The Calabi–Yau problem for minimal surfaces with Cantor ends

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Abstract. We show that every connected compact or bordered Riemann surface contains a Cantor set whose complement admits a complete conformal minimal immersion in $\mathbb{R}^3$ with bounded image. The analogous result holds for holomorphic immersions into any complex manifold of dimension at least 2, for holomorphic null immersions into $\mathbb{C}^n$ with $n \geq 3$, for holomorphic Legendrian immersions into an arbitrary complex contact manifold, and for superminimal immersions into any self-dual or anti-self-dual Einstein four-manifold.

1. Introduction

Let M be an open Riemann surface. It is classical [9, 20] that an immersion $M \to \mathbb{R}^n$ for $n \geq 3$ which is conformal (angle preserving) and harmonic parameterizes a minimal surface in $\mathbb{R}^n$; conversely, every immersed minimal surface in $\mathbb{R}^n$ arises in this way.

Let ds^2 denote the Euclidean metric on $\mathbb{R}^n$. An immersion $f \colon M \to \mathbb{R}^n$ is said to be *complete* if the Riemannian metric $f^* ds^2$ on M induces a complete distance function; equivalently, if the image of any divergent path in M by the map f is a path in $\mathbb{R}^n$ with infinite Euclidean length.

The *Calabi–Yau problem for minimal surfaces* (see [18], p. 170, and [21], p. 360) asks about the existence, conformal and asymptotic properties of complete immersed or embedded minimal surfaces with bounded images in $\mathbb{R}^n$ for $n \geq 3$. Pioneering constructions were given by Jorge and Xavier [17] in 1980 and Nadirashvili [19] in 1996. There were substantial developments since then, and a survey can be found in [9], Chapter 7.

In this paper we construct the first known examples of complete bounded minimal surfaces whose end is a *Cantor set*, that is, a compact, perfect, totally disconnected set. The following is a special case of our main result, Theorem 3.1. See also Theorem 4.1 for a generalization to a number of other geometries.

Theorem 1.1. *In every compact connected Riemann surface M, there is a Cantor set C whose complement admits a complete conformal minimal immersion $M \setminus C \to \mathbb{R}^3$ with bounded image. There also exist a Cantor set C in M and a complete conformal minimal embedding $M \setminus C \hookrightarrow \mathbb{R}^5$ with bounded image.*

Cantor sets, being of fractal nature, often serve as a challenging test case in geometric problems. Theorem 1.1 gives an affirmative answer to Problem 7.4.8 (B) in [9]. Note that the problem was posed incorrectly: whether there is a Cantor set $C \subset \mathbb{C}$ such that $\mathbb{C} \setminus C$ satisfies the conclusion of the theorem. This is impossible since a bounded harmonic map $\mathbb{C} \setminus C \to \mathbb{R}^n$ extends harmonically across the puncture at infinity. The correct question is answered affirmatively by Theorem 1.1 with $M = \mathbb{CP}^1$.

Remark 1.2. Theorem 1.1 also holds if M is a bordered Riemann surface of the form

$$(1.1) \qquad\qquad M = R \setminus \bigcup_i D_i,$$

where R is a compact connected Riemann surface and $\{D_i\}_i$ is a finite or countable collection of pairwise disjoint, smoothly bounded closed discs, diffeomorphic images of the unit disc $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \le 1\}$. (The discs D_i may cluster on one another, but M must be an open domain in R.) It was shown by Alarcón and the author [5] in 2021 that such M admits a bounded complete conformal minimal immersion in $\mathbb{R}^3$ and embedding in $\mathbb{R}^5$ extending continuously to $\overline{M}$ such that the image of the boundary $bM = \bigcup_i bD_i$ is a union of pairwise disjoint Jordan curves, the images of bD_i. (For finitely many discs D_i, this was proved beforehand in [1].) Together with our proof of Theorem 1.1 (see Section 3), this gives a Cantor set $C \subset M$ and a bounded complete conformal minimal immersion $M \setminus C \to \mathbb{R}^3$ (and an embedding into $\mathbb{R}^5$) which extends continuously to $bM = \bigcup_i bD_i$ and maps the curves bD_i to pairwise disjoint Jordan curves. (However, our proof does not give a continuous extension of the map to the Cantor set C.) The point is that we can simultaneously increase the intrinsic boundary distances at bM and at C. We also provide a precise control of the location of the image surface in $\mathbb{R}^n$; see Theorem 3.1.

Cantor sets which arise in the proof Theorem 1.1 are small modifications of the standard Cantor sets in the plane, and they have almost full measure in a surrounding rectangle. One may ask which Cantor sets in compact Riemann surfaces satisfy the conclusion of Theorem 1.1. In my opinion, this question is likely very difficult or even impossible to answer. Instead, I propose the following more reasonable problem.

Problem 1.3. Is there a Cantor set C in $\mathbb{CP}^1$ of zero area such that $\mathbb{CP}^1 \setminus C$ satisfies the conclusion of Theorem 1.1?

The first test case is to find a Cantor set with zero area whose complement $\mathbb{CP}^1 \setminus C$ admits a nonconstant bounded harmonic function.

2. Tools used in the proof

In this section we recall the prerequisites and tools which will be used in the proof. They are described in detail in the monograph [9], and we provide precise references.

The following quadric complex hypersurface in $\mathbb{C}^n$ for $n \ge 3$ is called the *null quadric*:

$$(2.1) \qquad \mathbf{A} = \{z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\}.$$

Let M be an open Riemann surface. Fix a nowhere vanishing holomorphic 1-form θ on M; such exists by the Oka–Grauert principle (see Theorem 5.3.1 in [14]). An immer-

sion $f = (f_1, \ldots, f_n): M \to \mathbb{R}^n$ is conformal and minimal (equivalently, conformal and harmonic) if and only if the $(1, 0)$-differential $\partial f = (\partial f_1, \ldots, \partial f_n)$ (the $\mathbb{C}$-linear part of the differential df) is holomorphic and satisfies the nullity condition $\sum_{i=1}^n (\partial f_i)^2 = 0$; equivalently, if and only if the map

$$h = 2 \, \partial f / \theta : M \to \mathbb{C}^n \setminus \{0\}$$

is holomorphic and assume values in the punctured null quadric $\mathbf{A}_* = \mathbf{A} \setminus \{0\}$.

The most important point in the development of the theory of minimal surfaces in Euclidean spaces, as presented in the monograph [9], is the fact that $\mathbf{A}_*$ is a complex homogeneous manifold for the complex orthogonal group $O_n(\mathbb{C})$, hence an Oka manifold; see Chapter 5 of [14] for the latter. Thus, there is an abundance of holomorphic maps $h: M \to \mathbf{A}_*$ from any open Riemann surface and, more generally, from any Stein manifold. Together with tools from convex integration theory and the method of dominating sprays, one can control the periods of the 1-form $h\theta$ over closed curves in a given open Riemann surface M. This yields many holomorphic maps $h: M \to \mathbf{A}_*$ which integrate to conformal minimal immersions $f: M \to \mathbb{R}^n$ by the Enneper–Weierstrass formula

$$f(p) = f(p_0) + \int_{p_0}^p 2 \Re(h\theta) \quad \text{for } p \in M,$$

where $p_0 \in M$ is a fixed reference point. Note that the integral is well-defined if and only if the 1-form $\Re(h\theta)$ has vanishing periods. See Theorem 2.3.4 in [9] for further details.

Similarly, a holomorphic immersion $f: M \to \mathbb{C}^n$ for $n \geq 3$ is a *holomorphic null curve* if and only if $df = h\theta$, where $h: M \to \mathbf{A}_*$ is a holomorphic map. If M is simply connected, then every conformal minimal immersion $M \to \mathbb{R}^n$ is the real part of a holomorphic null curve $M \to \mathbb{C}^n$, and vice versa. We can recover f from $h\theta$ by the formula

$$f(p) = f(p_0) + \int_{p_0}^p h\theta \quad \text{for } p \in M,$$

subject to the condition that the holomorphic 1-form $h\theta$ has vanishing periods.

Definition 2.1 (Definition 1.12.9 in [9]). Let M be a smooth surface. An *admissible set* in M is a compact set of the form $S = K \cup E$, where $K \subsetneq M$ is a finite union of pairwise disjoint compact domains in M with piecewise $\mathscr{C}^1$ boundaries and $E = \overline{S \setminus K}$ is a union of finitely many pairwise disjoint smooth Jordan arcs and closed Jordan curves meeting K only at their endpoints (if at all) and such that their intersections with bK are transverse.

We recall the notion of a generalized conformal minimal immersion and generalized holomorphic null curve. Denote by $\mathscr{A}^r(S, \mathbb{C}^n)$ the space of maps $S \to \mathbb{C}^n$ of class $\mathscr{C}^r$ which are holomorphic in the interior of an admissible set S in a Riemann surface.

Definition 2.2 (Definition 3.1.2 in [9]). Let $S = K \cup E$ be an admissible set in a Riemann surface M (see Definition 2.1), and let θ be a nowhere vanishing holomorphic 1-form on a neighbourhood of S in M. A *generalized conformal minimal immersion* $S \to \mathbb{R}^n$ ($n \geq 3$) of class $\mathscr{C}^r$ ($r \in \mathbb{N}$) is a pair $(f, h\theta)$, where $f: S \to \mathbb{R}^n$ is a $\mathscr{C}^r$ map whose restriction to the interior $\mathring{S} = \mathring{K}$ is a conformal minimal immersion, and the map $h \in \mathscr{A}^{r-1}(S, \mathbf{A}_*)$ satisfies the following two conditions:

(a) $h\theta = 2\partial f$ holds on K, and

(b)   for every smooth path α in M parameterizing a connected component of $E = \overline{S \setminus K}$
we have that $\Re(\alpha^*(h\theta)) = \alpha^*(df) = d(f \circ \alpha)$.

Note that the complex 1-form $h\theta$ in the above definition determines the 1-jet along S of a conformal harmonic extension of f. With an abuse of language, we shall sometimes call the map f itself a generalized conformal minimal immersion. The following is an analogue of this notion for holomorphic null curves.

Definition 2.3 (Definition 3.1.3 in [9]). Let $S = K \cup E$ and θ be as in Definition 2.2. A *generalized holomorphic null curve* $f: S \to \mathbb{C}^n$ ($n \geq 3$) of class $\mathscr{C}^r$ ($r \in \mathbb{N}$) is a pair $(f, h\theta)$ where $f \in \mathscr{A}^r(S, \mathbb{C}^n)$, $h \in \mathscr{A}^{r-1}(S, \mathbf{A}_*)$, and the following conditions hold:

(a) $h\theta = df = \partial f$ holds on K (hence $f: \overset{\circ}{K} \to \mathbb{C}^n$ is a holomorphic null curve), and

(b)   for any smooth path α in M parameterizing a connected component of E we have that $\alpha^*(h\theta) = \alpha^*(df) = d(f \circ \alpha)$.

The next result follows immediately from Lemma 3.5.4 in [9] and the proof of Theorem 3.6.1 in [9]; see equations (3.38) and (3.39) in [9]. The same argument applies to null curves in part (b). The last part of the proposition is very important in our construction.

Proposition 2.4. *Let $S = K \cup E$ be an admissible set in a Riemann surface M, and let θ be a nowhere vanishing holomorphic 1-form on a neighbourhood of S. Then, the following assertions hold for every pair of integers $n \geq 3$ and $r \geq 1$.*

(a)   *Every conformal minimal immersion $f: K \to \mathbb{R}^n$ of class $\mathscr{C}^r$ extends to a generalized conformal minimal immersion $(f, h\theta)$ of class $\mathscr{C}^r$ on S.*

(b)   *Every $\mathscr{C}^r$ map $f: K \to \mathbb{C}^n$ such that $f: \overset{\circ}{K} \to \mathbb{C}^n$ is a holomorphic null curve extends to a generalized holomorphic null curve $(f, h\theta)$ of class $\mathscr{C}^r$ on S.*

If $f(K)$ is contained in a connected open set Ω in $\mathbb{R}^n$ or $\mathbb{C}^n$, respectively, then the extension can be chosen such that $f(S) \subset \Omega$. More precisely, if E is an arc in M with the endpoints $E \cap K = \{p, q\} \in bK$ and L is an arc in $\mathbb{R}^n$ or $\mathbb{C}^n$ connecting the points $f(p)$ and $f(q)$, then the extension of f from K to $K \cup E$ can be chosen such that $f(E)$ is contained in any given neighbourhood of L.

Recall that a compact set S in an open Riemann surface M is said to be *Runge* in M if every holomorphic function on a neighbourhood of S can be approximated uniformly on S by holomorphic functions on M. By Runge's theorem for Riemann surfaces, this holds if and only if $M \setminus K$ has no relatively compact connected components, [12], Theorem 4. The following is a simplified version of the Mergelyan approximation theorem for conformal minimal surfaces and holomorphic null curves, given by Theorems 3.6.1 and 3.6.2 in [9].

Theorem 2.5. *Assume that M is an open Riemann surface, S is an admissible Runge set in M, $n \geq 3$ and $r \geq 1$ are integers, and $f: S \to \mathbb{R}^n$ is a generalized conformal minimal immersion of class $\mathscr{C}^r(S)$. Given $\varepsilon > 0$, there is a conformal minimal immersion $\tilde{f}: M \to \mathbb{R}^n$ satisfying $\|\tilde{f} - f\|_{\mathscr{C}^r(S)} < \varepsilon$. If $n \geq 5$, then $\tilde{f}$ can be chosen to be an injective immersion, and if $n = 4$, then $\tilde{f}$ can be chosen to be an immersion with simple double points.*

Likewise, a generalized holomorphic null curve $f : S \to \mathbb{C}^n$ of class $\mathscr{C}^r(S)$ with $r \geq 1$ can be approximated in $\mathscr{C}^r(S)$ by holomorphic null embeddings $M \hookrightarrow \mathbb{C}^n$.

Remark 2.6. Since an admissible set S is Runge in an open neighbourhood of itself in the ambient Riemann surface, Theorem 2.5 gives approximation of a generalized conformal minimal immersion on S by a conformal minimal immersion on a neighbourhood of S.

Given a compact, connected, smooth bordered surface M, an immersion $f : M \to \mathbb{R}^n$, and a point $p \in \mathring{M}$, we denote by $\mathrm{dist}_f(p, bM)$ the infimum of the lengths of piecewise $\mathscr{C}^1$ paths in M connecting p to bM in the Riemannian metric $f^* ds^2$. This number is called the *intrinsic radius* of M with respect to f.

Our last main tool is the following lemma, which says that the intrinsic radius a compact bordered Riemann surface can be arbitrarily large with respect to a conformal minimal immersion (or a holomorphic null curve) which is arbitrarily uniformly close to a given one. This lemma is at the core of the new construction methods in the Calabi–Yau problem for minimal surfaces, presented in Chapter 7 of [9]. The main ingredient in its proof is the Riemann–Hilbert problem for conformal minimal surfaces, which was first introduced in this subject in the paper [3] and was developed further in [1, 8] and in Chapter 6 of [9].

Lemma 2.7. *Assume that M is a compact bordered Riemann surface with piecewise smooth boundary and $f : M \to \mathbb{R}^n$ for $n \geq 3$ is a conformal minimal immersion of class $\mathscr{C}^1(M)$. Given a point $p_0 \in \mathring{M}$ and numbers $\varepsilon > 0$ (small) and $\tau > 0$ (big), there is a conformal minimal immersion $\tilde{f} : M \to \mathbb{R}^n$ of class $\mathscr{C}^1(M)$ such that*

$$|\tilde{f}(p) - f(p)| < \varepsilon \ \text{for all} \ p \in M \quad \text{and} \quad \mathrm{dist}_{\tilde{f}}(p_0, bM) > \tau.$$

The analogous result holds for holomorphic null immersions $M \to \mathbb{C}^n$.

Remark 2.8. Lemma 2.7 is a simplified version of Lemma 4.1 in [1]; a more precise version with interpolation is Lemma 7.3.1 in [9]. Although the boundary bM is assumed to be smooth in both results, piecewise smooth boundary of class $\mathscr{C}^{k,\alpha}$ for some $k \in \mathbb{N}$ and $0 < \alpha < 1$ suffices for the arguments (see Remark 7.4.2 in [9]). The intrinsic boundary distance at the finitely many corner points of bM can be enlarged by the method of exposing points (see Theorem 6.7.1 in [9]), which is an integral part of the proof of Lemma 7.3.1 in [9].

3. Proof of Theorem 1.1

We begin by recalling the standard construction of a Cantor set in the plane $\mathbb{C} = \mathbb{R}^2$.

Let $P = P_0 \subset \mathbb{C}$ be a closed rectangle. In the first step we remove from P an open horizontal strip of positive width around the straight line segment dividing P in two halves, top and bottom. This gives two smaller disjoint rectangles Q_1 and Q_2. Next, we remove from each of them an open vertical strip around the straight line segments dividing Q_1 and Q_2, respectively, in two halves, left and right. This gives four pairwise disjoint closed rectangles P_1^j ($j = 1, \ldots, 4$) of the same size, and we set $P_1 = \bigcup_{j=1}^4 P_1^j$. Thus, the passage from $P = P_0$ to P_1 amounts to removing a central cross from P.

We now repeat the same procedure for each of the rectangles P_1^j, removing a central cross in order to obtain four smaller pairwise disjoint rectangles. This gives rectangles P_2^j for $j = 1, 2, \ldots, 16$ of the second generation, and we set $P_2 = \bigcup_{j=1}^{16} P_2^j$. Continuing inductively, we find a decreasing sequence of compacts

$$(3.1) \qquad P = P_0 \supset P_1 \supset P_2 \supset \cdots \supset \bigcap_{i=0}^{\infty} P_i = C$$

whose intersection C is a Cantor set. The set P_i is the union of 4^i pairwise disjoint closed rectangles, obtained by removing a central cross from each of the rectangles in P_{i-1}.

In our proof of Theorem 1.1, the crosses removed at every step must be chosen fairly narrow. Furthermore, after removing the central cross from a given rectangle, we will slightly shrink each of the new rectangles towards its centre in order to ensure that the sequence of compacts in (3.1) is such that P_{i+1} is contained in the interior of P_i for every $i = 0, 1, 2, \ldots$. By choosing the width of the crosses and the amount of shrinking small enough at every step, we obtain a Cantor set C whose area is arbitrarily close to the area of the initial rectangle P. The first generation of this process is shown on Figure 1.
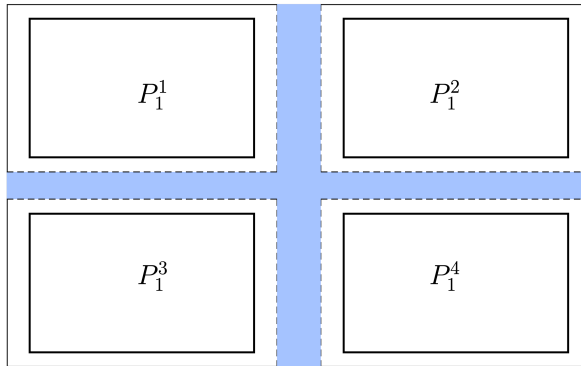
Figure 1. A central cross removed from a rectangle P.

Theorem 1.1 is a special case of the following result, which we shall now prove.

Theorem 3.1. *Assume that M is a compact connected Riemann surface, P is a compact rectangle in a holomorphic coordinate chart on M, and $f \colon M \setminus \mathring{P} \to \mathbb{R}^n$ for $n \geq 3$ is a conformal minimal immersion of class $\mathscr{C}^1(M \setminus \mathring{P})$. Given an open set $\Omega \subset \mathbb{R}^n$ containing $f(M \setminus \mathring{P})$, there are a Cantor set $C \subset P$ and a complete conformal minimal immersion $\tilde{f} \colon M \setminus C \to \Omega$ (embedding if $n \geq 5$) approximating f as closely as desired in $\mathscr{C}^1(M \setminus \mathring{P})$.*

The analogous conclusion holds if $M = R \setminus \bigcup_i D_i$ is an open Riemann surface of the form (1.1), $P \subset M$ is a rectangle as above, and $f \colon R \setminus (\cup_{j=1}^{m} \mathring{D}_j \cup \mathring{P}) \to \mathbb{R}^n$ for $n \geq 3$ and $m \in \mathbb{N}$ is a conformal minimal immersion of class $\mathscr{C}^1$ taking values in an open set

$\Omega \subset \mathbb{R}^n$. *In this case, there exist a Cantor set $C \subset P$ and a continuous map $\tilde{f} \colon \overline{M} \setminus C \to \Omega$ such that $\tilde{f} \colon M \setminus C \to \Omega \subset \mathbb{R}^n$ is a complete conformal minimal immersion (embedding if $n \geq 5$) which approximates f as closely as desired uniformly on $\overline{M} \setminus \overset{\circ}{P}$ and such that $\tilde{f}(bM)$ is a union of pairwise disjoint Jordan curves $\tilde{f}(bD_i)$.*

We emphasize that the Cantor set in our construction cannot be specified in advance.

Proof. For simplicity of exposition, we shall assume that M is a compact connected Riemann surface without boundary. The case when M is a bordered Riemann surface of the form (1.1) follows by combining the procedure explained below with the one in [5].

Write $P = P_0$. Note that $M_0 := M \setminus \overset{\circ}{P_0}$ is a compact domain in M with piecewise smooth boundary $bM_0 = bP_0$. Fix a base point $p_0 \in \overset{\circ}{M_0}$, which will be used to measure the intrinsic boundary distances. Since $f(M \setminus \overset{\circ}{P_0})$ is connected, we may assume that the neighbourhood Ω of $f(M \setminus \overset{\circ}{P_0})$ in the statement of the theorem is connected as well.

We now explain how to find the next compact domain $M_1 \subset M$ with piecewise smooth boundary satisfying $M_0 \subset \overset{\circ}{M_1}$, and a conformal minimal immersion $f_1 \colon M_1 \to \mathbb{R}^n$ which approximates $f_0 = f$ in $\mathscr{C}^1(M_0, \mathbb{R}^n)$ such that, for a given constant $c_1 > 0$, we have that

$$(3.2) \qquad f_1(M_1) \subset \Omega \quad \text{and} \quad \operatorname{dist}_{f_1}(p_0, bM_1) > c_1.$$

Let E denote the horizontal straight line segment dividing the rectangle P_0 in top and bottom halves. Then, $S = M_0 \cup E$ is an admissible set in M (see Definition 2.1).

By Proposition 2.4, we can extend f from M_0 across E to a generalized conformal minimal immersion $f \colon S \to \mathbb{R}^n$ of class $\mathscr{C}^1$ with $f(S) \subset \Omega$.

By the Mergelyan approximation theorem for conformal minimal immersions (see Theorem 2.5 and Remark 2.6), we can approximate f as closely as desired in $\mathscr{C}^1(S, \mathbb{R}^n)$ by a conformal minimal immersion $g \colon U \to \mathbb{R}^n$ on an open connected neighbourhood $U \subset M$ of S. By shrinking U around S if necessary, we may assume that $g(U) \subset \Omega$.

We now choose closed top and bottom rectangles $Q_1, Q_2 \subset \overset{\circ}{P_0} \setminus E$ as described above, first removing from P_0 a narrow horizontal strip around E and then shrinking each of the two rectangles by a small amount, so that $P_0 \setminus (\overset{\circ}{Q_1} \cup \overset{\circ}{Q_2}) \subset U$. Hence, the compact domain $M_1' = M \setminus (\overset{\circ}{Q_1} \cup \overset{\circ}{Q_2})$ in M with piecewise smooth boundary lies in the domain U of the map g and it contains $M_0 = M \setminus \overset{\circ}{P_0}$ in its interior.

Next, we repeat the same procedure with each of the two rectangles Q_1 and Q_2, splitting them in left and right parts by removing a narrow vertical band around their central segments E_1 and E_2, respectively. Using Proposition 2.4, we extend g across the arcs E_1 and E_2 to a generalized conformal minimal immersion on the admissible set $S' = M_1' \cup E_1 \cup E_2$, with range in Ω. By Theorem 2.5, we approximate g as closely as desired in $\mathscr{C}^1(S', \mathbb{R}^n)$ by a conformal minimal immersion $\tilde{g} \colon V \to \Omega \subset \mathbb{R}^n$ on a neighbourhood V of S'. Shrinking V around S', we may assume that $\tilde{g}(V) \subset \Omega$. Finally, pick closed rectangles P_1^j for $j = 1, \ldots, 4$ such that, setting

$$P_1 = \bigcup_{j=1}^{4} P_1^j \quad \text{and} \quad M_1 = M \setminus \overset{\circ}{P_1},$$

we have that $M_0 \subset \overset{\circ}{M_1}$ and M_1 is contained in V (the domain of $\tilde{g}$).

By Lemma 2.7, we can approximate $\tilde{g}$ as closely as desired in $\mathscr{C}^1(M_0, \mathbb{R}^n)$ by a conformal minimal immersion $f_1 \colon M_1 \to \Omega \subset \mathbb{R}^n$ satisfying condition (3.2).

This concludes the initial step. All subsequent steps are of the same kind. They follow the inductive construction of a Cantor set C, explained at the beginning of the section, the only difference being a small shrinking of rectangles in the next generation obtained by removing central crosses from the rectangles of the given generation. This shrinking was explained above when describing the initial step of the induction.

We now conclude the proof. Pick sequences $c_i > 0$ and $\varepsilon_i > 0$ such that $\lim_{i \to \infty} c_i = +\infty$ and $\lim_{i \to \infty} \varepsilon_i = 0$. By using the above procedure, we inductively construct

- a decreasing sequence of compact sets (3.1) such that P_{i+1} is contained in the interior of P_i for every $i = 0, 1, 2, \ldots$ and $C = \bigcap_{i=0}^{\infty} P_i$ is a Cantor set, and

- a sequence of conformal minimal immersions $f_i \colon M_i = M \setminus P_i \to \Omega \subset \mathbb{R}^n$,

such that the following conditions hold for every $i = 0, 1, 2, \ldots$:

$$(3.3) \qquad \text{(a)} \quad \| f_{i+1} - f_i \|_{\mathscr{C}^1(M_i)} < \varepsilon_i \quad \text{and} \quad \text{(b)} \quad \mathrm{dist}_{f_i}(p_0, bM_i) > c_i.$$

Note that $\overline{M}_i \subset M_{i+1}$ for every i and $\bigcup_{i=0}^{\infty} M_i = M \setminus C$. Assuming that the numbers ε_i converge to zero fast enough, condition (a) ensures that the sequence f_i converges to a conformal minimal immersion

$$\tilde{f} = \lim_{i \to \infty} f_i : M \setminus C \to \mathbb{R}^n$$

whose image is contained in the given neighbourhood Ω of $f(M \setminus P_0)$. Furthermore, if $n \geq 5$ then we may ensure that each f_i and their limit $\tilde{f}$ are injective. This is standard, see for example the proof of Theorem 3.6.1 in [9].

Conditions (a) and (b) in (3.3) together clearly imply that

$$\mathrm{dist}_{\tilde{f}}(p_0, bM_i) > c_i/2 \quad \text{for every } i \in \mathbb{N}$$

provided that the sequence ε_i goes to zero fast enough. Since the increasing sequence of domains M_i forms a normal exhaustion of $M \setminus C$, every divergent path in $M \setminus C$ emanating from p_0 must cross bM_i for every i, and hence it has infinite length with respect to the metric $\tilde{f}^* ds^2$ in $M \setminus C$. In other words, the immersion $\tilde{f}$ is complete. ∎

4. Generalization to other geometries

As mentioned in the introduction, the construction in the proof of Theorem 3.1 generalizes to several other geometries listed in the following theorem.

Theorem 4.1. *The analogue of Theorem* 3.1 *holds for the following classes of maps*:

(a) *Conformal harmonic immersions of nonorientable conformal surfaces into $\mathbb{R}^n$ for $n \geq 3$.*

(b) *Holomorphic immersions into an arbitrary complex manifold of dimension ≥ 2.*

(c) *Holomorphic null immersions into $\mathbb{C}^n$ with $n \geq 3$.*

(d) *Holomorphic Legendrian immersions into any complex contact manifold.*

(e) *Immersed oriented superminimal surfaces in any self-dual or anti-self-dual Einstein four-manifold.*

Remark 4.2. The statement of the theorem refers to completeness of immersed surfaces with respect to a Riemannian metric on the ambient manifold Y. Since the said surfaces are contained in a relatively compact subset of Y (indeed, in a small neighbourhood of the image of the original given compact surface), completeness does not depend on the choice of the metric on Y since any two metrics are comparable on a relatively compact domain.

Proof. The three crucial ingredients in the proof of Theorem 3.1 are the following:

(i)   existence of a suitable extension of an immersion in the given class from K to $S = K \cup E$, where S is an admissible set (cf. Proposition 2.4),

(ii)   the Mergelyan approximation theorem on admissible sets for immersions in the given class (see Theorem 2.5 for conformal minimal surfaces), and

(iii) the lemma on increasing the intrinsic radius (see Lemma 2.7).

These tools are available in all geometries listed in the theorem. Let us go case by case.

Case (a). *Nonorientable minimal surfaces.* The existence of an extension from K to $S = K \cup E$ is seen as in the orientable case, the Mergelyan approximation theorem is given by Theorem 4.4 in [8], and the analogue of Lemma 2.7 is given by Theorem 6.6 in [8].

Case (b). *Holomorphic immersions.* A holomorphic immersion $f \colon K \to Y$ clearly extends to a smooth immersion $f \colon S = K \cup E \to Y$ with range in a given neighbourhood of $f(K)$. The Mergelyan approximation theorem for manifold-valued maps on admissible sets holds by Corollary 9, p. 178, in [12] and Theorem 1.13.1 (b) in [9]. The lemma on increasing the intrinsic radius was shown in [2] for immersions into $\mathbb{C}^n$ with $n \geq 2$, which complete the proof for the case $Y = \mathbb{C}^n$ with $n \geq 2$. Combining these methods with the gluing techniques for holomorphic maps explained in [11] and [14], Chapter 5, implies the same result for immersions into any complex manifold of complex dimension at least two.

Case (c). *Holomorphic null immersions.* This case was developed in [1, 3] and is covered by Theorem 7.4.12 in [9]. The proof is almost the same as the proof of Theorem 3.1.

Case (d). *Holomorphic Legendrian immersions.* The case of immersions to Euclidean spaces $\mathbb{C}^{2n+1}$ with the standard contact form is developed in Section 6 of [7]. The generalization to Legendrian immersions into an arbitrary complex contact manifold follows by combining Theorem 1.3 in [4] with the Mergelyan approximation theorem in [13].

Case (e). *Superminimal surfaces in* (anti-) *self-dual Einstein four-manifolds.* This follows from case (d) by using the Bryant correspondence for Penrose twistor spaces over such manifolds; see [6, 15]. Given a self-dual or anti-self-dual Einstein four-manifold Y with nonzero scalar curvature, the total space X of the Penrose twistor bundle $\pi \colon X \to Y$ is a complex contact three-manifold, and the Bryant correspondence (see [10,16]) provides a bijective correspondence between holomorphic and antiholomorphic immersed Legendrian curves in X and oriented immersed superminimal surfaces in Y. In case that the

Einstein metric on Y has vanishing scalar curvature, the natural horizontal holomorphic distribution on X orthogonal to the fibres of $\pi\colon X \to Y$ is integrable (a holomorphic hypersurface foliation), so the result reduces to that in case (b); see [15] for the details. ∎

In this connection, we point out that Section 6 in the paper [6] describes an axiomatic approach to the Calabi–Yau property for a given class of immersions from compact manifolds with boundary. If the source manifolds are surfaces and we add to those axioms the content of Proposition 2.4 (i.e., the existence of a suitable extension of an immersion in the given class from K to $S = K \cup E$, where S is an admissible set) and the Mergelyan approximation theorem on admissible sets for maps in the given class, then the analogue of Theorem 3.1 holds for this class of immersions.

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