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Symmetric subcategories, tilting modules, and derived recollements

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Abstract. We introduce symmetric subcategories of abelian categories and show that the derived category of the endomorphism ring of any good tilting module over a ring is a recollement of the derived categories of the given ring and a symmetric subcategory of the module category of the endomorphism ring, in the sense of Beilinson–Bernstein–Deligne. Thus the kernel of the total left-derived tensor functor induced by a good tilting module is always triangle equivalent to the derived category of a symmetric subcategory of a module category. Explicit descriptions of symmetric subcategories associated to good 2-tilting modules over commutative Gorenstein local rings are presented.

1. Introduction

Finitely generated tilting modules have been applied successfully to understanding different aspects of algebraic structures and homological properties of groups, algebras and modules (for example, see [13,22,24,34]), while infinitely generated tilting modules have involved many important modules, such as adic modules, Fuchs divisible modules, generic modules, and Prüfer modules, but also have been of significant interest in studying derived categories and equivalences of general algebras and rings (see [1,6,7,14,16,37]). Further, they are intimately related to the famous finitistic dimension conjecture in the representation theory of algebras (see [3]).

In the general context of tilting theory, a central theme is to study relations between the derived module categories of the given algebras and the endomorphism algebras of tilting modules. For a finitely generated tilting module ${}_AT$ over a ring A with the endomorphism ring $B := \mathrm{End}_A(T)$, Happel showed in [22] (see also [18]) that the bounded derived categories of A and B are equivalent as triangulated categories. While for a good tilting module ${}_AT$ (not necessarily finitely generated) over a ring A, the derived category of B is generally a recollement of the derived category of A and the triangulated category $\mathrm{Ker}(T \otimes_B^{\mathbb{L}} -)$, the kernel of the derived tensor functor $T \otimes_B^{\mathbb{L}} -$ (see [7]). From this point

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of view, it seems to be of great interest to understand the category $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B -)$. In [14], $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B -)$ was strengthened as the derived module category of a ring C which is a universal localization of B if the projective dimension of ${}_AT$ is at most 1. For arbitrary good tilting modules, differential graded algebras were employed to characterise $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B -)$ as a derived category of a dg algebra in [8, 37]. Though derived categories of dg algebras are triangulated categories, they may not be equivalent to derived module categories of usual rings. In fact, necessary and sufficient conditions were given in [16] for $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B -)$ to be equivalent to the derived module category of a ring.

In this paper, we establish a new and intrinsic description of $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B -)$ for an arbitrary good tilting module T by introducing symmetric subcategories of the module category of B. This description generalises results in [14, 16], and is completely different from the approaches in [8, 37]. Moreover, our strategy of proofs is also different from the ones in the literature.

To state our results, we first recall the definition of tilting modules, which have been developed in different stages and can be traced back to the papers [11] and [4].

Definition 1.1 ([1, 13, 20, 24, 30]). Let $n\geq 0$ be a natural number and let A be a unitary ring. A left A-module T is called an *n-tilting* A-module if the following three conditions hold:

$(T1)$ proj.dim$({}_AT)\leq n$, that is, there is an exact sequence of A-modules

$$0\longrightarrow P_n\longrightarrow\cdots\longrightarrow P_1\longrightarrow P_0\longrightarrow {}_AT\longrightarrow 0$$

with all P_i projective.

$(T2)$ For any nonempty set α, $\mathrm{Ext}^j_A(T,T^{(\alpha)})=0$ for all $j\geq 1$, where $T^{(\alpha)}$ stands for the direct sum of α copies of T.

$(T3)$ There exists an exact sequence of A-modules

$$0\longrightarrow {}_AA\longrightarrow T_0\longrightarrow T_1\longrightarrow\cdots\longrightarrow T_n\longrightarrow 0$$

with T_i isomorphic to a direct summand of a direct sum of copies of T for all $0\leq i\leq n$.

Recall that an n-tilting A-module T is said to be *good* (see [7]) if each T_i in $(T3)$ is isomorphic to a direct summand of the direct sum of finitely many copies of T. Given an arbitrary tilting module T', one can always find a good tilting module T such that T and T' generate the same full subcategory of the category A-Mod of all left A-modules, that is, T and T' are equivalent, while they may have different endomorphism rings.

For a good tilting A-module T, it was shown in Theorem 2.2 of [7] that the total right-derived functor of $\mathrm{Hom}_A(T,-)$ induces an equivalence between the derived category of A and the quotient category of the derived category of B modulo its full triangulated subcategory $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B -)$.

The purpose of this article is to prove the following derived recollement theorem for arbitrary good tilting modules over rings, in which an explicit description of $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B -)$ is provided by symmetric subcategories. Roughly speaking, an n-symmetric subcategory of A-Mod is an exact subcategory closed under coproducts, products, extensions as well as kernels and cokernels of exact sequences of length n (see Section 3 for the precise definition). For an exact category $\mathcal{E}$, we denote by $\mathcal{D}(\mathcal{E})$ the unbounded derived category of $\mathcal{E}$, and for a ring A, we denote by $\mathcal{D}(A)$ the unbounded derived module category of A, that is, $\mathcal{D}(A) = \mathcal{D}(A\text{-Mod})$. As usual, by $\mathcal{D}^-(\mathcal{E})$ we denote the bounded-above derived category of $\mathcal{E}$.

Theorem 1.2. *Let A be a unitary ring, T a good n-tilting A-module with $n \geq 0$, and B the endomorphism ring of ${}_AT$. Then there exists an n-symmetric subcategory $\mathcal{E}$ of B-Mod such that, for $* \in \{\emptyset, -\}$, $\mathcal{D}^*(B)$ has the recollement*

$$\begin{tikzcd} \mathcal{D}^*(\mathcal{E}) \arrow[r] & \mathcal{D}^*(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "G"] & \mathcal{D}^*(A) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} ,$$

where $G := T \otimes^{\mathbb{L}}_B -$ is the total left-derived tensor functor defined by ${}_AT_B$.

Actually, for a tilting A-module T of projective dimension n, the n-symmetric subcategory in Theorem 1.2 can be described precisely as

$$\mathcal{E} := \{Y \in B\text{-Mod} \mid \mathrm{Tor}^B_i(T, Y) = 0 \text{ for all } i \geq 0\}.$$

As an immediate consequence of Theorem 1.2, the category $\mathcal{D}(\mathcal{E})$ in Theorem 1.2 is compactly generated.

For a finitely generated tilting module ${}_AT$, it is known (see [7]) that $\mathcal{E} = 0$, and one gets Happel's theorem, that is, $\mathcal{D}(B)$ and $\mathcal{D}(A)$ are triangle equivalent. For an infinitely generated good tilting module, Theorem 1.2 may serve as a counterpart of Happel's theorem since the three categories in Theorem 1.2 are of the same type. Further, compared with the descriptions in [37] and [8], the derived category of the symmetric subcategory $\mathcal{E}$ seems more intrinsic than the one of a differential graded algebra.

A crucial part in the proof of Theorem 1.2 is to show that the kernel of the total left-derived tensor functor ${}_AT \otimes^{\mathbb{L}}_B -$ can be realised as the derived category of the n-symmetric subcategory $\mathcal{E}$. Since $\mathcal{E}$ is not an abelian subcategory in general, the arguments and methods developed in [2, 8, 14, 16] do not work. To circumvent this obstacle here, we introduce two triangle functors from $\mathcal{D}(B)$ to $\mathcal{D}(\mathcal{E})$ and from $\mathcal{D}(B)$ to $\mathcal{D}(\mathcal{E}^\perp)$ (see Section 4.1), where $\mathcal{E}^\perp$ is the right perpendicular subcategory to $\mathcal{E}$ in B-Mod. We then employ the t-structure induced from the tilting module to realise the kernel of ${}_AT \otimes^{\mathbb{L}}_B -$ as the derived category of the symmetric subcategory $\mathcal{E}$. Notably, this method sheds some new light on when the kernel of ${}_AT \otimes^{\mathbb{L}}_B -$ is triangle equivalent to the derived module category of a ring, that is, when a good tilting module is homological (see [16]). We may state this observation as a corollary.

Corollary 1.3. *The following are equivalent for a good tilting A-module T.*

(1) *T is a homological tilting module.*

(2) *$\mathcal{E}$ is an abelian subcategory of B-Mod.*

(3) *$H^m(\mathrm{Hom}_A(P^\bullet, A) \otimes_A T) = 0$ for all $m \geq 2$, where the complex $P^\bullet$ is a deleted projective resolution of ${}_AT$, and H^m is the m-th cohomology functor.*

(4) *$(\mathcal{E}, \mathcal{E}^\perp)$ is a derived decomposition of the abelian category B-Mod, where*

$$\mathcal{E}^\perp := \{Y \in B\text{-Mod} \mid \mathrm{Ext}^n_B(X, Y) = 0, \forall X \in \mathcal{E}, n \geq 0\}.$$

The notion of derived decomposition of an abelian category was introduced in [17], and provides an approach to study the derived category of an abelian category by means of the derived categories of its abelian subcategories. The equivalence of (1) and (2) is known (see Proposition 6.2 in [8]), while the equivalence of (1) and (3) is proved in Theorem 1.1 of [16]. Only the condition (4) is new. Nevertheless, we will give a new and short proof of the corollary.

Note also that the pair $(\mathcal{E}, \mathcal{E}^{\perp})$ is Ext-orthogonal, but not always complete in B-Mod in the sense of Krause and Šťovíček (see [28]). It can be seen by Corollary 1.3 and Lemma 2.3 in [17] that the pair is a complete Ext-orthogonal pair in B-Mod if and only if $\mathcal{E}$ is an abelian subcategory of B-Mod.

Theorem 1.2 also provides ways to get recollements of derived categories of other types.

Corollary 1.4. *If A is a left coherent ring (that is, every finitely generated left ideal of A is finitely presented), and T is a good tilting A-module with $B := \mathrm{End}_A(T)$, then, for $* \in \{b, +, -, \emptyset\}$, there exists a recollement of derived categories*

$$\begin{tikzcd} \mathcal{D}^*(\mathcal{E}) \arrow[r] & \mathcal{D}^*(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "G"] & \mathcal{D}^*(A) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd},$$

where $\mathcal{E}$ is a symmetric subcategory of B-Mod and G is the total left-derived tensor functor defined by ${}_AT_B$.

Finally, we provide an explicit description of symmetric subcategories $\mathcal{E}$ associated with good 2-tilting modules constructed from commutative 2-Gorenstein rings. It turns out that $\mathcal{E}$ consists of certain 3-term exact complexes over the given Gorenstein ring. Specifically, let A be a commutative noetherian, 2-Gorenstein local ring. Then we consider the minimal injective coresolution of ${}_AA$:

$$0 \longrightarrow A \xrightarrow{\lambda} Q \xrightarrow{\alpha:=(\alpha_\mathfrak{p})_{\mathfrak{p}\in\mathcal{H}}} \bigoplus_{\mathfrak{p}\in\mathcal{H}} E(A/\mathfrak{p}) \xrightarrow{\beta:=(\beta_\mathfrak{p})_{\mathfrak{p}\in\mathcal{H}}} E(A/\mathfrak{m}) \longrightarrow 0.$$

Here Q is the total quotient ring of A, λ is the canonical inclusion, $\mathcal{H}$ is the set of all prime ideals of A with height 1, $\alpha_\mathfrak{p} \in \mathrm{Hom}_A(Q, E(A/\mathfrak{p}))$ and $\beta_\mathfrak{p} \in \mathrm{Hom}_A(E(A/\mathfrak{p}), E(A/\mathfrak{m}))$. Let $\mathcal{S}$ be a nonempty subset of $\mathcal{H}$ consisting of principal ideals of A which are generated by regular elements. Define $T_0 := \alpha^{-1}(T_1 \cap \mathrm{Ker}(\beta))$, $T_1 := \bigoplus_{\mathfrak{p}\in\mathcal{S}} E(A/\mathfrak{p})$ and $T_2 := E(A/\mathfrak{m})$. Then $T := T_0 \oplus T_1 \oplus T_2$ is a good 2-tilting A-module (see Lemma 5.3). Let $B_i = \mathrm{End}_A(T_i)$ for $i = 0, 1$. We define an abelian category $\mathcal{C}(A, T)$ that has objects: the 3-term complexes of A-modules $0 \to X^{-2} \to X^{-1} \to X^0 \to 0$ with $X^{-2} \in A$-Mod, $X^{-1} \in B_1$-Mod and $X^0 \in B_0$-Mod. Let $\mathcal{C}_{\mathrm{ac}}(A, T)$ be the full subcategory of $\mathcal{C}(A, T)$ consisting of all exact complexes. Then $\mathcal{C}_{\mathrm{ac}}(A, T)$ is a fully exact subcategory of $\mathcal{C}(A, T)$. The main result in Section 5 can be summarised as follows.

Theorem 1.5. *Suppose that A is a commutative, noetherian, 2-Gorenstein, complete local ring and $\mathcal{S}$ is a nonempty finite set consisting of principal ideals of A that are generated by regular elements of A. Let $\mathcal{E}$ be the 2-symmetric subcategory of B-Mod associated with the above-defined tilting module ${}_AT$. Then there exists an equivalence B-Mod $\to \mathcal{C}(A, T)$ of abelian categories which restricts to an equivalence $\mathcal{E} \to \mathcal{C}_{\mathrm{ac}}(A, T)$ of exact categories.*

Thus Theorem 1.5 gives an explicit description of $\mathcal{E}$ associated to the 2-tilting module T in terms of 3-term exact complexes of modules. Moreover, Theorem 1.5 is applicable to regular local, complete commutative rings of Krull dimension 2 by taking $\mathcal{S}$ to be any nonempty, finite set of prime ideals of height 1. This is due to the facts that every regular local commutative ring is a unique factorisation domain; and that a noetherian integral domain is a unique factorisation domain if and only if its prime ideals of height 1 are principal. For example, let A be the algebra $k[[x_1, x_2]]$ of formal power series over a field k in two variables x_1 and x_2, and $\mathcal{S} := \{Ax_1\}$.

This article is organised as follows. In Section 2, we recall definitions, notation and basic facts needed for proofs. In Section 3, we introduce m-symmetric subcategories and give methods to construct such subcategories. In Section 4, we prove the main result, Theorem 1.2, and its corollaries. In Section 5, we present an explicit description of symmetric subcategories associated to 2-tilting modules and end the section by three questions on further study of symmetric subcategories. For example, which n-symmetric subcategories of a module category can be characterized by n-weak tilting modules?

2. Preliminaries

In this section we briefly recall some definitions, facts and notation used in this paper. For unexplained notation employed in this paper, we refer the reader to [14, 16] and the references therein.

2.1. Semi-orthogonal decompositions, recollements and homotopy (co)limits

Let $\mathcal{C}$ be an additive category.

A full subcategory of $\mathcal{C}$ is always assumed to be closed under isomorphisms. For an object $X \in \mathcal{C}$, we write $\mathrm{add}(X)$ for the full subcategory of $\mathcal{C}$ consisting of all direct summands of finite coproducts of copies of X. If $\mathcal{C}$ admits coproducts (that is, coproducts indexed over sets exist in $\mathcal{C}$), we write $\mathrm{Add}\,(X)$ for the full subcategory of $\mathcal{C}$ consisting of all direct summands of coproducts of copies of X. Dually, we write $\mathrm{Prod}\,(X)$ for the full subcategory of $\mathcal{C}$ consisting of all direct summands of products of copies of X if $\mathcal{C}$ admits products (that is, products indexed over sets exist in $\mathcal{C}$).

We denote the composition of two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{C}$ by fg. We write f^* for $\mathrm{Hom}_{\mathcal{C}}(Z, f)\colon \mathrm{Hom}_{\mathcal{C}}(Z, X) \to \mathrm{Hom}_{\mathcal{C}}(Z, Y)$, and f_* for $\mathrm{Hom}_{\mathcal{C}}(f, Z)\colon \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$. While for two functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$, the composition of F and G is denoted by GF, which is a functor from $\mathcal{C}$ to $\mathcal{E}$. The kernel and image of F are defined as

$$\mathrm{Ker}(F) := \{X \in \mathcal{C} \mid FX \simeq 0\} \quad \text{and} \quad \mathrm{Im}(F) := \{Y \in \mathcal{D} \mid \exists\, X \in \mathcal{C},\ FX \simeq Y\},$$

respectively. Thus $\mathrm{Ker}(F)$ and $\mathrm{Im}(F)$ are closed under isomorphisms in $\mathcal{C}$.

For a complex $X^\bullet = (X^i, d_X^i)_{i\in\mathbb{Z}}$ over $\mathcal{C}$, the morphism $d_X^i\colon X^i \to X^{i+1}$ is called a *differential* of $X^\bullet$. For simplicity, we sometimes write $(X^i)_{i\in\mathbb{Z}}$ for $X^\bullet$ without mentioning d_X^i. For an integer n, we denote by $X^\bullet[n]$ the complex by shifting n degrees, namely $(X^\bullet[n])^i = X^{n+i}$ and $d^i_{X^\bullet[n]} = (-1)^n d_X^{n+i}$.

Let $\mathcal{C}(\mathscr{C})$ be the category of all complexes over $\mathscr{C}$ with chain maps, and let $\mathcal{K}(\mathscr{C})$ be the homotopy category of $\mathcal{C}(\mathscr{C})$. As usual, we denote by $\mathcal{C}^-(\mathscr{C})$ the category of bounded-above complexes over $\mathscr{C}$, and by $\mathcal{K}^-(\mathscr{C})$ the homotopy category of $\mathcal{C}^-(\mathscr{C})$. Note that $\mathcal{K}(\mathscr{C})$ and $\mathcal{K}^-(\mathscr{C})$ are triangulated categories. For a triangulated category, its shift functor is denoted by [1] universally.

Now, let $\mathcal{A}$ be an abelian category and let $X^\bullet \in \mathcal{C}(\mathcal{A})$. For $n \in \mathbb{Z}$, there are two left-truncated complexes

$$X^{\leq n}: \quad \cdots \longrightarrow X^{n-3} \xrightarrow{d_X^{n-3}} X^{n-2} \xrightarrow{d_X^{n-2}} X^{n-1} \xrightarrow{d_X^{n-1}} X^n \longrightarrow 0,$$
$$\tau^{\leq n} X^\bullet: \quad \cdots \longrightarrow X^{n-3} \xrightarrow{d_X^{n-3}} X^{n-2} \xrightarrow{d_X^{n-2}} X^{n-1} \xrightarrow{d_X^{n-1}} \mathrm{Ker}(d_X^n) \longrightarrow 0,$$

together with canonical chain maps

$$X^\bullet \to X^{\leq n+1} \to X^{\leq n} \quad \text{and} \quad \tau^{\leq n} X^\bullet \to \tau^{\leq n+1} X^\bullet \to X^\bullet.$$

Dually, right-truncated complexes $X^{\geq n}$ and $\tau^{\geq n} X^\bullet$ (by taking cokernels on the left) can be defined. Further, there are bi-truncated complexes for a pair of integers (n, m) with $n < m$:

$$X^{[n,m)}: \quad 0 \longrightarrow X^n \xrightarrow{d_X^n} X^{n+1} \longrightarrow \cdots \xrightarrow{d_X^{m-2}} X^{m-1} \xrightarrow{d_X^{m-1}} \mathrm{Ker}(d_X^m) \longrightarrow 0,$$
$$X^{(n,m]}: \quad 0 \longrightarrow \mathrm{Coker}(d_X^{n-1}) \xrightarrow{\overline{d_X^n}} X^{n+1} \longrightarrow \cdots \xrightarrow{d_X^{m-2}} X^{m-1} \xrightarrow{d_X^{m-1}} X^m \longrightarrow 0.$$

where $\overline{d_X^n}$ is induced from d_X^n.

Let $H^n(X^\bullet)$ be the cohomology of $X^\bullet$ in degree n. Then $H^n(-)$ is a functor from $\mathcal{C}(\mathcal{A})$ to $\mathcal{A}$ for all n.

Next, we recall the definition of semi-orthogonal decompositions of triangulated categories (for example, see Chapter 11 of [25]). Note that semi-orthogonal decompositions are also termed hereditary torsion pairs (see Chapter I.2 of [10]). They are closely related to Bousfield localizations (see Section 9.1 of [31]) and to t-structures of triangulated categories (see [9]).

Definition 2.1. Let $\mathcal{D}$ be a triangulated category. A pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{D}$ is called a *semi-orthogonal decomposition* of $\mathcal{D}$ if

(1) $\mathcal{X}$ and $\mathcal{Y}$ are triangulated subcategories of $\mathcal{D}$.

(2) $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

(3) For $D \in \mathcal{D}$, there is a triangle in $\mathcal{D}$,

$$X_D \xrightarrow{f_D} D \xrightarrow{g^D} Y^D \longrightarrow X_D[1],$$

such that $X_D \in \mathcal{X}$ and $Y^D \in \mathcal{Y}$.

The following is well known (for example, see Chapter I.2 of [10]), and will be used later.

Lemma 2.2. *Let* $(\mathcal{X}, \mathcal{Y})$ *be a semi-orthogonal decomposition of* $\mathcal{D}$. *The following hold.*

(1) *The inclusion* $\mathbf{i}: \mathcal{X} \to \mathcal{D}$ *has a right adjoint* $\mathbf{R}: \mathcal{D} \to \mathcal{X}$ *given by* $D \mapsto X_D$ *for each* $D \in \mathcal{D}$, *such that* f_D *is the counit adjunction morphism of* D. *Dually, the inclusion* $\mathbf{j}: \mathcal{Y} \to \mathcal{D}$ *has a left adjoint* $\mathbf{L}: \mathcal{D} \to \mathcal{Y}$ *given by* $D \mapsto Y^D$ *for each* $D \in \mathcal{D}$ *such that* g^D *is the unit adjunction morphism of* D.

(2) $\mathrm{Ker}(\mathbf{L}) = \mathcal{X}$, *and* $\mathbf{L}$ *induces a triangle equivalence* $\bar{\mathbf{L}}: \mathcal{D}/\mathcal{X} \to \mathcal{Y}$ *of which a quasi-inverse is the composition of* $\mathbf{j}$ *with the localization functor* $\mathcal{D} \to \mathcal{D}/\mathcal{X}$. *Dually,* $\mathrm{Ker}(\mathbf{R}) = \mathcal{Y}$, *and* $\mathbf{R}$ *induces a triangle equivalence* $\bar{\mathbf{R}}: \mathcal{D}/\mathcal{Y} \to \mathcal{X}$ *of which a quasi-inverse is the composition of* $\mathbf{i}$ *with the localization functor* $\mathcal{D} \to \mathcal{D}/\mathcal{Y}$.

Semi-orthogonal decompositions are intimately connected with recollements of triangulated categories introduced by Beilinson, Bernstein and Deligne in [9] for understanding derived categories of perverse sheaves over singular spaces. Later, this has widely been used in representation theories of algebraic groups, Lie algebras and associative algebras (see, for example, [6, 15, 19, 23]).

Definition 2.3 ([9]). Let $\mathscr{D}$, $\mathscr{D}'$ and $\mathscr{D}''$ be triangulated categories. The category $\mathscr{D}$ is called a *recollement* of $\mathscr{D}'$ and $\mathscr{D}''$ (or there is a recollement among $\mathscr{D}''$, $\mathscr{D}$ and $\mathscr{D}'$) if there are six triangle functors

$$\begin{tikzcd} \mathscr{D}'' \arrow[r, "i_*=i_!"] & \mathscr{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*"] & \mathscr{D}' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

satisfying the following four properties:

(1) The four pairs (i^*, i_*), $(i_!, i^!)$, $(j_!, j^!)$ and (j^*, j_*) of functors are adjoint.

(2) The three functors i_*, j_* and $j_!$ are fully faithful.

(3) $i^! j_* = 0$ (and thus also $j^! i_! = 0$ and $i^* j_! = 0$).

(4) There are two triangles for each object D in $\mathscr{D}$:

$$i_! i^!(D) \longrightarrow D \longrightarrow j_* j^*(D) \longrightarrow i_! i^!(D)[1]$$

and

$$j_! j^!(D) \longrightarrow D \longrightarrow i_* i^*(D) \longrightarrow j_! j^!(D)[1],$$

where $i_! i^!(D) \to D$ and $j_! j^!(D) \to D$ are counit adjunction morphisms, and $D \to j_* j^*(D)$ and $D \to i_* i^*(D)$ are unit adjunction morphisms.

If $\mathscr{D}$ is a recollement of $\mathscr{D}'$ and $\mathscr{D}''$, the pairs $(\mathrm{Im}(j_!), \mathrm{Im}(i_*))$ and $(\mathrm{Im}(i_*), \mathrm{Im}(j_*))$ are semi-orthogonal decompositions of $\mathcal{D}$ (see Lemma 2.6 in [14]).

Next, we recall the definition of homotopy colimits and limits in triangulated categories.

Definition 2.4 ([12, 31]). Let $\mathcal{D}$ be a triangulated category such that countable coproducts exist in $\mathcal{D}$. Let

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$$

be a sequence of objects and morphisms in $\mathcal{D}$. The *homotopy colimit* of this sequence, denoted by $\underrightarrow{\mathrm{Hocolim}}(X_n)$, is given, up to non-canonical isomorphism, by the triangle

$$\bigoplus_{n\geq 1} X_n \xrightarrow{(1-f_*)} \bigoplus_{n\geq 1} X_n \longrightarrow \underrightarrow{\mathrm{Hocolim}}(X_n) \longrightarrow \bigoplus_{n\geq 1} X_n[1],$$

where the morphism $(1-f_*)$ is induced by $(\mathrm{Id}_{X_i}, -f_i): X_i \to X_i \oplus X_{i+1} \subseteq \bigoplus_{n\geq 1} X_n$ for all $i \in \mathbb{N}$.

Dually, the *homotopy limit*, denoted by $\underleftarrow{\mathrm{Holim}}$, of a sequence of objects and morphisms in a triangulated category with countable products can be defined.

Now, we consider homotopy colimits and limits in derived categories of abelian categories. Let $\mathcal{A}$ be an abelian category. Recall that $\mathcal{A}$ *satisfies* AB4 if coproducts indexed over sets exist in $\mathcal{A}$ and coproducts of short exact sequences in $\mathcal{A}$ are exact. Dually, $\mathcal{A}$ *satisfies* AB4$'$ if products indexed over sets exist in $\mathcal{A}$ and products of short exact sequences in $\mathcal{A}$ are exact. An example of abelian categories satisfying both AB4 and AB4$'$ is the module category of a ring. A full subcategory $\mathcal{B}$ of the abelian category $\mathcal{A}$ is called an *abelian subcategory* of $\mathcal{A}$ if $\mathcal{B}$ is an abelian category and the inclusion $\mathcal{B} \to \mathcal{A}$ is an exact functor between abelian categories. This is equivalent to saying that $\mathcal{B}$ is closed under taking kernels and cokernels in $\mathcal{A}$.

The following result is known, see Lemma 6.1 in [32] and its dual.

Lemma 2.5. *Let $\mathcal{A}$ be an abelian category satisfying $AB4$ and $AB4'$. For a complex $X^\bullet \in \mathscr{C}(\mathcal{A})$, there are isomorphisms in $\mathcal{K}(\mathcal{A})$ (and also in $\mathcal{D}(\mathcal{A})$),*

$$X^\bullet \simeq \underrightarrow{\mathrm{Hocolim}}(X^{\geq -n}) \simeq \underrightarrow{\mathrm{Hocolim}}(X^{[-n,n+1)}) \simeq \underleftarrow{\mathrm{Holim}}(X^{\leq n}) \simeq \underleftarrow{\mathrm{Holim}}(X^{(-n,n+1]}),$$

where n runs over all natural numbers.

For the convenience of the reader, we mention the following fact on adjoint pairs of functors.

Lemma 2.6. *Let $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ be two functors of categories such that (F, G) forms a adjoint pair. Suppose that G is fully faithful.*

(1) *The counit $\epsilon: F \circ G \to \mathrm{Id}_{\mathcal{D}}$ is an isomorphism.*

(2) *G gives rise to an equivalence $\mathcal{D} \to \mathrm{Im}(G)$ with the quasi-inverse given by the restriction of F to $\mathrm{Im}(G)$. In particular, if $C \in \mathrm{Im}(G)$, then the unit $\eta_C: C \to GF(C)$ is an isomorphism.*

To judge the faithfulness of triangle functors, we need the following result in which part (1) follows from the dual statement of Lemma 10.3 in [27], while part (2) follows from [34], p. 446.

Lemma 2.7. *Let $\mathcal{D}$ and $\mathcal{D}'$ be triangulated categories.*

(1) *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are full triangulated subcategories of $\mathcal{D}$. If each morphism $X \to Y$ in $\mathcal{D}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ admits a factorisation $X \to N \to Y$ with $N \in \mathcal{X} \cap \mathcal{Y}$, then the canonical triangle functor $\mathcal{X}/(\mathcal{X} \cap \mathcal{Y}) \to \mathcal{D}/\mathcal{Y}$ is fully faithful.*

(2) *Let $F: \mathcal{D} \to \mathcal{D}'$ be a triangle functor. If F is full and sends nonzero objects to nonzero objects, then it is faithful.*

2.2. Derived categories of exact categories

An *exact category* (in the sense of Quillen) is by definition an additive category endowed with a class of conflations closed under isomorphism and satisfying certain axioms (see Section 4 in [27] for details). When the additive category is abelian, the class of conflations coincides with the class of short exact sequences.

Let $\mathcal{E}$ be an exact category and let $\mathcal{F}$ be a full subcategory of $\mathcal{E}$. Suppose that $\mathcal{F}$ is *closed under extensions* in $\mathcal{E}$, that is, for any conflation $0 \to X \to Y \to Z \to 0$ in $\mathcal{E}$ with both $X, Z \in \mathcal{F}$, we have $Y \in \mathcal{F}$. Then $\mathcal{F}$, endowed with the conflations in $\mathcal{E}$ having their terms in $\mathcal{F}$, is an exact category, and the inclusion $\mathcal{F} \subseteq \mathcal{E}$ is a fully faithful exact functor. In this case, $\mathcal{F}$ is called a *fully exact subcategory* of $\mathcal{E}$ (see Section 4 in [27]).

From now on, let $\mathcal{A}$ be an abelian category and $\mathcal{E}$ a fully exact subcategory of $\mathcal{A}$. A complex $X^\bullet \in \mathcal{C}(\mathcal{E})$ is said to be *strictly exact* if it is exact in $\mathcal{C}(\mathcal{A})$ and all of its boundaries belong to $\mathcal{E}$. Let $\mathcal{K}_{\mathrm{ac}}(\mathcal{E})$ be the full subcategory of $\mathcal{K}(\mathcal{E})$ consisting of those complexes which are isomorphic to strictly exact complexes. Then $\mathcal{K}_{\mathrm{ac}}(\mathcal{E})$ is a full triangulated subcategory of $\mathcal{K}(\mathcal{E})$ closed under direct summands. The *unbounded derived category* of $\mathcal{E}$, denoted by $\mathcal{D}(\mathcal{E})$, is defined to be the Verdier quotient of $\mathcal{K}(\mathcal{E})$ by $\mathcal{K}_{\mathrm{ac}}(\mathcal{E})$. Similarly, the bounded-below, bounded-above and bounded derived categories $\mathcal{D}^+(\mathcal{E})$, $\mathcal{D}^-(\mathcal{E})$ and $\mathcal{D}^b(\mathcal{E})$ can be defined. Observe that the canonical functor $\mathcal{D}^*(\mathcal{E}) \to \mathcal{D}(\mathcal{E})$ is fully faithful for $* \in \{+, -, b\}$.

If $F\colon \mathcal{E}_1 \to \mathcal{E}_2$ is an additive functor of exact categories, then F induces a functor $K(F)\colon \mathcal{K}(\mathcal{E}_1) \to \mathcal{K}(\mathcal{E}_2)$ of homotopy categories. Further, if F is an exact functor, that is, F sends conflations in $\mathcal{E}_1$ to the ones in $\mathcal{E}_2$, then F induces a functor $D(F)\colon \mathcal{D}(\mathcal{E}_1) \to \mathcal{D}(\mathcal{E}_2)$ of derived categories.

Let $\mathcal{F}$ and $\mathcal{E}$ be fully exact subcategories of $\mathcal{A}$ with $\mathcal{F} \subseteq \mathcal{E}$. Then $\mathcal{F}$ can be regarded as a fully exact subcategory of $\mathcal{E}$. We consider the following three conditions:

(a) If $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathcal{A}$, with $X \in \mathcal{E}$ and $Y, Z \in \mathcal{F}$, then $X \in \mathcal{F}$.

(b) Any exact sequence $0 \to E_1 \to E_0 \to F \to 0$ in $\mathcal{A}$, with $E_1, E_0 \in \mathcal{E}$ and $F \in \mathcal{F}$, fits into an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F_1 \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E_1 \arrow[r] & E_0 \arrow[r] & F \arrow[r] & 0
\end{tikzcd}$$

with $F_1, F_0 \in \mathcal{F}$.

(c) There is a natural number n such that, for each object $E \in \mathcal{E}$, there is a long exact sequence in $\mathcal{A}$,

$$0 \longrightarrow F_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} E \longrightarrow 0,$$

with $F_i \in \mathcal{F}$ and $\mathrm{Im}(f_i) \in \mathcal{E}$ for all $0 \le i \le n$.

The following result describes impacts of these conditions on derived categories.

Lemma 2.8. (1) *If* (a) *and* (b) *hold, then the inclusion* $\mathcal{F} \subseteq \mathcal{E}$ *induces a fully faithful triangle functor* $\mathcal{D}^-(\mathcal{F}) \to \mathcal{D}^-(\mathcal{E})$.

(2) *If $\mathcal{E}$ is closed under direct summands in $\mathcal{A}$ and if* (a) *and* (c) *hold, then the inclusion $\mathcal{F} \subseteq \mathcal{E}$ induces a triangle equivalence $\mathcal{D}(\mathcal{F}) \to \mathcal{D}(\mathcal{E})$ which restricts to an equivalence $\mathcal{D}^*(\mathcal{F}) \to \mathcal{D}^*(\mathcal{E})$ for any $* \in \{+, -, b\}$.*

Proof. (1) follows from the dual of Theorem 12.1 in [27] (see also Proposition A.2.1 in [33]), while (2) follows from Proposition A.5.6 in [33]. ∎

If all arrows in (a), (b) and (c) are reversed, we get dual conditions of (a), (b) and (c), respectively. So the dual version of Lemma 2.8 holds true.

2.3. Derived functors of the module categories of rings

Let R be a (unitary associative) ring. The full subcategories of projective and injective R-modules are denoted by R-Proj and R-Inj, respectively. If $M \in R$-Mod and I is a nonempty set, then $M^{(I)}$ and M^I denote the direct sum and product of I copies of M, respectively. The projective dimension and the endomorphism ring of M are denoted by proj.dim$({}_RM)$ and $\mathrm{End}_R(M)$, respectively.

A full subcategory $\mathcal{T}$ of R-Mod is called a *thick subcategory* if it is closed under direct summands in R-Mod and has the *two out of three property*: for any short exact sequence $0 \to X \to Y \to Z \to 0$ in R-Mod with two terms in $\mathcal{T}$, the third term belongs to $\mathcal{T}$ as well.

We write $\mathcal{C}(R)$, $\mathcal{K}(R)$ and $\mathcal{D}(R)$ for $\mathcal{C}(R\text{-Mod})$, $\mathcal{K}(R\text{-Mod})$ and $\mathcal{D}(R\text{-Mod})$, respectively, and regard R-Mod as the subcategory of $\mathcal{D}(R)$ consisting of all stalk complexes concentrated in degree zero.

Now we recall some definitions and basic facts on derived functors (see [26, 36] for more details). Let $\mathcal{K}(R)_P$ (respectively, $\mathcal{K}(R)_I$) be the smallest full triangulated subcategory of $\mathcal{K}(R)$, which contains all the bounded-above (respectively, bounded-below) complexes of projective (respectively, injective) R-modules, and is closed under arbitrary direct sums (respectively, direct products). Since $\mathcal{K}(R\text{-Proj})$ satisfies these two properties, we have $\mathcal{K}(R)_P \subseteq \mathcal{K}(R\text{-Proj})$. Let $\mathcal{K}_{\mathrm{ac}}(R)$ be the full subcategory of $\mathcal{K}(R)$ consisting of all exact complexes. Then $(\mathcal{K}(R)_P, \mathcal{K}_{\mathrm{ac}}(R))$ forms a semi-orthogonal decomposition of $\mathcal{K}(R)$ (see the dual version of Proposition 2.12 in [12]). Let ${}_p(-)\colon \mathcal{K}(R) \to \mathcal{K}(R)_P$ be a right adjoint of the inclusion $\mathcal{K}(R)_P \to \mathcal{K}(R)$. Then, by Lemma 2.2, the functor ${}_p(-)$ induces a triangle equivalence $\mathcal{D}(R) \xrightarrow{\simeq} \mathcal{K}(R)_P$ of which a quasi-inverse is the composition of the inclusion $\mathcal{K}(R)_P \to \mathcal{K}(R)$ with the localization functor $\mathcal{K}(R) \to \mathcal{D}(R)$. Moreover, for each $X^\bullet$ in $\mathcal{K}(R)$, the associated counit adjunction morphism $\alpha_{X^\bullet}\colon {}_pX^\bullet \to X^\bullet$ is a quasi-isomorphism. Thus $\alpha_{X^\bullet}$ or simply ${}_pX^\bullet$ is called a *projective resolution* of $X^\bullet$ in $\mathcal{D}(R)$. For example, if X is an R-module, then ${}_pX$ can be chosen as a deleted projective resolution of ${}_RX$. Dually, $(\mathcal{K}_{\mathrm{ac}}(R), \mathcal{K}(R)_I)$ is a semi-orthogonal decomposition of $\mathcal{K}(R)$ (see Proposition 2.12 in [12]). In particular, there exists a quasi-isomorphism $\beta_{X^\bullet}\colon X^\bullet \to {}_iX^\bullet$ in $\mathcal{K}(R)$ with ${}_iX^\bullet \in \mathcal{K}(R)_I$. The complex ${}_iX^\bullet$ is called the *injective resolution* of $X^\bullet$ in $\mathcal{D}(R)$. In particular, $\mathrm{Hom}_{\mathcal{K}(R)}(P^\bullet, X^\bullet) \simeq \mathrm{Hom}_{\mathcal{D}(R)}(P^\bullet, X^\bullet)$ and $\mathrm{Hom}_{\mathcal{K}(R)}(X^\bullet, I^\bullet) \simeq \mathrm{Hom}_{\mathcal{D}(R)}(X^\bullet, I^\bullet)$ for $P^\bullet \in \mathcal{K}(R)_P$, $I^\bullet \in \mathcal{K}(R)_I$ and $X^\bullet \in \mathcal{K}(R)$.

Let S be another ring. For a triangle functor $F\colon \mathcal{K}(R) \to \mathcal{K}(S)$, its *total left-derived functor* $\mathbb{L}F\colon \mathcal{D}(R) \to \mathcal{D}(S)$ is defined by $X^\bullet \mapsto F({}_pX^\bullet)$, and its *total right-derived functor* $\mathbb{R}F\colon \mathcal{D}(R) \to \mathcal{D}(S)$ is defined by $X^\bullet \mapsto F({}_iX^\bullet)$. Further, if $F(X^\bullet)$ is exact whenever $X^\bullet$ is exact, then F induces a triangle functor $D(F)\colon \mathcal{D}(R) \to \mathcal{D}(S)$, $X^\bullet \mapsto F(X^\bullet)$. In this case, up to natural isomorphism, $\mathbb{L}F = \mathbb{R}F = D(F)$, and $D(F)$ is called the *derived functor* of F.

Given a complex $M^\bullet$ of R-S-bimodules, we denote by $\mathbb{R}\mathrm{Hom}_R(M^\bullet, -)$ the total right-derived functor of $\mathrm{Hom}^\bullet_R(M^\bullet, -)$, and we denote by $M^\bullet \otimes^{\mathbb{L}}_S -$ the total left-derived functor of $M^\bullet \otimes^\bullet_S -$. Then $(M^\bullet \otimes^{\mathbb{L}}_S -, \mathbb{R}\mathrm{Hom}_R(M^\bullet, -))$ is an adjoint pair of triangle functors. In case of $Y \in S$-Mod and $X \in R$-Mod, we write $M^\bullet \otimes_S Y$ and $\mathrm{Hom}_R(M^\bullet, X)$ for $M^\bullet \otimes^\bullet_S Y$ and $\mathrm{Hom}^\bullet_R(M^\bullet, X)$, respectively.

3. Symmetric subcategories of abelian categories

Now we introduce n-symmetric subcategories of an abelian category for $n \geq 0$.

Recall that an abelian category is *complete* (respectively, *cocomplete*) if it has products (respectively, coproducts) indexed over sets; and *bicomplete* if it is both complete and cocomplete.

Definition 3.1. Let $n \in \mathbb{N}$, and let $\mathcal{A}$ be a bicomplete abelian category. An additive full subcategory $\mathcal{B}$ of $\mathcal{A}$ is said to be *n-symmetric* if

(1) $\mathcal{B}$ is closed under extensions, products and coproducts.

(2) For any exact sequence $0 \to X \to M_n \to \cdots \to M_1 \to M_0 \to Y \to 0$ in $\mathcal{A}$ with all $M_i \in \mathcal{B}$, we have $X, Y \in \mathcal{B}$.

Remark 3.2. Let $\mathcal{B}$ be an additive full subcategory of a bicomplete abelian category $\mathcal{A}$.

(1) If $\mathcal{B}$ is an n-symmetric subcategory of $\mathcal{A}$, then $\mathcal{B}$ is an exact, thick subcategory of $\mathcal{A}$. It is also $(n+1)$-symmetric.

(2) If $\mathcal{B}_i$ is an m_i-symmetric subcategories of $\mathcal{A}$ for $i = 1, 2$, then the category $\mathcal{B}_1 \cap \mathcal{B}_2$ is a $\max\{m_1, m_2\}$-symmetric subcategory of $\mathcal{A}$.

(3) If $\mathcal{B}$ is closed under extensions and satisfies Definition 3.1 (2), then $\mathcal{B}$ is an n-wide subcategory of $\mathcal{A}$ in the sense of Definition 4.1 in [29].

(4) $\mathcal{B}$ is 0-symmetric if and only if $\mathcal{B}$ is a Serre subcategory (that is, closed under subobjects, quotient objects and extensions) closed under coproducts and products if and only $\mathcal{B}$ is a localizing subcategory (that is, a Serre subcategory closed under coproducts) closed under products. $\mathcal{B}$ is 1-symmetric if and only *if* $\mathcal{B}$ is an abelian subcategory closed under extensions, coproducts and products.

In particular, if R is a unitary ring and $\mathcal{A} = R$-Mod, then a full subcategory $\mathcal{B}$ of $\mathcal{A}$ is 0-symmetric if and only if there is an ideal I of R with $I^2 = I$ such that

$$\mathcal{B} = (R/I)\text{-Mod} = \{M \in R\text{-Mod} \mid IM = 0\}.$$

This follows from Proposition 6.12 in [35].

A full subcategory $\mathcal{B}$ of $\mathcal{A}$ is 1-symmetric if and only if there is a ring epimorphism $\lambda\colon R \to S$ with $\mathrm{Tor}^R_1(S, S) = 0$ such that $\mathcal{B}$ coincides with the image of the induced fully faithful functor $\lambda_*\colon S\text{-Mod} \to R\text{-Mod}$, that is, $\mathcal{B} = \mathrm{Im}(\lambda_*) \simeq S$-Mod. This can be seen by Lemma 2.1 in [14].

Proposition 3.3. *Let $\mathcal{A}$ and $\mathcal{C}$ be bicomplete abelian categories satisfying AB4 and AB4′, and let $F\colon \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{C})$ be a triangle functor commuting with coproducts and products. If there are integers $s \le r$ such that $H^i(FX) = 0$ for all $X \in \mathcal{A}$, $r < i$, or $i < s$, then $\mathcal{E} := \mathcal{A} \cap \mathrm{Ker}(F)$ is an $(r-s)$-symmetric subcategory of $\mathcal{A}$.*

Proof. It is easy to see that $\mathcal{E}$ is an additive subcategory and closed under extensions, coproducts and products. Let $n := r - s$ and let $0 \to X \to M_n \to \cdots \to M_1 \to M_0 \to Y \to 0$ be an exact sequence in $\mathcal{A}$ with $M_j \in \mathcal{E}$ for all j. By assumption, we have $H^i(F(Y)) = 0$ for $i > r$ or $i < s$. Now, let $s \le i \le r$. Then $n + 1 + i \ge n + 1 + s = r + 1$. It follows from $F(M_j) = 0$ for $0 \le j \le n$ that $F(Y) \simeq F(X)[n+1]$ in $\mathcal{D}(\mathcal{C})$. This implies $H^i(F(Y)) \simeq H^i(F(X)[n+1]) \simeq H^{n+1+i}(F(X))$ for $i \in \mathbb{Z}$. In particular, $H^i(F(Y)) = 0$ by assumption. Thus $H^i(F(Y)) = 0$ for all $i \in \mathbb{Z}$ and $Y \in \mathrm{Ker}(F)$. This also shows $X \in \mathrm{Ker}(F)$. Thus $\mathcal{E}$ is an n-symmetric subcategory of $\mathcal{A}$. ∎

As an application of Proposition 3.3, we consider the module categories of rings.

Example 3.4. Let R and S be rings, and let ${}_RM_S$ be an R-S-bimodule.

(1) If M_S has a finite projective resolution of length n by finitely generated projective right S-modules, that is, there is an exact sequence

$$0 \to Q_n \to \cdots \to Q_1 \to Q_0 \to M_S \to 0$$

with all Q_j being finitely generated projective right S-modules, then $\mathrm{Ker}(M \otimes_S^{\mathbb{L}} -) \cap S\text{-Mod}$ is an n-symmetric subcategory of S-Mod. In this case, taking $F := M \otimes_S^{\mathbb{L}} -$, $s := -n$ and $r := 0$, we then get $H^i(F(X)) = \mathrm{Tor}^S_{-i}(M, X) = 0$ for $X \in S$-Mod and $r < i$ or $i < s$. Note that

$$\mathrm{Ker}(M \otimes_S^{\mathbb{L}} -) \cap S\text{-Mod} = \{Y \in S\text{-Mod} \mid \mathrm{Tor}_i^S(M, Y) = 0, \forall i \ge 0\}.$$

(2) If ${}_RM$ has a finite projective resolution of length n by finitely generated projective modules, then $\mathrm{Ker}(\mathbb{R}\mathrm{Hom}_R(M, -)) \cap R\text{-Mod}$ is an n-symmetric subcategory of R-Mod. In this case, taking $F := \mathbb{R}\mathrm{Hom}_R(M, -)$, $s := 0$ and $r := n$, we then get $H^i(F(X)) = \mathrm{Ext}^i_R(M, X) = 0$ for $X \in R$-Mod and $r < i$ or $i < s$. Clearly,

$$\mathrm{Ker}(\mathbb{R}\mathrm{Hom}_R(M, -)) \cap R\text{-Mod} = \{X \in R\text{-Mod} \mid \mathrm{Ext}^i_R(M, X) = 0, \forall i \ge 0\}.$$

(3) Let ${}_AT$ be a good n-tilting A-module ($n \ge 0$) with $B := \mathrm{End}_A(T)$. Then the category

$$\mathcal{E} := \mathrm{Ker}(T \otimes_B^{\mathbb{L}} -) \cap B\text{-Mod}$$

is always an n-symmetric subcategory of B-Mod. In fact, since ${}_AT$ is good, the right B-module T_B has a finite projective resolution of length n by finitely generated projective right B-modules. Now (3) follows from (1). Further, $\mathcal{E}$ is 0-symmetric if and only if ${}_AT$ is pure-projective if and only if the heart of the t-structure induced from the tilting module ${}_AT$ is a Grothendieck category by Theorems 5.12 and 7.5 in [6], and $\mathcal{E}$ is 1-symmetric if and only if ${}_AT$ is homological by Theorem 1.1 in [16].

4. Tilting modules and derived recollements

This entire section is devoted to proving Theorem 1.2. Roughly speaking, the strategy is to find a right adjoint of the derived functor $D(i)$ induced from the inclusion $i\colon \mathcal{E} \to B$-Mod, and then to show that $\operatorname{Ker}(G)$ is triangle equivalent to $\mathcal{D}(\mathcal{E})$. We first make a couple of preparations.

4.1. Derived functors induced by good tilting modules

Throughout this section, A denotes a unitary ring, T a *good* n-tilting A-module with $(T1)$, $(T2)$, and $(T3)$ for a natural number n, and $B := \operatorname{End}_A(T)$. Further, we define

$$G := {}_AT \otimes^{\mathbb{L}}_B - : \mathcal{D}(B) \longrightarrow \mathcal{D}(A) \quad \text{and} \quad H := \mathbb{R}\operatorname{Hom}_A(T, -) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$$

to be the total left- and right-derived functors of ${}_AT_B$, respectively. We write

$$\eta' : \mathrm{Id}_{B\text{-Mod}} \longrightarrow \operatorname{Hom}_A(T, T \otimes_B -) \quad \text{and} \quad \eta : \mathrm{Id}_{\mathcal{D}(B)} \longrightarrow H \circ G$$

for the unit adjunctions associated with the adjoint pairs $(T \otimes_B -, \operatorname{Hom}_A(T, -))$ and (G, H), respectively. Recall that the restriction of the localization functor $\mathcal{K}(B) \to \mathcal{D}(B)$ to $\mathcal{K}(B)_p$ is a triangle equivalence. A quasi-inverse of this equivalence is denoted by

$${}_p(-) : \mathcal{D}(B) \longrightarrow \mathcal{K}(B)_P.$$

Without loss of generality, we assume that ${}_pM$ is a deleted projective resolution of a B-module M. Moreover, let

$$\mathcal{E} := \operatorname{Ker}(T \otimes^{\mathbb{L}}_B -) \cap B\text{-Mod},$$

which is a fully exact subcategory of B-Mod.

The derived categories $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are related by the following recollement, which was implied by Theorem 2.2 in [7]. For a detailed explanation of the existence of the recollement, we refer to Lemma 5.3 in [16].

Theorem 4.1 ([7]). *There exists a recollement of triangulated categories*:

$$\begin{tikzcd} \operatorname{Ker}(G) \arrow[r] & \mathcal{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "G"] & \mathcal{D}(A) \arrow[l, bend right] \arrow[l, bend left, "H"] \end{tikzcd}.$$

In particular, $(\operatorname{Ker}(G), \operatorname{Im}(H))$ *is a semi-orthogonal decomposition of* $\mathcal{D}(B)$.

Observe that the modules ${}_AT$ and T_B have finite projective dimension. In fact, T_B has a finite projective resolution by finitely generated projective right B-modules since ${}_AT$ is good. The next result is deduced from Theorem 10.5.9 and Corollary 10.5.11 in [36].

Lemma 4.2. (1) *Let $\alpha_{Y^\bullet}\colon {}_pY^\bullet \to Y^\bullet$ be the projective resolution of the complex $Y^\bullet$ in $\mathcal{D}(B)$. If $\operatorname{Tor}_j^B(T, Y^i) = 0$ for all $i \in \mathbb{Z}$ and $j \geq 1$, then*

$$T \otimes_B \alpha_{Y^\bullet} : G(Y^\bullet) = T \otimes_B {}_pY^\bullet \longrightarrow T \otimes_B Y^\bullet$$

is an isomorphism in $\mathcal{D}(A)$.

(2) *Let* $\beta_{X^\bullet}\colon X^\bullet \to {}_iX^\bullet$ *be the injective resolution of the complex* $X^\bullet$ *in* $\mathcal{D}(A)$. *If* $\mathrm{Ext}_A^j(T, X^i) = 0$ *for all* $i \in \mathbb{Z}$ *and* $j \geq 1$, *then*

$$\mathrm{Hom}_A(T, \beta_{X^\bullet}) : \mathrm{Hom}_A(T, X^\bullet) \longrightarrow \mathrm{Hom}_A(T, {}_iX^\bullet) = H(X^\bullet)$$

is an isomorphism in $\mathcal{D}(B)$.

Combining Theorem 4.1 with Lemma 4.2, we have the following.

Lemma 4.3. (1) *For each* $P \in B$-Proj, *the unit adjunction morphism*

$$\eta'_P : P \longrightarrow \mathrm{Hom}_A(T, T \otimes_B P)$$

is injective with $\mathrm{Coker}(\eta'_P) \in \mathcal{E}$.

(2) *For each* $M^\bullet \in \mathcal{D}(B)$, *there is a commutative diagram of triangles in* $\mathcal{D}(B)$:

$$\begin{tikzcd}
\mathrm{Coker}(\eta'_{{}_pM^\bullet})[-1] \arrow[r] \arrow[d, equal] & {}_pM^\bullet \arrow[r, "\eta'_{{}_pM^\bullet}"] \arrow[d, "\simeq"', "\alpha_{M^\bullet}"] & \mathrm{Hom}_A(T, T \otimes_B {}_pM^\bullet) \arrow[r] \arrow[d, "\simeq"', "\mathrm{Hom}_A(T, \beta_{T\otimes_B {}_pM^\bullet})"] & \mathrm{Coker}(\eta'_{{}_pM^\bullet}) \arrow[d, equal] \\
\mathrm{Coker}(\eta'_{{}_pM^\bullet})[-1] \arrow[r] & M^\bullet \arrow[r, "\eta_{M^\bullet}"] & H \circ G(M^\bullet) \arrow[r] & \mathrm{Coker}(\eta'_{{}_pM^\bullet}),
\end{tikzcd}$$

where $\eta'_{{}_pM^\bullet} := (\eta'_{{}_pM^n})_{n\in\mathbb{Z}}$ *and* $\mathrm{Coker}(\eta'_{{}_pM^\bullet}) \in \mathrm{Ker}(G)$.

(3) *We have that*

$$\mathrm{Ker}(G) = \{\overline{Y}^\bullet \in \mathcal{D}(B) \mid \overline{Y}^\bullet \simeq Y^\bullet \text{ in } \mathcal{D}(B) \text{ with } Y^i \in \mathcal{E} \text{ for all } i \in \mathbb{Z}\}$$

and

$$\mathrm{Im}(H) = \{\overline{Z}^\bullet \in \mathcal{D}(B) \mid \overline{Z}^\bullet \simeq Z^\bullet \textit{ in } \mathcal{D}(B) \textit{ with } Z^i \in \mathrm{Hom}_A(T, \mathrm{Add}\,(T)) \textit{ for all } i \in \mathbb{Z}\},$$

where $\mathrm{Hom}_A(T, \mathrm{Add}\,(T)) := \{\mathrm{Hom}_A(T, T') \in B\text{-Mod} \mid T' \in \mathrm{Add}\,({}_AT)\}$.

Proof. (1) Since the canonical map $\mathrm{Hom}_A(T,T)^{(\alpha)} \to \mathrm{Hom}_A(T, T^{(\alpha)})$ is injective for any nonempty set α, the map η'_P is injective for any free B-module P, and thus η'_P is injective for any projective B-module P. By Lemma 4.2, $H \circ G(P) \simeq \mathrm{Hom}_A(T, T \otimes_B P)$. It follows from Theorem 4.1 that there exists a triangle

$$X_P^\bullet \longrightarrow P \xrightarrow{\eta'_P} \mathrm{Hom}_A(T, T \otimes_B P) \longrightarrow X_P^\bullet[1]$$

in $\mathcal{D}(B)$ with $X_P^\bullet \in \mathrm{Ker}(G)$. As η'_P is injective, we have $\mathrm{Coker}(\eta'_P) \simeq X_P^\bullet[1] \in \mathrm{Ker}(G)$, and therefore $\mathrm{Coker}(\eta'_P) \in \mathcal{E}$.

(2) Recall that ${}_pM^\bullet \in \mathcal{K}(B)_P \subseteq \mathcal{K}(B\text{-Proj})$ and $\mathrm{Ext}_A^j(T, T^{(\alpha)}) = 0$ for all $j \geq 1$ and nonempty sets α. Thus part (2) follows from Theorem 4.1, Definition 2.3(4) and Lemma 4.2(2).

(3) This is shown in Proposition 4.6 of [14] for good 1-tilting modules, but the proof there also works for good n-tilting modules. ∎

Thanks to Theorem 3.5 in [6], for any tilting module ${}_AU$, there exists a t-structure in $\mathcal{D}(A)$ associated with ${}_AU$ such that its heart is

$$\mathcal{H}(U) := \{X \in \mathcal{D}(A) \mid \mathrm{Hom}_{\mathcal{D}(A)}(U, X[n]) = 0 \text{ for all } n \neq 0\},$$

and that $\mathcal{H}(U)$ contains Add $({}_AU)$ and is an abelian category of which projective objects are isomorphic in $\mathcal{D}(A)$ to modules in Add $({}_AU)$.

In the sequel, for the good tilting A-module T, we set $\mathcal{H} := \mathcal{H}(T)$.

Lemma 4.4 (Proposition 5.5 in [6]). (1) *The restriction of H to $\mathcal{H}$ yields an exact and fully faithful functor*

$$H_T = \mathrm{Hom}_{\mathcal{H}}(T, -) : \mathcal{H} \longrightarrow B\text{-Mod}.$$

(2) $\mathrm{Im}(H_T) = \mathrm{Im}(H) \cap B$-Mod.

(3) *H_T has a left adjoint $F \colon B$-Mod $\to \mathcal{H}$ given by the composition of the restriction of G to B-Mod with the left adjoint of the inclusion $\mathcal{H} \to \mathcal{U}$, where*

$$\mathcal{U} := \{X \in \mathcal{D}(A) \mid \mathrm{Hom}_{\mathcal{D}(A)}(T, X[n]) = 0 \textit{ for all } n > 0\}.$$

Now, let $\mathcal{Y} := \mathrm{Im}(H_T)$. Then the functor H_T induces an equivalence $\mathcal{H} \simeq \mathcal{Y}$. Moreover, we have the following.

Corollary 4.5. (1) *The restriction of H_T to* Add $({}_AT)$ *coincides with the restriction of* $\mathrm{Hom}_A(T, -)$ *to* Add $({}_AT)$*, and the restriction of F, defined in Lemma 4.4*(3)*, to B*-Proj *coincides with the restriction of $T \otimes_B -$ to B*-Proj.

(2) *$\mathcal{Y}$ is an abelian subcategory of B*-Mod *closed under isomorphisms, extensions and direct products. Moreover, a B-module M is a projective object of $\mathcal{Y}$ if and only if $M \simeq \mathrm{Hom}_A(T, T')$ for some $T' \in$* Add $({}_AT)$.

Proof. (1) If $P \in B$-Proj, then $G(P) = T \otimes_B P \in$ Add $({}_AT) \subseteq \mathcal{U}$. Now, Corollary 4.5 follows from Lemma 4.4(1) and (3).

(2) By Lemma 4.4(1), we have the equivalence $H_T \colon \mathcal{H} \to \mathcal{Y}$, while the projective objects of $\mathcal{H}$ are objects of Add $({}_AT)$, up to isomorphism in $\mathcal{D}(A)$. Hence (2) follows. ■

In general, the category $\mathcal{E}$ is not an abelian subcategory of B-Mod. Nevertheless, we prove that $\mathcal{E}$ is a symmetric subcategory.

Lemma 4.6. (1) *$\mathcal{E}$ is an n-symmetric subcategory of B*-Mod*, and therefore a thick subcategory of B*-Mod.

(2) *If a B-module M is isomorphic in $\mathcal{D}(B)$ to a complex with all terms in $\mathcal{E}$, then M belongs to $\mathcal{E}$.*

(3) *Let $\mathcal{E}_0 := \{\mathrm{Coker}(\eta'_P) \mid P \in B$*-Proj$\}$. *Then a B-module $M \in \mathcal{E}$ if and only if there exists an exact sequence $0 \to M \to E_\infty \to E^1 \to 0$ of B-modules such that $E^1 \in \mathcal{E}_0$ and E_∞ admits a long exact sequence*

$$\cdots \longrightarrow E_i \longrightarrow \cdots \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_\infty \longrightarrow 0$$

with $E_i \in \mathcal{E}_0$ for all $i \geq 1$.

Proof. (1) follows from Example 3.4(3), while (2) is due to the characterization of $\mathrm{Ker}(G)$ in Lemma 4.3(3).

(3) The sufficiency of (3) follows from (1). To show the necessity of (3), we take $M \in \mathcal{E}$. Then $G(M) = 0$. Let ${}_pM := (P^{-i})_{i\in\mathbb{N}}$ be a deleted projective resolution of M. By Lemma 4.3(2), $M[1] \simeq \mathrm{Coker}(\eta'_{{}_pM})$ in $\mathcal{D}(B)$. Define $E_i := \mathrm{Coker}(\eta'_{P^{-i}})$ for each $i \in \mathbb{N}$. Then $\mathrm{Coker}(\eta'_{{}_pM})$ has the form

$$\cdots \longrightarrow E_i \longrightarrow E_{i-1} \longrightarrow \cdots \longrightarrow E_2 \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0 \longrightarrow 0,$$

where E_i is of the degree $-i$. Since P^{-i} lies in B-Proj, we have $E_i \in \mathcal{E}_0$ by Lemma 4.3(1). Note that $H^i(\mathrm{Coker}(\eta'_{{}_pM})) \simeq H^{i+1}(M) = 0$ for $i \neq -1$, and that $H^{-1}(\mathrm{Coker}(\eta'_{{}_pM})) \simeq H^0(M) = M$. Now, we write d_1 as the composite of the canonical surjection $\pi\colon E_1 \to \mathrm{Coker}(d_2)$ with $d'_1\colon \mathrm{Coker}(d_2) \to E_0$, due to $d_2d_1 = 0$. Then d'_1 is surjective and $\mathrm{Ker}(d'_1) \simeq M$. Let $E_\infty := \mathrm{Coker}(d_2)$ and $E^1 := E_0$. Then the necessity of (3) holds. ■

The category $\mathcal{E}$ can be regarded as a fully exact subcategory of B-Mod. Moreover, a complex $X^\bullet \in \mathcal{C}(\mathcal{E})$ is strictly exact (see Section 2.2) if and only if it is exact in $\mathcal{C}(B)$ by Lemma 4.6(2). This implies that $\mathcal{K}_{\mathrm{ac}}(\mathcal{E})$ is the full triangulated subcategory of $\mathcal{K}(\mathcal{E})$ consisting of exact complexes. Moreover, by Lemma 4.6(1), $\mathcal{K}_{\mathrm{ac}}(\mathcal{E})$ is closed under arbitrary direct sums and products in $\mathcal{K}(B)$, and therefore in $\mathcal{K}(\mathcal{E})$. Recall that the *unbounded derived category* $\mathcal{D}(\mathcal{E})$ of $\mathcal{E}$ is defined to be the Verdier quotient of $\mathcal{K}(\mathcal{E})$ by $\mathcal{K}_{\mathrm{ac}}(\mathcal{E})$. Then $\mathcal{D}(\mathcal{E})$ has direct sums and products, and the localization functor $\mathcal{K}(\mathcal{E}) \to \mathcal{D}(\mathcal{E})$ preserves direct sums and products by Lemma 1.5 in [12] and its dual statement.

The inclusions

$$i\colon \mathcal{E} \longrightarrow B\text{-Mod} \quad \text{and} \quad j : \mathcal{Y} \longrightarrow B\text{-Mod}$$

are exact functors between exact categories and induce automatically the derived functors:

$$D(i) : \mathcal{D}(\mathcal{E}) \longrightarrow \mathcal{D}(B) \quad \text{and} \quad D(j) : \mathcal{D}(\mathcal{Y}) \longrightarrow \mathcal{D}(B).$$

Moreover, by Lemma 4.3(1), there are another two additive functors between additive categories:

$$\begin{aligned} \mathrm{Hom}_A(T, T\otimes_B -) : B\text{-Proj} \longrightarrow \mathcal{Y}, &\quad X \mapsto \mathrm{Hom}_A(T, T\otimes_B X),\\ \mathrm{Coker}(\eta'_-) : B\text{-Proj} \longrightarrow \mathcal{E}, &\quad X \mapsto \mathrm{Coker}(\eta'_X), \end{aligned}$$

for $X \in B$-Proj. Here we fix an exact sequence

$$0 \longrightarrow X \xrightarrow{\eta'_X} \mathrm{Hom}_A(T, T\otimes_B X) \longrightarrow \mathrm{Coker}(\eta'_X) \longrightarrow 0$$

for each X. Since $\mathcal{K}(B)_P$ is a triangulated subcategory of $\mathcal{K}(B\text{-Proj})$, we can define the following derived functors $\Psi\colon \mathcal{D}(B) \to \mathcal{D}(\mathcal{Y})$ and $\Phi\colon \mathcal{D}(B) \to \mathcal{D}(\mathcal{E})$ between derived categories of exact categories, where

$$\Psi :\ \mathcal{D}(B) \xrightarrow{p(-)} \mathcal{K}(B)_P \xrightarrow{\mathrm{Hom}_A(T, T\otimes_B -)} \mathcal{K}(\mathcal{Y}) \xrightarrow{Q_{\mathcal{Y}}} \mathcal{D}(\mathcal{Y}),$$

$$\Phi :\ \mathcal{D}(B) \xrightarrow{p(-)} \mathcal{K}(B)_P \xrightarrow{\mathrm{Coker}(\eta'_-)} \mathcal{K}(\mathcal{E}) \xrightarrow{Q_{\mathcal{E}}} \mathcal{D}(\mathcal{E}),$$

with $Q_{\mathcal{Y}}$ and $Q_{\mathcal{E}}$ the localization functors. By Lemma 4.3(2), there is a commutative diagram of natural transformations among triangle endofunctors of $\mathcal{D}(B)$:

$$(\sharp)\qquad \begin{tikzcd} D(i)\circ\Phi[-1] \arrow[r] \arrow[d, equal] & \mathrm{Id}_{\mathcal{D}(B)} \arrow[r] \arrow[d, equal] & D(j)\circ\Psi \arrow[r] \arrow[d, "\simeq"] & D(i)\circ\Phi \arrow[d, equal] \\ D(i)\circ\Phi[-1] \arrow[r] & \mathrm{Id}_{\mathcal{D}(B)} \arrow[r, "\eta"] & H\circ G \arrow[r] & D(i)\circ\Phi. \end{tikzcd}$$

This yields a commutative diagram of triangles in $\mathcal{D}(B)$ if applied to an object in $\mathcal{D}(B)$.

Let $D(H_T)\colon \mathcal{D}(\mathcal{H}) \to \mathcal{D}(B)$ be the derived functor of H_T, and let $\mathbb{L}F\colon \mathcal{D}(B) \to \mathcal{D}(\mathcal{H})$ be the total left-derived functor of F. Since, by Lemma 4.4(3), (F, H_T) is an adjoint pair, we see that $(\mathbb{L}F, D(H_T))$ is an adjoint pair.

Lemma 4.7. (1) *Let $\overline{H_T}\colon \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{Y})$ be the equivalence induced by H_T. Then*

$$D(H_T) = D(j)\circ\overline{H_T},\quad \Psi = \overline{H_T}\circ\mathbb{L}F \quad \textit{and} \quad \mathrm{Im}(D(j)) = \mathrm{Im}(H).$$

Moreover, $(\Psi, D(j))$ is an adjoint pair and $D(j)$ is a fully faithful functor.

(2) *Let $\kappa\colon \mathrm{Ker}(G) \to \mathcal{D}(B)$ and $\nu\colon \mathrm{Im}(H) \to \mathcal{D}(B)$ be the inclusion functors. Then $D(i) = \kappa\circ\overline{D(i)}$ and $D(j) = \nu\circ\overline{D(j)}$, namely the following commutative diagrams exist:*

$$\begin{tikzcd} \mathcal{D}(\mathcal{E}) \arrow[rr, "D(i)"] \arrow[dr, "\overline{D(i)}"'] & & \mathcal{D}(B) \\ & \mathrm{Ker}(G) \arrow[ur, hook, "\kappa"'] & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathcal{D}(\mathcal{Y}) \arrow[rr, "D(j)"] \arrow[dr, "\overline{D(j)}"'] & & \mathcal{D}(B). \\ & \mathrm{Im}(H) \arrow[ur, hook, "\nu"'] & \end{tikzcd}$$

(3) $\mathrm{Ker}(D(i)) = 0$, $\mathrm{Ker}(\Phi) = \mathrm{Im}(H)$, *and Φ commutes with direct products. Moreover, Φ induces a triangle functor*

$$\overline{\Phi}:\ \mathcal{D}(B)/\mathrm{Im}(H) \longrightarrow \mathcal{D}(\mathcal{E})$$

which commutes with direct products.

(4) *The two compositions*

$$\mathrm{Ker}(G) \xrightarrow{\kappa} \mathcal{D}(B) \xrightarrow{Q} \mathcal{D}(B)/\mathrm{Im}(H) \quad \textit{and} \quad \mathcal{D}(B)/\mathrm{Im}(H) \xrightarrow{\overline{\Phi}[-1]} \mathcal{D}(\mathcal{E}) \xrightarrow{\overline{D(i)}} \mathrm{Ker}(G)$$

are quasi-inverse triangle equivalences, where Q denotes the localization functor. In particular, $\overline{D(i)}\circ\Phi[-1]\colon \mathcal{D}(B) \to \mathrm{Ker}(G)$ is a right adjoint of κ.

Proof. (1) Clearly, H_T induces an equivalence $H'_T\colon \mathcal{H} \to \mathcal{Y}$. Denote its derived functor by $\overline{H}_T\colon \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{Y})$. Hence $H_T = j\circ H'_T$. Since all functors in this equality are exact, we have $D(H_T) = D(j)\circ\overline{H_T}$. Note that $\mathbb{L}F$ is the composition of the functors:

$$\mathcal{D}(B) \xrightarrow{p(-)} \mathcal{K}(B)_P \hookrightarrow \mathcal{K}(B\text{-Proj}) \hookrightarrow \mathcal{K}(B) \xrightarrow{F} \mathcal{K}(\mathcal{H}) \xrightarrow{Q_{\mathcal{H}}} \mathcal{D}(\mathcal{H})\ .$$

By Corollary 4.5, the restriction of $F\colon \mathcal{K}(B) \to \mathcal{K}(\mathcal{H})$ to $\mathcal{K}(B\text{-Proj})$ coincides with that of ${}_AT\otimes_B -$, and has its image in $\mathcal{K}(\mathrm{Add}\,({}_AT))$, while the restriction of $H_T\colon \mathcal{K}(\mathcal{H}) \to$

$\mathcal{K}(\mathcal{Y})$ to $\mathcal{K}(\mathrm{Add}\,({}_AT))$ coincides with the restriction of $\mathrm{Hom}_A(T,-)$. It follows that $\Psi = \overline{H_T} \circ \mathbb{L}F$. Since $\overline{H_T}$ is an equivalence, $(\Psi, D(j))$ is an adjoint pair and $\mathrm{Im}(D(j)) = \mathrm{Im}(D(H_T))$. By Lemma 4.3(3), $\mathrm{Im}(H) \subseteq \mathrm{Im}(D(H_T))$.

To show $\mathrm{Im}(D(H_T)) \subseteq \mathrm{Im}(H)$, we apply the technique of homotopy limits in derived categories.

Let $M^\bullet \in \mathcal{C}(B)$. By Lemma 2.5, $M^\bullet \simeq \underleftarrow{\mathrm{Holim}}(M^{(-n,n+1]})$ in $\mathcal{D}(B)$ where $n \in \mathbb{N}$. Suppose $M^\bullet \in \mathcal{C}(\mathcal{Y})$. Then $M^{(-n,n+1]} \in \mathcal{C}^b(\mathcal{Y})$ by Corollary 4.5(2). Further, by Theorem 4.1, $\mathrm{Im}(H)$ is a triangulated subcategory of $\mathcal{D}(B)$ closed under direct products. It follows from $\mathcal{Y} \subseteq \mathrm{Im}(H)$ that $M^{(-n,n+1]} \in \mathrm{Im}(H)$ and further $M^\bullet \in \mathrm{Im}(H)$. This shows $\mathrm{Im}(D(H_T)) \subseteq \mathrm{Im}(H)$, Thus $\mathrm{Im}(H) = \mathrm{Im}(D(H_T)) = \mathrm{Im}(D(j))$.

Now, we show that the counit adjunction $\phi\colon \mathbb{L}F \circ D(H_T) \to \mathrm{Id}_{\mathcal{D}(\mathcal{H})}$ is an isomorphism. This implies that both $D(H_T)$ and $D(j)$ are fully faithful functors since $\overline{H_T}$ is an equivalence.

In fact, the adjoint pair (F, H_T) induces an adjoint pair $(K(F), K(H_T))$ of functors between $\mathcal{K}(\mathcal{H})$ and $\mathcal{K}(B)$. Let $\phi'\colon K(F) \circ K(H_T) \to \mathrm{Id}_{\mathcal{K}(\mathcal{H})}$ be its counit adjunction, and let $\psi'\colon \mathrm{Id}_{\mathcal{K}(B)} \to K(H_T) \circ K(F)$ be its unit adjunction. Now, let $X \in \mathcal{D}(\mathcal{H})$, let $Q^\bullet := {}_p(K(H_T)(X))$, and let $\alpha_{Q^\bullet}\colon Q^\bullet \to K(H_T)(X)$ be the projective resolution of $K(H_T)(X)$ in $\mathcal{D}(B)$. Then $\phi_X\colon \mathbb{L}F \circ D(H_T)(X) \to X$ is given by the composition of $K(F)(\alpha_{Q^\bullet})$, with $\phi'_X\colon K(F) \circ K(H_T)(X) \to X$. Since H_T is fully faithful by Lemma 4.4(1), the morphism $K(H_T)\colon \mathcal{K}(\mathcal{H}) \to \mathcal{K}(B)$ is fully faithful. This implies that ϕ'_X is an isomorphism by Lemma 2.6(1). Thus ϕ_X is an isomorphism in $\mathcal{D}(\mathcal{H})$ if and only if $K(F)(\alpha_{Q^\bullet})$ is a quasi-isomorphism. Since $H_T\colon \mathcal{H} \to B$-Mod is exact and fully faithful by Lemma 4.4(1), for each $U \in \mathcal{K}(\mathcal{H})$, we see that $U \in \mathcal{K}_{\mathrm{ac}}(\mathcal{H})$ if and only if $K(H_T)(U) \in \mathcal{K}_{\mathrm{ac}}(B)$. Consequently, $K(F)(\alpha_{Q^\bullet})$ is a quasi-isomorphism if and only if so is $K(H_T) \circ K(F)(\alpha_{Q^\bullet})$. Since $Q^\bullet \in \mathcal{K}(B)_P \subseteq \mathcal{K}(B\text{-Proj})$, we get $K(H_T)K(F)(Q^\bullet) = \mathrm{Hom}_A(T, T \otimes_B Q^\bullet)$ and $\psi'_{Q^\bullet} = \eta'_{Q^\bullet}$. Moreover, $H \circ G(Q^\bullet) \simeq \mathrm{Hom}_A(T, T \otimes_B Q^\bullet)$ in $\mathcal{D}(B)$ by Lemma 4.3(2), and therefore there exists a diagram

$$\begin{tikzcd}
Q^\bullet \arrow[r, equal] \arrow[d, "\eta_{Q^\bullet}"'] & Q^\bullet \arrow[r, "\alpha_{Q^\bullet}"] \arrow[d, "\psi'_{Q^\bullet}"'] & K(H_T)(X) \arrow[d, "\psi'_{H_T(X)}"', "\simeq"] \\
H \circ G(Q^\bullet) \arrow[r, "\simeq"] & K(H_T) \circ K(F)(Q^\bullet) \arrow[r, "K(H_T)\circ K(F)(\alpha_{Q^\bullet})"] & K(H_T)\circ K(F)\circ K(H_T)(X)
\end{tikzcd}$$

in which the left square is commutative in $\mathcal{D}(B)$ and the right square is commutative in $\mathcal{K}(B)$. Note that the isomorphism $\psi'_{H_T(X)}$ follows from Lemma 2.6(2) and the fact that $K(H_T)(X)$ lies in $\mathrm{Im}(K(H_T))$. Since $\mathrm{Im}(H) = \mathrm{Im}(D(H_T))$ by (1), we have $Q^\bullet \simeq K(H_T)(X) \in \mathrm{Im}(H)$. Recall that H is fully faithful. Thus $\eta_{Q^\bullet}$ is an isomorphism in $\mathcal{D}(B)$, and further, $K(H_T) \circ K(F)(\alpha_{Q^\bullet})$ is an isomorphism in $\mathcal{D}(B)$. As $K(H_T) \circ K(F)(\alpha_{Q^\bullet})$ is represented by a chain map, it is a quasi-isomorphism. This shows that ϕ is a natural isomorphism.

(2) This follows from (1) and Lemma 4.3(3).

(3) By Lemma 4.6(2), $X^\bullet \in \mathcal{C}(\mathcal{E})$ is strictly exact if and only if it is exact in $\mathcal{C}(B)$. This implies $\mathrm{Ker}(D(i)) = 0$, and therefore $\mathrm{Ker}(D(i) \circ \Phi) = \mathrm{Ker}(\Phi)$. By (1) and ($\sharp$), we have $\mathrm{Im}(H) = \mathrm{Ker}(\Phi)$. Thus Φ induces a triangle functor $\overline{\Phi}\colon \mathcal{D}(B)/\mathrm{Im}(H) \to \mathcal{D}(\mathcal{E})$. Since $\mathrm{Im}(H) \subseteq \mathcal{D}(B)$ is closed under direct products, $\mathcal{D}(B)/\mathrm{Im}(H)$ has direct products

and the localization functor Q in (4) commutes with direct products by the dual statement of Lemma 1.5 in [12]. As $\mathcal{D}(B)$ and $\mathcal{D}(B)/\mathrm{Im}(H)$ have the same objects, $\overline{\Phi}$ commutes with direct products if and only if so does Φ. Now let $\{M_i\}_{i\in I}$ be a family of objects in $\mathcal{D}(B)$ together with the projections $\pi_i\colon \prod_{i\in I} M_i \to M_i$, where I is a non-empty set. To prove that the canonical morphism $\pi\colon \Phi(\prod_{i\in I} M_i) \to \prod_{i\in I} \Phi(M_i)$ in $\mathcal{D}(\mathcal{E})$ induced from $\{\Phi(\pi_i)\}_{i\in I}$ is an isomorphism, it suffices to show that $D(i)(\pi)$ is an isomorphism in $\mathcal{D}(B)$. The reason reads as follows: π can be embedded into a triangle in $\mathcal{D}(\mathcal{E})$,

$$\Phi\Big(\prod_{i\in I} M_i\Big) \longrightarrow \prod_{i\in I} \Phi(M_i) \longrightarrow W \longrightarrow \Phi\Big(\prod_{i\in I} M_i\Big)[1],$$

and $D(i)(W) = 0$ if $D(i)(\pi)$ is an isomorphism. It then follows from $\mathrm{Ker}(D(i)) = 0$ that $W = 0$ and π is an isomorphism in $\mathcal{D}(\mathcal{E})$.

Indeed, since $D(i)$ commutes with direct products by Lemma 4.6(1), it is enough to show that the canonical morphism $D(i)\circ\Phi(\prod_{i\in I} M_i) \to \prod_{i\in I}(D(i)\circ\Phi(M_i))$ in $\mathcal{D}(B)$ induced from $\{D(i)\circ\Phi(\pi_i)\}_{i\in I}$ is an isomorphism, that is, $D(i)\circ\Phi$ commutes with direct products. But this follows from the diagram ($\sharp$) and the fact that H and G commute with direct products. Thus Φ and therefore $\overline{\Phi}$ commute with direct products.

(4) By Theorem 4.1, the pair $(\mathrm{Ker}(G), \mathrm{Im}(H))$ is a semi-orthogonal decomposition of $\mathcal{D}(B)$. By (3), $\mathrm{Ker}(\overline{D(i)}\circ\Phi[-1]) = \mathrm{Ker}(\Phi) = \mathrm{Im}(H)$. It follows from the diagram ($\sharp$) and Lemma 2.2 that $\overline{D(i)}\circ\Phi[-1]\colon \mathcal{D}(B) \to \mathrm{Ker}(G)$ is a right adjoint of κ and induces a triangle equivalence $\overline{D(i)}\circ\overline{\Phi}[-1]\colon \mathcal{D}(B)/\mathrm{Im}(H) \to \mathrm{Ker}(G)$. Thus (4) holds. ∎

Remark 4.8. The functors H and G restrict to functors between bounded-above derived categories since both ${}_AT$ and T_B have finite projective dimension. This implies that the pair $(\mathcal{D}^-(B)\cap\mathrm{Ker}(G), \mathcal{D}^-(B)\cap\mathrm{Im}(H))$ is a semi-orthogonal decomposition of $\mathcal{D}^-(B)$. Moreover, by definition, Φ and Ψ also restrict to functors between bounded-above derived categories:

$$\Phi^- : \mathcal{D}^-(B) \to \mathcal{D}^-(\mathcal{E}) \quad\text{and}\quad \Psi^- : \mathcal{D}^-(B) \to \mathcal{D}^-(\mathcal{Y}).$$

Thus Lemma 4.7 holds true for bounded-above derived categories.

4.2. Fully faithful triangle functors between derived categories

In this section we develop properties of triangle functors needed in our proofs. The main result of the section is that the triangle functor $\overline{D(i)}\colon \mathcal{D}(\mathcal{E}) \to \mathrm{Ker}(G)$ is an equivalence (see Proposition 4.17). To show this equivalence, we first introduce an intermediate category $\mathscr{E}$, a fully exact subcategory of B-Mod, and then prove that it contains all injective B-modules (see Lemma 4.12). This property enables us to establish a triangle equivalence from $\mathcal{D}(\mathscr{E})$ to $\mathcal{D}(B)$ and a fully faithful triangle functor from $\mathcal{D}^-(\mathcal{E})$ to $\mathcal{D}^-(\mathscr{E})$ which are induced from canonical inclusions of exact categories (see Lemma 4.15). It turns out that the restriction of the functor $\overline{D(i)}$ to $\mathcal{D}^-(\mathcal{E})$ is fully faithful. By the technique of homotopy colimits, we show finally that $\overline{D(i)}$ is a triangle equivalence.

By the diagram ($\sharp$), there exists a triangle in $\mathcal{D}(B)$ for each $M \in B$-Mod:

$$(\diamondsuit)\quad D(i)\circ\Phi[-1](M) \longrightarrow M \longrightarrow D(j)\circ\Psi(M) \longrightarrow D(i)\circ\Phi(M),$$

with $\Psi(M) \in \mathcal{C}^-(\mathcal{Y})$, $\Phi(M) \in \mathcal{C}^-(\mathcal{E})$ and $\Psi(M)^i = 0 = \Phi(M)^i$ for all $i \geq 1$.

Taking homologies of the triangle, we obtain a 5-term exact sequence of B-modules:

$$\varepsilon_M: \quad 0 \longrightarrow Y_M \xrightarrow{\epsilon_M^{-2}} X_M \xrightarrow{\epsilon_M^{-1}} M \xrightarrow{\epsilon_M^{0}} Y^M \longrightarrow X^M \longrightarrow 0,$$

where

$$Y_M := H^{-1}(D(j)\circ\Psi(M)), \quad X_M := H^{-1}(D(i)\circ\Phi(M)),$$
$$Y^M := H^{0}(D(j)\circ\Psi(M)), \quad X^M := H^{0}(D(i)\circ\Phi(M)).$$

To understand the functor $D(i)$, we further introduce the category $\mathscr{E}$ and other related perpendicular subcategories of B-Mod:

$$\begin{aligned}
\mathscr{E} &:= \{M \in B\text{-Mod} \mid X^M = 0\},\\
\mathscr{X} &:= \{X \in B\text{-Mod} \mid \mathrm{Ext}_B^n(X,Y)=0,\ Y\in\mathscr{Y},\ n=0,1\},\\
{}^{\perp}\mathscr{Y} &:= \{X \in B\text{-Mod} \mid \mathrm{Ext}_B^n(X,Y)=0,\ Y\in\mathscr{Y},\ n\geq 0\},\\
\mathcal{E}^{\perp} &:= \{Y \in B\text{-Mod} \mid \mathrm{Ext}_B^n(X,Y)=0,\ X\in\mathcal{E},\ n\geq 0\},\\
\mathrm{Coker}(\mathcal{E}) &:= \{\mathrm{Coker}(f) \in B\text{-Mod} \mid f: E_1\to E_0 \text{ with } E_1,E_0\in\mathcal{E}\}.
\end{aligned}$$

Then $\mathscr{X}$ is closed under extensions and cokernels in B-Mod, and both $\mathcal{E}^{\perp}$ and ${}^{\perp}\mathscr{Y}$ are thick subcategories of B-Mod. Moreover, we have the following result, which generalizes Lemma 5.8 in [6].

Lemma 4.9. *$\mathcal{E}^{\perp} = \mathscr{Y}$ and $\mathcal{E} = {}^{\perp}\mathscr{Y}$.*

Proof. We have $\mathcal{E} \subseteq \mathrm{Ker}(G)$ and $\mathscr{Y} \subseteq \mathrm{Im}(H)$. Since $\mathrm{Hom}_{\mathcal{D}(B)}(X^\bullet, Y^\bullet) = 0$ for any $X^\bullet \in \mathrm{Ker}(G)$ and $Y^\bullet \in \mathrm{Im}(H)$ by Theorem 4.1, we have $\mathscr{Y} \subseteq \mathcal{E}^{\perp}$ and $\mathcal{E} \subseteq {}^{\perp}\mathscr{Y}$. To show $\mathcal{E}^{\perp} \subseteq \mathscr{Y}$, we first show that if $Z \in \mathcal{E}^{\perp}$, then $\mathrm{Hom}_{\mathcal{D}(B)}(E^\bullet, Z) = 0$ for all $E^\bullet \in \mathcal{C}^-(\mathcal{E})$.

Indeed, let $\mathcal{D}_Z$ be the full subcategory of $\mathcal{D}(B)$ consisting of all complexes $U^\bullet$ such that $\mathrm{Hom}_{\mathcal{D}(B)}(U^\bullet[n], Z) = 0$ for all $n \in \mathbb{Z}$. Then $\mathcal{D}_Z$ is a triangulated subcategory of $\mathcal{D}(B)$ which is closed under direct sums and contains $\mathcal{E}$. Thus homotopy colimits in $\mathcal{D}(B)$ of sequences in $\mathcal{D}_Z$ belong to $\mathcal{D}_Z$ by Definition 2.4. By the first isomorphism in Lemma 2.5, each complex in $\mathcal{C}^-(\mathcal{E})$ can be obtained in $\mathcal{D}(B)$ from bounded complexes in $\mathcal{C}^b(\mathcal{E})$ by taking homotopy colimits. Hence $\mathcal{C}^-(\mathcal{E}) \subseteq \mathcal{D}_Z$, and therefore $\mathrm{Hom}_{\mathcal{D}(B)}(E^\bullet, Z) = 0$ for all $E^\bullet \in \mathcal{C}^-(\mathcal{E})$.

Since $\Phi(Z)[-1] \in \mathcal{C}^-(\mathcal{E})$, we have $\mathrm{Hom}_{\mathcal{D}(B)}(\Phi(Z)[-1], Z) = 0$. By the triangle $(\diamond)$, $D(j)\circ\Psi(Z) \simeq Z \oplus D(i)\circ\Phi(Z)$. However, $\mathrm{Hom}_{\mathcal{D}(B)}(D(i)\circ\Phi(Z), D(j)\circ\Psi(Z)) = 0$. Thus $D(i)\circ\Phi(Z) = 0$ and $Z \simeq D(j)\circ\Psi(Z) \in \mathrm{Im}(H)$. Further, by Lemma 4.4(2), we obtain $Z \in \mathrm{Im}(H) \cap B\text{-Mod} = \mathscr{Y}$. This shows $\mathcal{E}^{\perp} \subseteq \mathscr{Y}$, and therefore $\mathcal{E}^{\perp} = \mathscr{Y}$.

Since $\mathscr{Y}$ is an abelian category of B-Mod by Corollary 4.5(2), it is closed under cokernels in B-Mod. By the last isomorphism in Lemma 2.5, each complex in $\mathcal{C}^-(\mathscr{Y})$ can be obtained in $\mathcal{D}(B)$ from bounded complexes in $\mathcal{C}^b(\mathscr{Y})$ by taking homotopy limits. Similarly, if $Z \in {}^{\perp}\mathscr{Y}$, then $\mathrm{Hom}_{\mathcal{D}(B)}(Z, Y^\bullet) = 0$ for any $Y^\bullet \in \mathcal{C}^-(\mathscr{Y})$. Then the inclusion ${}^{\perp}\mathscr{Y} \subseteq \mathcal{E}$ follows from $(\diamond)$ and $\Psi(Z) \in \mathcal{C}^-(\mathscr{Y})$. Thus $\mathcal{E} = {}^{\perp}\mathscr{Y}$. ■

Lemma 4.10. *The following hold true for $M \in B$-Mod.*

(1) *$Y_M, Y^M \in \mathscr{Y}$ and $X^M \in \mathrm{Coker}(\mathcal{E})$. If $M \in \mathscr{E}$, then $X_M \in \mathrm{Coker}(\mathcal{E})$.*
(2) *$\mathscr{X} = \mathrm{Coker}(\mathcal{E}) = \{M \in \mathscr{E} \mid Y_M = 0 = Y^M\}$ and $\mathscr{Y} = \{M \in \mathscr{E} \mid X_M = 0\}$.*
(3) *$\mathscr{E}$ is closed under extensions and quotients in B-Mod.*

Proof. (1) Since $\mathcal{Y}$ is an abelian subcategory of B-Mod by Corollary 4.5(2), we have $H^i(\Psi(M)) \in \mathcal{Y}$ for any $i \in \mathbb{Z}$. Moreover, there is a right exact sequence of B-modules,

$$\Phi(M)^{-1} \xrightarrow{d^{-1}} \Phi(M)^0 \longrightarrow X^M \longrightarrow 0,$$

with $\Phi(M)^{-1}, \Phi(M)^0 \in \mathcal{E}$. This shows $X^M \in \mathrm{Coker}(\mathcal{E})$.

Now, let $M \in \mathscr{E}$. Then $X^M = 0$, which means that d^{-1} is surjective. Since $\mathcal{E}$ is closed under kernels of surjective homomorphisms in B-Mod by Lemma 4.6(1), $\mathrm{Ker}(d^{-1}) \in \mathcal{E}$. Similarly, there is a right exact sequence $\Phi(M)^{-2} \to \mathrm{Ker}(d^{-1}) \to X_M \to 0$ such that $\Phi(M)^{-2} \in \mathcal{E}$. Thus $X_M \in \mathrm{Coker}(\mathcal{E})$.

(2) Since $\mathcal{X}$ is closed under cokernels in B-Mod and $\mathcal{E} \subseteq \mathcal{X}$, we have $\mathrm{Coker}(\mathcal{E}) \subseteq \mathcal{X}$.

Let $M \in \mathcal{X}$. Since $Y^M \in \mathcal{Y}$ by (1), $\mathrm{Hom}_B(M, Y^M) = 0$ and $Y^M \simeq X^M$. As X^M is a quotient of $\Phi(M)^0 \in \mathcal{E}$, it follows from Lemma 4.9 that $\mathrm{Hom}_B(X^M, Y^M) = 0$. This forces $X^M = 0 = Y^M$, and thus $M \in \mathscr{E}$. Since $Y_M \in \mathcal{Y}$, we have $\mathrm{Ext}^1_B(M, Y_M) = 0$. Then $M \simeq Y_M \oplus X_M$. Note that $\mathrm{Hom}_B(M, Y_M) = 0$ for $M \in \mathcal{X}$ and $Y_M \in \mathcal{Y}$. Consequently, $Y_M = 0$ and $M \simeq X_M \in \mathrm{Coker}(\mathcal{E})$. Thus $\mathcal{X} \subseteq \{M \in \mathscr{E} \mid Y_M = 0 = Y^M\}$. Conversely, if $N \in \mathscr{E}$ and $Y_N = 0 = Y^N$, then $N \simeq X_N \in \mathrm{Coker}(\mathcal{E})$ by (1).

If $N \in \mathscr{E}$ with $X_N = 0$, then $N \simeq Y^N \in \mathcal{Y}$. Since $\mathcal{Y} \subseteq \mathrm{Im}(H)$, it follows from Lemma 4.7(1) that $N \in \mathcal{Y}$ if and only if $\Phi(N) = 0$ in $\mathcal{D}(B)$. This implies the last equality in (2).

(3) Let $0 \to U \to V \to W \to 0$ be an exact sequence of B-modules. Then this sequence induces a triangle

$$D(i) \circ \Phi(U) \longrightarrow D(i) \circ \Phi(V) \longrightarrow D(i) \circ \Phi(W) \longrightarrow D(i) \circ \Phi(U)[1]$$

in $\mathcal{D}(B)$. Taking homology H^0 of this triangle yields an exact sequence in B-Mod:

$$X^U \longrightarrow X^V \longrightarrow X^W \longrightarrow H^1(D(i) \circ \Phi(U)).$$

By the definition of Φ, $\Phi(U)$ is a complex with 0 at all positive degree, this implies $H^1(D(i) \circ \Phi(U)) = 0$. Hence $\mathscr{E}$ is closed under extensions and quotients in B-Mod. ∎

Corollary 4.11. (1) *For each $M \in \mathscr{E}$, there exists a 4-term exact sequence of B-modules,*

$$0 \longrightarrow Y_M \xrightarrow{\varepsilon_M^{-2}} X_M \xrightarrow{\varepsilon_M^{-1}} M \xrightarrow{\varepsilon_M^{0}} Y^M \longrightarrow 0,$$

with $Y_M, Y^M \in \mathcal{Y}$ and $X_M \in \mathcal{X}$.

(2) *ε_M^{-1} and ε_M^0 in* (1) *give rise to isomorphisms of abelian groups for $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$:*

$$(\varepsilon_M^{-1})^* : \mathrm{Hom}_B(X, X_M) \xrightarrow{\simeq} \mathrm{Hom}_B(X, M) \quad \textit{and}$$
$$(\varepsilon_M^{0})_* : \mathrm{Hom}_B(Y^M, Y) \xrightarrow{\simeq} \mathrm{Hom}_B(M, Y).$$

(3) *The inclusion $\mathcal{X} \to \mathscr{E}$ has a right adjoint $r\colon \mathscr{E} \to \mathcal{X}$ given by $M \mapsto X_M$ for any $M \in \mathscr{E}$. Moreover, if $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence in B-*Mod *with $M_i \in \mathscr{E}$ for $1 \le i \le 3$, then $r(M_1) \to r(M_2) \to r(M_3) \to 0$ is exact.*

Proof. (1) We have an exact sequence ϵ_M. Since $M \in \mathcal{E}$, ϵ_M can be shortened into 4-term sequence with the desired properties by Lemma 4.10(1)–(2).

(2) Since $X \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$, we have

$$\mathrm{Hom}_B(X, Y^M) = \mathrm{Hom}_B(X, Y_M) = \mathrm{Ext}^1_B(X, Y_M) = 0.$$

It then follows from the exact sequence in (1) that $(\varepsilon_M^{-1})^*$ is an isomorphism. This implies the first isomorphism in (2). Similarly, $(\varepsilon_M^0)_*$ is an isomorphism.

(3) If $M \in \mathcal{X}$, then ε_M^{-1} is an isomorphism by the proof of Lemma 4.10(2). Now, the isomorphism $(\varepsilon_M^{-1})^*$ in (2) gives an adjunction isomorphism of adjoint pairs of the functors.

Applying the functor $D(i) \circ \Phi\colon \mathcal{D}(B) \to \mathcal{D}(B)$ to the exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ in B-Mod, we obtain a triangle

$$D(i) \circ \Phi(M_1) \longrightarrow D(i) \circ \Phi(M_2) \longrightarrow D(i) \circ \Phi(M_3) \longrightarrow D(i) \circ \Phi(M_1)[1].$$

If we take H^{-1} on the triangle, then we gain an exact sequence of B-modules:

$$r(M_1) \longrightarrow r(M_2) \longrightarrow r(M_3) \longrightarrow X^{M_1}.$$

Now, it follows from $M_1 \in \mathcal{E}$ that $X^{M_1} = 0$, and thus the second part of (3) holds. ■

The following property of $\mathcal{E}$ is crucial in establishing an equivalence of derived categories of exact categories in Lemma 4.15 below.

Lemma 4.12. *If M is an injective B-module, then $M \in \mathcal{E}$.*

To show Lemma 4.12, we determine the image of an injective cogenerator for B-Mod under $H \circ G$. Let

$$(-)^\vee := \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) : \quad \mathbb{Z}\text{-Mod} \longrightarrow \mathbb{Z}\text{-Mod}.$$

Then $(-)^\vee$ induces an exact functor B^{op}-Mod $\to B$-Mod and $B^\vee$ is an injective cogenerator for B-Mod. Further, we define two natural transformations:

$$\begin{aligned} \theta &: T \otimes_B \mathrm{Hom}_A(-, T)^\vee \to \mathrm{Hom}_A(-, A)^\vee : \quad A\text{-Mod} \longrightarrow A\text{-Mod}, \\ \rho &: \mathrm{Hom}_A(-, A) \otimes_A T \longrightarrow \mathrm{Hom}_A(-, T) : \quad A\text{-Mod} \to B^{\mathrm{op}}\text{-Mod} \end{aligned}$$

given by

$$\begin{aligned} \theta_X &: {}_AT \otimes_B \mathrm{Hom}_A(X, T)^\vee \longrightarrow \mathrm{Hom}_A(X, A)^\vee : \quad t \otimes \sigma \mapsto [f \mapsto (f(\cdot t))\sigma], \\ \rho_X &: \mathrm{Hom}_A(X, A) \otimes_A T \longrightarrow \mathrm{Hom}_A(X, T) : \quad f \otimes t \mapsto f(\cdot t), \end{aligned}$$

where $X \in A$-Mod, $t \in T, \sigma \in \mathrm{Hom}_A(X, T)^\vee$, $f \in \mathrm{Hom}_A(X, A)$ and $(\cdot t) \in \mathrm{Hom}_A(A, T)$ is the right multiplication by t.

Lemma 4.13. (1) *θ is a natural isomorphism.*

(2) *If $X \in A$-Proj, then*

(i) $\mathrm{Tor}_j^B(T, \mathrm{Hom}_A(X, T)^\vee) = 0$ *for all* $j \geq 1$,

(ii) ρ_X *is injective and* $\mathrm{Tor}_j^A(\mathrm{Hom}_A(X, A), T) = 0$ *for all* $j \geq 1$.

Proof. (1) For $U \in A$-Mod and $g \in \mathrm{Hom}_A(U, T)$, we define

$$\theta_{U,X} : \mathrm{Hom}_A(U,T) \otimes_B \mathrm{Hom}_A(X,T)^\vee \longrightarrow \mathrm{Hom}_A(X,U)^\vee : \quad g \otimes \sigma \mapsto [f \mapsto (fg)\sigma].$$

Clearly, $\theta_{A,X} = \theta_X$ under the identification of $\mathrm{Hom}_A(A,T)$ with T. Moreover, if $U \in \mathrm{add}(T)$, then $\theta_{U,X}$ is an isomorphism. Note that T_B has a finitely generated projective resolution of length at most n:

$$(*) \quad 0 \longrightarrow \mathrm{Hom}_A(T_n,T) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(T_1,T) \longrightarrow \mathrm{Hom}_A(T_0,T) \longrightarrow T_B \longrightarrow 0$$

with $T_j \in \mathrm{add}({}_AT)$ for all $0 \le j \le n$. Now, θ_X is an isomorphism by the exact commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(T_1,T)\otimes_B\mathrm{Hom}_A(X,T)^\vee \arrow[r] \arrow[d, "\simeq"', "\theta_{T_1,X}"] & \mathrm{Hom}_A(T_0,T)\otimes_B\mathrm{Hom}_A(X,T)^\vee \arrow[r] \arrow[d, "\simeq"', "\theta_{T_0,X}"] & T\otimes_B\mathrm{Hom}_A(X,T)^\vee \arrow[r] \arrow[d, "\theta_X"] & 0 \\
\mathrm{Hom}_A(X,T_1)^\vee \arrow[r] & \mathrm{Hom}_A(X,T_0)^\vee \arrow[r] & \mathrm{Hom}_A(X,A)^\vee \arrow[r] & 0.
\end{tikzcd}$$

(2) If X is projective, then the sequence

$$0 \to \mathrm{Hom}_A(X,T_n)^\vee \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(X,T_1)^\vee \longrightarrow \mathrm{Hom}_A(X,T_0)^\vee \longrightarrow \mathrm{Hom}_A(X,A)^\vee \to 0$$

is exact. Now (i) follows if we apply $- \otimes_B \mathrm{Hom}_A(X,T)^\vee$ to $(*)$.

Note that (ii) holds if and only if the canonical map $A^\alpha \otimes_A T \to T^\alpha$ is injective and $\mathrm{Tor}_j^A(A^\alpha, T) = 0$ for any nonempty set α and for any $j \ge 1$. By Lemma 2.4 (3) in [16], the latter is equivalent to saying that ${}_AT$ is a strongly A-Mittag-Leffler A-module (that is, the m-th syzygy of ${}_AT$ is A-Mittag-Leffler for each $m \ge 0$). However, ${}_AT$ is always strongly A-Mittag-Leffler by Lemma 2.5 in [16]. Thus (ii) holds. ∎

Lemma 4.14. *Let $P^\bullet : 0 \to P_n \to \cdots \to P_1 \to P_0 \to 0$ be the deleted projective resolution of ${}_AT$ (see Definition 1.1), where P_i are of degree $-i$ for $0 \le i \le n$. Then*

(1) *$G(B^\vee) \simeq \mathrm{Hom}_A(P^\bullet, A)^\vee$ in $\mathcal{D}(A)$.*

(2) *Let $\rho^\bullet := (\rho_{P_i})_{0 \le i \le n} : \mathrm{Hom}_A(P^\bullet, A) \otimes_A T \longrightarrow \mathrm{Hom}_A(P^\bullet, T)$. Then $\rho^\bullet$ is an injective chain map and there is a commutative diagram of triangles in $\mathcal{D}(B)$:*

$$\begin{tikzcd}
\mathrm{Coker}(\rho^\bullet)^\vee \arrow[r] \arrow[d, equal] & \mathrm{Hom}_A(P^\bullet,T)^\vee \arrow[r, "\rho^{\bullet\vee}"] \arrow[d, "\simeq"] & (\mathrm{Hom}_A(P^\bullet,A)\otimes_A T)^\vee \arrow[r] \arrow[d, "\simeq"] & \mathrm{Coker}(\rho^\bullet)^\vee[1] \arrow[d, equal] \\
\mathrm{Coker}(\rho^\bullet)^\vee \arrow[r] & B^\vee \arrow[r, "\eta_{B^\vee}"] & H \circ G(B^\vee) \arrow[r] & \mathrm{Coker}(\rho^\bullet)^\vee[1].
\end{tikzcd}$$

Proof. (1) By $(T1)$ and $(T2)$ in Definition 1.1, there is an exact sequence of B-modules

$$0 \longrightarrow \mathrm{Hom}_A(P_n,T)^\vee \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(P_1,T)^\vee \longrightarrow \mathrm{Hom}_A(P_0,T)^\vee \longrightarrow B^\vee \longrightarrow 0,$$

where $P_i \in A$-Proj for $0 \le i \le n$. This means that $\mathrm{Hom}_A(P^\bullet,T)^\vee$ is quasi-isomorphic to $B^\vee$. Therefore $G(B^\vee) \simeq G(\mathrm{Hom}_A(P^\bullet,T)^\vee)$ in $\mathcal{D}(A)$. Moreover, by Lemma 4.13(2), $\mathrm{Tor}_j^B(T, \mathrm{Hom}_A(P_i,T)^\vee) = 0$ for $j \ge 1$. By Lemma 4.2(1), we obtain $G(\mathrm{Hom}_A(P^\bullet,T)^\vee) \simeq T \otimes_B \mathrm{Hom}_A(P^\bullet,T)^\vee$. Since we have $T \otimes_B \mathrm{Hom}_A(P^\bullet,T)^\vee \simeq \mathrm{Hom}_A(P^\bullet,A)^\vee$ by Lemma 4.13(1), we conclude that $G(B^\vee) \simeq \mathrm{Hom}_A(P^\bullet,A)^\vee$ in $\mathcal{D}(A)$.

(2) For any $X \in A$-Proj and $j \geq 1$, we have

$$\mathrm{Ext}_A^j(T, \mathrm{Hom}_A(X, A)^\vee) \simeq \mathrm{Tor}_j^A(\mathrm{Hom}_A(X, A), T)^\vee = 0$$

by Lemma 4.13(2). Since $P_i \in A$-Proj for all $0 \leq i \leq n$, Lemma 4.2(2) implies that

$$\mathrm{Hom}_A(T, \beta_{\mathrm{Hom}_A(P^\bullet, A)^\vee}): \ \mathrm{Hom}_A(T, \mathrm{Hom}_A(P^\bullet, A)^\vee) \longrightarrow H(\mathrm{Hom}_A(P^\bullet, A)^\vee)$$

is an isomorphism in $\mathcal{D}(B)$. Consequently, there are isomorphisms in $\mathcal{D}(B)$:

$$\begin{aligned} H \circ G(B^\vee) &\simeq H(\mathrm{Hom}_A(P^\bullet, A)^\vee) \simeq \mathrm{Hom}_A(T, \mathrm{Hom}_A(P^\bullet, A)^\vee) \\ &\simeq (\mathrm{Hom}_A(P^\bullet, A) \otimes_A T)^\vee. \end{aligned}$$

Clearly, $\rho^\bullet$ is injective by Lemma 4.13(3). Thus the sequence

$$0 \longrightarrow \mathrm{Hom}_A(P^\bullet, A) \otimes_A T \xrightarrow{\rho^\bullet} \mathrm{Hom}_A(P^\bullet, T) \longrightarrow \mathrm{Coker}(\rho^\bullet) \longrightarrow 0$$

is exact, and yields an triangle in $\mathcal{D}(B^{\mathrm{op}})$,

$$\mathrm{Coker}(\rho^\bullet)[-1] \longrightarrow \mathrm{Hom}_A(P^\bullet, A) \otimes_A T \xrightarrow{\rho^\bullet} \mathrm{Hom}_A(P^\bullet, T) \longrightarrow \mathrm{Coker}(\rho^\bullet),$$

and a triangle in $\mathcal{D}(B)$,

$$\mathrm{Coker}(\rho^\bullet)^\vee \longrightarrow \mathrm{Hom}_A(P^\bullet, T)^\vee \xrightarrow{\rho^{\bullet\vee}} (\mathrm{Hom}_A(P^\bullet, A) \otimes_A T)^\vee \longrightarrow \mathrm{Coker}(\rho^\bullet)^\vee[1].$$

Now, it follows from this triangle that (2) can be deduced by the following commutative diagram in $\mathcal{D}(B)$:

$$\begin{tikzcd}
\mathrm{Hom}_A(P^\bullet, T)^\vee \arrow[r, "\eta'_{\mathrm{Hom}_A(P^\bullet, T)^\vee}"] \arrow[d, "\simeq"] & \mathrm{Hom}_A\big(T, T \otimes_B \mathrm{Hom}_A(P^\bullet, T)^\vee\big) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & (\mathrm{Hom}_A(P^\bullet, A) \otimes_A T)^\vee \arrow[d] \\
B^\vee \arrow[r, "\eta_{B^\vee}"] & H \circ G(B^\vee) \arrow[r, equal] & H \circ G(B^\vee),
\end{tikzcd}$$

where the composition of the first row is $\rho^{\bullet\vee}$. ∎

Proof of Lemma 4.12. Recall that $X^M := H^0(D(i) \circ \Phi(M))$ for any B-module M. Suppose M is an injective B-module. Then $M \in \mathrm{Prod}\,(B^\vee)$. Since the functors $D(i)$, Φ and H^0 commute with direct products, we have $X^M \in \mathrm{Prod}\,(X^{B^\vee})$. In the following, we calculate $X^{B^\vee}$. In fact, by the diagram ($\sharp$) and Lemma 4.14(2), $D(i) \circ \Phi(B^\vee) \simeq \mathrm{Coker}(\rho^\bullet)^\vee[1]$ in $\mathcal{D}(B)$. This implies that

$$X^{B^\vee} \simeq H^0\big(\mathrm{Coker}(\rho^\bullet)^\vee[1]\big) = H^1\big(\mathrm{Coker}(\rho^\bullet)^\vee\big) \simeq \big(H^{-1}(\mathrm{Coker}(\rho^\bullet))\big)^\vee.$$

Since $H^{-1}(\mathrm{Coker}(\rho^\bullet)) = 0$ by Lemma 4.14(2), we have $X^{B^\vee} = 0$ and $\mathrm{Prod}\,(X^{B^\vee}) = 0$. Thus $X^M = 0$. ∎

Lemma 4.15. (1) *The inclusion* $\mathcal{E} \subseteq B$-Mod *induces a triangle equivalence* $\mathcal{D}(\mathcal{E}) \to \mathcal{D}(B)$ *which restricts to an equivalence* $\mathcal{D}^*(\mathcal{E}) \to \mathcal{D}^*(B)$ *for any* $* \in \{+, -, b\}$.

(2) *The inclusion* $\mathcal{E} \subseteq \mathscr{E}$ *induces a fully faithful triangle functor* $\mathcal{D}^-(\mathcal{E}) \to \mathcal{D}^-(\mathscr{E})$.

Proof. (1) By Lemma 4.10(3), $\mathscr{E}$ is closed under extensions and quotients in B-Mod. In particular, $\mathscr{E}$ is a fully exact subcategory of B-Mod, which is closed under direct summands. By Lemma 4.12, $\mathscr{E}$ contains all injective B-modules. Thus $\mathscr{E}$ satisfies the two properties:

(i) If $0 \to X \to Y \to Z \to 0$ is exact in B-Mod with $Y \in \mathscr{E}$, then $Z \in \mathscr{E}$.

(ii) For each B-module M, there exists a short exact sequence $0 \to M \to E_0 \to E_1 \to 0$ in B-Mod such that E_0 is injective and $E_1 \in \mathscr{E}$.

Now, (1) follows from the dual statement of Lemma 2.8(2) with $\mathscr{A} = B$-Mod.

(2) By Lemma 4.6(1), if $0 \to X \to Y \to Z \to 0$ is exact in B-Mod with $Z \in \mathcal{E}$, then $X \in \mathcal{E}$ if and only if $Y \in \mathcal{E}$. This implies that $\mathcal{E}$ is a fully exact subcategory of B-Mod. Clearly, $\mathcal{E} \subseteq \mathscr{X} \subseteq \mathscr{E}$ by Lemma 4.9 and Lemma 4.10(2).

Let $0 \to E_1 \to E_0 \xrightarrow{g} F \to 0$ be an exact sequence in B-Mod with $E_1, E_0 \in \mathscr{E}$ and $F \in \mathcal{E}$. By Corollary 4.11(3), we can apply the functor $r\colon \mathscr{E} \to \mathscr{X}$ to g and obtain a surjective map $r(g)\colon r(E_0) \to r(F)$. Moreover, the composition of $r(g)$ with the counit map $\varepsilon_F^{-1}\colon r(F) \to F$ coincides with the composition of the counit map $\varepsilon_{E_0}^{-1}\colon r(E_0) \to E_0$ with g. Note that ε_F^{-1} is an isomorphism for $F \in \mathcal{E} \subseteq \mathscr{X}$. Since $r(E_0) \in \mathscr{X} = \mathrm{Coker}(\mathcal{E})$ by Lemma 4.10(2), there exists a surjective map $f\colon F_0 \to r(E_0)$ with $F_0 \in \mathcal{E}$. Let $h := f\varepsilon_{E_0}^{-1}$. Then $hg = f\varepsilon_{E_0}^{-1}g = fr(g)\varepsilon_F^{-1} : F_0 \to F$, and therefore hg is surjective. Let $F_1 := \mathrm{Ker}(hg)$. Since $F_0, F \in \mathcal{E}$, we have $F_1 \in \mathcal{E}$. Moreover, there is an exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & F_1 \arrow[r] \arrow[d] & F_0 \arrow[r, "hg"] \arrow[d, "h"'] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E_1 \arrow[r] & E_0 \arrow[r, "g"] & F \arrow[r] & 0.
\end{tikzcd}$$

Now, (2) follows from Lemma 2.8(1). ∎

Lemma 4.16. (1) *The functor $D^-(i)\colon \mathcal{D}^-(\mathcal{E}) \to \mathcal{D}^-(B)$ is fully faithful and induces a triangle equivalence $\overline{D^-(i)}\colon \mathcal{D}^-(\mathcal{E}) \to \mathcal{D}^-(B) \cap \mathrm{Ker}(G)$.*

(2) *If the functor $\overline{\Phi}\colon \mathcal{D}(B)/\mathrm{Im}(H) \to \mathcal{D}(\mathcal{E})$ is dense, then the functor $\overline{D(i)}\colon \mathcal{D}(\mathcal{E}) \to \mathrm{Ker}(G)$ is a triangle equivalence.*

Proof. (1) Since $D^-(i)$ is the composition of the induced functor $\mathcal{D}^-(\mathcal{E}) \to \mathcal{D}^-(\mathscr{E})$ with the one $\mathcal{D}^-(\mathscr{E}) \to \mathcal{D}^-(B)$, Lemma 4.15 implies that $D^-(i)$ is fully faithful. By Lemmas 4.3(3) and 4.6(2), we have $\mathrm{Im}(D^-(i)) = \mathcal{D}^-(B) \cap \mathrm{Ker}(G)$. Consequently, the restriction $\overline{D^-(i)}\colon \mathcal{D}^-(\mathcal{E}) \to \mathcal{D}^-(B) \cap \mathrm{Ker}(G)$ of $\overline{D(i)}$ is a triangle equivalence.

(2) The triangle functor $\overline{D(i)}$ is always dense, thanks to Lemma 4.3(3). Moreover, by Lemma 4.7(2)–(3), $\mathrm{Ker}(\overline{D(i)}) = 0$, that is, $\overline{D(i)}$ sends nonzero objects of $\mathcal{D}(\mathcal{E})$ to nonzero objects of $\mathrm{Ker}(G)$. It follows from Lemma 2.7(2) that $\overline{D(i)}$ is an equivalence if and only if it is full. Now, let $\Sigma := \overline{D(i)} \circ \overline{\Phi}[-1]$, the composition of $\overline{\Phi}[-1]$ with $\overline{D(i)}$. Then Σ is a triangle equivalence by Lemma 4.7(4). This implies that, for any $X_1, X_2 \in \mathcal{D}(B)/\mathrm{Im}(H)$, the map

$$\mathrm{Hom}_{\mathcal{D}(B)/\mathrm{Im}(H)}(X_1, X_2) \longrightarrow \mathrm{Hom}_{\mathrm{Ker}(G)}(\Sigma(X_1), \Sigma(X_2))$$

induced from Σ is an isomorphism, and therefore the map

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{E})}(\overline{\Phi}[-1](X_1), \overline{\Phi}[-1](X_2)) \longrightarrow \mathrm{Hom}_{\mathrm{Ker}(G)}(\Sigma(X_1), \Sigma(X_2)),$$

induced from $\overline{D(i)}$, is surjective. So, if the functor $\overline{\Phi}$ (and thus also $\overline{\Phi}[-1]$) is dense, then $\overline{D(i)}$ must be full. In this case, $\overline{D(i)}$ is an equivalence. ■

Finally, we prove the following result, which is crucial to the proof of the main result, Theorem 1.2.

Proposition 4.17. *The functor $\overline{D(i)}\colon \mathcal{D}(\mathcal{E}) \to \mathrm{Ker}(G)$ is a triangle equivalence.*

Proof. By Lemma 4.16(2), to show the triangle equivalence, it suffices to show that the functor $\overline{\Phi}$ is dense, or equivalently, $\overline{\Phi}[-1]$ is dense. To check this point, we prove that the composition $\Theta\colon \mathcal{D}(\mathcal{E}) \to \mathcal{D}(\mathcal{E})$ of the triangle functors

$$\mathcal{D}(\mathcal{E}) \xrightarrow{\overline{D(i)}} \mathrm{Ker}(G) \xrightarrow{\kappa} \mathcal{D}(B) \xrightarrow{Q} \mathcal{D}(B)/\mathrm{Im}(H) \xrightarrow{\overline{\Phi}[-1]} \mathcal{D}(\mathcal{E})$$

is dense. For this purpose, we first prove that the restriction of Θ to $\mathcal{D}^-(\mathcal{E})$ is dense, and then show that Θ itself is dense by applying homotopy colimits of bounded-above complexes. This is implemented in 3 steps.

(1) The restriction $\Theta^-\colon \mathcal{D}^-(\mathcal{E}) \to \mathcal{D}^-(\mathcal{E})$ of Θ to $\mathcal{D}^-(\mathcal{E})$ is naturally isomorphic to the identity functor of $\mathcal{D}^-(\mathcal{E})$.

In fact, by Lemma 4.7(4) and Remark 4.8, the two compositions

$$\mathcal{D}^-(B) \cap \mathrm{Ker}(G) \xrightarrow{\kappa^-} \mathcal{D}^-(B) \xrightarrow{Q^-} \mathcal{D}^-(B)/(\mathcal{D}^-(B) \cap \mathrm{Im}(H)) \quad \text{and}$$

$$\mathcal{D}^-(B)/(\mathcal{D}^-(B) \cap \mathrm{Im}(H)) \xrightarrow{\overline{\Phi^-}[-1]} \mathcal{D}^-(\mathcal{E}) \xrightarrow{\overline{D^-(i)}} \mathcal{D}^-(B) \cap \mathrm{Ker}(G)$$

are quasi-inverse triangle equivalences, where $^-$ denotes universally the restriction of the involved functors to bounded-above derived categories. Since $\overline{D^-(i)}$ is a triangle equivalence by Lemma 4.16(1), the functor $\overline{\Phi^-}[-1]$ is also a triangle equivalence. This implies that the composition of the following functors:

$$\mathcal{D}^-(\mathcal{E}) \xrightarrow{\overline{D^-(i)}} \mathcal{D}^-(B) \cap \mathrm{Ker}(G) \xrightarrow{\kappa^-} \mathcal{D}^-(B) \xrightarrow{Q^-} \mathcal{D}^-(B)/(\mathcal{D}^-(B) \cap \mathrm{Im}(H))$$
$$\xrightarrow{\overline{\Phi^-}[-1]} \mathcal{D}^-(\mathcal{E})$$

is naturally isomorphic to the identity functor of $\mathcal{D}^-(\mathcal{E})$. Further, we claim that the inclusion $\mathcal{D}^-(B) \to \mathcal{D}(B)$ induces a fully faithful functor

$$\mathcal{D}^-(B)/(\mathcal{D}^-(B) \cap \mathrm{Im}(H)) \longrightarrow \mathcal{D}(B)/\mathrm{Im}(H).$$

By Lemma 2.7(1), it suffices to show that each morphism $f^\bullet : M^\bullet \to Y^\bullet$ in $\mathcal{D}(B)$ with $M^\bullet \in \mathcal{D}^-(B)$ and $Y^\bullet \in \mathrm{Im}(H)$ factorizes through an object $Z^\bullet \in \mathcal{D}^-(B) \cap \mathrm{Im}(H)$. Since $\mathcal{K}^-(B\text{-Proj})$ is equivalent to $\mathcal{D}^-(B)$, we can assume that $M^\bullet \in \mathcal{C}^-(B\text{-Proj})$ and $f^\bullet$ is represented by a chain map. As $\mathrm{Im}(H) = \mathrm{Im}(D(j))$ by Lemma 4.7(1), we have $Y^\bullet \in \mathcal{C}(\mathcal{Y})$ up to isomorphism in $\mathcal{D}(B)$. Then $f^\bullet$ factorizes through the left truncated complex

$$\tau^{\leq m} Y^\bullet : \quad \cdots \longrightarrow Y^i \longrightarrow Y^{i+1} \longrightarrow \cdots \longrightarrow Y^{m-1} \longrightarrow \mathrm{Ker}(d_Y^m) \longrightarrow 0$$

of $Y^\bullet$ at some degree m. Since $\mathcal{Y}$ is an abelian subcategory of B-Mod, $\mathrm{Ker}(d_Y^m) \in \mathcal{Y}$ and $\tau^{\leq m} Y^\bullet \in \mathcal{C}^-(\mathcal{Y})$, and therefore $\tau^{\leq m} Y^\bullet \in \mathcal{D}^-(B) \cap \mathrm{Im}(H)$.

Now, we construct the following commutative diagram of triangle functors:

$$\begin{tikzcd}
\mathcal{D}^-(\mathcal{E}) \arrow[r, "\overline{D^-(i)}", "\simeq"'] \arrow[d, hook] & \mathcal{D}^-(B) \cap \mathrm{Ker}(G) \arrow[r, "\kappa^-"] \arrow[d, hook] & \mathcal{D}^-(B) \arrow[r, "Q^-"] \arrow[d, hook] & \mathcal{D}^-(B)/(\mathcal{D}^-(B) \cap \mathrm{Im}(H)) \arrow[r, "\overline{\Phi^-}[-1]", "\simeq"'] \arrow[d, hook] & \mathcal{D}^-(\mathcal{E}) \arrow[d, hook] \\
\mathcal{D}(\mathcal{E}) \arrow[r, "\overline{D(i)}"] & \mathrm{Ker}(G) \arrow[r, "\kappa"] & \mathcal{D}(B) \arrow[r, "Q"] & \mathcal{D}(B)/\mathrm{Im}(H) \arrow[r, "\overline{\Phi}[-1]"] & \mathcal{D}(\mathcal{E})
\end{tikzcd}$$

of which the composition of all functors in the bottom row is exactly Θ. Then Θ^- is the composition of all functors in the top row of the diagram, and thus naturally isomorphic to the identity functor of $\mathcal{D}^-(\mathcal{E})$.

(2) Θ commutes with direct products.

This is obtained by showing that each component of Θ commutes with direct products. In fact, since $\mathcal{E}$ is closed under products in B-Mod by Lemma 4.6(1), $D(i)$ commutes with direct products. Clearly, the inclusion κ preserves direct products. Moreover, by the dual result of Lemma 1.5 in [12], if $\mathcal{S}$ is a colocalizing subcategory of a triangulated category $\mathcal{T}$ with direct products, then the triangulated category $\mathcal{T}/\mathcal{S}$ has direct products and the localization functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ preserves direct products. As $\mathrm{Im}(H)$ is a full triangulated subcategory of $\mathcal{D}(B)$ closed under products by Theorem 4.1, the localizing functor Q preserves direct products. Finally, $\overline{\Phi}[-1]$ commutes with direct products by Lemma 4.7(3).

(3) Θ is dense, and therefore $\overline{\Phi}[-1]$ is dense.

Let $X^\bullet \in \mathcal{C}(\mathcal{E})$. Then $X^{\leq n} \in \mathcal{C}^-(\mathcal{E})$ for $n \geq 0$ and $X^\bullet \simeq \underleftarrow{\mathrm{Holim}}(X^{\leq n})$ in $\mathcal{K}(B)$ by Lemma 2.5. Recall that $\mathcal{K}(\mathcal{E})$ is a triangulated subcategory of $\mathcal{K}(B)$ closed under direct products by Lemma 4.6(1). This implies $X^\bullet \simeq \underleftarrow{\mathrm{Holim}}(X^{\leq n})$ in $\mathcal{K}(\mathcal{E})$ and thus also in $\mathcal{D}(\mathcal{E})$ because the localization functor $\mathcal{K}(\mathcal{E}) \to \mathcal{D}(\mathcal{E})$ preserves direct products. As Θ commutes with products, it commutes with homotopy limits in $\mathcal{D}(\mathcal{E})$. Consequently,

$$\Theta(X^\bullet) = \Theta(\underleftarrow{\mathrm{Holim}}(X^{\leq n})) \simeq \underleftarrow{\mathrm{Holim}}(\Theta(X^{\leq n})).$$

Since the restriction of Θ to $\mathcal{D}^-(\mathcal{E})$ is isomorphic to the identity functor of $\mathcal{D}^-(\mathcal{E})$, it follows from $X^{\leq n} \in \mathcal{D}^-(\mathcal{E})$ that $\underleftarrow{\mathrm{Holim}}(\Theta(X^{\leq n})) \simeq \underleftarrow{\mathrm{Holim}}(X^{\leq n}) \simeq X^\bullet$ in $\mathcal{D}(\mathcal{E})$. Thus $\Theta(X^\bullet) \simeq X^\bullet$ in $\mathcal{D}(\mathcal{E})$. This shows that Θ is dense. ■

4.3. Proofs of main result and its corollaries

For the proof of Theorem 1.2, we start with the recollement in Theorem 4.1, substitute $\mathrm{Ker}(G)$ with $\mathcal{D}(\mathcal{E})$ by Proposition 4.17, and show that $(D(i), \Phi[-1])$ is an adjoint pair of functors.

We keep all notation introduced in the previous sections.

Proof of Theorem 1.2. By Proposition 4.17, $\overline{D(i)}\colon \mathcal{D}(\mathcal{E}) \to \mathrm{Ker}(G)$ is a triangle equivalence. Moreover, by Lemma 4.7(4), the inclusion $\kappa\colon \mathrm{Ker}(G) \to \mathcal{D}(B)$ has a right adjoint $\overline{D(i)} \circ \Phi[-1]\colon \mathcal{D}(B) \to \mathrm{Ker}(G)$. It follows that the pair $(D(i), \Phi[-1])$ of triangle functors between $\mathcal{D}(\mathcal{E})$ and $\mathcal{D}(B)$ is an adjoint pair. Now, we replace $\mathrm{Ker}(G)$ in the recollement of Theorem 4.1 by $\mathcal{D}(\mathcal{E})$ and obtain the recollement of the form

$$\begin{tikzcd} \mathcal{D}(\mathcal{E}) \arrow[r, "D(i)"] & \mathcal{D}(B) \arrow[l, bend right=50] \arrow[l, bend left=50, "{\Phi[-1]}"] \arrow[r, "G"] & \mathcal{D}(A) \arrow[l, bend right=50, "j_!"'] \arrow[l, bend left=50, "H"] \end{tikzcd}.$$

Note that the functors $D(i)$, $\Phi[-1]$, H and G can be restricted to functors between $\mathcal{D}^-$-derived categories by Remark 4.8. Therefore, to show that this recollement restricts to $\mathcal{D}^-$-derived categories, it suffices to show that the left adjoint $j_!\colon \mathcal{D}(A) \to \mathcal{D}(B)$ of G restricts to a functor $\mathcal{D}^-(A) \to \mathcal{D}^-(B)$.

For any $m \in \mathbb{Z}$ and $X \in \mathcal{D}(A)$, there are isomorphisms of abelian groups:

$$(\dagger) \quad \mathrm{Hom}_{\mathcal{D}(A)}(X, G(B^\vee)[m]) \simeq \mathrm{Hom}_{\mathcal{D}(B)}(j_!(X), B^\vee[m]) \simeq \mathrm{Hom}_{\mathcal{K}(B)}(j_!(X), B^\vee[m])$$
$$\simeq H^m(j_!(X)^\vee) \simeq (H^{-m}(j_!(X)))^\vee.$$

Let $X \in \mathcal{D}^-(A)$. Since $G(B^\vee) \in \mathcal{D}^b(A)$ by Lemma 4.14(1), there is an integer m_X such that $\mathrm{Hom}_{\mathcal{D}(A)}(X, G(B^\vee)[m]) = 0$ for all $m \le m_X$. Observe that the functor $(-)^\vee$ sends nonzero modules to nonzero modules. Thus $H^{-m}(j_!(X)) = 0$ for all $m \le m_X$. This implies $j_!(X) \in \mathcal{D}^-(B)$. ■

Recall that a ring epimorphism $\lambda\colon B \to C$ is said to be *homological* if $\mathrm{Tor}^B_m(C, C) = 0$ for all $m \ge 1$. This is equivalent to saying that the induced derived functor $D(\lambda_*)\colon \mathcal{D}(C) \to \mathcal{D}(B)$ is fully faithful, where $\lambda_*\colon C\text{-Mod} \to B\text{-Mod}$ is the restriction functor.

Definition 4.18 (see the Introduction of [16]). A good tilting module ${}_AT$ is said to be *homological* if there exists a homological ring epimorphism $\lambda\colon B \to C$ such that $D(\lambda_*)$ induces an equivalence $\mathcal{D}(C) \xrightarrow{\simeq} \mathrm{Ker}(G)$ of triangulated categories.

To prove Corollary 1.3, we recall the definition of derived decompositions of abelian categories in [17].

Definition 4.19 (Definition 1.1 in [17]). Let $\mathcal{A}$ be an abelian category, $\mathcal{B}$ and $\mathcal{C}$ full subcategories of $\mathcal{A}$. The pair $(\mathcal{B}, \mathcal{C})$ is called a *derived decomposition* of $\mathcal{A}$ if $\mathcal{B}$ and $\mathcal{C}$ are abelian subcategories of $\mathcal{A}$ such that

(1) the inclusions $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{C} \subseteq \mathcal{A}$ induce fully faithful functors $\mathcal{D}^b(\mathcal{B}) \hookrightarrow \mathcal{D}^b(\mathcal{A})$ and $\mathcal{D}^b(\mathcal{C}) \hookrightarrow \mathcal{D}^b(\mathcal{A})$, respectively.

(2) $(\mathcal{D}^b(\mathcal{B}), \mathcal{D}^b(\mathcal{C}))$ is a semi-orthogonal decomposition of $\mathcal{D}^b(\mathcal{A})$.

Proof of Corollary 1.3. The equivalence of (1) and (2) is shown in Proposition 6.2 of [8], while the one of (1) and (3) is proved in Theorem 1.1 of [16]. Here, we prefer to providing new and shorter proofs of these facts by applying results in the previous sections.

(1) $\Rightarrow$ (2): Suppose that T is homological. Then there is a homological ring epimorphism $\lambda : B \to C$ such that $\mathrm{Im}(D(\lambda_*)) = \mathrm{Ker}(G)$. Since $\mathrm{Im}(\lambda_*)$ is an abelian subcategory of B-Mod, we have $\mathrm{Im}(D(\lambda_*)) \cap B\text{-Mod} = \mathrm{Im}(\lambda_*)$. Further, by Proposition 4.17, the functor $\overline{D(i)} : D(\mathcal{E}) \to \mathrm{Ker}(G)$ is a triangle equivalence, and therefore $\mathrm{Im}(\overline{D(i)}) = \mathrm{Ker}(G)$. Due to Lemma 4.6(2), we have $\mathrm{Im}(\overline{D(i)}) \cap B\text{-Mod} = \mathcal{E}$, and therefore $\mathcal{E} = \mathrm{Im}(\lambda_*)$. Thus $\mathcal{E}$ is an abelian subcategory of B-Mod.

(2) $\Rightarrow$ (1): Suppose that $\mathcal{E}$ is an abelian subcategory of B-Mod. Then $\mathcal{E}$ is closed under direct sums, direct products, kernels and cokernels by Lemma 4.6(1). It follows that there exists a ring epimorphism $\lambda : B \to C$ such that λ_* induces an equivalence C-Mod $\xrightarrow{\simeq} \mathcal{E}$ of abelian categories (for example, see Lemma 2.1 in [14]). Since $\overline{D(i)}$ is fully faithful by Proposition 4.17, the functor $D(\lambda_*)$ is also fully faithful. Thus λ is homological. Further, it follows from $\mathrm{Im}(\overline{D(i)}) = \mathrm{Ker}(G)$ that $D(\lambda_*)$ induces an equivalence from $\mathcal{D}(C)$ to $\mathrm{Ker}(G)$. This shows (1).

(2) $\Rightarrow$ (3): Let $\rho^\bullet\colon \mathrm{Hom}_A(P^\bullet, A) \otimes_A T \to \mathrm{Hom}_A(P^\bullet, T)$ be the chain map defined in Lemma 5.1(2). Associated with $B^\vee$, there exists a triangle in $\mathcal{D}(B)$:

$$(\triangle): \quad X^\bullet[-1] \longrightarrow B^\vee \longrightarrow Y^\bullet \longrightarrow X^\bullet,$$

where

$$\begin{aligned} Y^\bullet &:= (\mathrm{Hom}_A(P^\bullet, A) \otimes_A T)^\vee \in \mathrm{Im}(H) = \mathrm{Im}(D(j)),\\ X^\bullet &:= \mathrm{Coker}(\rho^\bullet)^\vee[1] \in \mathrm{Ker}(G) = \mathrm{Im}(D(i)). \end{aligned}$$

Since $\mathcal{Y} \subseteq B$-Mod is an abelian subcategory, we have $H^s(Y^\bullet) \in \mathcal{Y}$ for any $s \in \mathbb{Z}$. Taking cohomologies on the triangle yields $H^{-s}(X^\bullet) \simeq H^{-s}(Y^\bullet)$ whenever $s \geq 2$. Moreover,

$$H^t(X^\bullet) \simeq H^{t+1}\big(\mathrm{Coker}(\rho^\bullet)^\vee\big) \simeq \big(H^{-t-1}(\mathrm{Coker}(\rho^\bullet))\big)^\vee = 0$$

for any $t \geq 0$. In particular, $H^0(X^\bullet) = 0$.

Suppose (2) holds. Then $H^s(X^\bullet) \in \mathcal{E}$ for any $s \in \mathbb{Z}$. By Lemma 4.9, $H^{-s}(X^\bullet) = 0 = H^{-s}(Y^\bullet)$ for any $s \geq 2$. It follows that $0 = H^{-m}(Y^\bullet) \simeq (H^m(\mathrm{Hom}_A(P^\bullet, A) \otimes_A T))^\vee$ for any $m \geq 2$. Now (3) holds by the fact that the functor $(-)^\vee$ sends nonzero modules to nonzero modules.

(3) $\Rightarrow$ (2): Suppose that $H^m(\mathrm{Hom}_A(P^\bullet, A) \otimes_A T) = 0$ for all $m \geq 2$, where $P^\bullet$ is a deleted projective resolution of ${}_AT$. We have to show that $\mathcal{E}$ is an abelian subcategory of B-Mod. In fact, by Lemma 4.6(1), $\mathcal{E}$ is an abelian subcategory of B-Mod if and only if it is closed under cokernels in B-Mod. Since the subcategory $\mathcal{X}$ of B-Mod, defined in Section 4.2, is always closed under cokernels in B-Mod, it is enough to show $\mathcal{E} = \mathcal{X}$.

Clearly, $\mathcal{E} \subseteq \mathcal{X}$. To show the converse, we first claim that each module $U \in \mathcal{X}$ can be embedded into a module belonging to $\mathcal{E}$. Actually, U can be embedded into its injective envelop J in B-Mod. By the first isomorphism in Corollary 4.11(2), U is isomorphic to a submodule of X_J. So, it suffices to show $X_I \in \mathcal{E}$ for any injective B-module I.

Obviously, $H^s(Y^\bullet) = 0$ for any $s \geq 1$. Combining this with (3), we have $H^{-s}(Y^\bullet) = 0$ for any $s \neq 0, 1$. This implies $H^{-t}(X^\bullet) = 0$ for any $t \neq 0, 1$. As $H^0(X^\bullet) = 0$, we get $X^\bullet \simeq H^{-1}(X^\bullet)[1]$ in $\mathcal{D}(B)$. Consequently, $H^{-1}(X^\bullet) \in \mathcal{E}$ by $X^\bullet \in \mathrm{Ker}(G)$. By Lemmas 4.3(2) and 4.14(2), the triangle $(\diamondsuit)$ associated with $B^\vee$ is isomorphic to the one in $(\triangle)$. This shows $X_{B^\vee} \simeq H^{-1}(X^\bullet) \in \mathcal{E}$. Recall that $D(i)$ and Φ commute with direct products by Lemma 4.7(3), and that Prod $(B^\vee)$ consists of all injective B-modules. Thus $X_I \in \mathrm{Prod}\,(X_{B^\vee}) \subseteq \mathcal{E}$ by the diagram $(\sharp)$ and Lemma 4.6(1).

It follows from the above claim that there is an injection $f_0\colon U \to E_0$ with $E_0 \in \mathcal{E}$. Since $\mathcal{X}$ contains $\mathcal{E}$ and is closed under cokernels in B-Mod, we have $\mathrm{Coker}(f_0) \in \mathcal{X}$. By iterating this construction, we can get an infinitely long exact sequence of B-modules:

$$0 \longrightarrow U \xrightarrow{f_0} E_0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_i \longrightarrow \cdots$$

such that $E_i \in \mathcal{E}$ for all $i \in \mathbb{N}$. Then $U \in \mathcal{E}$ by Lemma 4.6(2). Hence we have shown $\mathcal{X} \subseteq \mathcal{E}$. Thus (2) holds.

(2) $\Leftrightarrow$ (4). Note that $D^b(i)\colon \mathcal{D}^b(\mathcal{E}) \to \mathcal{D}^b(B)$ and $D^b(j)\colon \mathcal{D}^b(\mathcal{Y}) \to \mathcal{D}^b(B)$ are fully faithful by Proposition 4.17 and Lemma 4.7(1), respectively. Since proj.dim$({}_AT) < \infty$ and proj.dim$(T_B) < \infty$, the functors H and G can be regarded as functors between bounded derived categories. Thus $(\mathcal{D}^b(B) \cap \mathrm{Ker}(G), \mathcal{D}^b(B) \cap \mathrm{Im}(H))$ is a semi-orthogonal decomposition of $\mathcal{D}^b(B)$. Further, $\mathrm{Im}(\mathcal{D}^b(i)) = \mathcal{D}^b(B) \cap \mathrm{Ker}(G)$ by Lemmas 4.3(3) and 4.6(2), while $\mathrm{Im}(\mathcal{D}^b(j)) = \mathcal{D}^b(B) \cap \mathrm{Im}(H)$ by Lemma 4.3(3) and Corollary 4.5(2). Hence the pair $\big(\mathrm{Im}(\mathcal{D}^b(i)), \mathrm{Im}(\mathcal{D}^b(j))\big)$ is a semi-orthogonal decomposition of $\mathcal{D}^b(B)$. This implies that $(\mathcal{E}, \mathcal{Y})$ is a derived decomposition of B-Mod if and only if $\mathcal{E}$ is an abelian subcategory of B-Mod. Thus (2) and (4) are equivalent. ■

Proof of Corollary 1.4. Suppose that A is a left coherent ring. For $* \in \{b, +, -, \emptyset\}$, we have to show that there exists a recollement of derived categories

$$\begin{tikzcd} \mathcal{D}^*(\mathcal{E}) \arrow[r] & \mathcal{D}^*(B) \arrow[r, "G"] \arrow[l, bend right] \arrow[l, bend left] & \mathcal{D}^*(A). \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

Actually, by Lemma 4.6(2), up to triangle equivalence, we have the identifications of categories:

$$\begin{aligned} \mathcal{D}^b(\mathcal{E}) &= \{X^\bullet \in \mathcal{D}(\mathcal{E}) \mid H^i(X^\bullet) = 0, |i| >> 0\}, \\ \mathcal{D}^+(\mathcal{E}) &= \{X^\bullet \in \mathcal{D}(\mathcal{E}) \mid H^i(X^\bullet) = 0, i << 0\}, \\ \mathcal{D}^-(\mathcal{E}) &= \{X^\bullet \in \mathcal{D}(\mathcal{E}) \mid H^i(X^\bullet) = 0, i >> 0\}. \end{aligned}$$

Since proj.dim$({}_AT) \le n$ and proj.dim$(T_B) \le n$, the functors G and H in Theorem 1.2 restrict to functors at $\mathcal{D}^*$-level for $* \in \{b, +, -\}$. So we get the half recollement

$$\begin{tikzcd} \mathcal{D}^*(\mathcal{E}) \arrow[r] & \mathcal{D}^*(B) \arrow[r, "G"] \arrow[l, bend left] & \mathcal{D}^*(A) \arrow[l, bend left, "H"] \end{tikzcd}$$

To show Corollary 1.4, it suffices to show that the left adjoint $j_!\colon \mathcal{D}(A) \to \mathcal{D}(B)$ of the functor $G\colon \mathcal{D}(B) \to \mathcal{D}(A)$ restricts to a functor $\mathcal{D}^*(A) \to \mathcal{D}^*(B)$ whenever A is left coherent.

Since A is left coherent, the direct products of projective A^{op}-modules are flat. As the functor $(-)^\vee\colon A^{\mathrm{op}}\text{-Mod} \to A\text{-Mod}$ sends flat modules to injective ones, $\mathrm{Hom}_A(P^\bullet, A)^\vee$ is a bounded complex of injective A-modules. By Lemma 4.14(1), $\mathrm{Hom}_A(P^\bullet, A)^\vee \simeq G(B^\vee)$ in $\mathcal{D}(A)$. Moreover, it follows from (†) in the proof of Theorem 1.2 that

$$(H^{-m}(j_!(X)))^\vee \simeq \mathrm{Hom}_{\mathcal{D}(A)}(X, G(B^\vee)[m]).$$

Thus

$$(H^{-m}(j_!(X)))^\vee \simeq \mathrm{Hom}_{\mathcal{D}(A)}(X, \mathrm{Hom}_A(P^\bullet, A)^\vee[m]) \simeq \mathrm{Hom}_{\mathcal{K}(A)}(X, \mathrm{Hom}_A(P^\bullet, A)^\vee[m])$$

for any $X \in \mathcal{D}(A)$ and $m \in \mathbb{Z}$. Since the functor $(-)^\vee$ sends nonzero modules to nonzero modules, $X \in \mathcal{D}^*(A)$ if and only if $j_!(X) \in \mathcal{D}^*(B)$. Further, the right (left) adjoint functor from $\mathcal{D}(B)$ to $\mathcal{D}(\mathcal{E})$ of $D(i)$ restricts to $\mathcal{D}^*$-level follows from the first (second) triangle in Definition 2.3(4). ■

5. Examples from commutative 2-Gorenstein rings

In this section, we modify a known construction of tilting modules over commutative 2-Gorenstein rings and provide an explicit description of symmetric subcategories $\mathcal{E}$ associated with good 2-tilting modules constructed in this way.

Throughout this section, we let A be a commutative noetherian ring, and write $\mathrm{Spec}(A)$ for the spectrum of prime ideals of A. For $\mathfrak{p} \in \mathrm{Spec}(A)$, let $A_{\mathfrak{p}}$ be the localization of A at $\mathfrak{p}$, and let $k(\mathfrak{p})$ be the residue field $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$. Further, let $J_{\mathfrak{p}} := \varprojlim_i A_{\mathfrak{p}}/\mathfrak{p}^i A_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic completion of A. Then there is a canonical ring homomorphism $\eta_{\mathfrak{p}}\colon A \to J_{\mathfrak{p}}$. If A is local with the maximal ideal $\mathfrak{m}$ such that $\eta_{\mathfrak{m}}\colon A \to J_{\mathfrak{m}}$ is an isomorphism, then A is said to be *complete*. An element $0 \neq x \in A$ is called a *regular element* if it is not a zero-divisor in A.

Let M, N be A-modules. We denote by $E(M)$ and $\mathrm{inj.dim}(M)$ the injective envelope and injective dimension of M, respectively. The multiplication maps θ_M and $(\cdot g \cdot)$ for each $g \in \mathrm{Hom}_A(M, N)$ are defined as follows:

$$\theta_M : A \longrightarrow \mathrm{End}_A(M), \quad a \mapsto [(\cdot a) : m \mapsto ma],$$
$$(\cdot g \cdot) : \mathrm{End}_A(M) \otimes_A \mathrm{End}_A(N) \longrightarrow \mathrm{Hom}_A(M, N), \quad f \otimes h \mapsto fgh,$$

where $a \in A$, $m \in M$, $f \in \mathrm{End}_A(M)$ and $h \in \mathrm{End}_A(N)$. Clearly, $(\cdot g \cdot)$ is a homomorphism of A-A-bimodules.

The following properties of injective modules over commutative noetherian rings will be used later (see Sections 3.3 and 3.4 in [21] for proofs).

Lemma 5.1. *Let A be a commutative noetherian ring and $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(A)$.*

(1) $\mathrm{Hom}_A(E(A/\mathfrak{p}), E(A/\mathfrak{q})) \neq 0$ *if and only if* $\mathfrak{p} \subseteq \mathfrak{q}$.

(2) *If $a \in A \setminus \mathfrak{p}$, then the map $(a\cdot)\colon E(A/\mathfrak{p}) \to E(A/\mathfrak{p})$ multiplication by a is an automorphism.*

(3) *Let $E_n(A/\mathfrak{p}) := \{y \in E(A/\mathfrak{p}) \mid \mathfrak{p}^n y = 0\}$. Then $E(A/\mathfrak{p}) = \bigcup_{n\in\mathbb{N}} E_n(A/\mathfrak{p})$ and* $\mathrm{Hom}_{A_{\mathfrak{p}}}(k(\mathfrak{p}), E(A/\mathfrak{p})) \simeq k(\mathfrak{p})$ *as $k(\mathfrak{p})$-modules.*

(4) *There is a ring isomorphism $\phi_{\mathfrak{p}}\colon \mathrm{End}_A(E(A/\mathfrak{p})) \to J_{\mathfrak{p}}$ such that $\eta_{\mathfrak{p}} = \theta_{E(A/\mathfrak{p})}\phi_{\mathfrak{p}}$. In this sense, we identity* $\mathrm{End}_A(E(A/\mathfrak{p}))$ *with $J_{\mathfrak{p}}$.*

(5) $A_{\mathfrak{q}} \otimes_A E(A/\mathfrak{p}) = \begin{cases} E(A/\mathfrak{p}), & \mathfrak{p} \subseteq \mathfrak{q}, \\ 0, & \textit{otherwise}. \end{cases}$

(6) *Let E be an injective A-module. Then $E \simeq \bigoplus_{\mathfrak{h}\in\mathrm{Spec}(A)} E(A/\mathfrak{h})^{(X_{\mathfrak{h}})}$, where $X_{\mathfrak{h}}$ is a set of cardinality equal to the dimension of* $\mathrm{Hom}_{A_{\mathfrak{h}}}(k(\mathfrak{h}), A_{\mathfrak{h}} \otimes_A E)$ *over the field $k(\mathfrak{h})$.*

The next result is a special case of Lemma 6.5 (2) in [14].

Lemma 5.2. *Let $A \subseteq S$ be an extension of commutative rings, that is, A is a subring of S with the same identity. Suppose that the inclusion $A \to S$ is a ring epimorphism with* $\mathrm{Tor}_1^A(S, S) = 0$. *Then the map* $(\cdot\pi\cdot)\colon \mathrm{End}_A(S) \otimes_A \mathrm{End}_A(S/A) \to \mathrm{Hom}_A(S, S/A)$ *is an isomorphism, where $\pi\colon S \to S/A$ is the canonical surjection.*

The ring A is called an *n-Gorenstein* ring for a natural number $n \geq 0$ if $\mathrm{inj.dim}(A) = n$. For a module ${}_AM$ over an n-Gorenstein ring A, we see that $\mathrm{proj.dim}(M) < \infty$ if and only if inj.dim$(M) < \infty$, and these two dimensions are at most n (see Theorem 9.1.10 in [21]). So, each injective A-module has projective dimension at most n and each projective A-module has injective dimension at most n.

From now on, A denotes a commutative noetherian 2-Gorenstein local ring with the maximal ideal $\mathfrak{m}$. Let Q be the total quotient ring of A and let $\mathcal{H}$ be the set of all prime ideals of A with height 1. Then A has a minimal injective coresolution (for example, see Sections 1 and 6 in [5]):

$$0 \longrightarrow A \xrightarrow{\lambda} Q \xrightarrow{\alpha:=(\alpha_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{H}}} \bigoplus_{\mathfrak{p}\in\mathcal{H}} E(A/\mathfrak{p}) \xrightarrow{\beta:=(\beta_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{H}}} E(A/\mathfrak{m}) \longrightarrow 0,$$

where λ is the canonical inclusion and, for each $\mathfrak{p} \in \mathcal{H}$, $\alpha_{\mathfrak{p}} \in \mathrm{Hom}_A(Q, E(A/\mathfrak{p}))$ and $\beta_{\mathfrak{p}} \in \mathrm{Hom}_A(E(A/\mathfrak{p}), E(A/\mathfrak{m}))$. Set

$$E := \bigoplus_{\mathfrak{p}\in\mathcal{H}} E(A/\mathfrak{p}) \quad \text{and} \quad L := Q \oplus E \oplus E(A/\mathfrak{m}).$$

Then L is a good 2-tilting A-module (see [1, 16]).

Now, we modify the construction of L to get new 2-tilting A-modules. Let $\mathcal{S}$ be a nonempty subset of $\mathcal{H}$. We define $T := T_0 \oplus T_1 \oplus T_2$, where

$$T_0 := \alpha^{-1}(T_1 \cap \mathrm{Ker}(\beta)), \quad T_1 := \bigoplus_{\mathfrak{p}\in\mathcal{S}} E(A/\mathfrak{p}) \quad \text{and} \quad T_2 := E(A/\mathfrak{m}).$$

Let $f_2 \colon T_1 \to T_2$ be the restriction of β to T_1. Then $T_1 \subseteq E$, $A \subseteq T_0 \subseteq Q$, and there is an exact sequence of A-modules,

$$0 \longrightarrow A \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} T_2,$$

where f_0 is the inclusion, f_1 is induced by α and $f_1 := (f_{1,\mathfrak{p}})_{\mathfrak{p}\in\mathcal{S}}$ with $f_{1,\mathfrak{p}} \colon T_0 \to E(A/\mathfrak{p})$. Let $E_1 := \bigoplus_{\mathfrak{p}\in\mathcal{H}\setminus\mathcal{S}} E(A/\mathfrak{p})$ and let $\lambda_0 \colon T_0 \to Q$ be the inclusion. If f_2 is surjective, then there is an exact commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & A \arrow[r, "f_0"] \arrow[d, equal] & T_0 \arrow[r, "f_1"] \arrow[d, "\lambda_0"] & T_1 \arrow[r, "f_2"] \arrow[d] & T_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "\lambda"] & Q \arrow[r, "\alpha"] \arrow[d] & E \arrow[r, "\beta"] \arrow[d] & T_2 \arrow[r] & 0 \\
 & & E_1 \arrow[r, equal] \arrow[d] & E_1 \arrow[d] & & \\
 & & 0 & 0 & &
\end{tikzcd}$$

Lemma 5.3. (1) *If the map f_2 is surjective, then T is a good 2-tilting A-module with* $\operatorname{inj.dim}({}_AT) \leq 1$.

(2) *T_0 is a subring of Q with the same identity and isomorphic to the ring* $\operatorname{End}_A(T_0)$, *and* $\operatorname{Hom}_A(T_j, T_i) = 0$ *for any* $0 \leq i < j \leq 2$. *Thus, ${}_AT$ is not homological.*

(3) *If $\mathcal{S}$ contains a principal ideal of A generated by a regular element, then f_2 is surjective.*

Proof. (1) Recall that Q, E_1, T_1 and T_2 are injective A-modules. This forces $\operatorname{inj.dim}(T) = \operatorname{inj.dim}(T_0) \leq 1$. Since A is 2-Gorenstein, $\operatorname{proj.dim}({}_AT) \leq 2$. By Definition 1.1, to show (1), it suffices to show $\operatorname{Ext}^1_A(T_1 \oplus T_2, T_0^{(\delta)}) = 0 = \operatorname{Ext}^1_A(T_0, T_0^{(\delta)})$ for any nonempty set δ.

By Lemma 5.1 (1), $\operatorname{Hom}_A(T_1 \oplus T_2, Q) = 0 = \operatorname{Hom}_A(T_1 \oplus T_2, E_1)$. This implies that $\operatorname{Hom}_A(T_1 \oplus T_2, Q^{(\delta)}) = 0 = \operatorname{Hom}_A(T_1 \oplus T_2, E_1^{(\delta)})$. Applying $\operatorname{Hom}_A(T_1 \oplus T_2, -)$ to the exact sequence $0 \to T_0^{(\delta)} \to Q^{(\delta)} \to E_1^{(\delta)} \to 0$ yields both $\operatorname{Hom}_A(T_1 \oplus T_2, T_0^{(\delta)}) = 0$ and $\operatorname{Ext}^1_A(T_1 \oplus T_2, T_0^{(\delta)}) = 0$. Let $K := \operatorname{Ker}(f_2) = \operatorname{Im}(f_1)$. Then the exact sequence $0 \to K \to T_1 \to T_2 \to 0$ implies that the sequence

$$\operatorname{Ext}^1_A(T_1, T_0^{(\delta)}) \longrightarrow \operatorname{Ext}^1_A(K, T_0^{(\delta)}) \longrightarrow \operatorname{Ext}^2_A(T_2, T_0^{(\delta)})$$

is exact. As $\operatorname{inj.dim}(T_0) \leq 1$, there holds $\operatorname{Ext}^2_A(T_2, T_0^{(\delta)}) = 0$. Thus $\operatorname{Ext}^1_A(K, T_0^{(\delta)}) = 0$. It then follows from the exact sequence $0 \to A \to T_0 \to K \to 0$ that $\operatorname{Ext}^1_A(T_0, T_0^{(\delta)}) = 0$. This shows (1).

(2) Since $\operatorname{Hom}_A(T_1, Q) = 0$ and ${}_AQ$ is injective, we have $\operatorname{Hom}_A(K, Q) = 0$, and therefore $\operatorname{Hom}_A(K, T_0) = 0$. It follows from $\operatorname{Ext}^1_A(K, T_0) = 0$ that the map $(f_0)_*\colon \operatorname{End}_A(T_0) \to \operatorname{Hom}_A(A, T_0) \simeq T_0$ is an isomorphism. Since Q is injective and $\operatorname{Hom}_A(E_1, Q) = 0$, the map λ_0 induces an injective homomorphism of rings

$$\widetilde{\lambda}_0: \ \operatorname{End}_A(T_0) \longrightarrow \operatorname{End}_A(Q) \xrightarrow{\simeq} \operatorname{End}_Q(Q) \xrightarrow{\simeq} Q, \quad f \mapsto (1)f \ \text{ for } f \in \operatorname{End}_A(T_0).$$

This implies also that f is the multiplication map by $(1)f$. Moreover, λ_0 is the composition of the inverse $(f_0)_*^{-1} : T_0 \to \operatorname{End}_A(T_0)$ of $(f_0)_*$ with $\widetilde{\lambda}_0$. Thus T_0 is a subring of Q and $(f_0)_*^{-1}$ is a ring isomorphism sending $t_0 \in T$ to $(\cdot t_0)$. Since $\operatorname{Hom}_A(T_2, T_1 \oplus Q) = 0 = \operatorname{Hom}_A(T_1, Q)$ by Lemma 5.1 (1) and $T_0 \subseteq Q$, we have $\operatorname{Hom}_A(T_2, T_0) = 0 = \operatorname{Hom}_A(T_1, T_0)$. According to Corollary 1.2 in [16], the tilting A-module T is not homological.

(3) Suppose $\mathfrak{p} = Ax \in \mathcal{S}$ with $x \in A$. We show that $\beta_{\mathfrak{p}}\colon E(A/\mathfrak{p}) \to T_2$ is surjective, and then so is f_2.

Clearly, $\mathfrak{p}^n = Ax^n$ for $n \geq 1$. Since x is a regular element in A, we have ${}_A\mathfrak{p}^n \simeq {}_AA$ as A-modules. Then $\operatorname{proj.dim}(A/\mathfrak{p}^n) = 1$ and $\operatorname{Ext}^2_A(A/\mathfrak{p}^n, A) = 0$. Thus

$$\beta_n := \operatorname{Hom}_A(A/\mathfrak{p}^n, \beta) : \operatorname{Hom}_A(A/\mathfrak{p}^n, E) \longrightarrow \operatorname{Hom}_A(A/\mathfrak{p}^n, T_2)$$

is surjective. If $\mathfrak{p}^n \not\subseteq \mathfrak{q} \in \operatorname{Spec}(A)$ (or equivalently, $\mathfrak{p} \not\subseteq \mathfrak{q} \in \operatorname{Spec}(A)$), then it is clear that $\operatorname{Hom}_A(A/\mathfrak{p}^n, E(A/\mathfrak{q})) = 0$ by Lemma 5.1 (1). Consequently,

$$\beta_{\mathfrak{p},n} = \operatorname{Hom}_A(A/\mathfrak{p}^n, \beta_{\mathfrak{p}}) : \operatorname{Hom}_A(A/\mathfrak{p}^n, E(A/\mathfrak{p})) \longrightarrow \operatorname{Hom}_A(A/\mathfrak{p}^n, T_2)$$

is surjective.

Now, for any A-module Y, we identify $\mathrm{Hom}_A(A/\mathfrak{p}^n, Y)$ with $\{y \in Y \mid \mathfrak{p}^n y = 0\}$. Define

$$S_n := \{z \in T_2 \mid \mathfrak{p}^n z = 0\}.$$

Then we know that $\mathrm{Hom}_A(A/\mathfrak{p}^n, E(A/\mathfrak{p})) = E_n(A/\mathfrak{p})$ (see Lemma 5.1(3) for notation) and $\beta_{\mathfrak{p},n}\colon E_n(A/\mathfrak{p}) \to S_n$ is the restriction of $\beta_\mathfrak{p}$ to $E_n(A/\mathfrak{p})$. Since $\mathfrak{p} \subseteq \mathfrak{m}$, we have $E_n(A/\mathfrak{m}) \subseteq S_n$. It follows from Lemma 5.1(3) that $T_2 = \bigcup_{n\in\mathbb{N}} E_n(A/\mathfrak{m}) = \bigcup_{n\in\mathbb{N}} S_n$ and $E(A/\mathfrak{p}) = \bigcup_{n\in\mathbb{N}} E_n(A/\mathfrak{p})$. Thus $\beta_\mathfrak{p}$ is surjective. ■

Next, we describe the endomorphism ring $B := \mathrm{End}_A(T)$ of the tilting A-module T. Let $B_1 := \mathrm{End}_A(T_1)$. Then $B_1 = \prod_{\mathfrak{p}\in\mathcal{S}} J_\mathfrak{p}$ and $\theta_{T_1}\colon A \to B_1$ is given by $(\eta_\mathfrak{p})_{\mathfrak{p}\in\mathcal{S}}\colon A \to \prod_{\mathfrak{p}\in\mathcal{S}} J_\mathfrak{p}$ by (1) and (4) in Lemma 5.1. Also, each B_1-module can be regarded as an A-module via θ_{T_1} . Let C be the cokernel of θ_{T_1} and $\pi\colon B_1 \to C$ the canonical surjective map. We identify T_0 and Q with $\mathrm{End}_A(T_0)$ and $\mathrm{End}_A(Q)$, respectively.

Lemma 5.4. (1) *For $\mathfrak{p} \in \mathcal{S}$, the three maps in the sequence*

$$T_0 \otimes_A J_\mathfrak{p} \xrightarrow{\lambda_0\otimes 1} Q \otimes_A J_\mathfrak{p} \xrightarrow{(\cdot\alpha_\mathfrak{p}\cdot)} \mathrm{Hom}_A(Q, E(A/\mathfrak{p})) \xrightarrow{(\lambda_0)_*} \mathrm{Hom}_A(T_0, E(A/\mathfrak{p}))$$

are isomorphisms of T_0-$J_\mathfrak{p}$-bimodules, and their composition coincides with the map $(\cdot f_{1,\mathfrak{p}}\cdot)$.

(2) *If A is complete and $\mathcal{S}$ contains a principal ideal $\mathfrak{p}$ generated by a regular element in A, then $(\cdot\beta_\mathfrak{p})\colon J_\mathfrak{p} \to \mathrm{Hom}_A(E(A/\mathfrak{p}), T_2)$ is an isomorphism of $J_\mathfrak{p}$-modules.*

(3) *If A is complete and $\mathcal{S}$ consists of finitely many principal ideals of A generated by regular elements of A, then*

$$_{T_0}\mathrm{Hom}_A(T_0, T_1)_{B_1} \simeq T_0 \otimes_A B_1 \simeq Q \otimes_A B_1,$$
$$_{B_1}\mathrm{Hom}_A(T_1, T_2) \simeq B_1 \quad \textit{and} \quad {}_{T_0}\mathrm{Hom}_A(T_0, T_2) \simeq T_0 \otimes_A C.$$

Proof. (1) For any $\mathfrak{q} \in \mathcal{H} \setminus \mathcal{S}$, since $E(A/\mathfrak{q}) \otimes_A A_\mathfrak{p} = 0$ by Lemma 5.1(5), we have

$$E(A/\mathfrak{q}) \otimes_A J_\mathfrak{p} \simeq E(A/\mathfrak{q}) \otimes_A (A_\mathfrak{p} \otimes_A J_\mathfrak{p}) \simeq (E(A/\mathfrak{q}) \otimes_A A_\mathfrak{p}) \otimes_A J_\mathfrak{p} = 0.$$

This gives rise to $E_1 \otimes_A J_\mathfrak{p} = 0$. As $\mathrm{Hom}_A(E(A/\mathfrak{q}), E(A/\mathfrak{p})) = 0$ by Lemma 5.1(1), $\mathrm{Hom}_A(E_1, E(A/\mathfrak{p})) = 0$. Since $J_\mathfrak{p}$ is flat and $_AE(A/\mathfrak{p})$ is injective, λ_0 induces the first and third isomorphisms in (1). Now, we localize the minimal injective coresolution of A at $\mathfrak{p}$ and obtain the minimal injective coresolution of $_{A_\mathfrak{p}}A_\mathfrak{p}$:

$$0 \longrightarrow A_\mathfrak{p} \xrightarrow{\lambda'} Q_\mathfrak{p} \xrightarrow{\alpha'_\mathfrak{p}} E(A/\mathfrak{p}) \longrightarrow 0$$

by Lemma 5.1(5), where $Q_\mathfrak{p} \simeq A_\mathfrak{p} \otimes_A Q$. Since λ' is a ring epimorphism and $Q_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module, the map $(\cdot\alpha'_\mathfrak{p}\cdot)\colon Q_\mathfrak{p} \otimes_{A_\mathfrak{p}} J_\mathfrak{p} \to \mathrm{Hom}_{A_\mathfrak{p}}(Q_\mathfrak{p}, E(A/\mathfrak{p}))$ is an isomorphism of $Q_\mathfrak{p}$-$J_\mathfrak{p}$-bimodules by Lemma 5.2. Moreover, since $J_\mathfrak{p}$ and $E(A/\mathfrak{p})$ are $A_\mathfrak{p}$-modules, we obtain canonical isomorphisms $Q \otimes_A J_\mathfrak{p} \simeq Q_\mathfrak{p} \otimes_{A_\mathfrak{p}} J_\mathfrak{p}$ and $\mathrm{Hom}_{A_\mathfrak{p}}(Q_\mathfrak{p}, E(A/\mathfrak{p})) \simeq \mathrm{Hom}_A(Q, E(A/\mathfrak{p})$ as Q-$J_\mathfrak{p}$-bimodules. So, up to isomorphism,

$$(\cdot\alpha'_\mathfrak{p}\cdot) = (\cdot\alpha_\mathfrak{p}\cdot) : \; Q \otimes_A J_\mathfrak{p} \longrightarrow \mathrm{Hom}_A(Q, E(A/\mathfrak{p})).$$

It follows that $(\cdot\alpha_{\mathfrak{p}}\cdot)$ is an isomorphism of Q-$J_{\mathfrak{p}}$-bimodules. A straightforward check shows that the composition of three maps in (1) equals $(\cdot f_{1,\mathfrak{p}}\cdot)$.

(2) We keep the notation introduced in the proof of Lemma 5.3(3). Let $\mathfrak{p} = Ax \in \mathcal{S}$, with x a regular element in A, and let $D := A/\mathfrak{p}$ and $I := E_1(A/\mathfrak{p})$. Then $x \neq 0$ and D is a local ring with $\mathrm{Spec}(D) = \{\mathfrak{p}/\mathfrak{p}, \mathfrak{m}/\mathfrak{p}\}$. Clearly, there exists a minimal projective resolution $0 \to A \xrightarrow{\cdot x} A \to {}_A D \to 0$. By applying $\mathrm{Hom}_A(-, A)$ to the resolution, we get $\mathrm{Ext}^1_A(D, A) \simeq D$, while by applying $\mathrm{Hom}_A(D, -)$ to the minimal injective coresolution of A, we see from the proof of Lemma 5.3(3) that there is an exact sequence of D-modules:

$$0 \longrightarrow \mathrm{Ext}^1_A(D, A) \longrightarrow I \xrightarrow{\beta_{\mathfrak{p},1}} S_1 \longrightarrow 0,$$
$$\text{where } I = \mathrm{Hom}_A(D, E(A/\mathfrak{p})),\ S_1 = \mathrm{Hom}_A(D, T_2).$$

Note that the inclusions $D \subseteq I$ and $A/\mathfrak{m} \subseteq S_1$ are injective envelopes of D and $A/\mathfrak{m}$ in D-Mod, respectively. Consequently, D is a 1-Gorenstein ring with an injective coresolution $0 \to D \to I \xrightarrow{\beta_{\mathfrak{p},1}} S_1 \to 0$. This coresolution is isomorphic to the canonical one $0 \to D \to F \xrightarrow{\varphi} F/D \to 0$ with F the fraction field of D. Since $D \to F$ is a ring epimorphism with ${}_D F$ flat, it follows from Lemma 5.2 that $(\cdot\varphi\cdot)\colon \mathrm{End}_D(F) \otimes_D \mathrm{End}_D(F/D) \to \mathrm{Hom}_D(F, F/D)$ is an isomorphism of $\mathrm{End}_D(F)$-$\mathrm{End}_D(F/D)$-bimodules. Thus

$$(\cdot\beta_{\mathfrak{p},1}\cdot) : \mathrm{End}_D(I) \otimes_D \mathrm{End}_D(S_1) \longrightarrow \mathrm{Hom}_D(I, S_1)$$

is an isomorphism of $\mathrm{End}_D(I_1)$-$\mathrm{End}_D(S_1)$-bimodules.

Next, we consider the functor

$$\Phi := \mathrm{Hom}_A({}_A D_D, -) :\ A\text{-Mod} \longrightarrow D\text{-Mod}.$$

Clearly, $\Phi(M) = \{m \in M \mid \mathfrak{p}m = 0\}$, the annihilator of $\mathfrak{p}$ in M, denoted by $\mathrm{Ann}_M(\mathfrak{p})$, or $\mathrm{Ann}_M(x)$. This is both a submodule of ${}_A M$ and a D-module. Since a homomorphism $f\colon {}_A M \to {}_A N$ restricts to a homomorphism from $\mathrm{Ann}_M(x)$ to $\mathrm{Ann}_N(x)$, $\Phi(f)$ is just the restriction of f to annihilators. Thus $\Phi : A\text{-Inj} \to D\text{-Mod}$ is full, where A-Inj denotes the category of all injective A-modules, and there is an exact sequence of A-modules

$$0 \longrightarrow \Phi(M) \longrightarrow M \xrightarrow{\cdot x} M \longrightarrow 0$$

for each $M \in A$-Inj. Note that $\Phi(f) = 0$ if and only if $f \in x\mathrm{Hom}_A(M, N)$. Hence

$$\mathrm{Hom}_D(\Phi(M), \Phi(N)) \simeq \mathrm{Hom}_A(M, N)/x\mathrm{Hom}_A(M, N) \simeq D \otimes_A \mathrm{Hom}_A(M, N)$$

for $M, N \in A$-Inj.

Recall that $I = \Phi(E(A/\mathfrak{p}))$ and $S_1 = \Phi(T_2)$, and that $A \simeq J_{\mathfrak{m}}$, since A is complete. Then there are ring isomorphisms

$$\mathrm{End}_D(I) \simeq D \otimes_A \mathrm{End}_A(E(A/\mathfrak{p})) = D \otimes_A J_{\mathfrak{p}},$$
$$\mathrm{End}_D(S_1) \simeq D \otimes_A \mathrm{End}_A(T_2) = D \otimes_A J_{\mathfrak{m}} \simeq D \otimes_A A \simeq D.$$

Moreover, $\mathrm{Hom}_D(I, S_1) \simeq D \otimes_A \mathrm{Hom}_A(E(A/\mathfrak{p}), T_2)$ as D-modules. So, up to isomorphism, the isomorphism $(\cdot\beta_{\mathfrak{p},1}\cdot)$ is identified with

$$1 \otimes (\cdot\beta_{\mathfrak{p}}) : D \otimes_A J_{\mathfrak{p}} \longrightarrow D \otimes_A \mathrm{Hom}_A(E(A/\mathfrak{p}), T_2).$$

Since $J_{\mathfrak{p}}$ is a local ring with the Jacobson radical of $J_{\mathfrak{p}}$ generated by x, the map $(\cdot\beta_{\mathfrak{p}})$ is an isomorphism if $\mathrm{Hom}_A(E(A/\mathfrak{p}), T_2) \simeq J_{\mathfrak{p}}$ as $J_{\mathfrak{p}}$-modules. The reason reads as follows.

If $\gamma\colon \mathrm{Hom}_A(E(A/\mathfrak{p}), T_2) \to J_{\mathfrak{p}}$ is an isomorphism of $J_{\mathfrak{p}}$-modules, then $(\cdot\beta_{\mathfrak{p}})\gamma$ is an endomorphism of $J_{\mathfrak{p}}$. Suppose $(\cdot\beta_{\mathfrak{p}})\gamma$ is not an automorphism of $J_{\mathfrak{p}}$. Then its image must be contained in the Jacobson radical $xJ_{\mathfrak{p}}$ of the local ring $J_{\mathfrak{p}}$, and thus $(1 \otimes (\cdot\beta_{\mathfrak{p}}))(1 \otimes \gamma) = 1 \otimes ((\cdot\beta_{\mathfrak{p}})\gamma) = 0$, which is not an automorphism of $D \otimes_A J_{\mathfrak{p}}$, a contradiction. This means that $(\cdot\beta_{\mathfrak{p}})\gamma$ must be an automorphism, and therefore $(\cdot\beta_{\mathfrak{p}})$ is an isomorphism.

Now, we show $\mathrm{Hom}_A(E(A/\mathfrak{p}), T_2) \simeq J_{\mathfrak{p}}$ as $J_{\mathfrak{p}}$-modules. Note that

$$\mathrm{Hom}_A(E(A/\mathfrak{p}), T_2) \simeq \mathrm{Hom}_A(A_{\mathfrak{p}} \otimes_A E(A/\mathfrak{p}), T_2) \simeq \mathrm{Hom}_A(E(A/\mathfrak{p}), \mathrm{Hom}_A(A_{\mathfrak{p}}, T_2))$$

and that $\mathrm{Hom}_A(A_{\mathfrak{p}}, T_2)$ is an injective $A_{\mathfrak{p}}$-module. Clearly,

$$\mathrm{Spec}(A_{\mathfrak{p}}) = \{\mathfrak{q}A_{\mathfrak{p}} \mid \mathfrak{p} \supseteq \mathfrak{q} \in \mathrm{Spec}(A)\}.$$

By Lemma 5.1 (5)–(6), we have $\mathrm{Hom}_A(A_{\mathfrak{p}}, T_2) \simeq Z \oplus E(A/\mathfrak{p})^{(U)}$, where Z is a direct sum of copies of $E(A/\mathfrak{q})$ with $\mathfrak{q} \subsetneqq \mathfrak{p}$ and U is a set with $|U|$ equal to the $k(\mathfrak{p})$-dimension of $\mathrm{Hom}_{A_{\mathfrak{p}}}(k(\mathfrak{p}), \mathrm{Hom}_A(A_{\mathfrak{p}}, T_2))$. It follows from $\mathrm{Hom}_A(E(A/\mathfrak{p}), E(A/\mathfrak{q})) = 0$ that

$$\mathrm{Hom}_A(E(A/\mathfrak{p}), T_2) \simeq \mathrm{Hom}_A(E(A/\mathfrak{p}), E(A/\mathfrak{p})^{(U)}).$$

So we have to show $|U| = 1$, that is, $\mathrm{Hom}_{A_{\mathfrak{p}}}(k(\mathfrak{p}), \mathrm{Hom}_A(A_{\mathfrak{p}}, T_2)) \simeq k(\mathfrak{p})$.

Clearly, $k(\mathfrak{p}) = A_{\mathfrak{p}}/xA_{\mathfrak{p}}$, and $\mathrm{Hom}_{A_{\mathfrak{p}}}(k(\mathfrak{p}), E(A/\mathfrak{p})) \simeq k(\mathfrak{p})$, by Lemma 5.1 (3). Applying $\mathrm{Hom}_{A_{\mathfrak{p}}}(-, E(A/\mathfrak{p}))$ to the exact sequence $0 \to A_{\mathfrak{p}} \xrightarrow{\cdot x} A_{\mathfrak{p}} \to k(\mathfrak{p}) \to 0$, we obtain the minimal injective coresolution $0 \to k(\mathfrak{p}) \to E(A/\mathfrak{p}) \xrightarrow{\cdot x} E(A/\mathfrak{p}) \to 0$ of $k(\mathfrak{p})$ as an A-module. This yields an exact sequence of $J_{\mathfrak{p}}$-modules:

$$0 \longrightarrow \mathrm{Hom}_A(E(A/\mathfrak{p}), T_2) \xrightarrow{\cdot x} \mathrm{Hom}_A(E(A/\mathfrak{p}), T_2) \longrightarrow \mathrm{Hom}_A(k(\mathfrak{p}), T_2) \longrightarrow 0,$$

and therefore,

$$\begin{aligned}\mathrm{Hom}_A(k(\mathfrak{p}), T_2) &\simeq \mathrm{Hom}_A(E(A/\mathfrak{p}), T_2)/x\mathrm{Hom}_A(E(A/\mathfrak{p}), T_2)\\ &\simeq D \otimes_A \mathrm{Hom}_A(E(A/\mathfrak{p}), T_2).\end{aligned}$$

Since $k(\mathfrak{p}) \simeq J_{\mathfrak{p}}/xJ_{\mathfrak{p}} \simeq D \otimes_A J_{\mathfrak{p}}$ and since $D \otimes_A J_{\mathfrak{p}} \simeq D \otimes_A \mathrm{Hom}_A(E(A/\mathfrak{p}), T_2)$ by the isomorphism $1 \otimes (\cdot\beta_{\mathfrak{p}})$, it follows that

$$\mathrm{Hom}_{A_{\mathfrak{p}}}(k(\mathfrak{p}), \mathrm{Hom}_A(A_{\mathfrak{p}}, T_2)) \simeq \mathrm{Hom}_A(k(\mathfrak{p}), T_2) \simeq k(\mathfrak{p}).$$

Thus $\mathrm{Hom}_A(E(A/\mathfrak{p}), T_2) \simeq J_{\mathfrak{p}}$ as $J_{\mathfrak{p}}$-modules.

(3) Since $\mathcal{S}$ is a finite set, B_1 equals the direct sum $\bigoplus_{\mathfrak{p}\in\mathcal{S}} J_{\mathfrak{p}}$ of rings $J_{\mathfrak{p}}$. Then the maps

$$(\cdot f_1\cdot) : T_0 \otimes_A B_1 \longrightarrow \mathrm{Hom}_A(T_0, T_1), \quad (\cdot f_2) : B_1 \longrightarrow \mathrm{Hom}_A(T_1, T_2),$$
$$\text{and} \quad \lambda_0 \otimes 1 : T_0 \otimes_A B_1 \longrightarrow Q \otimes_A B_1$$

are isomorphisms by (1) and (2). It remains to show that $\operatorname{Hom}_A(T_0, T_2) \simeq T_0 \otimes_A C$ as T_0-modules.

Let $K := \operatorname{Im}(f_1)$ and let $g_1 \colon T_0 \to K$ be the canonical surjection. Since $\mathcal{S}$ consists of principal ideals of A, which are generated by regular elements in A, the map f_2 is surjective by Lemma 5.3(3). Thus the exact sequence $0 \to A \to T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} T_2 \to 0$ gives rise to a complex of T_0-modules:

$$0 \longrightarrow \operatorname{Hom}_A(T_0, A) \longrightarrow \operatorname{Hom}_A(T_0, T_0) \xrightarrow{f_1^*} \operatorname{Hom}_A(T_0, T_1) \xrightarrow{f_2^*} \operatorname{Hom}_A(T_0, T_2) \longrightarrow 0.$$

We claim that this complex is exact with $\operatorname{Hom}_A(T_0, A) = 0$; or equivalently, that $g_1^* \colon \operatorname{Hom}_A(T_0, T_0) \to \operatorname{Hom}_A(T_0, K)$ is an isomorphism and f_2^* is surjective.

Indeed, we consider the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f_0"] \arrow[d, "\theta_K"'] & T_0 \arrow[r, "g_1"] \arrow[d, "(\cdot g_1)"'] & K \arrow[r] \arrow[d, "\simeq"'] & 0 \\
0 \arrow[r] & \operatorname{End}_A(K) \arrow[r, "(g_1 \cdot)"] & \operatorname{Hom}_A(T_0, K) \arrow[r, "(f_0 \cdot)"] & \operatorname{Hom}_A(A, K) \arrow[r] & 0.
\end{tikzcd}$$

Recall that T_0 is identified with $\operatorname{End}_A(T_0)$. Then $(\cdot\, g_1) = g_1^*$, and therefore g_1^* is an isomorphism if and only if so is θ_K. Since T_1 is injective and $\operatorname{Hom}_A(T_2, T_1) = 0$ by Lemma 5.3(2), it follows that for any $h \in \operatorname{End}_A(K)$, there exists a unique pair (h_1, h_2) of homomorphisms such that the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "h"'] & T_1 \arrow[r, "f_2"] \arrow[d, "h_1"'] & T_2 \arrow[r] \arrow[d, "h_2"'] & 0 \\
0 \arrow[r] & K \arrow[r] & T_1 \arrow[r, "f_2"] & T_2 \arrow[r] & 0,
\end{tikzcd}$$

is commutative. This defines a ring homomorphism $\rho \colon \operatorname{End}_A(K) \to \operatorname{End}_A(T_2)$ sending h to h_2. Moreover, if h is the multiplication map by an element of A, then so are h_1 and h_2. This implies $\theta_K \rho = \theta_{T_2}$. Since A is complete, the map θ_{T_2} is an isomorphism by Lemma 5.1(4). Thus ρ is surjective. On the other hand, it follows from (2) and from Lemma 5.1(1) that $(\cdot f_2) := \operatorname{Hom}_A(T_1, f_2)$ is an isomorphism. This implies that $\operatorname{Hom}_A(T_1, K) = 0$ and ρ is injective. Thus ρ is an isomorphism, and so are both θ_K and $g_1^* = (\cdot g_1)$.

To see the surjection of f_2^*, let $Q^\bullet$ be the complex $0 \to T_0 \to T_1 \to T_2 \to 0$ of which T_0 is in degree 0. Since ${}_AT$ is a good tilting module by Lemma 5.3(1), $H(A) \simeq H(Q^\bullet) \simeq \operatorname{Hom}_A(T, Q^\bullet)$ in $\mathcal{D}(B)$ by Lemma 4.2(2). As H is fully faithful and $Q^\bullet \in \mathcal{C}^b(\operatorname{add}(T))$, we have

$$\operatorname{Hom}_{\mathcal{D}(A)}(A, A[i]) \simeq \operatorname{Hom}_{\mathcal{D}(B)}(\operatorname{Hom}_A(T, Q^\bullet), \operatorname{Hom}_A(T, Q^\bullet)[i]) \simeq \operatorname{Hom}_{\mathcal{K}(A)}(Q^\bullet, Q^\bullet[i])$$

for all $i \in \mathbb{Z}$. This shows $\operatorname{Hom}_{\mathcal{K}(A)}(Q^\bullet, Q^\bullet[2]) = 0$, and therefore

$$\operatorname{Hom}_A(T_0, T_2) = f_1 \operatorname{Hom}_A(T_1, T_2) + \operatorname{Hom}_A(T_0, T_1)\, f_2.$$

Now, it follows from the surjective map $(\cdot f_2)\colon B_1 = \mathrm{End}_A(T_1) \to \mathrm{Hom}_A(T_1, T_2)$ that f_2^* is surjective. This shows the claim.

Thus the sequence $0 \to \mathrm{Hom}_A(T_0, T_0) \to \mathrm{Hom}_A(T_0, T_1) \to \mathrm{Hom}_A(T_0, T_2) \to 0$ is exact. Now, we identify T_0 with $\mathrm{End}_A(T_0)$, and $\mathrm{Hom}_A(T_0, T_1)$ with $T_0 \otimes_A B_1$. Then f_1^* coincides with the map $T_0 \otimes \theta_{T_1}\colon T_0 \simeq T_0 \otimes_A A \to T_0 \otimes_A B_1$. It follows that $\mathrm{Hom}_A(T_0, T_2) \simeq \mathrm{Coker}(T_0 \otimes \theta_{T_1}) = T_0 \otimes_A C$. ■

As a consequence of Lemma 5.4(3) and its proof, we obtain the following result.

Corollary 5.5. *Suppose that A is complete and that $\mathcal{S}$ consists of finitely many principal ideals of A which are generated by regular elements of A. Then*

$$B \simeq \begin{pmatrix} T_0 & T_0 \otimes_A B_1 & T_0 \otimes_A C \\ 0 & B_1 & B_1 \\ 0 & 0 & A \end{pmatrix},$$

where the multiplication $(T_0 \otimes_A B_1) \times B_1 \to T_0 \otimes_A C$ is induced by the composition of the maps

$$\sigma : (T_0 \otimes_A B_1) \otimes_{B_1} B_1 \xrightarrow{\simeq} T_0 \otimes_A B_1 \xrightarrow{1 \otimes \pi} T_0 \otimes_A C.$$

In order to characterise the categories B-Mod and $\mathcal{E}$ associated with the good tilting A-module T, we introduce a new category $\mathcal{C}(A, T)$ as follows.

Definition 5.6. Objects of $\mathcal{C}(A, T)$ consist of three-term complexes

$$X^\bullet : \quad 0 \longrightarrow X^{-2} \xrightarrow{d_X^{-2}} X^{-1} \xrightarrow{d_X^{-1}} X^0 \longrightarrow 0$$

in $\mathcal{C}(A)$, with $X^{-1} \in B_1$-Mod and $X^0 \in T_0$-Mod, where B_1-modules and T_0-modules are regarded as A-modules via the ring homomorphisms θ_{T_1} and f_0, respectively.

Given two objects $X^\bullet$ and $Y^\bullet$ of $\mathcal{C}(A, T)$, a morphism in $\mathrm{Hom}_{\mathcal{C}(A,T)}(X^\bullet, Y^\bullet)$ is defined to be a chain map $f^\bullet := (f^{-2}, f^{-1}, f^0)\colon X^\bullet \to Y^\bullet$ in $\mathcal{C}(A)$ such that $f^{-1} \in \mathrm{Hom}_{B_1}(X^{-1}, Y^{-1})$ and $f^0 \in \mathrm{Hom}_{T_0}(X^0, Y^0)$.

Note that $\mathcal{C}(A, T)$ is an abelian category with the abelian structure induced from the ones of A-Mod, B_1-Mod and T_0-Mod in degrees $-2, -1$ and 0, respectively. Let $\mathcal{C}_{\mathrm{ac}}(A, T)$ be the full subcategory of $\mathcal{C}(A, T)$ consisting of all exact complexes. Then $\mathcal{C}_{\mathrm{ac}}(A, T)$ is a fully exact subcategory of $\mathcal{C}(A, T)$. For the convenience of the reader, we restate Theorem 1.5 here.

Theorem 5.7. *Suppose that A is complete and that $\mathcal{S}$ consists of finitely many principal ideals of A that are generated by regular elements of A. Let $\mathcal{E}$ be the 2-symmetric subcategory of B-*Mod *associated with the tilting module ${}_AT$. Then there is an equivalence B-*Mod *$\to \mathcal{C}(A, T)$ of abelian categories which restricts to an equivalence $\mathcal{E} \to \mathcal{C}_{\mathrm{ac}}(A, T)$ of exact categories.*

Proof. We first identify the ring $\mathrm{End}({}_AT)$ with the 3×3 upper triangular matrix ring $B := (B_{i,j})_{0 \le i,j \le 2}$ in Corollary 5.5. In particular, the maps f_1 and f_2 are identified with $1 \otimes 1 \in B_{0,1} = T_0 \otimes_A B_1$ and $1 \in B_{1,2} = B_1$, respectively. For $i \in \{0, 1, 2\}$, let $e_i \in B$ be the matrix with 1 at the (i, i)-entry, and 0 at all other entries.

For $M \in B$-Mod, let $M^{-i} := e_i M$ for $0 \le i \le 2$. Then $M^{-2} \in A$-Mod, $M^{-1} \in B_1$-Mod and $M^0 \in T^0$-Mod. Clearly, M^{-1} and M^0 can be regraded as A-modules via θ_{T_1} and f_0, respectively. Moreover, the B-module structure of M is determined by the triple $(\varphi_M, \phi_M, \psi_M)$ (or equivalently, by $(d_M^{-2}, d_M^{-1}, h_M)$)

$$\begin{aligned} \varphi_M &\in \mathrm{Hom}_{B_1}(B_1 \otimes_A M^{-2}, M^{-1}) \simeq \mathrm{Hom}_A(M^{-2}, M^{-1}) \ni d_M^{-2}, \\ \phi_M &\in \mathrm{Hom}_{T_0}\big((T_0 \otimes_A B_1) \otimes_{B_1} M^{-1}, M^0\big) \simeq \mathrm{Hom}_A(M^{-1}, M^0\big) \ni d_M^{-1}, \\ \psi_M &\in \mathrm{Hom}_{T_0}\big((T_0 \otimes_A C) \otimes_A M^{-2}, M^0\big) \simeq \mathrm{Hom}_A(C \otimes_A M^{-2}, M^0) \ni h_M, \end{aligned}$$

satisfying

$$(1 \otimes \varphi_M)\phi_M = (\sigma \otimes 1)\psi_M : (T_0 \otimes_A B_1) \otimes_{B_1} (B_1 \otimes_A M^{-2}) \longrightarrow M^0,$$

which is equivalent to the equality

$$(\pi \otimes 1)h_M = (1 \otimes d_M^{-2})\tau_{M^{-1}} d_M^{-1} : B_1 \otimes_A M^{-2} \longrightarrow M^0,$$

where $\tau_{M^{-1}}\colon B_1 \otimes_A M^{-1} \to M^{-1}$ is the multiplication map. Since the sequence

$$A \otimes_A M^{-2} \xrightarrow{\theta_{T_1} \otimes 1} B_1 \otimes_A M^{-2} \xrightarrow{\pi \otimes 1} C \otimes_A M^{-2} \longrightarrow 0$$

is exact, the existence and uniqueness of h_M are equivalent to the equality $d_M^{-2} d_M^{-1} = 0$. Thus M is completely determined by the complex

$$M^\bullet : 0 \longrightarrow M^{-2} \xrightarrow{d_M^{-2}} M^{-1} \xrightarrow{d_M^{-1}} M^0 \longrightarrow 0$$

in $\mathcal{C}(A)$ with $M^{-1} \in B_1$-Mod and $M^0 \in T^0$-Mod. Similarly, for B-modules M and N, a homomorphism $M \to N$ in B-Mod is determined by a chain map $f^\bullet\colon M^\bullet \to N^\bullet$ in $\mathcal{C}(A)$ such that f^{-1} and f^0 are homomorphisms of B_1-modules and T_0-modules, respectively. This implies that B-Mod and $\mathcal{C}(A, T)$ are equivalent as abelian categories.

Note that $M \in \mathcal{E}$ if and only if $T \otimes_B^{\mathbb{L}} M \simeq 0$ if and only if the complex is exact:

$$0 \to \mathrm{Hom}_A(T_2, T) \otimes_B M \xrightarrow{(f_2)_* \otimes 1} \mathrm{Hom}_A(T_1, T) \otimes_B M \xrightarrow{(f_1)_* \otimes 1} \mathrm{Hom}_A(T_0, T) \otimes_B M \to 0$$

which is obtained by applying $- \otimes_B M$ to the deleted projective resolution of T_B. Under the identification of $\mathrm{Hom}_A(T_i, T)$ with $e_i B$ for $0 \le i \le 2$, we see that $M \in \mathcal{E}$ if and only if $M^\bullet$ is exact. Thus $\mathcal{E}$ and $\mathcal{C}_{\mathrm{ac}}(A, T)$ are equivalent as exact categories. ■

Finally, we mention a few questions relevant to the results in this paper. Let A be a unitary ring.

(1) Which n-symmetric subcategories of A-Mod can be realised by n-weak tilting right A-modules? that is, under which conditions on an n-symmetric subcategory $\mathcal{B}$ of A-Mod is there an n-weak tilting right A-module M such that $\mathcal{B}$ is equivalent to

$$\{X \in A\text{-Mod} \mid \mathrm{Tor}_i^A(M, X) = 0, \forall\, i \ge 0\}$$

as exact categories? For the definition of weak tilting module over rings, we refer the reader to Definition 4.1 in p. 534 of [16].

(2) Let M be a right A-module and $\mathcal{B} := \{X \in A\text{-Mod} \mid \mathrm{Tor}_i^A(M, X) = 0, \forall\, i \geq 0\}$. Under which necessary and sufficient conditions on M_A does there exist a triangle equivalence between the triangulated categories $\mathcal{D}(\mathcal{B})$ and $\mathrm{Ker}(M \otimes_A^{\mathbb{L}} -)$?

Theorem 1.2 gives a sufficient condition to (2). Also, if $e^2 = e \in A$ such that AeA is a homological ideal in A, that is, the canonical surjection $A \to A/AeA$ is a homological ring epimorphism, then $M_A := eA$ gives rise to a desired equivalence in (2).

(3) Let $\mathcal{B}$ be an n-symmetric subcategory of A-Mod with $n \geq 2$. Does $\mathcal{D}(\mathcal{B})$ always have compact objects?

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