. \tag{4.7}$$

In the final step, we use that

$$\binom{\alpha}{2\lfloor\alpha\rfloor + 1} = O(1), \quad \text{as } \alpha \to \infty,$$

and, uniformly in $\alpha \in \mathbb{N} + \frac{1}{4}$ and $T \geq 0$,

$$\sum_{k=0}^{2\lfloor\alpha\rfloor} \left|\binom{\alpha}{k}\right| 2^{-k} \max\left\{1, T^k\right\} \ll \max\left\{1, T^{2\alpha}\right\}.$$

Next, we bound

$$\max\Big\{\left(2\pi^3\right)^{-\alpha}\Gamma(3\alpha+2)\max\big\{1,T^{5\alpha+1}\big\},(2\pi)^{-3\alpha}\Gamma\left(3\alpha+\frac{5}{2}\right)\max\big\{1,T^{5\alpha+\frac{7}{2}}\big\}\Big\}$$
$$\ll \left(2\pi^3\right)^{-\alpha}\Gamma\left(3\alpha+\frac{5}{2}\right)\max\big\{1,T^{5\alpha+\frac{7}{2}}\big\}$$

by comparing all factors and taking the maximum, and with (4.7) we conclude the proof of the lemma. ■

The following lemma deals with the integral in Proposition 4.3.

Proposition 4.7. *For some $C>0$, we have, uniformly in $\alpha\in\mathbb{N}+\frac{1}{4}$ and $T\geq 0$,*

$$\left|\int_{2\alpha+2-i\infty}^{2\alpha+2+i\infty}\frac{\Gamma(-\alpha+iT+z)\Gamma(-z)\zeta(-2\alpha+2iT+z)\zeta(-\alpha+iT-z)}{\Gamma(-\alpha+iT)}\,dz\right|$$
$$\ll C^{\alpha}\Gamma(3\alpha+3)\Gamma\left(3\alpha+\frac{21}{4}\right)\max\big\{1,T^{5\alpha+\frac{11}{2}}\big\}.$$

As the proof of Proposition 4.7 is rather technical, we split it in several lemmas. We start with the following.

Lemma 4.8. *There is an absolute constant $B>0$ such that we have, uniformly for $v\in\mathbb{R}$, $T\geq 0$, and $\alpha\in\mathbb{N}+\frac{1}{4}$,*

$$\left|\frac{\prod_{j=0}^{\lfloor\alpha\rfloor}(-\alpha+j+iT)\prod_{j=0}^{\lfloor\alpha\rfloor}\big(\alpha+1-j+i(T+v)\big)}{\prod_{j=0}^{2\lfloor\alpha\rfloor+2}(-2\alpha-2+j-iv)}\right| \ll \frac{B^{2\alpha+\frac{3}{2}}\max\big\{1,T^{2\alpha+\frac{3}{2}}\big\}}{\max\{1,|v|\}}. \tag{4.8}$$

Proof. To see the claim, note that while the numerator has $2\lfloor\alpha\rfloor+2$ factors, the denominator has $2\lfloor\alpha\rfloor+3$ factors. We combine factors

$$\frac{-\alpha+k+iT}{-2\alpha+2k-2-iv}\quad\text{and}\quad\frac{\alpha+1-k+i(T+v)}{-2\alpha+2k-1-iv}$$

for $0\leq k\leq\lfloor\alpha\rfloor$. We estimate, for $0\leq k\leq\lfloor\alpha\rfloor$ (with $m=k-\alpha\in\mathbb{Z}-\frac{1}{4}$),

$$\left|\frac{-\alpha+k+iT}{-2\alpha+2k-2-iv}\right|\ll\max\{1,T\}, \tag{4.9}$$

and (with $m=\alpha-k\in\mathbb{Z}+\frac{1}{4}$),

$$\left|\frac{\alpha+1-k+i(T+v)}{-2\alpha+2k-1-iv}\right|\leq\sup_{m\in\mathbb{Z}+\frac{1}{4}}\left|\frac{1+\frac{1}{m}+\frac{i(T+v)}{m}}{-2-\frac{1}{m}-\frac{iv}{m}}\right|$$
$$\leq\sqrt{\sup_{m\in\mathbb{Z}+\frac{1}{4}}\frac{\left(1+\frac{1}{m}\right)^2+\left(\frac{T+|v|}{m}\right)^2}{\left(2+\frac{1}{m}\right)^2+\left(\frac{v}{m}\right)^2}}.$$

We now distinguish several cases; recall that we assume $T \geq 0$.

Case 1. First let $0 < T \leq |v|$. Then we obtain

$$\sup_{m\in\mathbb{Z}+\frac{1}{4}} \frac{\left(1+\frac{1}{m}\right)^2+\left(\frac{T+v}{m}\right)^2}{\left(2+\frac{1}{m}\right)^2+\left(\frac{v}{m}\right)^2} \leq \sup_{m\in\mathbb{Z}+\frac{1}{4}} \frac{\left(1+\frac{1}{m}\right)^2+4\left(\frac{v}{m}\right)^2}{\left(2+\frac{1}{m}\right)^2+\left(\frac{v}{m}\right)^2}$$
$$\leq \sup_{m\in\mathbb{Z}+\frac{1}{4}} \frac{4\left(2+\frac{1}{m}\right)^2+\left(\frac{2v}{m}\right)^2}{\left(2+\frac{1}{m}\right)^2+\left(\frac{v}{m}\right)^2} = 4 \ll 1.$$

Case 2. Next we look at the case $T > |v|$. We obtain

$$\sup_{m\in\mathbb{Z}+\frac{1}{4}} \frac{\left(1+\frac{1}{m}\right)^2+\left(\frac{T+|v|}{m}\right)^2}{\left(2+\frac{1}{m}\right)^2+\left(\frac{v}{m}\right)^2} \leq \sup_{m\in\mathbb{Z}+\frac{1}{4}} \frac{\left(1+\frac{1}{m}\right)^2+\left(\frac{2T}{m}\right)^2}{\left(2+\frac{1}{m}\right)^2+\left(\frac{v}{m}\right)^2}.$$

Now, we again look at two different cases.

Case 2.1. If $0 \leq T \leq 1$, then we see

$$\sup_{m\in\mathbb{Z}+\frac{1}{4}} \frac{\left(1+\frac{1}{m}\right)^2+\left(\frac{2T}{m}\right)^2}{\left(2+\frac{1}{m}\right)^2+\left(\frac{v}{m}\right)^2} \leq \sup_{m\in\mathbb{Z}+\frac{1}{4}} \frac{\left(1+\frac{1}{m}\right)^2+\left(\frac{2}{m}\right)^2}{\left(2+\frac{1}{m}\right)^2} \ll 1.$$

Case 2.2. On the other hand, if $T \geq 1$, then we obtain

$$\sup_{m\in\mathbb{Z}+\frac{1}{4}} \frac{\left(1+\frac{1}{m}\right)^2+\left(\frac{2T}{m}\right)^2}{\left(2+\frac{1}{m}\right)^2+\left(\frac{v}{m}\right)^2} = T^2 \sup_{m\in\mathbb{Z}+\frac{1}{4}} \frac{\left(\frac{1}{T}+\frac{1}{mT}\right)^2+\left(\frac{2}{m}\right)^2}{\left(2+\frac{1}{m}\right)^2+\left(\frac{v}{m}\right)^2}$$
$$\leq T^2 \sup_{m\in\mathbb{Z}+\frac{1}{4}} \frac{\left(1+\frac{1}{m}\right)^2+\left(\frac{2}{m}\right)^2}{\left(2+\frac{1}{m}\right)^2} \ll T^2.$$

As a result, we obtain, uniformly for $\alpha \in \mathbb{N}+\frac{1}{4}$, $0 \leq k \leq \lfloor\alpha\rfloor$, $v \in \mathbb{R}$, and $T \geq 0$,

$$\left|\frac{\alpha+1-k+i(T+v)}{-2\alpha+2k-1-iv}\right| \ll \begin{cases} 1 & \text{if } T \leq |v| \\ \max\{1,T\} & \text{if } T > |v| \end{cases} \ll \max\{1,T\}. \tag{4.10}$$

The denominator of the left-hand side of (4.8) has one remaining factor, which is

$$\frac{1}{|-2\alpha+2\lfloor\alpha\rfloor-iv|} = \frac{1}{|-\frac{1}{2}-iv|} \ll \frac{1}{\max\{1,|v|\}}. \tag{4.11}$$

It follows from (4.9), (4.10) and (4.11) that the product (4.8) is uniformly bounded by $2\lfloor\alpha\rfloor+2$ factors $B\max\{1,T\}$, where B is some constant not depending on α, v, and T, and the factor $\frac{1}{\max\{1,|v|\}}$. Hence it is bounded by (using again $\lfloor\alpha\rfloor = \alpha-\frac{1}{4}$)

$$\frac{B^{2\lfloor\alpha\rfloor+2}\max\{1,T\}^{2\lfloor\alpha\rfloor+2}}{\max\{1,|v|\}} = \frac{B^{2\alpha+\frac{3}{2}}\max\left\{1,T^{2\alpha+\frac{3}{2}}\right\}}{\max\{1,|v|\}}.$$

This gives (4.8). ∎

For the following result, we define, for $T \geq 0$,

$$I_1(T) := \int_{-\infty}^{-T} \left|\zeta\left(-3\alpha - 2 + i(T - v)\right)\right| e^{\pi v}\, dv,$$
$$I_2(T) := \int_{-T}^{\infty} \left|\zeta\left(-3\alpha - 2 + i(T - v)\right)\right| \min\left\{1, e^{-\pi v}\right\} dv.$$

Lemma 4.9. *We have the uniform estimates*

$$\max\left\{1, T^{2\alpha+2}\right\} e^{\pi T} I_1(T) \ll \Gamma\left(3\alpha + \frac{21}{4}\right) \Gamma(3\alpha + 3) \max\left\{1, T^{5\alpha+\frac{21}{4}}\right\}, \tag{4.12}$$

$$\max\left\{1, T^{2\alpha+2}\right\} I_2(T) \ll \Gamma(3\alpha + 3)\, \Gamma\left(3\alpha + \frac{7}{2}\right) \max\left\{1, T^{5\alpha+\frac{11}{2}}\right\}. \tag{4.13}$$

Proof. We consider $I_1(T)$ and $I_2(T)$ separately. We obtain, substituting $u = T - v$ into $I_1(T)$,

$$e^{\pi T} I_1(T) = e^{2\pi T} \int_{2T}^{\infty} \left|\zeta\left(-3\alpha - 2 + iu\right)\right| e^{-\pi u}\, du.$$

With Proposition 4.2, we can bound

$$e^{\pi T} I_1(T) \ll \left(2\pi^2\right)^{-3\alpha} \Gamma(3\alpha + 3)\, \Gamma\left(3\alpha + \frac{7}{2}; 2\pi T\right) e^{2\pi T} + (2\pi)^{-3\alpha}\, \Gamma(3\alpha + 3). \tag{4.14}$$

Using the monotonicity of the Γ-function to estimate

$$\Gamma\left(3\alpha + \frac{7}{2}; 2\pi T\right) \ll \Gamma\left(3\alpha + \frac{17}{4}; 2\pi T\right)$$

(so that $3\alpha + \frac{17}{4} \in \mathbb{N}$), we use Theorem 2.2(2) to get

$$\Gamma\left(3\alpha + \frac{17}{4}; 2\pi T\right) \ll (2\pi)^{3\alpha}\, \Gamma\left(3\alpha + \frac{21}{4}\right) \max\left\{1, T^{3\alpha+\frac{13}{4}}\right\} e^{-2\pi T}.$$

Plugging this into (4.14), we conclude the proof of (4.12).

In $I_2(T)$, we split the integral at $v = 0$. First, with Proposition 4.2 we obtain

$$\int_{-T}^{0} \left|\zeta\left(-3\alpha - 2 + i(T - v)\right)\right| dv \ll \pi^{-3\alpha}\, \Gamma(3\alpha + 3) \max\left\{1, T^{3\alpha+\frac{7}{2}}\right\}. \tag{4.15}$$

For the contribution from $v \geq 0$, we substitute $u = T - v$. For the contribution from $u \leq 0$, we obtain, with Proposition 4.2,

$$e^{-\pi T} \int_{-\infty}^{0} \left|\zeta(-3\alpha - 2 + iu)\right| e^{\pi u}\, du \ll \left(2\pi^2\right)^{-3\alpha} \Gamma(3\alpha + 3)\, \Gamma\left(3\alpha + \frac{7}{2}\right) e^{-\pi T}. \tag{4.16}$$

For the contribution from $0 \leq u \leq T$, Proposition 4.2 again gives

$$e^{-\pi T} \int_{0}^{T} \left|\zeta(-3\alpha - 2 + iu)\right| e^{\pi u}\, du \ll (2\pi)^{-3\alpha}\, \Gamma(3\alpha + 3) \max\left\{1, T^{3\alpha+\frac{5}{2}}\right\}. \tag{4.17}$$

The estimates (4.15), (4.16) and (4.17) yield

$$I_2(T) \ll \Gamma(3\alpha+3)\,\Gamma\left(3\alpha+\frac{7}{2}\right)\max\left\{1, T^{3\alpha+\frac{7}{2}}\right\}.$$

Hence we obtain (4.13). ∎

We are now ready to prove Proposition 4.7.

Proof of Proposition 4.7. Choosing $M := 2\lfloor\alpha\rfloor+3$ and $s := -\alpha+iT$, the term including the integral in Proposition 4.3 equals, after substituting $z = 2\alpha+2+iv$,

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma\big(\alpha+2+i(T+v)\big)\,\Gamma(-2\alpha-2-iv)}{\Gamma(-\alpha+iT)}\\ &\qquad\times\zeta\big(2+i(2T+v)\big)\,\zeta\big(-3\alpha-2+i(T-v)\big)\,dv.\end{aligned}\tag{4.18}$$

First, we look at the Gamma factors in the integrand of (4.18). By Theorem 2.1 (1),

$$\begin{aligned}&\frac{\Gamma\big(\alpha+2+i(T+v)\big)\,\Gamma(-2\alpha-2-iv)}{\Gamma(-\alpha+iT)}\\ &=\frac{\Gamma\big(\frac{5}{4}+i(T+v)\big)\,\Gamma\big(\frac{1}{2}-iv\big)}{\Gamma\big(\frac{3}{4}+iT\big)}\,\frac{\prod_{j=0}^{\lfloor\alpha\rfloor}(-\alpha+j+iT)\prod_{j=0}^{\lfloor\alpha\rfloor}\big(\alpha+1-j+i(T+v)\big)}{\prod_{j=0}^{2\lfloor\alpha\rfloor+2}(-2\alpha-2+j-iv)}.\end{aligned}\tag{4.19}$$

The second factor on the right of (4.19) is estimated in Lemma 4.8. To bound the first factor in (4.19), we distinguish two cases.

First, assume $T+v\geq 0$. Then Theorem 2.1 (3), (4.19) and Lemma 4.8 yield

$$\left|\frac{\Gamma\big(\alpha+2+i(T+v)\big)\,\Gamma(-2\alpha-2-iv)}{\Gamma(-\alpha+iT)}\right| \ll B^{2\alpha+\frac{3}{2}}\max\left\{1, T^{2\alpha+2}\right\}\min\left\{1, e^{-\pi v}\right\}.\tag{4.20}$$

If $T+v<0$, then $v<-T\leq 0$, and with Theorem 2.1 (3), Lemma 4.8, and (4.19),

$$\left|\frac{\Gamma\big(\alpha+2+i(T+v)\big)\,\Gamma(-2\alpha-2-iv)}{\Gamma(-\alpha+iT)}\right| \ll B^{2\alpha+\frac{3}{2}}\max\left\{1, T^{2\alpha+2}\right\}e^{\pi T}e^{\pi v}.\tag{4.21}$$

Now we conclude, with $\left|\zeta\big(2+i(2T+v)\big)\right|\leq\zeta(2)$, and using (4.20) and (4.21), that (4.18) can be bounded by a constant times

$$B^{2\alpha+\frac{3}{2}}\max\left\{1, T^{2\alpha+2}\right\}\left(e^{\pi T}I_1(T)+I_2(T)\right).\tag{4.22}$$

Together with (4.22), (4.12) and (4.13), we conclude the proof of the claim. ∎

Now, we are ready to prove Proposition 4.4 and Theorem 4.1.

Proof of Proposition 4.4. Using Propositions 4.3 and 4.7, as well as Lemmas 4.5 and 4.6, the claim follows. ∎

Proof of Theorem 4.1. Using the triangle inequality, we obtain

$$|\mathcal{E}_n(z)| \ll \int_{-\infty}^{\infty} \left| \left(\frac{n^{\frac{3}{5}}}{X^2 z}\right)^{-k_n-\frac{1}{4}+iv} \Gamma\left(-k_n - \frac{1}{4} + iv\right) \zeta\left(-k_n + \frac{3}{4} + iv\right) \right.$$
$$\left. \times\, \omega\left(-k_n - \frac{1}{4} + iv\right) \right| dv.$$

With Theorem 2.1 (4), Theorem 2.4 (1), and Proposition 4.4, we obtain

$$\Gamma\left(-k_n - \frac{1}{4} + iv\right) \zeta\left(\frac{3}{4} - k_n + iv\right) \omega\left(-k_n - \frac{1}{4} + iv\right)$$
$$\ll C^{k_n} \Gamma\left(3k_n + \frac{15}{4}\right) \Gamma(3k_n + 6) \max\{1, |v|^{5k_n+7}\} e^{-\frac{\pi|v|}{2}}$$

for some constant $C > 0$. It follows that, for some constant $C_1 > 0$,

$$|\mathcal{E}_n(z)| \ll C_1^{k_n} n^{-\frac{3}{5}(k_n+\frac{1}{4})} |z|^{k_n+\frac{1}{4}} \Gamma(3k_n + 4) \Gamma(3k_n + 6) \Gamma(5k_n + 8). \tag{4.23}$$

Since

$$\Gamma(3k_n + 4) \Gamma(3k_n + 6) \Gamma(5k_n + 8) \ll k_n^{15} \Gamma(3k_n + 1)^2 \Gamma(5k_n + 1),$$

we obtain, with Theorem 2.1 (2) and for constants $C_2, C_3 > 0$,

$$\Gamma(3k_n + 4) \Gamma(3k_n + 6) \Gamma(5k_n + 8) \ll n^{C_2} C_3^{k_n} n^{\frac{2}{5} n^{\frac{2}{55}}}. \tag{4.24}$$

Note that

$$n^{-\frac{3k_n}{5}} \ll n^{\frac{3}{5}} n^{-\frac{3}{5} n^{\frac{2}{55}}},$$

since $k_n = \lfloor n^{\frac{2}{55}} \rfloor \geq n^{\frac{2}{55}} - 1$. Hence, with (4.23) and (4.24),

$$n^{-\frac{3}{5}(k_n+\frac{1}{4})} \Gamma(3k_n + 4) \Gamma(3k_n + 6) \Gamma(5k_n + 8) \ll n^{C_2+\frac{9}{20}} \left(\frac{C_3}{n^{\frac{1}{5}}}\right)^{n^{\frac{2}{55}}}. \tag{4.25}$$

Since $n^{C_2+\frac{9}{20}} \ll C_4^{n^{\frac{2}{55}}}$ for some $C_4 > 1$, and using (4.23) and (4.25), we conclude the proof of the theorem with $B := C_1 C_3 C_4$, $B^{k_n} = B^{\lfloor n^{\frac{2}{55}} \rfloor} \ll B^{n^{\frac{2}{55}}}$. ∎

5. The saddle point function

In this section, we investigate the derivatives of the sequence of functions

$$F_n(z) := \frac{3X^2}{z^{\frac{2}{3}}} - \frac{Y}{X n^{\frac{1}{10}} z^{\frac{1}{2}}} + 2X^2 z,$$

which is required for applying the saddle point method. It is useful to study the related function

$$\mathcal{F}(z;w) := -\frac{2X^2}{z^{\frac{5}{3}}} + \frac{Yw}{2Xz^{\frac{3}{2}}} + 2X^2,$$

which is holomorphic on $-i\mathbb{H}\times\mathbb{C}$; note that

$$F_n'(z) = \mathcal{F}\left(z;n^{-\frac{1}{10}}\right).$$

We look for solutions $S_n > 0$ for the equations $F_n'(S_n) = 0$. More generally, one can ask for real solutions $y = S(x) > 0$ of $\mathcal{F}(S(x),x) = 0$, where $x > 0$. We call $x \mapsto S(x)$ the *saddle point function*. The most important case for us, $x = n^{-\frac{1}{10}}$, is denoted by

$$S_n := S\left(n^{-\frac{1}{10}}\right). \tag{5.1}$$

The next lemma shows that this implicitly defined function exists for small $0 < x$ and can be extended to a holomorphic function around $x = 0$.

Lemma 5.1. *The function S is holomorphic near 0, and has a power series expansion*[1]

$$S(x) = \sum_{m=0}^{\infty} \varrho(m)\,x^m = 1 - \frac{3Y}{20X^3}\,x - \frac{3Y^2}{800\,X^6}\,x^2 - \frac{11Y^3}{64000\,X^9}\,x^3 + \frac{4959\,Y^5}{2048000000\,X^{15}}\,x^5 + O\left(x^6\right), \tag{5.2}$$

where $\rho(m) \in \mathbb{R}$ for all $m \geq 0$. In particular, $x \mapsto S(x)$, for x sufficiently small, is a real-valued function and the sequence S_n converges to 1 and satisfies $S_n < 1$ for n sufficiently large.

Proof. Let

$$\mathcal{H}(z;w) := \left(\mathcal{F}(z;w),w\right).$$

Note that $\mathcal{H}$ is holomorphic in $-i\mathbb{H}\times\mathbb{C}$. We obtain, for the Jacobi matrix,

$$J_{\mathcal{H}}(z;w) := \begin{pmatrix} \frac{\partial}{\partial z}\mathcal{F}(z;w) & \frac{\partial}{\partial w}\mathcal{F}(z;w) \\ \frac{\partial}{\partial z}w & \frac{\partial}{\partial w}w \end{pmatrix} = \begin{pmatrix} \frac{10X^2}{3z^{\frac{8}{3}}} - \frac{3wY}{4Xz^{\frac{5}{2}}} & \frac{Y}{2Xz^{\frac{3}{2}}} \\ 0 & 1 \end{pmatrix}.$$

In particular,

$$\det\left(J_{\mathcal{H}}(1;0)\right) = \frac{10}{3}\,X^2 \neq 0.$$

Therefore, due to the inverse mapping theorem (see [8], p. 27), $\mathcal{H}$ is locally biholomorphic around $(1,0)$ with inverse $\mathcal{H}^{-1} =: (\mathcal{H}_1^{-1},\mathcal{H}_2^{-1})$, i.e., $\mathcal{H}^{-1}$ can be expanded locally into a power series around $\mathcal{H}(1;0) = (0,0)$. Hence, we obtain

$$\mathcal{H}_1^{-1}(u;w) = \sum_{k,m\geq 0} \beta(k,m)\,u^k\,w^m \quad \text{and} \quad \mathcal{H}_2^{-1}(u;w) = w.$$

[1]Note that the fourth coefficient in (5.2) vanishes. We thank the referee for the comment that this fact could be interesting in itself. We do not have a deep reason for this phenomenon, but it could possibly be worthy of further investigation.

As a result, for real x sufficiently small, we find

$$\mathcal{H}^{-1}(0;x) = \big(S(x), x\big).$$

Hence we obtain

$$S(x) = \sum_{m=0}^{\infty} \beta(0,m)\, x^m = \sum_{m=0}^{\infty} \varrho(m)\, x^m,$$

and $\varrho(0) = 1$, since $\mathcal{H}^{-1}(0;0) = (1,0)$. Next note that all $\varrho(m)$ are real. Indeed, since $\overline{\mathcal{F}(z;x)} = \mathcal{F}(\bar{z};x)$, both $S(x)$ and $\overline{S(x)}$ are zeros of $z \mapsto \mathcal{F}(z;x)$. Now, because of biholomorphicity of $\mathcal{H}$ we obtain $S(x) = \overline{S(x)}$. One can use a computer or the formula for inverse power series in Corollary 11.2 on p. 437 of [4], to find the first few coefficients in (5.2). A straightforward calculation using (5.1) and (5.2) shows that $S_n \to 1$, as $n \to \infty$, and $S_n < 1$ for all n sufficiently large. ∎

6. Approximation of holomorphic functions

6.1. Error bounds for the asymptotic terms of G

In the next proposition, we give a criterion for the monotonicity of power series.

Proposition 6.1. *Let $a(n)$ be a weakly decreasing sequence of non-negative real numbers that converges monotonically to 0, and such that $a(0) \geq a(1) > a(2) \geq 0$. Then the function $P\colon (-1,1) \to \mathbb{R}$ defined by*

$$P(x) := \sum_{n=0}^{\infty} (-1)^n\, a(n)\, x^{2n}$$

has its global maximum $a(0)$ at $x = 0$, is increasing in the interval $[-2^{-\frac{1}{2}}, 0]$, and decreasing in the interval $[0, 2^{-\frac{1}{2}}]$.

Proof. By symmetry, it is enough to consider the case $x \geq 0$. Writing

$$P(x) = a(0) - \big(a(1) - a(2)x^2\big)x^2 - \big(a(3) - a(4)x^2\big)x^6 - \cdots$$

we see that $P(x) < P(0) = a(0)$ for all $0 < x < 1$, since

$$a(n) - a(n+1)x^2 \geq a(n) - a(n+1) \geq 0$$

and $a(1) - a(2)x^2 > 0$ for all $0 < x < 1$, since $a(1) > a(2)$. This gives the claim about the global maximum.

To show monotonicity, we prove that $P'(x) < 0$ for $0 < x < 2^{-\frac{1}{2}}$. Since $P(x)$ is given by a convergent real power series on $(-1,1)$, it extends to a holomorphic function in the inner unit disk and the power series converges uniformly and absolutely on compact subsets (note that any real power series $x \mapsto \sum_{n=0}^{\infty} a_n x^n$ extends to a complex power series $z \mapsto \sum_{n=0}^{\infty} a_n z^n$, and if the first converges in $(-\delta, \delta)$, the second has radius of convergence $R \geq \delta$). Hence, for all $|x| < 1$, the series $P(x)$ and all its derivatives converge absolutely. We have

$$P'(x) = 2\sum_{n=1}^{\infty}(-1)^n n\, a(n)\, x^{2n-1} = -2\sum_{n=1}^{\infty} T_n(x),$$

where

$$T_n(x) := (2n-1)\, a(2n-1)\, x^{4n-3} - 2n\, a(2n)\, x^{4n-1}.$$

Next we show that $T_n(x) \geq 0$ and also $T_1(x) > 0$ in the range $0 < x < 2^{-\frac{1}{2}}$, which proves that $P'(x) < 0$ in this region. To see this, we write

$$T_n(x) = (2n-1)\big(a(2n-1) - a(2n)\, x^{4n-3}\big) + a(2n)\big((2n-1)\, x^{4n-3} - 2n\, x^{4n-1}\big).$$

Since $a(n)$ is decreasing, the first term $(2n-1)(a(2n-1)-a(2n))x^{4n-3}$ is non-negative, and for $n = 1$ it is positive for $x > 0$, since $a(1) - a(2) > 0$. So we will deduce $T_1(x) > 0$ once we show that the second summand is non-negative in the interval $[0, 2^{-\frac{1}{2}}]$. We distinguish two cases.

If $a(2n) = 0$, then the second term vanishes and the claim follows. Thus we may assume that $a(2n) > 0$. In this case, the second term is positive for $x > 0$ if and only if the term

$$\frac{(2n-1)x^{4n-3} - 2n\, x^{4n-1}}{x^{4n-3}} = (2n-1) - 2nx^2$$

is positive. The quadratic function $x \mapsto (4n-2) - 4nx^2$ has its zeros at $\pm\sqrt{\frac{4n-2}{4n}}$, and is positive in $x = 0$. As a result, disregarding the choice of n, it is positive in the interval $[0, 2^{-\frac{1}{2}}]$, since

$$n \mapsto \sqrt{\frac{4n-2}{4n}} = \sqrt{1 - \frac{1}{2n}}$$

is a decreasing sequence with $2^{-\frac{1}{2}} \leq \sqrt{1 - \frac{1}{2n}}$ for all $n \geq 1$.

This proves the proposition. ∎

In the next lemma, we study the function

(6.1) $$f_n(x) := \frac{3X^2}{(S_n + ix)^{\frac{2}{3}}} - \frac{Y}{X n^{\frac{1}{10}} (S_n + ix)^{\frac{1}{2}}} + 2X^2(S_n + ix)$$

that will be important in the saddle point method. Here, S_n is the sequence of saddle points provided in (5.1).

Lemma 6.2. *For all n sufficiently large, $x \mapsto \mathrm{Re}(f_n(x))$ has its global maximum at $x = 0$ for $x \in \mathbb{R}$. In $[-\frac{1}{2}, 0]$ it is monotonically increasing, in $[0, \frac{1}{2}]$ it is monotonically decreasing, and for $|x| \geq \frac{1}{2}$ and real κ sufficiently small and not depending on x,*

(6.2) $$\big|\mathrm{Re}\big(f_n(x + i\kappa)\big)\big| \leq 4.8\, X^2.$$

Proof. Assume that n is sufficiently large such that $\frac{1}{2} < S_n < 1$. Let $|x| \le \frac{1}{2}$. We compute

$$\begin{aligned}
&\mathrm{Re}\big(f_n(x)\big)\\
&= \frac{3X^2}{S_n^{\frac{2}{3}}} \sum_{j=0}^{\infty} \binom{-\frac{2}{3}}{2j} (-1)^j \left(\frac{x}{S_n}\right)^{2j} - \frac{Y}{X n^{\frac{1}{10}} S_n^{\frac{1}{2}}} \sum_{j=0}^{\infty} \binom{-\frac{1}{2}}{2j} (-1)^j \left(\frac{x}{S_n}\right)^{2j} + 2X^2 S_n \\
&= \sum_{j=0}^{\infty} u_n(j)(-1)^j \left(\frac{x}{S_n}\right)^{2j}.
\end{aligned}$$

We first note that, for real values $\lambda \in \mathbb{R}$ and $j \in \mathbb{N}_0$,

$$\binom{\lambda}{2j+2} = \binom{\lambda}{2j} \frac{(\lambda - 2j)(\lambda - 1 - 2j)}{(2j+1)(2j+2)},$$

and as a result,

$$\begin{aligned}
u_n(j) - u_n(j+1) = &\frac{3X^2}{S_n^{\frac{2}{3}}} \binom{-\frac{2}{3}}{2j} \left(1 - \frac{\left(-\frac{2}{3} - 2j\right)\left(-\frac{5}{3} - 2j\right)}{(2j+1)(2j+2)}\right) \\
&- \frac{Y}{X n^{\frac{1}{10}} S_n^{\frac{1}{2}}} \binom{-\frac{1}{2}}{2j} \left(1 - \frac{\left(-\frac{1}{2} - 2j\right)\left(-\frac{3}{2} - 2j\right)}{(2j+1)(2j+2)}\right).
\end{aligned}$$

Note that we have

$$\begin{aligned}
R_1(j) &:= 1 - \frac{\left(-\frac{2}{3} - 2j\right)\left(-\frac{5}{3} - 2j\right)}{(2j+1)(2j+2)} = \frac{1}{3j} + O\big(j^{-2}\big), \quad \text{as } j \to \infty, \\
R_2(j) &:= 1 - \frac{\left(-\frac{1}{2} - 2j\right)\left(-\frac{3}{2} - 2j\right)}{(2j+1)(2j+2)} = \frac{1}{2j} + O\big(j^{-2}\big), \quad \text{as } j \to \infty.
\end{aligned}$$

Additionally, we see by comparing denominators and numerators, that $R_1(j), R_2(j) > 0$ for all $j \in \mathbb{N}$. By Lemma 2.3(3), there exist constants $m, M > 0$, such that for all $j \in \mathbb{N}$,

$$\frac{m}{j^{\frac{1}{3}}} \le \binom{-\frac{2}{3}}{2j} \quad \text{and} \quad \binom{-\frac{1}{2}}{2j} \le \frac{M}{j^{\frac{1}{2}}}.$$

Hence, we obtain

$$u_n(j) - u_n(j+1) \ge \frac{3mX^2}{S_n^{\frac{2}{3}}} \frac{R_1(j)}{j^{\frac{1}{3}}} - \frac{YM}{X n^{\frac{1}{10}} S_n^{\frac{1}{2}}} \frac{R_2(j)}{j^{\frac{1}{2}}}, \tag{6.3}$$

and since $R_1(j) \sim \frac{2}{3} R_2(j)$, the right-hand side of (6.3) is positive for $j \ge j_0$ for some $j_0 \in \mathbb{N}$. We note that j_0 does not increase if n increases (if n is sufficiently large, S_n converges to 1). Hence, if n is sufficiently large we may force $j_0 = 1$ so the right-hand side of (6.3) is positive for all $j \in \mathbb{N}$ and $u_n(0) > u_n(1)$, since S_n converges to 1. As a result, we obtain that $u_n(j) > u_n(j+1) > 0$ for all $j \in \mathbb{N}_0$, which proves that the sequence $u_n(j)$ is decreasing in j and is positive. Proposition 6.1 then gives the monotonicity and maximality of $\mathrm{Re}(f_n)$ at $x = 0$.

To show (6.2), we start with the formula

$$\mathrm{Re}\Big((S_n-\kappa+ix)^{-\frac{2}{3}}\Big)=\mathrm{Re}\Big(|S_n-\kappa+ix|^{-\frac{2}{3}}\, e^{-\frac{2}{3}i\,\mathrm{Arg}(S_n-\kappa+ix)}\Big).$$

For κ sufficiently small, we have that $S_n-\kappa\geq\frac{999}{1000}$ for all n sufficiently large. Since $|x|\geq\frac{1}{2}$ with $x\in\mathbb{R}$, $|S_n-\kappa+ix|$ is minimized by $|\frac{999}{1000}-\frac{i}{2}|$ for all n sufficiently large, thus $|S_n-\kappa+ix|^{-\frac{2}{3}}$ is maximized by $|\frac{999}{1000}-\frac{i}{2}|^{-\frac{2}{3}}$, and the maximum takes the value $0.9288\ldots\leq 0.93$. The same holds for $|S_n-\kappa+ix|^{-\frac{1}{2}}$, which is maximized by $|\frac{999}{1000}-\frac{i}{2}|^{-\frac{1}{2}}\leq 0.95$. Hence we obtain, with $\mathrm{Re}(z)=|z|\cos\big(\mathrm{Arg}(z)\big)$ and the triangle inequality,

$$\big|\mathrm{Re}\big(f_n(x+i\kappa)\big)\big|\leq 2.79\,X^2+2\,X^2+\frac{0.95Y}{Xn^{\frac{1}{10}}}\leq 4.8\,X^2,$$

where the last inequality holds for all n sufficiently large. ∎

We need another technical lemma.

Lemma 6.3. *We have, as $n\to\infty$,*

$$\sup_{-S_n\leq x\leq -n^{-\frac{7}{40}}}\left|\exp\left(n^{\frac{2}{5}}f_n(x)\right)\right| \ll \exp\left(n^{\frac{2}{5}}\left(\frac{3X^2}{S_n^{\frac{2}{3}}}+2X^2S_n-\frac{Y}{Xn^{\frac{1}{10}}S_n^{\frac{1}{2}}}-\frac{5X^2}{3n^{\frac{7}{20}}S_n^{\frac{8}{3}}}\right)\right).$$

Proof. We first note that

$$\frac{3X^2}{(S_n+ix)^{\frac{2}{3}}}=\frac{3X^2}{S_n^{\frac{2}{3}}}-\frac{2X^2ix}{S_n^{\frac{5}{3}}}-\frac{5X^2x^2}{3S_n^{\frac{8}{3}}}+\frac{40X^2ix^3}{27S_n^{\frac{11}{3}}}+O\big(x^4\big), \tag{6.4}$$

$$\begin{aligned}-\frac{Y}{Xn^{\frac{1}{10}}(S_n+ix)^{\frac{1}{2}}}=&-\frac{Y}{Xn^{\frac{1}{10}}S_n^{\frac{1}{2}}}+\frac{Yix}{2Xn^{\frac{1}{10}}S_n^{\frac{3}{2}}}+\frac{3Yx^2}{8Xn^{\frac{1}{10}}S_n^{\frac{5}{2}}}\\&-\frac{5Yix^3}{16Xn^{\frac{1}{10}}S_n^{\frac{7}{2}}}+O\big(x^4\big).\end{aligned} \tag{6.5}$$

The reader should note that, as both functions in (6.4) and (6.5) converge locally uniformly to holomorphic functions around $x=0$, so do all of their derivatives, and the O-terms are uniform in n. By Lemma 6.2, the function $x\mapsto|\exp(n^{\frac{2}{5}}f_n(x))|=\exp(n^{\frac{2}{5}}\,\mathrm{Re}(f_n(x)))$ is monotonically increasing in the interval $[-\frac{1}{2},0]$ if n is sufficiently large. Note that a locally monotonic real function f stays monotonic after composition with the exponential function and multiplication with a constant $m_f>0$ that even may depend on f. Hence, by Lemma 6.2, we obtain its supremum in the interval $[-\frac{1}{2},-n^{-\frac{7}{40}}]$ by just plugging in $-n^{-\frac{7}{40}}$. By using the Taylor expansions (6.4) and (6.5), we obtain

$$\mathrm{Re}\big(f_n(x)\big)=\left(\frac{3X^2}{S_n^{\frac{2}{3}}}+2X^2S_n-\frac{Y}{Xn^{\frac{1}{10}}S_n^{\frac{1}{2}}}\right)+\left(\frac{3Y}{8Xn^{\frac{1}{10}}S_n^{\frac{5}{2}}}-\frac{5X^2}{3S_n^{\frac{8}{3}}}\right)x^2+O\big(x^4\big).$$

Note that the terms with odd powers of x vanish by taking the real part and only considering real x. When putting $x = -n^{-\frac{7}{40}}$ into the above formula, we obtain

$$\begin{aligned}(6.6)\qquad n^{\frac{2}{5}}\operatorname{Re}\left(\frac{3X^2}{\left(S_n - in^{-\frac{7}{40}}\right)^{\frac{2}{3}}} - \frac{Y}{Xn^{\frac{1}{10}}\left(S_n - in^{-\frac{7}{40}}\right)^{\frac{1}{2}}}\right) + 2X^2S_n \\ = n^{\frac{2}{5}}\left(\frac{3X^2}{S_n^{\frac{2}{3}}} + 2X^2S_n - \frac{Y}{Xn^{\frac{1}{10}}S_n^{\frac{1}{2}}} - \frac{5X^2}{3n^{\frac{7}{20}}S_n^{\frac{8}{3}}}\right) + O(1).\end{aligned}$$

From this, we conclude with $|\exp(n^{\frac{2}{5}} f_n(x))| = \exp(n^{\frac{2}{5}}\operatorname{Re}(f_n(x)))$ and (6.6), that

$$\begin{aligned}\sup_{-\frac{1}{2}\le x\le -n^{-\frac{7}{40}}}\left|\exp\left(n^{\frac{2}{5}} f_n(x)\right)\right| \\ \ll \exp\left(n^{\frac{2}{5}}\left(\frac{3X^2}{S_n^{\frac{2}{3}}} + 2X^2S_n - \frac{Y}{Xn^{\frac{1}{10}}S_n^{\frac{1}{2}}} - \frac{5X^2}{3n^{\frac{7}{20}}S_n^{\frac{8}{3}}}\right)\right), \quad \text{as } n\to\infty.\end{aligned}$$

It remains to look at the range $[-S_n, -\frac{1}{2}]$. By (6.2) we see that, for all $x \le -\frac{1}{2}$, and as $n \to \infty$,

$$\begin{aligned}\exp\left(n^{\frac{2}{5}}\operatorname{Re}\left(f_n(x)\right)\right) \ll \exp\left(4.8X^2n^{\frac{2}{5}}\right) \\ \ll \exp\left(n^{\frac{2}{5}}\left(\frac{3X^2}{S_n^{\frac{2}{3}}} + 2X^2S_n - \frac{Y}{Xn^{\frac{1}{10}}S_n^{\frac{1}{2}}} - \frac{5X^2}{3n^{\frac{7}{20}}S_n^{\frac{8}{3}}}\right)\right),\end{aligned}$$

since $S_n \to 1$, and hence $3X^2S_n^{-\frac{2}{3}} + 2X^2S_n \to 5X^2$, as $n \to \infty$, so

$$4.8X^2 - \left(\frac{3X^2}{S_n^{\frac{2}{3}}} + 2X^2S_n - \frac{Y}{Xn^{\frac{1}{10}}S_n^{\frac{1}{2}}} - \frac{5X^2}{3n^{\frac{7}{20}}S_n^{\frac{8}{3}}}\right) \to -0.2X^2 < 0.$$

This proves the lemma. ∎

The following key lemma is due to Romik, and helps us to obtain estimates on the minor arcs.

Lemma 6.4 (Romik [11], equations (85) and (94)). *For all $\kappa > 0$, there exist constants $\beta, \delta > 0$, such that for all $0 < t < \beta$ and $\kappa t \le |u| \le \pi$, we have*

$$\left|G\left(e^{-t+iu}\right)\right| \le \exp\left(-\frac{\delta}{\sqrt{t}}\right) G\left(e^{-t}\right).$$

6.2. Approximation for the auxiliary asymptotic terms in G

Using Theorem 2.1 (2), we obtain by a straightforward calculation the following asymptotic bound for the coefficients ν_m defined in Theorem 3.3.

Lemma 6.5. *There exists a constant $C > 0$, such that we have, for $m \in \mathbb{N}$,*

$$\nu_m \ll C^m m^{3m}.$$

To find the asymptotic behavior of $r(n)$, we study the behavior of $G(e^{-z})$ near $z = 0$. We choose $k_n - \frac{1}{2} < \eta_n < k_n + \frac{1}{2}$, with $k_n = \lfloor n^{\frac{2}{55}} \rfloor$, in Theorem 3.3. In Section 4, the error integral from Theorem 3.3 was already bounded. The terms with negative exponents in z in the expansion of $\mathrm{Log}(G(e^{-z}))$ in Theorem 3.3 were studied in Lemmas 6.2 and 6.3. They are the main terms, as they, up to a logarithmic term, determine the growth of $G(e^{-z})$ next to $z = 0$. We now look at the remaining terms with positive exponents. Consider the function $H_n \colon B_{\frac{1}{2}}(0) \times B_\delta(0) \to \mathbb{C}$ defined by

$$H_n(w; z) := \exp\left(\sum_{m=0}^{k_n - 1} \nu_m \left(2X^2 (S(z) + iw)\right)^{m+\frac{1}{2}} z^{6m+3} \right), \tag{6.7}$$

where $S(z)$ is the saddle point function (5.2), that is holomorphic in some region $B_\delta(0)$. By choosing δ sufficiently small, we achieve $S(B_\delta(0)) \subset B_{\frac{1}{4}}(1)$, since $S(0) = 1$. Hence, for all $z \in B_\delta(0)$ and $w \in B_{\frac{1}{2}}(0)$, we obtain $\mathrm{Re}(S(z) + iw) > \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$. Since the principal branch of the square root is holomorphic in the right half-plane, H_n is holomorphic in $B_{\frac{1}{2}}(0) \times B_\delta(0)$ as a composition of holomorphic functions. As such, we can write $H_n(w; z)$ as a power series:

$$H_n(w; z) = \sum_{m=0}^{\infty} c_{n,z}(m)\, w^m = \sum_{m,\ell \geq 0} a_{n,m}(\ell)\, z^\ell w^m. \tag{6.8}$$

Our goal is to show the following proposition.

Proposition 6.6. *Fix $M \in \mathbb{N}_0$. For $|x| \leq n^{\frac{1}{40}}$ and n sufficiently large, we have*

$$H_n\left(\frac{x}{n^{\frac{1}{5}}}; n^{-\frac{1}{10}} \right) = 1 + \sum_{m=1}^{M} \frac{P_m^{[1]}(x)}{n^{\frac{m}{10}}} + O_M\left(n^{-\frac{M+1}{15}}\right),$$

where the $P_m^{[1]}$ are polynomials that do not depend on n and M, with each $P_m^{[1]}$ having degree at most $\frac{m}{2}$.

To prove Proposition 6.6, we need the following lemma.

Lemma 6.7. *Let $0 < \delta < 1$ be fixed. The sequence $w \mapsto H_n(w; n^{-\frac{1}{10}})$ of holomorphic functions converges uniformly on compact sets to the constant function 1 on $\mathbb{C}_\delta$.*

Proof. Since by Lemma 5.1 the sequence $S_n = S(n^{-\frac{1}{10}})$ converges to 1 as $n \to \infty$, we have for all $w \in \mathbb{C}_\delta$, $\mathrm{Re}(S_n + iw) > 0$ for n sufficiently large. For these n, the functions $w \mapsto H_n(w; n^{-\frac{1}{10}})$ are holomorphic in $\mathbb{C}_\delta$. Let $D \subset \mathbb{C}_\delta$ be compact. Then there exists a constant $C_D > 0$, such that $|S_n + iw| \leq S_n + |w| \leq C_D$ for all $w \in D$ and for all n sufficiently large. Using this, we obtain, with Lemma 6.5 and with the triangle inequality,

$$\left| \mathrm{Log}\left(H_n\left(w; n^{-\frac{1}{10}}\right) \right) \right| \leq \sum_{m=0}^{k_n - 1} \left| \nu_m \left(\frac{2 C_D X^2}{n^{\frac{3}{5}}} \right) \right|^{m+\frac{1}{2}} \ll n^{-\frac{3}{10}} \sum_{m=0}^{\infty} \frac{K_D^m}{n^{\frac{27m}{55}}} \ll_D n^{-\frac{3}{10}},$$

where the constant $K_D > 0$ only depends on D. The claim now follows. ∎

The next step is to study the coefficients $a_{n,m}(\ell)$ in (6.8).

Lemma 6.8. *For $\ell \in \mathbb{N}_0$, there exists $N_\ell \in \mathbb{N}$ such that, for all $m \in \mathbb{N}_0$, $a_{n,m}(\ell)$ is constant for $n \geq N_\ell$.*

Proof. The sequence $k_n = \lfloor n^{\frac{2}{55}} \rfloor$ increases monotonically, and thus we see, for $n_2 \geq n_1$,

$$\frac{H_{n_2}(w;z)}{H_{n_1}(w;z)} = \exp\left(\sum_{m=k_{n_1}}^{k_{n_2}-1} \nu_m \left(2X^2(S(z)+iw)\right)^{m+\frac{1}{2}} z^{6m+3} \right)$$
$$= 1 + O_{n_1,n_2,w}\left(z^{6k_{n_1}+3}\right).$$

For the last equality, note that if $z \mapsto f(z) = \sum_{n=m}^{\infty} a(n) z^n$ is holomorphic at $z = 0$, where $m \in \mathbb{N}_0$, so is the composition $z \mapsto \exp(f(z))$, and we can write

$$\exp\big(f(z)\big) = \exp\left(\sum_{n=m}^{\infty} a(n) z^n \right) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\sum_{n=m}^{\infty} a(n) z^n \right)^k .$$

This implies that

$$\exp\big(f(z)\big) = 1 + O_f\left(z^m\right).$$

Note that in the case $f(z) = H_{n_2}(w;z) H_{n_1}(w;z)^{-1}$, the Taylor coefficients $a(n)$ depend on n_1, n_2, and w. This finally shows that

$$H_{n_2}(w,z) - H_{n_1}(w,z) = O_{n_1,n_2,w}\left(z^{6k_{n_1}+3}\right). \tag{6.9}$$

On the other hand, we obtain with (6.8) that

$$H_{n_2}(w,z) - H_{n_1}(w,z) = \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} \left(a_{n_2,m}(\ell) - a_{n_1,m}(\ell)\right) z^\ell w^m. \tag{6.10}$$

From (6.9) and (6.10), it follows that $a_{n_2,m}(\ell) - a_{n_1,m}(\ell) = 0$ for all $m \in \mathbb{N}_0$ if $\ell < 6k_{n_1} + 3$. With $N_\ell := \ell^{55}$, the lemma follows. Indeed, we show that $a_{n_1,m}(\ell) = a_{n_2,m}(\ell)$ for all $m \in \mathbb{N}_0$ and $n_2 \geq n_1 \geq N_\ell$. For all $n_1 \geq N_\ell$, we obtain equivalently $n_1^{\frac{1}{55}} \geq \ell$. With this we have

$$\ell \leq n_1^{\frac{1}{55}} \leq \lfloor n_1^{\frac{1}{55}} \rfloor + 1 < 6\lfloor n_1^{\frac{2}{55}} \rfloor + 3 = 6k_{n_1} + 3,$$

since obviously $\lfloor n_1^{\frac{1}{55}} \rfloor \leq \lfloor n_1^{\frac{2}{55}} \rfloor$, and so $a_{n_1,m}(\ell) = a_{n_2,m}(\ell)$ for all $m \in \mathbb{N}_0$ and for $n_2 \geq n_1 \geq N_\ell$. ∎

Next, it is useful to have a uniform upper bound for the values $a_{n,m}(\ell)$. Using the Cauchy integral formula, we obtain the following bound.

Lemma 6.9. *We have, for $m, \ell \in \mathbb{N}_0$ and n sufficiently large,*

$$\left|a_{n,m}(\ell)\right| \ll 4^m n^{\frac{\ell}{30}}.$$

Proof. By the Cauchy integral formula, we have

$$a_{n,m}(\ell) = \frac{1}{(2\pi i)^2} \int_{|z|=n^{-\frac{1}{30}}} \int_{|w|=\frac{1}{4}} \frac{H_n(w;z)}{w^{m+1}\, z^{\ell+1}}\, dw\, dz,$$

where the integrals over closed contours are taken once and anti-clockwise. We bound

$$\left|H_n(w;z)\right| \le \exp\left(\sum_{m=0}^{k_n-1} \left| \nu_m \left(2X^2\left(S(z)+iw\right)\right)^{m+\frac{1}{2}} z^{6m+3} \right| \right).$$

Let $|z| = n^{-\frac{1}{30}}$, for n sufficiently large, and let $|w| = \frac{1}{4}$. Then there exists a constant $K > 0$, independent of w, z and n, such that

$$\left|S(z) + iw\right| \le \left|S(z)\right| + |w| \le K,$$

since the function $z \mapsto S(z)$ does not depend on w and n and is holomorphic locally around 0. Hence, we obtain the estimate

$$\left|2X^2\left(S(z)+iw\right)\right|^{m+\frac{1}{2}} \le \left(2X^2 K\right)^{m+\frac{1}{2}}.$$

With Lemma 6.5, we find, for all $0 \le m \le k_n - 1 = \lfloor n^{\frac{2}{55}} \rfloor - 1$, that $\nu_m \ll C^m n^{\frac{6m}{55}}$ uniformly in m and n, and hence

$$\begin{aligned} \left|H_n(w;z)\right| &\ll \exp\left(\sum_{m=0}^{k_n-1} \frac{\left(2X^2K\right)^{m+\frac{1}{2}} C^m n^{\frac{6m}{55}}}{n^{\frac{m}{5}+\frac{1}{10}}} \right) \\ &\ll \exp\left(\frac{\left(2X^2K\right)^{\frac{1}{2}}}{n^{\frac{1}{10}}} \sum_{m=0}^{\infty} \left(\frac{2X^2CK}{n^{\frac{1}{11}}} \right)^m \right) \ll \exp\left(\frac{B}{n^{\frac{1}{10}}} \right), \end{aligned}$$

with some constant $B > 0$ for n sufficiently large. We conclude that $H_n(w;z)$ is uniformly bounded for $|z| = n^{-\frac{1}{30}}$ and $|w| = \frac{1}{4}$, by $A > 0$, say. With this, and using the triangle inequality for integrals and the estimate with integrand supremum and curve lengths $2\pi n^{-\frac{1}{30}}$ and $\frac{\pi}{2}$, respectively, we conclude that

$$\begin{aligned} |a_{n,m}(\ell)| &\ll \int_{|z|=n^{-\frac{1}{30}}} \int_{|w|=\frac{1}{4}} \left| \frac{H_n(w;z)}{w^{m+1}\, z^{\ell+1}}\, dw\, dz \right| \\ &\ll n^{-\frac{1}{30}} \sup_{|w|=\frac{1}{4}, |z|=n^{-\frac{1}{30}}} \left| \frac{H_n(w;z)}{w^{m+1}\, z^{\ell+1}} \right| \le n^{-\frac{1}{30}} 4^m n^{\frac{\ell+1}{30}} A \ll 4^m n^{\frac{\ell}{30}}. \end{aligned}$$ ∎

We next approximate the coefficients $c_{n,z}$ defined in (6.8).

Lemma 6.10. *Fix $M \in \mathbb{N}_0$. Then for n sufficiently large, we have*

$$c_{n,n^{-\frac{1}{10}}}(m) = \sum_{\ell=0}^{M} \frac{a_m(\ell)}{n^{\frac{\ell}{10}}} + O\left(\frac{4^m}{n^{\frac{M+1}{15}}} \right),$$

where the $a_m(\ell)$ are independent on n, and the O-constant is independent of m, M and n.

Proof. By Lemma 6.8, for each $0 \le \ell \le M$ there exists a constant N_ℓ such that $a_{n,m}(\ell)$ is constant for $n \ge N_\ell$. Since M is fixed and independent of n, $R_M := \max_{0\le\ell\le M} N_\ell$ is also independent on n. Thus $a_{n,m}(\ell)$ does not depend on n if $n \ge R_M$, and for these n can simply be noted as $a_m(\ell)$. Hence, for all $n \ge R_M$ we have, by (6.8),

$$c_{n,z}(m) = \sum_{\ell=0}^{M} a_m(\ell)\, z^\ell + \sum_{\ell=M+1}^{\infty} a_{n,m}(\ell)\, z^\ell.$$

With Lemma 6.9 and (6.8) we then obtain, for n sufficiently large,

$$\left| \sum_{\ell=M+1}^{\infty} \frac{a_{n,m}(\ell)}{n^{\frac{\ell}{10}}} \right| \ll 4^m \sum_{\ell=M+1}^{\infty} n^{-\frac{\ell}{15}} = 4^m \frac{n^{-\frac{M+1}{15}}}{1-n^{-\frac{1}{15}}} = O\left(\frac{4^m}{n^{\frac{M+1}{15}}} \right).$$

This proves the lemma. ∎

By (6.8), the Cauchy integral formula, Lemmas 6.10 and 6.7, Proposition 6.6 follows.

6.3. Approximation for the main asymptotic term in G

The next step is to study the main term in Theorem 3.3. Similarly as in the case of H_n, we first interpret the saddle point parameter as an independent variable of a holomorphic function. Therefore, we define

$$f(w,z) := \exp\left(z^{-4} \left(\frac{3X^2}{(S(z)+iw)^{\frac{2}{3}}} - \frac{Yz}{X\big(S(z)+iw\big)^{\frac{1}{2}}} + 2X^2(S(z)+iw) \right) \right),$$

for $w \in \mathbb{C}_{\frac{1}{2}}$ and $z \in B_\delta(0)$, where δ is sufficiently small. Moreover, define

$$A(n) := \exp\left(A_1 n^{\frac{2}{5}} - A_2 n^{\frac{3}{10}} - A_3 n^{\frac{1}{5}} - A_4 n^{\frac{1}{10}} \right), \tag{6.11}$$

where the constants A_j were given in (1.8).

Lemma 6.11. *For z and x sufficiently small, we have the Laurent expansion*

$$\begin{aligned} z^{-4} &\left(\frac{3X^2}{\big(S(z)+ixz^2\big)^{\frac{2}{3}}} - \frac{Yz}{X\big(S(z)+ixz^2\big)^{\frac{1}{2}}} + 2X^2\big(S(z)+ixz^2\big) \right) \\ &= \frac{A_1}{z^4} - \frac{A_2}{z^3} - \frac{A_3}{z^2} - \frac{A_4}{z} - A_5 - \frac{5X^2}{3}x^2 + \sum_{\ell=1}^{\infty} P_{\ell+4}(x)\, z^\ell, \end{aligned} \tag{6.12}$$

where the $P_{\ell+4}$ are polynomials, and the constants A_j were defined in (1.8). For every fixed $M \in \mathbb{N}_{>1}$ and $|x| \le n^{\frac{1}{40}}$, we have the expansion, with n is sufficiently large,

$$\exp\left(n^{\frac{2}{5}} f_n\left(\frac{x}{n^{\frac{1}{5}}} \right) \right) = A(n)\exp(-A_5)\exp\left(-\frac{5X^2}{3}x^2 \right)\left(1 + \sum_{m=1}^{M} \frac{P_m^{[2]}(x)}{n^{\frac{m}{10}}} + O\left(n^{-\frac{3(M+1)}{80}} \right) \right),$$

where the $P_m^{[2]}$ are polynomials that do not depend on n, with each $P_m^{[2]}$ having degree at most $2m$.

Proof. We start with the Laurent series expansion

$$z^{-4}\left(\frac{3X^2}{\left(S(z)+iw\right)^{\frac{2}{3}}}-\frac{Yz}{X\left(S(z)+iw\right)^{\frac{1}{2}}}+2X^2\left(S(z)+iw\right)\right)=:z^{-4}\sum_{\ell,m\geq 0}a(\ell,m)\,w^\ell z^m.$$

Setting $w=xz^2$ gives

$$\begin{aligned}z^{-4}&\left(\frac{3X^2}{\left(S(z)+ixz^2\right)^{\frac{2}{3}}}-\frac{Yz}{X\left(S(z)+ixz^2\right)^{\frac{1}{2}}}+2X^2\left(S(z)+ixz^2\right)\right)\\&=z^{-4}\sum_{\ell,m\geq 0}a(\ell,m)\,x^\ell z^{2\ell} z^m=z^{-4}\sum_{\ell=0}^{\infty}P_\ell(x)\,z^\ell,\end{aligned}$$

where the polynomials P_ℓ are independent from x and z and satisfy $\deg(P_\ell)\leq\lfloor\frac{\ell}{2}\rfloor$. Expanding the first five terms of the above series explicitly, we get

$$\sum_{\ell=0}^{\infty}P_\ell(x)\,z^{\ell-4}=\frac{A_1}{z^4}-\frac{A_2}{z^3}-\frac{A_3}{z^2}-\frac{A_4}{z}-A_5-\frac{5X^2}{3}x^2+\sum_{\ell=5}^{\infty}P_\ell(x)\,z^{\ell-4}, \tag{6.13}$$

which proves (6.12). As a result, we have

$$\begin{aligned}\exp\left(n^{\frac{2}{5}}f_n\left(\frac{x}{n^{\frac{1}{5}}}\right)\right)&=f\left(\frac{x}{n^{\frac{1}{5}}},n^{-\frac{1}{10}}\right)\\&=A(n)\exp\left(-A_5-\frac{5X^2}{3}x^2+\sum_{\ell=1}^{\infty}\frac{P_{\ell+4}(x)}{n^{\frac{\ell}{10}}}\right).\end{aligned}$$

We write

$$\exp\left(\sum_{\ell=1}^{\infty}\frac{P_{\ell+4}(x)}{n^{\frac{\ell}{10}}}\right)=:1+\sum_{m=1}^{M}\frac{P_m^{[2]}(x)}{n^{\frac{m}{10}}}+\sum_{m=M+1}^{\infty}\frac{P_m^{[2]}(x)}{n^{\frac{m}{10}}}, \tag{6.14}$$

and we are left to show that, uniformly for $|x|\leq n^{\frac{1}{40}}$,

$$\sum_{m=M+1}^{\infty}\frac{P_m^{[2]}(x)}{n^{\frac{m}{10}}}=O\left(n^{-\frac{3(M+1)}{80}}\right), \tag{6.15}$$

and that $\deg(P_m^{[2]}(x))\leq 2m$. To obtain this claim we need a better understanding of the behavior of the P_ℓ. The strategy is to use (6.14) to write

$$\sum_{m=M+1}^{\infty}\frac{P_m^{[2]}(x)}{n^{\frac{m}{10}}}=\sum_{\substack{k\geq 0\\ m\geq M+1}}\lambda(k,m)\,x^k n^{-\frac{m}{10}}, \tag{6.16}$$

where the λ are defined through

$$\exp\Big(\sum_{\ell=1}^{\infty} P_{\ell+4}(x)\, z^{\ell}\Big) =: \sum_{k,m\geq 0} \lambda(k,m)\, x^k\, z^m.$$

We show that the power series on the right of (6.16) is $O(n^{-\frac{3(M+1)}{80}})$. We first study the convergence of (6.16), arguing with holomorphicity. With

$$z^{-4}\Bigg(\frac{3X^2}{\big(S(z)+ixz^2\big)^{\frac{2}{3}}} - \frac{Yz}{X\big(S(z)+ixz^2\big)^{\frac{1}{2}}} + 2X^2\big(S(z)+ixz^2\big)\Bigg)$$
$$= z^{-4}\sum_{\ell,m\geq 0} a(\ell,m)\, x^{\ell}\, z^{2\ell}\, z^m = z^{-4}\sum_{\ell=0}^{\infty} P_{\ell}(x)\, z^{\ell}$$

and (6.13), we see that the function

$$(x,z)\mapsto z^{-4}\Bigg(\frac{3X^2}{\big(S(z)+ixz^2\big)^{\frac{2}{3}}} - \frac{Yz}{X\big(S(z)+ixz^2\big)^{\frac{1}{2}}} + 2X^2\big(S(z)+ixz^2\big)\Bigg)$$
$$-\Big(\frac{A_1}{z^4} - \frac{A_2}{z^3} - \frac{A_3}{z^2} - \frac{A_4}{z} - A_5 - \frac{5X^2}{3}x^2\Big) = \sum_{\ell=1}^{\infty} P_{\ell+4}(x)\, z^{\ell} \tag{6.17}$$

is holomorphic around $(0,0)$. More precisely, there exists a constant $\kappa>0$ such that it is holomorphic for all $(x,z)\in\mathbb{C}^2$ with $|xz^2|<\kappa$, since $S(z)$ is holomorphic due to Lemma 5.1 and $\mathrm{Re}(S(z)+ixz^2)>0$ for κ sufficiently small. As a result, for $r_1,r_2>0$ satisfying $r_1r_2^2<\kappa$, we have holomorphicity in the domain

$$D_{r_1,r_2} := \big\{(w,z)\in\mathbb{C}^2 : |w|<r_1, |z|<r_2\big\}.$$

Hence (6.17) is analytic and the power series around $(0,0)$ converges absolutely for all $(x,z)\in D_{r_1,r_2}$ (see [5], pp. 314–315). Since we can choose $r_1,r_2>0$ arbitrarily with $r_1r_2^2<\kappa$, we have an absolutely convergent power series expansion

$$\exp\Big(\sum_{\ell=1}^{\infty} P_{\ell+4}(x)\, z^{\ell}\Big) = \sum_{k,m\geq 0} \lambda(k,m)\, x^k\, z^m, \quad \text{for } \big|xz^2\big|<\kappa. \tag{6.18}$$

By construction,

$$\sum_{m=M+1}^{\infty} P_m^{[2]}(x)\, z^m = \sum_{\substack{k\geq 0\\ m\geq M+1}} \lambda(k,m)\, x^k\, z^m, \quad \text{for } \big|xz^2\big|<\kappa. \tag{6.19}$$

Next we show that, for all positive integers m,

$$\lambda(k,m) = 0, \quad \text{if } k>2m. \tag{6.20}$$

By the identity theorem for power series and (6.19), for (6.20) it is sufficient to prove that $\deg(P_m^{[2]}) \leq 2m$ for all positive integers m. We observe, by the series composition

$$\exp\left(\sum_{\ell=1}^{\infty} P_{\ell+4}(x)\, z^{\ell}\right) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{\ell=1}^{\infty} P_{\ell+4}(x)\, z^{\ell}\right)^{k},$$

that the polynomials $P_m^{[2]}$ are sums of the form

$$P_m^{[2]}(x) = \sum_{\substack{k,\ell_1,\ldots,\ell_k\geq 1\\ \ell_1+\ell_2+\cdots+\ell_k=m}} a_{\ell_1,\ldots,\ell_k} \prod_{j=1}^{k} P_{\ell_j+4}(x),$$

where the $a_{\ell_1,\ldots,\ell_k}$ are complex coefficients. Since $\deg(P_{\ell+4}) \leq \lfloor \frac{\ell+4}{2} \rfloor$, it is sufficient to show that

$$\sum_{j=1}^{k} \left\lfloor \frac{\ell_j+4}{2} \right\rfloor \leq 2m, \tag{6.21}$$

for all $\ell_1,\ldots,\ell_k \in \mathbb{N}$ such that $\sum_{j=1}^{k} \ell_j = m$.

To show (6.21), we distinguish $\ell_j = 1$ and $\ell_j > 1$ as follows. Let r (with $0 \leq r \leq k$) denote the number of ones in a fixed collection $\ell_1,\ldots,\ell_k$ with $\sum_{j=1}^{k} \ell_j = m$. Without loss of generality, $\ell_j = 1$ for all $j \leq r$ and $\ell_j > 1$ for all $j > r$. We have

$$k - r \leq \frac{m-r}{2}. \tag{6.22}$$

As a result, we obtain with $r \leq m$ and (6.22),

$$\sum_{j=1}^{k} \left\lfloor \frac{\ell_j+4}{2} \right\rfloor \leq 2m.$$

This proves (6.20).

Equipped with (6.20), we can prove (6.15). With $x=1$ and $z=\sqrt{\frac{\kappa}{2}}$, one has, by (6.18),

$$\sum_{k,m\geq 0} |\lambda(k,m)| \left(\frac{\kappa}{2}\right)^{\frac{m}{2}} < \infty. \tag{6.23}$$

We also have, for all n sufficiently large, the inequality $n^{-\frac{1}{80}} \leq \sqrt{\frac{\kappa}{2}}$, since κ does not depend on n. When plugging $|x| \leq n^{\frac{1}{40}}$ and $z = n^{-\frac{1}{10}}$ into (6.19), we obtain, by (6.20) and (6.23),

$$\left| \sum_{m=M+1}^{\infty} \frac{P_m^{[2]}(x)}{n^{\frac{m}{10}}} \right| \ll n^{-\frac{3(M+1)}{80}}.$$

This proves (6.15). ∎

In a similar way as demonstrated in Lemma 6.11, we show the following.

Lemma 6.12. *Fix $M \in \mathbb{N}_{>1}$. For all n sufficiently large and real x with $|x| \le n^{\frac{1}{40}}$,*

$$\left(S_n + \frac{ix}{n^{\frac{1}{5}}}\right)^{-\frac{1}{3}} = 1 + \sum_{m=1}^{M} \frac{P_m^{[3]}(x)}{n^{\frac{m}{10}}} + O\left(n^{-\frac{7(M+1)}{80}}\right),$$

where the $P_m^{[3]}$ are polynomials that do not depend on n, with each $P_m^{[3]}$ having degree at most $\frac{m}{2}$.

6.4. Polynomial approximations

We now define polynomials $P_m^{[4]}$ as the coefficients in the formal series expansion

$$\prod_{j=1}^{3}\left(1 + \sum_{m=1}^{\infty} \frac{P_m^{[j]}(x)}{n^{\frac{m}{10}}}\right) =: \sum_{m=0}^{\infty} \frac{P_m^{[4]}(x)}{n^{\frac{m}{10}}}, \tag{6.24}$$

and turn this into an analytic formula by truncating the series on the left-hand side of (6.24) for $M \in \mathbb{N}$:

$$\prod_{j=1}^{3}\left(1 + \sum_{m=1}^{M} \frac{P_m^{[j]}(x)}{n^{\frac{m}{10}}}\right) =: \sum_{m=0}^{M} \frac{P_m^{[4]}(x)}{n^{\frac{m}{10}}} + \sum_{m=M+1}^{3M} \frac{P_{M,m}^{[4]}(x)}{n^{\frac{m}{10}}}.$$

Note that, for $k \in \mathbb{N}$,

$$\prod_{j=1}^{3}\left(1 + \sum_{m=1}^{M+k} \frac{P_m^{[j]}(x)}{n^{\frac{m}{10}}}\right) - \prod_{j=1}^{3}\left(1 + \sum_{m=1}^{M} \frac{P_m^{[j]}(x)}{n^{\frac{m}{10}}}\right) = O\left(n^{-\frac{M+1}{10}}\right).$$

We are interested in the behavior of the polynomials $P_m^{[4]}$.

Lemma 6.13. *For $j \in \{1,2,3\}$, we have for $1 \le m \le M$,*

$$\sup_{|x| \le n^{\frac{1}{40}}} \left|\frac{P_m^{[j]}(x)}{n^{\frac{m}{10}}}\right| = O_M(1), \quad \text{as } n \to \infty.$$

Setting $d_1 := \frac{1}{15}$, $d_2 := \frac{3}{80}$ and $d_3 := \frac{7}{80}$, the following lemma follows by a direct calculation.

Lemma 6.14. *We have, for $|x| \le n^{\frac{1}{40}}$,*

$$\prod_{j=1}^{3}\left(1 + \sum_{m=1}^{M} \frac{P_m^{[j]}(x)}{n^{\frac{m}{10}}} + O_M\left(n^{-d_j(M+1)}\right)\right)$$
$$= \sum_{m=0}^{M} \frac{P_m^{[4]}(x)}{n^{\frac{m}{10}}} + \sum_{m=M+1}^{3M} \frac{P_{M,m}^{[4]}(x)}{n^{\frac{m}{10}}} + O_M\left(n^{-\frac{3(M+1)}{80}}\right) = O_M(1).$$

Proof. By Lemma 6.13, the terms $P_m^{[j]}(x) n^{-\frac{m}{10}}$ (for $j \in \{1,2,3\}$) are uniformly bounded for $|x| \le n^{\frac{1}{40}}$. Hence, for $|x| \le n^{\frac{1}{40}}$,

$$P_m^{[j]}(x)\, n^{-\frac{m}{10}}\, O_M(n^a) = O_M(n^a) \quad \text{for all } a \in \mathbb{R}.$$

With
$$\prod_{j=1}^{3}\left(1+\sum_{m=1}^{M}\frac{P_m^{[j]}(x)}{n^{\frac{m}{10}}}\right)=\sum_{m=0}^{M}\frac{P_m^{[4]}(x)}{n^{\frac{m}{10}}}+\sum_{m=M+1}^{3M}\frac{P_{M,m}^{[4]}(x)}{n^{\frac{m}{10}}}$$
we conclude, after multiplying out the terms in Lemma 6.14, the first equality with

$$\prod_{j=1}^{3}\left(1+\sum_{m=1}^{M}\frac{P_m^{[j]}(x)}{n^{\frac{m}{10}}}+O_M\left(n^{-d_j(M+1)}\right)\right)$$
$$=\sum_{m=0}^{M}\frac{P_m^{[4]}(x)}{n^{\frac{m}{10}}}+\sum_{m=M+1}^{3M}\frac{P_{M,m}^{[4]}(x)}{n^{\frac{m}{10}}}+O_M\left(n^{-\frac{3(M+1)}{80}}\right). \tag{6.25}$$

The second equality in the lemma follows by Lemma 6.13 and (6.25) (again using that $d_1=\frac{1}{15}$, $d_2=\frac{3}{80}$, and $d_3=\frac{7}{80}$). This completes the proof of the lemma. ∎

We finish this section with the following lemma, which follows from Theorem 4.1.

Lemma 6.15. *Let $\kappa>0$ be sufficiently small. Then we have, for all $z\in B_\kappa(1)$,*
$$\exp\left(\mathcal{E}_n(z)\right)=1+O\left(\left(\frac{B}{n^{\frac{1}{5}}}\right)^{n^{\frac{2}{55}}}\right),$$
where $B>0$ is some constant.

7. Wright's circle method and the proof of Theorem 1.1

7.1. Overview of the strategy

We write for $n\in\mathbb{N}_0$, using Cauchy's theorem,
$$r(n)=\frac{1}{2\pi i}\int_{C_n}\frac{G(q)}{q^{n+1}}\,dq,$$
where C_n is the circle with radius $\exp\left(-2X^2n^{-\frac{3}{5}}\right)$ centered at 0 surrounding the origin counterclockwise exactly once, and where X was defined in (1.8). To estimate $r(n)$, we use Wright's circle method [14] and split C_n into two arcs: the major and the minor arc. The major arc C_n^{maj} is placed in a neighborhood of $q=1$, which is the dominant pole of $G(q)$. The idea is that the integral is then essentially determined by the contribution of the major arc. As we show in Propositions 7.3 and 7.4 below, the major arc produces the asymptotic main term, and the minor arc term is negligible. In view of Proposition 3.2(1) and Theorem 3.3, it is natural to choose the major arc in a way such that z (given through $q=e^{-z}$) lies in a cone $\mathcal{C}_\delta$ for some δ independent from n. This gives us sufficiently good control over the error term $E(\eta;z)$. We use the line segment $z=2X^2n^{-\frac{3}{5}}(1+iy)$, $-1\le y\le 1$. Define the minor arc as
$$C_n^{\min}:=\left\{q=e^{-2X^2n^{-\frac{3}{5}}+iv}:2X^2n^{-\frac{3}{5}}<|v|\le\pi\right\}.$$

The major arc is the complement of the minor arc in the above line segment. We obtain

$$r(n) = \frac{1}{2\pi i}\int_{C_n^{\mathrm{maj}}} \frac{G(q)}{q^{n+1}}\,dq + \frac{1}{2\pi i}\int_{C_n^{\mathrm{min}}} \frac{G(q)}{q^{n+1}}\,dq. \tag{7.1}$$

The key idea is to approximate $\mathrm{Log}(G(e^{-z}))$ by Theorem 3.3. However, since the asymptotic expansion of $\mathrm{Log}(G(e^{-z}))$ does not converge, we have to cut it off at a specific bound which we choose to be k_n. We abbreviate

$$\psi_n(w) := H_n\left(iS_n - iw; n^{-\frac{1}{10}}\right)\,\exp\big(\mathcal{E}_n(w)\big), \tag{7.2}$$

where S_n is the sequence of saddle points defined in (5.1), and the functions H_n and $\mathcal{E}_n$ were defined in (6.7) and (4.1), respectively. We obtain, by using Theorem 3.3 (choosing $k_n - \frac{1}{2} < \eta_n < k_n + \frac{1}{2}$) and letting $a_n := 2X^2 n^{-\frac{3}{5}}$,

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_n^{\mathrm{maj}}} \frac{G(q)}{q^{n+1}}\,dq &= \frac{a_n}{2\pi i}\int_{1-i}^{1+i} G(e^{-a_n w})\,e^{n a_n w}\,dw \\ &= \frac{2X^{\frac{4}{3}}}{i n^{\frac{2}{5}}}\int_{1-i}^{1+i} \frac{\exp\left(n^{\frac{2}{5}} f_n(iS_n - iw)\right)\psi_n(w)}{w^{\frac{1}{3}}}\,dw,\end{aligned} \tag{7.3}$$

where the function f_n was defined in (6.1). Note that $n^{\frac{2}{5}}$, na_n and $a_n^{-\frac{2}{3}}$ have the same power in n. In Subsection 7.3, we will find an asymptotic expansion for the integral in (7.3). In Subsection 7.4, we will show that the minor arc integral in (7.1) is negligible. Together, this will prove Theorem 1.1.

7.2. Modifying the path of integration and estimating the tails

The goal of this subsection is to prove the following result.

Proposition 7.1. *We have, as $n \to \infty$,*

$$\begin{aligned}&\int_{1-i}^{1+i} \exp\left(n^{\frac{2}{5}} f_n(iS_n - iw)\right)\psi_n(w)\,w^{\frac{1}{3}}\,dw \\ &\quad = \int_{S_n - in^{-\frac{7}{40}}}^{S_n + in^{-\frac{7}{40}}} \frac{\exp\left(n^{\frac{2}{5}} f_n(iS_n - iw)\right)\psi_n(w)}{w^{\frac{1}{3}}}\,dw + O\left(A(n)\exp\left(-\frac{5X^2}{3}n^{\frac{1}{20}}\right)\right),\end{aligned}$$

were ψ_n was defined in (7.2) and $A(n)$ was defined in (6.11).

We need the following lemma.

Lemma 7.2. *The functions $w \mapsto w^{-\frac{1}{3}}$ and $w \mapsto \psi_n(w)$ are both uniformly bounded on*

$$[(1-i)S_n, 1-i] \quad \text{and} \quad \left\{w \in \mathbb{C} : -S_n \le \mathrm{Im}(w) \le -n^{-\frac{7}{40}}, \mathrm{Re}(w) = S_n\right\}$$

for n sufficiently large.

Proof. First, one can show that, for n sufficiently large, the sets

$$\mathcal{S}_1 := \left[(1-i)S_n, 1-i\right] \quad \text{and} \quad \mathcal{S}_2 := \left\{w \in \mathbb{C} : -S_n \leq \mathrm{Im}(w) \leq -n^{-\frac{7}{40}}, \mathrm{Re}(w) = S_n\right\}$$

are contained in

$$K := \mathcal{C}_{\frac{\pi}{4}} \cap \left\{w \in \mathbb{C} : \frac{1}{2} \leq \mathrm{Re}(w) \leq 1\right\},$$

which is a compact subset of $\mathcal{C}_{\frac{\pi}{4}}$. Thus, it suffices to prove uniform boundedness of the functions claimed in the lemma in K. The function $w \mapsto w^{-\frac{1}{3}}$ is continuous on K, and takes its maximum in this region and is thus in particular bounded.

Since $\frac{1}{2} \leq S_n \leq 1$ for all n sufficiently large, we obtain that $\mathrm{Im}(iS_n - iw) \leq 1 - \frac{1}{2} = \frac{1}{2}$ for all $w \in K$. As a result, the compact set $iS_n - iK \subset \mathbb{C}_{\frac{1}{2}}$ for all n sufficiently large. By Lemma 6.7, the function $w \mapsto H_n(w; n^{-\frac{1}{10}})$ converges compactly to 1 on $\mathbb{C}_{\frac{1}{2}}$. This implies that, for every compact subset $\Upsilon \subset \mathbb{C}_{\frac{1}{2}}$, there exists a constant $C_\Upsilon > 0$ (only depending on Υ) such that $|H_n(z; n^{-\frac{1}{10}})| \leq C_\Upsilon$ for all $z \in \Upsilon$ and all n sufficiently large. We conclude that $w \mapsto H_n(iS_n - iw; n^{-\frac{1}{10}})$ is uniformly bounded on K (choose $\Upsilon = iS_n - iK$).

With Theorem 4.1, it follows that $w \mapsto \exp(\mathcal{E}_n(w))$ is uniformly bounded on K (note that the exponential of a bounded function is again bounded). Indeed, if $z \in K$, then we obtain

$$\left(\frac{B}{n^{\frac{1}{5}}}\right)^{n^{\frac{2}{55}}} |z|^{k_n + \frac{1}{4}} \ll \left(\sup_{z \in K} |z|\right)^{\frac{1}{4}} \left(\frac{\max\left\{1, \sup_{z\in K} |z|\right\} B}{n^{\frac{1}{5}}}\right)^{n^{\frac{2}{55}}} \ll \left(\frac{\sqrt{2}B}{n^{\frac{1}{5}}}\right)^{n^{\frac{2}{55}}} \ll 1.$$

The bound holds uniformly in n and $z \in K$.

As a result, if $x + iy \in \mathcal{C}_{\frac{\pi}{4}} \cap \{w \in \mathbb{C} : \frac{1}{2} \leq \mathrm{Re}(w) \leq 1\}$, we obtain

$$|x + iy| = \sqrt{x^2 + y^2} \leq \sqrt{2}x$$

(with equality if $x = |y|$). Since $\sqrt{2}x$ takes its maximum $\sqrt{2}$ for $x = 1$ in the range $\frac{1}{2} \leq x \leq 1$, we obtain $\sup_{z \in K} |z| = \sqrt{2}$. ■

We are ready to prove Proposition 7.1.

Proof of Proposition 7.1. Using the symmetry of the integrand under conjugation, we write

$$\int_{1-i}^{1+i} \frac{\exp\left(n^{\frac{2}{5}} f_n(iS_n - iw)\right)\psi_n(w)}{w^{\frac{1}{3}}} dw$$

$$= \int_{S_n - in^{-\frac{7}{40}}}^{S_n + in^{-\frac{7}{40}}} \frac{\exp\left(n^{\frac{2}{5}} f_n(iS_n - iw)\right)\psi_n(w)}{w^{\frac{1}{3}}} dw$$

$$+ O\left(\left(\int_{(1-i)S_n}^{S_n - in^{-\frac{7}{40}}} + \int_{1-i}^{(1-i)S_n}\right) \left|\frac{\exp\left(n^{\frac{2}{5}} f_n(iS_n - iw)\right)\psi_n(w)}{w^{\frac{1}{3}}} dw\right|\right), \tag{7.4}$$

where all curves are chosen as straight lines. We first observe, using (6.2), that for n sufficiently large,

$$\sup_{w\in[(1-i)S_n,1-i]}\left|\exp\left(n^{\frac{2}{5}} f_n(iS_n-iw)\right)\right| \leq \exp\left(4.8\,X^2 n^{\frac{2}{5}}\right).$$

By Lemma 7.2, we conclude

$$\int_{1-i}^{(1-i)S_n}\left|\frac{\exp\left(n^{\frac{2}{5}} f_n(iS_n-iw)\right)\psi_n(w)}{w^{\frac{1}{3}}}\,dw\right| \ll \exp\left(4.8\,X^2 n^{\frac{2}{5}}\right) = \exp\left(\left(A_1-0.2\,X^2\right)n^{\frac{2}{5}}\right). \tag{7.5}$$

In the same way, we obtain

$$\int_{S_n-iS_n}^{S_n-in^{-\frac{7}{40}}}\left|\frac{\exp\left(n^{\frac{2}{5}} f_n(iS_n-iw)\right)\psi_n(w)}{w^{\frac{1}{3}}}\,dw\right| \ll \sup_{\substack{-S_n\leq \mathrm{Im}(w)\leq -n^{-\frac{7}{40}}\\ \mathrm{Re}(w)=S_n}}\left|\exp\left(n^{\frac{2}{5}} f_n(iS_n-iw)\right)\right|.$$

Using Lemma 6.3, (6.12), and (5.1), we get

$$\sup_{\substack{-S_n\leq \mathrm{Im}(w)\leq -n^{-\frac{7}{40}}\\ \mathrm{Re}(w)=S_n}}\left|\exp\left(n^{\frac{2}{5}} f_n(iS_n-iw)\right)\right| \ll A(n)\exp\left(-\frac{5X^2}{3}\,n^{\frac{1}{20}}\right). \tag{7.6}$$

With (7.4), (7.5), and (7.6), this combines to

$$\int_{1-i}^{1+i}\frac{\exp\left(n^{\frac{2}{5}} f_n(iS_n-iw)\right)\psi_n(w)}{w^{\frac{1}{3}}}\,dw = \int_{S_n-in^{-\frac{7}{40}}}^{S_n+in^{-\frac{7}{40}}}\frac{\exp\left(n^{\frac{2}{5}} f_n(iS_n-iw)\right)\psi_n(w)}{w^{\frac{1}{3}}}\,dw + O\left(\exp\left(\left(A_1-0.2\,X^2\right)n^{\frac{2}{5}}\right)\right) + O\left(A(n)\exp\left(-\frac{5X^2}{3}\,n^{\frac{1}{20}}\right)\right).$$

The proposition now follows by comparing the error terms. ■

7.3. Expanding the main integral

In this section, we expand the major arc integral (7.3) to obtain a refined expression for $r(n)$. An important tool is the saddle point method. We set

$$C_m := 2X^{\frac{4}{3}}\exp(-A_5)\int_{-\infty}^{\infty} P_m^{[4]}(x)\exp\left(-\frac{5X^2x^2}{3}\right)dx, \tag{7.7}$$

where the polynomials $P_m^{[4]}$ were defined in Subsection 6.4. We prove the following result.

Proposition 7.3. *Let $M \in \mathbb{N}$. We obtain, as $n \to \infty$,*

$$r(n) = \frac{A(n)}{n^{\frac{3}{5}}} \left(\sum_{0 \le m \le \frac{3M}{8}} \frac{C_m}{n^{\frac{m}{10}}} + O_M\left(n^{-\frac{3(M+1)}{80}}\right)\right) + O\left(\int_{C_n^{\min}} \left| \frac{G(q)}{q^{n+1}} dq \right| \right) + O\left(A(n) \exp\left(-\frac{5X^2}{3} n^{\frac{1}{20}} \right)\right).$$

Proof. By (7.1), (7.3), and Proposition 7.1, we have, substituting $x = n^{\frac{1}{5}}(iS_n - iw)$,

$$r(n) = \frac{2X^{\frac{4}{3}}}{n^{\frac{3}{5}}} \int_{-n^{\frac{1}{40}}}^{n^{\frac{1}{40}}} \frac{\exp\left(n^{\frac{2}{5}} f_n\left(\frac{x}{n^{\frac{1}{5}}}\right)\right) H_n\left(\frac{x}{n^{\frac{1}{5}}}; n^{-\frac{1}{10}}\right) \exp\left(\mathcal{E}_n\left(S_n + \frac{ix}{n^{\frac{1}{5}}}\right)\right)}{\left(S_n + \frac{ix}{n^{\frac{1}{5}}}\right)^{\frac{1}{3}}} dx$$

$$+ O\left(\int_{C_n^{\min}} \left| \frac{G(q)}{q^{n+1}} dq \right| \right) + O\left(A(n) \exp\left(-\frac{5X^2}{3} n^{\frac{1}{20}} \right)\right). \tag{7.8}$$

Together with Lemmas 6.11, 6.12, 6.15, 6.14, and Proposition 6.6 (note that $S_n + ixn^{-\frac{1}{5}} \in B_\kappa(1)$ for all $|x| \le n^{\frac{1}{40}}$ for all n sufficiently large, where κ is chosen sufficiently small to satisfy Lemma 6.15), we find that the integral in (7.8), including the factor $2X^{\frac{4}{3}} n^{-\frac{3}{5}}$, equals

$$\frac{2X^{\frac{4}{3}} e^{-A_5} A(n)}{n^{\frac{3}{5}}} \int_{-n^{\frac{1}{40}}}^{n^{\frac{1}{40}}} \left(\sum_{m=0}^{M} \frac{P_m^{[4]}(x)}{n^{\frac{m}{10}}} + \sum_{m=M+1}^{3M} \frac{P_{M,m}^{[4]}(x)}{n^{\frac{m}{10}}} + O_M\left(n^{-\frac{3(M+1)}{80}}\right)\right)$$

$$\times \left(1 + O\left(\left(\frac{B}{n^{\frac{1}{5}}} \right)^{n^{\frac{2}{55}}} \right) \right) \exp\left(-\frac{5X^2 x^2}{3} \right) dx. \tag{7.9}$$

By Lemma 2.5, we obtain

$$\frac{2X^{\frac{4}{3}} e^{-A_5} A(n)}{n^{\frac{3}{5}}}$$

$$\times \int_{-n^{\frac{1}{40}}}^{n^{\frac{1}{40}}} \left(\sum_{m=0}^{M} \frac{P_m^{[4]}(x)}{n^{\frac{m}{10}}} + \sum_{m=M+1}^{3M} \frac{P_{M,m}^{[4]}(x)}{n^{\frac{m}{10}}} + O_M\left(n^{-\frac{3(M+1)}{80}}\right)\right) \exp\left(-\frac{5X^2 x^2}{3} \right) dx$$

$$= \frac{A(n)}{n^{\frac{3}{5}}} \left(\sum_{0 \le m \le \frac{3M}{8}} \frac{C_m}{n^{\frac{m}{10}}} + O_M\left(n^{-\frac{3(M+1)}{80}}\right)\right), \tag{7.10}$$

where the C_m's were defined in (7.7). The proof can be then concluded using (7.8), (7.9), (7.10), (7.7), and Lemma 6.14. ∎

7.4. The minor arc estimate

Lemma 6.4, Proposition 3.2(1), and Theorem 3.3 yield the following estimate on the minor arc.

Proposition 7.4 (Minor arc estimate). *There exists $\delta > 0$, independent from n, such that*

$$\int_{C_n^{\min}} \left| \frac{G(q)}{q^{n+1}} \, dq \right| = O\left(n^{\frac{1}{5}} \exp\left(A_1 n^{\frac{2}{5}} - \left(A_2 + \frac{\delta}{\sqrt{2}X} \right) n^{\frac{3}{10}} \right) \right), \quad \text{as } n \to \infty.$$

Proof. Due to the symmetry $G(\overline{q}) = \overline{G(q)}$, we have, with $q = e^{-z}$,

$$\int_{C_n^{\min}} \left| \frac{G(q)}{q^{n+1}} \, dq \right| = 2e^{2X^2 n^{\frac{2}{5}}} \int_{2X^2 n^{-\frac{3}{5}}}^{\pi} \left| G\left(e^{-2X^2 n^{-\frac{3}{5}} - iu} \right) \right| du. \tag{7.11}$$

By Lemma 6.4, choosing $\kappa = 1$, there exist constants $\beta, \delta > 0$ such that for $0 < t := 2X^2 n^{-\frac{3}{5}} < \beta$,

$$\left| G\left(e^{-\frac{2X^2 - iu}{n^{\frac{3}{5}}}} \right) \right| \leq \exp\left(-\frac{\delta}{\sqrt{2}X} n^{\frac{3}{10}} \right) G\left(e^{-\frac{2X^2}{n^{\frac{3}{5}}}} \right) \tag{7.12}$$

is valid for all $2X^2 n^{-\frac{3}{5}} \leq u \leq \pi$ and all n sufficiently large. We obtain, with $\eta = 1$ in Theorem 3.3 and Proposition 3.2(1),

$$G\left(e^{-\frac{2X^2}{n^{\frac{3}{5}}}} \right) \ll n^{\frac{1}{5}} \exp\left(3X^2 n^{\frac{2}{5}} - \frac{Y n^{\frac{3}{10}}}{X} \right) (1 + o(1)).$$

Hence, with (7.12),

$$\left| G\left(e^{-\frac{2X^2 + iu}{n^{\frac{3}{5}}}} \right) \right| = O\left(n^{\frac{1}{5}} \exp\left(3X^2 n^{\frac{2}{5}} - \left(\frac{Y}{X} + \frac{\delta}{\sqrt{2}X} \right) n^{\frac{3}{10}} \right) \right). \tag{7.13}$$

By plugging (7.13) into (7.11), we finally conclude the proof of the claim. ∎

7.5. Proof of Theorem 1.1

Theorem 1.1 now follows by a direct calculation using Propositions 7.3 and 7.4.

7.6. The constants C_j

We finally calculate the first constants C_0, C_1, and C_2 from Theorem 1.1. Since we have $P_0^{[4]} = 1$ by (6.14), we have, by (7.7),

$$C_0 = 2X^{\frac{4}{3}} \exp(-A_5) \int_{-\infty}^{\infty} \exp\left(-\frac{5X^2 u^2}{3} \right) du = \frac{2\sqrt{3\pi}\, X^{\frac{1}{3}}}{\sqrt{5}} \exp(-A_5),$$

where we use the well-known formula

$$\int_{-\infty}^{\infty} \exp(-bu^2) \, du = \sqrt{\frac{\pi}{b}}, \quad b > 0.$$

Note that this constant was already computed by Romik [11].

In the following, we provide the calculations to identify the constants C_1 and C_2. The proof of Theorem 1.1 tells us that we require the first terms of the expansions in Lemmas 6.11, 6.12, and Proposition 6.6 explicitly. For this, we make use of the Taylor approximation of the saddle point function, see also the terms in (5.2), whereas three non-trivial terms are sufficient. By series expansion, which can for instance be done with the help of suitable software, we get $P_1^{[1]}(x) = P_2^{[1]}(x) = 0$, using (6.7) and Proposition 6.6. Similarly, using (6.13) and (6.14) with the variable $z := n^{-\frac{1}{10}}$, we find

$$P_1^{[2]}(x) = -\frac{Y\left(35x^2X^2-6\right)}{120X^3} - \frac{4959Y^5}{102400000X^{13}},$$
$$P_2^{[2]}(x) = \frac{49x^4Y^2}{1152X^2} + \frac{40}{27}ix^3X^2 + x^2\left(\frac{11571Y^6}{819200000X^{14}} - \frac{19Y^2}{320X^4}\right) + \frac{57Y^6\left(431433Y^4-2293760000X^{10}\right)}{20971520000000000X^{26}}.$$

By Lemma 6.12 and a binomial theorem type expansion, again involving the first terms of the saddle point function expansion, we get

$$P_1^{[3]}(x) = \frac{Y}{20X^3} \quad \text{and} \quad P_2^{[3]}(x) = \frac{Y^2}{160X^6} - \frac{ix}{3}.$$

Now, using (6.24) and all the above polynomials, we obtain after multiplying out the parentheses,

$$P_1^{[4]}(x) = -\frac{Y\left(35x^2X^2-6\right)}{120X^3} - \frac{4959Y^5}{102400000X^{13}},$$
$$P_2^{[4]}(x) = \frac{1}{27}\left(40x^2X^2-9\right)ix + \frac{57Y^6\left(1015x^2X^2-622\right)}{4096000000X^{16}} + \frac{Y^2\left(245x^4X^4-426x^2X^2+36\right)}{5760X^6} + \frac{24591681Y^{10}}{20971520000000000X^{26}}.$$

We use these polynomials to calculate, with (7.7),

$$\int_{-\infty}^{\infty} P_1^{[4]}(x)\exp\left(-\frac{5X^2}{3}x^2\right)dx = -\sqrt{\frac{3\pi}{5}}\left(\frac{4959Y^5}{102400000X^{14}} - \frac{3Y}{80X^4}\right),$$
$$\int_{-\infty}^{\infty} P_2^{[4]}(x)\exp\left(-\frac{5X^2}{3}x^2\right)dx = \sqrt{\frac{3\pi}{5}}\left(\frac{24591681Y^{10}}{20971520000000000X^{27}} - \frac{7239Y^6}{1638400000X^{17}} - \frac{57Y^2}{12800X^7}\right).$$

In particular, we have

$$C_1 = -2X^{\frac{4}{3}} \exp(-A_5) \sqrt{\frac{3\pi}{5}} \left(\frac{4959\, Y^5}{102400000\, X^{14}} - \frac{3Y}{80\, X^4} \right), \tag{7.14}$$

$$C_2 = 2X^{\frac{4}{3}} \exp(-A_5) \sqrt{\frac{3\pi}{5}} \times \left(\frac{24591681\, Y^{10}}{20971520000000000\, X^{27}} - \frac{7239\, Y^6}{1638400000\, X^{17}} - \frac{57\, Y^2}{12800\, X^7} \right). \tag{7.15}$$

All numerical computations in this paper were done with standard packages, such as local series expansions, in Mathematica.

8. Open questions

Although Theorem 1.1 provides an infinite number of terms in the asymptotic expression for $r(n)$, there are still interesting open questions. It is natural to ask about the nature of the constants C_j, for example, an explicit formula as rational functions in X, Y, and rational zeta and Gamma values (up to a constant $\exp(-A_5)$). Another question refers to the nature of the error term. We show that the error is smaller than any inverse polynomial times the main term $A(n)$, but it could be interesting to refine the suggested method. Furthermore, it could be intersting to enwide these kind of results from SU(2) (partitions function) and SU(3) to general SU(n) with $n \geq 4$. For this, a detailed study of the related polynomial zeta function is required. We are planning to work on this in the future.

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