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Spans of translates in weighted ℓ^p spaces

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Abstract. We study the cyclic vectors and the spanning set of the circle for the $\ell^p_\beta(\mathbb{Z})$ spaces of all sequences $u = (u_n)_{n\in\mathbb{Z}}$ such that $(u_n(1+|n|)^\beta)_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$, with $p > 1$ and $\beta > 0$. The uniqueness set of the distribution on the circle whose Fourier coefficients are in $\ell^q_{-\beta}(\mathbb{Z})$ is the spanning set for the $\ell^p_\beta(\mathbb{Z})$ spaces, where q is the conjugate of p. Our characterizations are given in terms of the Hausdorff dimension and capacity.

1. Introduction and main results

Cyclic vectors are, amongst others, an important tool in the study of invariant subspaces and their characterization [1, 20, 25]. For the shift operator, the problem of cyclic vectors in the space of sequences $\ell^p(\mathbb{Z})$ goes back to the works of Wiener [26] for $p = 1$ and $p = 2$, Beurling [3] and Salem [24] for $1 < p < 2$, and Newman [18] for $p > 1$. This problem is still far from being resolved.

A vector $u \in \ell^p(\mathbb{Z})$ is called *cyclic* in $\ell^p(\mathbb{Z})$ if the linear span of its translates,

$$\{(u_{n+k})_{n\in\mathbb{Z}},\ k \in \mathbb{Z}\},$$

is dense in $\ell^p(\mathbb{Z})$. The Fourier transform of $u \in \ell^p(\mathbb{Z})$ is given by $\widehat{u}(t) = \sum_{n\in\mathbb{Z}} u_n e^{int}$, where the trigonometric series is to be interpreted as a distribution on the circle group $\mathbb{T} = \mathbb{R} \setminus 2\pi\mathbb{Z}$. For $u \in \ell^p(\mathbb{Z})$ with $1 \le p \le 2$, $\widehat{u}$ becomes a function. We denote by $\mathcal{Z}(\widehat{u})$ the zero set of $\widehat{u}$ in $\mathbb{T}$. Notice that for $u \in \ell^1(\mathbb{Z})$, the set $\mathcal{Z}(\widehat{u})$ is well-defined, since $\widehat{u}$ is continuous. The cyclicity can be viewed as an approximation problem or a uniqueness/removable singularities problem. Following Newman [18], a closed subset $E \subset \mathbb{T}$ is called *p-spanning* if every $u \in \ell^1(\mathbb{Z})$ with $\mathcal{Z}(\widehat{u}) \subset E$, is cyclic in $\ell^p(\mathbb{Z})$. On the other hand, E is called a *q-uniqueness* set if E does not support any non-vanishing distribution $\sum_{n\in\mathbb{Z}} c_n e^{int}$ with $(c_n)_{n\in\mathbb{Z}}$ in $\ell^q(\mathbb{Z})$. It is well known that E is p-spanning if and only if E is a q-uniqueness set, where q is the conjugate of p.

Wiener [26] characterized the cyclic vectors in $\ell^1(\mathbb{Z})$ and in $\ell^2(\mathbb{Z})$. Further, Beurling, Salem, and Newman [3, 18, 24] provided either necessary or sufficient conditions for u

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to be cyclic in $\ell^p(\mathbb{Z})$ for $p > 1$. These conditions were given in terms of the size (capacity and Hausdorff dimension) of the zero set of the Fourier transform $\widehat{u}$. However, Lev and Olevskii [14–16] showed that for $1 < p < 2$, the problem of cyclicity in $\ell^p(\mathbb{Z})$ is more complicated even for sequences in $\ell^1(\mathbb{Z})$: we cannot characterize the cyclicity of u in $\ell^p(\mathbb{Z})$ in terms of $\mathcal{Z}(\widehat{u})$ alone, which contradicts Wiener's conjecture.

We summarize the results of the previous works cited above. We denote by q the Hölder conjugate of $p \geq 1$, with $1/p + 1/q = 1$, and write $\dim(E)$ for the Hausdorff dimension of a subset $E \subset \mathbb{T}$.

(1) (Wiener): u is cyclic in $\ell^1(\mathbb{Z})$ if and only if $\widehat{u}$ has no zeros on $\mathbb{T}$.
(2) (Wiener): u is cyclic in $\ell^2(\mathbb{Z})$ if and only if $\widehat{u}$ is non-zero almost everywhere.
(3) (Beurling): Let $1 \leq p \leq 2$. If $\dim(E) < 2/q$, then E is a p-spanning.
(4) (Salem): Let $1 \leq p \leq 2$. For $2/q < \alpha \leq 1$, there exists $E \subset \mathbb{T}$ such that $\dim(E) = \alpha$ and E is not p-spanning
(5) (Newman): There exists a p-spanning set E for all $1 < p < 2$ such that $\dim(E) = 1$.
(6) (Lev and Olevskii): If $1 < p < 2$, there exist u and v in $\ell^1(\mathbb{Z})$ such that $\mathcal{Z}(\widehat{u}) = \mathcal{Z}(\widehat{v})$, u is not cyclic in $\ell^p(\mathbb{Z})$, and v is cyclic in $\ell^p(\mathbb{Z})$.

In this paper, we shall focus on the cyclic vectors on weighted $\ell^p(\mathbb{Z})$ spaces, namely $\ell^p_\beta(\mathbb{Z})$, the space of sequences $u = (u_n)_{n\in\mathbb{Z}}$ for which $(u_n(1 + |n|)^\beta) \in \ell^p(\mathbb{Z})$ for $p \geq 1$ and $\beta > 0$, endowed with the norm

$$\|u\|^p_{\ell^p_\beta(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} |u_n|^p \,(1 + |n|)^{p\beta}.$$

Note that the space $\ell^p_\beta(\mathbb{Z})$ (but not the norm) is invariant under translations. A vector $u \in \ell^p_\beta(\mathbb{Z})$ is called *cyclic* in $\ell^p_\beta(\mathbb{Z})$ if the linear span of $\{(u_{n+k})_{n\in\mathbb{Z}},\ k \in \mathbb{Z}\}$ is dense in $\ell^p_\beta(\mathbb{Z})$. For every closed subset E of $\mathbb{T}$, E is called (p,β)*-spanning* if every $u \in \ell^1(\mathbb{Z}) \cap \ell^p_\beta(\mathbb{Z})$ such that $\mathcal{Z}(\widehat{u}) \subset E$ is cyclic in $\ell^p_\beta(\mathbb{Z})$, and E is called a (q,β)*-uniqueness* set if E does not support any non-zero distribution $\sum_{n\in\mathbb{Z}} c_n e^{int}$ with $(c_n)_{n\in\mathbb{Z}}$ in $\ell^q_{-\beta}(\mathbb{Z})$. If E is a (q,β)-uniqueness set, then E is (p,β)-spanning, where $1/p + 1/q = 1$ (see Remark 2.5 below). Observe that the shift operator does not act as an isometry on $\ell^p_\beta(\mathbb{Z})$ unlike on $\ell^p(\mathbb{Z})$ spaces; this represents a difficulty for the study of cyclic vectors in $\ell^p_\beta(\mathbb{Z})$.

Notice that $\ell^p_\beta(\mathbb{Z})$ is a Banach algebra if and only if $\beta q > 1$ (see [6]). Hence, in this case we have an analogue of (1) in Wiener's theorem: a vector $u \in \ell^p_\beta(\mathbb{Z})$ is cyclic if and only if $\widehat{u}$ has no zeros on $\mathbb{T}$. Thus in the sequel of the paper we will only be interested in pairs (p,β) such that $\beta q < 1$.

Richter, Ross and Sundberg [23] gave a complete characterization of hyperinvariant subspaces of the weighted harmonic Dirichlet spaces $\ell^2_\beta(\mathbb{Z})$, $0 < \beta \leq 1/2$. Their characterization and a relation between capacity and Hausdorff dimension led to the result that $u \in \ell^1_\beta(\mathbb{Z})$ is cyclic in $\ell^2_\beta(\mathbb{Z})$ if and only if $\dim(\mathcal{Z}(\widehat{u})) \leq 1 - 2\beta$. Their result may be considered as an analog of Wiener's theorem about the cyclic vector of ℓ^2. Hence, we study the case of $p \neq 1$ and $p \neq 2$. Our main result for $1 < p < 2$ is the following (see Theorem 3.3).

Theorem A. *Let $1 < p < 2$ and $\beta > 0$ be such that $\beta q \le 1$, and let E be a closed subset of $\mathbb{T}$.*

(1) *If $\dim(E) < \frac{2}{q}(1-\beta q)$, then E is (p,β)-spanning.*

(2) *If $\dim(E) > 1-\beta q$, then E is not (p,β)-spanning.*

(3) *For $\frac{2}{q}(1-\beta q) \le \alpha \le 1$, there exists a closed subset $E \subset \mathbb{T}$ such that $\dim(E) = \alpha$ and E is not (p,β)-spanning.*

(4) *If $p = 2k/(2k-1)$ for some $k \in \mathbb{N} \setminus \{0\}$, there exists a (p,β)-spanning $E \subset \mathbb{T}$ such that $\dim(E) = 1-\beta q$.*

The property (4) shows that the constant $1-\beta q$ obtained in (2) is sharp. Indeed, on one hand, there is no cyclic vector u such that $\dim(\mathcal{Z}(\hat{u})) > 1-q\beta$, and on the other hand, we can find some cyclic vector u with $\dim(\mathcal{Z}(\hat{u})) = 1-\beta q$. However, this is only proved when $p = 2k/2k-1$ for some positive integer k. The proof is based on the construction of a closed subset E of $\mathbb{T}$ whose k-sums $E + \cdots + E$ are of zero capacity and of given Hausdorff dimension (see Lemma 3.2). The arithmetic structure of E allows us to reach the best constant $1-\beta q$ only for $p = 2k/(2k-1)$.

Next we will deal with the case $p > 2$. Newman in [18] showed that for all $\varepsilon > 0$, there exists a p-spanning set $E \subset \mathbb{T}$ which has a Lebesgue measure $|E| > 2\pi - \varepsilon$. The existence of q-uniqueness sets of arbitrary large measure for the spaces $\ell^q(\mathbb{Z})$, $1 < q < 2$, was established by Katznelson [11] (see also the Theorem in Chapter IV, Section 2.5, in [12]). Extensions of their results to a more general setting were given in [7], where they studied the uniqueness set of $\ell^q_{-\beta}(\mathbb{Z})$. We have the following result.

Theorem B. *Let $p > 2$ and $\beta > 0$ be such that $\beta q \le 1$.*

(1) *If $\beta > 1/2 - 1/p$, then every closed subset E of $\mathbb{T}$ of positive Lebesgue measure is not (p,β)-spanning.*

(2) *If $\beta < 1/2 - 1/p$, then for every $\varepsilon > 0$, there exists a (p,β)-spanning set $E \subset \mathbb{T}$ such that $|E| > 2\pi - \varepsilon$.*

Nikolski, in Corollary 6 of [21], considered the weighted space

$$\ell^p_\omega(\mathbb{Z}) = \Big\{(u_n)_{n\in\mathbb{Z}} \subset \mathbb{C} : \|u\|^p_\omega := \sum_{n\in\mathbb{Z}} |u_n|^p \omega_n^p < \infty\Big\}.$$

where $\omega_n = \log(e+|n|)^\gamma$, $\gamma > 0$. He showed that if $p > 2/(1-\gamma)$, $0 < \gamma < 1$, then there exists $E \subset \mathbb{T}$ with large Lebesgue measure which is a uniqueness set for the dual of $\ell^p_\omega(\mathbb{Z})$, which implies the cyclicity in $\ell^p_\omega(\mathbb{Z})$ of every $u \in \ell^1_\omega(\mathbb{Z})$ satisfying $\mathcal{Z}(\hat{u}) \subset E$. As a by-product of Theorem B, we show in Corollary 5.1 that the result of Nikolski remains valid for all $p > 2$ and $\gamma > 0$.

This paper is organized as follows. In the next section, we present the background and recall some properties of distribution spaces. Section 3 is devoted to the proof of Theorem A. We construct in Lemma 3.2 a Cantor type set of zero capacity whose k^{th} sum remains of zero capacity. Section 4 provides the proof of Theorem B based on the estimation of power sums of unimodular complex numbers (see Lemma 4.1). Finally, Section 5 is dedicated to some results on the ℓ^p spaces with logarithmic weights.

2. Notations and preliminaries

2.1. Background on ℓ^p weighted spaces

Let $1 \leq p < \infty$ and $\beta \in \mathbb{R}$. We denote by $\mathcal{D}'(\mathbb{T})$ the set of distributions on $\mathbb{T}$, and by $\mathcal{M}(\mathbb{T})$ the set of measures on $\mathbb{T}$. For $S \in \mathcal{D}'(\mathbb{T})$, we denote by $\widehat{S} = (\widehat{S}(n))_{n\in\mathbb{Z}}$ the sequence of Fourier coefficients of S, and we write $S = \sum_n \widehat{S}(n)e_n$, where $e_n(t) = e^{int}$. Notice that we use the same notation $\hat{u}$ and $\widehat{S}$ to denote respectively the Fourier transform of $u \in \ell^p$ and of $S \in \mathcal{D}'(\mathbb{T})$. The space $A^p_\beta(\mathbb{T})$ will be the set of all distributions $S \in \mathcal{D}'(\mathbb{T})$ such that $\widehat{S}$ belongs to $\ell^p_\beta(\mathbb{Z})$. We endow $A^p_\beta(\mathbb{T})$ with the norm

$$\|S\|_{A^p_\beta(\mathbb{T})} = \|\widehat{S}\|_{\ell^p_\beta} = \Big(\sum_{n\in\mathbb{Z}} |\widehat{S}(n)|^p(1+|n|)^{\beta p}\Big)^{1/p}.$$

We will write $A^p(\mathbb{T})$ for the space $A^p_0(\mathbb{T})$. By construction, the Fourier transform $u \to \hat{u}$ is an isometric isomorphism between $\ell^p_\beta(\mathbb{Z})$ and $A^p_\beta(\mathbb{T})$. We prefer to work with $A^p_\beta(\mathbb{T})$ rather than $\ell^p_\beta(\mathbb{Z})$. In this section, we establish some properties of $A^p_\beta(\mathbb{T})$ which will be needed to prove Theorem A and Theorem B.

For $1 \leq p < \infty$ and $\beta \geq 0$, we define the product of $f \in A^1_\beta(\mathbb{T})$ and $S \in A^p_\beta(\mathbb{T})$ by

$$fS = \sum_{n\in\mathbb{Z}} (\widehat{f} * \widehat{S})(n)\, e_n = \sum_{n\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} \widehat{f}(k)\,\widehat{S}(n-k)\Big)\, e_n,$$

and we see that $\|fS\|_{A^p_\beta(\mathbb{T})} \leq \|f\|_{A^1_\beta(\mathbb{T})}\|S\|_{A^p_\beta(\mathbb{T})}$. Note that if $S \in A^p_{-\beta}(\mathbb{T})$, we can also define the product $fS \in A^p_{-\beta}(\mathbb{T})$ by the same formula and obtain a similar inequality: $\|fS\|_{A^p_{-\beta}(\mathbb{T})} \leq \|f\|_{A^1_\beta(\mathbb{T})}\|S\|_{A^p_{-\beta}(\mathbb{T})}$.

For $p \neq 1$, the dual space of $A^p_\beta(\mathbb{T})$ can be identified with $A^q_{-\beta}(\mathbb{T})$, $1/p + 1/q = 1$, by the formula

$$\langle S, T\rangle = \sum_{n\in\mathbb{Z}} \widehat{S}(n)\,\widehat{T}(-n), \quad S \in A^p_\beta(\mathbb{T}),\ T \in A^q_{-\beta}(\mathbb{T}).$$

We need the following lemma, which gives us different inclusions between the $A^p_\beta(\mathbb{T})$ spaces.

Lemma 2.1. *Let $1 \leq r, s < \infty$ and $\beta, \gamma \in \mathbb{R}$.*

(1) *If $r \leq s$, then $A^r_\beta(\mathbb{T}) \subset A^s_\gamma(\mathbb{T})$ if and only if $\gamma \leq \beta$.*

(2) *If $r > s$, then $A^r_\beta(\mathbb{T}) \subset A^s_\gamma(\mathbb{T})$ if and only if $\beta - \gamma > 1/s - 1/r$.*

Proof. (1) Suppose that $r \leq s$. If $\gamma \leq \beta$, then $A^s_\beta(\mathbb{T}) \subset A^s_\gamma(\mathbb{T})$. Since $\|\cdot\|_{\ell^s} \leq \|\cdot\|_{\ell^r}$, we get $A^r_\beta(\mathbb{T}) \subset A^s_\gamma(\mathbb{T})$. Now suppose $\gamma > \beta$. Let $S \in \mathcal{D}'(\mathbb{T})$ such that $\widehat{S}(n)(1+|n|)^\beta = (1+m)^{-2/r}$ if $|n| = 2^m$ and $\widehat{S}(n) = 0$ otherwise. Then we have $S \in A^r_\beta(\mathbb{T}) \setminus A^s_\gamma(\mathbb{T})$.

(2) Suppose that $r > s$. If $\beta - \gamma > 1/s - 1/r$, then by Hölder's inequality, we obtain $A^r_\beta(\mathbb{T}) \subset A^s_\gamma(\mathbb{T})$. Now suppose that $\beta - \gamma < 1/s - 1/r$. Let $\varepsilon > 0$ be such that $\beta - \gamma + \varepsilon < 1/s - 1/r$, $\alpha = -1/s - \gamma + \varepsilon$, and let $S \in \mathcal{D}'(\mathbb{T})$ be such that $\widehat{S}(n) = n^\alpha$. We have $S \in A^r_\beta(\mathbb{T}) \setminus A^s_\gamma(\mathbb{T})$.

Suppose now that $\beta-\gamma = 1/s-1/r$, and let $S \in \mathcal{D}'(\mathbb{T})$ be such that $\widehat{S}(n)^r(1+|n|)^{\beta r} = 1/(1+|n|)\ln(1+|n|)^{1+\varepsilon}$, with $\varepsilon = r/s-1 > 0$. We have $S \in A^r_\beta(\mathbb{T}) \setminus A^s_\gamma(\mathbb{T})$, which proves that $A^r_\beta(\mathbb{T}) \not\subset A^s_\gamma(\mathbb{T})$. ∎

2.2. Cyclicity in $A^p_\beta(\mathbb{T})$

We denote by $\mathcal{P}(\mathbb{T})$ the set of trigonometric polynomials on $\mathbb{T}$. We say that $S \in A^p_\beta(\mathbb{T})$ is a cyclic vector in $A^p_\beta(\mathbb{T})$ if the set $\{PS,\, P \in \mathcal{P}(\mathbb{T})\}$ is dense in $A^p_\beta(\mathbb{T})$. It is clear that the cyclicity of S in $A^p_\beta(\mathbb{T})$ is equivalent to the cyclicity of the sequence $\widehat{S}$ in $\ell^p_\beta(\mathbb{Z})$. Moreover, for $1 \le p < \infty$ and $\beta \ge 0$, S is cyclic in $A^p_\beta(\mathbb{T})$ if and only if there exists a sequence (P_n) of trigonometric polynomials such that

$$\lim_{n\to\infty} \|1 - P_n S\|_{A^p_\beta(\mathbb{T})} = 0. \tag{2.1}$$

We obtain the first cyclicity results for the spaces $A^p_\beta(\mathbb{T})$ when $A^p_\beta(\mathbb{T})$ is a Banach algebra. More precisely, we have the following (see [6]).

Lemma 2.2. *Let $1 \le p < \infty$ and $\beta > 0$. Then $A^p_\beta(\mathbb{T})$ is a Banach algebra if and only if $\beta q > 1$. Moreover, when $\beta q > 1$, a vector $f \in A^p_\beta(\mathbb{T})$ is cyclic in $A^p_\beta(\mathbb{T})$ if and only if f has no zeros on $\mathbb{T}$.*

Let $f \in A^1_\beta(\mathbb{T})$ and $S \in \mathcal{D}'(\mathbb{T})$. We denote by $\mathcal{Z}(f)$ the zero set of the function f:

$$\mathcal{Z}(f) = \{\zeta \in \mathbb{T} : f(\zeta) = 0\}.$$

Lemma 2.3. *Let $1 \le p < \infty$ and $0 \le \beta < 1/2$. Let $f \in A^1_\beta(\mathbb{T})$ and $S \in A^p_{-\beta}(\mathbb{T})$. If for all $n \in \mathbb{Z}$, $\langle S, e_n f\rangle = 0$, then $\operatorname{supp}(S) \subset \mathcal{Z}(f)$.*

Proof. Recall that $e_n(t) = e^{int}$. We have

$$\langle S, e_n f\rangle = \langle fS, e_n\rangle = 0.$$

Hence $fS = 0$. Let $\varphi \in C^\infty(\mathbb{T})$ be such that $\operatorname{supp}(\varphi) \subset \mathbb{T} \setminus \mathcal{Z}(f)$. We claim that $\varphi/f \in A^1_\beta(\mathbb{T}) \subset A^q_\beta(\mathbb{T})$, where $1/p + 1/q = 1$. So we obtain

$$\langle S, \varphi\rangle = \langle fS, \varphi/f\rangle = 0,$$

which proves that $\operatorname{supp}(S) \subset \mathcal{Z}(f)$.

Now we prove the claim. Let $\varepsilon = \min\{|f(t)|, t \in \operatorname{supp}(\varphi)\} > 0$, and let $P \in \mathcal{P}(\mathbb{T})$ be such that $\|f - P\|_{A^1_\beta(\mathbb{T})} \le \varepsilon/3$. By the Cauchy–Schwarz and the Parseval inequalities,

$$\|g\|_{A^1_\beta(\mathbb{T})} \le \|g\|_\infty + 2^{1+\beta}\sqrt{\frac{1-\beta}{1-2\beta}}\,\|g'\|_\infty \tag{2.2}$$

for every $g \in C^1(\mathbb{T})$. As in [19], by applying (2.2) to φ/P^n, we see that

$$\frac{\varphi}{f} = \sum_{n\ge 1} \varphi\, \frac{(P-f)^{n-1}}{P^n} \in A^1_\beta(\mathbb{T}),$$

which finishes the proof. ∎

Lemma 2.4. *Let $1 < p < \infty$ and $f \in A^1_\beta(\mathbb{T})$ with $\beta \geq 0$. We have:*

(1) *If f is not cyclic in $A^p_\beta(\mathbb{T})$, then there exists $S \in A^q_{-\beta}(\mathbb{T}) \setminus \{0\}$ such that* $\operatorname{supp}(S) \subset \mathcal{Z}(f)$.

(2) *If there exists a nonzero measure $\mu \in A^q_{-\beta}(\mathbb{T})$ such that* $\operatorname{supp}(\mu) \subset \mathcal{Z}(f)$, *then f is not cyclic in $A^p_\beta(\mathbb{T})$.*

Proof. (1) If f is not cyclic in $A^p_\beta(\mathbb{T})$, by duality there exists $S \in A^q_{-\beta}(\mathbb{T}) \setminus \{0\}$ such that

$$\langle S, e_n f \rangle = 0, \quad \forall n \in \mathbb{Z}.$$

Thus, by Lemma 2.3, we have $\operatorname{supp}(S) \subset \mathcal{Z}(f)$.

(2) Let $\mu \in A^q_{-\beta}(\mathbb{T}) \cap \mathcal{M}(\mathbb{T}) \setminus \{0\}$ be such that $\operatorname{supp}(\mu) \subset \mathcal{Z}(f)$. Since μ is a measure on $\mathbb{T}$, we have $\langle \mu, e_n f \rangle = 0$ for all $n \in \mathbb{Z}$. So f is not cyclic in $A^p_\beta(\mathbb{T})$. ∎

Remark 2.5. If we suppose that $f \in A^1(\mathbb{T})$ (instead of $f \in A^1_\beta(\mathbb{T})$), then the result (1) of Lemma 2.4 remains valid. To prove this, it suffices to show that $A^1(\mathbb{T}) \cap A^p_\beta(\mathbb{T})$ is a Banach algebra and we can make the same proof of Lemma 2.4.

Now, let us show that the space $A^1(\mathbb{T}) \cap A^p_\beta(\mathbb{T})$, $p \geq 1$, $\beta > 0$, endowed with the norm

$$\|f\|_{A^1(\mathbb{T}) \cap A^p_\beta(\mathbb{T})} = \|f\|_{A^1(\mathbb{T})} + \|f\|_{A^p_\beta(\mathbb{T})}, \quad f \in A^1(\mathbb{T}) \cap A^p_\beta(\mathbb{T})$$

is a Banach algebra. Since

$$(1 + |k + l|)^\beta \leq (1 + |k|)^\beta + (1 + |l|)^\beta \quad \text{and} \quad \|fg\|_{A^p_\beta(\mathbb{T})} \leq \|f\|_{A^1(\mathbb{T})} \|g\|_{A^p_\beta(\mathbb{T})},$$

if we write $f = \sum a_n e_n$ and $g = \sum b_n e_n$, where $e_n(t) = e^{int}$, we get

$$\begin{aligned}
&\|fg\|_{A^p_\beta(\mathbb{T})} \\
&\leq \left(\sum_{n \in \mathbb{Z}} \Big[\sum_{m \in \mathbb{Z}} |a_{n-m} b_m| (1 + |n - m|)^\beta \Big]^p + \sum_{n \in \mathbb{Z}} \Big[\sum_{m \in \mathbb{Z}} |a_{n-m} b_m| (1 + |m|)^\beta \Big]^p \right)^{1/p} \\
&\leq \left(\sum_{n \in \mathbb{Z}} \Big[\sum_{m \in \mathbb{Z}} |b_m| \, |a_{n-m}| \, (1 + |n - m|)^\beta \Big]^p \right)^{1/p} \\
&\quad + \left(\sum_{n \in \mathbb{Z}} \Big[\sum_{m \in \mathbb{Z}} |a_{n-m}| \, |b_m| \, (1 + |m|)^\beta \Big]^p \right)^{1/p} \\
&\leq \|g\|_{A^1(\mathbb{T})} \, \|f\|_{A^p_\beta(\mathbb{T})} + \|f\|_{A^1(\mathbb{T})} \, \|g\|_{A^p_\beta(\mathbb{T})}, \quad \text{for } f, g \in A^1(\mathbb{T}) \cap A^p_\beta(\mathbb{T}),
\end{aligned}$$

which allows us to conclude.

Recall that $A^1_\beta(\mathbb{T})$ is a Banach algebra. Let I be a closed ideal in $A^1_\beta(\mathbb{T})$. We denote by $\mathcal{Z}_I$ the set of common zeros of the functions of I:

$$\mathcal{Z}_I = \bigcap_{f \in I} \mathcal{Z}(f).$$

We have the following result about spectral synthesis in $A^1_\beta(\mathbb{T})$, for the case $\beta = 0$, also called the Beurling–Pollard technique, see [9], pp. 121–123.

Lemma 2.6. *Let $0 \le \beta < 1/2$. Let I be a closed ideal in $A^1_\beta(\mathbb{T})$. If g is a Lipschitz function which vanishes on $\mathcal{Z}_I$, then $g \in I$.*

Proof. Notice first that since g is Lipschitz function, Bernstein's theorem (see [8], p. 13) gives that $g \in A^1_\beta(\mathbb{T})$. Let $I^\perp$ be the set of all S in the dual space of $A^1_\beta(\mathbb{T})$ satisfying $\langle S, f\rangle = 0$ for all $f \in I$. Hence, $S \in I^\perp$ and $\mathrm{supp}(S) \subset \mathcal{Z}_I$, see Remarque 1.3 in [4]. For $h > 0$, we set $S_h = S * \Delta_h$, where $\Delta_h \colon t \mapsto -|t|/h^2 + 1/h$ if $t \in [-h, h]$, and 0 otherwise. We have

$$\widehat{\Delta_h}(0) = 1/2\pi \quad \text{and} \quad \widehat{\Delta_h}(n) = \frac{1}{2\pi}\frac{4\sin(nh/2)^2}{(nh)^2} \quad \text{for } n \neq 0.$$

Since S is in the dual of $A^1_\beta(\mathbb{T})$, $S_h \in A^1(\mathbb{T})$. Moreover, we have $\mathrm{supp}(S_h) \subset \mathrm{supp}(S) + \mathrm{supp}(\Delta_h) \subset \mathcal{Z}^h_I := \mathcal{Z}_I + [-h, h]$. Let g be a Lipschitz function which vanishes on $\mathcal{Z}_I$. We have

$$|\langle S_h, g\rangle|^2 = \Big|\int_{\mathcal{Z}^h_I \setminus \mathcal{Z}(g)} S_h(x)g(x)dx\Big|^2 \le \Big(\sum_{n\in\mathbb{Z}} |\widehat{S}(n)\widehat{\Delta_h}(n)|^2\Big)\Big(\int_{\mathcal{Z}^h_I \setminus \mathcal{Z}(g)} |g(x)|^2 dx\Big).$$

Since $\mathcal{Z}_I \subset \mathcal{Z}_g$, for every $x \in \mathcal{Z}^h_I$, $|g(x)| \le ch$ for some positive constant c. Thus

$$|\langle S_h, g\rangle|^2 \le c^2\Big(\sum_{n\in\mathbb{Z}} \frac{\widehat{S}(n)^2}{1+n^2}\Big)\left(|\mathcal{Z}^h_I \setminus \mathcal{Z}(g)|\right).$$

Hence $\lim_{h\to 0}\langle S_h, g\rangle = 0$. By the dominated convergence theorem, we obtain that

$$\lim_{h\to 0}\langle S_h, g\rangle = \lim_{h\to 0}\sum_{n\in\mathbb{Z}} \widehat{S_h}(n)\widehat{g}(-n) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \widehat{S}(n)\widehat{g}(-n) = \frac{1}{2\pi}\langle S, g\rangle.$$

So $\langle S, g\rangle = 0$. Therefore, $g \in I$. ∎

We also need the following lemma, which is a consequence of Lemma 2.6. Newman gave a proof of this when $\beta = 0$ (see Lemma 2 in [18]).

Lemma 2.7. *Let $0 \le \beta < 1/2$ and consider a closed set $E \subset \mathbb{T}$. There exists a sequence of Lipschitz functions (f_n) which vanish on E such that*

$$\lim_{n\to\infty} \|f_n - 1\|_{A^p_\beta(\mathbb{T})} = 0$$

if and only if every $f \in A^1_\beta(\mathbb{T})$ satisfying $\mathcal{Z}(f) = E$ is cyclic in $A^p_\beta(\mathbb{T})$.

We finish this subsection with the following result of Newman (see the proof of Theorem 5 in [18]).

Lemma 2.8. *Let $p > 2$. Assume that for every $\varepsilon > 0$, there exists a Lipschitz function f such that $|\mathcal{Z}(f)| > 2\pi - \varepsilon$ and*

$$\|f - 1\|_{A^p_\beta(\mathbb{T})} \le \varepsilon.$$

Then for every ε, there exists a (p, β)-spanning closed set $E \subset \mathbb{T}$ with Lebesgue measure $|E| > 2\pi - \varepsilon$.

2.3. Generalized Cantor set and capacity

Given $E \subset \mathbb{T}$ and a non-decreasing continuous function h such that $h(0) = 0$, we define the *h-measure* of E by

$$H_h(E) = \lim_{\delta \to 0} \inf \Big\{ \sum_{i=0}^{\infty} h(|U_i|),\ E \subset \bigcup_{i=0}^{\infty} U_i,\ |U_i| \le \delta \Big\},$$

where each U_i is an open interval inside $\mathbb{T}$ and $|U_i|$ denotes its length.

We also define the *Hausdorff dimension* of a subset $E \subset \mathbb{T}$ as

$$\dim(E) = \inf\{\alpha \in (0,1), H_\alpha(E) = 0\} = \sup\{\alpha \in (0,1), H_\alpha(E) = \infty\},$$

where $H_\alpha = H_h$ for $h(t) = t^\alpha$ (see [9], pp. 23–30).

Let μ be a probability measure on $\mathbb{T}$ and let $\alpha \in (0,1)$. We define its *α-energy* by

$$I_\alpha(\mu) := \iint \frac{d\mu(t)\, d\mu(s)}{|t-s|^\alpha}.$$

Note that $I_\alpha(\mu) \in [0, +\infty]$. Simple calculations shows that

$$I_\alpha(\mu) \asymp \sum_{n \ge 1} \frac{|\widehat{\mu}(n)|^2}{(1+|n|)^{1-\alpha}}.$$

The *α-capacity* of a Borel set E is given by

$$C_\alpha(E) = \frac{1}{\{\inf\{I_\alpha(\mu),\ \mu \in \mathcal{M}_{\mathcal{P}}(E)\}},$$

where $\mathcal{M}_{\mathcal{P}}(E)$ is the set of all probability measures on $\mathbb{T}$ which are supported on a compact subset of E.

An important property which connects capacity and Hausdorff dimension is (see [9], p. 34) that

$$\dim(E) = \inf\{\alpha \in (0,1), C_\alpha(E) = 0\} = \sup\{\alpha \in (0,1), C_\alpha(E) > 0\}. \tag{2.3}$$

For $E \subset \mathbb{T}$, we denote by $A^p_\beta(E)$ the collection of $S \in A^p_\beta(\mathbb{T})$ such that $\operatorname{supp}(S) \subset E$, where $\operatorname{supp}(S)$ denotes the support of the distribution S. The following lemma is a direct consequence of the definition of capacity and the inclusion $A^q_{-\beta}(\mathbb{T}) \subset A^2_{(\alpha-1)/2}(\mathbb{T})$ when $q \ge 2$ and $0 \le \alpha < \frac{2}{q}(1-\beta q)$.

Lemma 2.9. *Let E be a Borel set, $\beta \ge 0$ and $q \ge 2$. If there exists $\alpha \in \mathbb{R}$ with $0 \le \alpha < \frac{2}{q}(1-\beta q)$ such that $C_\alpha(E) = 0$, then $A^q_{-\beta}(E) = \{0\}$.*

Let us recall Salem's theorem (see [24] and [9], pp. 106–110).

Theorem 2.10. *Let $0 < \alpha < 1$ and $q > 2/\alpha$. There exists a compact set $E \subset \mathbb{T}$ which satisfies $\dim(E) = \alpha$ and there exists a positive measure $\mu \in A^q(\mathbb{T}) \setminus \{0\}$ such that $\operatorname{supp}(\mu) \subset E$.*

The following theorem is due to Körner (see Theorem 1.2 in [13]).

Theorem 2.11. *Let* $h\colon [0,\infty) \to [0,\infty)$ *be an increasing continuous function such that* $h(0) = 0$*, and let* $\phi\colon [0,\infty) \to [0,\infty)$ *be a decreasing function. Suppose that*

(1) $\int_1^\infty \phi(x)^2 dx = \infty$,

(2) *there exist* $K_1, K_2 > 1$ *such that for all* $1 \le x \le y \le 2x$, $K_1\phi(2x) \le \phi(x) \le K_2\phi(y)$,

(3) *there exists* $\gamma > 0$ *such that* $\lim_{x\to\infty} x^{1-\gamma}\phi(x) = \infty$,

(4) *there exist* $0 < K_3 < K_4 < 1$ *such that for all* $t > 0$, $K_3 h(2t) \le h(t) \le K_4 h(2t)$.

Then there exists a probability measure μ *with support of Hausdorff* h*-measure zero such that*

$$|\widehat{\mu}(n)| \le \phi\Big(\frac{1}{h(|n|^{-1})}\Big)\Big(\ln\Big(\frac{1}{h(|n|^{-1})}\Big)\Big)^{1/2}, \quad \forall n \neq 0.$$

We finish this section by describing the construction of the generalized Cantor set. Let $(k_j)_{j\ge0}$ be a sequence of integers and let $(l_j)_{j\ge0}$ be a sequence of positive numbers such that $k_0 = 1$ and

$$k_j \ge 2, \quad \text{and} \quad k_j l_j < l_{j-1}, \quad j \ge 1.$$

Let $E_0 = [0, l_0]$. We dissect the interval $E_0 = [0, l_0]$ in $2k_1 - 1$ intervals of lengths respectively l_1 and $d_1 = (l_0 - k_1 l_1)/(k_1 - 1)$:

$$\begin{aligned}
&[0, l_1]; \quad]l_1, d_1 + l_1[;\\
&\quad\vdots\\
&[md_1 + ml_1, md_1 + (m+1)l_1]; \quad]md_1 + (m+1)l_1, (m+1)d_1 + (m+1)l_1[;\\
&\quad\vdots\\
&]l_0 - l_1 - d_1, l_0 - l_1[; \quad [l_0 - l_1, l_0].
\end{aligned}$$

We delete the $k_1 - 1$ open interval of length d_1 and we keep the k_1 equidistant closed intervals of length l_1. We set

$$E_1 = \bigcup_{m=0}^{k_1-1} [md_1 + ml_1, md_1 + (m+1)l_1].$$

Suppose that the set E_{n-1} , $n \ge 1$, has already been constructed, and that this set consists of p_{n-1} closed intervals of length l_{n-1}:

$$E_{n-1} = \bigcup_{j=1}^{p_{n-1}} [a_j, a_j + l_{n-1}].$$

We operate the same dissection on each of the intervals $[a_j, a_j + l_{n-1}]$ with the parameters (l_n, k_n) instead of (l_1, k_1), thus we obtain

$$E_n = \bigcup_{j=1}^{p_n} \bigcup_{s=0}^{k_n-1} [a_j + s(l_n + d_n), a_j + s(l_n + d_n) + l_n],$$

where $d_n = (l_{n-1} - k_n l_n)/(k_n - 1)$. The compact set

$$E = \bigcap_{n \geq 0} E_n$$

is called the generalized Cantor set.

Ohtsuka [22] obtained a criterion for vanishing $C_\alpha(E)$, see also [5]:

Theorem 2.12. *Let E be a generalized Cantor set. Then*

$$C_\alpha(E) = 0 \iff \sum_{n \geq 0} \frac{1}{(k_0 k_1 \ldots k_n)\, l_n^\alpha} = \infty.$$

3. Proof of Theorem A

3.1. Cyclicity and the set of all sums of k elements from $\mathcal{Z}(f)$

For $k \in \mathbb{N}$ and $E \subset \mathbb{T}$, let $k \times E$ denote the set of all sums of k elements from E,

$$k \times E = E + E + \cdots + E = \left\{ \sum_{n=1}^{k} x_n,\ x_n \in E \right\}.$$

We have the following result (the case $\beta = 0$ was considered by Newman in [18]).

Lemma 3.1. *Let $1 < p < 2$ and $\beta > 0$ be such that $\beta q \leq 1$, and let $f \in A^1_\beta(\mathbb{T})$.*

(a) *Let $k \in \mathbb{N} \setminus \{0\}$ be such that $k \leq q/2$. If $C_\alpha(k \times \mathcal{Z}(f)) = 0$ for some $\alpha < \frac{2}{q}(1 - \beta q)k$, then f is cyclic in $A^p_\beta(\mathbb{T})$.*

(b) *Let $k \in \mathbb{N} \setminus \{0\}$ be such that $q/2 \leq k \leq 1/(2\beta)$. If $C_\alpha(k \times \mathcal{Z}(f)) = 0$, where $\alpha = 1 - 2k\beta$, then f is cyclic in $A^p_\beta(\mathbb{T})$.*

Proof. Let $k \in \mathbb{N} \setminus \{0\}$. Suppose that f is not cyclic in $A^p_\beta(\mathbb{T})$. Then there exists $L \in A^q_{-\beta}(\mathbb{T})$, the dual of $A^p_\beta(\mathbb{T})$, such that $L(1) = 1$ and $L(Pf) = 0$, for all $P \in \mathcal{P}(\mathbb{T})$.

Since $\beta < 1/2$, by (2.2), we get $C^1(\mathbb{T}) \subset A^1_\beta(\mathbb{T}) \subset A^p_\beta(\mathbb{T})$. Moreover, by [17] (see also Lemma 5 in [18]), there exists $\phi \in L^2(\mathbb{T})$ such that

$$L(g) = \int_{\mathbb{T}} \big(g'(x)\phi(x) + g(x)\big)\, \mathrm{d}x, \quad g \in C^1(\mathbb{T}).$$

Since $L \in A^q_{-\beta}(\mathbb{T})$ which implies $(L(e_n))_{n\in\mathbb{Z}} \in \ell^q_{-\beta}(\mathbb{Z})$, we obtain

$$\sum_{n\in\mathbb{Z}} |n\widehat{\phi}(n)|^q (1 + |n|)^{-\beta q} < \infty. \tag{3.1}$$

Moreover, we have

$$\int_{\mathbb{T}} \big((e_n f)'(x)\, \phi(x) + (e_n f)(x)\big)\, \mathrm{d}x = 0, \quad n \in \mathbb{Z},$$

and hence $\langle \phi' - 1, e_n f \rangle = 0$, where ϕ' is defined in the sense of distributions. By (3.1), $\phi' - 1 \in A^q_{-\beta}(\mathbb{T})$, by Lemma 2.3, we get $\mathrm{supp}(\phi' - 1) \subset \mathcal{Z}(f)$.

For $m \in \mathbb{N}$, we denote by ϕ^{*m} the result obtained from convolving ϕ with itself m times. Using the fact that $S' * T = S * T'$ and $1 * S' = 0$ for any distributions S and T, we have

$$(\phi' - 1) * \big((\phi^{*(m-1)})^{(m-1)} + (-1)^{m-1}\big) = (\phi^{*m})^{(m)} + (-1)^m.$$

By induction and by the formula $\operatorname{supp}(T * S) \subset \operatorname{supp}(T) + \operatorname{supp}(S)$, we get that

$$\operatorname{supp}\big((\phi^{*m})^{(m)} + (-1)^m\big) \subset m \times \mathcal{Z}(f), \quad m \geq 1. \tag{3.2}$$

Note that $\widehat{(\phi^{*k})^{(k)}}(n) = i^k n^k \widehat{\phi}(n)^k$ for $k \geq 1$ and $n \in \mathbb{Z}$.

(a) Suppose that $0 < k \leq q/2$ and $C_\alpha(k \times \mathcal{Z}(f)) = 0$ for some $\alpha < \frac{2}{q}(1 - \beta q)k$. We rewrite (3.1) as

$$\sum_{n\in\mathbb{Z}} \big(|n\,\widehat{\phi}(n)|^k\big)^{q/k} (1 + |n|)^{-\frac{q}{k}\beta k} < \infty.$$

Setting $q' = q/k \geq 2$ and $\beta' = \beta k$, we have $(\phi^{*k})^{(k)} \in A^{q'}_{-\beta'}(\mathbb{T})$. By (3.2) and Lemma 2.9, we obtain that $(\phi^{*k})^{(k)} = (-1)^{k-1}$. This contradicts the fact that $\widehat{(\phi^{*k})^{(k)}}(0) = 0$.

(b) Now suppose that $k \geq q/2$ and $C_\alpha(k \times \mathcal{Z}(f)) = 0$, where $\alpha = 1 - 2k\beta$. Since $q \leq 2k$, we have by (3.1),

$$\sum_{n\in\mathbb{Z}} |n\,\widehat{\phi}(n)|^{2k} (1 + |n|)^{-2k\beta} < \infty.$$

So $(\phi^{*k})^{(k)} \in A^2_{-k\beta}(\mathbb{T})$ and $(\phi^{*k})^{(k)} = (-1)^{k-1}$, which contradicts $\widehat{(\phi^{*k})^{(k)}}(0) = 0$. ■

3.2. Construction of generalized Cantor set

We need to compute the capacity of the Minkowski sum of some Cantor type subset of $\mathbb{T}$. We denote by $[x]$ the integer part of $x \in \mathbb{R}$. For $\lambda \in [0, 1]$ and $k \in \mathbb{N} \setminus \{0\}$, we define

$$K^k_\lambda = \{m \in \mathbb{N},\ \exists j \in \mathbb{N},\ m \in [2^j, 2^j(1 + \lambda + 1/j) - k + 1]\},$$

and we set, in $\mathbb{R}/\mathbb{Z} \simeq [0, 1[$,

$$S^k_\lambda = \Big\{x = \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}},\ (x_i) \in \{0,1\}^{\mathbb{N}} \text{ such that } i \in K^k_\lambda \Rightarrow x_i = 0\Big\}.$$

We denote $K_\lambda = K^1_\lambda$ and $S_\lambda = S^1_\lambda$. We have the following lemma.

Lemma 3.2. *For all $k \geq 1$, we have*

(1) $k \times S_\lambda \subset S^k_\lambda$,

(2) $C_\alpha(S^k_\lambda) = 0$ *if and only if* $\alpha \geq (1 - \lambda)/(1 + \lambda)$,

(3) $\dim(k \times S_\lambda) = (1 - \lambda)/(1 + \lambda)$ *and* $C_{(1-\lambda)/(1+\lambda)}(k \times S_\lambda) = 0$.

Proof. (1) We prove this by induction. If $k = 1$, we have $S_\lambda = S_\lambda^1$. We suppose the result true for $k-1$ for some $k \geq 2$, and we will show that $k \times S_\lambda \subset S_\lambda^k$. Observe that we have

$$k \times S_\lambda \subset (k-1) \times S_\lambda + S_\lambda \subset S_\lambda^{k-1} + S_\lambda.$$

Let $x \in S_\lambda^{k-1}$, $y \in S_\lambda$ and $z = x + y$. Denote by (x_i), (y_i) and (z_i) their binary decompositions. Let $m \in K_\lambda^k$. Then there exists $j \in \mathbb{N}$ such that $m \in [2^j, 2^j(1+\lambda+1/j) - k + 1]$. Since $m \in K_\lambda^k$ and $m, m+1 \in K_\lambda^{k-1} \subset K_\lambda$, we get $x_m = y_m = x_{m+1} = y_{m+1} = 0$. Therefore, we write

$$z = x + y = \sum_{i=0}^{m-1} \frac{x_i + y_i}{2^{i+1}} + \sum_{i=m+2}^{\infty} \frac{x_i + y_i}{2^{i+1}}.$$

For infinitely many $i \geq m+2$, we have $x_i + y_i < 2$, and hence

$$\sum_{i=m+2}^{\infty} \frac{x_i + y_i}{2^{i+1}} < \frac{1}{2^{m+1}}.$$

Denoting by $[s]$ the integer part of s, we have

$$[2^{m+1} z] = 2[2^m z] = 2^{m+1} \sum_{i=0}^{m-1} \frac{x_i + y_i}{2^{i+1}}.$$

Therefore, we obtain by the uniqueness of the decomposition that

$$z_m = [2^{m+1} z] - 2[2^m z] = 0.$$

This proves that $z = x + y \in S_\lambda^k$ and $k \times S_\lambda \subset S_\lambda^k$.

(2) We will first show that the set S_λ^k is a generalized Cantor set. Let

$$\nu_j = [2^j(1+\lambda+1/j) - k + 1] + 1$$

and N_0, depending only on k and λ, be such that for all $j \geq N_0$, $2^j < \nu_j < 2^{j+1}$. We set for $N \geq N_0$,

$$l_N = \sum_{j=N}^{\infty} \Big(\frac{1}{2^{\nu_j}} - \frac{1}{2^{2^{j+1}}} \Big).$$

Since

$$2^j(1+\lambda+1/j) - k + 1 < \nu_j \leq 2^j(1+\lambda+1/j) - k + 2,$$

we have

$$\sum_{j=N}^{\infty} \frac{1}{2^{2^j(1+\lambda+\frac{1}{j})}} \Big(\frac{1}{2^{2-k}} - \frac{1}{2^{2^j(1-\lambda-\frac{1}{j})}} \Big) \leq l_N \leq \sum_{j=N}^{\infty} \frac{1}{2^{2^j(1+\lambda+\frac{1}{j})}} \Big(\frac{1}{2^{1-k}} - \frac{1}{2^{2^j(1-\lambda-\frac{1}{j})}} \Big).$$

There exists $C \geq 1$ such that, for all $j \geq N$,

$$\frac{1}{C} \leq \frac{1}{2^{2-k}} - \frac{1}{2^{2^j(1-\lambda-1/j)}} \leq \frac{1}{2^{1-k}} - \frac{1}{2^{2^j(1-\lambda-1/j)}} \leq C.$$

And for $N \geq N_0$,

$$\frac{1}{2^{2^N(1+\lambda+1/N)}} \leq \sum_{j=N}^{\infty} \frac{1}{2^{2^j(1+\lambda+1/j)}} \leq \frac{1}{2^{2^N(1+\lambda+1/N)}} + \sum_{j=0}^{\infty} \Big(\frac{1}{2^{2^{N+1}(1+\lambda)}}\Big)^{2^j}$$
$$\leq \frac{1}{2^{2^N(1+\lambda+1/N)}} + \sum_{j=0}^{\infty} \Big(\frac{1}{2^{2^{N+1}(1+\lambda)}}\Big)^{j+1} \leq \frac{1}{2^{2^N(1+\lambda+1/N)}} + \frac{2}{2^{2^{N+1}(1+\lambda)}}$$
$$\leq \frac{3}{2^{2^N(1+\lambda+1/N)}}.$$

Hence we obtain that l_N is comparable to $2^{-2^N(1+\lambda+1/N)}$, that is,

$$\frac{1}{C\,2^{2^N(1+\lambda+1N)}} \leq l_N \leq \frac{3C}{2^{2^N(1+\lambda+1/N)}}. \tag{3.3}$$

Moreover, we have

$$l_N = \frac{1}{2^{\nu_N}} - \sum_{j=N+1}^{\infty} \Big(\frac{1}{2^{2^j}} - \frac{1}{2^{\nu_j}}\Big) < \frac{1}{2^{\nu_N}} \leq \frac{1}{2^{2^N}}. \tag{3.4}$$

For $N \geq N_0$, we set

$$E_N = \Big\{ \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + l_N z,\ z \in [0,1],\ x_i \in \{0,1\},\ i \in K_\lambda^k \Rightarrow x_i = 0 \Big\}.$$

Observe that we can write E_N as a union of disjoint intervals:

$$E_N = \bigcup_{\substack{(x_i)\in\{0,1\}^{2^N} \\ i\in K_\lambda^k \Rightarrow x_i=0}} E_N^{(x_i)},$$

where

$$E_N^{(x_i)} = \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + l_N[0,1[.$$

Since by (3.4), $l_N < 1/2^{2^N}$, the intervals $E_N^{(x_i)}$ are disjoint:

$$E_N^{(x_i)} \cap E_N^{(x_i')} = \emptyset, \quad (x_i) \neq (x_i').$$

For fixed $N \geq N_0$, let $(x_i)_{0\leq i\leq 2^N-1} \in \{0,1\}^{2^N}$ and $(y_i)_{0\leq i\leq 2^{N+1}-1} \in \{0,1\}^{2^{N+1}}$. We claim that:

$E_{N+1}^{(y_i)} \subset E_N^{(x_i)}$ if and only if $x_i = y_i$ for all $0 \leq i < 2^N$, and $y_i = 0$ for all $2^N \leq i < \nu_N$.

Indeed, suppose that $E_{N+1}^{(y_i)} \subset E_N^{(x_i)}$ and let $u \in E_{N+1}^{(y_i)}$. We have

$$u = \sum_{i=0}^{2^{N+1}-1} \frac{y_i}{2^{i+1}} + l_{N+1} z_2 = \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + l_N z_1,$$

where $z_1, z_2 \in [0,1[$. By (3.4), $l_N < 1/2^{\nu_N}$, and using the uniqueness of the binary representation, we obtain $x_i = y_i$ for all $0 \le i < 2^N$ and $y_i = 0$ for all $2^N \le i < \nu_N$. Now suppose $x_i = y_i$ for all $0 \le i < 2^N$ and $y_i = 0$ for all $2^N \le i < \nu_N$. Let $u \in E_{N+1}^{(y_i)}$. We write

$$u = \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + \sum_{i=\nu_N}^{2^{N+1}-1} \frac{y_i}{2^{i+1}} + l_{N+1} z, \quad z \in [0,1[.$$

Since

$$\sum_{i=\nu_N}^{2^{N+1}-1} \frac{1}{2^{i+1}} + l_{N+1} = \frac{1}{2^{\nu_N}} - \frac{1}{2^{2^{N+1}}} + l_{N+1} = l_N,$$

we get

$$\sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} \le \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + \sum_{i=Z_N}^{2^{N+1}-1} \frac{y_i}{2^{i+1}} + l_{N+1} z \le \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + l_N,$$

and $u \in E_N^{(x_i)}$. This concludes the proof of the claim.

By the claim, for fixed (x_i) and for $N \ge N_0$, we have the following properties:

(i) the interval $E_N^{(x_i)}$ contains precisely

$$k_{N+1} = \#\big\{(y_i)_{\nu_N \le i \le 2^{N+1}-1} \colon y_i \in \{0,1\}\big\} = 2^{2^{N+1}-\nu_N}$$

intervals of the form $E_{N+1}^{(y_i)}$.

(ii) The intervals of the form $E_{N+1}^{(y_i)}$ contained in $E_N^{(x_i)}$ are equidistant intervals of length l_{N+1}; the distance of two consecutive intervals of the form $E_{N+1}^{(y_i)}$ is equal to $1/(2^{2^{N+1}} - l_{N+1})$.

(iii) Writing $E_N^{(x_i)} = [a,b]$, there exist (y_i) and (z_i) such that $E_{N+1}^{(y_i)} = [a, a + l_{N+1}]$ and $E_{N+1}^{(z_i)} = [b - l_{N+1}, b]$.

Finally, we can write S_λ^k as

$$S_\lambda^k = \bigcap_{N \ge N_0} E_N.$$

This shows that S_λ^k is a generalized Cantor set. By Theorem 2.12, we have for $0 < \alpha < 1$ that $C_\alpha(S_\lambda^k) = 0$ if and only if

$$\sum_{N=N_0}^{\infty} \frac{1}{(k_{N_0} \cdots k_{N-1})\, l_N^\alpha} = \infty,$$

where $k_{N_0} = 1$. Since

$$2^{(k-2)(N-N_0)+(2^N-2^{N_0})(1-\lambda)-\sigma_N} \le k_{N_0} \cdots k_{N-1} \le 2^{(k-1)(N-N_0)+(2^N-2^{N_0})(1-\lambda)-\sigma_N},$$

where

$$\sigma_N = \sum_{j=N_0}^{N-1} \frac{2^j}{j},$$

we have, by (3.3), that $C_\alpha(S_\lambda^k) = 0$ if and only if

$$\sum_{N=N_0}^{\infty} 2^{2^N(\alpha(1+\lambda)-(1-\lambda))+\alpha 2^N/N+\sigma_N-(k-1)(N-N_0)+2^{N_0}(1-\lambda)} = \infty.$$

Therefore, $C_\alpha(S_\lambda^k) = 0$ if and only if $\alpha \geq (1-\lambda)/(1+\lambda)$.

Finally, (3) follows from (1), and (2) by the capacity property. ∎

3.3. Proof of Theorem A

We are now ready to prove Theorem A. It follows immediately from the following theorem, stated in $A^p_\beta(\mathbb{T})$ spaces.

Theorem 3.3. *Let $1 < p < 2$ and $\beta > 0$ be such that $\beta q \leq 1$.*

(1) *If $f \in A^1_\beta(\mathbb{T})$ and* $\dim(\mathcal{Z}(f)) < \frac{2}{q}(1-\beta q)$, *then f is cyclic in $A^p_\beta(\mathbb{T})$.*

(2) *If $f \in A^1_\beta(\mathbb{T})$ and $C_{1-\beta q}(\mathcal{Z}(f)) > 0$, then f is not cyclic in $A^p_\beta(\mathbb{T})$.*

(3) *For $\frac{2}{q}(1-\beta q) \leq \alpha \leq 1$, there exists a closed set $E \subset \mathbb{T}$ such that* $\dim(E) = \alpha$ *and every $f \in A^1_\beta(\mathbb{T})$ satisfying $\mathcal{Z}(f) = E$ is not cyclic in $A^p_\beta(\mathbb{T})$.*

(4) *Let $k = [q/2]$. For all $\varepsilon > 0$, there exists a closed set $E \subset \mathbb{T}$ such that*

$$\dim(E) \geq \max\left(\tfrac{2}{q}(1-\beta q)k - \varepsilon,\ 1 - 2(k+1)\beta\right) \tag{3.5}$$

and every $f \in A^1_\beta(\mathbb{T})$ satisfying $\mathcal{Z}(f) = E$ is cyclic in $A^p_\beta(\mathbb{T})$. Furthermore, if $p = 2k/(2k-1)$ for some $k \in \mathbb{N} \setminus \{0\}$, E can be chosen such that $\dim(E) = 1 - \beta q$.

Proof. (1) Note that, by (2.3), $\dim(\mathcal{Z}(f)) < \frac{2}{q}(1-\beta q)$ if and only if there exists $\alpha < \frac{2}{q}(1-\beta q)$ such that $C_\alpha(\mathcal{Z}(f)) = 0$. If $C_\alpha(\mathcal{Z}(f)) = 0$, by Lemma 2.9, there is no $S \in A^q_{-\beta}(\mathbb{T}) \setminus \{0\}$ such that $\operatorname{supp}(S) \subset \mathcal{Z}(f)$. So, by Lemma 2.4 (1), f is cyclic in $A^p_\beta(\mathbb{T})$.

(2) Suppose that $C_{1-\beta q}(\mathcal{Z}(f)) > 0$. There exists a probability measure μ of energy $I_{1-\beta q}(\mu) < \infty$, such that $\operatorname{supp}(\mu) \subset \mathcal{Z}(f)$. So $\mu \in A^2_{-\beta q/2}(\mathbb{T}) \setminus \{0\}$. Since $|\widehat{\mu}(n)| \leq 1$ for all $n \in \mathbb{Z}$ and $q \geq 2$, we have $\mu \in A^q_{-\beta}(\mathbb{T})$. By Lemma 2.4 (2), f is not cyclic in $A^p_\beta(\mathbb{T})$.

(3) Suppose that $\frac{2}{q}(1-\beta q) < \alpha \leq 1$. There exists $\varepsilon > 0$ such that $\frac{2}{q}(1-\beta q) + \varepsilon < \alpha$. Let q' be such that $2/q - 2\beta + \varepsilon = 2/q'$. Since $\beta > 1/q - 1/q'$, by Lemma 2.1, $A^{q'}(\mathbb{T}) \subset A^q_{-\beta}(\mathbb{T})$. By Theorem 2.10, as q' satisfies $q' > 2/\alpha$, there exist a closed subset $E \subset \mathbb{T}$ such that $\dim(E) = \alpha$ and a non-zero positive measure $\mu \in A^{q'}(\mathbb{T}) \subset A^q_{-\beta}(\mathbb{T})$ such that $\operatorname{supp}(\mu) \subset E$. Now (3) follows from Lemma 2.4(2).

Now if $\alpha = \frac{2}{q}(1-\beta q)$ and $\gamma > 2/q$, then, by Theorem 2.11 with $\phi(t) = (t\ln(et))^{-1/2}$ for $t \geq 1$ and $h(t) = \frac{t^\alpha}{\ln(e/t)^\gamma}$ for $t \in [0,\infty)$, there exists a probability measure μ with support of Hausdorff h-measure zero such that

$$|\widehat{\mu}(n)| \leq \phi\Big(\frac{1}{h(|n|^{-1})}\Big)\Big(\ln\Big(\frac{1}{h(|n|^{-1})}\Big)\Big)^{1/2} \leq (|n|^\alpha \ln(e|n|)^\gamma)^{-1/2},$$

for $n \neq 0$. So

$$\begin{aligned}\sum_{n\neq 0} |\widehat{\mu}(n)|^q (1+|n|)^{-\beta q} &\leq C\sum_{n\neq 0} |n|^{-\alpha q/2-\beta q}\ln(e|n|)^{-\gamma q/2}\\ &\leq C\sum_{n\neq 0}\frac{1}{|n|\ln(e|n|)^{\gamma q/2}} < \infty,\end{aligned}$$

with C a positive constant. Hence, $\mu \in A^q_{-\beta}(\mathbb{T})$. We set $E = \mathrm{supp}(\mu)$. By Lemma 2.4 the result is proved.

(4) Let $k = [q/2]$. Suppose first $\frac{2}{q}(1-\beta q)k > 1-2(k+1)\beta$ and let $0 < \varepsilon' < \varepsilon$ satisfy $1-2(k+1)\beta \leq \frac{2}{q}(1-\beta q)k - \varepsilon'$. Consider the set S_λ, where λ satisfies

$$\frac{2}{q}(1-\beta q)k - \varepsilon' < \frac{1-\lambda}{1+\lambda} < \frac{2}{q}(1-\beta q)k.$$

By Lemma 3.2 (3), we have $\dim(S_\lambda) = (1-\lambda)/(1+\lambda)$ and $C_{(1-\lambda)/(1+\lambda)}(k\times S_\lambda) = 0$. Therefore, by Lemma 3.1 (a), every $f \in A^1_\beta(\mathbb{T})$ such that $\mathcal{Z}(f) = S_\lambda$ is cyclic in $A^p_\beta(\mathbb{T})$.

Now, suppose $\frac{2}{q}(1-\beta q)k \leq 1-2(k+1)\beta$. We consider S_λ, where

$$\frac{1-\lambda}{1+\lambda} = 1-2(k+1)\beta.$$

By Lemma 3.2 (3), we have

$$\dim(S_\lambda) = \frac{1-\lambda}{1+\lambda} = 1-2(k+1)\beta \quad \text{and} \quad C_{(1-\lambda)/(1+\lambda)}((k+1)\times S_\lambda) = 0.$$

Thus, by Lemma 3.1 (b), every $f \in A^1_\beta(\mathbb{T})$ such that $\mathcal{Z}(f) = S_\lambda$ is cyclic in $A^p_\beta(\mathbb{T})$.

Suppose now that $p = 2k/(2k-1)$ for some $k \in \mathbb{N}\setminus\{0\}$. As before, we consider S_λ, where

$$\frac{1-\lambda}{1+\lambda} = 1-2k\beta = 1-\beta q.$$

Again by Lemma 3.1 (b), every $f \in A^1_\beta(\mathbb{T})$ such that $\mathcal{Z}(f) = S_\lambda$ is cyclic in $A^p_\beta(\mathbb{T})$.

Note that the set E which was considered in Theorem 3.3 (4) satisfies $C_\alpha(E) = 0$, where

$$\alpha \geq \max\big(\tfrac{2}{q}(1-\beta q)k - \varepsilon,\ 1-2(k+1)\beta\big). \qquad \blacksquare$$

4. Proof of Theorem B

4.1. Some power sum

To prove Theorem B, we need the following lemmas.

Lemma 4.1. *Let R be a prime power and m a positive integer. We set $k = R^{m+1}$ and $N = (R-1)(R^{m+1}+1)$. Then there exist N^{th} roots of unity $z_1, \dots, z_k$ such that*

$$\Big|\sum_{j=1}^{k} z_j^r\Big| \le \sqrt{k}, \quad r = 1, \dots, N-1.$$

Proof. The proof is inspired from a result by Andersson, see Lemma 1 in [2]. Let $F = \{x_j,\ 1 \le j \le k\}$ be a finite field of order k, and let E be an extension field of F of order k^2. Let ω be an element that generates the multiplicative group E^*, and let χ be a multiplicative character on E of order $k^2 - 1$. We set

$$z_j = \chi^d(\omega + x_j), \quad 1 \le j \le k,$$

where $d = \sum_{j=0}^{m} R^j$. Since $Nd = k^2 - 1$, the z_j are N^{th} roots of unity. For $1 \le r \le N-1$, the characters χ^{rd} are non-trivial on E, thus by Theorem 1 in [10] we get

$$\Big|\sum_{j=1}^{k} z_j^r\Big| = \Big|\sum_{j=1}^{k} \chi^{rd}(\omega + x_j)\Big| \le \sqrt{k}, \quad r = 1, \dots, N-1. \qquad \blacksquare$$

Lemma 4.2. *With the notation of Lemma 4.1, we set*

$$c_n = \frac{\left(\sum_{j=1}^{k} z_j^n\right)}{k} \left(\frac{\sin(\pi n/N)}{\pi n/N}\right)^2, \quad n \in \mathbb{Z}.$$

Then

$$\sum_{n \in \mathbb{Z}\setminus\{0\}} |c_n|^p (1 + |n|)^{\beta p} \le \frac{N^{1+\beta p}}{k^{p/2}}.$$

Proof. We have

$$c_{r+N\ell} = \frac{\left(\sum_{j=1}^{k} z_j^r\right)}{k} \left(\frac{\sin(\pi r/N)}{\pi r/N + \pi \ell}\right)^2$$

and

$$\begin{aligned}\sum_{n \in \mathbb{Z}\setminus\{0\}} |c_n|^p (1 + |n|)^{\beta p} &= \sum_{r=1}^{N-1} \sum_{\ell \in \mathbb{Z}} |c_{r+N\ell}|^p (1 + |N\ell + r|)^{\beta p} \\ &= \sum_{r=1}^{N-1} \frac{\left|\sum_{j=1}^{k} z_j^r\right|^p}{k^p} \sum_{\ell \in \mathbb{Z}} \frac{|\sin(\pi r/N)|^{2p}}{\pi^{2p}|r/N + \ell|^{2p}} (1 + |N\ell + r|)^{\beta p}.\end{aligned}$$

To estimate
$$\frac{|\sin(\pi r/N)|^{2p}}{\pi^{2p}|r/N+\ell|^{2p}}\,(1+|N\ell+r|)^{\beta p},$$
we will consider two cases.

Case 1. $N\geq 2r$. In this case,
$$\frac{\sin^2\big((r/N)\pi)\big)}{\pi^2(r/N+\ell)^2}\,(1+|r+\ell N|)^{\beta}\leq\frac{(r/N)^{2-\beta}}{(|\ell|-1/2)^2}\,(r/N+r^2/N+|\ell|r)^{\beta}$$
$$\leq\frac{1}{2^{2-\beta}}\,\frac{1}{(|\ell|-1/2)^2}\,(1/2+r/2+|\ell|r)^{\beta}\leq\frac{1}{2^{2-\beta}}\,\frac{(|\ell|+1/2)^{\beta}}{(|\ell|-1/2)^2}\,(1+r)^{\beta}. \tag{4.1}$$
Therefore,
$$\sum_{\ell\in\mathbb{Z}}\frac{|\sin(\pi r/N)|^{2p}}{\pi^{2p}|r/N+\ell|^{2p}}\,(1+|N\ell+r|)^{\beta p}\leq\frac{1}{2^{(2-\beta)p}}\sum_{\ell\in\mathbb{Z}}\frac{(|\ell|+1/2)^{\beta p}}{(|\ell|-1/2)^{2p}}\,(1+r)^{\beta p}. \tag{4.2}$$

Case 2. $N\leq 2r$. For $|r+\ell N|<N$,
$$\frac{\sin^2((r/N)\pi))}{\pi^2(r/N+\ell)^2}\,(1+|r+\ell N|)^{\beta}\leq(1+N)^{\beta}\leq 2^{\beta}(1+r)^{\beta}.$$
We remark that there are at most two integers of the form $r+\ell N$ with $|r+\ell N|<N$. Thus,
$$\sum_{\ell\,:\,|r+\ell N|<N}\frac{|\sin(\pi r/N)|^{2p}}{\pi^{2p}|r/N+\ell|^{2p}}\,(1+|N\ell+r|)^{\beta p}\leq 2^{1+\beta p}(1+r)^{\beta p}. \tag{4.3}$$

Assume now that $|r+\ell N|\geq N$ and $\ell\in\mathbb{Z}$. We note that in this case $|r/N+\ell|=|r+\ell N|/N\geq 1$. We have
$$\frac{\sin^2(r\pi/N)}{\pi^2(r/N+\ell)^2}\,(1+|r+\ell N|)^{\beta}\leq\frac{1}{(r/N+\ell)^2\,\pi^2}\,N^{\beta}\,(1/N+|r/N+\ell|)^{\beta}$$
$$\leq\frac{2^{2\beta}}{\pi^2\,|r/N+\ell|^{2-\beta}}\,r^{\beta}.$$
Then we get
$$\sum_{\ell\in\mathbb{Z}:\,|r+\ell N|\geq N}\frac{|\sin(\pi r/N)|^{2p}}{\pi^{2p}|r/N+\ell|^{2p}}\,(1+|N\ell+r|)^{\beta p} \tag{4.4}$$
$$\leq\sum_{\ell\in\mathbb{Z}:\,|r/N+\ell|\geq 1}\frac{2^{2\beta p}}{\pi^{2p}|r/N+\ell|^{(2-\beta)p}}\,r^{\beta p}\leq\frac{2^{2\beta p+1}}{\pi^{2p}}\sum_{\ell\geq 1}\frac{1}{\ell^{(2-\beta)p}}\,r^{\beta p}.$$
Combining (4.3) and (4.4), we obtain
$$\sum_{\ell\in\mathbb{Z}}\frac{|\sin(\pi r/N)|^{2p}}{\pi^{2p}\,|r/N+\ell|^{2p}}(1+|N\ell+r|)^{\beta p}\leq\underbrace{\max\Big(2^{1+\beta p},\,\frac{2^{2\beta p+1}}{\pi^{2p}}\sum_{k\geq 1}\frac{1}{k^{(2-\beta)p}}\Big)}_{c_{\beta,p}}(1+r)^{\beta p}.$$

Therefore,

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} |c_n|^p (1+|n|)^{\beta p} = \sum_{r=1}^{N-1} \frac{|\sum_{j=1}^k z_j^r|^p}{k^p} \sum_{\ell\in\mathbb{Z}} \frac{|\sin(\pi r/N)|^{2p}}{\pi^{2p}|r/N+\ell|^{2p}} (1+|N\ell+r|)^{\beta p}$$
$$\le c_{\beta,p} \sum_{r=1}^{N-1} \frac{|\sum_{j=1}^k z_j^r|^p}{k^p} (1+r)^{\beta p} \le c_{\beta,p} \frac{N^{1+\beta p}}{k^{p/2}}. \qquad \blacksquare$$

4.2. Proof of Theorem B

To prove (1), we suppose that $p \ge 2$ and $\beta > 1/2 - 1/p$. It suffices to check that the characteristic function of E, χ_E, is in $\ell^q_{-\beta}$, the dual space of ℓ^p_β. By Hölder's inequality,

$$\sum_{n\in\mathbb{Z}} |\widehat{\chi_E}(n)|^q (1+|n|)^{-\beta q} \le \Big(\sum_{n\in\mathbb{Z}} |\widehat{\chi_E}(n)|^2\Big)^{q/2} \Big(\sum_{n\in\mathbb{Z}} (1+|n|)^{-\frac{2\beta q}{2-q}}\Big)^{\frac{2-q}{q}}.$$

The sums $\sum_{n\in\mathbb{Z}} |\widehat{\chi_E}(n)|^2$ and $\sum_{n\in\mathbb{Z}}(1+|n|)^{-\frac{2\beta q}{2-q}}$ converge since χ_E is in $L^2(\mathbb{T})$ and $\beta > 1/q - 1/2$.

In order to prove (2), using notations from Lemmas 4.1 and 4.2, we first define

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}.$$

The function f is the sum of k triangles each with base $4\pi/N$ and height N/k, then f is a Lipschitz function and its support has measure $4\pi k/N$.

We recall that $k = R^{m+1}$ and $N = (R-1)(R^{m+1}+1)$. Let $0 \le \beta < 1/2 - 1/p$ and choose m such that $\beta < \frac{1}{2}\frac{m+1}{m+2} - \frac{1}{p}$. We have

$$\frac{N^{1+\beta p}}{k^{p/2}} \sim \frac{R^{(m+2)(1+\beta p)}}{R^{\frac{m+1}{2}p}} \sim R^{p(m+2)(\beta+\frac{1}{p}-\frac{1}{2}\frac{m+1}{m+2})}.$$

Since

$$p(m+2)\Big(\beta + \frac{1}{p} - \frac{1}{2}\frac{m+1}{m+2}\Big) < 0,$$

we have that

$$\frac{N^{1+\beta p}}{k^{p/2}} \to 0, \quad \text{as } R \to \infty.$$

By Lemma 4.1, we have

$$\|f-1\|^p_{A^p_\beta(\mathbb{T})} \le \frac{N^{1+\beta p}}{k^{p/2}}.$$

On the other hand, $k/N \to 0$ as $R \to \infty$, and hence, for every $\varepsilon > 0$, there exists a Lipschitz function f such that $\|f-1\|_{A^p_\beta(\mathbb{T})} < \varepsilon$ and with support of measure less than ε. Finally, Lemma 2.8 concludes the proof.

5. Remarks

We say that $(\omega_n) \in \mathbb{R}$ is a *weight* if there exists a constant $C > 0$ such that $w_n \geq 1$ and $\omega_{n+k} \leq C\omega_n \omega_k$ for all $k, n \in \mathbb{Z}$. For a weight ω and $1 \leq p < \infty$, we set

$$A^p_\omega(\mathbb{T}) = \Big\{ f \in \mathcal{C}(\mathbb{T}) : \|f\|^p_{A^p_\omega(\mathbb{T})} = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^p \omega_n^p < \infty \Big\}.$$

Note that

$$\|fS\|_{A^p_\omega(\mathbb{T})} \leq \|f\|_{A^1_\omega(\mathbb{T})} \|S\|_{A^p_\omega(\mathbb{T})} \quad \text{for } f \in A^1_\omega(\mathbb{T}) \text{ and } S \in A^p_\omega(\mathbb{T}).$$

Hence we have the same result as (2.1) to characterize cyclicity in $A^p_\omega(\mathbb{T})$ by norm.

When $\omega_n = O((1+|n|)^\varepsilon)$ for all $\varepsilon > 0$, for instance, by letting $\omega_n = \ln(e+|n|)^\gamma$, where $\gamma \geq 0$, we can show the same result as Lemma 2.7. By noting that

$$A^p_\beta(\mathbb{T}) \subset A^p_\omega(\mathbb{T}) \subset A^p(\mathbb{T})$$

for all $p \geq 1$ and $\beta > 0$, we obtain the following result by Theorem A and Theorem B:

Corollary 5.1. *Let $\omega = (\omega_n)_{n\in\mathbb{Z}}$ be a weight such that* $\lim_{n\to+\infty} \frac{\log \omega_n}{\log n} = 0$.

(1) *Let $1 < p < 2$.*
   (a) *If $f \in A^1_\omega(\mathbb{T})$ and $\dim(\mathcal{Z}(f)) < 2/q$, then f is cyclic in $A^p_\omega(\mathbb{T})$.*
   (b) *For $2/q < \alpha \leq 1$, there exists a closed subset $E \subset \mathbb{T}$ such that $\dim(E) = \alpha$ and every $f \in A^1_\omega(\mathbb{T})$ satisfying $\mathcal{Z}(f) = E$ is not cyclic in $A^p_\omega(\mathbb{T})$.*
   (c) *For all $0 < \varepsilon < 1$, there exists a closed subset $E \subset \mathbb{T}$ such that $\dim(E) = 1-\varepsilon$ and every $f \in A^1_\omega(\mathbb{T})$ satisfying $\mathcal{Z}(f) = E$ is cyclic in $A^p_\omega(\mathbb{T})$.*

(2) *Let $p > 2$. For every $\varepsilon > 0$, there exists a closed subset $E \subset \mathbb{T}$ such that $|E| > 2\pi - \varepsilon$ and every $u \in A^1_\beta(\mathbb{Z})$ satisfying $\mathcal{Z}(\widehat{u}) = E$ is cyclic in $A^p_\beta(\mathbb{Z})$.*

Proof. (1) Suppose that $1 < p < 2$.

(a) Let $f \in A^1_\omega(\mathbb{T})$ be such that $\dim(\mathcal{Z}(f)) < 2/q$. Then there exists $0 < \beta < 1/2$ such that $\dim(\mathcal{Z}(f)) < \frac{2}{q}(1-\beta q)$. By Theorem 3.3(1), every $g \in A^1_\beta(\mathbb{T})$ satisfying $\mathcal{Z}(g) = \mathcal{Z}(f)$ is cyclic in $A^p_\beta(\mathbb{T})$. Therefore, by Lemma 2.7, there exists a sequence of Lipschitz functions (f_n) which are zero on $\mathcal{Z}(f)$ such that

$$\lim_{n\to\infty} \|f_n - 1\|_{A^p_\beta(\mathbb{T})} = 0.$$

Moreover, $\omega_n = O((1+|n|)^\beta)$, therefore,

$$\lim_{n\to\infty} \|f_n - 1\|_{A^p_\omega(\mathbb{T})} = 0.$$

Again by Lemma 2.7 in $A^p_\omega(\mathbb{T})$, we obtain that f is cyclic in $A^p_\omega(\mathbb{T})$.

(b) By Theorem 2.10, there exists a closed set $E \subset \mathbb{T}$ such that $\dim(E) = \alpha$ and every $f \in A^1(\mathbb{T})$ satisfying $\mathcal{Z}(f) = E$ is not cyclic in $A^p(\mathbb{T})$. Let $f \in A^1_\omega(\mathbb{T})$ be such that $\mathcal{Z}(f) = E$. Since $f \in A^1(\mathbb{T})$, f is not cyclic in $A^p(\mathbb{T})$. However, $\|\cdot\|_{A^p(\mathbb{T})} \le \|\cdot\|_{A^p_\omega(\mathbb{T})}$, therefore f is not cyclic in $A^p_\omega(\mathbb{T})$.

(c) Let $0 < \varepsilon < 1$ and $\beta > 0$ be such that $1 - 2([q/2] + 1)\beta \ge 1 - \varepsilon$. By Theorem 3.3 (4), there exists a closed set $E \subset \mathbb{T}$ such that

$$\dim(E) \ge 1 - 2([q/2] + 1)\beta \ge 1 - \varepsilon,$$

and every $f \in A^1_\beta(\mathbb{T})$ satisfying $\mathcal{Z}(f) = E$ is cyclic in $A^p_\beta(\mathbb{T})$. Since $A^p_\beta(\mathbb{T}) \subset A^p_\omega(\mathbb{T})$, by Lemma 2.7, we get our result.

(2) If $p > 2$, then the result immediately follows from Theorem B. ■

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