Rev. Mat. Iberoam. (Online first)
DOI 10.4171/RMI/1361

Published by EMS Press

Interior estimates for the Monge–Ampère type fourth order equations

Ling Wang and Bin Zhou

Abstract. In this paper, we give several new approaches to study interior estimates for a class of fourth order equations of Monge–Ampère type. First, we prove interior estimates for the homogeneous equation in dimension two by using the partial Legendre transform. As an application, we obtain a new proof of the Bernstein theorem without using Caffarelli–Gutiérrez's estimate, including the Chern conjecture on affine maximal surfaces. For the inhomogeneous equation, we also obtain a new proof in dimension two by an integral method relying on the Monge–Ampère Sobolev inequality. This proof works even when the right-hand side is singular. In higher dimensions, we obtain the interior regularity in terms of integral bounds on the second derivatives and the inverse of the determinant.

1. Introduction

We study the regularity of the following fourth order equations of Monge–Ampère type:

$$U^{ij} w_{ij} = f, \tag{1.1}$$

where $\{U^{ij}\}$ is the cofactor matrix of D^2u of an unknown uniformly convex function, and

$$w = \begin{cases} [\det D^2 u]^{-(1-\theta)}, & \theta \geq 0,\ \theta \neq 1, \\ \log \det D^2 u, & \theta = 1. \end{cases} \tag{1.2}$$

When $\theta = 1/(n+2)$, this is the *affine mean curvature equation* in affine geometry [7]. When $\theta = 0$, it is *Abreu's equation* arising from the problem of extremal metrics on toric manifolds in Kähler geometry [1], and is equivalent to

$$\sum_{i,j} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = f,$$

where $\{u^{ij}\}$ is the inverse matrix of D^2u. The regularity of (1.1) has been extensively studied before, see [4, 5, 10, 18, 19, 32–35]. This equation is usually treated as a system of

---

2020 Mathematics Subject Classification: Primary 35J96; Secondary 35J30, 35B45, 35B65.
Keywords: Monge–Ampère equation, linearized Monge–Ampère equation, fourth order equation, partial Legendre transform, degenerate elliptic equations.

a Monge–Ampère equation and a linearized Monge–Ampère equation. Therefore, in previous works, its regularity relies heavily on Caffarelli–Gutiérrez's deep result on the interior regularity of the linearized Monge–Ampère equation [3], which was later extended by [13, 14, 24] to the the boundary and to higher order estimates. In this paper, we investigate the interior estimates of (1.1) by several new approaches. We will mainly concentrate on the case $\theta \in [0,1]$ due to the interesting geometric background.

We first consider the case of the homogeneous equation

$$U^{ij} w_{ij} = 0, \tag{1.3}$$

where w is given by (1.2). We apply the partial Legendre transform to give a new proof of the interior estimates of (1.3) in dimension two.

Theorem 1.1. *Assume $n = 2$ and $\theta \in [0,1]$. Let $\Omega \subset \mathbb{R}^2$ be a convex domain and let u be a smooth convex solution to equation (1.3) on Ω satisfying*

$$0 < \lambda \le \det D^2 u \le \Lambda. \tag{1.4}$$

Then for any $\Omega' \Subset \Omega$, there exists a constant $C > 0$, depending on $\sup_\Omega |u|$, λ, Λ, θ and $\operatorname{dist}(\Omega', \partial\Omega)$, such that

$$\|u\|_{C^{4,\alpha}(\Omega')} \le C.$$

The partial Legendre transform for the fourth order equation was first used in [25], where the authors deal with the second boundary value problem. After the partial Legendre transform, equation (1.3) becomes a quasi-linear second order equation (see (2.3)) for the determinant. The main ingredient in our proof is an interior integral gradient estimate (Theorem 2.2). When $\theta \in [0, 1/4]$, condition (1.4) holds by the determinant estimates and arguments of strict convexity [32, 34]. By Theorem 1.1 and a rescaling argument as in Theorem 2.1 of [32], we obtain a new proof of the following Bernstein theorem [17, 32, 34] without using Caffarelli–Gutiérrez's theory.

Theorem 1.2. *Assume $n = 2$ and $0 \le \theta \le 1/4$. Let u be an entire smooth uniformly convex solution to (1.3) on $\mathbb{R}^2$. Then u is a quadratic polynomial.*

In the case of the inhomogeneous equation and in higher dimensions, the partial Legendre transform does not work. We will investigate the interior regularity by an integral method motivated by De Giorgi–Nash–Moser's theory. Consider the inhomogeneous equation with general right-hand side term

$$U^{ij} w_{ij} = f(x, u, Du, D^2 u), \tag{1.5}$$

where w given by (1.2) with $\theta \in [0,1]$. This equation is introduced by [21, 22] in the study of convex functionals with a convexity constraint related to the Rochet–Choné model for the monopolist problem in economics. It is said to be singular since the right-hand side term depends on $D^2 u$. A typical example considered in [21, 22, 25] is

$$f = \operatorname{div}(|\nabla u|^{p-2} \nabla u) + f^0(x, u). \tag{1.6}$$

Note that when $f \in L^\infty(\Omega)$, once we have the determinant estimate (1.4), we can use Caffarelli–Gutiérrez's theory to get the interior regularity. The assumption on f can be weakened to $f \in L^{n/2+\varepsilon}(\Omega)$ by [23]. Hence for f defined as (1.6), when $n = 2$ and $p \geq 2$, we can obtain the interior regularity of u directly by using interior $W^{2,1+\varepsilon}$-estimates of the Monge–Ampère equations [9,29]. To settle the more singular case $1 < p < 2$, Le [20] established the interior estimate of the linearized Monge–Ampère equation with right-hand side term in divergence form in dimension two. One of the main tools in [20] is the Monge–Ampère Sobolev inequality (see Lemma 3.2). In this paper, we will use the Monge–Ampère Sobolev inequality and the $W^{2,1+\varepsilon}$-estimates for the Monge–Ampère equation directly in the fourth order equation to obtain a $W^{2,p}$-estimate of the solution (Theorem 3.3). Then we can apply the regularity theory of second order elliptic equation of divergence type to obtain a new proof for the interior estimates of (1.5).

Theorem 1.3. *Assume $n = 2$ and $\theta \in [0, 1]$. Let $\Omega \subset \mathbb{R}^2$ be a convex domain. Assume*

$$f = \operatorname{div}(g) + h,$$

where $g := (g^1(x), g^2(x)) : \Omega \to \mathbb{R}^2$ is a bounded vector function and $h \in L^q(\Omega)$ for some $q > n/2$. Suppose u is a smooth convex solution to equation (1.5) on Ω satisfying (1.4). Then for any $\Omega' \Subset \Omega$, there exists a constant $C > 0$, depending on $\sup_\Omega |u|$, λ, Λ, θ, $\|g\|_{L^\infty(\Omega)}$, $\|h\|_{L^q(\Omega)}$ and $\operatorname{dist}(\Omega', \partial\Omega)$, such that

$$\|u\|_{W^{4,q}(\Omega')} \leq C.$$

Remark 1.4. (1) It is clear that the above theorem applies to the case (1.6) for any $p > 1$ in dimension two. The higher dimensional case is still open.

(2) When $f \in W^{1,q}(\Omega)$ and $\theta = 0$ or 1, we give another new proof in Section 3 inspired by [6] for the complex setting. More precisely, we get a C^2-estimate of u in terms of the $W^{2,p}$-bound (Theorem 3.4), which makes (1.5) become a uniformly elliptic equation. Then the classical theory of uniformly elliptic equations can be applied.

In higher dimensions, the interior regularity and the Bernstein theorem are still widely open. In fact, according to the counterexample in [32] for the affine maximal surface equation, there may be no interior estimates if no further assumptions are made. More precisely, (1.4) may not hold. We give a partial result by assuming integral bounds on the second derivatives and the inverse of the determinant.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain and let u be a smooth uniformly convex solution to equation (1.3) with $\theta \in [0, 1]$ on Ω. Assume that $p, q > 0$ satisfy $n/p + 1/q < 2/n$. Then for any $\Omega' \Subset \Omega$, there exists a constant C, depending only on p, q, θ, $\sup_\Omega |u|$, $\|u\|_{W^{2,p}(\Omega)}$, $\|(\det D^2u)^{-1}\|_{L^q(\Omega)}$ and $\operatorname{dist}(\Omega', \partial\Omega)$, such that*

$$\|u\|_{C^{4,\alpha}(\Omega')} \leq C.$$

As an application, we obtain a Liouville type theorem in higher dimensions.

Corollary 1.6. *Let u be an entire smooth uniformly convex solution to (1.1) with $\theta \in [0, 1]$ on $\mathbb{R}^n$. Suppose there are $p, q, C > 0$ such that $n/p + 1/q < 2/n$ and*

$$\int_{B_R(0)} |D^2u|^p + (\det D^2u)^{-q}\, dx \leq CR^n, \quad \forall R > 0. \tag{1.7}$$

Then u is a quadratic polynomial.

The structure of the paper is as follows. In Section 2.1, we apply partial the Legendre transform to (1.1) in dimension two to derive a new equation. The key interior gradient estimate (Theorem 2.2) for the new equation is established in Section 2.2. Then we prove Theorem 1.1 in Section 2.3 with this key estimate. In Section 3, we first derive the $W^{2,p}$-estimate of u (Theorem 3.3), and then we prove Theorem 1.3. Section 4 is devoted to some study on interior regularity in higher dimensions.

2. The homogeneous equation in dimension two

In this section, we present a new proof for the interior estimate for the homogeneous equation without Caffarelli–Gutiérrez's theory.

2.1. The new equation under partial Legendre transform

We first focus on the dimension two case. Write $u(x) = u(x_1, x_2)$. The partial Legendre transform in the x_1-variable is

$$u^\star(\xi, \eta) = x_1 u_{x_1}(x_1, x_2) - u(x_1, x_2), \tag{2.1}$$

where

$$y = (\xi, \eta) = \mathcal{P}(x_1, x_2) := (u_{x_1}, x_2) \in \mathcal{P}(\Omega) =: \Omega^\star.$$

We have

$$\frac{\partial(\xi,\eta)}{\partial(x_1,x_2)} = \begin{pmatrix} u_{x_1x_1} & u_{x_1x_2} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \frac{\partial(x_1,x_2)}{\partial(\xi,\eta)} = \begin{pmatrix} 1/u_{x_1x_1} & -u_{x_1x_2}/u_{x_1x_1} \\ 0 & 1 \end{pmatrix}.$$

Hence,

$$u^\star_\xi = x_1, \quad u^\star_{\xi\xi} = \frac{1}{u_{x_1x_1}}, \quad u^\star_\eta = -u_{x_2}, \quad u^\star_{\eta\eta} = -\frac{\det D^2 u}{u_{x_1x_1}}, \quad u^\star_{\xi\eta} = -\frac{u_{x_1x_2}}{u_{x_1x_1}}. \tag{2.2}$$

The partial Legendre transform has been used widely in the study of the Monge–Ampère equation [8, 11, 12, 24, 26]. Here we apply it to equation (1.3).

In order to derive the equation under the partial Legendre transform, we consider the associated functionals of (1.3):

$$A_\theta(u) = \begin{cases} \int_\Omega [\det D^2 u]^\theta \, dx, & \theta > 0, \ \theta \neq 1, \\ \int_\Omega \log\det D^2 u \, dx, & \theta = 0, \\ \int_\Omega \det D^2 u \, \log\det D^2 u \, dx, & \theta = 1. \end{cases}$$

The case of $\theta = 0$ is essentially included in [25].

Proposition 2.1. *Let u be a uniformly convex solution to (1.3) in Ω. Then in $\Omega^\star = \mathcal{P}(\Omega)$, its partial Legendre transform $u^\star$ satisfies*

$$w^\star w^\star_{\xi\xi} + w^\star_{\eta\eta} + (\theta - 1) {w^\star_\xi}^2 + \frac{\theta - 2}{w^\star} {w^\star_\eta}^2 = 0, \tag{2.3}$$

Here, $w^\star = -u^\star_{\eta\eta}/u^\star_{\xi\xi}$.

Proof. As

$$\det D^2 u = -\frac{u^\star_{\eta\eta}}{u^\star_{\xi\xi}} \quad \text{and} \quad dx dy = u^\star_{\xi\xi}\, d\xi\, d\eta,$$

we have

$$\begin{aligned} A_\theta(u) &= \int_{\Omega^\star} \Big(-\frac{u^\star_{\eta\eta}}{u^\star_{\xi\xi}}\Big)^\theta u^\star_{\xi\xi}\, d\xi\, d\eta \\ &= \int_{\Omega^\star} (-u^\star_{\eta\eta})^\theta {u^\star_{\xi\xi}}^{1-\theta}\, d\xi\, d\eta =: A^\star_\theta(u^\star), \quad \theta \in (0,1); \\ A_0(u) &= \int_{\Omega^\star} \log\Big(-\frac{u^\star_{\eta\eta}}{u^\star_{\xi\xi}}\Big) u^\star_{\xi\xi}\, d\xi\, d\eta =: A^\star_0(u^\star); \\ A_1(u) &= \int_{\Omega^\star} \Big(-\frac{u^\star_{\eta\eta}}{u^\star_{\xi\xi}}\Big) \log\Big(-\frac{u^\star_{\eta\eta}}{u^\star_{\xi\xi}}\Big) u^\star_{\xi\xi}\, d\xi\, d\eta =: A^\star_1(u^\star). \end{aligned}$$

Since u is maximal with respect to the functional A_θ, $u^\star$ is maximal with respect to the functional $A^\star_\theta$. It suffices to derive the Euler–Lagrange equation of $A^\star_\theta$. See [32, 34] for the case of the Legendre transform.

First, we consider $\theta \in (0,1)$. For $\varphi \in C_0^\infty(\Omega^\star)$, by integration by parts,

$$\begin{aligned} \frac{dA^\star_\theta(u^\star + t\varphi)}{dt}\Big|_{t=0} &= \int_{\Omega^\star} [(1-\theta)(-u^\star_{\eta\eta})^\theta (u^\star_{\xi\xi})^{-\theta}\varphi_{\xi\xi} - \theta(-u^\star_{\eta\eta})^{\theta-1} {u^\star_{\xi\xi}}^{1-\theta}\varphi_{\eta\eta}]\, d\xi\, d\eta \\ &= \int_{\Omega^\star} \big\{[(1-\theta)(-u^\star_{\eta\eta})^\theta (u^\star_{\xi\xi})^{-\theta}]_{\xi\xi} - [\theta(-u^\star_{\eta\eta})^{\theta-1} {u^\star_{\xi\xi}}^{1-\theta}]_{\eta\eta}\big\}\varphi\, d\xi\, d\eta. \end{aligned}$$

Denote $w^\star = -u^\star_{\eta\eta}/u^\star_{\xi\xi}$. Then the equation, after the transformation, becomes

$$-\theta({w^\star}^{\theta-1})_{\eta\eta} + (1-\theta)({w^\star}^\theta)_{\xi\xi} = 0.$$

After simplification, this is (2.3). Similarly, for $\varphi \in C_0^\infty(\Omega^*)$,

$$\begin{aligned} \frac{dA^\star_0(u^\star + t\varphi)}{dt}\Big|_{t=0} &= \int_{\Omega^\star} -\frac{u^\star_{\xi\xi}}{u^\star_{\eta\eta}}\Big(-\frac{\varphi_{\eta\eta} u^\star_{\xi\xi} - u^\star_{\eta\eta}\varphi_{\xi\xi}}{{u^\star_{\xi\xi}}^2}\Big) u^\star_{\xi\xi} + \log\Big(-\frac{u^\star_{\eta\eta}}{u^\star_{\xi\xi}}\Big)\varphi_{\xi\xi}\, d\xi\, d\eta \\ &= \int_{\Omega^\star} \frac{\varphi_{\eta\eta} u^\star_{\xi\xi} - u^\star_{\eta\eta}\varphi_{\xi\xi}}{u^\star_{\eta\eta}} + \log\Big(-\frac{u^\star_{\eta\eta}}{u^\star_{\xi\xi}}\Big)\varphi_{\xi\xi}\, d\xi\, d\eta, \end{aligned}$$

and the equation, after the transformation, becomes

$$-[(w^\star)^{-1}]_{\eta\eta} + (\log w^\star)_{\xi\xi} = 0.$$

After simplification, we obtain (2.3). Finally,

$$\begin{aligned} \frac{dA^\star_1(u^\star + t\varphi)}{dt}\Big|_{t=0} &= \int_{\Omega^\star} (1 + \log w^\star)\Big(-\frac{\varphi_{\eta\eta}u^\star_{\xi\xi} - u^\star_{\eta\eta}\varphi_{\xi\xi}}{{u^\star_{\xi\xi}}^2}\Big)u^\star_{\xi\xi} + w^\star \log w^\star \varphi_{\xi\xi}\, d\xi\, d\eta \\ &= -\int_{\Omega^\star} (1 + \log w^\star)(w^\star\varphi_{\xi\xi} - \varphi_{\eta\eta}) + w^\star \log w^\star \varphi_{\xi\xi}\, d\xi\, d\eta. \end{aligned}$$

so the equation becomes

$$-w^{\star}_{\xi\xi} - (\log w^{\star})_{\eta\eta} = 0,$$

which is equivalent to (2.3). ∎

2.2. The interior gradient estimate of (2.3)

For simplicity, we change notations in this section and write (2.3) as

$$uu_{xx} + u_{yy} = (1-\theta)u_x^2 + \frac{2-\theta}{u}u_y^2. \tag{2.4}$$

It is easy to see that this is a quasi-linear equation with right-hand side depending on the gradient. We prove the following interior gradient estimate.

Theorem 2.2. *Assume u is a solution to (2.4) with $\theta \in [0,1]$ on $B_R := B_R(0)$ and satisfies $0 < \lambda \le u \le \Lambda$. Then there exist $\alpha, C > 0$ depending on λ, Λ, R and θ, such that*

$$\int_{B_R} |\nabla u|^3 (R^2 - x^2 - y^2)^\alpha \, dV \le C. \tag{2.5}$$

Proof. Let $w = v\phi\eta$, where

$$v = \sqrt{u_x^2 + u_y^2 + 1} \quad \text{and} \quad \eta = (R^2 - x^2 - y^2)^\alpha, \quad \alpha > 3,$$

and $\phi = \phi(u)$ is a positive function of u to be determined later.

A direct calculation yields

$$\begin{aligned} &uw_{xx} + w_{yy} \\ &\quad = (uv_{xx} + v_{yy})\phi\eta + 2[uv_x(\phi\eta)_x + v_y(\phi\eta)_y] + [u(\phi\eta)_{xx} + (\phi\eta)_{yy}]v. \end{aligned} \tag{2.6}$$

It is clear that

$$v_x = v^{-1}(u_x u_{xx} + u_y u_{xy}), \quad v_y = v^{-1}(u_x u_{xy} + u_y u_{yy}), \tag{2.7}$$

$$v_{xx} = -v^{-3}(u_x u_{xx} + u_y u_{xy})^2 + v^{-1}(u_{xx}^2 + u_x u_{xxx} + u_{xy}^2 + u_y u_{xxy}), \tag{2.8}$$

$$v_{yy} = -v^{-3}(u_x u_{xy} + u_y u_{yy})^2 + v^{-1}(u_{xy}^2 + u_x u_{xyy} + u_{yy}^2 + u_y u_{yyy}). \tag{2.9}$$

Differentiating the equation, we have

$$u_x u_{xx} + uu_{xxx} + u_{xyy} = 2(1-\theta)u_x u_{xx} - \frac{2-\theta}{u^2}u_x u_y^2 + \frac{2(2-\theta)}{u}u_y u_{xy}, \tag{2.10}$$

$$u_y u_{xx} + uu_{xxy} + u_{yyy} = 2(1-\theta)u_x u_{xy} - \frac{2-\theta}{u^2}u_y u_y^2 + \frac{2(2-\theta)}{u}u_y u_{yy}. \tag{2.11}$$

By the Cauchy inequality, we have

$$\begin{aligned} (u_x u_{xx} + u_y u_{xy})^2 &\le (v^2 - 1)(u_{xx}^2 + u_{xy}^2), \\ (u_x u_{xy} + u_y u_{yy})^2 &\le (v^2 - 1)(u_{xy}^2 + u_{yy}^2). \end{aligned} \tag{2.12}$$

Then by (2.7), (2.8), (2.9), (2.10), (2.11) and (2.12), the first term in (2.6) satisfies

$$\begin{aligned}
&uv_{xx} + v_{yy}\\
&= u\,[v^{-1}(u_{xx}^2 + u_{xy}^2) - v^{-3}(u_x u_{xx} + u_y u_{xy})^2] + v^{-1}u(u_x u_{xxx} + u_y u_{xxy})\\
&\quad + [v^{-1}(u_{xy}^2 + u_{yy}^2) - v^{-3}(u_x u_{xy} + u_y u_{yy})^2] + v^{-1}(u_x u_{xyy} + u_y u_{yyy}))\\
&\geq v^{-1}\big[2(1-\theta)u_x(u_x u_{xx} + u_y u_{xy}) + \tfrac{2(2-\theta)}{u} u_y(u_x u_{xy} + u_y u_{yy}) - (v^2-1)u_{xx}\big]\\
&\quad - v^{-1}\,\tfrac{2-\theta}{u^2}\,(u_x^2 u_y^2 + u_y^4) + v^{-3}\big[u(u_{xx}^2 + u_{xy}^2) + u_{xy}^2 + u_{yy}^2\big]\\
&= 2(1-\theta)u_x v_x + \tfrac{2(2-\theta)}{u}\,u_y v_y - \tfrac{2-\theta}{u^2}\,\tfrac{v^2-1}{v}\,u_y^2 - \tfrac{v^2-1}{v}\,u_{xx}\\
&\quad + v^{-3}\big[u(u_{xx}^2 + u_{xy}^2) + u_{xy}^2 + u_{yy}^2\big]\\
&\geq 2(1-\theta)u_x v_x + \tfrac{2(2-\theta)}{u}\,u_y v_y - \tfrac{2-\theta}{u^2}\,\tfrac{v^2-1}{v}\,u_y^2 - v u_{xx} + v^{-1}u_{xx} + v^{-3}uu_{xx}^2\\
&\geq 2(1-\theta)u_x v_x + \tfrac{2(2-\theta)}{u}\,u_y v_y - \tfrac{2-\theta}{u^2}\,\tfrac{v^2-1}{v}\,u_y^2 - v u_{xx} - Cv,
\end{aligned}$$

where $C > 0$ depends on λ. By integration by parts,

$$\begin{aligned}
&\int_{B_R} (uv_{xx} + v_{yy})\phi\eta\, dV\\
&\geq -\int_{B_R} 2(1-\theta)[u_{xx}v\phi\eta + u_x v(\phi\eta)_x]\, dV\\
&\quad - \int_{B_R} 2(2-\theta)\Big[\frac{u_{yy}}{u}\,v\phi\eta - \frac{u_y^2}{u^2}\,v\phi\eta + \frac{1}{u}\,u_y\,v(\phi\eta)_y\Big]\, dV\\
&\quad - \int_{B_R} \Big(\frac{2-\theta}{u^2}\,\frac{v^2-1}{v}\,u_y^2\phi\eta + vu_{xx}\phi\eta + C\phi\eta v\Big)\, dV\\
&= \int_{B_R} u_{xx}\,v\phi\eta\, dV - \int_{B_R} 2(2-\theta)\,\frac{1}{u}\Big[(1-\theta)u_x^2 + \frac{2-\theta}{u}\,u_y^2\Big]v\phi\eta\, dV\\
&\quad - \int_{B_R} \Big[2(1-\theta)u_x\,v(\phi\eta)_x + 2(2-\theta)\Big(\frac{u_y v(\phi\eta)_y}{u} - \frac{u_y^2 v\phi\eta}{u^2}\Big)\Big]\, dV\\
&\quad - \int_{B_R} \Big[\frac{2-\theta}{u^2}\Big(v - \frac{1}{v}\Big)u_y^2\phi\eta + C\phi\eta v\Big]\, dV\\
&= \int_{B_R} u_{xx}\,v\phi\eta\, dV - \int_{B_R} \Big[2(1-\theta)\phi' + \frac{2(2-\theta)(1-\theta)}{u}\,\phi\Big]u_x^2\,v\eta\, dV\\
&\quad - \int_{B_R} \Big[2(2-\theta)\,\frac{\phi'}{u} + (2-\theta)(3-2\theta)\,\frac{1}{u^2}\,\phi\Big]u_y^2\,v\eta\, dV\\
&\quad - \int_{B_R} \Big[2(1-\theta)u_x v\phi\eta_x + \frac{2(2-\theta)}{u}u_y v\phi\eta_y + C\phi\eta v\Big]\, dV + \int_{B_R} \frac{2-\theta}{u^2}\,v^{-1}u_y^2\phi\eta\, dV\\
&\geq \int_{B_R} u_{xx}\,v\phi\eta\, dV - \int_{B_R} \Big[2(1-\theta)\phi' + \frac{2(2-\theta)(1-\theta)}{u}\,\phi\Big]u_x^2\,v\eta\, dV\\
&\quad - \int_{B_R} \Big[2(2-\theta)\,\frac{\phi'}{u} + (2-\theta)(3-2\theta)\,\frac{1}{u^2}\,\phi\Big]u_y^2\,v\eta\, dV\\
&\quad - C\Big[\int_{B_R} |\nabla u|^2(R^2 - x^2 - y^2)^{\alpha-1}\, dV + \int_{B_R} |\nabla u|(R^2 - x^2 - y^2)^{\alpha}\, dV + 1\Big].
\end{aligned}$$

Here $C > 0$ depends on λ, Λ, R and θ. For the third term in (2.6),

$$
\begin{aligned}
&[u(\phi\eta)_{xx} + (\phi\eta)_{yy}]v \\
&= v\big[\phi''(u)(uu_x^2 + u_y^2)\eta + \phi'(u)(uu_{xx} + u_{yy})\eta + 2\phi'(u)(uu_x\eta_x + u_y\eta_y) \\
&\quad + \phi \cdot (u\eta_{xx} + \eta_{yy})\big] \\
&= v\Big[\phi''(u)(uu_x^2 + u_y^2)\eta + \phi'(u)\Big((1-\theta)u_x^2 + \frac{2-\theta}{u}u_y^2\Big)\eta + 2\phi'(u)(uu_x\eta_x + u_y\eta_y) \\
&\quad + \phi \cdot (u\eta_{xx} + \eta_{yy})\Big] \\
&\le \Big[(\phi''u + (1-\theta)\phi')u_x^2 + \Big(\phi'' + \phi'\frac{2-\theta}{u}\Big)u_y^2\Big]v\eta \\
&\quad + C\big[|\nabla u|^2(R^2 - x^2 - y^2)^{\alpha-1} + |\nabla u|(R^2 - x^2 - y^2)^{\alpha-2}\big].
\end{aligned}
$$

Here $C > 0$ depends on λ, Λ, R and θ. Integrating by parts the second term in (2.6),

$$
2\int_{B_R} uv_x(\phi\eta)_x + v_y(\phi\eta)_y \, dV = -2\int_{B_R} u_x(\phi\eta)_x v \, dV - 2\int_{B_R} [u(\phi\eta)_{xx} + (\phi\eta)_{yy}]v \, dV,
$$

and hence,

$$
\begin{aligned}
&\int_{B_R} uw_{xx} + w_{yy} \, dV \\
&\quad = \int_{B_R} (uv_{xx} + v_{yy})\phi\eta \, dV - 2\int_{B_R} u_x(\phi\eta)_x v \, dV - \int_{B_R} [u(\phi\eta)_{xx} + (\phi\eta)_{yy}]v \, dV \\
&\quad \ge \int_{B_R} u_{xx} v\phi\eta \, dV + \int_{B_R} \Big[(-5+3\theta)\phi' - \frac{2(2-\theta)(1-\theta)}{u}\phi - \phi''u\Big]u_x^2 v\eta \, dV \\
&\qquad + \int_{B_R} \Big[-3(2-\theta)\frac{\phi'}{u} - (2-\theta)(3-2\theta)\frac{\phi}{u^2} - \phi''\Big]u_y^2 v\eta \, dV \\
&\qquad - C\Big[\int_{B_R} |\nabla u|^2(R^2 - x^2 - y^2)^{\alpha-1} \, dV + \int_{B_R} |\nabla u|(R^2 - x^2 - y^2)^{\alpha-2} \, dV + 1\Big].
\end{aligned}
\tag{2.13}
$$

Note that the left-hand side term satisfies

$$
\int_{B_R} uw_{xx} + w_{yy} \, dV = \int_{B_R} u_{xx} v\phi\eta \, dV. \tag{2.14}
$$

Now we choose

$$
\phi(u) = Au^{\theta-2} - \frac{u}{2\theta^2 - 9\theta + 9},
$$

with

$$
A \ge \frac{\Lambda^{3-\theta}}{2\theta^2 - 9\theta + 9} + 1.
$$

Then it is clear that $\phi(u) > 0$. Furthermore, since $\theta \in [0, 1]$, we have

$$
(-5+3\theta)\phi' - \frac{2(2-\theta)(1-\theta)}{u}\phi - \phi''u = 1
$$

and

$$-3(2-\theta)\frac{\phi'}{u}-(2-\theta)(3-2\theta)\frac{\phi}{u^2}-\phi''=\frac{2(2-\theta)(3-\theta)}{(2\theta^2-9\theta+9)u}\geq C_0>0.$$

Combining them with (2.13) and (2.14), we obtain

$$\begin{aligned}&\int_{B_R}|\nabla u|^3(R^2-x^2-y^2)^\alpha\,dV\\ &\leq C_1\int_{B_R}|\nabla u|^2(R^2-x^2-y^2)^{\alpha-1}\,dV+C_2\int_{B_R}|\nabla u|(R^2-x^2-y^2)^{\alpha-2}-dV+C_3\\ &\leq C_1'\Big(\int_{B_R}|\nabla u|^3(R^2-x^2-y^2)^\alpha\,dV\Big)^{2/3}+C_2'\Big(\int_{B_R}|\nabla u|^3(R^2-x^2-y^2)^\alpha\,dV\Big)^{1/3}+C_3.\end{aligned}$$

Hence, (2.5) follows. ■

2.3. Proof of Theorem 1.1

In order to use the partial Legendre transform, we first recall the modulus of convexity. For a convex function on $\mathbb{R}^n$, the *modulus of convexity* m_u of u is defined by

$$m_u(t)=\inf\{u(x)-\ell_z(x):|x-z|>t\}, \tag{2.15}$$

where $t>0$ and ℓ_z is the supporting function of u at z. For a strictly convex function, m_u must be a positive function. A result of Heinz [16] implies that in two dimensions, when $\det D^2u\geq\lambda>0$, there exists a positive function $C(t)>0$ depending on λ such that $m_u(t)\geq C(t)>0$. Now for the partial Legendre transform (2.1), we consider the mapping

$$(\xi,\eta)=\mathcal{P}(x,y)=(u_x,y):B_R(0)\to\mathbb{R}^2. \tag{2.16}$$

The following important property is revealed in [26].

Lemma 2.3 (Lemma 2.1 in [26]). *There exists a constant $\delta>0$ depending on the modulus of convexity m_u defined in* (2.15)*, such that $B_\delta(0)\subset\mathcal{P}(B_R(0))$.*

Proof of Theorem 1.1. For any $p\in\Omega$, let $R=\operatorname{dist}(p,\partial\Omega)/2$. Without loss of generality, we assume $\mathcal{P}(p)=0$. It is clear that $\sup_{B_R(p)}|Du|\leq C$ for some constant C depending on R and $\sup_\Omega|u|$. By Lemma 2.3, there exists $\delta>0$ such that $B_\delta(0)\subset\mathcal{P}(B_R(p))$. According to Proposition 2.1, $u^\star$ satisfies (2.3) in $B_\delta(0)$ with

$$0<\lambda\leq w^\star=-\frac{u^\star_{\eta\eta}}{u^\star_{\xi\xi}}\leq\Lambda.$$

By Theorem 2.2,

$$\|w^\star\|_{W^{1,3}(B_{7\delta/8}(0))}\leq C.$$

Note that $n=2$. By the Sobolev theorem, we have the C^α estimate of $w^\star$. And by the interior $W^{2,p}$-estimate of the uniformly elliptic equation (2.3), we have the estimate

$$\|w^\star\|_{W^{2,3/2}(B_{3\delta/4}(0))}\leq C,$$

which implies the $W^{1,6}$-estimate of $w^\star$. Again by the interior $W^{2,p}$-estimate of the uniformly elliptic equation (2.3), we have

$$\|w^\star\|_{W^{2,3}(B_{\delta/2}(0))} \le C,$$

which implies the $C^{1,\alpha}$ estimate of $w^\star$. Then by the Schauder estimate of (2.3), we have

$$\|w^\star\|_{C^{2,\alpha}(B_{\delta/4}(0))} \le C$$

and all the higher order estimates of $u^\star$. Transforming back by the partial Legendre transform, we obtain the lower bound of $u_{x_1x_1}$ by (2.2). Since we can do partial Legendre transforms of u in any direction, we can obtain the lower bound for the smallest eigenvalue of D^2u, which implies the boundedness of D^2u by (1.4). Then we have all the higher order estimates of u. ∎

3. The inhomogeneous equations in dimension two

In this section, we will study the interior estimate for the inhomogeneous equation (1.5).

We first recall the regularity theory of the second order elliptic equation in divergence form,

$$D_j(a_{ij}(x)D_iu) = D_ig^i + h \quad \text{in } \Omega, \tag{3.1}$$

where $g = (g^1, \dots, g^n)$ is a vector valued function and $\{a_{ij}(x)\}$ satisfies

$$0 < \lambda(x)|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda(x)|\xi|^2, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}$$

and can be discontinuous. When $\{a_{ij}(x)\}$ is uniformly elliptic, a fundamental regularity theory was established by De Giorgi, Nash, Moser, etc. In [28,31], the classical De Giorgi–Nash–Moser theory was extended to degenerate linear elliptic equation with $\lambda(x)^{-1} \in L^p(\Omega)$ and $\Lambda(x) \in L^q(\Omega)$, where $1/p + 1/q < 2/n$. We denote by S_n^+ the set of $n \times n$ nonnegative definite matrices.

Theorem 3.1 (Theorem 4.2 in [31]). *Let $\{a_{ij}\}: \Omega \to S_n^+$ be such that $\lambda^{-1} \in L^p_{\mathrm{loc}}(\Omega)$ for some $p > n$. Let $g \in L^\infty(\Omega; \mathbb{R}^n)$ and $h \in L^q(\Omega)$. Assume p and q satisfy $1/p + 1/q < 2/n$. Suppose that u is a subsolution (supersolution) to (3.1) in $B_R = B_R(y) \subset \Omega$ and that $u \le 0$ ($u \ge 0$) on ∂B_R. Then*

$$\sup_{B_R} u(-u) \le C\|\lambda^{-1}\|_{L^p(B_R)}\big(\|g\|_{L^\infty(B_R)}R^{1-n/p} + \|h\|_{L^q(B_R)}R^{2-n/q-n/p}\big).$$

In [27], the author investigated the linearized Monge–Ampère equation with right-hand side $f = D_ig^i + h$, i.e., $a_{ij}(x) = U^{ij}$ in (3.1). Under the stronger assumption that $\det D^2u$ is sufficiently close to a positive constant, he can use the $W^{2,p}$-estimate of the Monge–Ampère equation and Theorem 3.1 to obtain the interior regularity. It was later shown in [20] that when $n = 2$ we only need the determinant to be bounded from above and below.

Now we turn to the fourth order equation (1.5). By the divergence free property of U^{ij}, i.e., $\sum_i D_i U^{ij} = 0$, we can rewrite equation (1.5) in divergence form:

$$D_j(U^{ij} D_i w) = D_i g^i + h. \tag{3.2}$$

Note that in dimension two, it holds

$$\frac{\det D^2 u}{\Delta u} I \le U^{ij} \le \Delta u\, I.$$

In view of Theorem 3.1, it suffices to get the L^p-bound of Δu for sufficiently large p. In the following, we will use an integral method directly in (1.5) to derive it. Instead of the classical Sobolev inequality, we will need the following Monge–Ampère Sobolev inequality.

Lemma 3.2 (Proposition 2.6 in [20]). *Assume $n = 2$. Let u be a smooth, strictly convex function defined in a neighborhood of a bounded domain $\Omega \subset \mathbb{R}^2$. Suppose u satisfies*

$$0 < \lambda \le \det D^2 u \le \Lambda. \tag{3.3}$$

Then for any $\chi > 2$ there exists a constant $C > 0$, depending only on λ, Λ and χ, such that

$$\Big(\int_\Omega |v|^\chi\, dx\Big)^{1/\chi} \le C\Big(\int_\Omega U^{ij} v_i v_j\, dx\Big)^{1/2} \quad \text{for all } v \in C_0^\infty(\Omega).$$

The above inequality is the two-dimensional counterpart of the Monge–Ampère Sobolev inequality in higher dimensions derived by Tian and Wang [30].

Theorem 3.3. *Assume $n = 2$ and $\theta \in [0, 1]$. Let $g \in L^\infty(\Omega; \mathbb{R}^n)$ and $h \in L^q(\Omega)$. Let u be a uniformly convex smooth solution to equation (3.2) in B_R satisfying (3.3). Then for any $p \ge 1$, there exists a constant $C > 0$ depending on λ, Λ, R, θ, $\|g\|_{L^\infty(B_R)}$, $\|h\|_{L^q(B_R)}$ and p such that*

$$\|\Delta u\|_{L^p(B_{R/2})} \le C.$$

Proof. We first consider the case $\theta \in [0, 1)$. Denote

$$v = \Delta u, \quad w = (\det D^2 u)^{-(1-\theta)}, \quad \rho = R^2 - |x|^2.$$

We consider

$$z = v^\beta \varphi \rho^\alpha,$$

where $\alpha, \beta > 0$ are constants and $\varphi = \varphi(w)$ is a positive function to be determined. Then

$$\begin{aligned} z_i &= \beta v_i v^{\beta-1} \varphi \rho^\alpha + v^\beta (\varphi\rho^\alpha)_i, \\ z_{ij} &= \beta(\beta-1) v_i v_j v^{\beta-2} \varphi\rho^\alpha + \beta v_{ij} v^{\beta-1} \varphi\rho^\alpha + \beta v_i v^{\beta-1} (\varphi\rho^\alpha)_j \\ &\quad + \beta v_j v^{\beta-1} (\varphi\rho^\alpha)_i + v^\beta (\varphi\rho^\alpha)_{ij}. \end{aligned}$$

By integration by parts with (1.5) and choosing $\alpha > 2$, we have

$$
\begin{aligned}
0 &= \int_{B_R} U^{ij} z_{ij}\, dx \\
&= \beta(\beta-1)\int_{B_R} U^{ij}\, v_i\, v_j\, \varphi\, \rho^\alpha\, v^{\beta-2}\, dx + \beta\int_{B_R} U^{ij}\, v_{ij}\, \varphi\, \rho^\alpha\, v^{\beta-1}\, dx \\
&\quad + 2\beta\int_{B_R} U^{ij}\, v_i\, w_j\, \varphi'\, \rho^\alpha\, v^{\beta-1}\, dx + 2\alpha\beta\int_{B_R} U^{ij}\, v_i\, \rho_j\, \varphi\, \rho^{\alpha-1}\, v^{\beta-1}\, dx \\
&\quad + \int_{B_R} U^{ij} w_i w_j \varphi'' \rho^\alpha v^\beta\, dx + 2\alpha\int_{B_R} U^{ij} w_i \rho_j \varphi' \rho^{\alpha-1} v^\beta\, dx + \int_{B_R} f\varphi'\rho^\alpha v^\beta\, dx \\
&\quad + \alpha\int_{B_R} U^{ij}\, \rho_{ij}\, \varphi\, \rho^{\alpha-1}\, v^\beta\, dx + \alpha(\alpha-1)\int_{B_R} U^{ij}\, \rho_i\, \rho_j\, \varphi\, \rho^{\alpha-2}\, v^\beta\, dx.
\end{aligned}
\tag{3.4}
$$

Note that for any m,

$$
\begin{aligned}
u^{ij}(u_{mm})_{ij} &= u^{il}\, u^{kj}\, u_{mij}\, u_{mkl} + (\ln\det D^2u)_{mm} \\
&= u^{il}\, u^{kj}\, u_{mij}\, u_{mkl} + \frac{1}{1-\theta}\frac{w_m^2}{w^2} - \frac{1}{1-\theta}\frac{w_{mm}}{w}.
\end{aligned}
$$

We have

$$
\begin{aligned}
\int_{B_R} U^{ij}\, v_{ij}\, \varphi\, \rho^\alpha\, v^{\beta-1}\, dx &= \int_{B_R} \det D^2u\, u^{il}\, u^{kj}\, u_{mij}\, u_{mkl}\, \varphi\, \rho^\alpha\, v^{\beta-1}\, dx \\
&\quad + \frac{1}{1-\theta}\int_{B_R} \det D^2u\, \frac{|\nabla w|^2}{w^2}\, \varphi\, \rho^\alpha\, v^{\beta-1}\, dx \\
&\quad - \frac{1}{1-\theta}\int_{B_R} \det D^2u\, \frac{\triangle w}{w}\, \varphi\, \rho^\alpha\, v^{\beta-1}\, dx.
\end{aligned}
$$

By integration by parts, we have

$$
\begin{aligned}
\int_{B_R} \det D^2u\, \frac{\triangle w}{w}\, \varphi\, \rho^\alpha\, v^{\beta-1}\, dx &= \int_{B_R} w^{-\frac{2-\theta}{1-\theta}}\, \triangle w\, \varphi\, \rho^\alpha\, v^{\beta-1}\, dx \\
= \frac{2-\theta}{1-\theta}\int_{B_R} w^{-\frac{3-2\theta}{1-\theta}}\, |\nabla w|^2\, \varphi\, \rho^\alpha\, v^{\beta-1}\, dx &- \int_{B_R} w^{-\frac{2-\theta}{1-\theta}}\, |\nabla w|^2\, \varphi'\, \rho^\alpha\, v^{\beta-1}\, dx \\
- \alpha\int_{B_R} w^{-\frac{2-\theta}{1-\theta}}\, w_i\, \rho_i\, \varphi\, \rho^{\alpha-1}\, v^{\beta-1}\, dx &- (\beta-1)\int_{B_R} w^{-\frac{2-\theta}{1-\theta}}\, w_i\, v_i\, \varphi\, \rho^\alpha\, v^{\beta-2}\, dx.
\end{aligned}
\tag{3.5}
$$

By changing coordinates at each point, say x_0, we can assume that $D^2u(x_0)$ is diagonal. Then

$$
u^{il}\, u^{kj}\, u_{mij}\, u_{mkl} = \frac{u_{mij}^2}{u_{ii}u_{jj}} \geq \frac{u_{mmi}^2}{u_{ii}u_{mm}} \geq \frac{v_i^2}{u_{ii}\, v} = \frac{u^{ij}\, v_i\, v_j}{v},
\tag{3.6}
$$

where we used the Cauchy inequality in the second inequality. Hence by (3.5) and (3.6), we have

$$\begin{aligned}&\int_{B_R} U^{ij} v_{ij} \varphi\rho^\alpha v^{\beta-1}\,dx\\ &\geq \int_{B_R} U^{ij} v_i v_j \varphi\rho^\alpha v^{\beta-2}\,dx + \frac{1}{1-\theta}\int_{B_R} w^{-\frac{2-\theta}{1-\theta}}\Big(\varphi' - \frac{\varphi w^{-1}}{1-\theta}\Big)|\nabla w|^2\rho^\alpha v^{\beta-1}\,dx\\ &\quad+\frac{\alpha}{1-\theta}\int_{B_R} w^{-\frac{2-\theta}{1-\theta}} w_i\rho_i\varphi\rho^{\alpha-1}v^{\beta-1}\,dx + \frac{\beta-1}{1-\theta}\int_{B_R} w^{-\frac{2-\theta}{1-\theta}} w_i v_i\varphi\rho^\alpha v^{\beta-2}\,dx.\end{aligned}\tag{3.7}$$

Then putting (3.7) into (3.4) yields

$$\begin{aligned}0 \geq{}& \beta^2\int_{B_R} U^{ij} v_i v_j\varphi\rho^\alpha v^{\beta-2}\,dx + \frac{\beta}{1-\theta}\int_{B_R} w^{-\frac{2-\theta}{1-\theta}}(\varphi' - \frac{\varphi w^{-1}}{1-\theta})|\nabla w|^2\rho^\alpha v^{\beta-1}\,dx\\ &+\frac{\alpha\beta}{1-\theta}\int_{B_R} w^{-\frac{2-\theta}{1-\theta}} w_i\rho_i\varphi\rho^{\alpha-1}v^{\beta-1}\,dx + \frac{\beta(\beta-1)}{1-\theta}\int_{B_R} w^{-\frac{2-\theta}{1-\theta}} w_i v_i\varphi\rho^\alpha v^{\beta-2}\,dx\\ &+2\beta\int_{B_R} U^{ij} v_i w_j\varphi'\rho^\alpha v^{\beta-1}\,dx + 2\alpha\beta\int_{B_R} U^{ij} v_i\rho_j\varphi\rho^{\alpha-1}v^{\beta-1}\,dx\\ &+\int_{B_R} U^{ij} w_i w_j\varphi''\rho^\alpha v^\beta\,dx + \int_{B_R} f\varphi'\rho^\alpha v^\beta\,dx + 2\alpha\int_{B_R} U^{ij} w_i\rho_j\varphi'\rho^{\alpha-1}v^\beta\,dx\\ &+\alpha\int_{B_R} U^{ij}\rho_{ij}\varphi\rho^{\alpha-1}v^\beta\,dx + \alpha(\alpha-1)\int_{B_R} U^{ij}\rho_i\rho_j\varphi\rho^{\alpha-2}v^\beta\,dx.\end{aligned}\tag{3.8}$$

Now we choose $\varphi(w) = e^{e^{Aw}}$, where $A > 0$ is to be determined later. Then we know that

$$\varphi'(w) = Ae^{Aw}\varphi, \quad \varphi''(w) = (A^2e^{Aw} + A^2e^{2Aw})\,\varphi.\tag{3.9}$$

We will estimate the right-hand side of (3.8) term by term.

By (3.9), the two terms of the third line in (3.8) satisfy

$$\begin{aligned}\Big|2\beta\int_{B_R} U^{ij} v_i w_j\varphi'\rho^\alpha v^{\beta-1}\,dx\Big| &= \Big|\int_{B_R} 2\beta Ae^{Aw} U^{ij} v_i w_j\varphi\rho^\alpha v^{\beta-1}\,dx\Big|\\ &\leq \int_{B_R}\beta^2\frac{e^{Aw}}{1/2 + e^{Aw}} U^{ij} v_i v_j\varphi\rho^\alpha v^{\beta-2}\,dx\\ &\quad+\int_{B_R}\Big(\frac{A^2}{2}e^{Aw} + A^2e^{2Aw}\Big)U^{ij} w_i w_j\varphi\rho^\alpha v^\beta\,dx,\end{aligned}$$

and

$$\begin{aligned}\Big|2\alpha\beta\int_{B_R} U^{ij} v_i\rho_j\varphi\rho^{\alpha-1}v^{\beta-1}\,dx\Big| &\leq \int_{B_R}\frac{\beta^2}{8(1+2e^{Aw})} U^{ij} v_i v_j\varphi\rho^\alpha v^{\beta-2}\,dx\\ &\quad+\int_{B_R} 8(1+2e^{Aw})\alpha^2 U^{ij}\rho_i\rho_j\varphi\rho^{\alpha-2}v^\beta\,dx.\end{aligned}$$

Note that

$$|\nabla\rho|^2 \le \frac{\triangle u}{\det D^2u}\,U^{ij}\rho_i\rho_j \quad \text{and} \quad |\nabla v|^2 \le \frac{\triangle u}{\det D^2u}\,U^{ij}v_iv_j.$$

The two terms of the second line in (3.8) satisfy

$$\begin{aligned}
&\left|\frac{\alpha\beta}{1-\theta}\int_{B_R} w^{-\frac{2-\theta}{1-\theta}} w_i\rho_i\varphi\rho^{\alpha-1}v^{\beta-1}\,dx\right| \\
&\le \int_{B_R}\frac{\beta}{(1-\theta)^2}w^{-\frac{3-2\theta}{1-\theta}}|\nabla w|^2\varphi\rho^\alpha v^{\beta-1}\,dx + \int_{B_R}\frac{1}{4}\alpha^2\beta w^{-\frac{1}{1-\theta}}|\nabla\rho|^2\varphi\rho^{\alpha-2}v^{\beta-1}\,dx \\
&\le \int_{B_R}\frac{\beta}{(1-\theta)^2}w^{-\frac{3-2\theta}{1-\theta}}|\nabla w|^2\varphi\rho^\alpha v^{\beta-1}\,dx + \int_{B_R}\frac{1}{4}\alpha^2\beta U^{ij}\rho_i\rho_j\varphi\rho^{\alpha-2}v^{\beta}\,dx,
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{\beta(\beta-1)}{1-\theta}\int_{B_R} w^{-\frac{2-\theta}{1-\theta}} w_i v_i\varphi\rho^\alpha v^{\beta-2}\,dx\right|
&\le \int_{B_R}\frac{(\beta-1)^2}{3(1-\theta)^2}(1+2e^{Aw})w^{-\frac{3-2\theta}{1-\theta}}|\nabla w|^2\varphi\rho^\alpha v^{\beta-1}\,dx \\
&\quad + \int_{B_R}\frac{3\beta^2 w^{-\frac{1}{1-\theta}}}{4(1+2e^{Aw})}|\nabla v|^2\varphi\rho^\alpha v^{\beta-3}\,dx \\
&\le \int_{B_R}\frac{(\beta-1)^2}{3(1-\theta)^2}(1+2e^{Aw})w^{-\frac{3-2\theta}{1-\theta}}|\nabla w|^2\varphi\rho^\alpha v^{\beta-1}\,dx \\
&\quad + \int_{B_R}\frac{3\beta^2}{4(1+2e^{Aw})}U^{ij}v_iv_j\varphi\rho^\alpha v^{\beta-2}\,dx.
\end{aligned}$$

By (3.9), the last term of the fourth line in (3.8) satisfies

$$\begin{aligned}
&\left|2\alpha\int_{B_R}U^{ij}w_i\rho_j\varphi'\rho^{\alpha-1}v^\beta\,dx\right| \\
&\qquad \le \int_{B_R}\frac{A^2}{2}e^{Aw}U^{ij}w_iw_j\varphi\rho^\alpha v^\beta\,dx + \int_{B_R}2\alpha^2e^{Aw}U^{ij}\rho_i\rho_j\varphi\rho^{\alpha-2}v^\beta\,dx.
\end{aligned}$$

Note that $f = \operatorname{div}(g) + h$, where g is a bounded vector field. We denote

$$|g| = \sqrt{\textstyle\sum_i (g^i)^2}.$$

Then by integration by parts, we have

$$\begin{aligned}
&\left|\int_{B_R} f\varphi'\rho^\alpha v^\beta\,dx\right| \\
&= \left|\int_{B_R} h\varphi'\rho^\alpha v^\beta\,dx - \int_{B_R} g^i(w_i\varphi''\rho^\alpha v^\beta + \alpha\rho_i\varphi'\rho^{\alpha-1}v^\beta + \beta v_i\varphi'\rho^\alpha v^{\beta-1})\,dx\right| \\
&\le \int_{B_R}\frac{1}{2}w^{-\frac{3-2\theta}{1-\theta}}|\nabla w|^2\varphi\rho^\alpha v^{\beta-1}\,dx + \int_{B_R}\frac{1}{2}(A^2e^{Aw}+A^2e^{2Aw})^2 w^{\frac{3-2\theta}{1-\theta}}|g|^2\varphi\rho^\alpha v^{\beta+1}\,dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_{B_R}\frac{\alpha}{2}w^{-\frac{1}{1-\theta}}|\nabla\rho|^2Ae^{Aw}\varphi\rho^{\alpha-2}v^{\beta-1}\,dx+\int_{B_R}\frac{\alpha}{2}w^{\frac{1}{1-\theta}}|g|^2Ae^{Aw}\varphi\rho^{\alpha}v^{\beta+1}\,dx\\
&+\int_{B_R}|h|\varphi'\rho^{\alpha}v^{\beta}\,dx+\int_{B_R}\frac{\beta^2w^{-\frac{1}{1-\theta}}}{16(1+2e^{Aw})}|\nabla v|^2\varphi\rho^{\alpha}v^{\beta-3}\,dx\\
&+\int_{B_R}4(1+2e^{Aw})A^2e^{2Aw}w^{\frac{1}{1-\theta}}|g|^2\varphi\rho^{\alpha}v^{\beta+1}\,dx\\
\leq&\int_{B_R}\frac{1}{2}w^{-\frac{3-2\theta}{1-\theta}}|\nabla w|^2\varphi\rho^{\alpha}v^{\beta-1}\,dx+\int_{B_R}\frac{1}{2}(A^2e^{Aw}+A^2e^{2Aw})^2w^{\frac{3-2\theta}{1-\theta}}|g|^2\varphi\rho^{\alpha}v^{\beta+1}dx\\
&+\int_{B_R}\frac{\alpha}{2}U^{ij}\rho_i\rho_jAe^{Aw}\varphi\rho^{\alpha-2}v^{\beta}\,dx+\int_{B_R}\frac{\alpha}{2}w^{\frac{1}{1-\theta}}|g|^2Ae^{Aw}\varphi\rho^{\alpha}v^{\beta+1}\,dx\\
&+\int_{B_R}|h|\varphi'\rho^{\alpha}v^{\beta}\,dx+\int_{B_R}\frac{\beta^2}{16(1+2e^{Aw})}U^{ij}v_iv_j\varphi\rho^{\alpha}v^{\beta-2}\,dx\\
&+\int_{B_R}4(1+2e^{Aw})A^2e^{2Aw}w^{\frac{1}{1-\theta}}|g|^2\varphi\rho^{\alpha}v^{\beta+1}\,dx.
\end{aligned}
$$

Hence, (3.8) reduces to

$$
\begin{aligned}
0\geq&\int_{B_R}\frac{\beta^2}{16(1+2e^{Aw})}U^{ij}v_iv_j\varphi\rho^{\alpha}v^{\beta-2}\,dx\\
&+\alpha\int_{B_R}U^{ij}\rho_{ij}\varphi\rho^{\alpha-1}v^{\beta}\,dx-\int_{B_R}|h|\varphi'\rho^{\alpha}v^{\beta}\,dx\\
&+\int_{B_R}\Big(\frac{\beta Awe^{Aw}}{1-\theta}-\frac{(\beta-1)^2}{3(1-\theta)^2}(1+2e^{Aw})-\frac{2\beta}{(1-\theta)^2}-\frac{1}{2}\Big)\\
&\quad\times w^{-\frac{3-2\theta}{1-\theta}}|\nabla w|^2\varphi\rho^{\alpha}v^{\beta-1}\,dx\\
&+\int_{B_R}\Big(\alpha(\alpha-1)-(8+18e^{Aw}+\frac{1}{4}\beta)\alpha^2-\frac{\alpha}{2}Ae^{Aw}\Big)U^{ij}\rho_i\rho_j\varphi\rho^{\alpha-2}v^{\beta}\,dx\\
&-\int_{B_R}\frac{1}{2}(A^2e^{Aw}+A^2e^{2Aw})^2w^{\frac{3-2\theta}{1-\theta}}|g|^2\varphi\rho^{\alpha}v^{\beta+1}\,dx\\
&-\int_{B_R}\frac{\alpha}{2}w^{\frac{1}{1-\theta}}|g|^2Ae^{Aw}\varphi\rho^{\alpha}v^{\beta+1}\,dx\\
&-\int_{B_R}4(1+2e^{Aw})A^2e^{2Aw}w^{\frac{1}{1-\theta}}|g|^2\varphi\rho^{\alpha}v^{\beta+1}\,dx.
\end{aligned}
$$

Now we choose A sufficiently large such that

$$
\frac{\beta}{1-\theta}Awe^{Aw}-\frac{(\beta-1)^2}{3(1-\theta)^2}(1+2e^{Aw})-\frac{2\beta}{(1-\theta)^2}-\frac{1}{2}>0,
$$

i.e.,

$$
(3\beta(1-\theta)Aw-2(\beta-1)^2)e^{Aw}-(\beta-1)^2-6\beta-\frac{3(1-\theta)^2}{2}>0.
$$

Note that
$$U^{ij}\rho_i\rho_j \le v^{n-1}|\nabla\rho|^2 \quad\text{and}\quad |U^{ij}\rho_{ij}| \le 2n\, v^{n-1}.$$
Then we have
$$\begin{aligned}\int_{B_R} & U^{ij} v_i v_j \,\rho^\alpha\, v^{\beta-2}\,dx \\ &\le C\int_{B_R}\rho^\alpha v^{\beta+1}\,dx + C\int_{B_R}\rho^{\alpha-2} v^{\beta+n-1}\,dx + C\int_{B_R}|h|\,\rho^\alpha v^\beta\,dx.\end{aligned}$$
Since
$$\begin{aligned}U^{ij}&(v^{\beta/2}\rho^{\alpha/2})_i(v^{\beta/2}\rho^{\alpha/2})_j \\ &= \frac{\beta^2}{4}U^{ij}v_iv_j\,\rho^\alpha v^{\beta-2} + \frac{\alpha^2}{4}U^{ij}\rho_i\rho_j\,\rho^{\alpha-2}v^\beta + \frac{\alpha\beta}{2}U^{ij}v_i\,\rho_j\,\rho^{\alpha-1}v^{\beta-1} \\ &\le \frac{\beta^2}{2}U^{ij}v_iv_j\,\rho^\alpha v^{\beta-2} + \frac{\alpha^2}{2}U^{ij}\rho_i\rho_j\,\rho^{\alpha-2}v^\beta,\end{aligned}$$
we have
$$\begin{aligned}\int_{B_R} & U^{ij}(v^{\beta/2}\rho^{\alpha/2})_i(v^{\beta/2}\rho^{\alpha/2})_j\,dx \\ &\le C\int_{B_R}\rho^\alpha v^{\beta+1}\,dx + C\int_{B_R}\rho^{\alpha-2}v^{\beta+n-1}\,dx + C\int_{B_R}|h|\,\rho^\alpha v^\beta\,dx \\ &\le C\int_{B_R}v^{\beta+n-1}\,dx + C\Big(\int_{B_R}|h|^q\,dx\Big)^{1/q}\Big(\int_{B_R}(v^{\beta/2}\rho^{\alpha/2})^{\frac{2q}{q-1}}\,dx\Big)^{1-1/q}.\end{aligned}$$
Choosing $\chi \ge \frac{2q}{q-1}$ and using Lemma 3.2 with $n = 2$, we have

(3.10)
$$\Big[\int_{B_R}(v^{\beta/2}\rho^{\alpha/2})^\chi\,dx\Big]^{2/\chi} \le C\int_{B_R}v^{\beta+1}\,dx + C.$$

By the interior $W^{2,1+\varepsilon}$-estimates for Monge–Ampère equation [9, 29] with (3.3), we know that there is a small $\varepsilon_0 > 0$, that depends only on λ and Λ, such that $\|v\|_{L^{1+\varepsilon_0}} \le C(\lambda,\Lambda)$. Then in (3.10), we choose $\beta = \varepsilon_0$ and $\chi = 2p/\varepsilon_0$ for $p \in (\frac{q}{q-1}\varepsilon_0, +\infty)$ to get $\|\Delta u\|_{L^p(B_{R/2})} \le C$.

For $\theta = 1$, we know that $w = \ln\det D^2u$, and we can obtain the L^p-bound of Δu following the same method used above. Alternatively, we can apply the Legendre transform to u for the case $\theta = 1$ to get the estimate of D^2u by the strictly convexity of u with condition (3.3). ∎

Now we establish higher estimates of u in terms of the $W^{2,p}$-estimate. By chaining together a sequence of balls, in a standard fashion, we know $\Delta u \in L^p_{\text{loc}}(\Omega)$ for any fixed $p \in [1,+\infty)$. Then by Theorem 3.1 and the same arguments as in Proposition 6.1 of [27] or Theorem 1.3 in [20], we get the Hölder continuity of w and all the higher order estimates of u.

As we mentioned in the introduction, there is another approach to establish the C^2-estimate from $W^{2,p}$ when $f \in W^{1,q}(B_1)$, which holds in any dimension.

Theorem 3.4. *Let u be a uniformly convex smooth solution to equation (1.5) in B_1 with $\theta = 0$ or 1. Assume that $f \in W^{1,q}(B_1)$ for some $q > n/2$. Suppose u satisfies (3.3). Assume that $\|\Delta u\|_{L^{p_n}(B_1)}$ is bounded for some $p_n > n(n-1)/2$. Then there exists a constant $C > 0$, depending on λ, Λ, $\|\Delta u\|_{L^{p_n}(B_1)}$ and $\|f\|_{W^{1,q}(B_1)}$, such that*

$$\sup_{B_{1/2}} \Delta u \le C.$$

Proof. The proof is inspired by [6]. We consider the case $\theta = 1$. The case $\theta = 0$ then follows by using the Legendre transform. Denote $v = u^{kl} w_k w_l$, where $\{u^{kl}\}$ is the inverse matrix of $D^2 u$. By direct calculations,

$$\begin{aligned}
v_i &= -u^{ks} u^{tl} u_{sti} w_k w_l + 2u^{kl} w_{ik} w_l,\\
v_{ij} &= u^{kp} u^{rs} u^{tl} u_{prj} u_{sti} w_k w_l + u^{ks} u^{lp} u^{rt} u_{prj} u_{sti} w_k w_l - u^{ks} u^{tl} u_{stij} w_k w_l\\
&\quad - 2u^{ks} u^{tl} u_{sti} w_{jk} w_l - 2u^{ks} u^{tl} u_{stj} w_{ik} w_l + 2u^{kl} w_{ijk} w_l + 2u^{kl} w_{ik} w_{jl}.
\end{aligned}$$

For any $x_0 \in B_1$, we can choose a coordinate transformation so that $u_{ij}(x_0) = u_{ii}(x_0)\delta_{ij}$. Then

$$\begin{aligned}
u^{ij} v_{ij}(x_0) &= 2u^{ii} u^{kk} u^{rr} u^{ll} u_{kri} u_{lri} w_k w_l - u^{ii} u^{kk} u^{ll} u_{iikl} w_k w_l\\
&\quad - 4u^{ii} u^{kk} u^{ll} u_{kli} w_{ik} w_l + 2u^{ii} u^{kk} w_{iik} w_k + 2u^{ii} u^{kk} w_{ik}^2. \qquad (3.11)
\end{aligned}$$

Differentiating $\ln\det D^2 u = w$ twice, we have

$$u^{ij} u_{ijkl} - u^{is} u^{tj} u_{ijk} u_{stl} = (\ln\det D^2 u)_{kl} = w_{kl}. \qquad (3.12)$$

Note that (1.5) can be written as $u^{ij} w_{ij} = e^{-w} f$. Differentiating the equation respect to x_k-direction directly yields

$$-u^{ip} u^{rj} u_{prk} w_{ij} + u^{ij} w_{ijk} = e^{-w}(f_k - w_k f). \qquad (3.13)$$

Inserting (3.12) and (3.13) into (3.11), we have

$$\begin{aligned}
u^{ij} v_{ij}(x_0) &= u^{kk} u^{ll} u^{ii} u^{jj} u_{kji} u_{lji} w_k w_l - 2u^{ii} u^{kk} u^{ll} u_{kli} w_{ik} w_l + u^{ii} u^{kk} w_{ik}^2\\
&\quad + u^{ii} u^{kk} w_{ik} w_{ik} - u^{kk} u^{ll} w_{kl} w_k w_l + 2u^{kk} e^{-w}(f_k - w_k f) w_k\\
&= u^{ii} u^{kk} \Big(\sum_l u^{ll} u_{ikl} w_l\Big)^2 - 2u^{ii} u^{kk} \Big(\sum_l u^{ll} u_{ikl} w_l\Big) w_{ik} + u^{ii} u^{kk} w_{ik}^2\\
&\quad + u^{ii} u^{kk} w_{ik}^2 - u^{kk} u^{ll} w_{kl} w_k w_l + 2u^{kk} e^{-w}(f_k - w_k f) w_k\\
&= u^{ii} u^{kk} |w_{ik} - \sum_l u^{ll} u_{ikl} w_l|^2 + u^{ii} u^{kk} w_{ik}^2 - u^{kk} u^{ll} w_{kl} w_k w_l\\
&\quad + 2u^{kk} e^{-w}(f_k - w_k f) w_k. \qquad (3.14)
\end{aligned}$$

Next, we compute

$$\begin{aligned}
&u^{ij} (e^{\frac{1}{2}w} v)_{ij}(x_0)\\
&= \frac{1}{4} e^{\frac{1}{2}w} u^{ii} w_i^2 u^{kk} w_k^2 + \frac{1}{2} e^{\frac{1}{2}w} u^{ii} w_{ii} v + e^{\frac{1}{2}w} u^{ii} w_i v_i + e^{\frac{1}{2}w} u^{ii} v_{ii}. \qquad (3.15)
\end{aligned}$$

Note that

$$
\begin{aligned}
u^{ii} w_i v_i &= -u^{ii} u^{kk} u^{ll} u_{ikl} w_k w_l w_i + 2u^{ii} u^{kk} w_{ki} w_k w_i \\
&= u^{ii} u^{kk} w_k w_i (w_{ki} - \sum_l u^{ll} u_{ikl} w_l) + u^{ii} u^{kk} w_{ki} w_k w_i .
\end{aligned} \tag{3.16}
$$

Combining (3.14), (3.15) and (3.16), we have

$$
\begin{aligned}
u^{ij} (e^{w/2} v)_{ij} (x_0) = u^{ii} u^{kk} \Big| w_{ik} - \sum_l u^{ll} u_{ikl} w_l + \frac{1}{2} w_i w_k \Big|^2 e^{w/2} + u^{ii} u^{kk} w_{ik}^2 e^{w/2} \\
- \frac{3}{2} e^{-w/2} f v + 2e^{-w/2} u^{kk} f_k w_k .
\end{aligned} \tag{3.17}
$$

By (3.12), we have

$$
u^{ij} (\Delta u)_{ij} (x_0) = u^{ii} u^{jj} u_{ijk}^2 + \Delta w. \tag{3.18}
$$

Let $z = e^{w/2} v + \Delta u$. Note that

$$
\begin{aligned}
\Delta w = \sum_k w_{kk} &\le \frac{1}{2} (u^{kk})^2 w_{kk}^2 e^{w/2} + \frac{1}{2} u_{kk}^2 e^{-w/2} \\
&\le \frac{1}{2} (u^{kk})^2 w_{kk}^2 e^{w/2} + \frac{1}{2} (\Delta u)^2 e^{-w/2}.
\end{aligned}
$$

Then by (3.17) and (3.18), we have

$$
\begin{aligned}
u^{ij} z_{ij} (x_0) &\ge -C|f|v - C|f_k|v - C|f_k| + u^{kk} u^{ll} w_{kl}^2 e^{w/2} + \Delta w \\
&\ge -C(|f| + |f_k|)v - C|f_k| + (u^{kk})^2 w_{kk}^2 e^{w/2} - \frac{1}{2} (u^{kk})^2 w_{kk}^2 e^{w/2} \\
&\quad - \frac{1}{2} (\Delta u)^2 e^{-w/2} \\
&\ge -C(|f| + |f_k| + \Delta u) z,
\end{aligned} \tag{3.19}
$$

where $C = C(\lambda, \Lambda)$. In the last inequality we used

$$
z \ge \Delta u \ge n (\det D^2 u)^{1/n} \ge n \lambda^{1/n}.
$$

Since (3.19) is valid at every point in B_1, by (3.3), we have the following inequality:

$$
D_j (U^{ij} D_i z) \ge -gz - C \Delta u z \quad \text{in } B_1, \tag{3.20}
$$

where $g = C(|f| + |f_k|) \in L^q(B_1)$.

Next, we derive the upper bound of z by integration and iteration. Let $\eta \in C_0^\infty(B_1)$ be a cutoff function. Multiplying (3.20) by $\varphi = \eta^2 z^{\beta - 1}$ with $\beta \ge 2$ and by integration by parts, we have

$$
\begin{aligned}
(\beta-1)\int_{B_1} U^{ij} z_i z_j \eta^2 z^{\beta-2}\,dx
&\le -2\int_{B_1} U^{ij} z_i \eta_j \eta z^{\beta-1}\,dx + C\int_{B_1}\Delta u\cdot\eta^2 z^\beta\,dx + \int_{B_1} g\eta^2 z^\beta\,dx\\
&\le \frac{\beta-1}{2}\int_{B_1} U^{ij} z_i z_j \eta^2 z^{\beta-2}\,dx + \frac{2}{\beta-1}\int_{B_1} U^{ij}\eta_i\eta_j z^\beta\,dx\\
&\quad + C\int_{B_1}\Delta u\cdot\eta^2 z^\beta\,dx + \int_{B_1} g\eta^2 z^\beta\,dx,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\int_{B_1} U^{ij}(\eta z^{\beta/2})_i(\eta z^{\beta/2})_j\,dx
\le C\beta\Big(\int_{B_1} U^{ij}\eta_i\eta_j z^\beta\,dx + \int_{B_1}\Delta u\cdot\eta^2 z^\beta\,dx + \int_{B_1} g\eta^2 z^\beta\,dx\Big).
\end{aligned}
$$

Then by the Monge–Ampère Sobolev inequality [30] and $(U^{ij}) \le (\Delta u)^{n-1} I$, we have

$$
\begin{aligned}
&\|\eta z^{\beta/2}\|^2_{L^p(B_1)}\\
&\le C_0\beta\Big(\int_{B_1}(\Delta u)^{n-1}|D\eta|^2 z^\beta\,dx + \int_{B_1}\Delta u\cdot\eta^2 z^\beta\,dx + \int_{B_1} g\eta^2 z^\beta\,dx\Big),
\end{aligned}
\tag{3.21}
$$

where $p = 2^* = \frac{2n}{n-2}$ for $n > 2$ and $p > 2$ for $n = 2$. Then by Hölder's inequality,

$$
\int_{B_1} g\eta^2 z^\beta\,dx \le \Big(\int_{B_1} g^q\,dx\Big)^{1/q}\Big(\int_{B_1}|\eta z^{\beta/2}|^{\frac{2q}{q-1}}\Big)^{1-1/q}.
$$

Since $q > n/2$, we have $2 < \frac{2q}{q-1} < 2^*$. By Hölder's inequality and Young's inequality, we obtain

$$
\|\eta z^{\beta/2}\|_{L^{\frac{2q}{q-1}}(B_1)} \le \varepsilon\|\eta z^{\beta/2}\|_{L^p(B_1)} + C(n,q)\,\varepsilon^{-\frac{n}{2q-n}}\|\eta z^{\beta/2}\|_{L^2(B_1)}.
$$

Then we choose $\varepsilon = (4C_0\|g\|_{L^q(B_1)}\beta + 1)^{-1/2}$. By (3.21),

$$
\begin{aligned}
&\|\eta z^{\beta/2}\|^2_{L^p(B_1)}\\
&\le C\beta^\alpha\Big(\int_{B_1}(\Delta u)^{n-1}|D\eta|^2 z^\beta\,dx + \int_{B_1}\Delta u\cdot\eta^2 z^\beta\,dx + \int_{B_1}\eta^2 z^\beta\,dx\Big),
\end{aligned}
\tag{3.22}
$$

where $\alpha = \frac{2q}{2q-n}$. Then by Hölder's inequality, we have

$$
\int_{B_1}(\Delta u)^{n-1}|D\eta|^2 z^\beta\,dx \le \|\Delta u\|_{L^{pn}(B_1)}\cdot\||D\eta|z^{\beta/2}\|^2_{L^{\frac{2pn}{pn-n+1}}(B_1)},
\tag{3.23}
$$

$$
\int_{B_1}\Delta u\cdot\eta^2 z^\beta\,dx \le \|\Delta u\|_{L^{pn}(B_1)}\cdot\|\eta z^{\beta/2}\|^2_{L^{\frac{2pn}{pn-1}}(B_1)}.
\tag{3.24}
$$

Combining (3.22), (3.23) and (3.24), we get

$$\begin{aligned}
&\|\eta z^{\beta/2}\|_{L^p(B_1)} \\
&\quad\le C\beta^{\alpha/2}\Big(\||D\eta| z^{\beta/2}\|_{L^{\frac{2p_n}{p_n-n+1}}(B_1)} + \|\eta z^{\beta/2}\|_{L^{\frac{2p_n}{p_n-1}}(B_1)} + \|\eta z^{\beta/2}\|_{L^2(B_1)}\Big)\\
&\quad\le C\beta^{\alpha/2}\Big(\||D\eta| z^{\beta/2}\|_{L^{\frac{2p_n}{p_n-n+1}}(B_1)} + \|\eta z^{\beta/2}\|_{L^{\frac{2p_n}{p_n-n+1}}(B_1)}\Big).
\end{aligned}$$

Now for any $0 < r < R \le 1$, we choose a cutoff function $\eta \in C_0^\infty(B_R)$ such that

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ in } B_r \quad \text{and} \quad |D\eta| \le \frac{2}{R-r}.$$

Then we obtain

$$\|z^{\beta/2}\|_{L^p(B_r)} \le \frac{C\beta^{\alpha/2}}{R-r}\|z^{\beta/2}\|_{L^{\frac{2p_n}{p_n-n+1}}(B_R)}.$$

By the assumption $p_n > \frac{n(n-1)}{2}$, we know $p > \frac{2p_n}{p_n-n+1}$. Denote

$$\chi = p\,\frac{p_n-n+1}{2p_n} > 1.$$

Then

$$\|z\|_{L^{\frac{\beta p_n}{p_n-n+1}\chi}(B_r)} \le \frac{C^{2/\beta}\beta^{\alpha/\beta}}{(R-r)^{2/\beta}}\|z\|_{L^{\frac{\beta p_n}{p_n-n+1}}(B_R)}. \tag{3.25}$$

We iterate (3.25) to get the desired estimate. Set

$$\beta_i = 2\chi^i \quad \text{and} \quad R_i = r + \frac{R-r}{2^i}, \quad i = 0, 1, 2, \dots,$$

i.e.,

$$\beta_i = \chi\beta_{i-1} \quad \text{and} \quad R_{i-1} - R_i = \frac{R-r}{2^i}, \quad i = 1, 2, \dots$$

By (3.25),

$$\begin{aligned}
&\|z\|_{L^{\frac{2p_n}{p_n-n+1}\chi^{i+1}}(B_{R_{i+1}})}\\
&\quad\le C^{\sum_{j=0}^i \frac{2}{\beta_j}}\cdot\prod_{j=0}^i \beta_j^{\alpha/\beta_j}\cdot 4^{\sum_{j=0}^i \frac{j}{\beta_j}}\frac{1}{(R-r)^{\sum_{j=0}^i (2/\beta_j)}}\cdot\|z\|_{L^{\frac{2p_n}{p_n-n+1}}(B_R)}.
\end{aligned}$$

Letting $i \to \infty$, by Young's inequality, we have

$$\begin{aligned}
\|z\|_{L^\infty(B_r)} &\le \frac{C}{(R-r)^{\frac{\chi}{\chi-1}}}\|z\|_{L^{\frac{2p_n}{p_n-n+1}}(B_R)} = \frac{C}{(R-r)^{\frac{\chi}{\chi-1}}}\|z\|_{L^1(B_R)}^{\frac{p_n-n+1}{2p_n}}\cdot\|z\|_{L^\infty(B_R)}^{\frac{p_n+n-1}{2p_n}}\\
&\le \frac{1}{2}\|z\|_{L^\infty(B_R)} + \frac{C}{(R-r)^{\frac{\chi}{\chi-1}\cdot\frac{2p_n}{p_n-n+1}}}\|z\|_{L^1(B_R)}.
\end{aligned}$$

Set $f(t) = \|z\|_{L^\infty(B_t)}$ for $t \in (0, 1]$. Then for any $0 < r < R \le \overline{R} < 1$,

$$f(r) \le \frac{1}{2} f(R) + \frac{C}{(R-r)^{\frac{\chi}{\chi-1}\cdot\frac{2pn}{pn-n+1}}} \|z\|_{L^1(B_{\overline{R}})}.$$

We apply Lemma 3.5 below to get

$$f(r) \le \frac{C}{(R-r)^{\frac{\chi}{\chi-1}\cdot\frac{2pn}{pn-n+1}}} \|z\|_{L^1(B_{\overline{R}})}. \tag{3.26}$$

It remains to show $\|z\|_{L^1(B_{\overline{R}})} \le C$. It is clear that $\Delta u \in L^1(B_{\overline{R}})$, hence it is suffices to estimate the integral of v. Let $\eta \in C_0^\infty(B_1)$ be a cutoff function such that $\eta \equiv 1$ in $B_{\overline{R}}$. Multiplying (1.5) by $\varphi = \eta^2 w$ and integrating by parts, we have

$$\int_{B_1} U^{ij} w_i w_j \eta^2 \, dx + 2\int_{B_1} U^{ij} w_i \eta_j \eta w \, dx = \int_{B_1} f w \eta^2 \, dx.$$

Then by the Cauchy inequality, we get

$$\begin{aligned}\frac{1}{2}\int_{B_1} U^{ij} w_i w_j \eta^2 \, dx &\le 2\int_{B_1} U^{ij} \eta_i \eta_j w^2 \, dx + \int_{B_1} |f| w \eta^2 \, dx \\ &\le 2\int_{B_1} (\Delta u)^{n-1} |D\eta|^2 w^2 \, dx + \int_{B_1} |f| w \eta^2 \, dx.\end{aligned}$$

Hence

$$\int_{B_{\overline{R}}} u^{kl} w_k w_l \, dx \le C$$

follows by $\Delta u \in L^{p_n}(B_1)$, $f \in L^q(B_1)$. Then we complete the proof by choosing $r = 1/2$ and $R = \overline{R}$ in (3.26). ∎

Lemma 3.5 (Lemma 4.3 in [15]). *Let $f(t) \ge 0$ be bounded in $[\tau_0, \tau_1]$ with $\tau_0 \ge 0$. Suppose that for $\tau_0 \le t < s \le \tau_1$ we have*

$$f(t) \le \beta f(s) + \frac{A}{(s-t)^\alpha} + B$$

for some $\beta \in [0, 1)$. Then for any $\tau_0 \le t < s \le \tau_1$, there holds

$$f(t) \le C(\alpha, \beta)\Big\{\frac{A}{(s-t)^\alpha} + B\Big\}.$$

Remark 3.6. Note that for $n = 2$, we have already got an L^p-bound of Δu in Theorem 3.3 for all $p \in [1, +\infty)$. However, it is still unknown how to obtain L^p-bound of Δu when $n \ge 3$ by the integral method used in Theorem 3.3. We also do not know how to obtain higher estimates for other exponents θ by the above method.

4. An interior estimate in higher dimensions

In this section, we will prove Theorem 1.5. By [2,3], it suffices to get the interior estimates on the upper and lower bound of det D^2u. Note that there exists a constant $c_n > 0$ such that

$$\frac{\det D^2 u}{c_n \cdot \Delta u} I \le U^{ij} \le c_n (\Delta u)^{n-1} I. \tag{4.1}$$

In view of (3.2) and (4.1), we first consider the following degenerate linear elliptic equation:

$$-D_j(a_{ij}(x)D_i u) + c(x)u = f(x) \quad \text{in } \Omega. \tag{4.2}$$

Lemma 4.1. *Assume $\{a_{ij}(x)\}$ satisfies*

$$\frac{d(x)}{\lambda(x)} \le a_{ij}(x) \le \lambda(x)^{n-1} \quad \textit{in } B_1, \tag{4.3}$$

where $\lambda(x) \in L^p(B_1)$, $d^{-1} \in L^q(B_1)$. Assume $c(x), f(x) \in L^{p_0}(B_1)$. Let $u \in W^{1,p}(B_1)$ be a subsolution in the following sense:

$$\int_{B_1} a_{ij} D_i u D_j \varphi + cu\varphi \, dx \le \int_{B_1} f\varphi \, dx \quad \textit{for any } \varphi \in W_0^{1,p}(B_1) \textit{ and } \varphi \ge 0. \tag{4.4}$$

Suppose p, q and p_0 satisfy $n/p + 1/q < 2/n$ and $p_0 \ge p/(n-1)$. Then

$$\sup_{B_{1/2}} u \le C \left(\|u^+\|_{L^1(B_1)} + \|f\|_{L^{p_0}(B_1)} \right),$$

where C depends only on p, $\|\lambda\|_{L^p(B_1)}$, $\|d^{-1}\|_{L^q(B_1)}$ and $\|c\|_{L^{p_0}(B_1)}$.

Proof. For some $k > 0$ and $m > 0$, set $\bar{u} = u^+ + k$ and

$$\bar{u}_m = \begin{cases} \bar{u}, & u < m, \\ k + m, & u \ge m. \end{cases}$$

Let $\eta \in C_0^\infty(B_1)$. We choose a test function $\varphi = \eta^2(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) \in W_0^{1,p}(B_1)$ for some $\beta \ge 0$ to be determined later. Substituting φ into (4.4), we have

$$\begin{aligned} \beta \int_{B_1} a_{ij} D_i \bar{u} \, D_j \bar{u}_m \, \bar{u}_m^{\beta-1} \, \bar{u} \, \eta^2 \, dx + \int_{B_1} a_{ij} D_i \bar{u} \, D_j \bar{u} \, \bar{u}_m^\beta \, \eta^2 \, dx \\ \le -2 \int_{B_1} a_{ij} D_i \bar{u} D_j \eta \, \bar{u}_m^\beta \, \bar{u} \, \eta \, dx + \int_{B_1} \left(|c| \eta^2 \bar{u}_m^\beta \bar{u}^2 + |f| \eta^2 \bar{u}_m^\beta \bar{u} \right) dx. \end{aligned}$$

Note that $D\bar{u} = D\bar{u}_m$ in $\{u < m\}$ and $D\bar{u}_m = 0$ in $\{u \geq m\}$. By the Cauchy inequality, we have

$$\begin{aligned}\beta \int_{B_1} & a_{ij}\, D_i \bar{u}_m\, D_j \bar{u}_m\, \bar{u}_m^{\beta-1}\, \bar{u}\, \eta^2\, dx + \int_{B_1} a_{ij}\, D_i \bar{u}\, D_j \bar{u}\, \bar{u}_m^{\beta}\, \eta^2\, dx \\ &\leq \frac{1}{2} \int_{B_1} a_{ij}\, D_i \bar{u}\, D_j \bar{u}\, \bar{u}_m^{\beta}\, \eta^2\, dx + 4 \int_{B_1} a_{ij}\, D_i \eta\, D_j \eta\, \bar{u}_m^{\beta}\, \bar{u}^2\, dx \\ &\quad + \int_{B_1} \big(|c|\, \eta^2\, \bar{u}_m^{\beta}\, \bar{u}^2 + |f|\, \eta^2\, \bar{u}_m^{\beta}\, \bar{u}\big)\, dx.\end{aligned}$$

Then by (4.3),

$$\begin{aligned}\beta \int_{B_1} & \frac{d}{\lambda(x)}\, |D\bar{u}_m|^2\, \bar{u}_m^{\beta}\, \eta^2\, dx + \frac{1}{2} \int_{B_1} \frac{d}{\lambda(x)}\, |D\bar{u}|^2\, \bar{u}_m^{\beta}\, \eta^2\, dx \\ &\leq \beta \int_{B_1} a_{ij}\, D_i \bar{u}_m\, D_j \bar{u}_m\, \bar{u}_m^{\beta-1}\, \bar{u}\, \eta^2\, dx + \frac{1}{2} \int_{B_1} a_{ij}\, D_i \bar{u}\, D_j \bar{u}\, \bar{u}_m^{\beta}\, \eta^2\, dx \\ &\leq 4 \int_{B_1} \lambda(x)^{n-1}\, |D\eta|^2\, \bar{u}_m^{\beta}\, \bar{u}^2\, dx + \int_{B_1} c_0\, \eta^2\, \bar{u}_m^{\beta}\, \bar{u}^2\, dx, \end{aligned} \tag{4.5}$$

where $c_0 = |c| + |f|/k$. Choose $k = \|f\|_{L^{p_0}(B_1)}$ if f is not identically 0. Otherwise choose arbitrary $k > 0$ and let $k \to 0^+$. Let $w = \bar{u}_m^{\beta/2}\, \bar{u}$. There holds

$$|Dw|^2 = \left|\frac{\beta}{2}\, \bar{u}_m^{\beta/2-1}\, \bar{u}_m D\bar{u}_m + \bar{u}_m^{\beta/2}\, D\bar{u}\right|^2 \leq (1+\beta)(\beta\, \bar{u}_m^{\beta} |D\bar{u}_m|^2 + \bar{u}_m^{\beta} |D\bar{u}|^2).$$

Therefore, by (4.5),

$$\int_{B_1} \frac{d}{\lambda(x)} |Dw|^2 \eta^2\, dx \leq 8(1+\beta) \int_{B_1} \lambda(x)^{n-1} |D\eta|^2 w^2\, dx + 2(1+\beta) \int_{B_1} c_0 w^2 \eta^2\, dx.$$

By

$$|D(w\eta)|^2 \leq 2|Dw|^2\eta^2 + 2|D\eta|^2 w^2,$$

we have

$$\begin{aligned}\int_{B_1} & \frac{d}{\lambda(x)} |D(w\eta)|^2\, dx \\ &\leq \int_{B_1} \Big(\frac{2d}{\lambda(x)} + 16(1+\beta)\lambda(x)^{n-1}\Big) |D\eta|^2 w^2\, dx + 4(1+\beta) \int_{B_1} c_0 w^2 \eta^2\, dx \\ &\leq 18(1+\beta) \int_{B_1} \lambda(x)^{n-1} |D\eta|^2 w^2\, dx + 4(1+\beta) \int_{B_1} c_0 w^2 \eta^2\, dx. \end{aligned} \tag{4.6}$$

Next, we deal with the $\lambda(x)$ in the above estimate. By the assumptions $\lambda \in L^p(B_1)$ and $d^{-1} \in L^q(B_1)$, we have $\lambda d^{-1} \in L^{\frac{pq}{p+q}}(B_1)$. By Hölder's inequality, we have

$$\|D(w\eta)\|^2_{L^{\frac{2pq}{pq+p+q}}(B_1)} \leq \|\lambda d^{-1}\|_{L^{\frac{pq}{p+q}}(B_1)} \cdot \int_{B_1} \frac{1}{\lambda(x)d^{-1}} |D(w\eta)|^2 \, dx, \tag{4.7}$$

$$\int_{B_1} \lambda(x)^{n-1}|D\eta|^2 w^2 \, dx \leq \|\lambda\|_{L^p(B_1)} \cdot \|wD\eta\|^2_{L^{\frac{2p}{p-n+1}}(B_1)}, \tag{4.8}$$

$$\int_{B_1} c_0 w^2\eta^2 \, dx \leq \|c_0\|_{L^{p_0}(B_1)} \cdot \Big(\int_{B_1} (\eta w)^{\frac{2p_0}{p_0-1}} \, dx\Big)^{1-1/p_0}. \tag{4.9}$$

Combining (4.6)–(4.9), we get

$$\begin{aligned}\|D(w\eta)\|_{L^{\frac{2pq}{pq+p+q}}(B_1)} &\leq C(1+\beta)^{1/2}\Big(\|wD\eta\|_{L^{\frac{2p}{p-n+1}}(B_1)} + \|w\eta\|_{L^{\frac{2p_0}{p_0-1}}(B_1)}\Big)\\ &\leq C(1+\beta)^{1/2}\Big(\|wD\eta\|_{L^{\frac{2p}{p-n+1}}(B_1)} + \|w\eta\|_{L^{\frac{2p}{p-n+1}}(B_1)}\Big)\end{aligned}$$

for C depending on $\|\lambda\|_{L^p(B_1)}$, $\|d^{-1}\|_{L^q(B_1)}$, $\|c\|_{L^{p_0}(B_1)}$. Here we used $p_0 \geq (n-1)/p$. By the Sobolev inequality,

$$\|w\eta\|_{L^\alpha(B_1)} \leq C(1+\beta)^{1/2}\Big(\|wD\eta\|_{L^{\frac{2p}{p-n+1}}(B_1)} + \|w\eta\|_{L^{\frac{2p}{p-n+1}}(B_1)}\Big),$$

where

$$\frac{1}{\alpha} = \frac{pq+p+q}{2pq} - \frac{1}{n}. \tag{4.10}$$

Now for any $0 < r < R \leq 1$, we choose a cutoff function $\eta \in C_0^\infty(B_R)$ such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_r \quad \text{and} \quad |D\eta| \leq \frac{2}{R-r}.$$

Then we obtain

$$\|w\|_{L^\alpha(B_r)} \leq \frac{C(1+\beta)^{1/2}}{R-r} \|w\|_{L^{\frac{2p}{p-n+1}}(B_R)}.$$

By (4.10) and the assumption $n/p + 1/q < 2/n$, we have $\alpha > \frac{2p}{p-n+1}$. We can do the iteration as follows.

Recalling the definition of w, we have

$$\|\bar{u}_m^{\beta/2}\,\bar{u}\|_{L^\alpha(B_r)} \leq \frac{C(1+\beta)^{1/2}}{R-r} \|\bar{u}_m^{\beta/2}\,\bar{u}\|_{L^{\frac{2p}{p-n+1}}(B_R)}.$$

Set $\gamma = \beta + 2 \geq 2$. By $\bar{u}_m \leq \bar{u}$, we obtain

$$\|\bar{u}_m^{\gamma/2}\|_{L^\alpha(B_r)} \leq \frac{C\gamma^{1/2}}{R-r} \|\bar{u}^{\gamma/2}\|_{L^{\frac{2p}{p-n+1}}(B_R)}.$$

Letting $m \to \infty$, we get

$$\|\bar{u}^{\gamma/2}\|_{L^{\alpha}(B_r)} \le \frac{C\gamma^{1/2}}{R-r} \|\bar{u}^{\gamma/2}\|_{L^{\frac{2p}{p-n+1}}(B_R)},$$

i.e.,

$$\|\bar{u}\|_{L^{\gamma\alpha/2}(B_r)} \le \frac{(C\gamma)^{1/\gamma}}{(R-r)^{2/\gamma}} \|\bar{u}\|_{L^{\frac{\gamma}{2}\frac{2p}{p-n+1}}(B_R)}.$$

Denote

$$\chi = \alpha \cdot \frac{p-n+1}{2p} > 1.$$

Then

$$\|\bar{u}\|_{L^{\frac{\gamma p}{p-n+1}\chi}(B_r)} \le \frac{(C\gamma)^{1/\gamma}}{(R-r)^{2/\gamma}} \|\bar{u}\|_{L^{\frac{\gamma p}{p-n+1}}(B_R)}. \tag{4.11}$$

We iterate (4.11) to get the desired estimate. Set

$$\gamma_i = 2\chi^i \quad \text{and} \quad R_i = r + \frac{R-r}{2^i}, \quad i = 0, 1, 2, \ldots,$$

i.e.,

$$\gamma_i = \chi\gamma_{i-1} \quad \text{and} \quad R_{i-1} - R_i = \frac{R-r}{2^i}, \quad i = 1, 2, \ldots$$

By (4.11),

$$\|\bar{u}\|_{L^{\frac{2p}{p-n+1}\chi^{i+1}}(B_{R_{i+1}})} \le C^{\sum_{j=0}^{i}\frac{1}{\gamma_j}} \cdot \prod_{j=0}^{i} \gamma_j^{1/\gamma_j} \cdot 4^{\sum_{j=0}^{i}\frac{j}{\gamma_j}} \frac{1}{(R-r)^{\sum_{j=0}^{i}\frac{2}{\gamma_j}}} \cdot \|\bar{u}\|_{L^{\frac{2p}{p-n+1}}(B_R)}.$$

Letting $i \to \infty$, by Young's inequality, we have

$$\begin{aligned}
\|\bar{u}\|_{L^{\infty}(B_r)} &\le \frac{C}{(R-r)^{\frac{\chi}{\chi-1}}} \|\bar{u}\|_{L^{\frac{2p}{p-n+1}}(B_R)} \\
&= \frac{C}{(R-r)^{\frac{\chi}{\chi-1}}} \|\bar{u}\|_{L^1(B_R)}^{\frac{p-n+1}{2p}} \cdot \|\bar{u}\|_{L^{\infty}(B_R)}^{\frac{p+n-1}{2p}} \\
&\le \frac{1}{2}\|\bar{u}\|_{L^{\infty}(B_R)} + \frac{C}{(R-r)^{\frac{\chi}{\chi-1}\cdot\frac{2p}{p-n+1}}} \|\bar{u}\|_{L^1(B_R)}.
\end{aligned}$$

Set $f(t) = \|\bar{u}\|_{L^{\infty}(B_t)}$ for $t \in (0, 1]$. Then for any $0 < r < R \le 1$,

$$f(r) \le \frac{1}{2} f(R) + \frac{C}{(R-r)^{\frac{\chi}{\chi-1}\cdot\frac{2p}{p-n+1}}} \|\bar{u}\|_{L^1(B_1)}.$$

We apply Lemma 3.5 to get

$$f(r) \le \frac{C}{(R-r)^{\frac{\chi}{\chi-1}\cdot\frac{2p}{p-n+1}}} \|\bar{u}\|_{L^1(B_1)}.$$

The lemma follows by choosing $r = 1/2$ and $R = 1$. ∎

Now we can use Lemma 4.1 to obtain the interior estimates for (1.1). For simplicity, we only consider the homogeneous equation.

Proof of Theorem 1.5. Denote $a_{ij} = U^{ij}$, $\lambda = \Delta u$ and $d = \det D^2u$. Firstly, we consider the case $0 \le \theta < 1$. We apply Lemma 4.1 to equation (1.3), which yields

$$\sup_{B_{1/2}} w \le C\|w\|_{L^1(B_1)} \le C\|w\|_{L^{\frac{q}{1-\theta}}(B_1)}.$$

Since $w = (\det D^2u)^{-(1-\theta)} \in L^{\frac{q}{1-\theta}}(B_1)$, we know $\sup_{B_{1/2}}\{(\det D^2u)^{-1}\} \le C$. For the upper bound of the determinant, we set $w = 1/v$. A direct calculation yields

$$w_i = -\frac{v_i}{v^2} \quad \text{and} \quad w_{ij} = \frac{2v_iv_j}{v^3} - \frac{v_{ij}}{v^2}.$$

Then v satisfies

$$0 = U^{ij}w_{ij} = \frac{2}{v^3}U^{ij}v_iv_j - \frac{1}{v^2}U^{ij}v_{ij} \ge -\frac{1}{v^2}U^{ij}v_{ij},$$

i.e.,

$$-D_i(U^{ij}D_jv) \le 0 \quad \text{in } B_1.$$

Similarly, we use Lemma 4.1 to obtain

$$\sup_{B_{1/2}} v \le C\|v\|_{L^1(B_1)} \le C\|v\|_{L^{\frac{p}{n(1-\theta)}}(B_1)},$$

which implies $\sup_{B_{1/2}} \det D^2u \le C$.

Next, we consider the case $\theta = 1$. Write $w = \log d$, where $d \in L^{p/n}(B_1)$. Then we have

$$w_i = \frac{d_i}{d} \quad \text{and} \quad w_{ij} = -\frac{d_id_j}{d^2} + \frac{d_{ij}}{d},$$

which yields

$$0 = U^{ij}w_{ij} = -\frac{1}{d^2}U^{ij}d_id_j + \frac{1}{d}U^{ij}d_{ij} \le \frac{1}{d}U^{ij}d_{ij},$$

i.e.,

$$-D_i(U^{ij}D_jd) \le 0 \quad \text{in } B_1.$$

Similarly, write $w = -\log z$ with $z = (\det D^2u)^{-1} \in L^q(B_1)$. We have

$$w_i = -\frac{z_i}{z} \quad \text{and} \quad w_{ij} = \frac{z_iz_j}{z^2} - \frac{z_{ij}}{z},$$

which means

$$0 = U^{ij}w_{ij} = \frac{1}{z^2}U^{ij}z_iz_j - \frac{1}{z}U^{ij}z_{ij} \ge -\frac{1}{z}U^{ij}z_{ij},$$

i.e.,

$$-D_i(U^{ij}D_jz) \le 0 \quad \text{in } B_1.$$

Then we use Lemma 4.1 to obtain the bounds of $\det D^2u$.

Once we have the determinant estimates, all the interior estimates follow. ∎

Remark 4.2. To get the upper bound of $\det D^2u$, we only need $u \in W^{2,p}(B_1)$ with $p > n^2/2$. In fact, by taking $\det D^2u = v^{1/(1-\theta)}$ $(0 \le \theta < 1)$, where v defined as in the proof of Theorem 1.5, condition (4.3) in Lemma 4.1 becomes

$$\frac{v^{1/(1-\theta)}}{\lambda(x)} \le a_{ij}(x) \le \lambda(x)^{n-1}$$

Then the upper bound can be obtained by similar arguments as in Lemma 4.1 without assumption on $(\det D^2u)^{-1}$.

Remark 4.3. We can also consider the inhomogeneous equation

$$D_j(U^{ij}D_iw) = f \quad \text{in } B_1, \tag{4.12}$$

where $f \in L^{p_0}(B_1)$. For the lower bound of $\det D^2u$, if we assume $p_0 \ge p/(n-1)$, we can apply Lemma 4.1 to (4.12) directly for all $\theta \in [0,1)$. However, for the upper bound of $\det D^2u$, $v = 1/w$ satisfies

$$-D_j(U^{ij}D_iv) \le fv^2 \quad \text{in } B_1. \tag{4.13}$$

Even we assume $p_0 = \infty$, we only know $fv \in L^{\frac{p}{n(1-\theta)}}(B_1)$. Then we can only apply Lemma 4.1 to the equation (4.13) to get the upper bound of $\det D^2u$ when $1/n \le \theta < 1$. Hence, we have all the higher interior estimates for $\theta \in [1/n, 1)$. Note that the case $\theta = 0$ (Abreu's equation) and $\theta = 1/(n+2)$ (affine mean curvature equation) are not included.

Finally, we prove a Liouville type theorem.

Proof of Corollary 1.6. For u in any $B_R \subset \mathbb{R}^n$, define $u_R(x) = \frac{1}{R^2}u(Rx)$. Then we know that $U_R^{ij}(w_R)_{ij} = 0$ in B_1. Since u satisfies (1.7), we know u_R satisfies

$$\int_{B_1} |D^2u_R|^p + (\det D^2u_R)^{-q}\,dx \le C.$$

Applying Theorem 1.5 to u_R, we find that there exists a $C > 0$ independent of R, such that $\|u_R\|_{C^{4,\alpha}(B_{1/2})} \le C$. In particular, we know that $\|D^3u_R\|_{L^\infty(B_{1/2})} \le C$. Hence

$$\|D^3u(Rx)\|_{L^\infty(B_{1/2})} \le \frac{C}{R},$$

i.e.,

$$\|D^3u(x)\|_{L^\infty(B_{R/2})} \le \frac{C}{R}.$$

Let $R \to +\infty$, we have $D^3u \equiv 0$, which means u is a quadratic function. ∎

Acknowledgements. The authors would like to thank Guohuan Qiu for some valuable discussions.

Funding. This research is partially supported by National Key R&D Program of China SQ2020YFA0712800 and NSFC grants 11822101.

References

[1] Abreu, M.: Kähler geometry of toric varieties and extremal metrics. *Int. J. Math.* **9** (1998), no. 6, 641–651.

[2] Caffarelli, L. A.: Interior $W^{2,p}$ estimates for solutions of the Monge–Ampère equation. *Ann. Math.* **131** (1990), no. 1, 135–150.

[3] Caffarelli, L. A. and Gutiérrez, C. E.: Properties of solutions of the linearized Monge–Ampère equation. *Amer. J. Math.* **119** (1997), no. 2, 423–465.

[4] Chau, A. and Weinkove, B.: Monge–Ampère functionals and the second boundary value problem. *Math. Res. Lett.* **22** (2015), no. 4, 1005–1022.

[5] Chen, B., Han, Q., Li, A.-M. and Sheng, L.: Interior estimates for the n-dimensional Abreu's equation. *Adv. Math.* **251** (2014), 35–46.

[6] Chen, X. X. and Cheng, J. R.: On the constant scalar curvature Kähler metrics (I). A priori estimates. *J. Amer. Math. Soc.* **34** (2021), no. 4, 909–936.

[7] Chern, S. S.: Affine minimal hypersurfaces. In *Minimal submanifolds and geodesics (Proc. Japan-United States Sem., Tokyo, 1977)*, pp. 17–30. North-Holland, Amsterdam-New York, 1979.

[8] Daskalopoulos, P. and Savin, O.: On Monge–Ampère equations with homogeneous right-hand sides. *Comm. Pure Appl. Math.* **62** (2009), no. 5, 639–676.

[9] De Philippis, G., Figalli, A. and Savin, O.: A note on interior $W^{2,1+\varepsilon}$ estimates for the Monge–Ampère equation. *Math. Ann.* **3**57 (2013), no. 1, 11–22.

[10] Donaldson, S. K.: Interior estimates for solutions of Abreu's equation. *Collect. Math.* **56** (2005), no. 2, 103–142.

[11] Figalli, A.: *The Monge–Ampère equation and its applications.* Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2017.

[12] Guan, P. and Phong, D. H.: Partial Legendre transforms of non-linear equations. *Proc. Amer. Math. Soc.* **140** (2012), no. 11, 3831–3842.

[13] Gutiérrez, C. E. and Nguyen, T.: Interior gradient estimates for solutions to the linearized Monge–Ampère equation. *Adv. Math.* **228** (2011), no. 4, 2034–2070.

[14] Gutiérrez, C. E. and Nguyen, T.: Interior second derivative estimates for solutions to the linearized Monge–Ampère equation. *Trans. Amer. Math. Soc.* **367** (2015), no. 7, 4537–4568.

[15] Han, Q. and Lin, F. H.: *Elliptic partial differential equations*. Second edition. Courant Lecture Notes in Mathematics 1, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2011.

[16] Heinz, H.: Über die Differential ungleichung $0 < \alpha \leq rt - s^2 \leq \beta < \infty$. *Math. Z.* **72** (1959), 107–126.

[17] Jia, F. and Li, A. M.: A Bernstein property of some fourth order partial differential equations. *Results Math.* **56** (2009), no. 1-4, 109–139.

[18] Le, N. Q.: Global second derivative estimates for the second boundary value problem of the prescribed affine mean curvature and Abreu's equations. *Int. Math. Res. Not. IMRN* (2013), no. 11, 2421–2438.

[19] Le, N. Q.: $W^{4,p}$ solution to the second boundary value problem of the prescribed affine mean curvature and Abreu's equations. *J. Differential Equations* **260** (2016), no. 5, 4285–4300.

[20] Le, N. Q.: Hölder regularity of the 2D dual semigeostrophic equations via analysis of linearized Monge–Ampère equations. *Comm. Math. Phys.* **360** (2018), no. 1, 271–305.

[21] Le, N. Q.: Singular Abreu equations and minimizers of convex functionals with a convexity constraint. *Comm. Pure Appl. Math.* **73** (2020), no. 10, 2248–2283.

[22] Le, N. Q.: On singular Abreu equations in higher dimensions. *J. Anal. Math.* **144** (2021), no. 1, 191–205.

[23] Le, N. Q. and Nguyen, T.: Global $W^{1,p}$-estimates for solutions to the linearized Monge–Ampère equations. *J. Geom. Anal.* **27** (2017), no. 3, 1751–1788.

[24] Le, N. Q. and Savin, O.: Boundary regularity for solutions to the linearized Monge–Ampère equations. *Arch. Ration. Mech. Anal.* **210** (2013), no. 3, 813–836.

[25] Le, N. Q. and Zhou, B.: Solvability of a class of singular fourth order equations of Monge–Ampère type. *Ann. PDE* **7** (2021), no. 2, Paper no. 13, 32 pp.

[26] Liu, J. K.: Interior C^2 estimate for Monge–Ampère equations in dimension two. *Proc. Amer. Math. Soc.* **149** (2021), no. 6, 2479–2486.

[27] Loeper, G.: On the regularity of the polar factorization for time dependent maps. *Calc. Var. Partial Differential Equations* **22** (2005), no. 3, 343–374.

[28] Murthy, M. R. V. and Stampacchia, G.: Boundary value problems for some degenerate-elliptic operators. *Ann. Mat. Pura Appl.* **80** (1968), 1–122.

[29] Schmidt, T.: $W^{2,1+\epsilon}$-estimates for the Monge–Ampère equation. *Adv. Math.* **240** (2013), 672–689.

[30] Tian, G.-J. and Wang, X.-J.: A class of Sobolev type inequalities. *Meth. Appl. Anal.* **15** (2008), no. 2, 263–276.

[31] Trudinger, N. S.: Linear elliptic operators with measurable coefficients. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* **27** (1973), 265–308.

[32] Trudinger, N. S. and Wang, X.-J.: The Bernstein problem for affine maximal hypersurfaces. *Invent. Math.* **140** (2000), no. 2, 399–422.

[33] Trudinger, N. S. and Wang, X.-J.: The affine plateau problem. *J. Amer. Math. Soc.* **18** (2005), no. 2, 253–289.

[34] Zhou, B.: The Bernstein theorem for a class of fourth order equations. *Calc. Var. Partial Differential Equations* **43** (2012), no. 1-2, 25–44.

[35] Zhou, B.: The first boundary value problem for Abreu's equation. *Int. Math. Res. Not.* (2012), no. 7, 1439–1484.

Received January 7, 2022; revised May 28, 2022. Published online July 27, 2022.

Ling Wang
School of Mathematical Sciences, Peking University, Beijing 100871, China;
lingwang@stu.pku.edu.cn

Bin Zhou
School of Mathematical Sciences, Peking University, Beijing 100871, China;
bzhou@pku.edu.cn