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Restricted families of projections onto planes: The general case of nonvanishing geodesic curvature

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Abstract. It is shown that if $\gamma\colon [a,b] \to S^2$ is C^3 with $\det(\gamma, \gamma', \gamma'') \neq 0$, and if $A \subseteq \mathbb{R}^3$ is a Borel set, then $\dim \pi_\theta(A) \geq \min\{2, \dim A, \dim A/2 + 3/4\}$ for a.e. $\theta \in [a,b]$, where π_θ denotes projection onto the orthogonal complement of $\gamma(\theta)$ and "dim" refers to Hausdorff dimension. This partially resolves a conjecture of Fässler and Orponen in the range $1 < \dim A \leq 3/2$, which was previously known only for non-great circles. For $3/2 < \dim A < 5/2$, this improves the known lower bound for this problem.

1. Introduction

Let S^2 be the unit sphere in $\mathbb{R}^3$. Given a curve $\gamma\colon [a,b] \to S^2$ and $\theta \in [a,b]$, let π_θ be the orthogonal projection onto $\gamma(\theta)^\perp \subseteq \mathbb{R}^3$, given by

$$\pi_\theta(x) = x - \langle x, \gamma(\theta)\rangle \gamma(\theta), \quad x \in \mathbb{R}^3.$$

Let $\dim A$ denote the Hausdorff dimension of a set $A \subseteq \mathbb{R}^3$.

Theorem 1.1. *Let $\gamma\colon [a,b] \to S^2$ be C^3 with $\det(\gamma, \gamma', \gamma'')$ nonvanishing. If $A \subseteq \mathbb{R}^3$ is an analytic set, then*

$$\dim \pi_\theta(A) \geq \min\Big\{2, \dim A, \frac{\dim A}{2} + \frac{3}{4}\Big\},$$

for a.e. $\theta \in [a,b]$.

This partially resolves Conjecture 1.6 from [3] in the range $\dim A \leq 3/2$, for projections onto planes. In the range $1 < \dim A \leq 3/2$, this was previously known in the special case of non-great circles; due to Orponen and Venieri [15]. In the range $3/2 < \dim A < 5/2$, Theorem 1.1 improves and generalises the previous best known lower bound from [5], which was also specific to non-great circles.

Denote the s-dimensional Hausdorff measure in Euclidean space by $\mathcal{H}^s$. In $\mathbb{R}^2$, the classical Marstrand projection theorem [11] states that if $A \subseteq \mathbb{R}^2$ is a Borel set and P_e

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denotes orthogonal projection onto the 1-dimensional subspace through $e \in S^1$, then for $\dim A \le 1$,

$$\dim P_e(A) = \dim A, \quad \mathcal{H}^1\text{-a.e. } e \in S^1,$$

and for $\dim A > 1$,

$$\mathcal{H}^1(P_e(A)) > 0, \quad \mathcal{H}^1\text{-a.e. } e \in S^1.$$

This was generalised to higher dimensions by Mattila. In $\mathbb{R}^3$, there are two versions of the Martrand–Mattila projection theorem; one for lines and one for planes. The version for lines is analogous to the above with S^1 replaced by S^2. For planes, it states that if $A \subseteq \mathbb{R}^3$ is a Borel set, then if $\dim A \le 2$,

$$\dim \pi_v(A) = \dim A, \quad \mathcal{H}^2\text{-a.e. } v \in S^2, \tag{1.1}$$

and if $\dim A > 2$, then

$$\mathcal{H}^2(\pi_v(A)) > 0, \quad \mathcal{H}^2\text{-a.e. } v \in S^2,$$

where π_v denotes projection onto $v^\perp$. Restricted projection families can be formed by constraining v to move along a one-dimensional curve $\gamma\colon [a,b] \to S^2$, and the restricted projection problem asks whether (1.1) still holds with a natural 1-dimensional measure replacing the surface measure $\mathcal{H}^2$ on S^2.

Without the assumption that $\det(\gamma, \gamma', \gamma'')$ is nonvanishing, the equality

$$\dim \pi_\theta(A) = \dim A, \quad \text{a.e. } \theta \in [a,b], \tag{1.2}$$

can only hold (in general) for $\dim A \le 1$, and was proved in this range by Järvenpää, Järvenpää, Ledrappier, and Leikas [6] using the energy method of Kaufman [8]. Considering the counterexample where γ is a great circle contained in a plane P and $A \subseteq P$ shows that some extra assumption on γ is necessary for (1.2) to hold in general for $1 < \dim A \le 2$. The following conjecture is due to Fässler and Orponen; ρ_θ denotes projection onto the 1-dimensional subspace through $\gamma(\theta)$.

Conjecture 1.2 ([3], Conjecture 1.6). *Let $\gamma\colon [a,b] \to S^2$ be a C^3 curve with $\det(\gamma, \gamma', \gamma'')$ nonvanishing, and let $A \subseteq \mathbb{R}^3$ be an analytic set. Then*

$$\dim \rho_\theta(A) = \min\{\dim A, 1\}, \quad \text{a.e. } \theta \in [a,b],$$

and

$$\dim \pi_\theta(A) = \min\{\dim A, 2\}, \quad \text{a.e. } \theta \in [a,b].$$

For projections onto planes, progress was made by Fässler–Orponen [3], Oberlin–Oberlin [13], Orponen [14], Orponen–Venieri [15], and by the author in [5]. The condition $\dim A > 5/2$ remains the best known sufficient condition that ensures $\mathcal{H}^2(\pi_\theta(A)) > 0$ for a.e. $\theta \in [a,b]$; this is due to Oberlin–Oberlin [13]. A comparison with some of the previous bounds is shown in Figure 1.

The improvement over [5] in Theorem 1.1 stems from Definition 4.1, the application of which is the main novelty of this work. Definition 4.1 reformulates the projection problem as an averaged inequality over collections of "bad" tubes. The proof of Theorem 1.1 follows a similar approach to [5] (which used ideas from [4, 9, 13, 15]), and proceeds by

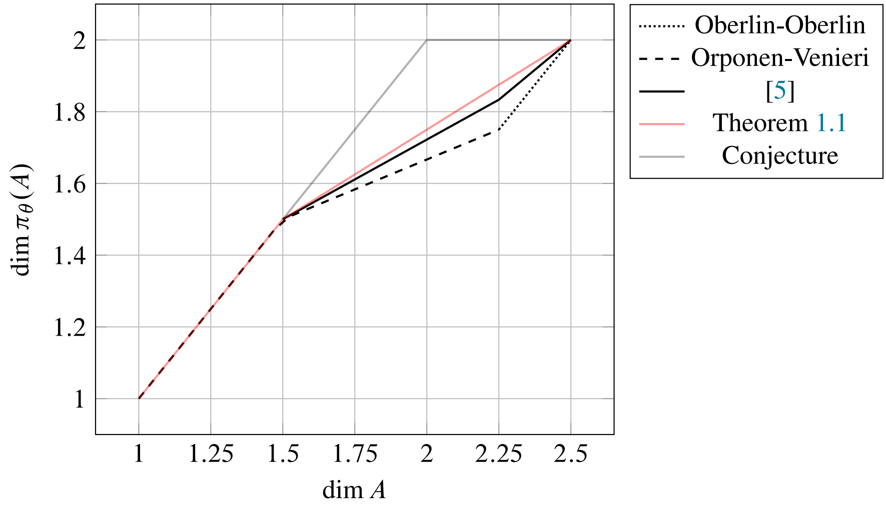

Figure 1. The current and some of the previous a.e. lower bounds for $\dim \pi_\theta(A)$, in the range $\dim A \in (1, 5/2)$. The Orponen–Venieri theorem and the bound from [5] were for the special case of non-great circles.

splitting an integral into a "good" and "bad" part. A self-similarity in the proof allows the "bad" part to be bounded by re-using Definition 4.1, which circumvents the appeal to the Orponen–Venieri lemma (Lemma 2.3 in [15]). Since the use of this lemma was the only step in [5] specific to non-great circles, this allows the proof to be generalised to curves in S^2 of nonvanishing geodesic curvature. The bound on the "good" part uses the decoupling theorem for the cone in $\mathbb{R}^3$, from [1].

Section 4 contains the proof of Theorem 1.1. Section 2 is a proof of the refined Strichartz inequality, needed in Section 4. Section 3 contains a derivation of the wave packet decomposition needed in Section 4, and is independent of Section 2. Section 5 is a discussion of some related problems.

2. Refined Strichartz inequality

Definition 2.1. Given a C^3 curve $\gamma : [a, b] \to S^2$ with $\det(\gamma, \gamma', \gamma'')$ nonvanishing, for each $R \geq 1$ let

$$\Theta_R = \left\{ \frac{j}{10^5 B^{10} R^{1/2}} : j \in \mathbb{Z} \right\} \cap [a, b],$$

where $B \geq 1$ is the smallest constant such that

$$|\det(\gamma, \gamma', \gamma'')| \geq B^{-1} \quad \text{and} \quad \|\gamma\|_{C^3[a,b]} \leq B.$$

For each $\theta \in \Theta_R$, let

$$\tau(\theta) = \left\{ x_1 \gamma(\theta) + x_2 \frac{\gamma'(\theta)}{|\gamma'(\theta)|} + x_3 \frac{(\gamma \times \gamma')(\theta)}{|(\gamma \times \gamma')(\theta)|} : 1 \leq x_1 \leq 2, |x_2| \leq R^{-1/2}, |x_3| \leq R^{-1} \right\}.$$

Let

$$\mathcal{P}_{R^{-1}}(\Gamma(\gamma)) = \{\tau(\theta) : \theta \in \Theta_R\}.$$

Given $\delta > 0$ and $\tau \in \mathcal{P}_{R^{-1}}(\Gamma(\gamma))$, let

$$T_{\tau,0} = \Big\{x_1 \frac{(\gamma \times \gamma')(\theta_\tau)}{|(\gamma \times \gamma')(\theta_\tau)|} + x_2 \frac{\gamma'(\theta_\tau)}{|\gamma'(\theta_\tau)|} + x_3\, \gamma(\theta_\tau) : \\ |x_1| \leq R^{1+\delta},\ |x_2| \leq R^{1/2+\delta},\ |x_3| \leq R^{1/2+\delta}\Big\}.$$

Theorem 2.2. *Let $\gamma\colon [a,b] \to S^2$ be a C^3 curve with $\det(\gamma, \gamma', \gamma'')$ nonvanishing. Let $B \geq 1$ be such that*

$$|\det(\gamma, \gamma', \gamma'')| \geq B^{-1}, \tag{2.1}$$

and

$$\|\gamma\|_{C^3[a,b]} \leq B. \tag{2.2}$$

Then for any $A \geq 1$ and $\epsilon > 0$, there exists $\delta_0 > 0$ such that the following holds for all $0 < \delta < \delta_0$. Let $R \geq 1$ and suppose that

$$f = \sum_{T \in \mathbb{W}} f_T, \quad \textit{where} \quad \mathbb{W} \subseteq \bigcup_{\tau \in \mathcal{P}_{R^{-1}}(\Gamma(\gamma))} \mathbb{T}_\tau,$$

and each $\mathbb{T}_\tau$ is an A-overlapping set of translates of $T_{\tau,0}$ intersecting $B(0,R)$. Assume that for all $T \in \mathbb{W}$,

$$\|f_T\|_{L^\infty(B(0,R)\setminus T)} + \sup_{q \in [1,2]} \|\widehat{f_T}\|_{L^q(\mathbb{R}^3 \setminus \tau(T))} \leq A\, R^{-10000} \|f_T\|_2, \tag{2.3}$$

with $\|f_T\|_2$ constant over $T \in \mathbb{W}$ up to a factor of 2. Let Y be a disjoint union of $R^{1/2}$-balls in $B(0,R)$, each of which intersects at most M sets $2T$ with $T \in \mathbb{W}$. Then for $2 \leq p \leq 6$,

$$\|f\|_{L^p(Y)} \leq C_{\epsilon,\delta,A}\, B^{10^{10}/\epsilon} R^\epsilon \left(\frac{M R^{-3/2}}{|\mathbb{W}|}\right)^{1/2-1/p} \left(\sum_{T \in \mathbb{W}} \|f_T\|_2^2\right)^{1/2}.$$

Proof. Assume that $[a,b] = [-1,1]$. Fix $A \geq 1$, $\epsilon \in (0, 1/2)$, $\delta_0 = \epsilon^{100}$, $\delta \in (0, \delta_0)$,

$$R \geq \min\left\{B^{10^3/\epsilon}, 2^{10^5/\epsilon}\right\},$$

and assume inductively that the theorem holds with $[a,b] = [-1,1]$ for all $\widetilde{R} \leq R^{3/4}$, for all curves γ satisfying (2.1) and (2.2), and for all $B \geq 1$.

For each $\tau \in \mathcal{P}_{R^{-1}}(\Gamma(\gamma))$, let $\kappa = \kappa(\tau) \in \mathcal{P}_{R^{-1/2}}(\Gamma(\gamma))$ be the element of $\mathcal{P}_{R^{-1/2}}(\Gamma(\gamma))$ which minimises $|\theta_\tau - \theta_\kappa|$. For each κ, let

$$\square_{\kappa,0} = \Big\{x_1 \frac{(\gamma \times \gamma')(\theta_\kappa)}{|(\gamma \times \gamma')(\theta_\kappa)|} + x_2 \frac{\gamma'(\theta_\kappa)}{|\gamma'(\theta_\kappa)|} + x_3\, \gamma(\theta_\kappa) : \\ |x_1| \leq R^{1+\delta},\ |x_2| \leq R^{3/4+\delta},\ |x_3| \leq R^{1/2+\delta}\Big\},$$

and

$$\mathbb{P}_\kappa = \left\{\Box = a\gamma(\theta_\kappa) + b\,\frac{\gamma'(\theta_\kappa)}{|\gamma'(\theta_\kappa)|} + \Box_{\kappa,0} : a \in \tfrac{1}{10} R^{1/2+\delta}\,\mathbb{Z},\ b \in \tfrac{1}{10} R^{3/4+\delta}\,\mathbb{Z}\right\}.$$

Let $\mathbb{P} = \bigcup_{\kappa \in \mathcal{P}_{R^{-1/2}}(\Gamma(\gamma))} \mathbb{P}_\kappa$. Given any τ and corresponding $\kappa = \kappa(\tau)$,

$$|\langle (\gamma \times \gamma')(\theta_\tau), \gamma'(\theta_\kappa)\rangle| \leq B^{-7} R^{-1/4}, \tag{2.4}$$

and

$$|\langle (\gamma \times \gamma')(\theta_\tau), \gamma(\theta_\kappa)\rangle| \leq B^{-7} R^{-1/2}. \tag{2.5}$$

It follows that for each $T \in \mathbb{T}_\tau$, there are ~ 1 sets $\Box \in \mathbb{P}_{\kappa(\tau)}$ with $T \cap 10\Box \neq \emptyset$, and moreover $T \subseteq 100\Box$ whenever $T \cap 10\Box \neq \emptyset$. For each such T, let $\Box = \Box(T) \in \mathbb{P}_\kappa$ be some choice such that $T \cap 10\Box \neq \emptyset$. For each τ, let

$$S_{\tau,0} = \Big\{x_1 \frac{(\gamma \times \gamma')(\theta_\tau)}{|(\gamma \times \gamma')(\theta_\tau)|} + x_2 \frac{\gamma'(\theta_\tau)}{|\gamma'(\theta_\tau)|} + x_3\,\gamma(\theta_\tau) : \\ |x_1| \leq R^{1+\delta/2},\ |x_2| \leq R^{1/2+\delta/2},\ |x_3| \leq R^{1/4+\delta/2}\Big\},$$

and

$$\mathbb{S}_\tau = \left\{S = a\gamma(\theta_\tau) + b\,\frac{\gamma'(\theta_\tau)}{|\gamma'(\theta_\tau)|} + S_{\tau,0} : a \in \tfrac{1}{10} R^{1/4+\delta/2}\mathbb{Z},\ b \in \tfrac{1}{10} R^{1/2+\delta/2}\mathbb{Z}\right\}.$$

For each $T \in \mathbb{W}$ and $S \in \mathbb{S}_{\tau(T)}$, let

$$f_S = f_{S,T} := \eta_S f_T,$$

where $\{\eta_S\}_{S \in \mathbb{S}_\tau}$ is a smooth partition of unity such that

$$\sup_{q \in [1,\infty]} \|\eta_S\|_{L^q(100\Box \setminus 0.99S)} \lesssim R^{-10^6},$$

where the implicit constant is absolute, with $\widehat{\eta_S}$ supported in

$$\Big\{x_1\,\gamma(\theta_\tau) + x_2 \frac{\gamma'(\theta_\tau)}{|\gamma'(\theta_\tau)|} + x_3 \frac{(\gamma \times \gamma')(\theta_\tau)}{|(\gamma \times \gamma')(\theta_\tau)|} : \\ |x_1| \leq 10^{-3},\ |x_2| \leq 10^{-3} R^{-1/2},\ |x_3| \leq 10^{-3} R^{-1}\Big\}.$$

This partition can be constructed using the Poisson summation formula. By dyadic pigeonholing and by (2.3), there are dyadic numbers μ and ν such that

$$\|f\|_{L^p(Y)} \lesssim (\log R)^2 \Big\| \sum_{\Box \in \mathbb{P}} \sum_{(S,T) \in \mathbb{W}_\Box} f_{S,T} \Big\|_{L^p(Y)} + R^{-1000} \Big(\sum_{T \in \mathbb{W}} \|f_T\|_2^2 \Big)^{1/2}, \tag{2.6}$$

where, for each $\square$, $\mathbb{W}_\square$ is a subset of

$$\big\{(S,T) : T \in \mathbb{W},\ \square = \square(T),\ S \in \mathbb{S}_{\tau(T)},\ \|\eta_S f_T\|_2 \in [\nu, 2\nu)\big\},$$

such that for any $(S_0, T_0) \in \mathbb{W}_\square$,

$$|\{(S, T_0) \in \mathbb{W}_\square\}| \in [\mu, 2\mu),$$

and with the property that

$$(S,T) \in \mathbb{W}_\square \ \Rightarrow\ S \cap T \neq \emptyset.$$

The dyadic range of ν was constrained; relying on the tail term in (2.6) to handle the contribution from those $f_{S,T}$ with $\|f_{S,T}\|_2 \leq R^{-2000}\|f_T\|_2$. Hence

$$\|f_{S,T}\|_{L^\infty(\mathbb{R}^3 \setminus 0.99S)} + \sup_{q\in[1,2]} \|\widehat{f_{S,T}}\|_{L^q(\mathbb{R}^3\setminus 1.1\tau(T))} \lesssim A\,R^{-7000}\,\|f_{S,T}\|_2,$$

for all $\square$ and $(S,T) \in \mathbb{W}_\square$, where the implicit constant is absolute.

For each κ and $\square \in \mathbb{P}_\kappa$, let $\{Q_\square\}_{Q_\square}$ be a finitely overlapping cover of $100\square$ by translates of the ellipsoid

$$\Big\{x_1\,\gamma(\theta_\kappa) + x_2\,\frac{\gamma'(\theta_\kappa)}{|\gamma'(\theta_\kappa)|} + x_3\,\frac{(\gamma\times\gamma')(\theta_\kappa)}{|(\gamma\times\gamma')(\theta_\kappa)|} : \\ \big(|x_1|^2 + (|x_2|\,R^{-1/4})^2 + (|x_3|\,R^{-1/2})^2\big)^{1/2} \leq R^{1/4+\delta/2}\Big\}.$$

Using Poisson summation again, let $\{\eta_{Q_\square}\}_{Q_\square\in\mathcal{Q}_\square}$ be a smooth partition of unity such that on $10^3\square$,

$$\sum_{Q_\square\in\mathcal{Q}_\square} \eta_{Q_\square} = 1,$$

and such that each $\eta_{Q_\square}$ satisfies

$$\|\eta_{Q_\square}\|_\infty \lesssim 1, \quad \|\eta_{Q_\square}\|_{L^\infty(\mathbb{R}^3\setminus Q_\square)} \lesssim R^{-10000},$$

and

$$|\eta_{Q_\square}(x)| \lesssim \operatorname{dist}(x, Q_\square)^{-10000} \quad \forall x \in \mathbb{R}^3,$$

with $\widehat{\eta_{Q_\square}}$ supported in

$$\Big\{\xi_1\,\gamma(\theta_\kappa) + \xi_2\,\frac{\gamma'(\theta_\kappa)}{|\gamma'(\theta_\kappa)|} + \xi_3\,\frac{(\gamma\times\gamma')(\theta_\kappa)}{|(\gamma\times\gamma')(\theta_\kappa)|} : |\xi_1| \leq R^{-1/4},\ |\xi_2| \leq R^{-1/2},\ |\xi_3| \leq R^{-3/4}\Big\}.$$

By dyadic pigeonholing,

$$\Big\|\sum_{\square}\sum_{(S,T)\in\mathbb{W}_\square} f_{S,T}\Big\|_{L^p(Y)} \lesssim \log R\,\Big\|\sum_{\square}\sum_{(S,T)\in\mathbb{W}_\square} \eta_{Y_\square} f_{S,T}\Big\|_{L^p(Y)} \\ + A\,R^{-1000}\Big(\sum_{T\in\mathbb{W}} \|f_T\|_2^2\Big)^{1/2},$$

where, for each $\square$, $Y_\square$ is a union over a subset of the sets $Q_\square$, and $\eta_{Y_\square}$ is the corresponding sum over $\eta_{Q_\square}$, such that each $Q_\square \subseteq Y_\square$ intersects a number $\# \in [M'(\square), 2M'(\square))$ different sets $3S$ with $(S,T) \in \mathbb{W}_\square$, up to a factor of 2. By pigeonholing again,

$$\Big\| \sum_{\square} \sum_{(S,T)\in\mathbb{W}_\square} \eta_{Y_\square} f_{S,T} \Big\|_{L^p(Y)} \lesssim (\log R)^2 \Big\| \sum_{\square\in\mathbb{B}} \sum_{(S,T)\in\mathbb{W}_\square} \eta_{Y_\square} f_{S,T} \Big\|_{L^p(Y)},$$

where $|\mathbb{W}_\square|$ and $M' = M'(\square)$ are constant over $\square \in \mathbb{B}$ up to a factor of 2. By one final pigeonholing step,

$$\Big\| \sum_{\square\in\mathbb{B}} \sum_{(S,T)\in\mathbb{W}_\square} \eta_{Y_\square} f_{S,T} \Big\|_{L^p(Y)} \lesssim \log R \Big\| \sum_{\square\in\mathbb{B}} \sum_{(S,T)\in\mathbb{W}_\square} \eta_{Y_\square} f_{S,T} \Big\|_{L^p(Y')},$$

where Y' is a union over $R^{1/2}$-balls $Q \subseteq Y$ such that each ball $2Q$ intersects a number $\# \in [M'', 2M'')$ of the sets $Y_\square$ in a set of strictly positive Lebesgue measure, as $\square$ varies over $\mathbb{B}$. Fix $Q \subseteq Y'$. By the decoupling theorem for generalised C^3 cones (see Exercise 12.5 in [2]), followed by Hölder's inequality,

$$\begin{aligned}
&\Big\| \sum_{\square\in\mathbb{B}} \sum_{(S,T)\in\mathbb{W}_\square} \eta_{Y_\square} f_{S,T} \Big\|_{L^p(Q)} \\
&\qquad \leq C_\epsilon B^{100} R^{\epsilon/100} (M'')^{1/2-1/p} \Big(\sum_{\square\in\mathbb{B}} \Big\| \sum_{(S,T)\in\mathbb{W}_\square} \eta_{Y_\square} f_{S,T} \Big\|_{L^p(2Q)}^p \Big)^{1/p} \\
&\qquad\quad + R^{-900} \Big(\sum_{T\in\mathbb{W}} \|f_T\|_2^2 \Big)^{1/2}.
\end{aligned}$$

Summing over Q gives

$$\begin{aligned}
\|f\|_{L^p(Y)} &\lesssim C_\epsilon (\log R)^{100} B^{100} R^{\epsilon/100} (M'')^{1/2-1/p} \\
&\quad \times \Big(\sum_{\square\in\mathbb{B}} \Big\| \sum_{(S,T)\in\mathbb{W}_\square} f_{S,T} \Big\|_{L^p(Y_\square)}^p \Big)^{1/p} + A R^{-800} \Big(\sum_{T\in\mathbb{W}} \|f_T\|_2^2 \Big)^{1/2}.
\end{aligned}$$

This will be bounded using the inductive assumption, following a Lorentz rescaling.

For each $\theta \in [-1,1]$, define the Lorentz rescaling map $L = L_\theta$ at θ by

$$\begin{aligned}
&L\Big[x_1 \gamma(\theta) + x_2 \frac{\gamma'(\theta)}{|\gamma'(\theta)|} + x_3 \frac{(\gamma\times\gamma')(\theta)}{|(\gamma\times\gamma')(\theta)|} \Big] \\
&\qquad = x_1 \gamma(\theta) + R^{1/4} x_2 \frac{\gamma'(\theta)}{|\gamma'(\theta)|} + R^{1/2} x_3 \frac{(\gamma\times\gamma')(\theta)}{|(\gamma\times\gamma')(\theta)|}.
\end{aligned}$$

Let

$$\widetilde{\gamma}(\phi) = \frac{L(\gamma(\phi))}{|L(\gamma(\phi))|}, \quad \phi \in [-1,1].$$

Then for any $\phi \in [-1,1]$,

$$\widetilde{\gamma}'(\phi) = \frac{\pi_{\widetilde{\gamma}(\phi)^\perp}(L(\gamma'(\phi)))}{|L(\gamma(\phi))|},$$

and

$$\widetilde{\gamma}''(\phi) = \frac{\pi_{\widetilde{\gamma}(\phi)^{\perp}}(L(\gamma''(\phi)))}{|L(\gamma(\phi))|} - \frac{\langle L(\gamma(\phi)), L(\gamma'(\phi))\rangle\, \pi_{\widetilde{\gamma}(\phi)^{\perp}}(L(\gamma'(\phi)))}{|L(\gamma(\phi))|^3}.$$

Hence

$$\det(\widetilde{\gamma}, \widetilde{\gamma}', \widetilde{\gamma}'') = \frac{1}{|L\circ\gamma|^3}\det(L\circ\gamma, L\circ\gamma', L\circ\gamma'') = \frac{R^{3/4}}{|L\circ\gamma|^3}\det(\gamma,\gamma',\gamma'').$$

Let $\varepsilon = (10^5 B^{10})^{-1}$, and for fixed $\theta \in [-1+\varepsilon, 1-\varepsilon]$, let

$$\widehat{\gamma}(\phi) = \widetilde{\gamma}(\theta + R^{-1/4}\phi), \quad \phi \in [-\varepsilon, \varepsilon].$$

The assumption that $\|\gamma\|_{C^3[-1,1]} \leq B$ yields

$$1 \leq |L(\gamma(\phi))| \leq 1 + 10B\varepsilon, \quad \forall \phi \in [\theta - \varepsilon R^{-1/4}, \theta + \varepsilon R^{-1/4}].$$

Similarly,

$$\begin{aligned} |L(\gamma'(\phi)) - L(\gamma'(\theta))| &\leq 10\varepsilon B R^{1/4}, \quad \forall \phi \in [\theta - \varepsilon R^{-1/4}, \theta + \varepsilon R^{-1/4}],\\ |L(\gamma''(\phi)) - L(\gamma''(\theta))| &\leq 10\varepsilon B R^{1/4}, \quad \forall \phi \in [\theta - \varepsilon R^{-1/4}, \theta + \varepsilon R^{-1/4}], \end{aligned}$$

and

$$|L\gamma'''(\phi)| \leq BR^{1/2}, \quad \forall \phi \in [\theta - \varepsilon R^{-1/4}, \theta + \varepsilon R^{-1/4}].$$

It follows that

$$|\det(\widehat{\gamma}, \widehat{\gamma}', \widehat{\gamma}'')| \geq (2B)^{-1}$$

on $[-\varepsilon, \varepsilon]$, and that

$$\|\widehat{\gamma}\|_{C^3[-\varepsilon,\varepsilon]} \leq 2B$$

(the calculation for the third derivative is omitted).

For each $\square \in \mathbb{B}$, given $(S,T) \in \mathbb{W}_{\square}$, let $g_{S,T} = f_{S,T} \circ L$, where $L = L_{\theta_{\kappa(\square)}}$. Then

$$\Big\| \sum_{(S,T)\in\mathbb{W}_{\square}} f_{S,T} \Big\|_{L^p(Y_{\square})} \leq R^{3/(4p)} \Big\| \sum_{(S,T)\in\mathbb{W}_{\square}} g_{S,T} \Big\|_{L^p(L^{-1}Y_{\square})}. \tag{2.7}$$

The inequalities (2.4) and (2.5) imply that for each $(S,T) \in \mathbb{W}_{\square}$, the set $L^{-1}(S)$ is a equivalent (up to a factor 1.01) to a box of length $R^{1/2+\delta/2}$ in its longest direction parallel to $L^{-1}(\gamma\times\gamma')(\theta_\tau)$ and of length $R^{1/4+\delta/2}$ in its other two directions. The ellipsoids $Q_{\square}$ are rescaled to $R^{1/4+\delta/2}$-balls $L^{-1}(Q_{\square})$. Moreover, it will be shown that

$$\begin{aligned} L(\tau) \subseteq \Big\{ &x_1\, \widetilde{\gamma}(\theta_\tau) + x_2 \frac{\widetilde{\gamma}'(\theta_\tau)}{|\widetilde{\gamma}'(\theta_\tau)|} + x_3 \frac{(\widetilde{\gamma}\times\widetilde{\gamma}')(\theta_\tau)}{|(\widetilde{\gamma}\times\widetilde{\gamma}')(\theta_\tau)|} : \\ &1 \leq x_1 \leq 2.01,\ |x_2| \leq (1.01)R^{-1/4},\ |x_3| \leq R^{-1/2} \Big\}. \end{aligned} \tag{2.8}$$

To prove this, let

$$x = x_1\, \gamma(\theta_\tau) + x_2 \frac{\gamma'(\theta_\tau)}{|\gamma'(\theta_\tau)|} + x_3 \frac{(\gamma\times\gamma')(\theta_\tau)}{|(\gamma\times\gamma')(\theta_\tau)|} \in \tau,$$

where

$$x_1 \in [1,2], \quad |x_2| \le R^{-1/2} \quad \text{and} \quad |x_3| \le R^{-1}.$$

The vector $(\widetilde{\gamma} \times \widetilde{\gamma}')(\theta_\tau)$ is parallel to $L^{-1}((\gamma \times \gamma')(\theta_\tau))$, since $L^{-1}((\gamma \times \gamma')(\theta_\tau))$ is orthogonal to $\widetilde{\gamma}(\theta_\tau)$ and $\widetilde{\gamma}'(\theta_\tau)$. The inequality

$$|L^{-1}((\gamma \times \gamma')(\theta_\tau))| \ge R^{-1/2}\, |(\gamma \times \gamma')(\theta_\tau)|$$

gives

$$\left|\left\langle Lx, \frac{L^{-1}((\gamma \times \gamma')(\theta_\tau))}{|L^{-1}((\gamma \times \gamma')(\theta_\tau))|}\right\rangle\right| \le R^{-1/2}. \tag{2.9}$$

Moreover,

$$\begin{aligned}
&\left|\left\langle Lx, \frac{\pi_{L(\gamma(\theta_\tau))^\perp}(L(\gamma'(\theta_\tau)))}{|\pi_{L(\gamma(\theta_\tau))^\perp}(L(\gamma'(\theta_\tau)))|}\right\rangle\right| \\
&\qquad = \left|\left\langle x_2\, L\Big(\frac{\gamma'(\theta_\tau)}{|\gamma'(\theta_\tau)|}\Big) + x_3\, L\Big(\frac{(\gamma \times \gamma')(\theta_\tau)}{|(\gamma \times \gamma')(\theta_\tau)|}\Big), \frac{\pi_{L(\gamma(\theta_\tau))^\perp}(L(\gamma'(\theta_\tau)))}{|\pi_{L(\gamma(\theta_\tau))^\perp}(L(\gamma'(\theta_\tau)))|}\right\rangle\right| \\
&\qquad \le (1.01) R^{-1/4}.
\end{aligned} \tag{2.10}$$

For the direction $L(\gamma(\theta_\tau))$,

$$\left\langle Lx, \frac{L(\gamma(\theta_\tau))}{|L(\gamma(\theta_\tau))|}\right\rangle = x_1\, |L(\gamma(\theta_\tau))| + O(R^{-1/4}). \tag{2.11}$$

Combining (2.9), (2.10) and (2.11) gives (2.8).

Inductively applying the theorem at scale $R^{1/2}$ therefore gives

$$\begin{aligned}
(2.7) &\lesssim C_{\epsilon,\delta,A}\, B^{10^{10}/\epsilon}\, R^{3\epsilon/4}\, R^{3/(4p)} \Big(\frac{M' R^{-3/4}}{|\mathbb{W}_\square|}\Big)^{1/2-1/p} \Big(\sum_{(S,T)\in \mathbb{W}_\square} \|g_{S,T}\|_2^2\Big)^{1/2} \\
&= C_{\epsilon,\delta,A}\, B^{10^{10}/\epsilon}\, R^{3\epsilon/4} \Big(\frac{M' R^{-3/2}}{|\mathbb{W}_\square|}\Big)^{1/2-1/p} \Big(\sum_{(S,T)\in \mathbb{W}_\square} \|f_{S,T}\|_2^2\Big)^{1/2},
\end{aligned}$$

for each $\square \in \mathbb{B}$. Hence

$$\begin{aligned}
&\|f\|_{L^p(Y)} \\
&\le C_{\epsilon,\delta,A}\, B^{10^{10}/\epsilon}\, R^{4\epsilon/5} \Big(\frac{M' M'' R^{-3/2}}{|\mathbb{W}_\square|}\Big)^{1/2-1/p} \Big(\sum_{\square\in\mathbb{B}} \Big(\sum_{(S,T)\in \mathbb{W}_\square} \|f_{S,T}\|_2^2\Big)^{p/2}\Big)^{1/p}.
\end{aligned}$$

Using $\|f_{S,T}\|_2^2 \lesssim \|f_T\|_2^2/\mu$, this is

$$\lesssim C_{\epsilon,\delta,A}\, B^{10^{10}/\epsilon}\, R^{4\epsilon/5} \Big(\frac{M' M'' R^{-3/2}}{|\mathbb{W}|\mu}\Big)^{1/2-1/p} \Big(\frac{|\mathbb{B}||\mathbb{W}_\square|}{|\mathbb{W}|\mu}\Big)^{1/p} \Big(\sum_{T\in\mathbb{W}} \|f_T\|_2^2\Big)^{1/2}.$$

The second bracketed term is $\lesssim 1$, since

$$|\mathbb{W}|\mu = \sum_{T\in\mathbb{W}} \mu \geq \sum_{\square\in\mathbb{B}} \sum_{\substack{T\in\mathbb{W}:\\ \square=\square(T)}} \mu \sim \sum_{\square\in\mathbb{B}} \sum_{\substack{T\in\mathbb{W}:\\ \square=\square(T)}} \sum_{\substack{(S,T')\in\mathbb{W}_\square:\\ T'=T}} 1 = |\mathbb{B}||\mathbb{W}_\square|.$$

It remains to show that $M'M'' \lesssim \mu M$. Let $Q \subseteq Y'$ be any $R^{1/2}$-ball. By definition of μ and M,

$$\mu M \gtrsim \sum_{\substack{T\in\mathbb{W}:\\ 2T\cap Q\neq\emptyset}} \sum_{\substack{\square\in\mathbb{B}:\\ \square=\square(T)}} \sum_{\substack{(S,T')\in\mathbb{W}_\square\\ T'=T}} 1 = \sum_{\square\in\mathbb{B}} \sum_{\substack{T\in\mathbb{W}:\\ \square=\square(T)\\ 2T\cap Q\neq\emptyset}} \sum_{\substack{(S,T')\in\mathbb{W}_\square:\\ T'=T}} 1 = \sum_{\square\in\mathbb{B}} \sum_{\substack{(S,T)\in\mathbb{W}_\square:\\ 2T\cap Q\neq\emptyset}} 1.$$

By definition of M' and M'',

$$\begin{aligned}
M'M'' &\sim \sum_{\substack{\square\in\mathbb{B}:\\ m(Y_\square\cap 2Q)>0}} M' \leq \sum_{\substack{\square\in\mathbb{B}:\\ m(Y_\square\cap 2Q)>0}} \sum_{Q_\square\subseteq Y_\square} M' \frac{m(Q_\square\cap 2Q)}{m(Y_\square\cap 2Q)} \\
&\sim \sum_{\substack{\square\in\mathbb{B}:\\ m(Y_\square\cap 2Q)>0}} \sum_{Q_\square\subseteq Y_\square} \sum_{\substack{(S,T)\in\mathbb{W}_\square:\\ Q_\square\cap 3S\neq\emptyset}} \frac{m(Q_\square\cap 2Q)}{m(Y_\square\cap 2Q)} \\
&= \sum_{\substack{\square\in\mathbb{B}:\\ m(Y_\square\cap 2Q)>0}} \sum_{(S,T)\in\mathbb{W}_\square} \sum_{\substack{Q_\square\subseteq Y_\square:\\ Q_\square\cap 3S\neq\emptyset}} \frac{m(Q_\square\cap 2Q)}{m(Y_\square\cap 2Q)} \\
&\leq \sum_{\substack{\square\in\mathbb{B}:\\ m(Y_\square\cap 2Q)>0}} \sum_{\substack{(S,T)\in\mathbb{W}_\square:\\ 2T\cap Q\neq\emptyset}} \sum_{Q_\square\subseteq Y_\square} \frac{m(Q_\square\cap 2Q)}{m(Y_\square\cap 2Q)} \qquad (2.12) \\
&\lesssim \sum_{\square\in\mathbb{B}} \sum_{\substack{(S,T)\in\mathbb{W}_\square:\\ 2T\cap Q\neq\emptyset}} 1 \lesssim \mu M.
\end{aligned}$$

The inequality (2.12) above follows from the observation that if $Q_\square \cap 2Q \neq \emptyset$, and if $(S,T) \in \mathbb{W}_\square$ is such that $Q_\square \cap 3S \neq \emptyset$, then $2T \cap Q \neq \emptyset$. To prove this, recall that $L^{-1}(S)$ is equivalent (up to a factor 1.01) to a box of length $R^{1/2+\delta/2}$ in its longest direction and of length $R^{1/4+\delta/2}$ in its other two directions. The sets $L^{-1}(Q_\square)$ are $R^{1/4+\delta/2}$-balls. Therefore $Q_\square \cap 3S \neq \emptyset$ implies that $Q_\square \subseteq 100S \subseteq 1.5T$, which yields that $2T \cap Q \neq \emptyset$ since $Q_\square \cap 2Q \neq \emptyset$. ∎

3. Wave packet decomposition

Throughout this section, assume that $\gamma\colon [a,b] \to S^2$ is C^2 with unit speed, that

$$|\gamma(\theta) - \gamma(\phi)| \leq 2^{-100} \quad \text{and} \quad |\gamma'(\theta) - \gamma'(\phi)| \leq 2^{-100}, \tag{3.1}$$

for all $\theta, \phi \in [a,b]$, that

$$\gamma_1' \neq 0, \qquad \text{where } \gamma = (\gamma_1, \gamma_2, \gamma_3), \tag{3.2}$$

and that

$$\det(\gamma, \gamma', \gamma'') > 0. \tag{3.3}$$

To simplify notation, the convention $\mathbb{N} = \{0, 1, 2, \dots\}$ will be assumed.

Definition 3.1. Let $\gamma\colon [a, b] \to S^2$ be a C^2 unit speed curve satisfying (3.1), (3.2) and (3.3). Let $\varrho > 0$ and $\varepsilon \in (0, 1]$. For each $k \geq 0$, let

$$\Theta_k = \left\{a + \varrho 2^{-k/2} l : l \in \mathbb{N} \cap [0, (b-a) \cdot \varrho^{-1} 2^{k/2})\right\}. \tag{3.4}$$

For each $k \in [0, j] \cap \mathbb{N}$, if $k < j$ then for each $\theta \in \Theta_k$, let

$$\begin{aligned}\tau^+(\theta, j, k) = \big\{&\lambda_1 (\gamma \times \gamma')(\theta) + \lambda_2 \gamma'(\theta) + \lambda_3 \gamma(\theta) : \\ &2^{j-2} \leq \lambda_1 \leq 2^{j+2},\ |\lambda_2| \leq \varepsilon 2^{-k/2+j},\ -2^{-k+j+2} \leq \lambda_3 \leq -2^{-k+j-2}\big\},\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\tau^-(\theta, j, k) = \big\{&\lambda_1 (\gamma \times \gamma')(\theta) + \lambda_2 \gamma'(\theta) + \lambda_3 \gamma(\theta) : \\ &-2^{j+2} \leq \lambda_1 \leq -2^{j-2},\ |\lambda_2| \leq \varepsilon 2^{-k/2+j},\ 2^{-k+j-2} \leq \lambda_3 \leq 2^{-k+j+2}\big\}.\end{aligned} \tag{3.6}$$

If $k = j$ then for each $\theta \in \Theta_j$, let

$$\begin{aligned}\tau^+(\theta, j, j) = \big\{&\lambda_1 (\gamma \times \gamma')(\theta) + \lambda_2 \gamma'(\theta) + \lambda_3 \gamma(\theta) : \\ &2^{j-2} \leq \lambda_1 \leq 2^{j+2},\ |\lambda_2| \leq \varepsilon 2^{j/2},\ |\lambda_3| \leq 4\big\},\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\tau^-(\theta, j, j) = \big\{&\lambda_1 (\gamma \times \gamma')(\theta) + \lambda_2 \gamma'(\theta) + \lambda_3 \gamma(\theta) : \\ &-2^{j+2} \leq \lambda_1 \leq -2^{j-2},\ |\lambda_2| \leq \varepsilon 2^{j/2},\ |\lambda_3| \leq 4\big\}.\end{aligned} \tag{3.8}$$

Let

$$\Lambda^+_{j,k} = \{\tau^+(\theta, j, k) : \theta \in \Theta_k\}, \quad \Lambda^-_{j,k} = \{\tau^-(\theta, j, k) : \theta \in \Theta_k\},$$
$$\Lambda_{j,k} = \Lambda^+_{j,k} \cup \Lambda^-_{j,k},$$

and

$$\Lambda = \bigcup_{j=0}^{\infty} \bigcup_{k=0}^{j} \Lambda_{j,k}.$$

Given $J \geq 0$, let

$$\Lambda^J = \bigcup_{j \geq J} \bigcup_{J \leq k \leq j} \Lambda_{j,k},$$

and

$$\Lambda^{J,o} = \bigcup_{j > 2J} \bigcup_{J < k \leq j} \left\{\tau \in \Lambda_{j,k} : \operatorname{dist}(\theta_\tau : \{a, b\}) > 100 m^{-1} \max\{\varepsilon, \varrho\} 2^{-k/2}\right\},$$

where m is the minimum of $|\det(\gamma, \gamma', \gamma'')|$ on $[a, b]$.

Given $\delta > 0$, for each j, k and each $\tau \in \Lambda_{j,k}$ let

$$\begin{aligned}\mathbb{T}_\tau = \mathbb{T}_\tau^\delta = \Big\{ T = a_1\gamma(\theta_\tau) + a_2\gamma'(\theta_\tau) + a_3(\gamma\times\gamma')(\theta_\tau) \\ + \big\{\lambda_1\gamma(\theta_\tau) + \lambda_2\gamma'(\theta_\tau) + \lambda_3(\gamma\times\gamma')(\theta_\tau) : \\ |\lambda_2|, |\lambda_3| \leq 2^{k/2-j+k\delta},\ |\lambda_1| \leq 2^{k-j+k\delta}\big\}, \\ (a_2, a_3) \in 2^{-10-j+k/2+k\delta}\mathbb{Z}^2,\ a_1 \in 2^{-10+k-j+k\delta}\mathbb{Z}\Big\}.\end{aligned}$$

The parameter J should be thought of as morally equal to 1, and is only used to exclude low frequency pieces. For any $\tau \in \Lambda$ and constant $C \geq 0$, let $C\tau$ be the box with the same centre as τ but with side lengths scaled by C.

Lemma 3.2. *Let $\gamma\colon [a,b] \to S^2$ be a C^2 unit speed curve satisfying* (3.1), (3.2) *and* (3.3). *Let $\varepsilon \in (0,1]$ and $\varrho > 0$. There exists a constant C_γ such that, if $\tau_1 \in \Lambda_{j_1,k_1}$, $\tau_2 \in \Lambda_{j_2,k_2}$, and*

$$1.01\tau_1 \cap 1.01\tau_2 \neq \emptyset, \tag{3.9}$$

then

$$|j_1 - j_2| + |k_1 - k_2| \leq C \quad \text{and} \quad |\theta_{\tau_1} - \theta_{\tau_2}| \leq C2^{-k_1/2},$$

and such that if $\tau_1 \in \Lambda^+_{j_1,k_1}$ and $\tau_2 \in \Lambda^-_{j_2,k_2}$, or if $\tau_1 \in \Lambda^-_{j_1,k_1}$ and $\tau_2 \in \Lambda^+_{j_2,k_2}$, then

$$k_1 + k_2 \leq C.$$

Moreover, there exists a constant K, depending only on γ and ε, such that if $k_1, k_2 \geq K$ and (3.9) *holds, then*

$$|\theta_{\tau_1} - \theta_{\tau_2}| \leq C\varepsilon 2^{-k_1/2}. \tag{3.10}$$

Proof. If $1.01\tau_1 \cap 1.01\tau_2 \neq \emptyset$, let

$$\begin{aligned}\lambda_1(\gamma\times\gamma')(\theta_{\tau_1}) + \lambda_2\gamma'(\theta_{\tau_1}) + \lambda_3\gamma(\theta_{\tau_1}) \\ = \mu_1(\gamma\times\gamma')(\theta_{\tau_2}) + \mu_2\gamma'(\theta_{\tau_2}) + \mu_3\gamma(\theta_{\tau_2})\end{aligned} \tag{3.11}$$

be a point in the intersection, where each side satisfies the conditions in any of (3.5), (3.6), (3.7) or (3.8), multiplied by the factor 1.01. Then

$$|j_1 - j_2| \leq 5, \tag{3.12}$$

by comparing norms on either side. By symmetry, it may be assumed that $k_1 \leq k_2$. If $\operatorname{sgn}\lambda_1 \neq \operatorname{sgn}\mu_1$, then by (3.1),

$$|\lambda_1(\gamma\times\gamma')(\theta_{\tau_1}) + \lambda_2\gamma'(\theta_{\tau_1}) + \lambda_3\gamma(\theta_{\tau_1}) - \mu_1(\gamma\times\gamma')(\theta_{\tau_2})| \geq 2^{j_1-3},$$

and therefore $k_2 \leq 100$ by (3.11) and the triangle inequality. This shows that all conclusions of the lemma hold if $\operatorname{sgn}\lambda_1 \neq \operatorname{sgn}\mu_1$ (assuming that $K \geq 1000$ and that $C \geq 2^{100}\max\{1, b-a\}$), so assume that $\operatorname{sgn}\lambda_1 = \operatorname{sgn}\mu_1$. By (3.11) and (3.12),

$$|\lambda_1(\gamma\times\gamma')(\theta_{\tau_1}) - \mu_1(\gamma\times\gamma')(\theta_{\tau_2})| \leq 2^{j_1-k_1/2+8}. \tag{3.13}$$

The inequality

$$|v - \lambda w| \geq |v - w|/2, \quad \forall \lambda \in [0,1] \quad \forall v, w \in S^2 \text{ with } \langle v, w\rangle \geq 0,$$

together with (3.1), (3.13) and the assumption $\operatorname{sgn} \lambda_1 = \operatorname{sgn} \mu_1$, gives

$$|(\gamma \times \gamma')(\theta_{\tau_1}) - (\gamma \times \gamma')(\theta_{\tau_2})| \leq 2^{-k_1/2+15}. \tag{3.14}$$

By (3.2), (3.14), the mean value theorem and the identity

$$(\gamma \times \gamma')' = -\det(\gamma, \gamma', \gamma'')\gamma',$$

it follows that

$$|\theta_{\tau_1} - \theta_{\tau_2}| \leq \frac{2^{-k_1/2+15}}{\min|\gamma_1' \det(\gamma, \gamma', \gamma'')|}. \tag{3.15}$$

A similar argument gives that

$$|\theta_{\tau_1} - \theta_{\tau_2}| \lesssim \varepsilon 2^{-k_1/2},$$

provided k_1 and k_2 are sufficiently large depending on ε and γ.

If $k_1 = j_1$ then the lemma follows, so assume that $k_1 < j_1$. By the generalised mean value theorem and the assumption that γ is C^2,

$$\gamma(\theta) = \gamma(\phi) + (\theta - \phi)\gamma'(\phi) + \frac{1}{2}(\theta - \phi)^2 \gamma''(\phi) + o(|\theta - \phi|^2) \tag{3.16}$$

for any $\theta, \phi \in [a, b]$, where the rate of decay to zero in the error term is uniform in ϕ and θ. Hence

$$\langle \gamma(\theta), (\gamma \times \gamma')(\phi)\rangle = \frac{1}{2}(\theta - \phi)^2 \det(\gamma(\phi), \gamma'(\phi), \gamma''(\phi)) + o(|\theta - \phi|^2).$$

Using (3.11), (3.15), letting $\theta = \theta_{\tau_1}$ and $\phi = \theta_{\tau_2}$, and taking the dot product of both sides of (3.11) with $\gamma(\theta_{\tau_1})$, gives

$$\left|\lambda_3 - \frac{\mu_1}{2}(\theta_{\tau_1} - \theta_{\tau_2})^2 \det(\gamma(\theta_{\tau_2}), \gamma'(\theta_{\tau_2}), \gamma''(\theta_{\tau_2})) - \mu_1\, o(|\theta_{\tau_1} - \theta_{\tau_2}|^2)\right| \leq C_\gamma\, 2^{j_2 - k_1/2 - k_2/2},$$

for some constant C_γ which may depend on γ. Since $|\lambda_3| \gtrsim |2^{j_1-k_1}|$ and $\operatorname{sgn} \lambda_3 = -\operatorname{sgn} \lambda_1 = -\operatorname{sgn} \mu_1$, and since the decay to zero in the error term is uniform, the left-hand side is $\gtrsim 2^{j_1-k_1}$ provided k_1 and k_2 are sufficiently large depending only on γ. This implies that $k_2 \lesssim k_1$, and therefore $|k_1 - k_2| \leq C$ provided that C is sufficiently large (depending on γ). ■

Lemma 3.3. *There exist constants $\varrho > 0$, $\varepsilon \in (0,1]$ and $J_1 \geq 0$ such that for all $J \geq J_1$, there is a partition of unity $\{\psi_\tau\}_{\tau\in\Lambda^J}$ subordinate to the cover $\{1.01\tau : \tau \in \Lambda^J\}$ of*

$$\bigcup_{\tau \in \Lambda^{J,o}} \tau,$$

such that for each $\tau \in \Lambda^{J,o} \cap \Lambda_{j,k}$, the function ψ_τ is smooth and satisfies

$$\left|\left(\frac{d}{dt}\right)^l \psi_\tau(x+tv)\right| \lesssim_l |2^{-j}\langle v, (\gamma \times \gamma')(\theta_\tau)\rangle|^l + |2^{-j+k/2}\langle v, \gamma'(\theta_\tau)\rangle|^l + |2^{-j+k}\langle v, \gamma(\theta_\tau)\rangle|^l, \tag{3.17}$$

for all $l \in \mathbb{N}$, $t \in \mathbb{R}$ and $x, v \in \mathbb{R}^3$.

Given any $\delta > 0$, for every $\tau \in \Lambda$ there exists a smooth partition of unity $\{\eta_T\}_{T \in \mathbb{T}_\tau}$ subordinate to the cover $\mathbb{T}_\tau = \mathbb{T}_\tau^\delta$ of $\mathbb{R}^3$, such that

$$\left|\left(\frac{d}{dt}\right)^l \eta_T(x+tv)\right| \lesssim_l |2^{j-k-k\delta}\langle v, \gamma(\theta_\tau)\rangle|^l + |2^{j-k/2-k\delta}\langle v, \gamma'(\theta_\tau)\rangle|^l + |2^{j-k/2-k\delta}\langle v, (\gamma \times \gamma')(\theta_\tau)\rangle|^l,$$

for all $l \in \mathbb{N}$, $t \in \mathbb{R}$, $x, v \in \mathbb{R}^3$ and $T \in \mathbb{T}_\tau$.

Proof. For $\tau \in \Lambda^+$, let

$$g_\tau(x) = g_0(4.005 + \langle x, \gamma(\theta_\tau)\rangle),$$

and for $\tau \in \Lambda^-$, let

$$g_\tau(x) = g_0(4.005 - \langle x, \gamma(\theta_\tau)\rangle),$$

where g_0 is a smooth function on $\mathbb{R}$ with $0 \leq g_0 \leq 1$, $g_0(x) = 1$ for $x \geq 1/1000$ and $g(x) = 0$ for $x \leq 0$. Choose J_1 large enough and ε small enough to ensure that if $J \geq J_1$, if $\tau \in \Lambda^J$ and if $\tau' \in \Lambda \cap \Lambda_{j,j}$, then $g_{\tau'}(x) = 1$ for all $x \in \tau \cap (1.01)\tau'$; such a choice of J_1 and ε exists by the angle condition (3.10) in Lemma 3.2, and by (3.16).

By translating and rescaling a fixed bump function on the unit cube, for each $\tau \in \Lambda$ let f_τ be a smooth bump function which is equal to 1 on τ, nonzero in the interior of 1.01τ, with $f_\tau \geq 1/100$ on 1.009τ and with $f_\tau = 0$ outside 1.01τ. Let $J \geq J_1$. If $\tau \in \Lambda_{j,k} \cap \Lambda^J$ with $k < j$, let

$$\psi_\tau(x) = \begin{cases} \frac{f_\tau(x)}{\sum_{\tau' \in \Lambda^J} f_{\tau'}(x)} & x \in (1.01\tau)^o, \\ 0 & x \in \mathbb{R}^3 \setminus (1.01\tau)^o. \end{cases} \tag{3.18}$$

For $\tau \in \Lambda_{j,j} \cap \Lambda^J$, let

$$\psi_\tau(x) = \begin{cases} \frac{f_\tau(x) g_\tau(x)}{\sum_{\tau' \in \Lambda^J} f_{\tau'}(x)} & x \in (1.01\tau)^o, \\ 0 & x \in \mathbb{R}^3 \setminus (1.01\tau)^o. \end{cases} \tag{3.19}$$

Then $\sum_{\tau \in \Lambda^J} \psi_\tau(x) = 1$ for $x \in \bigcup_{\tau \in \Lambda^J} \tau$, by the choice of J_1 and ε.

It will be shown that if the constant ϱ in (3.4) is small enough, and if J_1 is large enough, then for any $\tau \in \Lambda_{j,k} \cap \Lambda^{J,o}$ with $k < j$,

$$1.01\tau \subseteq \bigcup_{\tau' \in \Lambda^J} \tau'. \tag{3.20}$$

If $\tau \in \Lambda^+$ (which can be assumed; the argument for $\tau \in \Lambda^-$ being similar), this follows from the following argument. Given

$$\begin{aligned}1.01\tau \ni x &= \lambda_1(\gamma \times \gamma')(\theta_\tau) + \lambda_2\gamma'(\theta_\tau) + \lambda_3\gamma(\theta_\tau)\\ &= \lambda_1(\gamma\times\gamma')\Big(\theta_\tau + \frac{\lambda_2}{\lambda_1 \det(\gamma(\theta_\tau),\gamma'(\theta_\tau),\gamma''(\theta_\tau))}\Big) + O(2^{j-k}),\end{aligned}$$

let

$$\theta' = \theta_\tau + \frac{\lambda_2}{\lambda_1 \det(\gamma(\theta_\tau),\gamma'(\theta_\tau),\gamma''(\theta_\tau))}.$$

Choose j' such that $2^{j'-1} \leq \lambda_1 \leq 2^{j'+1}$, and define k' by

$$2^{j'-k'} < \operatorname{dist}(x, \operatorname{span}\{(\gamma\times\gamma')(\theta'), \gamma'(\theta')\}) \leq 2^{j'-k'+1}.$$

If J_1 is chosen sufficiently large and ε is sufficiently small, then (by (3.16)) the parameter k' is well-defined and satisfies $|k-k'| \lesssim 1$, $k' \geq 0$, and moreover

$$\langle x, \gamma(\theta')\rangle < 0.$$

If $k' \leq j'$, choose $\theta_{\tau'} \in \Theta_{k'}$ such that

$$|\theta' - \theta_{\tau'}| \leq 2^{-k'/2}\varrho.$$

Then (by (3.16)), if J_1 is sufficiently large and then ϱ is chosen sufficiently small (depending on ε),

$$\begin{gathered}2^{j'-2} \leq \langle x, (\gamma\times\gamma')(\theta_{\tau'})\rangle \leq 2^{j'+2},\\ |\langle x, \gamma'(\theta_{\tau'})\rangle| \leq \varepsilon 2^{j'-k'/2},\\ -2^{j'-k'+2} \leq \langle x, \gamma(\theta_{\tau'})\rangle \leq -2^{j'-k'-2}.\end{gathered}$$

By letting $\tau' \in \Lambda_{j',k'} \cap \Lambda^+$ be the box corresponding to the angle $\theta_{\tau'}$, this proves (3.20). If $k' \geq j'$, the above argument still works by taking $\theta_{\tau'} \in \Theta_{j'}$ instead. The covering property in (3.20) implies that the denominator in the definition of ψ_τ in (3.18) is bounded away from zero on the support of the numerator, and therefore (by Lemma 3.2) ψ_τ is smooth and satisfies the inequalities in (3.17) whenever $\tau \in \Lambda^{J,o} \cap \Lambda_{j,k}$ with $k < j$. For the first part of the lemma, it remains to prove (3.17) in the case $k = j$.

For the case $k = j$, it will be shown that if the constant ϱ in (3.4) is small enough, and if J_1 is large enough, then for any $\tau \in \Lambda_{j,j} \cap \Lambda^{J,o}$,

$$(1.01\tau) \setminus \{x : \langle x, \gamma(\theta_\tau)\rangle > 4.005\} \subseteq \bigcup_{\tau' \in \Lambda^J} 1.009\tau'. \tag{3.21}$$

To see this, given $x \in (1.01\tau) \setminus \{x : \langle x, \gamma(\theta_\tau)\rangle > 4.005\}$, write

$$\begin{aligned}x &= \lambda_1(\gamma \times \gamma')(\theta_\tau) + \lambda_2\gamma'(\theta_\tau) + \lambda_3\gamma(\theta_\tau)\\ &= \lambda_1(\gamma\times\gamma')\Big(\theta_\tau + \frac{\lambda_2}{\lambda_1 \det(\gamma(\theta_\tau),\gamma'(\theta_\tau),\gamma''(\theta_\tau))}\Big) + O(1).\end{aligned}$$

Choose j' such that $2^{j'-1} \leq \lambda_1 \leq 2^{j'+1}$. If

$$-4 \leq \langle x, \gamma(\theta_\tau)\rangle \leq 4.005, \tag{3.22}$$

then let $k' = j'$ and choose $\theta_{\tau'} \in \Theta_{j'}$ such that

$$|\theta' - \theta_{\tau'}| \leq 2^{-j'/2}\varrho.$$

Then (by (3.16)), if J_1 is sufficiently large and then ϱ is chosen sufficiently small (depending on ε),

$$2^{j'-2} \leq \langle x, (\gamma \times \gamma')(\theta_{\tau'})\rangle \leq 2^{j'+2},$$
$$|\langle x, \gamma'(\theta_{\tau'})\rangle| \leq \varepsilon\, 2^{j'/2},$$
$$-4.036 \leq \langle x, \gamma(\theta_{\tau'})\rangle \leq 4.036.$$

If (3.22) does not hold, then

$$-4.04 \leq \langle x, \gamma(\theta_\tau)\rangle \leq -4,$$

In this case, let $k' = j' - 1$, and choose $\theta_{\tau'} \in \Theta_{k'}$ such that

$$|\theta' - \theta_{\tau'}| \leq 2^{-k'/2}\varrho.$$

Then (by (3.16)), if J_1 is sufficiently large and then ϱ is chosen sufficiently small (depending on ε),

$$2^{j'-2} \leq \langle x, (\gamma \times \gamma')(\theta_{\tau'})\rangle \leq 2^{j'+2},$$
$$|\langle x, \gamma'(\theta_{\tau'})\rangle| \leq \varepsilon\, 2^{j'-k'/2},$$
$$-2^{j'-k'+2} \leq \langle x, \gamma(\theta_{\tau'})\rangle \leq -2^{j'-k'-2}.$$

In either case, by letting $\tau' \in \Lambda_{j',k'} \cap \Lambda^+$ be the cap corresponding to angle $\theta_{\tau'}$, this proves (3.21). This implies that the denominator in the definition of ψ_τ in (3.19) is bounded away from zero on the support of the numerator, and therefore (by Lemma 3.2) ψ_τ is smooth and satisfies (3.17) whenever $\tau \in \Lambda_{j,j} \cap \Lambda^{J,o}$. This proves the first part of the lemma.

The second part of the lemma is straightforward. ∎

Definition 3.4. Let $\delta > 0$. Let $\varepsilon, \varrho > 0$ and $J_1 \geq 0$ be parameters ensuring the existence of the partition of unity in Lemma 3.3, and let $J \geq J_1$. Given a box $\tau \in \Lambda_{j,k} \cap \Lambda^J$ and $T \in \mathbb{T}_\tau = \mathbb{T}_\tau^\delta$, define

$$M_T f = \eta_T (f * \widecheck{\psi_\tau}),$$

for each Schwartz function f.

Let ϕ be a bump function equal to 1 on $B_3(0,1)$ which vanishes outside $B_3(0,2)$.

Lemma 3.5. *Let $\varepsilon, \varrho > 0$ and $J_1 \geq 0$ be parameters ensuring the existence of the partition of unity in Lemma 3.3, and let $J \geq J_1$. Let $j_0 \in \mathbb{N}$, let*

$$\phi_{j_0}(x) = 2^{3j_0}\phi(2^{j_0}x), \quad x \in \mathbb{R}^3,$$

let $\alpha \in [0,3]$, and let $\epsilon, \delta, \alpha_0' > 0$. For any finite Borel measure μ on $B_3(0,1)$, and any $J \geq J_1$, there is a decomposition

$$\mu * \phi_{j_0} = \mu_g + \mu_b,$$

where

$$\mu_g = \mu_{g,j_0,J,\alpha,\epsilon,\delta,\alpha_0'} \quad \textit{and} \quad \mu_b = \mu_{b,j_0,J,\alpha,\epsilon,\delta,\alpha_0'}$$

are complex-valued continuous functions supported in $B_3(0, 2^{J\delta})$*, and*

$$\mu_b = \sum_{j>2J} \sum_{\substack{k\in[j\epsilon,j] \\ k>J}} \sum_{\tau\in\Lambda_{j,k}} \sum_{T\in\mathbb{T}_{\tau,b}} M_T(\mu * \phi_{j_0}), \tag{3.23}$$

where

$$\mathbb{T}_{\tau,b} := \left\{T \in \mathbb{T}_\tau : \mu(4T) \geq 2^{10-k\alpha_0'/2-\alpha(j-k)}\right\}, \quad \mathbb{T}_{\tau,g} = \mathbb{T}_\tau \setminus \mathbb{T}_{\tau,b},$$

and the sum in (3.23) *converges in* $L^\infty(\mathbb{R}^3)$.

In the proof of the main theorem, only the behaviour of μ on tubes of radius at least 2^{-j_0} is considered, so there is no loss in convolving μ with the bump function above, and this (crucially) localises the frequencies to the ball of radius $\approx 2^{j_0}$.

The precise exponent in the "bad" part μ_b is defined as above in such a way that the average L^1 norm of the measures $\pi_{\theta\#}\mu_b$ can be controlled by re-using Definition 4.1, using the strategy below.

Proof of Lemma 3.5. Most of the lemma follows by defining μ_b as in (3.23) and by defining $\mu_g = \mu * \phi_{j_0} - \mu_b$; the only nontrivial thing to check is that the sum in (3.23) converges in L^∞. For this it suffices to show that for any T with $\tau(T) \in \Lambda_{j,k}$ and $j \geq j_0$,

$$\|M_T(\mu * \phi_{j_0})\|_{L^\infty} \lesssim_N 2^{(j_0-j)N},$$

for any $N \geq 0$. By Hausdorff–Young, the definition of M_T, and the assumption that μ is finite, it suffices to show that for $j \geq 2j_0$,

$$\left\|\psi_\tau \cdot \widehat{\phi_{j_0}}\right\|_1 \lesssim_N 2^{-jN}. \tag{3.24}$$

By the Schwartz property of ϕ,

$$\left\|\psi_\tau \cdot \widehat{\phi_{j_0}}\right\|_1 \leq \left\|\widehat{\phi_{j_0}}\right\|_{L^1(1.01\tau)} \lesssim_N \frac{m(\tau)}{2^{(j-j_0)N}},$$

where $m(\tau)$ denotes the Lebesgue measure of τ. By replacing N with $3N$, this gives (3.24) and proves the lemma. ∎

Lemma 3.6. *Let* $\delta > 0$. *Let* $\varepsilon, \varrho > 0$ *and* $J_1 \geq 0$ *be parameters ensuring the existence of the partition of unity in Lemma* 3.3. *If* $J \geq J_1$ *is sufficiently large, and if* $\tau \in \Lambda_{j,k} \cap \Lambda^{J,o}$*, then*

$$\|M_T(\mu * \phi_{j_0})\|_1 \lesssim_N 2^{3j\delta}\mu(2T) + \min\{2^{-JN}, 2^{-(j-j_0)N}\}\mu(\mathbb{R}^3),$$

for any $T \in \mathbb{T}_\tau$ *and any* $N \geq 1$.

Proof. If $j > j_0 + J/10$, the inequality follows by Cauchy–Schwarz, Plancherel and the Schwartz decay of ϕ. Assume then that $j \leq j_0 + J/10$. By the definition of $M_T(\mu * \phi_{j_0})$,

$$\begin{aligned}\|M_T\mu\|_1 \leq &\int_{1.5T}\int_{y+2^{j\delta}\hat{\tau}} \left|\widetilde{\psi_\tau}(x-y)\right| dx\, d(\mu * \phi_{j_0})(y)\\ &+ \int_{1.5T}\int_{T\setminus(y+2^{j\delta}\hat{\tau})} \left|\widetilde{\psi_\tau}(x-y)\right| dx\, d(\mu * \phi_{j_0})(y)\\ &+ \int_{\mathbb{R}^3\setminus(1.5T)}\int_{T} \left|\widetilde{\psi_\tau}(x-y)\right| dx\, d(\mu * \phi_{j_0})(y),\end{aligned}$$

where $\hat{\tau}$ is the "dual" box to τ centred at the origin; with axes parallel to τ but reciprocal side lengths. The first integral is $\lesssim 2^{3j\delta}(\mu * \phi_{j_0})(1.5T)$, which is smaller than $2^{3j\delta}\mu(2T)$ since $k > J$ and $j \leq j_0 + J/10$. The second integral is $\lesssim_N 2^{-jN}(\mu * \phi_{j_0})(1.5T)$ by Lemma 3.3 and repeated integration by parts. Similarly, by Lemma 3.3 and repeated integration by parts, the third integral is

$$\lesssim_{N'} \int_{\mathbb{R}^3\setminus(1.5T)} \operatorname{dist}(y,T)^{-N'}\, d\mu(y) \lesssim 2^{-jN}\mu(\mathbb{R}^3),$$

if N' is chosen large enough. This proves the lemma. ∎

Lemma 3.7. *Let $\delta > 0$. Let $\varepsilon, \varrho > 0$ and $J_1 \geq 0$ be parameters ensuring the existence of the partition of unity in Lemma 3.3, and let $J \geq J_1$. Then there exists $K_1 \geq 0$ such that if $\tau \in \Lambda_{j,k} \cap \Lambda^J$ and*

$$|\theta - \theta_\tau| \geq 2^{-k(1/2-\delta)}, \quad \theta \in [a,b], \tag{3.25}$$

then for $k \geq K_1$, $N \geq 1$, $T \in \mathbb{T}_\tau$ and for $f \in L^1(\mathbb{R}^3)$,

$$\|\pi_{\theta\#} M_T f\|_{L^1(\mathbb{R}^3,\mathcal{H}^2)} \lesssim_N 2^{-kN} m(\tau)\|f\|_1.$$

Proof. By identifying the complex measure $\pi_{\theta\#}M_T f$ with its Radon–Nikodym derivative with respect to $\mathcal{H}^2$,

$$\pi_{\theta\#}M_T f(x) = \int_{\mathbb{R}^3} f(y)\Big[\int_{\mathbb{R}^3}\psi_\tau(\xi)e^{2\pi i\langle\xi,x-y\rangle}\Big[\int_{\mathbb{R}}\eta_T(x+t\gamma(\theta))e^{2\pi i t\langle\xi,\gamma(\theta)\rangle}\,dt\Big]d\xi\Big]dy,$$

for any $x \in \gamma(\theta)^\perp$. It therefore suffices to show that

$$\left|\int_{\mathbb{R}}\eta_T(x+t\gamma(\theta))\,e^{2\pi i t\langle\xi,\gamma(\theta)\rangle}\,dt\right| \lesssim_N 2^{-kN}, \quad \forall \xi \in \tau,\ \forall x \in \mathbb{R}^3.$$

By repeated integration by parts, it suffices to show that for all $t \in \mathbb{R}$, $l \geq 1$, $\xi \in \tau$ and $x \in \mathbb{R}^3$,

$$\left|\left(\frac{d}{dt}\right)^l \eta_T(x+t\gamma(\theta))\right| \lesssim_l 2^{-lk\delta}\,|\langle\xi,\gamma(\theta)\rangle|^l. \tag{3.26}$$

Define $\varepsilon = |\theta - \theta_\tau|$. The assumed lower bound (3.25) on ε, together with (3.16) and the assumption that γ is C^2 with $\det(\gamma, \gamma', \gamma'')$ nonvanishing, yields

$$|\langle \xi, \gamma(\theta)\rangle| \gtrsim 2^j \varepsilon^2, \quad \forall \xi \in \tau,$$

provided k is sufficiently large. Hence to prove (3.26) it suffices to show that

$$\left|\left(\frac{d}{dt}\right)^l \eta_T(x + t\gamma(\theta))\right| \lesssim_l (2^{j-k\delta}\varepsilon^2)^l \quad \forall l \geq 1.$$

By Lemma 3.3 and (3.16),

$$\begin{aligned}
&\left|\left(\frac{d}{dt}\right)^l \eta_T(x + t\gamma(\theta))\right| \\
&\quad \lesssim_l \left|\langle \gamma(\theta), \gamma(\theta_\tau)\rangle\, 2^{j-k-k\delta}\right|^l + \left|\langle \gamma(\theta), \gamma'(\theta_\tau)\rangle\, 2^{j-k/2-k\delta}\right|^l \\
&\qquad + \left|\langle \gamma(\theta), (\gamma \times \gamma')(\theta_\tau)\rangle\, 2^{j-k/2-k\delta}\right|^l \\
&\quad \lesssim (2^{j-k-k\delta})^l + (\varepsilon\, 2^{j-k/2-k\delta})^l + (\varepsilon^2\, 2^{j-k/2-k\delta})^l \lesssim (\varepsilon^2\, 2^{j-k\delta})^l,
\end{aligned}$$

where the last inequality follows from the assumed lower bound (3.25) on ε. ∎

4. Proof of the main theorem

For a Borel measure μ on $\mathbb{R}^3$ and $\alpha \in [0, 3]$, let

$$c_\alpha(\mu) = \sup_{\substack{x \in \mathbb{R}^3 \\ r > 0}} \frac{\mu(B(x, r))}{r^\alpha}.$$

Definition 4.1. Let $\gamma \colon [a, b] \to S^2$ be a function and let $\alpha \in [0, 3]$. Define $\alpha_0 = \alpha_0(\alpha, \gamma)$ to be the supremum over all $\alpha^* \geq 0$ such that there exists $\delta = \delta(\alpha, \alpha^*, \gamma) > 0$ and $C = C(\alpha, \alpha^*, \gamma) > 0$ such that

$$\int_a^b (\pi_{\theta\#}\mu)\Big(\bigcup_{D \in \mathbb{D}_\theta} D\Big)\, d\theta \leq C\mu(\mathbb{R}^3) R^{-\delta}, \tag{4.1}$$

for all Borel measures μ on the unit ball with $c_\alpha(\mu) \leq 1$, for any $R \geq 1$, and for any collection of sets $\{\mathbb{D}_\theta : \theta \in [a, b]\}$ such that the integrand of (4.1) is measurable, where, for each $\theta \in [a, b]$, $\mathbb{D}_\theta$ is a disjoint set of at most $\mu(\mathbb{R}^3) R^{\alpha^*/2}$ discs in $\pi_\theta(\mathbb{R}^3)$ of radius $R^{-1/2}$.

The proof of the main theorem will be broken up into several separate lemmas. First, Lemma 4.2 deals with the contribution from the "bad" part of the measure, whilst Lemmas 4.3, 4.4, 4.5, 4.6 and 4.7 deal with the "good" part. Lemma 4.8 converts everything into a lower bound for α_0 in Definition 4.1, which is then used to obtain the main theorem.

Lemma 4.2. *Suppose that $\gamma \colon [a, b] \to S^2$ is a C^2 unit speed curve satisfying (3.1), (3.2) and (3.3) on $[a, b]$, and let $[\widetilde{a}, \widetilde{b}] \subseteq (a, b)$. Let $\alpha \in [0, 3]$ and $\epsilon \in (0, 1/100)$. If $\alpha_0' \in [0, \alpha_0(\alpha, 1, \gamma|_{[\widetilde{a}, \widetilde{b}]}))$, then there exists $\delta' > 0$ such that for any $\delta \in (0, \delta']$, there is a positive* *integer J_0 such that for all $j_0 \geq J \geq J_0$ with $J \in [(j_0\epsilon)/1000, 1000 j_0 \epsilon]$ and for all Borel measures μ on the unit ball with $c_\alpha(\mu) \leq 1$,*

$$\int_{\widetilde{a}}^{\widetilde{b}} \int |\pi_{\theta\#}\mu_b| \, d\mathcal{H}^2 \, d\theta \leq 2^{-J\delta'} \mu(\mathbb{R}^3),$$

where $\mu_b = \mu_{b,j_0,J,\alpha,\epsilon,\delta,\alpha_0'}$ is defined by (3.23) with respect to $\gamma\colon [a,b] \to S^2$.

Proof. Let $\delta'' = \delta''(\alpha, \alpha_0') \in (0, 1/100)$ be an exponent that works in (4.1) with α^* replaced by $\alpha_0' + 100\delta'$ and with $A = 1$, for some positive δ', and after taking δ' smaller if necessary assume that $\delta' \leq (\delta''\epsilon^2)/100$. Let $\delta \in (0, \delta']$ be given. Let j_0 and J be such that $j_0 \geq J \geq J_0$, where J_0 is implicity chosen sufficiently large (depending on δ) so that the argument below holds. By Lemma 3.5,

$$\begin{aligned}
&\int_{\widetilde{a}}^{\widetilde{b}} \int |\pi_{\theta\#}\mu_b| \, d\mathcal{H}^2 \, d\theta && (4.2)\\
&\leq \int_{\widetilde{a}}^{\widetilde{b}} \sum_{j>2J} \sum_{\substack{k\in[j\epsilon,j]\\ k>J}} \sum_{\tau\in\Lambda_{j,k}} \sum_{T\in\mathbb{T}_{\tau,b}} \int |\pi_{\theta\#} M_T(\mu * \phi_{j_0})| \, d\mathcal{H}^2 \, d\theta \\
&= \int_{\widetilde{a}}^{\widetilde{b}} \sum_{j>2J} \sum_{\substack{k\in[j\epsilon,j]\\ k>J}} \sum_{\substack{\tau\in\Lambda_{j,k}:\\ |\theta_\tau-\theta|<2^{k(-1/2+\delta)}}} \sum_{T\in\mathbb{T}_{\tau,b}} \int |\pi_{\theta\#} M_T(\mu * \phi_{j_0})| \, d\mathcal{H}^2 \, d\theta && (4.3)\\
&+ \int_{\widetilde{a}}^{\widetilde{b}} \sum_{j>2J} \sum_{\substack{k\in[j\epsilon,j]\\ k>J}} \sum_{\substack{\tau\in\Lambda_{j,k}:\\ |\theta_\tau-\theta|\geq 2^{k(-1/2+\delta)}}} \sum_{T\in\mathbb{T}_{\tau,b}} \int |\pi_{\theta\#} M_T(\mu * \phi_{j_0})| \, d\mathcal{H}^2 \, d\theta. && (4.4)
\end{aligned}$$

By Lemma 3.7, the contribution from (4.4) is

$$\lesssim_{\delta,\epsilon} 2^{-J} \mu(\mathbb{R}^3).$$

If J_0 is sufficiently large, then by Lemma 3.6 the contribution from (4.3) is

$$\begin{aligned}
&\lesssim 2^{-J}\mu(\mathbb{R}^3) + \sum_{j>2J} \sum_{\substack{k\in[j\epsilon,j]\\ k>J}} \int_{\widetilde{a}}^{\widetilde{b}} \sum_{\substack{\tau\in\Lambda_{j,k}:\\ |\theta_\tau-\theta|<2^{k(-1/2+\delta)}}} \sum_{T\in\mathbb{T}_{\tau,b}} 2^{3j\delta} \mu(2T) \, d\theta \\
&\lesssim 2^{-J}\mu(\mathbb{R}^3) + \sum_{j>2J} \sum_{\substack{k\in[j\epsilon,j]\\ k>J}} 2^{10j\delta} (\mathcal{H}^1 \times \mu)(B_{j,k}), && (4.5)
\end{aligned}$$

where $\mathcal{H}^1$ is the Lebesgue measure on $[a,b]$,

$$B_{j,k} = \left\{(\theta, x) \in [\widetilde{a}, \widetilde{b}] \times \mathbb{R}^3 : x \in B_{j,k}(\theta)\right\},$$

and

$$B_{j,k}(\theta) = \bigcup_{\substack{\tau\in\Lambda_{j,k}:\\ |\theta_\tau-\theta|<2^{k(-1/2+\delta)}}} \bigcup_{T\in\mathbb{T}_{\tau,b}} 2T.$$

For fixed j and k, let $\{B_l\}_l$ be a finitely overlapping cover of $B_3(0,1)$ by balls of radius $2^{-(j-k)}$. For each θ and l, let

$$B_{j,k,l}(\theta) = \bigcup_{\substack{\tau\in\Lambda_{j,k}:\\ |\theta_\tau-\theta|<2^{k(-1/2+\delta)}}} \bigcup_{\substack{T\in\mathbb{T}_{\tau,b}:\\ 2T\cap B_l\neq\emptyset}} 2T,$$

and let

$$B_{j,k,l} = \left\{(\theta,x)\in[\widetilde{a},\widetilde{b}]\times\mathbb{R}^3 : x\in B_{j,k,l}(\theta)\right\}.$$

Let $\mu_{j,k}$ be the pushforward of μ under $x\mapsto 2^{j-k-2k\delta}x$. Then

$$\begin{aligned}(\mathcal{H}^1\times\mu)(B_{j,k}) &\leq \sum_l (\mathcal{H}^1\times\mu)(B_{j,k,l})\\ &= \sum_l 2^{-\alpha(j-k-2k\delta)}\,(\mathcal{H}^1\times\widetilde{\mu}_{j,k,l})(B'_{j,k,l}),\end{aligned} \tag{4.6}$$

where

$$B'_{j,k,l} = \left\{(\theta,x)\in[\widetilde{a},\widetilde{b}]\times\mathbb{R}^3 : x\in 2^{j-k-2k\delta}B_{j,k,l}(\theta)\right\},$$

and $\widetilde{\mu}_{j,k,l} = 2^{\alpha(j-k-2k\delta)}\cdot\mu_{j,k}\chi_{\widetilde{B}_l}$, where

$$\widetilde{B}_l = \{2^{j-k-2k\delta}b_l + y : |y|\leq 1\},$$

with b_l the centre of B_l.

Up to translation and finite overlaps, $B'_{j,k,l}$ and $\widetilde{\mu}_{j,k,l}$ satisfy the conditions of Definition 4.1; for each θ, the set $2^{j-k-2k\delta}B_{j,k,l}(\theta)$ is contained in a union of tubes of radius $2^{-k/2}$ parallel to $\gamma(\theta)$, with the number of tubes $\lesssim 2^{k(\alpha_0'+100\delta')/2}\widetilde{\mu}_{j,k,l}(\mathbb{R}^3)$, such that each tube overlaps $\lesssim 2^{10k\delta}$ of the others. Moreover, $c_\alpha(\widetilde{\mu}_{j,k,l})\leq 1$ and $\widetilde{\mu}_{j,k,l}$ is supported in a ball of radius 1. Hence

$$(\mathcal{H}^1\times\widetilde{\mu}_{j,k,l})(B'_{j,k,l}) \lesssim \widetilde{\mu}_{j,k,l}(\mathbb{R}^3)\,2^{-k\delta''/4} \leq 2^{\alpha(j-k-2k\delta)}\,\mu_{j,k}(\widetilde{B}_l)\,2^{-k\delta''/4},$$

Putting this into (4.6) yields

$$(\mathcal{H}^1\times\mu)(B_{j,k}) \lesssim 2^{-k\delta''/4}\,\mu(\mathbb{R}^3),$$

Substituting this into (4.5) and then (4.2) gives

$$\int_{\widetilde{a}}^{\widetilde{b}}\int |\pi_{\theta\#}\mu_b|\,d\mathcal{H}^2\,d\theta \lesssim 2^{-(J\epsilon\delta'')/8}\,\mu(\mathbb{R}^3),$$

and hence

$$\int_{\widetilde{a}}^{\widetilde{b}}\int |\pi_{\theta\#}\mu_b|\,d\mathcal{H}^2\,d\theta \leq 2^{-J\delta'}\,\mu(\mathbb{R}^3),$$

provided J_0 is sufficiently large. This proves the lemma. ∎

Lemma 4.3. *Let $\gamma : [a,b] \to S^2$ be a C^2 unit speed curve satisfying (3.1), (3.2) and (3.3) on $[a,b]$. Then there exists a constant $\sigma > 0$ depending on γ, and for each $\epsilon \in (0,1)$ an integer $J_0 \geq 0$ depending on γ and ϵ, such that for any $\tau \in \Lambda^J \cap \Lambda_{j,k}$ with $J \geq J_0$ and $k \in [j\epsilon, j]$,*

$$\text{dist}(1.01\tau, \eta_1\gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta)) \geq \sigma \max(|\eta|^{1-10\epsilon}, 2^{j(1-10\epsilon)}), \tag{4.7}$$

for all $\eta \in \mathbb{R}^2$ with $|\eta_1| \geq |\eta_2|^{1-\epsilon}$ and for all $\theta \in [a,b]$ with $|\theta - \theta_\tau| \leq \sigma$.

Proof. If either $|\eta| < 2^{j-10}$ or $|\eta| > 2^{j+10}$, this is immediate, so it may be assumed that $2^{j-10} \leq |\eta| \leq 2^{j+10}$. Let $x = \lambda_1(\gamma \times \gamma')(\theta_\tau) + \lambda_2\gamma'(\theta_\tau) + \lambda_3\gamma(\theta_\tau) \in 1.01\tau$, where $|\lambda_1| \sim 2^j$, $|\lambda_2| \lesssim 2^{j-k/2}$ and $|\lambda_3| \sim 2^{j-k}$. Suppose first that $|\theta_\tau - \theta| \leq 2^{-3j\epsilon}$. If $k \leq 5j\epsilon$, then $|\theta_\tau - \theta| \ll 2^{-k/2}$ and hence $|\langle x, \gamma(\theta)\rangle| \gtrsim 2^{j-k} \geq 2^{j(1-6\epsilon)}$, which implies that

$$\text{dist}(x, \eta_1\gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta)) \gtrsim \max(|\eta|^{1-10\epsilon}, 2^{j(1-10\epsilon)}). \tag{4.8}$$

If $k > 5j\epsilon$, then $|\langle x, \gamma'(\theta)\rangle| \leq 2^{j(1-2\epsilon)}$, which, due to the condition $|\eta_1| \geq |\eta_2|^{1-\epsilon}$, implies (4.8). It remains to consider the possibility that $|\theta - \theta_\tau| > 2^{-3j\epsilon}$, in which case

$$\langle \lambda_1(\gamma \times \gamma')(\theta_\tau), \gamma(\theta)\rangle = \frac{\lambda_1}{2}\left[(\theta_\tau - \theta)^2 \det(\gamma(\theta_\tau), \gamma'(\theta_\tau), \gamma''(\theta_\tau)) + o(|\theta_\tau - \theta|^2)\right],$$

which gives $|\langle x, \gamma(\theta)\rangle| \gtrsim 2^{j(1-6\epsilon)}$ for $|\theta - \theta_\tau| < c_\gamma$, and this implies (4.8). ∎

Lemma 4.4. *Let $\gamma: [a,b] \to S^2$ be a C^2 unit speed curve satisfying (3.1), (3.2) and (3.3) on $[a,b]$, let $[\widetilde{a}, \widetilde{b}] \subseteq (a,b)$, and assume that $|\widetilde{b} - \widetilde{a}| \leq \sigma$, where σ is a constant that works in Lemma 4.3. Let $\epsilon \in (0,1)$, $\delta > 0$, let $\alpha \in [0,3]$ and let $\alpha_0' > 0$. Then there exists a positive integer J_0 such that for all $j_0 \geq J \geq J_0$ and for all finite Borel measures μ on the unit ball,*

$$\int_{\widetilde{a}}^{\widetilde{b}} \int_{\{|\eta_1| \geq |\eta_2|^{1-\epsilon}\} \cap B(0, 2^{j_0(1+\delta)})} |\widehat{\mu_g}(\eta_1\gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta))|^2 \, d\eta \, d\theta$$
$$\leq 8\mu(\mathbb{R}^3)^2 + 8\int_{\widetilde{a}}^{\widetilde{b}} \int_{\{|\eta_1| \geq |\eta_2|^{1-\epsilon}\} \cap B(0, 2^{j_0(1+\delta)})} |\widehat{\mu}(\eta_1\gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta))|^2 \, d\eta \, d\theta,$$

where $\mu_g = \mu_{g, j_0, J, \alpha, \epsilon, \delta, \alpha_0'}$ is defined by Lemma 3.5.

Proof. This follows from the triangle inequality, Lemma 4.3, the definition of μ_b, and the rapid decay of $\mathcal{F}(M_T(\mu * \phi_{j_0}))$ outside $1.01\tau(T)$. ∎

Lemma 4.5. *Let $\alpha \in [0,3]$, $\alpha_0' \in (0,3]$ and let $\epsilon \in (0,1/2)$. Suppose that $\gamma: [a,b] \to S^2$ is a C^3 unit speed curve satisfying (3.1), (3.2) and (3.3), on $[a,b]$, and let $[\widetilde{a}, \widetilde{b}] \subseteq (a,b)$. Let*

$$A_{j,k} = [B(0, 2^{j+1}) \setminus B(0, 2^j)] \cap \{2^{j-k/2} \leq |\eta_1| < 2^{j+1-k/2}\}, \quad k < j,$$

and

$$A_{j,j} = [B(0, 2^{j+1}) \setminus B(0, 2^j)] \cap \{|\eta_1| < 2^{(j+1)/2}\}.$$

Then there exists $\delta_0 \in (0, \epsilon^{100})$*, and for any* $\delta \in (0, \delta_0]$ *a* $J_0 \geq 0$*, such that if* $J \geq J_0$*,* $j_0(1+\delta) \geq j \geq 3J$ *and* $j\epsilon \leq k \leq j$*, then for any Borel measure* μ *on* $B_3(0,1)$ *with* $c_\alpha(\mu) \leq 1$*,*

$$\int_{\tilde{a}}^{\tilde{b}} \int_{A_{j,k}} |\widehat{\mu_g}(\eta_1\gamma'(\theta)+\eta_2(\gamma\times\gamma')(\theta))|^2 \, d\eta \, d\theta \leq \mu(\mathbb{R}^3)\, 2^{100j\epsilon + j(2-\alpha)+k(-1/2+2\alpha/3-\alpha_0'/3)}, \tag{4.9}$$

where $\mu_b = \mu_{b,j_0,J,\alpha,\epsilon,\delta,\alpha_0'}$ *is defined by* (3.23) *with respect to* $\gamma\colon [a,b] \to S^2$*.*

Proof. Suppose first that $k < j$. By the wave packet decomposition and Lemma 3.2, there is a constant C such that the left-hand side of (4.9) is

$$\leq C\, 2^{-100j} \mu(\mathbb{R}^3)^2 + \int_{\tilde{a}}^{\tilde{b}} \int_{A_{j,k}} \Bigg| \sum_{|j'-j|\leq C} \sum_{\substack{|k'-k|\leq C \\ k' \leq j'}} \sum_{\tau \in \Lambda_{j',k'}} \sum_{T \in \mathbb{T}_{\tau,g}} \mathcal{F}(M_T(\mu * \phi_{j_0}))(\eta_1\gamma'(\theta) + \eta_2(\gamma\times\gamma')(\theta)) \Bigg|^2 d\eta\, d\theta. \tag{4.10}$$

Let $\tilde{\mu}$ be defined by setting $\widehat{\tilde{\mu}}$ equal to the function inside the modulus signs above. Let $\{B_m\}_m$ be a finitely overlapping cover of $\mathbb{R}^3$ by balls of radius 2^{k-j}, and let $\{\vartheta_m\}_m$ be a corresponding subordinate smooth partition of unity. Then by changing variables (see (4.22)–(4.24) below) and by Plancherel,

$$(4.10) \lesssim C 2^{-100j} \mu(\mathbb{R}^3)^2 + 2^{k/2-j} \sum_m \int_{\mathbb{R}^3} |\widehat{\tilde{\mu}\vartheta_m}|^2. \tag{4.11}$$

For each m and for arbitrarily large N, the integral in (4.11) satisfies

$$\int_{\mathbb{R}^3} |\widehat{\tilde{\mu}\vartheta_m}|^2 \lesssim C_N\, 2^{-kN} \mu(\mathbb{R}^3)^2 + \sum_{|j'-j|\leq C} \sum_{\substack{|k'-k|\leq C \\ k'\leq j'}} \sum_{\tau\in\Lambda_{j',k'}} \sum_{\substack{T\in\mathbb{T}_{\tau,g} \\ T\cap B_m \neq \emptyset}} \int |M_T(\mu * \phi_{j_0})|^2, \tag{4.12}$$

which follows from the "essential orthogonality" of wave packets.

A similar inequality will be shown in the case $k = j$. By the wave packet decomposition and Lemma 3.2, the left-hand side of (4.9) is in this case

$$\leq C\, 2^{-100j} \mu(\mathbb{R}^3)^2 + \int_{\tilde{a}}^{\tilde{b}} \int_{A_{j,j}} \Bigg| \sum_{|j'-j|\leq C} \sum_{\substack{|k'-j|\leq C \\ k' \leq j'}} \sum_{\tau \in \Lambda_{j',k'}} \sum_{T \in \mathbb{T}_{\tau,g}} \mathcal{F}(M_T(\mu * \phi_{j_0}))(\eta_1\gamma'(\theta) + \eta_2(\gamma\times\gamma')(\theta)) \Bigg|^2 d\eta\, d\theta.$$

Let $\widetilde{\mu}$ be defined by setting $\widehat{\widetilde{\mu}}$ equal to the function inside the modulus signs above. By the finite overlapping property of the sets τ,

$$\begin{aligned}(4.13)\quad &\int_{\widetilde{a}}^{\widetilde{b}} \int_{A_{j,j}} |\widehat{\widetilde{\mu}}(\eta_1\gamma'(\theta) + \eta_2(\gamma\times\gamma')(\theta))|^2 \, d\eta \, d\theta \\ &\leq C\,2^{-100j}\mu(\mathbb{R}^3)^2 + \sum_{|j'-j|\leq C} \sum_{\substack{|k'-j|\leq C \\ k'\leq j'}} \sum_{\tau\in\Lambda_{j',k'}} \\ &\times \int_{\widetilde{a}}^{\widetilde{b}} \int \Bigg| \sum_{\substack{T\in\mathbb{T}_{\tau,g} \\ T\cap B(0,10)\neq\emptyset}} \sum_{T'\subseteq T} \mathcal{F}(M_{T'}(\mu*\phi_{j_0}))(\eta_1\gamma'(\theta)+\eta_2(\gamma\times\gamma')(\theta))\Bigg|^2 d\eta\, d\theta,\end{aligned}$$

where the sets T' cover T with planks of dimensions $\approx 1\times 2^{-j/2}\times 2^{-j}$, with long direction parallel to $\gamma(\theta_{\tau(T)})$, medium direction parallel to $\gamma'(\theta_{\tau(T)})$ and short direction parallel to $(\gamma\times\gamma')(\theta_{\tau(T)})$, and $M_{T'}\mu = \eta_{T'}M_T\mu$, where $\{\eta_{T'}\}_{T'}$ is a smooth partition of unity subordinate to the cover $\{T'\}_{T'}$. By the 2-dimensional Plancherel theorem followed by the uncertainty principle (bounding the L^2 norm by the L^∞ norm, followed by Hausdorff–Young and Cauchy–Schwarz),

$$\begin{aligned}(4.13) &\lesssim C\,2^{-100j}\mu(\mathbb{R}^3)^2 \\ &\quad + 2^{10j\delta - j/2} \sum_{|j'-j|\leq C} \sum_{\substack{|k'-j|\leq C \\ k'\leq j'}} \sum_{\tau\in\Lambda_{j',k'}} \sum_{\substack{T\in\mathbb{T}_{\tau,g} \\ T\cap B(0,10)\neq\emptyset}} \int |M_T(\mu*\phi_{j_0})|^2.\end{aligned}$$

This shows that (4.11)–(4.12) holds also in the case $k = j$, although possibly with a $2^{10j\delta}$ loss, and with $\{B_m\}_m$ equal to the cover of $B(0,1)$ by the single ball $B(0,10)$ in that case. The remainder of the proof will therefore cover both cases simultaneously ($k \leq j$).

Applying Plancherel to the non-negligible term in the right-hand side of (4.12) gives

$$\begin{aligned}(4.14)\quad &\sum_{|j'-j|\leq C} \sum_{\substack{|k'-k|\leq C \\ k'\leq j'}} \sum_{\tau\in\Lambda_{j',k'}} \sum_{\substack{T\in\mathbb{T}_{\tau,g} \\ T\cap B_m\neq\emptyset}} \int |M_T(\mu*\phi_{j_0})|^2 \\ &= \int \sum_{|j'-j|\leq C} \sum_{\substack{|k'-k|\leq C \\ k'\leq j'}} \sum_{\tau\in\Lambda_{j',k'}} \sum_{\substack{T\in\mathbb{T}_{\tau,g} \\ T\cap B_m\neq\emptyset}} \left([\eta_T M_T(\mu*\phi_{j_0})] * \widetilde{\psi_\tau}\right) d(\mu*\phi_{j_0}).\end{aligned}$$

Let ν be the restriction of $\mu*\phi_{j_0}$ to $2^{10k\delta}B_m$. By Cauchy–Schwarz, the right-hand side of (4.14) is

$$\begin{aligned}(4.15)\quad &\leq C_N\,2^{-kN}\mu(\mathbb{R}^3)^2 + \mu(2^{100k\delta}B_m)^{1/2} \\ &\times \left(\int \Bigg| \sum_{|j'-j|\leq C} \sum_{\substack{|k'-k|\leq C \\ k'\leq j'}} \sum_{\tau\in\Lambda_{j',k'}} \sum_{\substack{T\in\mathbb{T}_{\tau,g} \\ T\cap B_m\neq\emptyset}} [\eta_T M_T(\mu*\phi_{j_0})] * \widetilde{\psi_\tau} \Bigg|^2 d\nu \right)^{1/2}.\end{aligned}$$

Let

$$f_T = [\eta_T M_T(\mu * \phi_{j_0})] * \widecheck{\psi_\tau} \quad \text{and} \quad f = \sum_{|j'-j|\leq C} \sum_{\substack{|k'-k|\leq C \\ k'\leq j'}} \sum_{\tau\in\Lambda_{j',k'}} \sum_{\substack{T\in\mathbb{T}_{\tau,g} \\ T\cap B_m\neq\emptyset}} f_T.$$

The integral in (4.15) satisfies

$$\int |f|^2 d\nu \lesssim \int |f|^2\, d(\nu * \zeta),$$

where $\zeta(x) = \frac{2^{3j}}{1+2^{jN}|x|^N}$ for some very large N. This follows from the uncertainty principle since $\widehat{f}$ is supported in a ball of radius $\lesssim 2^j$. By dyadic pigeonholing, there is a subset

$$\mathbb{W} \subseteq \bigcup_{|j'-j|\leq C} \bigcup_{\substack{|k'-k|\leq C \\ k'\leq j'}} \bigcup_{\tau\in\Lambda_{j',k'}} \{T \in \mathbb{T}_{\tau,g} : T\cap B_m \neq \emptyset\},$$

such that $\|f_T\|_2$ is constant up to a factor of 2 as T varies over $\mathbb{W}$, and

$$\int |f|^2\, d(\nu*\zeta) \lesssim \log(2^j)^2 \int \Big|\sum_{T\in\mathbb{W}} f_T\Big|^2 d(\nu*\zeta) + 2^{-100j} \sum_{|j'-j|\leq C} \sum_{\substack{|k'-k|\leq C \\ k'\leq j'}} \sum_{\tau\in\Lambda_{j',k'}} \sum_{\substack{T\in\mathbb{T}_{\tau,g} \\ T\cap B_m\neq\emptyset}} \|f_T\|_2^2.$$

By pigeonholing again and by Hölder's inequality, there is a disjoint union Y of balls Q of radius $2^{-j+k/2}$, such that

$$\int \Big|\sum_{T\in\mathbb{W}} f_T\Big|^2 d(\nu*\zeta) \lesssim \log(2^k)\, \|f\|_{L^6(Y)}^2 \Big(\int_Y (\nu*\zeta)^{3/2}\Big)^{2/3}, \tag{4.16}$$

and such that each $Q\subseteq Y$ intersects a number $\# \in [M, 2M)$ boxes $3T$ as T varies over $\mathbb{W}$, for some dyadic number M. By rescaling and then applying the refined Strichartz inequality (Theorem 2.2) with $p = 6$, the first factor in (4.16) satisfies

$$\|f\|_{L^6(Y)} \leq C_{\epsilon,\delta}\, 2^{j-k/2+k\epsilon} \Big(\frac{M}{|\mathbb{W}|}\Big)^{1/3} \Big(\sum_{T\in\mathbb{W}} \|f_T\|_2^2\Big)^{1/2}.$$

For the second factor in (4.16), the assumed inequality $c_\alpha(\mu) \leq 1$ implies that $\|\nu*\zeta\|_\infty \lesssim 2^{j(3-\alpha)}$. Hence

$$\int_Y (\nu*\zeta)^{3/2} \lesssim \frac{2^{\frac{j}{2}(3-\alpha)}}{M} \sum_{T\in\mathbb{W}} (\nu*\zeta)(3.5T) \leq \frac{2^{\frac{j}{2}(3-\alpha)}}{M} \sum_{T\in\mathbb{W}} \mu(4T) + C_N 2^{-jN}\mu(\mathbb{R}^3)$$
$$\lesssim \frac{2^{\frac{j}{2}(3-\alpha)}|\mathbb{W}|2^{-k\alpha_0'/2-\alpha(j-k)}}{M} + C_N 2^{-jN}.$$

Hence,

$$(4.16) \lesssim \log(2^k)\, 2^{j(3-\alpha)+k(-1+2\alpha/3-\alpha_0'/3+\epsilon)} \sum_{T\in\mathbb{W}} \|f_T\|_2^2. \tag{4.17}$$

By Plancherel, $\|f_T\|_2 \le \|M_T(\mu * \phi_{j_0})\|_2$ for every T. Assuming the tail terms are not dominant, substituting into (4.17) and then into (4.14) yields

$$\begin{aligned}\sum_{|j'-j|\le C} \sum_{\substack{|k'-k|\le C\\ k'\le j'}} \sum_{\tau\in\Lambda_{j',k'}} \sum_{\substack{T\in\mathbb{T}_{\tau,g}\\ T\cap B_m\neq\emptyset}} \int |M_T(\mu*\phi_{j_0})|^2 &\\ \lesssim \mu(2^{100k\delta}B_m)^{1/2}\, 2^{10j\epsilon+\frac{1}{2}[j(3-\alpha)+k(-1+2\alpha/3-\alpha_0'/3)]}&\\ \times\left(\sum_{|j'-j|\le C} \sum_{\substack{|k'-k|\le C\\ k'\le j'}} \sum_{\tau\in\Lambda_{j',k'}} \sum_{\substack{T\in\mathbb{T}_{\tau,g}\\ T\cap B_m\neq\emptyset}} \int |M_T(\mu*\phi_{j_0})|^2\right)^{1/2}&.\end{aligned}$$

By cancelling the common factors, this yields

$$\begin{aligned}\sum_{|j'-j|\le C} \sum_{\substack{|k'-k|\le C\\ k'\le j'}} \sum_{\tau\in\Lambda_{j',k'}} \sum_{\substack{T\in\mathbb{T}_{\tau,g}\\ T\cap B_m\neq\emptyset}} \int |M_T(\mu*\phi_{j_0})|^2 &\\ \lesssim \mu(2^{100k\delta}B_m)\, 2^{20j\epsilon+j(3-\alpha)+k(-1+2\alpha/3-\alpha_0'/3)}&.\end{aligned}$$

By substituting back into (4.11)–(4.12) and summing over m, this proves the lemma (the m for which the tail terms dominate make a negligible contribution to the sum). ■

Lemma 4.6. *Let $\alpha\in[0,3]$, $\alpha_0'>0$, and let $\epsilon,\delta>0$ with $\delta<\epsilon/1000$. Let $\gamma\colon[a,b]\to S^2$ be a C^2 unit speed curve satisfying (3.1), (3.2) and (3.3). Then there is a constant C and a positive integer J_0 such that for all $j_0\ge J\ge J_0$ and for all finite Borel measures μ on the unit ball,*

$$\int_a^b \int_{B(0,2^{3J})} |\widehat{\mu_g}(\eta_1\gamma'(\theta)+\eta_2(\gamma\times\gamma')(\theta))|^2\, d\eta\, d\theta \le C\,2^{CJ}\mu(\mathbb{R}^3)^2.$$

Proof. This follows from the trivial bound on each $M_T(\mu*\phi_{j_0})$, and the rapid decay of $\mathcal{F}(M_T(\mu*\phi_{j_0}))$ outside $1.01\tau(T)$, for each τ. ■

Lemma 4.7. *Suppose that $\gamma\colon[a,b]\to S^2$ is a C^3 unit speed curve satisfying (3.1), (3.2) and (3.3) on $[a,b]$. Let $[\widetilde{a},\widetilde{b}]\subseteq(a,b)$ be such that $|\widetilde{b}-\widetilde{a}|\le\sigma$, where σ is a constant that works in Lemma 4.3. Let $\alpha\in[0,3]$ and $\epsilon>0$. If $\alpha_0'\in(0,\alpha_0(\alpha,1,\gamma|_{[\widetilde{a},\widetilde{b}]}))$, then there exists $\delta_0>0$, and for any $\delta\in(0,\delta_0]$ a $J_0\ge0$, such that for all $j_0\ge 3J$ and $J\ge J_0$ with $J\in[(j_0\epsilon)/1000, 1000j_0\epsilon]$ and for all Borel measures μ on the unit ball with $c_\alpha(\mu)\le1$,*

$$\int_{\widetilde{a}}^{\widetilde{b}} \int |\pi_{\theta\#}\mu_g|^2\, d\mathcal{H}^2\, d\theta \le \mu(\mathbb{R}^3)\, 2^{j_0(\max\{0,2-\alpha,3/2-\alpha/3-\alpha_0'/3\}+10^4\epsilon)}. \tag{4.18}$$

Proof. By Plancherel,

$$\begin{aligned}(4.19)\qquad &\int_{\tilde{a}}^{\tilde{b}} \int |\pi_{\theta\#}\mu_g|^2 \, d\mathcal{H}^2 \, d\theta \\ &= \int_{\tilde{a}}^{\tilde{b}} \int_{B(0,2^{j_0(1+\delta)})} |\widehat{\mu_g}(\eta_1\gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta))|^2 \, d\eta \, d\theta \\ &\quad + \int_{\tilde{a}}^{\tilde{b}} \int_{\mathbb{R}^2 \setminus B(0,2^{j_0(1+\delta)})} |\widehat{\mu_g}(\eta_1\gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta))|^2 \, d\eta \, d\theta.\end{aligned}$$

The inequality

$$\int_{\tilde{a}}^{\tilde{b}} \int_{\mathbb{R}^2 \setminus B(0,2^{j_0(1+\delta)})} |\widehat{\mu_g}(\eta_1\gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta))|^2 \, d\eta \, d\theta \lesssim \mu(\mathbb{R}^3)^2$$

follows straightforwardly from the rapid decay of $\widehat{\phi_{j_0}}$ outside $B(0, 2^{j_0})$; see pp. 13–14 in [5] for a more detailed calculation of a similar inequality. The other term in (4.19) can be written as

$$\begin{aligned}(4.20)\qquad &\int_{\tilde{a}}^{\tilde{b}} \int_{B(0,2^{j_0(1+\delta)})} |\widehat{\mu_g}(\eta_1\gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta))|^2 \, d\eta \, d\theta \\ &= \int_{\tilde{a}}^{\tilde{b}} \int_{\{|\eta_1| \geq |\eta_2|^{1-\epsilon}\} \cap B(0,2^{j_0(1+\delta)})} |\widehat{\mu_g}(\eta_1\gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta))|^2 \, d\eta \, d\theta \\ &\quad + \int_{\tilde{a}}^{\tilde{b}} \int_{\{|\eta_1| < |\eta_2|^{1-\epsilon}\} \cap B(0,2^{j_0(1+\delta)})} |\widehat{\mu_g}(\eta_1\gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta))|^2 \, d\eta \, d\theta.\end{aligned}$$

By Lemma 4.4, the first term satisfies

$$\begin{aligned}(4.21)\qquad &\int_{\tilde{a}}^{\tilde{b}} \int_{\{|\eta_1| \geq |\eta_2|^{1-\epsilon}\} \cap B(0,2^{j_0(1+\delta)})} |\widehat{\mu_g}(\eta_1\gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta))|^2 \, d\eta \, d\theta \\ &\lesssim \mu(\mathbb{R}^3)^2 + \int_{\tilde{a}}^{\tilde{b}} \int_{\{|\eta_1| \geq |\eta_2|^{1-\epsilon}\} \cap B(0,2^{j_0(1+\delta)})} |\widehat{\mu}(\eta_1\gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta))|^2 \, d\eta \, d\theta.\end{aligned}$$

The change of variables

$$\xi = \xi(\eta, \theta) = \eta_1\gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta)$$

has Jacobian

$$\begin{aligned}(4.22)\qquad \left|\frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(\eta_1, \eta_2, \theta)}(\eta_1, \eta_2, \theta)\right| &= |\eta_1| \, |\det((\gamma \times \gamma')(\theta), \gamma'(\theta), \gamma''(\theta))| \\ (4.23)\qquad &= |\eta_1| \, |\langle [\gamma' \times (\gamma \times \gamma')](\theta), \gamma''(\theta) \rangle| \\ &= |\eta_1| \, |\langle \gamma(\theta), \gamma''(\theta) \rangle| \\ (4.24)\qquad &= |\eta_1|.\end{aligned}$$

The line (4.24) above follows from the assumption that γ is a curve in S^2 with unit speed, whilst (4.22) and (4.23) use the scalar triple product formula $\det(a,b,c) = \langle a, b \times c\rangle$ and the identity

$$(\gamma \times \gamma')' = -\det(\gamma, \gamma', \gamma'')\gamma'.$$

Applying this change of variables to (4.21) gives

$$\begin{aligned}
(4.21) &\lesssim \mu(\mathbb{R}^3)^2 + \int_{B(0,2^{j_0(1+\delta)})} |\xi|^{\epsilon-1}|\widehat{\mu}(\xi)|^2\, d\xi \\
&\leq \mu(\mathbb{R}^3)^2 + \begin{cases} 2^{j_0(1+\delta)(2+2\epsilon-\alpha)} I_{\alpha-\epsilon}(\mu) & \alpha \leq 2 \\ 2^{j_0(1+\delta)2\epsilon} I_{2-\epsilon}(\mu) & \alpha > 2 \end{cases} \\
&\lesssim \mu(\mathbb{R}^3)^2 + 2^{j_0(1+\delta)(\max\{0,2-\alpha\}+2\epsilon)}\mu(\mathbb{R}^3),
\end{aligned}$$

which is much smaller than the right-hand side of (4.18). This bounds the first term in (4.20).

It remains to bound the second term in (4.20). This satisfies

$$\begin{aligned}
&\int_{\widetilde{a}}^{\widetilde{b}} \int_{\{|\eta_1|<|\eta_2|^{1-\epsilon}\}\cap B(0,2^{j_0(1+\delta)})} |\widehat{\mu_g}(\eta_1\gamma'(\theta) + \eta_2(\gamma\times\gamma')(\theta))|^2\, d\eta\, d\theta \\
&\quad \leq \sum_{j\in[3J, j_0(1+\delta)]} \sum_{k\in[j\epsilon, j]} \int_{\widetilde{a}}^{\widetilde{b}} \int_{A_{j,k}} |\widehat{\mu_g}(\eta_1\gamma'(\theta) + \eta_2(\gamma\times\gamma')(\theta))|^2\, d\eta\, d\theta \\
&\qquad + \int_{\widetilde{a}}^{\widetilde{b}} \int_{B(0,2^{3J})} |\widehat{\mu_g}(\eta_1\gamma'(\theta) + \eta_2(\gamma\times\gamma')(\theta))|^2\, d\eta\, d\theta \\
&\quad \lesssim \mu(\mathbb{R}^3)\, 2^{j_0(\max\{0,2-\alpha,3/2-\alpha/3-\alpha_0'/3\}+200\epsilon)},
\end{aligned}$$

by Lemmas 4.5 and 4.6. This covers the final case and finishes the proof of the lemma. ■

Lemma 4.8. *Let* $\gamma\colon [a,b] \to S^2$ *be a* C^3 *unit speed curve with* $\det(\gamma,\gamma',\gamma'') \neq 0$ *on* $[a,b]$, *let* $[\widetilde{a},\widetilde{b}] \subseteq (a,b)$, *and let* $\alpha \in (0,3]$. *Then for any* $\alpha^* > 0$ *with*

$$\alpha^* < \min\left\{2, \alpha, \frac{\alpha}{3} + \frac{1}{2} + \frac{\alpha_0(\alpha, 1, \gamma|_{[\widetilde{a},\widetilde{b}]})}{3}\right\},$$

there exist $\delta'', C > 0$ *such that*

$$\int_{\widetilde{a}}^{\widetilde{b}} (\pi_{\theta\#}\mu)\Big(\bigcup_{D\in\mathbb{D}_\theta} D\Big)\, d\theta \leq C\mu(\mathbb{R}^3)R^{-\delta''}, \tag{4.25}$$

for all Borel measures μ *on the unit ball with* $c_\alpha(\mu) \leq 1$, *for any* $R \geq 1$, *and for any collection of sets* $\{\mathbb{D}_\theta : \theta \in [\widetilde{a},\widetilde{b}]\}$ *such that the integrand of* (4.25) *is measurable, where, for each* $\theta \in [\widetilde{a},\widetilde{b}]$, $\mathbb{D}_\theta$ *is a disjoint set of at most* $\mu(\mathbb{R}^3)R^{\alpha^*/2}$ *discs in* $\pi_\theta(\mathbb{R}^3)$ *of radius* $R^{-1/2}$.

As a corollary, $\alpha_0(\alpha, 1, \gamma|_{[\widetilde{a},\widetilde{b}]}) \geq \min\{2, \alpha, \alpha/2 + 3/4\}$.

Proof. By localisation, it may be assumed that γ satisfies (3.1), (3.2) and (3.3) on $[a,b]$, and that $|b-a| \leq \sigma$ where σ is a constant that works in Lemma 4.3. Let α, α^*, R, μ and the sets $\mathbb{D}_\theta$ be given. Define

$$\epsilon = \frac{\min\{\alpha, \alpha/3 + 1/2 + \alpha_0/3, 2\} - \alpha^*}{10^{10}}.$$

By the "good-bad" decomposition (Lemma 3.5) with $\alpha_0' := \alpha_0 - \epsilon$ and $\delta \ll \epsilon$, J defined by $R^\epsilon \in [2^J, 2^{J+1})$ and j_0 defined by $R^{1/2} \in [2^{j_0}, 2^{j_0+1})$,

$$\int_{\tilde{a}}^{\tilde{b}} (\pi_{\theta\#}\mu)\Big(\bigcup_{D\in\mathbb{D}_\theta} D\Big)\, d\theta$$
$$\lesssim \int_{\tilde{a}}^{\tilde{b}} \int |\pi_{\theta\#}\mu_b|\, d\mathcal{H}^2\, d\theta + \sup_{\theta\in[\tilde{a},\tilde{b}]} \mathcal{H}^2\Big(\bigcup_{D\in\mathbb{D}_\theta} D\Big)^{1/2} \Big(\int_{\tilde{a}}^{\tilde{b}} \int |\pi_{\theta\#}\mu_g|^2\, d\mathcal{H}^2\, d\theta\Big)^{1/2}$$

$$\lesssim \int_{\tilde{a}}^{\tilde{b}} \int |\pi_{\theta\#}\mu_b|\, d\mathcal{H}^2\, d\theta + \mu(\mathbb{R}^3)^{1/2} R^{\alpha^*/4-1/2}\Big(\int_{\tilde{a}}^{\tilde{b}} \int |\pi_{\theta\#}\mu_g|^2 d\mathcal{H}^2\, d\theta\Big)^{1/2}. \tag{4.26}$$

By Lemma 4.2,

$$\int_{\tilde{a}}^{\tilde{b}} \int |\pi_{\theta\#}\mu_b|\, d\mathcal{H}^2\, d\theta \leq \mu(\mathbb{R}^3)\, R^{-\epsilon\delta'}, \tag{4.27}$$

provided R is sufficiently large. By Lemma 4.7,

$$\mu(\mathbb{R}^3)^{1/2} R^{\alpha^*/4-1/2}\Big(\int_{\tilde{a}}^{\tilde{b}} \int |\pi_{\theta\#}\mu_g|^2\, d\mathcal{H}^2\, d\theta\Big)^{1/2} \lesssim \mu(\mathbb{R}^3) R^{\frac{1}{2}\max\{\alpha^*-2, \alpha^*-\alpha, \alpha^*-1/2-\alpha/3-\alpha_0/3\}+10^5\epsilon}. \tag{4.28}$$

Applying (4.27) and (4.28) to (4.26) yields (4.25) and proves the first part of the lemma, provided δ'' is chosen sufficiently small, R is taken sufficiently large, and the constant $C(\alpha, \alpha^*, \gamma)$ is then taken large enough to handle the small values of R.

The last part of the lemma follows directly from the first part, and from Definition 4.1 which defines α_0. ■

Proof of Theorem 1.1. Assume without loss of generality that $\dim A > 0$ and that A is a subset of the unit ball. Let $\epsilon > 0$ be small, let $\alpha = \dim A - \epsilon$ and (using Frostman's lemma) let μ be a nonzero, finite Borel measure on A with $c_\alpha(\mu) \leq 1$. Suppose that $E \subseteq [a+\epsilon, b-\epsilon]$ is a compact set such that

$$\dim \pi_\theta \operatorname{supp} \mu \leq s := \min\left\{2, \alpha, \frac{\alpha}{2} + \frac{3}{4}\right\} - 10\epsilon,$$

for every $\theta \in E$. Let $\varepsilon > 0$ be small. For each $\theta \in E$, let[1] $\{B(\pi_\theta(x_j(\theta)), r_j(\theta))\}_{j=1}^\infty$ be a covering of $\pi_\theta \operatorname{supp} \mu$ by discs of dyadic radii smaller than ε, with each $x_j(\theta) \in \operatorname{supp} \mu$,

[1]Issues of measurability will be ignored since they can be easily adjusted for.

such that

$$\sum_j r_j(\theta)^{s+\epsilon} < 1. \tag{4.29}$$

It may be additionally assumed that $\pi_\theta(x_j(\theta)) \in \pi_\theta \operatorname{supp} \mu$ for every j and θ. For each $\theta \in E$ and each $k \geq |\log_2 \varepsilon|$, let

$$D_k(\theta) = \bigcup_{j : r_j(\theta) = 2^{-k}} B(\pi_\theta(x_j(\theta)), r_j(\theta)).$$

Then for each $\theta \in E$,

$$1 \leq \sum_{k \geq |\log_2 \varepsilon|} (\pi_{\theta\#}\mu)(D_k(\theta)).$$

By the Besicovitch covering theorem, for each $\theta \in E$ there is a disjoint subcollection $\{B(\pi_\theta(x_j(\theta)), r_j(\theta))\}_{j \in I}$, and corresponding subsets $D_k(\theta)' \subseteq D_k(\theta)$, such that

$$1 \lesssim \sum_{k \geq |\log_2 \varepsilon|} (\pi_{\theta\#}\mu)(D_k'(\theta)),$$

and hence

$$\mathcal{H}^1(E) \lesssim \sum_{k \geq |\log_2 \varepsilon|} \int_E (\pi_{\theta\#}\mu)(D_k'(\theta))\, d\theta.$$

By (4.29), for each θ and k, the set $D_k(\theta)'$ is the union of at most $2^{k(s+\epsilon)}$ disjoint discs of radius 2^{-k}. By Lemma 4.8, there exists a $\delta > 0$ independent of ε such that

$$\mathcal{H}^1(E) \lesssim \sum_{k \geq |\log_2 \varepsilon|} 2^{-k\delta} \lesssim \varepsilon^\delta.$$

Letting $\varepsilon \to 0$ gives $\mathcal{H}^1(E) = 0$. Hence

$$\dim \pi_\theta(A) \geq \dim \pi_\theta \operatorname{supp} \mu \geq \min\left\{2, \alpha, \frac{\alpha}{2} + \frac{3}{4}\right\} - 10\epsilon$$

for a.e. $\theta \in [a + \epsilon, b - \epsilon]$. The theorem follows by letting $\epsilon \to 0$ along a countable sequence. ∎

5. Further improvement and related problems

A related problem is the family of projections $\rho_\theta(x) = \langle x, \gamma(\theta)\rangle\gamma(\theta)$ onto lines, where γ is a smooth curve in S^2 with $\det(\gamma, \gamma', \gamma'')$ nonvanishing. For non-great circles, this was resolved by Käenmäki–Orponen–Venieri in [7]. I do not know if the method of proof here would also work on this problem; I would guess that at least a different kind of refined Strichartz inequality would be needed, with “slabs” in place of “tubes”. One application, due to Liu [10], of the Käenmäki–Orponen–Venieri projection theorem is to give the sharp lower bound of 3 for the Hausdorff dimension of Kakeya sets in the first Heisenberg group $\mathbb{H}$. It would be interesting if Theorem 1.1 could be analogously applied to generalised Besicovitch sets in $\mathbb{H}$ (e.g., sets containing a left translate of every vertical subgroup).

I do not know if Theorem 1.1 holds with C^3 replaced by C^2. The only step in the proof of Theorem 1.1 which seems to make crucial use of the C^3 assumption is the decoupling theorem for generalised cones, though the statement and some steps in the proof of the refined Strichartz inequality would likely become more technical if γ were only assumed to be C^2. As far as I am aware, the decoupling theorem for generalised cones has only been proved in the literature with C^3 assumptions (see e.g. [16]).

Although the proof of Theorem 1.1 has some similarities with the proof of the lower bound for the distance set problem in [4], it makes crucial use of the fact that the projections π_θ are linear. Since the distance function is nonlinear, the recursive method of bounding the "bad" part in the proof of Theorem 1.1 does not seem to work on the distance set problem.

Another problem on restricted projections comes from the family of maps $S_g(x, y) = x - gy$ from $\mathbb{R}^{2n}$ to $\mathbb{R}^n$, where $n \geq 2$ is fixed and g ranges over $O(n)$. In [12], Mattila proved that

$$\dim S_g(A) \geq \max\{\min\{\dim A, n-1\}, \min\{\dim A - 1, n\}\}, \quad \text{a.e. } g \in O(n), \tag{5.1}$$

and asked whether this can be replaced by $\dim S_g(A) \geq \min\{n, \dim A\}$ for a.e. $g \in O(n)$. A counterexample (obtained with help from A. Barron) is the set $A = \{(x, x) : x \in \mathbb{R}^n\}$; since every $g \in O(n)$ with $\det g = (-1)^{n+1}$ has 1 as an eigenvalue, the set $S_g(A)$ has dimension at most $n - 1$ whenever $\det g = (-1)^{n+1}$. By taking direct sums with Cantor subsets of the plane $\{(x, -x) : x \in \mathbb{R}^n\}$, this can be modified to show that (5.1) is sharp for all values of $\dim A$. It would be interesting to know whether a better lower bound is possible if the requirement "a.e. $g \in O(n)$" is weakened to "with probability at least 1/2".

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