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Transport equation in generalized Campanato spaces

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Abstract. In this paper we study the transport equation in $\mathbb{R}^n \times (0,T)$, $T>0$, $n \geq 2$,

$$\partial_t f + v \cdot \nabla f = g, \quad f(\cdot,0) = f_0 \quad \text{in } \mathbb{R}^n,$$

in generalized Campanato spaces $\mathcal{L}^s_{q(p,N)}(\mathbb{R}^n)$. The critical case is particularly interesting, and is applied to the local well-posedness problem for the incompressible Euler equations in a space close to the Lipschitz space in our companion paper [Ann. Inst. H. Poincaré Anal. Non Linéaire 38 (2021), no. 2, 201–241]. In the critical case $s=q=N=1$, we have the embeddings $B^1_{\infty,1}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n) \hookrightarrow C^{0,1}(\mathbb{R}^n)$, where $B^1_{\infty,1}(\mathbb{R}^n)$ and $C^{0,1}(\mathbb{R}^n)$ are the Besov and Lipschitz spaces, respectively. For $f_0 \in \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$, $v \in L^1(0,T;\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)))$ and $g \in L^1(0,T;\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)))$, we prove the existence and uniqueness of solutions to the transport equation in $L^\infty(0,T;\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n))$ such that

$$\|f\|_{L^\infty(0,T;\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)))} \leq C\big(\|v\|_{L^1(0,T;\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)))}, \|g\|_{L^1(0,T;\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)))}\big).$$

Similar results for the other cases are also proved.

1. Introduction

Let $0 < T < +\infty$ and $Q = \mathbb{R}^n \times \mathbb{R}_+$n with $n \in \mathbb{N}$, $n \geq 2$. We consider the transport equation

$$\begin{cases} \partial_t f + (v\cdot\nabla) f = g & \text{in } Q, \\ v = v_0 & \text{on } \mathbb{R}^n \times \{0\}, \end{cases} \tag{1.1}$$

where $f = f(x_1,\dots,x_n,t)$ is unknown, while $v = (v_1,\dots,v_n) = v(x,t)$ represents a given drift velocity, and $g = g(x_1,\dots,x_n,t)$ is a given function.

Our aim in this paper is to obtain estimates of solutions to (1.1) in generalized Campanato spaces. Our proof relies on a key estimate in terms of local oscillation of solutions. As a byproduct we get existence of solutions in Besov spaces and Tribel–Lizorkin spaces, which can be estimated by the data belonging to these spaces.

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One of the main motivations to study the transport equation in such generalized Campanato spaces is to apply it to prove local well-posedness of the incompressible Euler equations in function spaces embedded in the Lipschitz space, which includes linearly growing functions at spatial infinity. We refer to [1] for the study of the transport equations for the non-Lipschitz vector fields.

For recent developments of the local well-posedness/ill-posedness of the Euler equations in various critical function spaces embedded in $C^{0,1}(\mathbb{R}^n)$, we refer to [2, 4, 5, 8, 12–14, 17, 18]). We also refer to [9] for the study of transport equation with drift velocity in a less regular space. For the application of our new function spaces in the critical case to the Euler equations, see our companion paper [7]. Obviously, our results could be also useful for many other equations which can be written in the form (1.1).

Let us introduce the function spaces we will use throughout the paper. Let $N \in \mathbb{N} \cup \{0, -1\}$. By $\mathcal{P}_N$ (respectively $\dot{\mathcal{P}}_N$) we denote the space of all polynomial (respectively all homogenous polynomials) of degree less than or equal to N. We equip the space $\mathcal{P}_N$ with the norm $\|P\|_{(p)} = \|P\|_{L^p(B(1))}$. Note that, since $\dim(\mathcal{P}_N) < +\infty$, all norms $\|\cdot\|_{(p)}$, $1 \le p \le \infty$, are equivalent. For notational convenience, for $N = -1$, we use the convention $\mathcal{P}_{-1} = \{0\}$, which consists of the trivial polynomial $P \equiv 0$.

Let $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $1 \le p \le +\infty$. For $x_0 \in \mathbb{R}^n$ and $0 < r < \infty$, we define the oscillation

$$\operatorname*{osc}_{p,N}(f; x_0, r) := |B(r)|^{-1/p} \inf_{P \in \mathcal{P}_N} \|f - P\|_{L^p(B(x_0,r))}. \tag{1.2}$$

From our convention above, we note that in the case $N = -1$, we have

$$\operatorname*{osc}_{p,-1}(f; x_0, r) := |B(r)|^{-1/p} \|f\|_{L^p(B(x_0,r))}.$$

Then we define, for $1 \le q, p \le +\infty$ and $s \in (-\infty, N+1]$, the spaces

$$\mathcal{L}^s_{q(p,N)}(\mathbb{R}^n) = \Big\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) \ \Big| \ |f|_{\mathcal{L}^s_{q(p,N)}} := \Big\| \Big(\sum_{j\in\mathbb{Z}} \big(2^{-sj} \operatorname*{osc}_{p,N}(f; \cdot, 2^j)\big)^q \Big)^{1/q} \Big\|_{L^\infty} < +\infty \Big\},$$

when $0 < q < +\infty$, and

$$\mathcal{L}^s_{\infty(p,N)}(\mathbb{R}^n) = \Big\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) \ \Big| \ |f|_{\mathcal{L}^s_{\infty(p,N)}} := \Big\| \sup_{j\in\mathbb{Z}} \big(2^{-sj} \operatorname*{osc}_{p,N}(f; \cdot, 2^j)\big) \Big\|_{L^\infty} < +\infty \Big\}.$$

Furthermore, by $\mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n)$, $k \in \mathbb{N}$, we denote the space of all $f \in W^{k,p}_{\mathrm{loc}}(\mathbb{R}^n)$ such that $D^k f \in \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n)$. The space $\mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n)$ will be equipped with the norm

$$\|f\|_{\mathcal{L}^{k,s}_{q(p,N)}} = |D^k f|_{\mathcal{L}^s_{q(p,N)}} + \|f\|_{L^p(B(1))}, \quad f \in \mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n).$$

According to the characterization theorem of the Triebel–Lizorkin spaces in terms of oscillation, we have

$$\begin{cases} f \in F^s_{r,q}(\mathbb{R}^n) \iff \|f\|_{L^{\min\{r,q\}}} + \Big\| \Big(\displaystyle\sum_{j=-\infty}^{0} \big(2^{-sj} \operatorname*{osc}_{p,N}(f; \cdot, 2^j)\big)^q \Big)^{1/q} \Big\|_{L^r} < +\infty, \\ 0 < r < +\infty, \quad 0 < q \le \infty, \quad s > \Big(\dfrac{1}{r} - \dfrac{1}{p}\Big)_+, \quad s > \Big(\dfrac{1}{q} - \dfrac{1}{p}\Big)_+ \end{cases}$$

(cf. [16, Theorem, Chap. 1.7.3]), and we could regard the spaces $\mathcal{L}^s_{q(p,N)}(\mathbb{R}^n)$ as an extension of the limit case of $F^s_{r,q}(\mathbb{R}^n)$ as $r \to +\infty$.

In fact, for $q = +\infty$ and $s > 0$, we get the usual Campanato spaces with the isomorphism relation (cf. [6, 11])

$$\mathcal{L}^{n+ps,p}_N(\mathbb{R}^n) \cong \mathcal{L}^s_{\infty(p,N)}(\mathbb{R}^n).$$

Furthermore, in the case $N = 0$, $s = 0$ and $q = \infty$, we get the space of bounded mean oscillation, i.e.,

$$\mathcal{L}^0_{\infty(p,0)}(\mathbb{R}^n) \cong \mathrm{BMO}.$$

For $N = -1$ and $s \in (-n/p, 0)$, the above space coincides with the usual Morrey space $\mathcal{M}^{n+ps}(\mathbb{R}^n)$.

We note that the oscillation introduced in (1.2) is attained by a unique polynomial $P_* \in \mathcal{P}_N$.

According to Theorem 3.6 (see Section 3 below), for the spaces $\mathcal{L}^1_{1(p.1)}(\mathbb{R}^n)$, we have the following embedding properties:

$$B^{1+n/r}_{r,1} \hookrightarrow \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{0,1}_{1(p,0)}(\mathbb{R}^n) \hookrightarrow C^{0,1}(\mathbb{R}^n) \quad \text{for all } r > 0.$$

Accordingly,

$$\|\nabla u\|_\infty \le c\|u\|_{\mathcal{L}^1_{1(p,1)}}.$$

Furthermore, for every $f \in \mathcal{L}^k_{1(p,k)}(\mathbb{R}^n)$, $k \in \{0, 1\}$, there exists a unique $\dot{P}^k_\infty(f) \in \dot{\mathcal{P}}_1$ such that for all $x_0 \in \mathbb{R}^n$,

$$f \text{ converge asymptotically to } \dot{P}^k_\infty(f) \text{ as } |x| \to +\infty.$$

The precise meaning of this asymptotic limit will be given in Section 3 below.

We are now in a position to present our first main result.

Theorem 1.1 (The case $N = 0$). *Let $0 < T < +\infty$. Let $s \in (-n/q, 0)$, $1 < p < +\infty$, $1 \le q < +\infty$. Let $v \in L^1(0, T; L^p_{\mathrm{loc}}(\mathbb{R}^n))$, with*

$$\int_0^T \|\nabla v(\tau)\|_\infty \, d\tau < +\infty. \tag{1.3}$$

Then, for every $f_0 \in \mathcal{L}^s_{q(p,0)}(\mathbb{R}^n)$ and $g \in L^1(0, T; \mathcal{L}^s_{q(p,0)}(\mathbb{R}^n))$, there exists a unique solution $f \in L^\infty(0, T; \mathcal{L}^s_{q(p,0)}(\mathbb{R}^n))$ to the transport equation (1.1). Furthermore, for almost all $t \in (0, T)$,

$$|f(t)|_{\mathcal{L}^s_{q(p,0)}} \le c\Big\{|f_0|_{\mathcal{L}^s_{q(p,0)}} + \int_0^T |g(\tau)|_{\mathcal{L}^s_{q(p,0)}} \, d\tau\Big\} \exp\Big(c\int_0^T \|\nabla v(\tau)\|_\infty \, d\tau\Big). \tag{1.4}$$

The second main result deals with the case $N = 1$.

Theorem 1.2 (The case $N = 1$ and $s = 1$). *Let $0 < T < +\infty$ and $1 < p < +\infty$, $1 \le q \le +\infty$. Let $v \in L^1(0, T; \mathcal{L}^1_{q(p,1)}(\mathbb{R}^n))$ satisfy (1.3) and*

$$\int_0^T \sup_{x_0 \in \mathbb{R}^n} \Big(\sum_{j=-\infty}^{0} (-j)^{q-1} 2^{-jq} \operatorname*{osc}_{p,1}(v(\tau); x_0, 2^j)\Big)^{1/q} d\tau < +\infty. \tag{1.5}$$

Let $f_0 \in \mathcal{L}^1_{q(p,1)}(\mathbb{R}^n)$ *and* $g \in L^1(0,T;\mathcal{L}^1_{q(p,1)}(\mathbb{R}^n))$ *satisfy the condition*

$$\sup_{x_0\in\mathbb{R}^n} \operatorname*{osc}_{p,0}(f_0;x_0,1) + \int_0^T \sup_{x_0\in\mathbb{R}^n} \operatorname*{osc}_{p,0}(g(\tau);x_0,1)\,d\tau < +\infty. \tag{1.6}$$

Then there exists a unique solution $f \in L^\infty(0,T;\mathcal{L}^1_{q(p,1)}(\mathbb{R}^n))$ *to the transport equation* (1.1). *Furthermore, for all* $t \in (0,T)$,

$$|f(t)|_{\tilde{\mathcal{L}}^1_{q(p,1)}} \le c\Big\{|f_0|_{\tilde{\mathcal{L}}^1_{q(p,1)}} + \int_0^T |g(\tau)|_{\tilde{\mathcal{L}}^1_{q(p,1)}}\,d\tau\Big\}\exp\Big(c\int_0^T C(\tau)\,d\tau\Big), \tag{1.7}$$

where we set

$$C(\tau) = \|\nabla v(\tau)\|_\infty + \sup_{x_0\in\mathbb{R}^n}\Big(\sum_{j\in\mathbb{Z}}(j^-)^{q-1}2^{-jq}\operatorname*{osc}_{p,1}(v(\tau);x_0,2^j)\Big)^{1/q},$$

$j^- = -\min\{j,0\}$, *and* $|z|_{\tilde{\mathcal{L}}^1_{q(p,0)}}$ *stands for the semi norm*

$$|z|_{\tilde{\mathcal{L}}^1_{q(p,1)}} = |z|_{\mathcal{L}^1_{q(p,1)}} + \sup_{x_0\in\mathbb{R}^n}|\nabla \dot{P}^1_{x_0,1}(z)|.$$

Our third main result concerns the case $s > 1$.

Theorem 1.3 (The case $N \ge 1$ and $s > 1$). *Let* $0 < T < +\infty$, $1 < s < +\infty$, $1 < p < +\infty$ *and* $1 \le q \le +\infty$. *Set* $N = [s]$. *Let* $v \in L^1(0,T;\mathcal{L}^s_{q(p,N)}(\mathbb{R}^n))$ *satisfy* (1.3), *and let* $f_0 \in \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n)$ *and* $g \in L^1(0,T;\mathcal{L}^s_{q(p,N)}(\mathbb{R}^n))$ *satisfy the condition*

$$\|\nabla f_0\|_\infty + \int_0^T \|\nabla g(\tau)\|_\infty\,d\tau < +\infty.$$

Then there exists a unique solution $f \in L^\infty(0,T;\mathcal{L}^s_{q(p,N)}(\mathbb{R}^n))$ *to the transport equation* (1.1), *together with the estimate*

$$|f(t)|_{\mathcal{L}^s_{q(p,N)}} \le c\Big\{|f_0|_{\tilde{\mathcal{L}}^s_{q(p,N)}} + \int_0^T |g(\tau)|_{\tilde{\mathcal{L}}^s_{q(p,N)}}\Big\}\exp\Big(c\int_0^T \|v(\tau)\|_{\tilde{\mathcal{L}}^s_{q(p,0)}}\,d\tau\Big), \tag{1.8}$$

where $|z|_{\tilde{\mathcal{L}}^s_{q(p,0)}}$ *stands for the semi norm defined by*

$$|z|_{\tilde{\mathcal{L}}^s_{q(p,N)}} = |z|_{\mathcal{L}^s_{q(p,N)}} + \|\nabla z\|_\infty.$$

From Theorem 1.2 we get the following corollary for the special case $s = q = N = 1$, which will be useful for our future application to the Euler equations in the critical spaces.

Corollary 1.4. *Let* $0 < T < +\infty$ *and* $1 < p < +\infty$. *Let* $v \in L^1(0,T;\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n))$, $f_0 \in \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$ *and* $g \in L^1(0,T;\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n))$. *Then there exists a unique solution* $f \in L^\infty(0,T;\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n))$ *to the transport equation* (1.1). *Furthermore, for all* $t \in (0,T)$,

$$\|f(t)\|_{\mathcal{L}^1_{1(p,1)}} \le C\Big\{1 + \int_0^T |v(\tau)|_{\mathcal{L}^1_{1(p,1)}}\,d\tau\Big\}\exp\Big(c\int_0^T \|\nabla v(\tau)\|_\infty\,d\tau\Big), \tag{1.9}$$

where

$$C = c\Big(\|f_0\|_{\mathcal{L}^1_{1(p,1)}} + \int_0^T \|g(\tau)\|_{\mathcal{L}^1_{1(p,1)}}\,d\tau\Big),$$

with $c = \text{const} > 0$ *depending on* n *and* p.

Remark 1.5. First, using the well-known characterization of $B^1_{\infty,1}(\mathbb{R}^n)$ in terms of oscillation, we easily verify the embeddings

$$B^1_{\infty,1}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \hookrightarrow \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n).$$

Indeed, referring to the theorem of Chapter 1.7.3 in [16], we see that

$$v \in B^1_{\infty,1}(\mathbb{R}^n) \iff \sum_{j=-\infty}^{0} 2^{-j} \big\| \operatorname*{osc}_{p,1}(v;\cdot,2^j) \big\|_{L^\infty} + \|v\|_{L^\infty} < +\infty.$$

This shows, for $x \in \mathbb{R}^n$, that

$$\sum_{j\in\mathbb{Z}} 2^{-j} \operatorname*{osc}_{p,1}(v;x,2^j) = \sum_{j=-\infty}^{0} 2^{-j} \big\| \operatorname*{osc}_{p,1}(v;\cdot,2^j) \big\|_{L^\infty} + \sum_{j=1}^{\infty} 2^{-j} \operatorname*{osc}_{p,1}(v;x,2^j).$$

On the other hand, it is readily seen that $\operatorname{osc}_{p,1}(v;x,2^j) \le 2\|v\|_{L^\infty}$. Accordingly, the second sum on the right-hand side is bounded by $\|v\|_{L^\infty}$. This yields

$$\begin{aligned}\|v\|_{\mathcal{L}^1_{1(p,1)}} &= \sum_{j\in\mathbb{Z}} 2^{-j} \operatorname*{osc}_{p,1}(v;x,2^j) + \|v\|_{L^p(B(1))} \le \sum_{j\in\mathbb{Z}} 2^{-j} \operatorname*{osc}_{p,1}(v;x,2^j) + c\|v\|_{L^\infty} \\ &\le c \sum_{j=-\infty}^{0} 2^{-j} \big\| \operatorname*{osc}_{p,1}(v;\cdot,2^j) \big\|_{L^\infty} + c\|v\|_{L^\infty} \le c\|v\|_{B^1_{\infty,1}}.\end{aligned}$$

Secondly, according to [15], p. 85 (see also [2]), we have the embedding

$$B^1_{\infty,1}(\mathbb{R}^n) \hookrightarrow C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$

On the other hand, there exists a function $f \in \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$ which is not in $C^1(\mathbb{R}^n)$ (see Appendix B). This clearly shows that $\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$ contains less regular functions than $B^1_{\infty,1}(\mathbb{R}^n)$.

Thirdly, since $\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$ contains linearly growing functions at infinity, in particular, polynomials of degree less than or equal to one, $\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$ is strictly bigger than $B^1_{\infty,1}(\mathbb{R}^n)$ in terms of asymptotic behaviors as infinity. We also note that the use of our generalized Campanato spaces to handle the bounded domain problem is quite convenient as in the case of usual Campanato spaces.

2. Preliminary lemmas

Let $X = \{X_j\}_{j\in\mathbb{Z}}$ be a sequence of non-negative real numbers. Given $s \in \mathbb{R}$ and $0 < q < +\infty$, we denote

$$\{2^{js}\} \cdot X := \{2^{js} X_j\}_{j\in\mathbb{Z}}, \quad X^q := \{X_j^q\}_{j\in\mathbb{Z}}.$$

We define $S_{\alpha,q}\colon X = \{X_j\}_{j\in\mathbb{Z}} \mapsto Y = \{Y_j\}_{j\in\mathbb{Z}}$, where

$$Y_j = (S_{\alpha,q}(X))_j = 2^{j\alpha} \Big(\sum_{i=j}^{\infty} (2^{-i\alpha} X_i)^q \Big)^{1/q}, \quad j \in \mathbb{Z}.$$

From the above definition, for $\alpha = 0$, it follows that

$$\|S_{0,q}(X)\|_{\ell^\infty} = \|X\|_{\ell^\infty} \le \|X\|_{\ell^q} \quad \text{for all } X \in \ell^q.$$

Clearly, for all $\alpha, \beta \in \mathbb{R}$,

$$2^{\beta j}(S_{\alpha,q}(X))_j = S_{\alpha+\beta,q}(\{2^{\beta i}X_i\})_j, \quad j \in \mathbb{Z}. \tag{2.1}$$

Given $X = \{X_j\}_{j\in\mathbb{Z}}$, $Y = \{Y_j\}_{j\in\mathbb{Z}}$, we denote $X \le Y$ if $X_j \le Y_j$ for all $j \in \mathbb{Z}$. Throughout this paper, we frequently make use of the following lemma, which could be regarded as a generalization of the result in [3].

Lemma 2.1. *For all $\beta < \alpha$ and $0 < p \le q \le +\infty$,*

$$S_{\beta,q}(S_{\alpha,p}(X)) \le \frac{1}{1-2^{-(\alpha-\beta)}} S_{\beta,q}(X). \tag{2.2}$$

Proof. We first observe

$$\begin{aligned}(S_{\beta,q}(S_{\alpha,p}X))_j &= 2^{j\beta}\Big\{\sum_{i=j}^{\infty} 2^{-i\beta q}(S_{\alpha,p}X)_i^q\Big\}^{1/q} \\ &= 2^{j\beta}\Big\{\sum_{i=j}^{\infty} 2^{-i\beta q}\Big[2^{i\alpha}\Big(\sum_{l=i}^{\infty}(2^{-\alpha l}X_l)^p\Big)^{1/p}\Big]^q\Big\}^{1/q} \\ &= 2^{j\beta}\Big\{\sum_{i=j}^{\infty} 2^{i(\alpha-\beta)q}\Big(\sum_{l=i}^{\infty} 2^{-(\alpha-\beta)pl}2^{-\beta pl}X_l^p\Big)^{q/p}\Big\}^{1/q} \\ &= \big(S_{0,q}(S_{\alpha-\beta,p}(\{2^{-\beta i}X_i\})\big)_j.\end{aligned} \tag{2.3}$$

The case $p = 1, \beta = 0$. Let X be sequence with $X_j = 0$ except finite $j \in \{m, m+1, \ldots\}$. With the aid of Hölder's inequality, we get

$$\begin{aligned}(S_{0,q}(S_{\alpha,1}(X)))_j^q &= \sum_{i=j}^{\infty}\Big(2^{i\alpha}\sum_{l=i}^{\infty}2^{-\alpha l}X_l\Big)^q = \sum_{i=j}^{\infty}2^{iq\alpha}\sum_{l=i}^{\infty}2^{-\alpha l}X_l\Big(\sum_{l=i}^{\infty}2^{-\alpha l}X_l\Big)^{q-1} \\ &= \sum_{i=j}^{\infty}2^{iq\alpha}\sum_{l=0}^{\infty}2^{-\alpha(l+i)}X_{l+i}\Big(\sum_{l=i}^{\infty}2^{-\alpha l}X_l\Big)^{q-1} \\ &= \sum_{l=0}^{\infty}2^{-\alpha l}\sum_{i=j}^{\infty}X_{l+i}\Big(2^{i\alpha}\sum_{l=i}^{\infty}2^{-\alpha l}X_l\Big)^{q-1} \\ &= \sum_{l=0}^{\infty}2^{-\alpha l}\sum_{i=j}^{\infty}X_{l+i}S_{\alpha,1}(X)_i^{q-1} \\ &\le \sum_{l=0}^{\infty}2^{-\alpha l}\Big(\sum_{i=j}^{\infty}X_{l+i}^q\Big)^{1/q}\Big(\sum_{i=j}^{\infty}(S_{\alpha,1}(X))_i^q\Big)^{(q-1)/q} \\ &\le \frac{1}{1-2^{-\alpha}}(S_{0,q}(X))_j(S_{0,q}(S_{\alpha,1}(X)))_j^{q-1},\end{aligned}$$

where we used the fact that $(\sum_{i=j}^{\infty}X_{l+i}^q)^{1/q} \le (\sum_{i=j}^{\infty}X_i^q)^{1/q} = (S_{0,q}X)_j$ for all $l \ge 0$. Dividing both sides by $(S_{0,q}(S_{\alpha,1}(X)))_j^{q-1}$, we get (2.2).

In the general case $S_{0,q}(X)_j < +\infty$, we obtain, from (2.2) for the truncated sequence, the property $S_{0,q}(S_{\alpha,1}(X))_j < +\infty$. This shows (2.2) for the general case.

The case $0 < p \le q \le +\infty$, $\beta < \alpha$. Recalling the definition of $S_{\alpha,p}(X)$, we find

$$S_{\alpha,p}(X)_j = \Big(2^{j\alpha p}\sum_{i=j}^{\infty} 2^{-i\alpha p} X_i^p\Big)^{1/p} = (S_{\alpha p,1}(\{X_i^p\}))_j^{1/p}, \quad j \in \mathbb{Z}. \tag{2.4}$$

From (2.4), first with $\alpha - \beta$ in place of α, and then for $\beta = 0$ and $p = q$, we obtain the following two identities for $j \in \mathbb{Z}$:

$$\big(S_{\alpha-\beta,p}(\{2^{-\beta i} X_i\})\big)_j = \big(S_{(\alpha-\beta)p,1}(\{2^{-\beta p i} X_i^p\}\big)_j^{1/p}, \tag{2.5}$$

$$\big[S_{0,q}\big(S_{(\alpha-\beta)p,1}(\{2^{-\beta p i} X_i^p\})\big)\big]_j^{1/p} = \big[S_{0,1}\big(\big(S_{(\alpha-\beta)p,1}(\{2^{-\beta p i} X_i^p\})\big)^{q/p}\big)\big]_j^{1/q}. \tag{2.6}$$

Applying $S_{0,q}$ to both sides of (2.5) and then using (2.6) together with (2.1), and the inequality from the first part of the proof, we arrive at

$$\begin{aligned}
\big(S_{0,q}\big(S_{\alpha-\beta,p}(\{2^{-\beta i} X_i\})\big)\big)_j &= \big[S_{0,q}\big(S_{(\alpha-\beta)p,1}(\{2^{-\beta p i} X_i^p\})\big)\big]_j^{1/p} \\
&= \big[S_{0,1}\big(\big(S_{(\alpha-\beta)p,1}(\{2^{-\beta p i} X_i^p\})\big)^{q/p}\big)\big]_j^{1/q} \\
&= \big(S_{0,q/p}\big(S_{(\alpha-\beta)p,1}(\{2^{-\beta p i} X_i^p\})\big)\big)_j^{1/p} \\
&\le \frac{1}{(1-2^{-(\alpha-\beta)p})^{1/p}}\big(S_{0,q/p}(\{2^{-\beta p i} X_i^p\})\big)_j^{1/p} \\
&\le \frac{1}{1-2^{-(\alpha-\beta)}}\, 2^{-\beta j}\,(S_{\beta,q}(X))_j,
\end{aligned}$$

where we used the fact $(1-x^a)^{1/a} \ge 1 - x$ for all $0 < x < 1$ and $a > 1$. Combining this with (2.3), we have (2.2). ■

3. Properties of the spaces $\mathcal{L}^s_{q(p,N)}(\mathbb{R}^n)$

In this section our objective is to provide important properties of the space $\mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n)$ such as embedding properties, equivalent norms, interpolations properties and product estimates. First, let us recall the definition of the generalized mean for distributions $f \in \mathcal{S}'$, where $\mathcal{S}$ denotes the usual Schwarz class of rapidly decaying functions. For $f \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$, we define the convolution

$$f * \varphi(x) = \langle f, \varphi(x - \cdot)\rangle, \quad x \in \mathbb{R}^n,$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing. Below we use the notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then $f * \varphi \in C^\infty(\mathbb{R}^n)$, and for every multi-index $\alpha \in \mathbb{N}_0^n$,

$$D^\alpha(f * \varphi) = f * (D^\alpha \varphi) = (D^\alpha f) * \varphi.$$

Given $x_0 \in \mathbb{R}^n$, $0 < r < +\infty$ and $f \in \mathcal{S}'$, we define the mean

$$[f]^\alpha_{x,r} = f * D^\alpha \varphi_r(x).$$

where $\varphi_r(y) = r^{-n}\varphi(r^{-1}(y))$, and $\varphi \in C_c^\infty(B(1))$ stands for Friedrich's mollifying kernel such that $\int_{\mathbb{R}^n} \varphi\, dx = 1$ and $\varphi \geq 0$. Note that for $f \in L^1_{\rm loc}(\mathbb{R}^n)$, we get

$$[f]^0_{x,r} = \int_{\mathbb{R}^n} f(x-y)\varphi_r(y)\, dy = \int_{B(x,r)} f(y)\varphi_{x,r}(-y)\, dy,$$

where $\varphi_{x,r} = \varphi_r(\cdot + x)$. Furthermore, from the above definition, it follows that

$$[f]^\alpha_{x,r} = (D^\alpha f) * \varphi_r(x) = [D^\alpha f]^0_{x,r}. \tag{3.1}$$

For $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$, we immediately get

$$[f]^\alpha_{x,r} \leq c r^{-|\alpha|-n} \|f\|_{L^1(B(x,r))} \quad \text{for all } x \in \mathbb{R}^n, r > 0. \tag{3.2}$$

Lemma 3.1. *Let $x_0 \in \mathbb{R}^n$, $0 < r < +\infty$ and $N \in \mathbb{N}_0$. For every $f \in \mathcal{S}'$, there exists a unique polynomial $P^N_{x_0,r}(f) \in \mathcal{P}_N$ such that*

$$[f - P^N_{x_0,r}(f)]^\alpha_{x_0,r} = 0 \quad \textit{for all } |\alpha| \leq N. \tag{3.3}$$

Proof. Set $L = \binom{n+N}{N}$. Clearly, $\dim \mathcal{P}_N = L$. We define the mapping $T_N \colon \mathcal{P}_N \to \mathbb{R}^L$ by

$$(T_N Q)_\alpha = [Q]^\alpha_{x_0,r}, \quad |\alpha| \leq N,\ Q \in \mathcal{P}_N.$$

In order to prove the assertion of the lemma, it will be sufficient to show that T_N is injective, since by $\dim \mathcal{P}_N = L$, this implies that T_N is also surjective. In fact, this can be proved by induction over N. For $N = 0$, we see this by the fact that

$$(T_0 1)_0 = [1]^0_{x_0,r} = 1.$$

This T_0 stands for the identity in $\mathcal{P}_0 \cong \mathbb{R}$. Assume that T_{N-1} is injective. Let $Q = \sum_{|\alpha|\leq N} a_\alpha x^\alpha \in \mathcal{P}_N$ be such that $T_N(Q) = 0$. Using (3.1), for $|\alpha| = N$, this implies

$$0 = [Q]^\alpha_{x_0,r} = \Big[\sum_{|\beta|\leq N} a_\beta D^\alpha x^\beta\Big]^0_{x_0,r} = [\alpha! a_\alpha]^0_{x_0,r} = \alpha! a_\alpha.$$

Here, we used the formula $D^\alpha x^\beta = \alpha!\delta_{\alpha\beta}$ for all $|\beta| \leq N$.

Accordingly, $Q \in \mathcal{P}_{N-1}$, and we have $T_{N-1}(Q) = T_N(Q) = 0$. By our assumption, it follows that $Q = 0$. This proves that T_N is injective and thus surjective. ■

Lemma 3.2. (1) *Let $f \in \mathcal{S}'$. Then, for all $|\beta| \leq N$,*

$$P^{N-|\beta|}_{x_0,r}(D^\beta f) = D^\beta P^N_{x_0,r}(f). \tag{3.4}$$

(2) *The mapping $P^N_{x_0,r} \colon L^p(B(x_0,r)) \to \mathcal{P}_N$, $1 \leq p \leq +\infty$, defines a projection, i.e.,*

$$P^N_{x_0,r}(Q) = Q \quad \textit{for all } Q \in \mathcal{P}_N, \tag{3.5}$$

$$\|P^N_{x_0,r}(f)\|_{L^p(B(x_0,4r))} \leq c\|P^N_{x_0,r}(f)\|_{L^p(B(x_0,r))} \leq c\|P^N_{0,1}\|_p \|f\|_{L^p(B(x_0,r))}, \tag{3.6}$$

where

$$\|P^N_{0,1}\|_p = \sup_{\substack{g \in L^p(B(1))\\ g\neq 0}} \frac{\|P^N_{0,1}(g)\|_{L^p(B(1))}}{\|g\|_{L^p(B(1))}} = \sup_{\substack{g \in L^p(B(x_0,r))\\ g\neq 0}} \frac{\|P^N_{x_0,r}(g)\|_{L^p(B(x_0,r))}}{\|g\|_{L^p(B(x_0,r))}}.$$

(3) *For all* $f \in W^{j,p}(B(x_0,r))$, $1 \le p < +\infty$, $1 \le j \le N+1$,

$$\|f - P^N_{x_0,r}(f)\|_{L^p(B(x_0,r))} \le c r^j \sum_{|\alpha|=j} \|D^\alpha f - D^\alpha P^N_{x_0,r}(f)\|_{L^p(B(x_0,r))}. \tag{3.7}$$

Proof. (1) Let $\gamma \in \mathbb{N}_0^n$ be a multi-index with $|\gamma| \le N - |\beta|$. Obviously, $|\beta + \gamma| \le N$. From the definition of $P^N_{x_0,r}$, observing (3.3), and employing (3.1), we find

$$\begin{aligned}[P^{N-|\beta|}_{x_0,r}(D^\beta f)]^\gamma_{x_0,r} = [D^\beta f]^\gamma_{x_0,r} = D^\beta f * D^\gamma \varphi_r(x_0) = f * D^{\beta+\gamma}\varphi_r(x_0) \\ = [f]^{\beta+\gamma}_{x_0,r} = [P^N_{x_0,r}(f)]^{\beta+\gamma}_{x_0,r} = [D^\beta P^N_{x_0,r}(f)]^\gamma_{x_0,r}.\end{aligned}$$

As we have seen in the proof of Lemma 3.1, the mapping $T_{N-|\beta|}\colon \mathcal{P}_{N-|\beta|} \to \mathcal{P}_{N-|\beta|}$ is injective. This yields (3.4).

(2) We show that $P^N_{x_0,r}$ is a projection, i.e., $P^N_{x_0,r}(Q) = Q$ for all $Q \in \mathcal{P}_N$. Indeed, given $Q \in \mathcal{P}_N$, by the definition of $P^N_{x_0,r}$ (3.3), it follows that

$$[Q - P^N_{x_0,r}(Q)]^\alpha_{x_0,r} = 0 \quad \text{for all } |\alpha| \le N.$$

Consequently, $T_N(Q - P^N_{x_0,r}(Q)) = 0$. Since T_N is injective, we get $P^N_{x_0,r}(Q) = Q$. Inequality (3.6) can be verified by a standard scaling and translation argument.

(3) We prove (3.7) by induction over j. For $j = 1$, (3.7) follows from the usual Poincaré inequality, since $[f - P^N_{x_0,r}(f)]^0_{x_0,r} = 0$. Assume (3.7) holds for $j-1$. Thus,

$$\|f - P^N_{x_0,r}(f)\|_{L^p(B(x_0,r))} \le c r^{j-1} \sum_{|\alpha|=j-1} \|D^\alpha f - D^\alpha P^N_{x_0,r}(f)\|_{L^p(B(x_0,r))}. \tag{3.8}$$

Thanks to (3.5), for all $|\alpha| = j-1$,

$$D^\alpha P^N_{x_0,r}(f) = P^{N-j+1}_{x_0,r}(D^\alpha f).$$

Hence, $[D^\alpha f - D^\alpha P^N_{x_0,r}(f)]^0_{x_0,r} = 0$. An application of the Poincaré inequality gives

$$\|D^\alpha f - D^\alpha P^N_{x_0,r}(f)\|_{L^p(B(x_0,r))} \le c r \|DD^\alpha f - DD^\alpha P^N_{x_0,r}(f)\|_{L^p(B(x_0,r))}. \tag{3.9}$$

Combining (3.8) and (3.9), we get (3.7). ■

Remark 3.3. From (3.7), with $j = N+1$, we get the generalized Poincaré inequality

$$\|f - P^N_{x_0,r}(f)\|_{L^p(B(x_0,r))} \le c r^{N+1} \|D^{N+1} f\|_{L^p(B(x_0,r))}$$

for all $f \in W^{N+1,p}(B(x_0,r))$.

Corollary 3.4. *For all* $x_0 \in \mathbb{R}^n$, $0 < r < +\infty$, $N \in \mathbb{N}_0$, *and* $1 \le p < +\infty$,

$$\|f - P^N_{x_0,r}(f)\|_{L^p(B(x_0,r))} \le c \inf_{Q \in \mathcal{P}_N} \|f - Q\|_{L^p(B(x_0,r))} = c r^{n/p} \operatorname*{osc}_{p,N}(f; x_0, r). \tag{3.10}$$

Proof. Let $Q \in \mathcal{P}_N$ be arbitrarily chosen. In view of (3.5), we find

$$f - P^N_{x_0,r}(f) = f - Q - P^N_{x_0,r}(f - Q).$$

Hence, applying the triangle inequality, along with (3.6), we get

$$\begin{aligned}\|f - P^N_{x_0,r}(f)\|_{L^p(B(x_0,r))} &\le \|f - Q\|_{L^p(B(x_0,r))} + \|P^N_{x_0,r}(f - Q)\|_{L^p(B(x_0,r))}\\ &\le c\|f - Q\|_{L^p(B(x_0,r))}.\end{aligned}$$

This shows the validity of (3.10). ∎

In our discussion below and in the sequel of the paper, it will be convenient to work with smooth functions. Using the standard mollifier, we get the following estimate in $\mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n)$ for the mollification.

Lemma 3.5. *Let* $\varepsilon > 0$*. Given* $f \in \mathcal{S}'$*, we define the mollification*

$$f_\varepsilon(x) = [f]^0_{x,\varepsilon} = f * \varphi_\varepsilon(x), \quad x \in \mathbb{R}^n.$$

(1) *For all* $f \in \mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n)$ *and all* $\varepsilon > 0$*,*

$$|f_\varepsilon|_{\mathcal{L}^{k,s}_{q(p,N)}} \le c|f|_{\mathcal{L}^{k,s}_{q(p,N)}}. \tag{3.11}$$

(2) *Let* $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ *be such that for all* $0 < \varepsilon < 1$*,*

$$|f_\varepsilon|_{\mathcal{L}^{k,s}_{q(p,N)}} \le c_0. \tag{3.12}$$

Then $f \in \mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n)$ *and we have* $|f|_{\mathcal{L}^{k,s}_{q(p,N)}} \le c_0$*.*

Proof. (1) We may restrict ourself to the case $k = 0$. Let $x_0 \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, and set $0 < r < +\infty$. By the definition of $P^N_{x_0,r}(f)$ (cf. (3.3)), together with (3.1), it follows that for all $|\alpha| \le N$ and for almost all $y \in \mathbb{R}^n$,

$$f * D^\alpha \varphi_r(x_0 - y) = [f]^\alpha_{x_0-y,r} = [P^N_{x_0-y,r}(f)]^\alpha_{x_0-y,r} = P^N_{x_0-y,r}(f) * D^\alpha \varphi_r(x_0 - y).$$

Multiplying both sides by $\varphi_{0,\varepsilon}(y)$, integrating the result over $\mathbb{R}^n$ and applying Fubini's theorem, we get, for all $|\alpha| \le N$,

$$\begin{aligned}[f_\varepsilon]^\alpha_{x_0,r} &= (f * \varphi_\varepsilon * D^\alpha\varphi_r)(x_0) = (f * D^\alpha\varphi_r * \varphi_\varepsilon)(x_0)\\ &= \int_{\mathbb{R}^n} (f * D^\alpha\varphi_r)(x_0 - y)\varphi_\varepsilon(y)\,dy\\ &= \int_{\mathbb{R}^n} P^N_{x_0-y,r}(f) * D^\alpha\varphi_r(x_0 - y)\varphi_\varepsilon(y)\,dy\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} P^N_{x_0-y,r}(f)(x) D^\alpha\varphi_r(x_0 - y - x)\varphi_\varepsilon(y)\,dx\,dy\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} P^N_{x_0-y,r}(f)(x-y) D^\alpha\varphi_r(x_0 - x)\varphi_\varepsilon(y)\,dx\,dy\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} P^N_{x_0-y,r}(f)(x-y)\varphi_\varepsilon(y)\,dy D^\alpha\varphi_r(x_0 - x)\,dx\\ &= \Big[\int_{\mathbb{R}^n} P^N_{x_0-y,r}(f)(x-y)\varphi_\varepsilon(y)\,dy\Big]^\alpha_{x_0,r}.\end{aligned}$$

This shows that

$$P^N_{x_0,r}(f_\varepsilon)(x) = \int_{\mathbb{R}^n} P^N_{x_0-y,r}(f)(x-y)\varphi_\varepsilon(y)\,dy$$
$$= \int_{\mathbb{R}^n} P^N_{x_0-\varepsilon y,r}(f)(x-\varepsilon y)\varphi(y)\,dy, \quad x \in \mathbb{R}^n.$$

Accordingly,

$$|f_\varepsilon(x) - P^N_{x_0,2^j}(f_\varepsilon)(x)|^p \le \Big(\int_{\mathbb{R}^n} |f(x-\varepsilon y) - P^N_{x_0-\varepsilon y,2^j}(f)(x-\varepsilon y)|\varphi(y)\,dy\Big)^p.$$

Integrating both sides over $B(x_0,2^j)$ and multiplying with $1/|B(2^j)|$, using Jensen's inequality with respect to the probability measure $\varphi\,dy$, we find

$$\begin{aligned}\operatorname*{osc}_{p,N}(f_\varepsilon;x_0,2^j) &\le \Big(\fint_{B(x_0,2^j)}\Big(\int_{\mathbb{R}^n}|f(x-\varepsilon y) - P_{x_0-\varepsilon y,2^j}(f)(x-\varepsilon y)|\varphi(y)\,dy\Big)^p dx\Big)^{1/p}\\ &= \int_{\mathbb{R}^n}\Big(\fint_{B(x_0-\varepsilon y,2^j)}|f(x) - P_{x_0-\varepsilon y,2^j}(f)(x)|^p\,dx\Big)^{1/p}\varphi(y)\,dy\\ &\le c\int_{B(1)}\operatorname*{osc}_{p,N}(f;x_0-\varepsilon y;2^j)\varphi(y)\,dy.\end{aligned}$$

Multiplying both sides by 2^{-js}, applying the ℓ^q norm to both sides of the resultant inequality, and using Minkowski's inequality, we are led to

$$\begin{aligned}\Big(\sum_{j\in\mathbb{Z}}(2^{-js}\operatorname*{osc}_{p,N}(f_\varepsilon;x_0,2^j))^q\Big)^{1/q} &\le c\int_{B(1)}\Big(\sum_{j\in\mathbb{Z}}(2^{-js}\operatorname*{osc}_{p,N}(f;x_0-\varepsilon y;2^j))^q\Big)^{1/q}\varphi(y)\,dy\\ &\le c|f|_{\mathcal{L}^s_{q(p,N)}}.\end{aligned}$$

Taking the supremum over all $x_0 \in \mathbb{R}^n$ in the above inequality shows (3.11).

(2) Let $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ satisfy (3.12). This implies that $f \in W^{k,p}_{\mathrm{loc}}(\mathbb{R}^n)$. Let $x_0 \in \mathbb{R}^n$ and $l,m \in \mathbb{Z}$, $l < m$. From the absolutely continuity of the Lebesgue measure, together with (3.12), it follows that

$$\sum_{j=l}^m (2^{-js}\operatorname*{osc}_{p,N}(D^k f;x_0,2^j))^q = \lim_{\varepsilon\searrow 0}\sum_{j=l}^m(2^{-js}\operatorname*{osc}_{p,N}(D^k f_\varepsilon;x_0,2^j))^q \le c_0^q.$$

This shows that $\{2^{-sj}\operatorname{osc}_{p,N}(D^k f;x_0,2^j)\}_{j\in\mathbb{Z}} \in \ell^q$, and its sum is bounded by c_0. Accordingly, $f \in \mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n)$, and we have $|f|_{\mathcal{L}^{k,s}_{q(p,N)}} \le c_0$. ■

We are are now in a position to prove the following embedding properties. First, let us introduce the definition of the projection to the space of homogenous polynomials, $\dot{P}^N_{x_0,r}\colon \mathcal{S}' \to \dot{\mathcal{P}}_N$, defined by

$$\dot{P}^N_{x_0,r}(f)(x) = \sum_{|\alpha|=N}\frac{1}{\alpha!}[f]^\alpha_{x_0,r}x^\alpha, \quad x \in \mathbb{R}^n.$$

Clearly, for all $f \in \mathcal{S}'$,

(3.13) $$D^\alpha \dot P^N_{x_0,r}(f) = \dot P^{N-|\alpha|}_{x_0,r}(D^\alpha f) \quad \text{for all } |\alpha| \le k.$$

Theorem 3.6. (1) *For every $N \in \mathbb{N}_0$, the following embedding holds true:*

(3.14) $$\begin{cases} \mathcal{L}^N_{1(p,N)}(\mathbb{R}^n) \hookrightarrow C^{N-1,1}(\mathbb{R}^n) & \text{if } N \ge 1, \\ \mathcal{L}^0_{1(p,0)}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) & \text{if } N = 0. \end{cases}$$

(2) *For every $f \in \mathcal{L}^N_{1(p,N)}(\mathbb{R}^n)$, there exists a unique $\dot P^N_\infty \in \dot{\mathcal{P}}_N$ such that for all $x_0 \in \mathbb{R}^n$,*

$$\lim_{r\to\infty} \dot P^N_{x_0,r}(f) \to \dot P^N_\infty(f) \quad \text{in } \mathcal{P}_N.$$

Furthermore, $\dot P^N_\infty \colon \mathcal{L}^N_{1(p,N)}(\mathbb{R}^n) \to \dot{\mathcal{P}}_N$ is a projection, with the property

(3.15) $$D^\alpha \dot P^N_\infty(f) = \dot P^{N-|\alpha|}_\infty(D^\alpha f) \quad \text{for all } |\alpha| \le N.$$

(3) *For all $g, f \in \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$,*

(3.16) $$\dot P^1_\infty(g\partial_k f) = \dot P^1_\infty(g)\partial_k \dot P^1_\infty(f) = \dot P^1_\infty(g)\dot P^0_\infty(\partial_k f), \quad k = 1, \dots, n.$$

In addition, for $g \in C^{0,1}(\mathbb{R}^n;\mathbb{R}^n)$ and for all $f \in \mathcal{L}^0_{1(p,0)}(\mathbb{R}^n)$,

(3.17) $$\dot P^0_\infty(g\partial_k f) := \lim_{r\to\infty} P^0_{0,r}(g\partial_k f) = 0, \quad k = 1, \dots, n,$$

where $g\,\partial_k f = \partial_k(gf) - \partial_k g f \in \mathcal{S}'$.

(4) *For all $v \in \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n;\mathbb{R}^n)$, with $\nabla \cdot v = 0$ almost everywhere in $\mathbb{R}^n$, and all $f \in \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$,*

(3.18) $$\dot P^0_\infty(\nabla v \cdot \nabla f) = \dot P^0_\infty(\nabla v) \cdot \dot P^0_\infty(\nabla f).$$

Proof. (1) Let $\varepsilon > 0$ be arbitrarily chosen. Let $f \in \mathcal{L}^N_{1(p,N)}(\mathbb{R}^n)$. Set $f_\varepsilon = f * \varphi_\varepsilon$. By Lemma 3.5, we get $f_\varepsilon \in \mathcal{L}^N_{1(p,N)}(\mathbb{R}^n)$, and we have

$$|f_\varepsilon|_{\mathcal{L}^N_{1(p,N)}} \le c|f|_{\mathcal{L}^N_{1(p,N)}}.$$

Let $x_0 \in \mathbb{R}^n$ be fixed. Let $j \in \mathbb{Z}$. Clearly, $f_\varepsilon \in C^\infty(\mathbb{R}^n)$. Let $\alpha \in \mathbb{N}^n_0$ be a multi-index with $|\alpha| = N$. Then

$$D^\alpha P^N_{x_0,2^j}(f_\varepsilon) = P^0_{x_0,2^j}(D^\alpha(f_\varepsilon)) = [D^\alpha(f_\varepsilon)]^0_{x_0,2^j} = D^\alpha \dot P^N_{x_0,2^j}(f_\varepsilon).$$

Let $m \in \mathbb{Z}$. Since $D^\alpha f_\varepsilon$ is continuous, we have

$$D^\alpha f_\varepsilon(x) = \lim_{j\to-\infty} [D^\alpha f_\varepsilon]^0_{x,2^j} \quad \text{for all } x \in \mathbb{R}^n.$$

Using the triangle inequality along with (3.5) and (3.11), and using (3.2), we get

(3.19) $$\begin{aligned} |D^\alpha f_\varepsilon(x) - [D^\alpha f_\varepsilon]^0_{x,2^m}| &= \Big| \sum_{j=-\infty}^{m} [D^\alpha f_\varepsilon]^0_{x,2^{j-1}} - [D^\alpha f_\varepsilon]^0_{x,2^j} \Big| \\ &\le \sum_{j=-\infty}^{m} \big|[D^\alpha f_\varepsilon]^0_{x,2^{j-1}} - [D^\alpha f_\varepsilon]^0_{x,2^j}\big| \\ &= \sum_{j=-\infty}^{m} \big|[f_\varepsilon]^\alpha_{x,2^{j-1}} - [f_\varepsilon]^\alpha_{x,2^j}\big| \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=-\infty}^{m} \left| [f_\varepsilon - P^N_{x,2^j}(f_\varepsilon)]^\alpha_{x,2^{j-1}} - [f_\varepsilon - P^N_{x,2^j}(f_\varepsilon)]^\alpha_{x,2^j} \right| \\
&= \sum_{j=-\infty}^{m} \left| [f_\varepsilon - P^N_{x,2^j}(f_\varepsilon)]^\alpha_{x,2^{j-1}} \right| \le c \sum_{j=-\infty}^{m} 2^{-jN} \operatorname*{osc}_{p,N}(f_\varepsilon; x, 2^j) \\
&\le c|f_\varepsilon|_{\mathcal{L}^N_{1(p,N)}} \le c|f|_{\mathcal{L}^N_{1(p,N)}}.
\end{aligned}
$$

Thus, $\{D^N f_\varepsilon\}$ is bounded in $L^\infty(B(r))$ for all $0 < r < +\infty$. By means of the Banach–Alaoglu theorem and Cantor's diagonalization principle, we get a sequence $\varepsilon_k \searrow 0$ as $k \to +\infty$ and $f \in W^{N,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ such that for all $0 < r < +\infty$,

$$
D^N f_{\varepsilon_k} \to D^N f \quad \text{weakly-}* \text{ in } L^\infty(B(r)) \text{ as } k \to +\infty.
$$

Furthermore, from (3.19), we get, for almost all $x \in \mathbb{R}^n$ and all $m \in \mathbb{Z}$,

$$
|D^N f(x)| \le c|f|_{\mathcal{L}^N_{1(p,N)}} + \sum_{|\alpha|=N} |[f]^\alpha_{x,2^m}|. \tag{3.20}
$$

Let $x_0 \in \mathbb{R}^n$ be fixed. We now choose $m \in \mathbb{Z}$ such that $2^{m-1} \le |x_0| < 2^m$. Then, noting that $B(x_0, 2^m) \subset B(2^{m+1})$, and employing (3.5) and (3.2), we get

$$
\begin{aligned}
\left| |[f]^\alpha_{x_0,2^m}| - |[f]^\alpha_{0,2^m}| \right| &\le [f]^\alpha_{x_0,2^m} - [f]^\alpha_{0,2^m} \\
&= \left| [f - P^N_{0,2^{m+1}}]^\alpha_{x_0,2^m} - [f - P^N_{0,2^{m+1}}]^\alpha_{0,2^m} \right| \\
&\le c2^{-mN} \operatorname*{osc}_{p,N}(f; 0, 2^{m+1}) \le c|f|_{\mathcal{L}^N_{1(p,N)}}.
\end{aligned}
$$

Similarly, for all $j \in \mathbb{Z}$, we get

$$
\left| |[f]^\alpha_{0,2^m}| - |[f]^\alpha_{0,2^j}| \right| \le c|f|_{\mathcal{L}^N_{1(p,N)}}.
$$

Thus, combining the two inequalities we have just obtained, and using the triangle inequality, we find, for all $j \in \mathbb{Z}$,

$$
\sum_{|\alpha|=N} |[f]^\alpha_{x,2^m}| \le c|f|_{\mathcal{L}^N_{1(p,N)}} + \sum_{|\alpha|=N} |[f]^\alpha_{0,2^j}|.
$$

This, together with (3.20), for all $j \in \mathbb{Z}$, yields

$$
\|D^N f\|_\infty \le c|f|_{\mathcal{L}^N_{1(p,N)}} + c \sum_{|\alpha|=N} |[f]^\alpha_{0,2^j}| \le c|f|_{\mathcal{L}^N_{1(p,N)}} + c\|\dot{P}^N_{0,2^j}(f)\|. \tag{3.21}
$$

This completes the proof of (3.14).

(2) Let $x_0 \in \mathbb{R}^n$. Let $m, l \in \mathbb{Z}$, $l < m$. Noting that $\dot{P}^N_{x_0,2^j}(Q) = Q$ for all $Q \in \dot{\mathcal{P}}_N$ and $\dot{P}^N_{x_0,2^j}(Q) = 0$ for all $Q \in \mathcal{P}_{N-1}$, we get the following identity for all $j, k \in \mathbb{Z}$:

$$
\dot{P}^N_{x_0,2^j}(P^N_{x_0,2^k}(f)) = \dot{P}^N_{x_0,2^k}(f).
$$

Using the triangle inequality together with the above identity, (3.2) and (3.10), we estimate

$$\|\dot P^N_{x_0,2^l}(f)-\dot P^N_{x_0,2^m}(f)\|\le\sum_{j=l+1}^{m}\|\dot P^N_{x_0,2^{j-1}}(f)-\dot P^N_{x_0,2^j}(f)\|$$

$$\le c\sum_{j=l+1}^{m}2^{-jN-jn/p}\|\dot P^N_{x_0,2^{j-1}}(f-P^N_{x_0,2^j}(f))-\dot P^N_{x_0,2^j}(f-P^N_{x_0,2^j}(f))\|_{L^p(B(x_0,2^j))}$$

$$\le c\sum_{j=l+1}^{m}2^{-jN}\operatorname*{osc}_{p,N}(f;x_0,2^j).$$

Owing to $f\in\mathcal{L}^N_{1(p,N)}(\mathbb{R}^n)$, the right-hand side of the above inequality tends to zero as $m,l\to+\infty$. This shows that $\{\dot P^N_{x_0,2^m}(f)\}$ is a Cauchy sequence in $\dot P_N$ and converges to a unique limit $\dot P^N_{\infty,x_0}$. We claim that

$$\dot P^N_{\infty,x_0}=\dot P^N_{\infty,0}=:\dot P^N_\infty(f). \tag{3.22}$$

In fact, for $m\in\mathbb{Z}$ such that $|x_0|\le 2^m$, we obtain

$$\begin{aligned}&\|\dot P^N_{x_0,2^m}(f)-\dot P^N_{0,2^m}(f)\|\\&\quad\le c2^{-mN-mn/p}\|\dot P^N_{x_0,2^m}(f-P^N_{0,2^{m+1}}(f))-\dot P^N_{0,2^m}(f-P^N_{0,2^{m+1}}(f))\|_{L^p(B(x_0,2^m))}\\&\quad\le c2^{-mN}\operatorname*{osc}_{p,N}(f;0,2^{m+1})\to 0\quad\text{as } m\to+\infty.\end{aligned}$$

Consequently, (3.22) must hold. Identity (3.15) is an immediate consequence of (3.13).

(3) Now, let $g,f\in\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$. Let $x_0\in\mathbb{R}^n$. Let $\alpha\in\mathbb{N}^n_0$ with $|\alpha|=1$. We first show that $\{[g\partial_k f]^\alpha_{x_0,2^j}\}_{j\in\mathbb{N}}$, $k\in\{1,\dots,n\}$, is a Cauchy sequence. Let $j\in\mathbb{N}$ be fixed. We easily calculate

$$\begin{aligned}&[g\partial_k f]^\alpha_{x_0,2^{j-1}}-[g\partial_k f]^\alpha_{x_0,2^j}\\&\quad=\big[g\partial_k f-P^1_{x_0,2^j}(g)[\partial_k f]^0_{x_0,2^j}\big]^\alpha_{x_0,2^{j-1}}-\big[g\partial_k f-P^1_{x_0,2^j}(g)[\partial_k f]^0_{x_0,2^j}\big]^\alpha_{x_0,2^j}.\end{aligned}$$

Furthermore, applying integration by parts, we get

$$\begin{aligned}&\big[g\partial_k f-P^1_{x_0,2^j}(g)[\partial_k f]^0_{x_0,2^j}\big]^\alpha_{x_0,2^{j-1}}\\&\quad=\big[(g-P^1_{x_0,2^j}(g)([\partial_k f]^0_{x_0,2^j})\big]^\alpha_{x_0,2^{j-1}}+\big[g\cdot(\partial_k f-[\partial_k f]^0_{x_0,2^j})\big]^\alpha_{x_0,2^{j-1}}\\&\quad=-\int_{\mathbb{R}^n}(g-P^1_{x_0,2^j}(g))([\partial_k f]^0_{x_0,2^j})D^\alpha\varphi_{x_0,2^{j-1}}\,dx\\&\qquad-\int_{\mathbb{R}^n}g(\partial_k f-[\partial_k f]^0_{x_0,2^j})D^\alpha\varphi_{x_0,2^{j-1}}\,dx\\&\quad=-\int_{\mathbb{R}^n}(g-P^1_{x_0,2^j}(g))([\partial_k f]^0_{x_0,2^j})D^\alpha\varphi_{x_0,2^{j-1}}\,dx\\&\qquad+\int_{\mathbb{R}^n}\partial_k g\big(f-P^1_{x_0,2^j}(f)-[f-P^1_{x_0,2^j}(f)]^1_{x_0,2^j}\big)D^\alpha\varphi_{x_0,2^{j-1}}\,dx\\&\qquad+\int_{\mathbb{R}^n}g\big(f-P^1_{x_0,2^j}(f)-[f-P^1_{x_0,2^j}(f)]^1_{x_0,2^j}\big)\partial_k D^\alpha\varphi_{x_0,2^{j-1}}\,dx.\end{aligned}$$

This, together with (3.10), yields

$$\begin{aligned}&\big[g\partial_k f - P^1_{x_0,2^j}(g)[\partial_k f]^0_{x_0,2^j}\big]^\alpha_{x_0,2^{j-1}}\\&\quad\le c\|\nabla f\|_\infty 2^{-j}\operatorname*{osc}_{p,1}(v;x_0,2^j)+c\|\nabla v\|_\infty 2^{-j}\operatorname*{osc}_{p,1}(f;x_0,2^j).\end{aligned}$$

By an analogous reasoning, we find

$$\begin{aligned}&\big[g\partial_k f - P^1_{x_0,2^j}(g)[\partial_k f]^0_{x_0,2^j}\big]^\alpha_{x_0,2^{j}}\\&\quad\le c\|\nabla f\|_\infty 2^{-j}\operatorname*{osc}_{p,1}(v;x_0,2^j)+c\|\nabla v\|_\infty 2^{-j}\operatorname*{osc}_{p,1}(f;x_0,2^j).\end{aligned}$$

Let $l,m\in\mathbb{Z}$ with $l<m$ be arbitrarily chosen. Using the triangle inequality together with the two estimates we have just obtained, we estimate

$$\begin{aligned}&\big|[g\partial_k f]^\alpha_{x_0,2^l}-[g\partial_k f]^\alpha_{x_0,2^m}\big|\\&=\Big|\sum_{j=l+1}^{m}[g\partial_k f]^\alpha_{x_0,2^{j-1}}-[g\partial_k f]^\alpha_{x_0,2^j}\Big|\\&\le c\|\nabla f\|_\infty\sum_{j=l+1}^{m}2^{-j}\operatorname*{osc}_{p,1}(g;x_0,2^j)+c\|\nabla g\|_\infty\sum_{j=l+1}^{m}2^{-j}\operatorname*{osc}_{p,1}(f;x_0,2^j).\end{aligned}$$

Since $g,f\in\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$, the right-hand side converges to zero as $l,m\to+\infty$. Thus, $\{[g\partial_k f]^\alpha_{x_0,2^l}\}$ is a Cauchy sequence, and has a unique limit, say a_{x_0}. Let $j\in\mathbb{N}$ be such that $2^j\ge|x_0|$. Thus, $B(x_0,2^j)\subset B(2^{j+1})$. Reasoning as above, we estimate

$$\begin{aligned}&\big|[g\partial_k f]^0_{x_0,2^j}-[g\partial_k f]^\alpha_{0,2^{j+1}}\big|\\&\quad= c\|\nabla f\|_\infty 2^{-j}\operatorname*{osc}_{p,1}(g;0,2^{j+1})+c\|\nabla g\|_\infty 2^{-j}\operatorname*{osc}_{p,1}(f;0,2^{j+1}).\end{aligned}$$

Since the right-hand side converges to zero as $j\to+\infty$, we get $a_{x_0}=a_0$. By setting $[g\partial_k f]^\alpha_\infty=a_0$, we complete the proof of (3.16).

Next we prove (3.17). Let $g\in C^{0,1}(\mathbb{R}^n)$ and $f\in\mathcal{L}^0_{1(p,0)}(\mathbb{R}^n)$. Applying integration by parts and the product rule, we calculate

$$\begin{aligned}[g\partial_k f]^0_{x_0,r}&=-\int_{B(x_0,r)}\partial_k g(y)(f(y)-[f]^0_{x_0,r})\varphi_r(x_0-y)\,dy\\&\quad+\int_{B(x_0,r)}g(y)(f(y)-[f]^0_{x_0,r})\partial_k\varphi_r(x_0-y)\,dy.\end{aligned}$$

Applying Hölder's inequality, we easily get

$$|[g\partial_k f]^0_{x_0,r}|\le c\|\nabla g\|_\infty\operatorname*{osc}_{p,0}(f;x_0,r)+cr^{-1}\|g\|_{L^\infty(B(x_0,r))}\operatorname*{osc}_{p,0}(f;x_0,r).$$

By noting that $r^{-1}\|g\|_{L^\infty(B(x_0,r))}\le c|g(x_0)|+c\|\nabla g\|_\infty$, and by using the fact that $\mathrm{osc}_{p,0}(f;x_0,r)\to 0$ as $r\to+\infty$, we obtain (3.17).

It remains to show the identity (3.18). Let $v\in\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n;\mathbb{R}^n)$ with $\nabla\cdot v=0$ and $f\in\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$. Using (3.15) together with $\nabla\cdot v=0$ and (3.16), we obtain

$$\begin{aligned}[\partial_k v\cdot\nabla f]^0_{x_0,r}&=\partial_j P^1_{x_0,r}((\partial_k v_j)f)=\partial_j\dot P^1_{x_0,r}((\partial_k v_j)f)\\&\to\partial_j\dot P^1_\infty(\partial_k v_j)\dot P^1_\infty(f)=\dot P^0_\infty(\partial_k v)\cdot\dot P^0_\infty(\nabla f)\quad\text{as } r\to+\infty.\end{aligned}$$

This shows that

$$\dot{P}^0_\infty(\partial_k v\cdot\nabla f)=\lim_{r\to\infty}[\partial_k v\cdot\nabla f]^0_{x_0,r}=\dot{P}^0_\infty(\partial_k v)\cdot\dot{P}^0_\infty(\nabla f),$$

which completes the proof of the Lemma. ■

Next we prove the following norm equivalence which is similar to the properties of the known Campanato space.

Lemma 3.7. *Let $1\le p<+\infty$, $1\le q\le+\infty$, $N,N'\in\mathbb{N}_0$, $N<N'$, $s\in[-n/p,N+1)$. If $f\in\mathcal{L}^{k,s}_{q(p,N')}(\mathbb{R}^n)$ satisfies*

$$\lim_{m\to\infty}\dot{P}^L_{0,2^m}(D^k f)=0\quad \textit{for all } L=N+1,\dots,N', \tag{3.23}$$

then $f\in\mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n)$ and

$$|f|_{\mathcal{L}^{k,s}_{q(p,N')}}\le|f|_{\mathcal{L}^{k,s}_{q(p,N)}}\le c|f|_{\mathcal{L}^{k,s}_{q(p,N')}}. \tag{3.24}$$

Proof. We may restrict ourself to the case $k=0$. First, let us prove that for all $s\in[-n/p,N)$ and for all $f\in\mathcal{L}^{s}_{q(p,N)}(\mathbb{R}^n)$ such that

$$\lim_{m\to\infty}\dot{P}^N_{0,2^m}(f)=0, \tag{3.25}$$

it follows that $f\in\mathcal{L}^{s}_{q(p,N-1)}(\mathbb{R}^n)$, together with the estimate

$$|f|_{\mathcal{L}^{s}_{q(p,N-1)}}\le c|f|_{\mathcal{L}^{s}_{q(p,N)}}. \tag{3.26}$$

Let $x_0\in\mathbb{R}^n$, $0<r<+\infty$. Noting that $P^N_{x_0,2r}(f)-\dot{P}^N_{x_0,2r}(f)\in\mathcal{P}_{N-1}$, we see that

$$\dot{P}^N_{x_0,r}(P^N_{x_0,2r}(f))=\dot{P}^N_{x_0,r}(P^N_{x_0,2r}(f)-\dot{P}^N_{x_0,2r}(f))+\dot{P}^N_{x_0,r}(\dot{P}^N_{x_0,2r}(f))=\dot{P}^N_{x_0,2r}(f).$$

By a scaling argument and the triangle inequality, we infer

$$\begin{aligned}
&r^{-N-n/p}\|\dot{P}^N_{x_0,r}(f)\|_{L^p(B(x_0,r))}-(2r)^{-N-n/p}\|\dot{P}^N_{x_0,2r}(f)\|_{L^p(B(x_0,2r))}\\
&\quad=\|\dot{P}^N_{x_0,r}(f)\|-\|\dot{P}^N_{x_0,2r}(f)\|\le\|\dot{P}^N_{x_0r}(f)-\dot{P}^N_{x_0,2r}(f)\|\\
&\quad=\|\dot{P}^N_{x_0,r}(f-P^N_{x_0,2r}(f)\|\le c(2r)^{-N-n/p}\|f-P^N_{x_0,2r}(f)\|_{L^p(B(x_0,2r))}\\
&\quad\le cr^{-N}\operatorname*{osc}_{p,N}(f;x_0,2r).
\end{aligned}$$

Let $j,m\in\mathbb{Z}$, $j<m$. Using the above estimate, we deduce that

$$\begin{aligned}
&\left|2^{-jN-jn/p}\|\dot{P}^N_{x_0,2^j}(f)\|_{L^p(B(x_0,2^j))}-2^{-mN-mn/p}\|\dot{P}^N_{x_0,2^N}(f)\|_{L^p(B(x_0,2^m))}\right|\\
&\quad\le c\sum_{i=j}^{m-1}2^{-iN}\operatorname*{osc}_{p,N}(f;x_0,2^{i+1})\le c2^N\sum_{i=j}^{m-1}2^{-iN}\operatorname*{osc}_{p,N}(f;x_0,2^i).
\end{aligned}$$

Observing (3.25), we see that

$$\lim_{m\to\infty}2^{-mN-mn/p}\|\dot{P}^N_{x_0,2^N}(f)\|_{L^p(B(x_0,2^m))}=\lim_{m\to\infty}\|\dot{P}^N_{x_0,2^m}(f)\|=0.$$

Thus, letting $m \to +\infty$ in the above estimate, we arrive at

$$2^{jN}\|\dot{P}^N_{x_0,2^j}(f)\| = 2^{-jn/p}\|\dot{P}^N_{x_0,2^j}(f)\|_{L^p(B(x_0,2^j))} \le c2^{jN}\sum_{i=j}^{\infty}2^{-iN}\underset{p,N}{\operatorname{osc}}(f;x_0,2^i) = c\big(S_{N,1}\big(\underset{p,N}{\operatorname{osc}}(f;x_0)\big)\big)_j, \tag{3.27}$$

where $\operatorname{osc}_{p,N}(f;x_0)$ stands for a sequence defined as

$$\underset{p,N}{\operatorname{osc}}(f;x_0)_i = \underset{p,N}{\operatorname{osc}}(f;x_0,2^i), \quad i \in \mathbb{Z}.$$

Using the triangle inequality together with (3.27), we obtain

$$\begin{aligned}
&\underset{p,N-1}{\operatorname{osc}}(f;x_0,2^j) \\
&= 2^{-jn/p}\inf_{P\in\mathcal{P}_{N-1}}\|f-P\|_{L^p(B(x_0,2^j))} \\
&\le c2^{-jn/p}\|f-P^N_{x_0,2^j}(f)+\dot{P}^N_{x_0,2^j}(f)\|_{L^p(B(x_0,2^j))} \\
&\le c2^{-jn/p}\|f-P^N_{x_0,2^j}(f)\|_{L^p(B(x_0,2^j))}+c2^{-jn/p}\|\dot{P}^N_{x_0,2^j}(f)\|_{L^p(B(x_0,2^j))} \\
&\le c\underset{p,N}{\operatorname{osc}}(f;x_0,2^j)+2^{jN}\|\dot{P}^N_{x_0,2^j}(f)\| \\
&\le c\underset{p,N}{\operatorname{osc}}(f;x_0,2^j)+c\big(S_{N,1}\big(\underset{p,N}{\operatorname{osc}}(f;x_0)\big)\big)_j.
\end{aligned} \tag{3.28}$$

Noting that $\operatorname{osc}_{p,N}(f;x_0,2^j) \le S_{N,1}(\operatorname{osc}_{p,N}(f;x_0,2^j))$, from (3.28), we infer

$$\underset{p,N-1}{\operatorname{osc}}(f;x_0)_j = \underset{p,N-1}{\operatorname{osc}}(f;x_0,2^j) \le c\big(S_{N,1}\big(\underset{p,N}{\operatorname{osc}}(f;x_0)\big)\big)_j, \quad j \in \mathbb{Z}. \tag{3.29}$$

Applying $S_{s,q}$ to both sides of (3.29), and using Lemma 2.1, we get the inequality

$$|f|_{\mathcal{L}^s_{q(p,N-1)}} = \sup_{x_0\in\mathbb{R}^n} S_{s,q}\big(\underset{p,N-1}{\operatorname{osc}}(f;x_0)\big) \le c\sup_{x_0\in\mathbb{R}^n} S_{s,q}\big(\underset{p,N}{\operatorname{osc}}(f;x_0)\big) = |f|_{\mathcal{L}^s_{q(p,N')}},$$

which implies (3.26). We are now in a position to apply (3.26) iteratively, replacing N by $N+1$ to get

$$|f|_{\mathcal{L}^s_{q(p,N)}} \le c|f|_{\mathcal{L}^s_{q(p,N+1)}} \le \cdots \le c|f|_{\mathcal{L}^s_{q(p,N')}}.$$

This completes the proof of the lemma. ∎

Remark 3.8. For all $f \in \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n)$, $1 \le p < +\infty$, $1 \le q \le +\infty$, $s \in [-n/p, N+1)$, the condition (3.23) is fulfilled, and therefore (3.24) holds for all $f \in \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n)$ under the assumptions on p, q, s, N and N' of Lemma 3.7. To verify this fact, we observe, for $f \in \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n)$, that

$$\sup_{m\in\mathbb{Z}} 2^{-Nm}\underset{p,N}{\operatorname{osc}}(f,0,2^m) \le |f|_{\mathcal{L}^s_{q(p,N)}}.$$

Then, for $L \in \mathbb{N}$, $L > N$, we estimate for the multi-index α, with $|\alpha| = L$,

$$\begin{aligned}
|D^\alpha \dot{P}^L_{0,2^m}(f)| &= |D^\alpha \dot{P}^L_{0,2^m}((f-P^N_{0,2^m}))| \le c2^{-Lm}\underset{p,N}{\operatorname{osc}}(f,0,2^m) \\
&\le c2^{m(N-L)}|f|_{\mathcal{L}^s_{q(p,N)}} \to 0 \quad \text{as } m \to +\infty.
\end{aligned}$$

Hence, (3.23) is fulfilled.

Remark 3.9. For $q = \infty$, the space $\mathcal{L}^s_{\infty(p,N)}(\mathbb{R}^n)$ coincides with the usual Campanato space. Therefore, the statement of Lemma 3.7 is well known and its proof can be found in [11], p. 75.

A careful inspection of the proof of Lemma 3.7 gives the following.

Corollary 3.10. *Let $N, N' \in \mathbb{N}_0$, $N < N'$. Let $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ satisfy (3.23) with $k = 0$. Then, for all $x_0 \in \mathbb{R}^n$ and $j \in \mathbb{Z}$,*

$$\operatorname*{osc}_{p,N}(f; x_0, 2^j) \le c\big(S_{N+1,1}\big(\operatorname*{osc}_{p,N'}(f; x_0)\big)\big)_j. \tag{3.30}$$

Proof. Set $k = N' - N$. Using (3.29) with N' in place of N, we find

$$\operatorname*{osc}_{p,N'-1}(f; x_0, 2^j) \le c\big(S_{N',1}\big(\operatorname*{osc}_{p,N'}(f; x_0)\big)\big)_j, \quad j \in \mathbb{Z}.$$

Iterating this inequality k-times and applying Lemma 2.1, we arrive at

$$\begin{aligned}\operatorname*{osc}_{p,N}(f; x_0) = \operatorname*{osc}_{p,N'-k}(f; x_0) &\le c S_{N+1,1}\big(S_{N+2,1} \dots S_{N',1}\big(\operatorname*{osc}_{p,N'}(f; x_0)\big)\big)\\ &\le c S_{N+1,1}\big(\operatorname*{osc}_{p,N'}(f; x_0)\big).\end{aligned}$$

Whence, (3.30). ■

We have the following growth properties of functions in $\mathcal{L}^s_{q(p,N)}(\mathbb{R}^n)$ as $|x| \to +\infty$

Lemma 3.11. *Let $N \in \mathbb{N}_0$. Let also $f \in \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n)$, $1 \le q \le +\infty$, $1 \le p < +\infty$, $s \in [N, N+1)$.*

(1) *For $s \in (N, N+1)$,*

$$|f(x)| \le c(1 + |x|^s)\|f\|_{\mathcal{L}^s_{q(p,N)}} \quad \text{for all } x \in \mathbb{R}^n. \tag{3.31}$$

(2) *For $s = N$,*

$$|f(x)| \le c(1 + \log(1 + |x|)^{1/q'}|x|^N)\|f\|_{\mathcal{L}^N_{q(p,N)}} \quad \text{for all } x \in \mathbb{R}^n. \tag{3.32}$$

Here $q' = q/(q-1)$, $c = \text{const} > 0$, depending on q, p, s, N and n.

Proof. (1) Let $s \in (N, N+1)$. Let $x_0 \in \mathbb{R}^n$. Let $j \in \mathbb{N}_0$, such that $2^j \le 1 + |x_0| \le 2^{j+1}$. Let α be a multi-index with $|\alpha| = N$. Verifying that

$$D^\alpha f(x_0) = \lim_{i \to -\infty} D^\alpha \dot{P}^N_{x_0,2^i}(f)$$

and using the triangle inequality, we find

$$\begin{aligned}|D^\alpha f(x_0)| &\le \sum_{i=-\infty}^{j} |D^\alpha \dot{P}^N_{x_0,2^i}(f) - D^\alpha \dot{P}^N_{x_0,2^{i-1}}(f)| + |D^\alpha \dot{P}^N_{x_0,2^j}(f)|\\ &\le c \sum_{i=-\infty}^{j} 2^{-iN} \operatorname*{osc}_{p,N}(f; x_0, 2^i) + |D^\alpha \dot{P}^N_{x_0,2^j}(f)|.\end{aligned}$$

With the aid of Hölder's inequality, we find

$$\sum_{i=-\infty}^{j} 2^{-iN} \operatorname*{osc}_{p,N}(f;x_0,2^i) = \sum_{i=-\infty}^{j} 2^{-i(N-s)}2^{-is} \operatorname*{osc}_{p,N}(f;x_0,2^i) \le c2^{j(s-N)}|f|_{\mathcal{L}^s_{q(p,N)}} \le c(1+|x_0|^{s-N})|f|_{\mathcal{L}^s_{q(p,N)}}.$$

On the other hand,

$$\begin{aligned}|D^\alpha \dot{P}^N_{x_0,2^j}(f)| &= |D^\alpha \dot{P}^N_{x_0,2^j}(f - P^N_{0,2^{j+1}}(f)| \\ &\quad + |D^\alpha(P^N_{0,2^{j+1}}(f) - P^N_{0,1}(f))| + |D^\alpha P^N_{0,1}(f)| \\ &\le 2^{-jN-n/p}\|f - P^N_{0,2^{j+1}}(f)\|_{L^p(x_0,2^{j+1})} \\ &\quad + c\sum_{i=0}^{j} 2^{-i(N-s)}2^{-is} \operatorname*{osc}_{p,N}(f;0,2^i) + c\|f\|_{L^p(B(1))} \\ &\le \operatorname*{osc}_{p,N}(f;0,2^{j+1}) + c\sum_{i=0}^{j} 2^{-i(N-s)}2^{-is} \operatorname*{osc}_{p,N}(f;0,2^i) + c\|f\|_{L^p(B(1))} \\ &\le c(1+|x_0|^{s-N})\|f\|_{\mathcal{L}^s_{q(p,N)}}.\end{aligned}$$

Accordingly,

$$\|D^N f(x)\| \le c(1+|x|^{s-N})\|f\|_{\mathcal{L}^s_{q(p,N)}}.$$

This implies (3.31).

(2) We now consider the case $s = N$. Let $x_0 \in \mathbb{R}^n$. As above, we choose $j \in \mathbb{N}_0$ such that $2^j \le 1 + |x_0| < 2^{j+1}$.

In this case, we first claim

$$\|D^N \dot{P}^N_{x_0,1}(f)\| \le (\log(1+|x_0|))^{1/q'}\|f\|_{\mathcal{L}^N_{q(p,N)}}. \tag{3.33}$$

Indeed, arguing as above, using the triangle inequality along with Hölder's inequality, we get

$$\begin{aligned}\|D^N \dot{P}^N_{x_0,1}(f)\| &\le \sum_{i=1}^{j}\|D^N \dot{P}^N_{x_0,2^i}(f)\| - \|D^N \dot{P}^N_{x_0,2^{i-1}}(f)\| + \|D^N \dot{P}^N_{x_0,2^j}(f)\| \\ &\le \sum_{i=1}^{j} 2^{-Ni} \operatorname*{osc}_{p,N}(f;x_0,2^i) + \|D^N \dot{P}^N_{x_0,2^j}(f)\| \\ &\le \sum_{i=1}^{j+1} 2^{-Ni} \operatorname*{osc}_{p,N}(f;x_0,2^i) + \|D^N \dot{P}^N_{0,2^{j+1}}(f)\| \\ &\le cj^{1/q'}|f|_{\mathcal{L}^N_{q(p,N)}} + \|D^N \dot{P}^N_{0,2^{j+1}}(f)\|.\end{aligned}$$

Similarly,

$$\|D^N \dot{P}^N_{0,2^{j+1}}(f)\| \le cj^{1/q'}|f|_{\mathcal{L}^N_{q(p,N)}} + \|D^N \dot{P}^N_{0,1}(f)\|.$$

Combining the two inequalities we have just obtained, we get (3.33).

Let $i \in \mathbb{Z}$. Then, by the triangle inequality together with (3.33), we find

$$
\begin{aligned}
2^{-n/p-iN}\|\dot{P}^N_{x_0,2^i}(f)\|_{L^p(x_0,2^i)} &\le c\|D^N\dot{P}^N_{x_0,2^i}(f)\|\\
&\le c\sum_{l=i}^{1}\big(\|D^N\dot{P}^N_{x_0,2^l}(f)\| - \|D^N\dot{P}^N_{x_0,2^{l-1}}(f)\|\big) + c\|D^N\dot{P}^N_{x_0,1}(f)\|\\
&\le c\sum_{l=i}^{1}\big(\|D^N\dot{P}^N_{x_0,2^l}(f) - D^N\dot{P}^N_{x_0,2^{l-1}}(f)\|\big) + c\|D^N\dot{P}^N_{x_0,1}(f)\|\\
&\le c\sum_{l=i}^{1}2^{-Nl}\operatorname*{osc}_{p,N}(f;x_0,2^l) + c\|D^N\dot{P}^N_{x_0,1}(f)\|\\
&\le c|i|^{1/q'}|f|_{\mathcal{L}^N_{q(p,N)}} + c(\log(1+|x_0|))^{1/q'}\|f\|_{\mathcal{L}^N_{q(p,N)}}.
\end{aligned}
$$

This shows that

$$
\begin{aligned}
(3.34)\quad 2^{-i(N-1)}&\operatorname*{osc}_{p,N-1}(f;x_0,2^i)\\
&\le 2^{-i(N-1)}\operatorname*{osc}_{p,N}(f;x_0,2^i) + 2^{-n/p}\|\dot{P}^N_{x_0,2^i}(f)\|_{L^p(x_0,2^i)}\\
&\le 2^{-i(N-1)}\operatorname*{osc}_{p,N}(f;x_0,2^i) + c2^i\big(|i|^{1/q'} + (\log(1+|x_0|))^{1/q'}\big)\|f\|_{\mathcal{L}^N_{q(p,N)}}.
\end{aligned}
$$

Summing both sides from $i = -\infty$ to $i = 1$ and applying Hölder's inequality, we get

$$
(3.35)\qquad \sum_{i=-\infty}^{1}2^{-i(N-1)}\operatorname*{osc}_{p,N-1}(f;x_0,2^i) \le c\big(1 + (\log(1+|x_0|))^{1/q'}\big)\|f\|_{\mathcal{L}^N_{q(p,N)}}.
$$

Let α be a multi-index with $|\alpha| = N-1$. Since $D^\alpha f(x_0) = \lim_{i\to-\infty} D^\alpha\dot{P}^{N-1}_{x_0,2^i}(f)$, using the triangle inequality together with (3.35), we infer

$$
\begin{aligned}
|D^\alpha f(x_0)| &\le |D^\alpha\dot{P}^{N-1}_{x_0,2^i}(f)| + c\sum_{i=-\infty}^{1}|D^\alpha\dot{P}^{N-1}_{x_0,2^i}(f) - D^\alpha\dot{P}^{N-1}_{x_0,2^{i-1}}(f)|\\
&\le |D^\alpha\dot{P}^{N-1}_{x_0,2^i}(f)| + c\sum_{i=-\infty}^{1}2^{-(N-1)}\operatorname*{osc}_{p,N-1}(f;x_0,2^i)\\
&\le \|D^{N-1}\dot{P}^{N-1}_{x_0,2^i}(f)\| + c\big(1 + (\log(1+|x_0|))^{1/q'}\big)\|f\|_{\mathcal{L}^N_{q(p,N)}}.
\end{aligned}
$$

Arguing as above, using the triangle inequality and (3.34), we find

$$
\begin{aligned}
\|D^{N-1}\dot{P}^{N-1}_{x_0,2^i}(f)\| &\le c\sum_{i=0}^{j}2^{-(N-1)i}\operatorname*{osc}_{p,N-1}(f,x_0,2^i) + \|D^{N-1}\dot{P}^{N-1}_{x_0,2^j}(f)\|\\
&\le c\sum_{i=0}^{j}2^{-(N-1)i}\operatorname*{osc}_{p,N-1}(f,x_0,2^i) + \sum_{i=0}^{j+1}2^{-(N-1)i}\operatorname*{osc}_{p,N-1}(f,0,2^i) + \|D^{N-1}\dot{P}^{N-1}_{0,1}(f)\|\\
&\le c2^j j^{1/q'}\|f\|_{\mathcal{L}^N_{q(p,N)}} \le c(1+\log(1+|x_0|)^{1/q'}|x_0|)\|f\|_{\mathcal{L}^N_{q(p,N)}}.
\end{aligned}
$$

Combining the above inequalities, we obtain

$$|D^{N-1} f(x_0)| \le (1 + \log(1 + |x_0|)^{1/q'} |x_0|) \|f\|_{\mathcal{L}^N_{q(p,N)}}.$$

This yields (3.32). ∎

Using Poincaré's inequality and Lemma 3.7, we get the following embedding.

Lemma 3.12. *Let* $0 \le s < \infty$, $k \in \mathbb{N}_0$, $1 < p < +\infty$, $1 \le q \le +\infty$. *For* $N = [s]$, *the following hold*:

(1) *For* $q = \infty$ *and* $s \notin \mathbb{N}$,

$$\mathcal{L}^{k,s}_{\infty(p,N)}(\mathbb{R}^n) \cong C^{k+[s],s-[s]}(\mathbb{R}^n). \tag{3.36}$$

(2) *For* $q = \infty$ *and* s *an integer, i.e.,* $s = N$, *we have*

$$\mathcal{L}^{k,N}_{\infty(p,N)}(\mathbb{R}^n) \cong \mathrm{BMO}_{k+N}(\mathbb{R}^n), \tag{3.37}$$

where

$$\begin{aligned} \mathrm{BMO}_N &= \Big\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \;\Big|\; \sup_{x_0 \in \mathbb{R}^n} \sup_{j \in \mathbb{Z}} \big(2^{-Nj} \operatorname*{osc}_{1,N}(f; x_0, 2^j)\big) < +\infty \Big\}. \\ &= \{ f \in W^{N,1}_{\mathrm{loc}}(\mathbb{R}^n) \mid \nabla^N f \in \mathrm{BMO} \}. \end{aligned}$$

(3) *For* $1 \le q < \infty$,

$$\mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{k+s}_{q(p,N+k)}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{k,s}_{\infty(p,N)}(\mathbb{R}^n). \tag{3.38}$$

Proof. (1) For $k = 0$, note that the space $\mathcal{L}^s_{\infty(p,N)}(\mathbb{R}^n)$ coincides with the Campanato space $\mathcal{L}^{n+ps,p}_N(\mathbb{R}^n)$, where $N = [s]$, which is isomorphic to $C^{[s],s-[s]}(\mathbb{R}^n)$ (cf. Chapter III of [11] and [6]). For $k \ge 1$ and $f \in \mathcal{L}^{k,s}_{\infty(p,N)}(\mathbb{R}^n)$, we get $D^k f \in \mathcal{L}^s_{\infty(p,N)}(\mathbb{R}^n) \cong C^{[s],s-[s]}(\mathbb{R}^n)$, which shows (3.36).

(2) For $k = 0$ and $s = N$, the space $\mathcal{L}^s_{\infty(p,N)}(\mathbb{R}^n)$ coincides with the Campanato space $\mathcal{L}^{n+pN,p}_N(\mathbb{R}^n)$. According to Chapter III of [11], this space coincides with the space BMO_N. For $k \in \mathbb{N}$, we argue as above to verify (3.37).

(3) Let $\mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n)$. Using the Poincaré inequality (3.7) with $j = k$, we find that $\mathrm{osc}_{p,N+k}(f; x_0, 2^j) \le c 2^{jk} \mathrm{osc}_{p,N}(D^k f; x_0, 2^j)$. Accordingly,

$$\big\| \{2^{-(s+k)j} \operatorname*{osc}_{p,N+k}(f; x_0, 2^j)\}_{j \in \mathbb{Z}} \big\|_{\ell^q} \le c \big\| \{2^{-sj} \operatorname*{osc}_{p,N}(D^k f; x_0, 2^j)\}_{j \in \mathbb{Z}} \big\|_{\ell^q},$$

where

$$\operatorname*{osc}_{p,N}(f; x_0) = \big\{ \operatorname*{osc}_{p,N}(f; x_0, 2^j) \big\}_{j \in \mathbb{Z}}.$$

Taking the supremum over all $x_0 \in \mathbb{R}^n$ on both sides of the above estimate, we get the first embedding.

It remains to show the second embedding. We first notice that $\mathcal{L}^{k+s}_{q(p,N+k)}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{k+s}_{\infty(p,N+k)}(\mathbb{R}^n)$. Indeed,

$$2^{-(s+k)j} \operatorname*{osc}_{p,N+k}(f; x_0, 2^j) \le 2^{-(s+k)j} \big(S_{k+s,q}\big(\operatorname*{osc}_{p,N+k}(f; x_0)\big)\big)_j \le |f|_{\mathcal{L}^{k+s}_{q(p,N+k)}}.$$

Taking the supremum over all $j \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^n$, we get the embedding

$$\mathcal{L}^{k+s}_{q(p,N+k)}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{k+s}_{\infty(p,N+k)}(\mathbb{R}^n).$$

On the other hand, for $s \in (N, N+1)$, from (3.36), it follows that $\mathcal{L}^{k+s}_{\infty(p,N+k)}(\mathbb{R}^n) \cong C^{k+N,s-N}(\mathbb{R}^n) \cong \mathcal{L}^{k,s}_{\infty(p,N)}(\mathbb{R}^n)$. For $s = N$, using (3.37), we also get $\mathcal{L}^{k+N}_{\infty(p,N+k)}(\mathbb{R}^n) \cong \mathcal{L}^{k,N}_{\infty(p,N)}(\mathbb{R}^n)$. This shows the desired embedding. ∎

Using the Gagliardo–Nirenberg inequalities, we can get the interpolation properties. First let us recall the Gagliardo–Nirenberg inequalities.

Lemma 3.13. *Let $j, N \in \mathbb{N}_0, 0 \le j < k$. Let $1 \le p, p_0, p_1 \le +\infty$, and $\theta \in [j/N, 1]$, satisfying*

$$\frac{1}{p} = \frac{j}{n} + \frac{1-\theta}{p_0} + \Big(\frac{1}{p_1} - \frac{k}{n}\Big)\theta.$$

Then, for all $f \in L^{p_0}(B(1)) \cap W^{k,p_1}(B(1))$,

$$\|D^j f\|_{L^p(B(1))} \le c\|f\|^{1-\theta}_{L^{p_0}(B(1))}\|f\|^{\theta}_{W^{k,p_1}(B(1))}.$$

Notice that, using the generalized Poincaré inequality, under the assumption of Lemma 3.13, for all $f \in L^{p_0}(B(1)) \cap W^{k,p_1}(B(1))$, and $N \in \mathbb{N}_0$, $N \ge k-1$, the following inequality holds:

$$\begin{aligned} &\|D^j(f - P^N_{0,1}(f))\|_{L^p(B(1))} \\ &\quad \le c\|f - P^N_{0,1}(f))\|^{1-\theta}_{L^{p_0}(B(1))}\|D^k f - D^k P^N_{0,1}(f)\|^{\theta}_{L^{p_1}(B(1))}. \end{aligned} \tag{3.39}$$

By a standard scaling and translation argument, we deduce from (3.39) that for all $x_0 \in \mathbb{R}^n$, $0 < r < +\infty$, $N \in \mathbb{N}_0$, $N \ge k-1$, and for all $f \in L^{p_0}(B(x_0, r)) \cap W^{k,p_1}(B(x_0, r))$, the following inequality holds:

$$\begin{aligned} \|D^j f - P^{N-j}_{x_0,r}(D^j f)\|_{L^p(B(x_0,r))} &= \|D^j(f - P^N_{x_0,r}(f))\|_{L^p(B(x_0,r))} \\ &\le c\|f - P^N_{x_0,r}(f))\|^{1-\theta}_{L^{p_0}(B(x_0,r))}\|D^k f - P^{N-k}_{x_0,r}(D^k f)\|^{\theta}_{L^{p_1}(B(x_0,r))}. \end{aligned} \tag{3.40}$$

Theorem 3.14. *Let $j, k, N \in \mathbb{N}_0$, $0 \le j < k \le N+1$. Let $1 \le p, p_0, p_1 < +\infty$, $1 \le q, q_0, q_1 \le +\infty$, $-\infty < s, s_0, s_1 < N+1$, and $\theta \in [j/N, 1]$, satisfying*

$$\frac{1}{p} = \frac{j}{n} + \frac{1-\theta}{p_0} + \Big(\frac{1}{p_1} - \frac{k}{n}\Big)\theta, \tag{3.41}$$

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \tag{3.42}$$

$$s + j = (1-\theta)s_0 + \theta(s_1 + k). \tag{3.43}$$

Then, for all $\mathcal{L}^{s_0}_{q_0(p_0,N)}(\mathbb{R}^n) \cap \mathcal{L}^{k,s_1}_{q_1(p_1,N)}(\mathbb{R}^n)$,

$$\|f\|_{\mathcal{L}^{j,s}_{q(p,N-j)}} \le c\|f\|^{1-\theta}_{\mathcal{L}^{s_0}_{q_0(p_0,N)}} \|f\|^{\theta}_{\mathcal{L}^{k,s_1}_{q_1(p_1,N-k)}}. \tag{3.44}$$

Proof. Observing (3.41) and (3.42), thanks to (3.40), we find

$$\begin{aligned} 2^{-sl} \operatorname*{osc}_{p,N-j}(D^j f; x_0, 2^l) &\le c 2^{-ls_0(1-\theta)-ls_1\theta} \operatorname*{osc}_{p_0,N}(f; x_0, 2^l)^{1-\theta} \operatorname*{osc}_{p_1,N-k}(D^k f; x_0, 2^l)^{\theta} \\ &= c\big[2^{-ls_0} \operatorname*{osc}_{p_0,N}(f; x_0, 2^l)\big]^{1-\theta} \big[2^{-ls_1} \operatorname*{osc}_{p_1,N-k}(D^k f; x_0, 2^l)\big]^{\theta}. \end{aligned}$$

According to (3.43), we may apply the ℓ^q norm to both sides of the above inequality and use Hölder's inequality. This gives

$$\begin{aligned} &\Big(\sum_{l\in\mathbb{Z}} \big(2^{-sl} \operatorname*{osc}_{p,N-j}(D^j f; x_0, 2^l)\big)^q\Big)^{1/q} \\ &\quad \le c\Big(\sum_{l\in\mathbb{Z}} \big(2^{-ls_0} \operatorname*{osc}_{p_0,N}(f; x_0, 2^l)\big)^{q_0}\Big)^{(1-\theta)/q_0} \Big(\sum_{l\in\mathbb{Z}} \big(2^{-ls_1} \operatorname*{osc}_{p_1,N-k}(D^k f; x_0, 2^l)\big)^{q_1}\Big)^{\theta/q_1}. \end{aligned}$$

Taking the supremum over all $x_0 \in \mathbb{R}^n$, we get the assertion (3.44). ∎

Remark 3.15. Consider the special case

$$\begin{cases} N = k, \quad p = p_0 = p_1, \quad \theta = j/k, \quad s = s_0 = s_1 = 0, \\ 1 \le q < +\infty, \quad q_0 = +\infty, \quad q_1 = qk/j. \end{cases}$$

Then (3.44) reads

$$\|f\|_{\mathcal{L}^{j,0}_{q(p,k-j)}} \le c\|f\|^{1-j/k}_{\mathcal{L}^{0}_{\infty(p,k)}} \|f\|^{j/k}_{\mathcal{L}^{k,0}_{qk/j(p,0)}} \le c\|f\|^{1-j/k}_{\mathrm{BMO}} \|f\|^{j/k}_{\mathcal{L}^{k,0}_{q(p,0)}}.$$

Under the assumption

$$\lim_{m\to\infty} \dot{P}^L_{0,2^m}(D^j u) = 0 \quad \text{for all } L = 1, \dots, k-j,$$

we estimate the term on the left-hand side using (3.24) with $N = 0$ and $N' = k - j$. This yields

$$\|f\|_{\mathcal{L}^{j,0}_{q(p,0)}} \le c\|f\|^{1-j/k}_{\mathrm{BMO}} \|f\|^{j/k}_{\mathcal{L}^{k,0}_{q(p,0)}}.$$

We are now in a position to prove the following product estimate.

Theorem 3.16. *Let* $1 < p < +\infty$. *Let* $N \in \mathbb{N}_0$ *and* $s \in (-\infty, N+1)$. *Then, for all* $f, g \in \mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$,

$$\|fg\|_{\mathcal{L}^{k,s}_{q(p,N)}} \le c\big(\|f\|_{\infty}\|g\|_{\mathcal{L}^{k,s}_{q(p,N)}} + \|g\|_{\infty}\|f\|_{\mathcal{L}^{k,s}_{q(p,N)}}\big). \tag{3.45}$$

Proof. Let $\alpha, \beta \in N^n_0$ two multi-index both are not zero with $|\alpha + \beta| = k$. Set $|\alpha| = j$.

Using the triangle inequality, we see that

$$
\begin{aligned}
(3.46)\qquad & \|D^\alpha f D^\beta g - P^{N+k-j}_{x_0,r}(D^\alpha f) P^{N+j}_{x_0,r}(D^\beta g)\|_{L^p(B(x_0,r))} \\
&\le c\|(D^\alpha f - P^{N+k-j}_{x_0,r}(D^\alpha f))(D^\beta g - P^{N+j}_{x_0,r}(D^\beta g))\|_{L^p(B(x_0,r))} \\
&\quad + c\|(D^\alpha f - P^{N+k-j}_{x_0,r}(D^\alpha f)) P^{N+j}_{x_0,r}(D^\beta g)\|_{L^p(B(x_0,r))} \\
&\quad + c\|P^{N+k-j}_{x_0,r}(D^\alpha f))(D^\beta g - P^{N+j}_{x_0,r}(D^\beta g))\|_{L^p(B(x_0,r))} \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}
$$

Using Hölder's inequality together with the Gagliardo–Nirenberg inequality (3.40), we estimate

$$
\begin{aligned}
\mathrm{I} &\le c\|D^\alpha f - P^{N+k-j}_{x_0,r}(D^\alpha f)\|_{L^{kp/j}(B(x_0,r))}\|D^\beta g - P^{N+j}_{x_0,r}(D^\beta g)\|_{L^{kp/(k-j)}(B(x_0,r))} \\
&= c\|D^\alpha f - D^\alpha P^{N+k}_{x_0,r}(f)\|_{L^{kp/j}(B(x_0,r))}\|D^\beta g - D^\beta P^{N+k}_{x_0,r}(g)\|_{L^{kp/(k-j)}(B(x_0,r))} \\
&\le c\|D^j(f - P^{N+k}_{x_0,r}(f))\|_{L^{kp/j}(B(x_0,r))}\|D^{k-j}(g - P^{N+k}_{x_0,r}(g))\|_{L^{kp/(k-j)}(B(x_0,r))} \\
&\le c\|f - P^{N+k}_{x_0,r}(f)\|^{1-j/k}_{L^\infty(B(x_0,r))}\|D^k(f - P^{N+k}_{x_0,r}(f))\|^{j/k}_{L^p(B(x_0,r))} \\
&\qquad \times c\|g - P^{N+k}_{x_0,r}(g)\|^{j/k}_{L^\infty(B(x_0,r))}\|D^k(g - P^{N+k}_{x_0,r}(g))\|^{1-j/k}_{L^p(B(x_0,r))} \\
&\le c\|f\|^{1-j/k}_{L^\infty(B(x_0,r))}\|D^k f - P^{N}_{x_0,r}(D^k f))\|^{j/k}_{L^p(B(x_0,r))} \\
&\qquad \times \|g\|^{j/k}_{L^\infty(B(x_0,r))}\|D^k g - P^{N}_{x_0,r}(D^k g))\|^{1-j/k}_{L^p(B(x_0,r))}.
\end{aligned}
$$

Applying Young's inequality, we obtain

$$
\begin{aligned}
\mathrm{I} &\le c\|f\|_{L^\infty(B(x_0,r))}\|D^k g - P^{N}_{x_0,r}(D^k g)\|_{L^p(B(x_0,r))} \\
&\quad + c\|g\|_{L^\infty(B(x_0,r))}\|D^k f - P^{N}_{x_0,r}(D^k f)\|_{L^p(B(x_0,r))}.
\end{aligned}
$$

In order to estimate II, we make use of the inequality

$$
\|P^{N+j}_{x_0,r}(D^\beta g)\|_{L^\infty(B(x_0,r))} \le c r^{-(k-j)}\|g\|_{L^\infty(B(x_0,r))},
$$

which can be proved by a standard scaling argument. Together with Poincaré's inequality, we find

$$
\begin{aligned}
\mathrm{II} &\le c r^{k-j}\|D^{k-j}(D^\alpha f - P^{N+k-j}_{x_0,r}(D^\alpha f))\|_{L^p(B(x_0,r))} r^{-(k-j)}\|g\|_{L^\infty(B(x_0,r))} \\
&\le c\|g\|_{L^\infty(B(x_0,r))}\|D^k f - P^{N}_{x_0,r}(D^k f)\|_{L^p(B(x_0,r))}.
\end{aligned}
$$

By an analogous reasoning, we get

$$
\mathrm{III} \le c\|f\|_{L^\infty(B(x_0,r))}\|D^k g - P^{N}_{x_0,r}(D^k g)\|_{L^p(B(x_0,r))}.
$$

Inserting the estimates of I, II and III into the right-hand side of (3.46), we arrive at

$$
\begin{aligned}
(3.47)\qquad & \|D^\alpha f D^\beta g - P^{N+k-j}_{x_0,r}(D^\alpha f) P^{N+j}_{x_0,r}(D^\beta g)\|_{L^p(B(x_0,r))} \\
&\le c\|f\|_{L^\infty(B(x_0,r))}\|D^k g - P^{N}_{x_0,r}(D^k g)\|_{L^p(B(x_0,r))} \\
&\quad + c\|g\|_{L^\infty(B(x_0,r))}\|D^k f - P^{N}_{x_0,r}(D^k f)\|_{L^p(B(x_0,r))}.
\end{aligned}
$$

Let $\gamma \in \mathbb{N}_0$ be a multi-index with $|\gamma| = k$. Using the Leibniz formula, we compute

$$D^\gamma(fg) = \sum_{\alpha+\beta=\gamma} \Big(\frac{\gamma!}{\alpha!\beta!}\Big) D^\alpha f\, D^\beta g.$$

Thus, employing Corollary 3.4, and using the triangle inequality together with (3.47), we obtain

$$\begin{aligned}
&\|D^\gamma(fg) - P^{2N+k}_{x_0,r}(D^\gamma(fg))\|_{L^p(B(x_0,r))}\\
&\le c \inf_{Q\in\mathcal{P}_{2N+k}} \|D^\gamma(fg) - Q\|_{L^p(B(x_0,r))}\\
&\le c \Big\| D^\gamma(fg) - \sum_{\alpha+\beta=\gamma} \Big(\frac{\gamma!}{\alpha!\beta!}\Big) P^{N+k-j}_{x_0,r}(D^\alpha f) P^{N+j}_{x_0,r}(D^\beta g)\Big\|_{L^p(B(x_0,r))}\\
&= c \Big\| \sum_{\alpha+\beta=\gamma} \Big(\frac{\gamma!}{\alpha!\beta!}\Big)(D^\alpha f D^\beta g - P^{N+k-j}_{x_0,r}(D^\alpha f) P^{N+j}_{x_0,r}(D^\beta g))\Big\|_{L^p(B(x_0,r))}\\
&\le c\|f\|_\infty \|D^k g - P^N_{x_0,r}(D^k g)\|_{L^p(B(x_0,r))} + c\|g\|_\infty \|D^k f - P^N_{x_0,r}(D^k f)\|_{L^p(B(x_0,r))}.
\end{aligned}$$

This yields the product estimate

$$\operatorname*{osc}_{p,2N+k}(D^k(fg); x_0, r) \le c\|f\|_\infty \operatorname*{osc}_{p,N}(D^k g; x_0, r) + c\|g\|_\infty \operatorname*{osc}_{p,N}(D^k f; x_0, r). \tag{3.48}$$

In (3.48), we insert $r = 2^j$, $j \in \mathbb{Z}$, and multiply this by 2^{-sj}. Then, applying the ℓ^q norm to both sides of (3.48), we are led to

$$\|fg\|_{\mathcal{L}^{k,s}_{q(p,2N+k)}} \le c\big(\|f\|_\infty \|g\|_{\mathcal{L}^{k,s}_{q(p,N)}} + \|g\|_\infty \|f\|_{\mathcal{L}^{k,s}_{q(p,N)}}\big).$$

Verifying (3.23) for $N' = 2N + k$, we are in a position to apply Lemma 3.7 with $N' = 2N + k$. This gives (3.45). ∎

4. Proof of the main theorems

We start with the following energy identity for solutions to the transport equation. Let $1 < p < +\infty$, $x_0 \in \mathbb{R}$ and $0 < r < +\infty$. We denote $\varphi_{x_0,r} = \varphi(r^{-1}(x_0 - \cdot))$. We define the following minimal polynomial $P^{N,*}_{x_0,r}(f)$, $f \in L^p(B(x_0, r))$, by

$$\|(f - P^{N,*}_{x_0,r}(f))\varphi_{x_0,r}\|_p = \min_{Q\in\mathcal{P}_N} \|(f - Q)\varphi_{x_0,r}\|_p. \tag{4.1}$$

The existence and uniqueness of such polynomial is shown in the appendix.

We recall the notation $\varphi_{x_0,r} = r^{-n}\varphi(r^{-1}(x_0 - \cdot))$. We have the following.

Lemma 4.1. *Given $v \in L^1(0, T; C^{0,1}(\mathbb{R}^n; \mathbb{R}^n))$ and $g \in L^1(0, T; L^p_{\mathrm{loc}}(\mathbb{R}^n))$, let $f \in L^\infty(0, T; C^{0,1}(\mathbb{R}^n)) \cap C([0, T]; L^p_{\mathrm{loc}}(\mathbb{R}^n))$ be a weak solution to the transport equation*

$$\partial_t f + (v \cdot \nabla) f = g \quad \text{in } Q_T. \tag{4.2}$$

Let $N \in \mathbb{N}_0$. *Define,*

$$L = \begin{cases} 2N-1 & \textit{if } N \geq 1, \\ 0 & \textit{if } N = 0. \end{cases}$$

Then, for all $t \in [0,T]$,

$$(4.3)\quad e(t) = e(0) + \int_0^t v \cdot \nabla \varphi_{x_0,r} |f - P^{L,*}_{x_0,r}(f)|^p \varphi^{p-1}_{x_0,r} e(\tau)^{1-p}\, dx\, d\tau$$

$$+ \frac{1}{p}\int_0^t \int_{B(x_0,r)} \nabla \cdot v |f - P^{L,*}_{x_0,r}(f)|^p \varphi^p_{x_0,r} e(\tau)^{1-p}\, dx\, d\tau$$

$$+ \int_0^t \int_{B(x_0,r)} v \cdot \nabla P^{L,*}_{x_0,r}(f(\tau)) \cdot |f - P^{L,*}_{x_0,r}(f(\tau))|^{p-2} (f - P^{L,*}_{x_0,r}(f)) \varphi^p_{x_0,r} e(\tau)^{1-p}\, dx\, d\tau$$

$$+ \int_0^t \int_{B(x_0,r)} (g - P^N_{x_0,r}(g)) |f - P^{L,*}_{x_0,r}(f)|^{p-2}(f - P^{L,*}_{x_0,r}(f)) \varphi^p_{x_0,r} e(\tau)^{1-p}\, dx\, d\tau$$

$$= e(0) + \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},$$

where

$$e(\tau) = \|(f(\tau) - P^{L,*}_{x_0,r}(f(\tau)))\varphi_{x_0,r}\|_p, \quad \tau \in [0,T].$$

In addition, the following inequality holds for all $t \in [0,T]$:

$$(4.4)\quad \operatorname*{osc}_{p,L}\Big(f(t); x_0, \frac{r}{2}\Big) \leq c \operatorname*{osc}_{p,L}(f(0); x_0, r)$$

$$+ c r^{-1} \int_0^t \|v(\tau)\|_{L^\infty(B(x_0,r))} \operatorname*{osc}_{p,N}(f(\tau); x_0, 2r)\, d\tau$$

$$+ c \int_0^t \|\nabla \cdot v(\tau)\|_{L^\infty(B(x_0,r))} \operatorname*{osc}_{p,N}(f(\tau); x_0, 2r)\, d\tau$$

$$+ \delta_{N0}\, c \int_0^t \operatorname*{osc}_{p,N}(v(\tau); x_0, r) \|\nabla P^N_{x_0,r}(f(\tau))\|_{L^\infty(B(x_0,r))}\, d\tau$$

$$+ c \int_0^t \operatorname*{osc}_{p,N}(g(\tau); x_0, r)\, d\tau,$$

where $\delta_{N0} = 0$ *if* $N = 0$ *and* 1 *otherwise.*

Proof. Let $x_0 \in \mathbb{R}^n$, $0 < r < +\infty$ be fixed. For $\delta \geq 0$, we define

$$F_\delta(z) = (\delta + |z|^2)^{(p-2)/2} z, \quad z \in \mathbb{R}^n.$$

Let $N \in \mathbb{N}_0$. Set $L = 0$ if $N = 0$ and $L = 2N-1$ if $L \geq 1$. For $\delta > 0$, by $P^{L,\delta}_{x_0,r}(f(\tau)) \in \mathscr{P}_L$, $0 \leq \tau \leq T$, we denote the minimal polynomial, defined in Appendix A (cf. Lemma A.1), such that

$$(4.5)\quad \int_{B(x_0,r)} F_\delta(f(\tau) - P^{L,\delta}_{x_0,r}(f(\tau)) \cdot Q \varphi^p_{x_0,r}\, dx = 0 \quad \text{for all } \tau \in [0,T] \text{ and all } Q \in \mathscr{P}_L.$$

Furthermore, for all $\tau \in [0,T]$,

$$(4.6)\qquad P^{L,\delta}_{x_0,r}(f(\tau)) \to P^{L,*}_{x_0,r}(f(\tau)) \quad \text{in } L^p(B(x_0,r)) \text{ as } \delta \searrow 0.$$

According to (A.6), the function $s \mapsto P^{L,\delta}_{x_0,r}(f(s))$ is differentiable for $\delta > 0$, and from (4.2), we get

$$\partial_t(f - P^{L,\delta}_{x_0,r}(f)) + (v\cdot\nabla)(f - P^{L,\delta}_{x_0,r}(f)) + (v\cdot\nabla)P^{L,\delta}_{x_0,r}(f) = g - \partial_t P^{L,\delta}_{x_0,r}(f) \quad \text{in } Q_T. \tag{4.7}$$

First, let us verify that $\partial_t P^{L,\delta}_{x_0,r}(f(\tau)) \in \mathcal{P}_L$ for all $\tau \in [0,T]$. In fact, for any multi-index $\alpha \in \mathbb{N}_0$ with $|\alpha| = L+1$, recalling that $P^{L,\delta}_{x_0,r}(f) \in \mathcal{P}_L$, we get $D^\alpha \partial_t P^{L,\delta}_{x_0,r}(f) = \partial_t D^\alpha P^{L,\delta}_{x_0,r}(f) = 0$. This shows the claim.

We multiply (4.7) by $F_\delta(f(\tau) - P^{L,\delta}_{x_0,r}(f(\tau))\varphi^p_{x_0,r}$, integrate over $B(x_0,r)$ and apply integration by parts. This, together with (4.5), yields

$$\begin{aligned}
&\partial_t \big\|(\delta + |f(\tau) - P^{L,\delta}_{x_0,r}(f(\tau))|^2)^{1/2}\varphi_{x_0,r}\big\|_p \big\|(\delta + |f(\tau) - P^{L,\delta}_{x_0,r}(f(\tau))|^2)^{1/2}\varphi_{x_0,r}\big\|_p^{p-1}\\
&= \frac{1}{p}\partial_t \big\|(\delta + |f(\tau) - P^{L,\delta}_{x_0,r}(f(\tau))|^2)^{1/2}\varphi_{x_0,r}\big\|_p^p\\
&= \int_{B(x_0,r)} v(\tau)\cdot\nabla\varphi_{x_0,r}(\delta + |f(\tau) - P^{L,\delta}_{x_0,r}(f(\tau))|^2)^{p/2}\varphi^{p-1}_{x_0,r}\,dx\\
&\quad + \frac{1}{p}\int_{B(x_0,r)} \nabla\cdot v(\tau)(\delta + |f(\tau) - P^{L,\delta}_{x_0,r}(f(\tau))|^2)^{p/2}\varphi^{p}_{x_0,r}\,dx\\
&\quad + \int_{B(x_0,r)} v(\tau)\cdot\nabla P^{L,\delta}_{x_0,r}(f(\tau))F_\delta(f(\tau) - P^{L,\delta}_{x_0,r}(f(\tau)))\varphi^{p}_{x_0,r}\,dx\\
&\quad + \int_{B(x_0,r)} (g(\tau) - P^{N}_{x_0,r}(g(\tau)))F_\delta(f(\tau) - P^{L,\delta}_{x_0,r}(f(\tau)))\varphi^{p}_{x_0,r}\,dx.
\end{aligned}$$

In the last line we used identity (4.5) for $Q = P^N_{x_0,r}(g(\tau))$.

Multiplying both sides of the above identity by $e_\delta(\tau)^{1-p}$, where

$$e_\delta(\tau) := \big\|(\delta + |f(\tau) - P^{N,\delta}_{x_0,r}(f(\tau))|^2)^{1/2}\varphi_{x_0,r}\big\|_p,$$

integrating the result over $(0,t)$, $t \in [0,T]$, with respect to τ, and applying integration by parts, we find

$$\begin{aligned}
e_\delta(t) &= e_\delta(0)\\
&\quad + \int_0^t\int_{B(x_0,r)} v(\tau)\cdot\nabla\varphi_{x_0,r}(\delta + |f(\tau) - P^{L,\delta}_{x_0,r}(f(\tau))|^2)^{p/2}\varphi^{p-1}_{x_0,r}\,e_\delta(\tau)^{1-p}\,dx\,d\tau\\
&\quad + \frac{1}{p}\int_0^t\int_{B(x_0,r)} \nabla\cdot v(\tau)(\delta + |f(\tau) - P^{L,\delta}_{x_0,r}(f(\tau))|^2)^{p/2}\varphi^{p}_{x_0,r}\,e_\delta(\tau)^{1-p}\,dx\,d\tau\\
&\quad + \int_0^t\int_{B(x_0,r)} v(\tau)\cdot\nabla P^{L,\delta}_{x_0,r}(f(\tau))F_\delta(f(\tau) - P^{L,\delta}_{x_0,r}(f(\tau)))\varphi^{p}_{x_0,r}\,e_\delta(\tau)^{1-p}\,dx\,d\tau\\
&\quad + \int_0^t\int_{B(x_0,r)} (g(\tau) - P^{N}_{x_0,r}(g(\tau)))F_\delta(f(\tau) - P^{L,\delta}_{x_0,r}(f(\tau)))\varphi^{p}_{x_0,r}e_\delta(\tau)^{1-p}\,dx\,d\tau.
\end{aligned}$$

In the above identity, letting $\delta \to 0$ and making use of (4.6), we obtain (4.4).

(2) Using the the triangle inequality, we estimate

$$\begin{aligned}
\mathrm{I} &\le c\int_0^t \|\nabla\varphi_{x_0,r}\cdot v(\tau)\|_\infty \|f(\tau)-P^{L,*}_{x_0,r}(f(\tau))\|_{L^p(B(x_0,r))} e(\tau)^{p-1}e(\tau)^{1-p}\,d\tau\\
&\le c\int_0^t \|\nabla\varphi_{x_0,r}\cdot v(\tau)\|_\infty \|f(\tau)-P^{L,*}_{x_0,r}(f(\tau))\|_{L^p(B(x_0,r))}\,d\tau\\
&\le c\int_0^t \|\nabla\varphi_{x_0,r}\cdot v(\tau)\|_\infty \|(f(\tau)-P^{L,*}_{x_0,2r}(f(\tau)))\varphi_{x_0,2r}\|_p\,d\tau\\
&\quad + c\int_0^t \|\nabla\varphi_{x_0,r}\cdot v(\tau)\|_\infty \|P^{L,*}_{x_0,r}(f(\tau))-P^{L,*}_{x_0,2r}(f(\tau))\|_{L^p(B(x_0,r))}\,d\tau = \mathrm{I}_1+\mathrm{I}_2.
\end{aligned}$$

Thanks to the minimizing property (4.1), we get

$$\mathrm{I}_1 \le cr^{-1}\int_0^t \|v(\tau)\|_{L^\infty(B(x_0,r))}\|f(\tau)-P^L_{x_0,2r}(f(\tau))\|_{L^p(B(x_0,2r))}\,d\tau.$$

On the other hand, for estimating I_2, making use of (A.9), we see that for all $\tau\in[0,T]$,

$$P^{L,*}_{x_0,r}(f(\tau))-P^{L,*}_{x_0,2r}(f(\tau)) = P^{L,*}_{x_0,r}(f(\tau)-P^L_{x_0,2r}(f(\tau)))-P^{L,*}_{x_0,2r}(f(\tau)-P^L_{x_0,2r}f(\tau)).$$

This, together with (A.7) and (A.1), yields

$$\begin{aligned}
&\|P^{L,*}_{x_0,r}(f(\tau))-P^{L,*}_{x_0,2r}(f(\tau))\|_{L^p(B(x_0,r))}\\
&\le \|P^{L,*}_{x_0,r}(f(\tau)-P^L_{x_0,2r}f(\tau))\|_{L^p(B(x_0,r))}+\|P^{N,*}_{x_0,2r}(f(\tau)-P^L_{x_0,2r}(f(\tau)))\|_{L^p(B(x_0,r))}\\
&\le c\|f(\tau)-P^L_{x_0,2r}(f(\tau))\|_{L^p(B(x_0,2r))}.
\end{aligned}$$

Consequently, I_2 has the same estimate as I_1, which gives

$$\mathrm{I} \le cr^{-1}\int_0^t \|v(\tau)\|_{L^\infty(B(x_0,r))}\|f(\tau)-P^L_{x_0,2r}(f(\tau))\|_{L^p(B(x_0,2r))}\,d\tau.$$

Using (A.1), we immediately get

$$\begin{aligned}
\mathrm{II} &\le c\int_0^t \|\nabla\cdot v(\tau)\|_{L^\infty(B(x_0,r))}\|(f(\tau)-P^{L,*}_{x_0,r}(f(\tau)))\varphi_{x_0,r}\|_{L^p(B(x_0,r))}\,d\tau\\
&\le c\int_0^t \|\nabla\cdot v(\tau)\|_{L^\infty(B(x_0,r))}\|f(\tau)-P^L_{x_0,2r}(f(\tau))\|_{L^p(B(x_0,2r))}\,d\tau.
\end{aligned}$$

We proceed with the estimation of III. Clearly, for $N=0$, since $P^{L,*}_{x_0,r}(f(\tau))=\mathrm{const}$ for all $\tau\in[0,T]$, the integral III vanishes. Thus, only the case $N>0$ remains. Let $\tau\in[0,T]$ be fixed. Making use of (4.5) with $\delta=0$, we find

$$\begin{aligned}
&\int_{B(x_0,r)} v(\tau)\cdot\nabla P^{L,*}_{x_0,r}(f(\tau))\cdot|f(\tau)-P^{L,*}_{x_0,r}(f(\tau))|^{p-2}(f(\tau)-P^{L,*}_{x_0,r}(f(\tau)))\varphi^p_{x_0,r}\,dx\\
&=\int_{B(x_0,r)} v(\tau)\cdot\nabla(P^{L,*}_{x_0,r}(f(\tau))-P^N_{x_0,r}(f(\tau)))\cdot F_0(f(\tau)-P^{L,*}_{x_0,r}(f(\tau)))\varphi^p_{x_0,r}\,dx\\
&\quad+\int_{B(x_0,r)} v(\tau)\cdot\nabla P^N_{x_0,r}(f(\tau))\cdot F_0(f(\tau)-P^{L,*}_{x_0,r}(f(\tau)))\varphi^p_{x_0,r}\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{B(x_0,r)} v(\tau)\cdot\nabla(P^{L,*}_{x_0,r}(f(\tau)) - P^N_{x_0,r}(f(\tau)))\cdot F_0(f(\tau) - P^{L,*}_{x_0,r}(f(\tau)))\varphi^p_{x_0,r}\,dx\\
&\quad+ \int_{B(x_0,r)} (v(\tau) - P^N_{x_0,r}(v(\tau)))\cdot\nabla P^N_{x_0,r}(f(\tau))\cdot F_0(f(\tau) - P^{L,*}_{x_0,r}(f(\tau)))\varphi^p_{x_0,r}\,dx\\
&= J_1 + J_2.
\end{aligned}
$$

Using the fact that $P^{L,*}_{x_0,r}(Q) = P^N_{x_0,r}(Q) = Q$ for all $Q \in \mathcal{P}_N$, for $Q = P^N_{x_0,r}(f(\tau))$ and for all $\tau \in (0,t)$, we get

$$
\|\nabla(P^{L,*}_{x_0,r}(f(\tau)) - P^N_{x_0,r}(f(\tau))\|_{L^p(B(x_0,r))} \le cr^{-1}\|f(\tau) - P^N_{x_0,r}(f(\tau))\|_{L^p(B(x_0,r))}.
$$

Then Hölder's inequality yields

$$
J_1 \le cr^{-1}\|v(\tau)\|_{L^\infty(B(x_0,r))}\|f(\tau) - P^N_{x_0,r}(f(\tau))\|_{L^p(B(x_0,r))}e(\tau)^{p-1}.
$$

Similarly,

$$
J_2 \le c\|v(\tau) - P^N_{x_0,r}(v(\tau))\|_{L^p(B(x_0,r))}\|\nabla P^N_{x_0,r}(f(\tau))\|_{L^\infty(B(x_0,r))}e(\tau)^{p-1}.
$$

Inserting the estimates of J_1 and J_2 into the integral III, we obtain

$$
\begin{aligned}
\mathrm{III} &\le cr^{-1}\int_0^t \|v(\tau)\|_{L^\infty(B(x_0,r))}\|f(\tau) - P^N_{x_0,2r}(f(\tau))\|_{L^p(B(x_0,2r))}\,d\tau\\
&\quad+ c\int_0^t \|v(\tau) - P^N_{x_0,r}(v(\tau))\|_{L^p(B(x_0,r))}\|\nabla P^N_{x_0,r}(f(\tau))\|_{L^\infty(B(x_0,r))}\,d\tau.
\end{aligned}
$$

To estimate IV, we use Hölder's inequality. This leads to

$$
\mathrm{IV} \le \int_0^t \|g(\tau) - P^N_{x_0,r}(g(\tau))\|_{L^p(B(x_0,r))}\,d\tau.
$$

Inserting the estimates of I, II, III and IV into the right-hand side of (4.3), we find

$$
\begin{aligned}
e(t) &\le e(0) + cr^{-1}\int_0^t \|v(\tau)\|_{L^\infty(B(x_0,r))}\|f(\tau) - P^N_{x_0,2r}(f(\tau))\|_{L^p(B(x_0,2r))}\,d\tau\\
&\quad+ c\int_0^t \|\nabla\cdot v(\tau)\|_{L^\infty(B(x_0,r))}\|f(\tau) - P^N_{x_0,2r}(f(\tau))\|_{L^p(B(x_0,2r))}\,d\tau\\
&\quad+ c\int_0^t \|v(\tau) - P^N_{x_0,r}(v(\tau))\|_{L^p(B(x_0,r))}\|\nabla P^N_{x_0,r}(f(\tau))\|_{L^\infty(B(x_0,r))}\,d\tau\\
&\quad+ c\int_0^t \|g(\tau) - P^N_{x_0,r}(g(\tau))\|_{L^p(B(x_0,r))}\,d\tau.
\end{aligned} \tag{4.8}
$$

Noting that

$$
\|f(t) - P^L_{x_0,r/2}(f(t))\|_{L^p(B(x_0,r/2))} \le c\|(f(t) - P^{L,*}_{x_0,r}(f(t)))\varphi_{x_0,r}\|_p = ce(t),
$$

and using (A.1), recalling that $L = 2N-1$, inequality (4.4) follows from (4.8). ■

Remark 4.2. Given $v \in L^1(0,T;C^{0,1}(\mathbb{R}^n;\mathbb{R}^n))$, and $\pi \in L^1(0,T;W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n))$, let $f \in L^\infty(0,T;C^{0,1}(\mathbb{R}^n;\mathbb{R}^n))$ with $\nabla\cdot f = 0$ be a weak solution to the system

$$\partial_t f + (v\cdot\nabla) f = -\nabla\pi \quad \text{in } Q_T.$$

Then, repeating the proof of Lemma 4.1 for the case $p = 2$ and $N = 1$ in the vector valued case, we find

$$\begin{aligned}
e(t) &= e(0) + \int_0^t v\cdot\nabla\varphi_{x_0,r}|f - P^{1,*}_{x_0,r}(f)|^2\varphi_{x_0,r}\,e(\tau)^{-1}\,dx\,d\tau \\
&\quad + \frac{1}{2}\int_0^t\int_{B(x_0,r)} \nabla\cdot v|f - P^{1,*}_{x_0,r}(f)|^2\varphi^2_{x_0,r}\,e(\tau)^{-1}\,dx\,d\tau \\
&\quad + \int_0^t\int_{B(x_0,r)} v\cdot\nabla P^{1,*}_{x_0,r}(f)\cdot(f - P^{1,*}_{x_0,r}(f))\varphi^2_{x_0,r}\,e(\tau)^{-1}\,dx\,d\tau \\
&\quad + \int_0^t\int_{B(x_0,r)} (\nabla\pi - P^1_{x_0,r}(\nabla\pi))(f - P^{1,*}_{x_0,r}(f))\varphi^2_{x_0,r}\,e(\tau)^{-1}\,dx\,d\tau \\
&= e(0) + \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},
\end{aligned} \tag{4.9}$$

where

$$e(\tau) = \|(f(\tau) - P^{1,*}_{x_0,r}(f(\tau)))\varphi_{x_0,r}\|_2, \quad \tau\in[0,T].$$

The integrals I, II and III can be estimated as in the proof of Lemma 4.1. For the estimation of IV, we proceed as follows.

Assume that the mollifier $\varphi \in C^\infty_c(B(1))$ is radial symmetric. Let $u \in L^1(B(x_0,r))$. It can be checked easily that the minimal polynomial $P^{1,*}_{x_0,r}(u)$ is given by

$$\begin{aligned}
P^{1,*}_{x_0,r}(u)(x) &= \frac{1}{\int_{\mathbb{R}^n}\varphi^2_{x_0,r}\,dy}\int_{\mathbb{R}^n} u\varphi^2_{x_0,r}\,dy \\
&\quad + \frac{n}{\int_{\mathbb{R}^n}\varphi^2_{x_0,r}|x_0 - y|^2\,dy}\int_{\mathbb{R}^n} u\varphi^2_{x_0,r}(y_i - x_{0,i})\,dy(x_i - x_{0,i}).
\end{aligned}$$

For $u = (u_1,\dots,u_n)$, with $\nabla\cdot u = 0$ almost everywhere in $B(x_0,r)$, recalling that φ is radially symmetric, by Gauss' theorem, we get

$$\nabla\cdot P^{1,*}_{x_0,r}(u)(x) = \frac{n}{\int_{\mathbb{R}^n}\varphi^2_{x_0,r}|x_0 - y|^2\,dy}\int_{B(x_0,r)} u\cdot(y - x_0)\varphi^2_{x_0,r}\,dy = 0.$$

Using integration by parts together with $\nabla\cdot P^{1,*}_{x_0,r}(f(\tau)) = 0$, and applying the Sobolev–Poincaré inequality, we get

$$\begin{aligned}
&\int_{B(x_0,r)} (\nabla\pi(\tau) - P^1_{x_0,r}(\nabla\pi(\tau)))(f(\tau) - P^{1,*}_{x_0,r}(f(\tau)))\varphi^2_{x_0,r}\,e(\tau)^{-1}\,dx \\
&\quad = -2\int_{B(x_0,r)} (\pi(\tau) - P^2_{x_0,r}(\pi(\tau)))(f(\tau) - P^{1,*}_{x_0,r}(f(\tau)))\varphi_{x_0,r}\cdot\nabla\varphi_{x_0,r}\,e(\tau)^{-1}\,dx \\
&\quad \le cr^{-1}\Big(\int_{B(x_0,r)} |\nabla\pi(\tau) - P^1_{x_0,r}(\nabla\pi(\tau))|^{2n/(n+2)}\,dx\Big)^{(n+2)/(2n)} \\
&\quad \le cr^{n/2}\operatorname*{osc}_{2n/(n+2),1}(\nabla\pi(\tau); x_0, r).
\end{aligned}$$

This yields

$$\mathrm{IV} \le c r^{n/2} \int_0^t \operatorname*{osc}_{2n/(n+2),1} (\nabla \pi(\tau); x_0, r)\, d\tau.$$

Inserting the estimates of I, II, III and IV into the right-hand side of (4.9), and arguing as in the proof of Lemma 4.1, we arrive at

$$\begin{aligned}\operatorname*{osc}_{2,1}\Big(f(t); x_0, \frac{r}{2}\Big) \le c \operatorname*{osc}_{2,1}(f(0); x_0, r) &+ c r^{-1} \int_0^t \|v(\tau)\|_{L^\infty(B(x_0,r))} \operatorname*{osc}_{2,1}(f(\tau); x_0, 2r)\, d\tau \\ &+ c \int_0^t \|\nabla \cdot v(\tau)\|_{L^\infty(B(x_0,r))} \operatorname*{osc}_{2,1}(f(\tau); x_0, 2r)\, d\tau \\ &+ c \int_0^t \operatorname*{osc}_{2,1}(v(\tau); x_0, r) |\nabla P^1_{x_0,r}(f(\tau)|\, d\tau \\ &+ c \int_0^t \operatorname*{osc}_{2n/(n+2),1}(\nabla \pi(\tau); x_0, r)\, d\tau.\end{aligned}$$

Proofs of the main theorems

1. Existence and uniqueness in terms of particle trajectories. Assume $f_0 \in \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n)$, $g \in L^1(0, T; \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n))$, and $\nabla v \in L^1(0, T; L^\infty(\mathbb{R}^n))$. Let $(x, t) \in Q_T$ be fixed. By $X_t(x, \cdot)$, we denote the unique solution to the ODE

$$\frac{d}{d\tau} X_t(x, \tau) = v(X_t(x, \tau), \tau), \quad \tau \in [0, T], \qquad X_t(x, t) = x, \tag{4.10}$$

which is ensured by Carathéodory's theorem. We define the flow map $\Phi_{t,\tau} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi_{t,\tau}(x) = X_t(x, \tau), \quad x \in \mathbb{R}^n,\ \tau, t \in [0, T].$$

By the uniqueness of this flow, we get the inverse formula

$$\Phi^{-1}_{t,\tau}(x) = \Phi_{\tau,t}(x).$$

Furthermore, from (4.10), we deduce that

$$\frac{d}{d\tau} \Phi_{t,\tau}(x) = v(\Phi_{t,\tau}(x), \tau), \quad \tau \in [0, T], \qquad \Phi_{t,t}(x) = x. \tag{4.11}$$

Let $(x, t) \in Q_T$. We set $y = \Phi_{t,0}(x)$, which is equivalent to $x = \Phi_{0,t}(y)$. We define

$$f(x, t) = f_0(y) + \int_0^t g(\Phi_{0,s}(y), s)\, ds. \tag{4.12}$$

Recalling that $f(t)$ is Lipschitz for almost all $t \in (0, T)$, we see that f is differentiable with respect to time almost everywhere in $(0, T)$. Recalling the inverse formula, we have $x = \Phi_{0,t}(y)$. Consequently, for $y \in \mathbb{R}^n$ fixed, from (4.12), we get

$$f(\Phi_{0,t}(y), t) = f_0(y) + \int_0^t g(\Phi_{0,s}(y), s)\, ds \quad \text{for all } t \in (0, T). \tag{4.13}$$

Differentiating (4.13) with respect to t, and observing (4.11), we obtain

$$\partial_t f(\Phi_{0,t}(y),t) + (v(\Phi_{0,t}(y),t)\cdot\nabla) f(\Phi_{0,t}(y),t) = g(\Phi_{0,t}(y),t).$$

This shows that f solves (1.1) in Q_T. In addition, by verifying that $\Phi_{0,0}(x) = x$, we get, from (4.13),

$$f(x,0) = f_0(x) \quad \text{for all } x \in \mathbb{R}^n.$$

This solution is also unique. In fact, assume that there is another solution $\bar{f}$ that solves (1.1). If we set $w = f - \bar{f}$, then w solves (1.1) with homogenous data. In other words, for every $y \in \mathbb{R}^n$, the function $Y(t) = w(\Phi_{0,t}(y),t)$ solves the ODE

$$\dot{Y} = 0, \quad Y(0) = 0,$$

which implies $Y \equiv 0$, and thus $w(\Phi_{0,t}(y),t) = 0$. With $y = \Phi_{t,0}(x)$, we get $w(x,t) = 0$ for all $(x,t) \in Q_T$.

2. Growth of the solution as $|x| \to +\infty$. Applying ∇_x to both sides of (4.11), and using the chain rule, we find that

$$\frac{d}{d\tau}\nabla\Phi_{s,\tau}(x) = \nabla v(\Phi_{s,\tau}(x),\tau)\cdot\nabla\Phi_{s,\tau}(x).$$

Integration with respect to τ over (s,t) yields

$$\nabla\Phi_{s,t}(x) = \mathbb{I} + \int_s^t \nabla v(\Phi_{s,\tau}(x),\tau)\cdot\nabla\Phi_{s,\tau}(x)\,d\tau,$$

where $\mathbb{I}$ stands for the unit matrix. Thus, for all $s,t \in (0,T)$,

$$|\nabla\Phi_{s,t}(x)| \le 1 + \int_s^t \|\nabla v(\tau)\|_\infty |\nabla\Phi_{s,\tau}(x)|\,d\tau.$$

By means of Gronwall's lemma, it follows that for all $s,t \in (0,T)$,

$$|\nabla\Phi_{s,t}(x)| \le \exp\Big(\int_s^t \|\nabla v(\tau)\|_\infty\,d\tau\Big). \tag{4.14}$$

From definition (4.10), we deduce that

$$\begin{aligned}\nabla f(x,t) &= \nabla f_0(\Phi_{t,0}(x))\cdot\nabla\Phi_{t,0}(x) + \int_0^t \nabla_x g(\Phi_{0,\tau}(\Phi_{t,0}(x)),\tau)\,d\tau\\ &= \nabla f_0(\Phi_{t,0}(x))\cdot\nabla\Phi_{t,0}(x)\\ &\quad + \int_0^t \nabla g(\Phi_{0,\tau}(\Phi_{t,0}(x)),\tau)\cdot\nabla\Phi_{0,\tau}(\Phi_{t,0}(x))\cdot\nabla\Phi_{t,0}(x)\,d\tau.\end{aligned}$$

Thus, for $\nabla f_0 \in L^\infty(\mathbb{R}^n)$ and $g \in L^1(0,T;L^\infty(\mathbb{R}^n))$, in view of (4.14), we get, for all $t \in (0,T)$,

$$\|\nabla f(t)\|_\infty \le \Big(\|\nabla f_0\|_\infty + \int_0^T \|\nabla g(\tau)\|_\infty\Big)\exp\Big(2\int_0^T \|\nabla v(\tau)\|_\infty\,d\tau\Big). \tag{4.15}$$

Using integration by parts, from (4.11), we get, for all $s, t \in (0, T)$,

$$\Phi_{s,t}(x) - x = \Phi_{s,t} - \Phi_{s,s}(x) = \int_s^t v(\Phi_{s,\tau}(y), \tau) - v(0, \tau)\, d\tau + \int_s^t v(0, \tau)\, d\tau.$$

This leads to the inequality

$$|\Phi_{s,t}(x)| \le |x| + \int_0^T |v(0, \tau)|\, d\tau + \int_s^t \|\nabla v(\tau)\|_\infty |\Phi_{s,\tau}(y)|\, d\tau.$$

By means of Gronwall's lemma, we find, for all $s, t \in (0, T)$,

$$|\Phi_{s,t}(x)| \le \Big(|x| + \int_0^T |v(0, \tau)|\, d\tau\Big) \exp\Big(\int_0^T \|\nabla v(\tau)\|_\infty\, d\tau\Big) \le c(1 + |x|). \tag{4.16}$$

Let $x \in \mathbb{R}^n$ and $t \in (0, T)$. For $N = 0$, $s \in [0, 1)$, using Lemma 3.11, we get

$$|f_0(x)| \le c(1 + |x|^s)\|f_0\|_{\mathcal{L}^s_{q(p,N)}}, \tag{4.17}$$

$$|g(x, \tau)| \le c(1 + |x|^s)\|g(\tau)\|_{\mathcal{L}^s_{q(p,N)}}. \tag{4.18}$$

For $N = 1$, $s = 1$ and $1 < q \le \infty$, we get, by Lemma 3.11,

$$|f_0(x)| \le c(1 + \log(1 + |x|)^{1/q'}|x|)\|f_0\|_{\mathcal{L}^s_{q(p,N)}}, \tag{4.19}$$

$$|g(x, \tau)| \le c(1 + \log(1 + |x|)^{1/q'}|x|)\|g(\tau)\|_{\mathcal{L}^s_{q(p,N)}}, \tag{4.20}$$

with $q' = q/(q-1)$. In the remaining cases, for $\nabla f_0 \in L^\infty(\mathbb{R}^n)$, $\nabla g \in L^1(0, T; L^\infty(\mathbb{R}^n))$, we find

$$|f_0(x)| \le c(1 + |x|)(\|f_0\|_{\mathcal{L}^s_{q(p,N)}} + \|\nabla f_0\|_\infty), \tag{4.21}$$

$$|g(x, \tau)| \le c(1 + |x|)(\|g(\tau)\|_{\mathcal{L}^s_{q(p,N)}} + \|\nabla g(\tau)\|_\infty). \tag{4.22}$$

Setting $y = \Phi_{t,0}(x)$, we get, from (4.13),

$$|f(x, t)| \le |f_0(y)| + \int_0^t |g(\Phi_{0,s}(y), s)|\, ds.$$

Employing (4.17)–(4.22), together with (4.16), we see that for all $(x, t) \in Q_T$,

$$|f(x, t)| \le c \begin{cases} (1 + |x|^{\min\{s,1\}}) & \text{if } s \ne 1, \\ (1 + \log(1 + |x|)^{1/q'}|x|) & \text{if } s = 1, \end{cases} \tag{4.23}$$

where c stands for a constant depending on s, q, p, N, n, and f_0, g and v.

3. Local energy estimation. Let $x_0 \in \mathbb{R}^n$. Let $\xi \in C^2([0, T]; \mathbb{R}^n)$ be a solution to the ODE

$$\dot{\xi}(\tau) = v(x_0 + \xi(\tau), \tau), \quad \tau \in [0, T]. \tag{4.24}$$

We set

$$F(x, \tau) = f(x + \xi(\tau), \tau), \quad V(x, \tau) = v(x + \xi(\tau), \tau) - \dot{\xi}(\tau),$$
$$G(x, s) = g(x + \xi(\tau), \tau), \quad (x, s) \in Q_T.$$

It is readily seen that V solves the transport equation

$$\partial_t F + (V\cdot\nabla)F = G \quad \text{in } Q_T. \tag{4.25}$$

In particular, from (4.24), we infer

$$V(x_0,\tau) = 0 \quad \text{for all } \tau\in[0,T]. \tag{4.26}$$

Set $L = 2N-1$ if $N>0$ and $L=0$ if $N=0$. According to (4.4) of Lemma 4.1 with $r=2^{j+1}$, $j\in\mathbb{Z}$, and noting that in view of (4.26), we have $2^{-j}\|V(\tau)\|_{L^\infty(B(x_0,2^{j+1}))}\le c\|\nabla v(\tau)\|_\infty$, we find

$$\begin{aligned}
&\operatorname*{osc}_{p,L}(F(t);x_0,2^j)\\
&\quad\le c\operatorname*{osc}_{p,L}(f_0(\cdot+\xi(0));x_0,2^{j+1}) + c\int_0^t \|\nabla v(\tau)\|_\infty \operatorname*{osc}_{p,N}(F(\tau);x_0,2^{j+2})\,d\tau\\
&\qquad+\delta_{N0}\,c\int_0^t \operatorname*{osc}_{p,N}(V(\tau);x_0,2^{j+1})\|\nabla P^N_{x_0,r}(F(\tau))\|_{L^\infty(B(x_0,2^{j+1}))}\,d\tau\\
&\qquad+c\int_0^t \operatorname*{osc}_{p,N}(G(\tau);x_0,2^{j+1})\,d\tau,
\end{aligned}\tag{4.27}$$

where $\delta_{N0}=0$ if $N=0$ and 1 otherwise.

Proof of (1.4) *in Theorem* 1.1. Inequality (4.27) gives

$$\begin{aligned}
\operatorname*{osc}_{p,0}(F(t);x_0,2^j) \le c\operatorname*{osc}_{p,0}(F(0);x_0,2^{j+1})&\\
+c\int_0^t\|\nabla v(\tau)\|_\infty \operatorname*{osc}_{p,0}(F(\tau);x_0,2^{j+2})\,d\tau&\\
+c\int_0^t \operatorname*{osc}_{p,0}(G(\tau);x_0,2^{j+1})\,d\tau.&
\end{aligned}\tag{4.28}$$

Observing (4.23), since $s<1$, we get $S_{1,1}(\operatorname{osc}_{p,0}(f(\tau);x_0))<+\infty$. Thus, applying $S_{1,1}$ to both sides of (4.28), we obtain

$$\begin{aligned}
S_{1,1}\big(\operatorname*{osc}_{p,0}(F(t);x_0)\big)\le cS_{1,1}\big(\operatorname*{osc}_{p,0}(F(0);x_0)\big)&\\
+c\int_0^t\|\nabla v(\tau)\|_\infty S_{1,1}\big(\operatorname*{osc}_{p,0}(F(\tau);x_0)\big)\,d\tau&\\
+c\int_0^t S_{1,1}\big(\operatorname*{osc}_{p,0}(G(\tau);x_0)\big)\,d\tau.&
\end{aligned}\tag{4.29}$$

Applying Gronwall's lemma, we deduce, from (4.29),

$$\begin{aligned}
&\operatorname*{osc}_{p,0}(F(t);x_0)\le S_{1,1}\big(\operatorname*{osc}_{p,0}(F(t);x_0)\big)\\
&\le c\Big\{S_{1,1}\big(\operatorname*{osc}_{p,0}(F(0);x_0)\big)+\int_0^t S_{1,1}\big(\operatorname*{osc}_{p,0}(G(\tau);x_0)\big)\,d\tau\Big\}\exp\Big(c\int_0^t\|\nabla v(\tau)\|_\infty\,d\tau\Big).
\end{aligned}\tag{4.30}$$

Let $t \in [0, T]$. Clearly, the constant in (4.30) is independent of the choice of the characteristic for ξ. Therefore, we may choose ξ such that $\xi(t) = 0$, which implies $F(t) = f(t)$. Hence, we may replace $F(t)$ by $f(t)$ on the left-hand side of (4.30). Afterwards, with the help of Lemma 2.1, we are in a position to operate $S_{s,q}$ to both sides of (4.30), verifying $F(0) = f_0(\cdot - \xi(0))$, that yields

$$\Big(S_{s,q}\big(\operatorname*{osc}_{p,0}(f(t); x_0)\big)\Big)_j \le c\Big\{\Big(S_{s,q}\big(\operatorname*{osc}_{p,0}(f_0(\cdot - \xi(0); x_0))\big)\Big)_j + \int_0^t \Big(S_{s,q}\big(\operatorname*{osc}_{p,0}(G(\tau); x_0)\big)\Big)_j \, d\tau\Big\} \exp\Big(c \int_0^t \|\nabla v(\tau)\|_\infty \, d\tau\Big).$$

Multiplying both sides by 2^{-js}, we get

$$\Big(\sum_{i=j}^{\infty}\big(2^{-si} \operatorname*{osc}_{p,1}(f(t); x_0; 2^i)\big)^q\Big)^{1/q} \le c\Big\{|f_0|_{\mathcal{L}^s_{q(p,0)}} + \int_0^t |G(\tau)|_{\mathcal{L}^s_{q(p,0)}} \, d\tau\Big\} \exp\Big(c \int_0^t \|\nabla v(\tau)\|_\infty \, d\tau\Big). \tag{4.31}$$

Passing $j \to -\infty$ and taking the supremum over $x_0 \in \mathbb{R}^n$ in (4.31), we get (1.4). ■

Proof of (1.7) *in Theorem* 1.2. Recalling that $V(x_0, \tau) = 0$ for all $\tau \in [0, T]$, we see that $2^{-j}\|V(\tau)\|_{L^\infty(B(x_0,2^{j+1}))} \le c\|\nabla v(\tau)\|_\infty$ and $2^{-j} \operatorname{osc}_{p,0}(V(\tau); x_0, 2^{j+1}) \le c\|\nabla v(\tau)\|_\infty$. Thus, (4.13) leads to

$$\begin{aligned}\operatorname*{osc}_{p,1}(F(t); x_0, 2^j) &\le c \operatorname*{osc}_{p,1}(F(0); x_0, 2^{j+1}) + c \int_0^t \|\nabla v(\tau)\|_\infty \operatorname*{osc}_{p,1}(F(\tau); x_0, 2^{j+2}) \, d\tau \\ &\quad + c \int_0^t \operatorname*{osc}_{p,1}(V(\tau); x_0, 2^{j+1}) |\nabla \dot{P}^1_{x_0,2^{j+1}}(F(\tau))| \, d\tau \\ &\quad + c \int_0^t \operatorname*{osc}_{p,1}(G(\tau); x_0, 2^{j+1}) \, d\tau.\end{aligned} \tag{4.32}$$

For $j \ge 0$, using the triangle inequality, we get

$$\begin{aligned}|\nabla \dot{P}^1_{x_0,2^j}(F(\tau))| &\le c \sum_{i=0}^{j} 2^{-i} \operatorname*{osc}_{p,1}(F(\tau); x_0, 2^i) + |\nabla \dot{P}^1_{x_0,1}(F(\tau))| \\ &\le c2^{-j}\Big(S_{3,1}\big(\operatorname*{osc}_{p,1}(F(\tau); x_0)\big)\Big)_j + |\nabla \dot{P}^1_{x_0,1}(F(\tau))|.\end{aligned}$$

For $j < 0$, using the triangle inequality along with Hölder's inequality, we find

$$\begin{aligned}|\nabla \dot{P}^1_{x_0,2^j}(F(\tau))| &\le c \sum_{i=0}^{j} 2^{-i} \operatorname*{osc}_{p,1}(F(\tau); x_0, 2^i) + |\nabla \dot{P}^1_{x_0,1}(F(\tau))| \\ &\le (-j)^{1/q'}\Big(\sum_{i=j}^{0} 2^{-iq}\big(\operatorname*{osc}_{p,1}(F(\tau); x_0, 2^i)\big)^q\Big)^{1/q} + |\nabla \dot{P}^1_{x_0,1}(F(\tau))|.\end{aligned}$$

Summing up the above estimates, we arrive at

$$(4.33)\quad \operatorname*{osc}_{p,1}(V(\tau);x_0,2^{j+1})|\nabla \dot P^1_{x_0,2^{j+1}}(F(\tau))| \\ \le 2^{-j}\operatorname*{osc}_{p,1}(V(\tau);x_0,2^{j+1})\big(S_{3,1}\big(\operatorname*{osc}_{p,1}(F(\tau);x_0)\big)\big)_j\, c(j^-)^{1/q'}\operatorname*{osc}_{p,1}(V(\tau);x_0,2^{j+1}) \\ \times\Big\{\Big(\sum_{i=-\infty}^{0}2^{-iq}\big(\operatorname*{osc}_{p,1}(F(\tau);x_0,2^i)\big)^q\Big)^{1/q}+|\nabla \dot P^1_{x_0,1}(F(\tau))|\Big\},$$

where $j^-=-\min\{j,0\}$. Applying the operator $S_{2,1}$ to the both sides of the above inequality, and making use of Lemma 2.1, with $p=q=1$, $\alpha=3$ and $\beta=2$, we obtain

$$S_{2,1}\big(\big\{\operatorname*{osc}_{p,1}(V(\tau);x_0,2^{i+1})|\nabla \dot P^1_{x_0,2^{i+1}}(F(\tau))|\big\}\big) \\ \le c|v(\tau)|_{\mathcal{L}^1_{q(p,1)}}S_{2,1}\big(\operatorname*{osc}_{p,1}(F(\tau);x_0)\big)+cS_{2,1}\big(\big\{(i^-)^{1/q'}\operatorname*{osc}_{p,1}(V(\tau);x_0,2^i)\big\}\big) \\ \times\Big\{\Big(\sum_{i=-\infty}^{0}2^{-iq}\big(\operatorname*{osc}_{p,1}(F(\tau);x_0,2^i)\big)^q\Big)^{1/q}+|\nabla \dot P^1_{x_0,1}(F(\tau))|\Big\}.$$

Observing (4.23), all sums in the above estimates are finite. Again appealing to (4.16), we are in a position to apply $S_{2,1}$ to both sides of (4.32) to get

$$S_{2,1}\big(\operatorname*{osc}_{p,1}(F(t);x_0)\big) \\ \le cS_{2,1}\big(\operatorname*{osc}_{p,1}(F(0);x_0)\big)+c\int_0^t(\|\nabla v(\tau)\|_\infty+|v(\tau)|_{\mathcal{L}^1_{q(p,1)}})S_{2,1}\big(\operatorname*{osc}_{p,1}(F(\tau);x_0)\big)\,d\tau \\ +c\int_0^t S_{2,1}\big(\big\{(i^-)^{1/q'}\operatorname*{osc}_{p,1}(V(\tau);x_0,2^i)\big\}\big)\Big\{\Big(\sum_{i=-\infty}^{0}2^{-iq}\big(\operatorname*{osc}_{p,1}(F(\tau);x_0,2^i)\big)^q\Big)^{1/q} \\ +|\nabla \dot P^1_{x_0,1}(F(\tau))|\Big\}\,d\tau+c\int_0^t S_{2,1}\big(\operatorname*{osc}_{p,1}(G(\tau);x_0)\big)\,d\tau.$$

Applying Gronwall's lemma, we are led to

$$(4.34)\quad \operatorname*{osc}_{p,1}(F(t);x_0)\le S_{2,1}\big(\operatorname*{osc}_{p,1}(F(t);x_0)\big) \\ \le\Big\{cS_{2,1}\big(\operatorname*{osc}_{p,1}(f_0(\cdot+\xi(0);x_0))\big)+c\int_0^t\Big[S_{2,1}\big(\big\{(i^-)^{1/q'}\operatorname*{osc}_{p,1}(V(\tau);x_0,2^i)\big\}\big) \\ \times\Big(\sum_{i=-\infty}^{0}2^{-iq}\big(\operatorname*{osc}_{p,1}(F(\tau);x_0,2^i)\big)^q\Big)^{1/q}+c|\nabla \dot P^1_{x_0,1}(F(\tau))|\Big]\,d\tau \\ +c\int_0^t S_{2,1}\big(\operatorname*{osc}_{p,1}(G(\tau);x_0)\big)\,d\tau\Big\}\exp\int_0^T(\|\nabla v(\tau)\|_\infty+|v(\tau)|_{\mathcal{L}^1_{q(p,1)}})\,d\tau.$$

Observing (1.5), using Lemma 2.1, we may apply $S_{1,q}$ to both sides of (4.34). Accordingly,

$$\sup_{t\in[0,T]}S_{1,q}\big(\operatorname*{osc}_{p,1}(F(t);x_0)\big)<+\infty.$$

For given $t \in [0,T]$, we may choose ξ such $\xi(t)=0$. Thus, the same holds for $f(t)$ in place of $F(t)$. Now, we are able to apply $S_{1,q}$ to both sides of (4.33), which yields

$$\begin{aligned}
&S_{1,q}\big(\{\operatorname*{osc}_{p,1}(V(\tau);x_0,2^{i+1})|\nabla \dot{P}^1_{x_0,2^{i+1}}(F(\tau))|\}\big)\\
&\quad\le c|v(\tau)|_{\mathcal{L}^1_{q(p,1)}} S_{1,q}\big(\operatorname*{osc}_{p,1}(F(\tau);x_0)\big)\, c S_{1,q}\big(\{(i^-)^{1/q'}\operatorname*{osc}_{p,1}(V(\tau);x_0,2^i)\}\big)\\
&\qquad\times\Big\{\Big(\sum_{i=-\infty}^{\infty}2^{-iq}\big(\operatorname*{osc}_{p,1}(F(\tau);x_0,2^i)\big)^q\Big)^{1/q}+|\nabla \dot{P}^1_{x_0,1}(F(\tau))|\Big\}.
\end{aligned}$$

Applying $S_{1,q}$ to both sides of (4.32) multiplying the result by 2^{-j} and letting $j\to-\infty$, we infer

$$\begin{aligned}
&\Big(\sum_{i=-\infty}^{\infty}2^{-iq}\big(\operatorname*{osc}_{p,1}(F(t);x_0,2^i)\big)^q\Big)^{1/q}\\
&\le c|f_0|_{\mathcal{L}^1_{q(p,1)}}+c\int_0^t\|\nabla v(\tau)\|_\infty\Big(\sum_{i=-\infty}^{\infty}2^{-iq}\big(\operatorname*{osc}_{p,1}(F(t);x_0,2^i)\big)^q\Big)^{1/q}d\tau\\
&+c\int_0^t\Big[\Big(\sum_{i=-\infty}^{\infty}(i^-)^{q-1}\big(2^{-i}\operatorname*{osc}_{p,1}(V(\tau);x_0,2^i)\big)^q\Big)^{1/q}\Big(\sum_{i=-\infty}^{\infty}2^{-iq}\big(\operatorname*{osc}_{p,1}(F(\tau);x_0,2^i)\big)^q\Big)^{1/q}\\
&+|\nabla \dot{P}^1_{x_0,1}(F(\tau))|\Big]d\tau+c\int_0^t|g(\tau)|_{\mathcal{L}^1_{q(p,1)}}d\tau.
\end{aligned}\tag{4.35}$$

Next, we require to estimate $|\nabla \dot{P}^1_{x_0,1}(F(\tau))|$ by the initial data f_0 and g. We apply $\dot{P}^1_{x_0,1}$ to both sides (4.25). This gives

$$\partial_t \dot{P}^1_{x_0,1}(F)+\dot{P}^1_{x_0,1}(V\cdot\nabla F)=\dot{P}^1_{x_0,1}(G)\quad\text{in } Q_T.\tag{4.36}$$

Noting that $\dot{P}^1_{x_0,1}(P^1_{x_0,1}(V)\cdot\nabla \dot{P}^1_{x_0,1}(F))=P^1_{x_0,1}(V)\cdot\nabla \dot{P}^1_{x_0,1}(F)$, and applying ∇ to both sides of (4.36), we infer

$$\begin{aligned}
&\frac{d}{dt}\nabla \dot{P}^1_{x_0,1}(F)+(\nabla \dot{P}^1_{x_0,1}(V))\cdot\nabla \dot{P}^1_{x_0,1}(F)\\
&\qquad=\nabla \dot{P}^1_{x_0,1}\big(P^1_{x_0,1}(V)\cdot\nabla \dot{P}^1_{x_0,1}(F)-V\cdot\nabla F\big)+\nabla \dot{P}^1_{x_0,1}(G)\quad\text{in } [0,T].
\end{aligned}\tag{4.37}$$

On the other hand,

$$\begin{aligned}
&\nabla \dot{P}^1_{x_0,1}\big(P^1_{x_0,1}(V)\cdot\nabla \dot{P}^1_{x_0,1}(F)-V\cdot\nabla F\big)\\
&\quad=\nabla \dot{P}^1_{x_0,1}\big((P^1_{x_0,1}(V)-V)\cdot\nabla \dot{P}^1_{x_0,1}(F)\big)+\nabla \dot{P}^1_{x_0,1}\big(V\cdot\nabla(\dot{P}^1_{x_0,1}(F)-F)\big)\\
&\quad=\nabla \dot{P}^1_{x_0,1}\big((P^1_{x_0,1}(V)-V)\cdot\nabla \dot{P}^1_{x_0,1}(F)\big)-\nabla \dot{P}^1_{x_0,1}\big(\nabla\cdot V(P^1_{x_0,1}(F)-F)\big).
\end{aligned}$$

Inserting this identity into the right-hand side of (4.37), and then multiplying the result by $\nabla \dot{P}^1_{x_0,1}(F)/|\nabla \dot{P}^1_{x_0,1}(F)|$, we get the following differential inequality:

$$\frac{d}{dt}|\nabla \dot{P}^1_{x_0,1}(F)|\le c\|\nabla v\|_\infty|\nabla \dot{P}^1_{x_0,1}(F)|+c\|\nabla v\|_\infty\operatorname*{osc}_{p,1}(F,x_0;1)+|\nabla \dot{P}^1_{x_0,1}(G)|.$$

Integrating this inequality over $(0,t)$ and applying integration by parts, we obtain

$$
\begin{aligned}
(4.38)\quad & |\nabla \dot P^1_{x_0,1}(F(t))| \\
&\le |\nabla \dot P^1_{x_0,1}(F(0))| + c\int_0^t \|\nabla v(\tau)\|_\infty |\nabla \dot P^1_{x_0,1}(F(\tau))|\,d\tau \\
&\quad + \int_0^t \|\nabla v(\tau)\|_\infty \operatorname*{osc}_{p,1}(F(\tau),x_0;1)\,d\tau + \int_0^t |\nabla \dot P^1_{x_0,1}(G(\tau))|\,d\tau \\
&\le \|f_0\|_{\tilde{\mathcal{L}}^1_{q(p,1)}} + c\int_0^t \|\nabla v(\tau)\|_\infty \operatorname*{osc}_{p,1}(F(\tau),x_0;1)\,d\tau + \int_0^t \|g(\tau)\|_{\tilde{\mathcal{L}}^1_{q(p,1)}}\,d\tau,
\end{aligned}
$$

where $|z|_{\tilde{\mathcal{L}}^1_{q(p,0)}}$ stands for the semi norm

$$
|z|_{\tilde{\mathcal{L}}^1_{q(p,1)}} = |z|_{\mathcal{L}^1_{q(p,1)}} + \sup_{x_0\in\mathbb{R}^n} |\nabla \dot P^1_{x_0,1}(z)|.
$$

Combining (4.35) and (4.38), we arrive at

$$
\begin{aligned}
&\Big(\sum_{i=-\infty}^{\infty} 2^{-iq}\big(\operatorname*{osc}_{p,1}(F(t);x_0,2^i)\big)^q\Big)^{1/q} + |\nabla \dot P^1_{x_0,1}(F(t))| \\
&\le c|f_0|_{\tilde{\mathcal{L}}^1_{q(p,1)}} + c\int_0^t \|\nabla v(\tau)\|_\infty \Big(\sum_{i=-\infty}^{\infty} 2^{-iq}\big(\operatorname*{osc}_{p,1}(F(t);x_0,2^i)\big)^q\Big)^{1/q} d\tau \\
&+ c\int_0^t \Big(\sum_{i=-\infty}^{\infty} (i^-)^{q-1}(2^{-i}\operatorname*{osc}_{p,1}(V(\tau);x_0,2^i))^q\Big)^{1/q} \Big\{\Big(\sum_{i=-\infty}^{\infty} 2^{-iq}\big(\operatorname*{osc}_{p,1}(F(\tau);x_0,2^i)\big)^q\Big)^{1/q} \\
&+ |\nabla \dot P^1_{x_0,1}(F(\tau))|\Big\}\,d\tau + c\int_0^t |g(\tau)|_{\tilde{\mathcal{L}}^1_{q(p,1)}}\,d\tau.
\end{aligned}
$$

Applying Gronwall's lemma and for given $t\in[0,T]$, choosing ξ such that $\xi(t)=0$, and taking the supremum over $x_0\in\mathbb{R}^n$, we obtain the desired estimate (1.7). ∎

Proof of (1.8) *in Theorem* 1.3. We first define

$$
\chi(x_0,t) = \sup_{j\in\mathbb{Z}} 2^{-j}\operatorname*{osc}_{p,0}(F(t);x_0,2^j), \qquad (x_0,t)\in\mathbb{R}^n\times[0,T].
$$

Clearly, thanks to (4.23), $\chi(x_0,t)$ is finite. Noting that $\|\nabla P^N_{x_0,2^{j+1}}(F(\tau))\|_{L^\infty(B(x_0,2^{j+1}))} \le c\chi(x_0,\tau)$, from (4.27) with $L=2N-1$, we get

$$
\begin{aligned}
(4.39)\qquad \operatorname*{osc}_{p,2N-1}(F(t);x_0,2^j) &\le c\operatorname*{osc}_{p,N}(F(0);x_0,2^{j+1}) \\
&\quad + c\int_0^t \|\nabla v(\tau)\|_\infty \operatorname*{osc}_{p,N}(F(\tau);x_0,2^{j+2})\,d\tau \\
&\quad + c2^{-j}\int_0^t \operatorname*{osc}_{p,N}(V(\tau);x_0,2^{j+1})\chi(x_0,\tau)\,d\tau \\
&\quad + c\int_0^t \operatorname*{osc}_{p,N}(G(\tau);x_0,2^{j+1})\,d\tau.
\end{aligned}
$$

First let us estimate the term $\operatorname{osc}_{p,0}(F(t); x_0, 2^{j+1})$. In view of (4.28), with $j+1$ in place of j, and recalling that $\nabla f_0 \in L^\infty(\mathbb{R}^n)$, $g \in L^1(0,T; L^\infty(\mathbb{R}^n))$, we see that

$$\begin{aligned} &\operatorname*{osc}_{p,0}(F(t); x_0, 2^{j+1}) \\ &\quad \le c2^j \|\nabla f_0\|_\infty + c\int_0^t \|\nabla v(\tau)\|_\infty \operatorname*{osc}_{p,0}(F(\tau); x_0, 2^{j+3})\, d\tau + c\int_0^t 2^j \|\nabla g(\tau)\|_\infty \, d\tau. \end{aligned} \tag{4.40}$$

Multiplying both sides of (4.40) by 2^{-j} and taking the supremum over all $j \in \mathbb{Z}$, using the triangle inequality, we obtain

$$\chi(x_0, t) \le c\|\nabla f_0\|_\infty + c\int_0^t \|\nabla v(\tau)\|_\infty \chi(x_0, \tau)\, d\tau + c\int_0^t \|\nabla g(\tau)\|_\infty \, d\tau. \tag{4.41}$$

Thanks to (4.23), we have $S_{N+1,1}(\operatorname{osc}_{p,N} F(t); x_0) < +\infty$ for all $t \in [0,T]$. Applying $S_{N+1,1}$ to both sides of (4.39), and using Corollary 3.10 with $N' = 2N-1$, we get

$$\begin{aligned} &\operatorname*{osc}_{p,N}(F(t); x_0) \le S_{N+1,1}\big(\operatorname*{osc}_{p,2N-1}(F(t); x_0)\big) \\ &\quad \le c S_{N+1,1}\big(\operatorname*{osc}_{p,N}(F(0); x_0)\big) + c\int_0^t \|\nabla v(\tau)\|_\infty S_{N+1,1}\big(\operatorname*{osc}_{p,N}(F(\tau); x_0)\big)\, d\tau \\ &\quad\quad + c\int_0^t S_{N+1,1}\big(\operatorname*{osc}_{p,N}(V(\tau); x_0)\big)\chi(x_0,\tau)\, d\tau + c\int_0^t S_{N+1,1}\big(\operatorname*{osc}_{p,N}(G(\tau); x_0)\big)\, d\tau. \end{aligned} \tag{4.42}$$

Next, once more using (4.23), we see that $S_{s,q}(\operatorname{osc}_{p,N}(F(t); x_0)) < +\infty$, for all $t \in [0,T]$. Thus, we apply $S_{s,q}$ to both sides of (4.42) and use Lemma 2.1. This combined with (4.41) gives

$$\begin{aligned} &2^{-js}\big(S_{s,q}\big(\operatorname*{osc}_{p,N}(F(t); x_0)\big)\big)_j + \chi(x_0, t) \le c|f_0|_{\mathcal{L}^s_{q(p,N)}} + \chi(x_0, 0) \\ &\quad + c\int_0^t \big(|v(\tau)|_{\mathcal{L}^s_{q(p,1)}} + \|\nabla v(\tau)\|_\infty\big)\big[2^{-js}\big(S_{s,q}\big(\operatorname*{osc}_{p,N}(F(t); x_0)\big)\big)_j + \chi(x_0,\tau)\big]\, d\tau \\ &\quad + c\int_0^t \big(|g(\tau)|_{\mathcal{L}^s_{q(p,N)}} + \|\nabla g(\tau)\|_\infty\big)\, d\tau. \end{aligned} \tag{4.43}$$

By virtue of Gronwall's lemma, we deduce, from (4.43),

$$\begin{aligned} &2^{-js}\big(S_{s,q}\big(\operatorname*{osc}_{p,N}(F(t); x_0)\big)\big)_j + \chi(x_0, t) \\ &\quad \le c\Big\{|f_0|_{\mathcal{L}^s_{q(p,N)}} + \|\nabla f_0\|_\infty + \int_0^T \big(|g(\tau)|_{\mathcal{L}^s_{q(p,N)}} + \|\nabla g(\tau)\|_\infty\big)\, d\tau\Big\} \\ &\quad\quad \times \exp\Big(\int_0^T \big(|v(\tau)|_{\mathcal{L}^s_{q(p,1)}} + \|\nabla v(\tau)\|_\infty\big)\, d\tau\Big). \end{aligned}$$

Whence, (1.8). ■

Proof of (1.9) *in Corollary* 1.4. In view of Theorem 3.6, we have $\nabla f_0 \in L^\infty(\mathbb{R}^n)$, $\nabla g \in L^1(0,T; L^\infty(\mathbb{R}^n))$. More precisely, (3.21) yields

$$\|\nabla f_0\|_\infty \le c\|f_0\|_{\mathcal{L}^1_{1(p,1)}}, \qquad \int_0^T \|\nabla g(\tau)\|_\infty \, d\tau \le c\|g\|_{L^1(0,T;\mathcal{L}^1_{1(p,1)})}.$$

In particular, this shows that condition (1.6) of Theorem 1.2 is fulfilled. Furthermore, since $v \in L^1(0,T;\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n))$, condition (1.5) of Theorem 1.2 is also satisfied. Now, we are in a position to apply Theorem 1.2, which yields $f \in L^\infty(0,T;\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n))$. This allows to apply $S_{1,1}$ to both sides of (4.27). This, together with Gronwall's lemma and the inequality $|\nabla \dot{P}^1_{x_0,2^{j+1}}(F(\tau))| \le c2^{-j}\|\nabla f(\tau)\|_\infty$, yields

$$(4.44)\quad \big(S_{1,1}\big(\operatorname*{osc}_{p,1}(F(t);x_0)\big)\big)_j \\ \le c\Big\{\big(S_{1,1}\big(\operatorname*{osc}_{p,1}(F(0);x_0)\big)\big)_j + \int_0^t \big(S_{1,1}\big(\operatorname*{osc}_{p,1}(G(\tau);x_0)\big)\big)_j\,d\tau \\ + \int_0^t \big(S_{1,1}\big(\operatorname*{osc}_{p,1}(V(\tau);x_0)\big)\big)_j \|\nabla f(\tau)\|_\infty\,d\tau\Big\}\exp\Big(c\int_0^t \|\nabla v(\tau)\|_\infty\,d\tau\Big).$$

Choosing ξ so that $\xi(t) = 0$, multiplying both sides by 2^{-j}, letting $j \to -\infty$, and taking the supremum over $x_0 \in \mathbb{R}^n$, we deduce, from (4.44),

$$(4.45)\quad |f(t)|_{\mathcal{L}^1_{1(p,1)}} \le c\Big\{\|f_0\|_{\mathcal{L}^1_{1(p,1)}} + \int_0^t \|g(\tau)\|_{\mathcal{L}^1_{1(p,1)}}\,d\tau \\ + \int_0^t \|v(\tau)\|_{\mathcal{L}^1_{1(p,1)}}\|\nabla f(\tau)\|_\infty\,d\tau\Big\}\exp\Big(c\int_0^t \|\nabla v(\tau)\|_\infty\,d\tau\Big).$$

Combining (4.45) and (4.15), along with (4.23) in order to estimate $\|f(t)\|_{L^p(B(1))}$, we get the desired estimate (1.9). ■

Below we prove the uniqueness parts of Theorems 1.1–1.3 and Corollary1.4. In fact we prove the stronger version of it, namely, the strong-weak uniqueness.

Proof of strong-weak uniqueness. Let $\bar{f} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ be a weak solution to (1.1). Then $w = f - \bar{f}$ solves the transport equation with homogenous data:

$$\partial_t w + (v\cdot\nabla)w = 0 \quad \text{in } Q_T, \qquad w = 0 \quad \text{on } \mathbb{R}^n \times \{0\},$$

in a weak sense, i.e., for all $t \in (0,T)$ and all $\varphi \in L^\infty(0,t;W^{1,2}(\mathbb{R}^n)) \cap W^{1,1}(0,t;L^2(\mathbb{R}^n))$ with $\operatorname{supp}(\varphi) \Subset \mathbb{R}^n \times [0,t]$,

$$(4.46)\quad -\int_0^t\int_{\mathbb{R}^n} w\partial_t\varphi + (v\cdot\nabla)\varphi w + \nabla\cdot v\varphi w\,dx\,ds = -\int_{\mathbb{R}^n} w(t)\varphi(t)\,dx.$$

Let $\psi \in C^\infty_c(\mathbb{R}^n)$ be a given function. Using the method of characteristics, for every $\varepsilon > 0$, we get a solution $\varphi^\varepsilon \in L^\infty(0,t;W^{1,2}(\mathbb{R}^n)) \cap W^{1,1}(0,t;L^2(\mathbb{R}^n))$ of the the following dual problem:

$$(4.47)\quad \partial_t\varphi^\varepsilon + v\cdot\nabla\varphi^\varepsilon + \nabla\cdot v_\varepsilon\varphi^\varepsilon = 0 \quad \text{in } Q_t, \qquad \varphi^\varepsilon(t) = \psi \quad \text{in } \mathbb{R}^n.$$

Noting that $\|\nabla v_\varepsilon(\tau)\|_\infty \le \|\nabla v(\tau)\|_\infty$, and using Gronwall's lemma, we see that $\|\varphi^\varepsilon\|_1 + \|\varphi^\varepsilon\|_\infty \le c$, with a constant $c > 0$ independent of $\varepsilon > 0$. Since $v(0,\cdot)$, $\|\nabla v(\cdot)\|_\infty \in L^1(0,T)$, using (4.16), we get a number $0 < R < +\infty$ such that $\operatorname{supp}(\varphi^\varepsilon) \subset B(R) \times [0,t]$.

In (4.46), putting $\varphi = \varphi_\varepsilon$, and using (4.47), we infer

$$\int_{\mathbb{R}^n} w(t)\psi \, dx = \int_0^t \int_{\mathbb{R}^n} w\partial_t \varphi_\varepsilon + (v \cdot \nabla)\varphi_\varepsilon w + \nabla \cdot v\varphi^\varepsilon w \, dx \, ds = \int_0^t \int_{B(R)} \nabla \cdot (v - v_\varepsilon)\varphi^\varepsilon w \, dx \, ds. \tag{4.48}$$

Noting that $\nabla \cdot (v(s) - v_\varepsilon(s)) \to 0$ in $L^2(B(R))$ as $\varepsilon \searrow 0$ for almost all $s \in (0,t)$, with the aid of Vitali's convergence theorem [10], p. 180, it follows that

$$\int_0^t \int_{B(R)} \nabla \cdot (v - v_\varepsilon)\varphi^\varepsilon w \, dx \, ds \to 0 \quad \text{as } \varepsilon \searrow 0.$$

Letting $\varepsilon \searrow 0$ in (4.48), we deduce that $\int_{\mathbb{R}^n} w(t)\psi dx = 0$. Whence, $w \equiv 0$. This shows the uniqueness. ■

A. Minimal polynomials

Let $1 < p < +\infty$. Let $x_0 \in \mathbb{R}^n$ and $0 < r < +\infty$ be fixed. Set $\phi = \varphi(r^{-1}(x_0 - \cdot))$, where $\varphi \in C_c^\infty(B(1))$, being radial symmetric, stands for the standard mollifier. For $\delta \geq 0$, we define the functional $J_\delta \colon L^p(B(x_0,r)) \to \mathbb{R}$ by

$$J_\delta(f) = \int_{B(x_0,)} (\delta + |f|^2)^{p/2} \phi^p \, dx, \quad f \in L^p(B(x_0,r)).$$

Recall that $\mathcal{P}_N$, $N \in \mathbb{N}_0$, denotes the space of all polynomial of degree less that or equal to N. Since J_δ is strict convex and lower semi continuous with $J_\delta(f) \to +\infty$ as $\|f\|_{L^p(B(x_0,r))} \to +\infty$, for each $f \in L^p(B(x_0,r))$, there exists a unique $P^{N,\delta}_{x_0,r}(f) \in \mathcal{P}_N$ with

$$J_\delta(P^{N,\delta}_{x_0,r}(f) - f) = \min_{P \in \mathcal{P}_N} J_\delta(P - f). \tag{A.1}$$

Clearly, the mapping $J_{\delta,f} \colon P \mapsto J_\delta(P - f)$ is differentiable as a function from $\mathcal{P}_N$ into $\mathbb{R}$. Since the first variation must vanish at each minimizer, we get

$$\langle DJ_{\delta,f}(P^{N,\delta}_{x_0,r}(f), P)\rangle = 0 \quad \text{for all } P \in \mathcal{P}_N.$$

This shows that

$$\int_{B(x_0,r)} F_\delta(P^{N,\delta}_{x_0,r}(f) - f) \cdot P\phi^p \, dx = 0 \quad \text{for all } P \in \mathcal{P}_N, \tag{A.2}$$

where

$$F_\delta(u) = (\delta + |u|^2)^{(p-2)/2} u, \quad u \in \mathbb{R}^n.$$

It is well known that F_δ is monotone and continuously differentiable for each $\delta > 0$. Furthermore, there exists a constant $c > 0$ independent of δ such that for all $u, v \in \mathbb{R}^m$,

$$(F_\delta(u) - F_\delta(v))(u - v) \geq c(p-1)(\delta + |u| + |u - v|)^{p-2}|u - v|^2, \tag{A.3}$$

$$|F_\delta(u) - F_\delta(v)| \leq cp(\delta + |u| + |u - v|)^{(p-2)/2}|u - v|. \tag{A.4}$$

We now define the mapping $G_\delta \colon L^p(B(x_0,r)) \times \mathcal{P}_N \to (\mathcal{P}_N)'$ by

$$\langle G_\delta(f,P), Q\rangle = \int_{B(x_0,r)} F_\delta(f(x)-P)\cdot Q\phi^2(x)\,dx, \quad f \in L^p(B(x_0,r),\ p, Q \in \mathcal{P}_N.$$

Clearly, (A.2) is equivalent to

$$G_\delta(f, P^{N,\delta}_{x_0,r}(f)) = 0. \tag{A.5}$$

We obtain the following properties of G_δ.

Lemma A.1. (1) *For every $f \in L^p(B(x_0,r))$, the mapping $G_\delta(f,\cdot)\colon \mathcal{P}_N \to (\mathcal{P}_N)'$ is strictly monotone and bijective, and for $\delta > 0$, strongly monotone and a C^1 diffeomorphism.*

(2) *For $\delta > 0$, the mapping $f \mapsto P^{N,\delta}_{x_0,r}(f) : L^p(B(x_0,r)) \to \mathcal{P}_N$ is Fréchet differentiable, and its derivative, for $f \in L^p(B(x_0,r))$, is given by*

$$DP^{N,\delta}_{x_0,r}(f) = -[D_2G_\delta(f, P^{N,\delta}_{x_0,r}(f))]^{-1} \circ D_1G_\delta(f, P^{N,\delta}_{x_0,r}(f)), \tag{A.6}$$

where $D_1G_\delta(f,P) \in \mathcal{L}(L^p(B(x_0,r)), (\mathcal{P}_N)')$ stands for the derivative with respect to the first variable, while $D_2G_\delta(f,P) \in \mathcal{L}(\mathcal{P}_N, (\mathcal{P}_N)')$ stands for the derivative with respect to the second variable. Furthermore, for every $f \in L^p(B(x_0,r))$,

$$\|P^{N,\delta}_{x_0,r}(f)\|^p_{L^p(B(x_0,r/2))} \le 2^p \int_{B(x_0,r)} (\delta + |f|^2)^{p/2}\phi^p\,dx. \tag{A.7}$$

(3) *For all $f \in L^p(B(x_0,r))$,*

$$P^{N,\delta}_{x_0,r}(f) \to P^{N,*}_{x_0,r}(f) \quad \text{in } \mathcal{P}_N \text{ as } \delta \searrow 0, \tag{A.8}$$

where $P^{N,*}_{x_0,r}(f) = P^{N,0}_{x_0,r}(f)$.

Proof. (1) Observing (A.3), for all $f \in L^p(B(x_0,r))$ and $P, Q \in \mathcal{P}_N$, we get

$$\begin{aligned}&\langle (G_\delta(f,P) - G_\delta(f,Q)), (P-Q)\rangle \\ &\quad\ge c(p-1)\int_{B(x_0,r)} (\delta + |P-f| + |P-Q|)^{p-2}|P-Q|^2\phi^2\,dx.\end{aligned}$$

This immediately shows that $G_\delta(f,\cdot)$ is strictly monotone, and for $\delta > 0$, strongly monotone. Here we have used the fact that $\|P\|_{L^2(B(x_0,r))}$ defines an equivalent norm on $\mathcal{P}_N$. Furthermore, if $\delta > 0$, we see that $G_\delta(f,\cdot)\colon \mathcal{P}_N \to (\mathcal{P}_N)'$ is continuously differentiable and coercive, i.e.,

$$\frac{\langle G_\delta(f,P), P\rangle}{\|P\|} \to 0 \quad \text{as } \|P\| \to +\infty.$$

Applying the theory of monotone operators, we see that $G_\delta(f,\cdot)$ is bijective and a C^1 diffeomorphism.

(2) Let $\delta > 0$ and $f \in L^p(B(x_0,r))$. Let $P^{N,\delta}_{x_0,r}(f) \in \mathcal{P}_N$ denote the minimizer of the functional $J_\delta(\cdot - f)$ in $\mathcal{P}_N$. In view of (A.5), we have $G_\delta(f, P^{N,\delta}_{x_0,r}(f)) = 0$. Since D_2G_δ is an isomorphism from $\mathcal{P}_N$ into $(\mathcal{P}_N)'$, by the implicit function theorem we infer that the mapping $P^{N,\delta}_{x_0,r}\colon L^p(B(x_0,r)) \to \mathcal{P}_N$ is Fréchet differentiable, and we have (A.6).

Proof of (A.7). Since J_δ is convex and recalling the minimizing property of $P^{N,\delta}_{x_0,r}(f)$, we get

$$J_\delta\Big(\frac{P^{N,\delta}_{x_0,r}(f)}{2}\Big) \le \frac{1}{2}(J_\delta(P^{N,\delta}_{x_0,r}(f)-f)+J_\delta(f)) \le J_\delta(f).$$

This shows that

$$2^{-p}\int_{B(x_0,r)} |P^{N,\delta}_{x_0,r}(f)|^p\,dx \le J_\delta(f).$$

Whence, (A.7).

(3) Now, let $\delta_k \searrow 0$ as $k \to +\infty$. By (A.7) we see that $\{P^{N,\delta_k}_{x_0,r}(f)\}$ is bounded. Thus, there exists a subsequence, and $P^{N,*}_{x_0,r}(f) \in \mathcal{P}_N$ such that $P^{N,\delta_{k_j}}_{x_0,r}(f) \to P^{N,*}_{x_0,r}$ in $\mathcal{P}_N$ as $j \to +\infty$. Since

$$F_{\delta^{k_j}}(f(x)-P^{N,\delta_{k_j}}_{x_0,r}(f)) \to F_0(f(x)-P^{N,*}_{x_0,r}(f)) \quad \text{as } j \to +\infty,$$

for all $x \in B(x_0,r)$, by Lebesgue's theorem of dominated convergence, it follows that $0 = G_{\delta_{k_j}}(P^{N,\delta_{k_j}}_{x_0,r}(f)) \to G_0(f, P^{N,*}_{x_0,r})$. Since $G_0(f,\cdot)$ is strictly monotone, the zero is unique, and thus $P^{N,*}_{x_0,r}(f) = P^{N,0}_{x_0,r}(f)$. Thus, the convergence property (A.8) is verified.

Furthermore, in (A.7) letting $\delta \searrow 0$, we see that

$$\|P^{N,*}_{x_0,r}(f)\|_{L^p(B(x_0,r/2))} \le 2\|f\phi\|_{L^p(B(x_0,r))}.$$

This completes the proof of the lemma. ■

Remark A.2. The mapping $P^{N,\delta}_{x_0,r}\colon L^p(B(x_0,r)) \to \mathcal{P}_N$ fulfills the projection property

$$P^{N,\delta}_{x_0,r}(Q) = Q \quad \text{for all } Q \in \mathcal{P}_N. \tag{A.9}$$

In fact, this follows immediately from (A.1) by setting $f = Q$ therein.

B. Example of a function in $\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n) \setminus C^1(\mathbb{R}^n)$

The following example shows that $\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$ is not in $C^1(\mathbb{R}^n)$. For simplicity, we only consider the case $n=1$, since the general case $n \in \mathbb{N}$ can be reduced to $n=1$. We define

$$f(x) = \int_0^x u(y)\,dy, \quad x \in \mathbb{R},$$

where

$$u(x) = \begin{cases} 1-2^{2m}|x-2^{-m}| & \text{if } x \in I_m,\ m \in \mathbb{N},\ m \ge 2, \\ 0 & \text{otherwise,} \end{cases}$$

and $I_m = [2^{-m}-2^{-2m}, 2^{-m}+2^{-2m})$, $m \in \mathbb{N}$.

Proof of $f \in \mathcal{L}^1_{1(p,1)}(\mathbb{R})$. Thanks to (3.38), it will be sufficient to show that $u \in \mathcal{L}^0_{1(p,0)}(\mathbb{R})$. In what follows, we estimate $\operatorname{osc}_{p,0}(u;x,r)$ for $x \in [0,1]$ and $0 < r < +\infty$.

We start with the case $x = 0$. Clearly, there exists a unique $m \in \mathbb{Z}$ such that $2^{-m-1} < r \le 2^{-m}$. With this m, we get

$$\operatorname*{osc}_{p,0}(u;0,r) \le 2\Big(\frac{1}{2r}\int_{-r}^{r} |u(y)|^p\,dy\Big)^{1/p} \le 2\Big(\frac{1}{2r}\int_{0}^{r} \sum_{j=\max\{2,m\}}^{\infty} \chi_{I_j}\,dy\Big)^{1/p}$$
$$\le c r^{-1/p}\Big(\sum_{j=\max\{2,m\}}^{\infty} 2^{-2j}\Big)^{1/p} \le c\min\{2^{-m/p},2^{m/p}\}.$$

This yields

$$\text{(B.1)}\qquad \sum_{j=-\infty}^{+\infty} \operatorname*{osc}_{p,0}(u;0,2^j) = \sum_{j=-\infty}^{1} \operatorname*{osc}_{p,0}(u;0,2^j) + \sum_{j=2}^{\infty} \operatorname*{osc}_{p,0}(u;0,2^j)$$
$$\le c\sum_{j=-\infty}^{1} 2^{j/p} + c\sum_{j=2}^{\infty} 2^{-j/p} \le c < +\infty.$$

Let $x \in (0,1]$. Then there exists $m \in \mathbb{N}$ such that $2^{-m} < x \le 2^{-m+1}$. Let $0 < r < +\infty$. We consider the following three cases.

(1) For $2^{-m-1} < r < +\infty$, by the triangle inequality, we get

$$\operatorname*{osc}_{p,0}(u;x,r) \le c\operatorname*{osc}_{p,0}(u;0,8r).$$

(2) For $2^{-2m} < r \le 2^{-m-1}$, again by the triangle inequality, we find

$$\operatorname*{osc}_{p,0}(u;x,r) \le 2\Big(\frac{1}{2r}\int_{x-r}^{x+r} |u|^p\,dy\Big)^{1/p} \le 2\Big(\frac{1}{2r}\int_{0}^{1} (\chi_{I_{m+1}} + \chi_{I_m} + \chi_{I_{m-1}})\,dy\Big)^{1/p}$$
$$\le c r^{-1/p} 2^{-2m/p}.$$

(3) For $0 < r \le 2^{-2m}$, using Poincaré's inequality, we obtain

$$\operatorname*{osc}_{p,0}(u;x,r) \le c r\Big(\frac{1}{2r}\int_{x-r}^{x+r} |u'(y)|^p\,dy\Big)^{1/p}$$
$$\le c r^{1-1/p}\Big(\int_0^1 (2^{2(m+1)}\chi_{I_{m+1}} + 2^{2m}\chi_{I_m} + 2^{2(m-1)}\chi_{I_{m-1}})^p\,dy\Big)^{1/p}$$
$$\le c r^{1/p'} 2^{2m/p'},$$

where $p' = p/(p-1)$.

Using the estimates above together with (B.1), we obtain

$$\sum_{j\in\mathbb{Z}} \operatorname*{osc}_{p,0}(u;x,2^j) = \sum_{j=-m+1}^{\infty} \operatorname*{osc}_{p,0}(u;x,2^j) + \sum_{j=-2m+1}^{-m} \operatorname*{osc}_{p,0}(u;x,2^j) + \sum_{j=-\infty}^{-2m} \operatorname*{osc}_{p,0}(u;x,2^j)$$
$$\le c\sum_{j=-m+1}^{\infty} \operatorname*{osc}_{p,0}(u;0,2^{j+3}) + c2^{-2m/p}\sum_{j=-2m+1}^{-m} 2^{-j/p} + 2^{2m/p'}\sum_{j=-\infty}^{-2m} 2^{j/p'} \le c,$$

where the c stands for an absolute constant. Accordingly,

$$\sup_{x\in[0,1]} \sum_{j\in\mathbb{Z}} \operatorname*{osc}_{p,0}(u;x,2^j) < +\infty. \tag{B.2}$$

For $x < 0$, there exists $m \in \mathbb{Z}$ such that $-2^{m+1} < x \le -2^m$. Using the triangle inequality, together with (B.1), we easily see that

$$\sum_{j\in\mathbb{Z}} \operatorname*{osc}_{p,0}(u;x,2^j) \le \sum_{j=m}^{\infty} \operatorname*{osc}_{p,0}(u;x,2^j) \le c \sum_{j\in\mathbb{Z}} \operatorname*{osc}_{p,0}(u;0,2^j) \le c.$$

Similarly, with the aid of (B.2), we get

$$\sum_{j\in\mathbb{Z}} \operatorname*{osc}_{p,0}(u;x,2^j) \le c \sum_{j\in\mathbb{Z}} \operatorname*{osc}_{p,0}(u;1,2^j) \le c$$

for all $x \ge 1$. This shows that $u \in \mathcal{L}^0_{1(p,0)}(\mathbb{R}^n)$, and thus $f \in \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$ but $f \notin C^1(\mathbb{R})$ since u is not continuous in 0. ∎

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