

$SL_2(\mathbb{R})$ -developments and signature asymptotics for planar paths with bounded variation

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Abstract. The signature transform, defined by the formal tensor series of global iterated path integrals, is a homomorphism between the path space and the tensor algebra that has been studied in geometry, control theory, number theory as well as stochastic analysis. An elegant isometry conjecture states that the length of a bounded variation path γ can be recovered from the asymptotics of its normalised signature:

$$\text{Length}(\gamma) = \lim_{n \rightarrow \infty} \left\| n! \int_{0 < t_1 < \dots < t_n < T} d\gamma_{t_1} \otimes \dots \otimes d\gamma_{t_n} \right\|^{1/n}.$$

This property depends on a key topological non-degeneracy notion known as tree-reducedness (namely, with no tree-like pieces). Existing arguments have relied crucially on γ having a continuous derivative under the unit speed parametrisation. In this article, we prove the above isometry conjecture for planar paths by assuming only local bounds on the angle of γ' (which ensures the absence of tree-like pieces). Our technique is based on lifting the path onto the special linear group $SL_2(\mathbb{R})$ and analysing the behaviour of the associated angle dynamics at a microscopic level.

1. Introduction

The *signature transform* (or simply the *signature*) of a multidimensional path $\gamma: [0, L] \rightarrow \mathbb{R}^d$ is the formal tensor series

$$S(\gamma) \triangleq \sum_{n=0}^{\infty} \int_{0 < t_1 < \dots < t_n < L} d\gamma_{t_1} \otimes \dots \otimes d\gamma_{t_n}$$

formed by the global iterated path integrals of all orders. Such a transformation was originally introduced by K. T. Chen [8] to construct a cohomology theory on loop spaces over manifolds, which had led to far-reaching applications in geometry and algebraic topology. It also played an essential role in Dyson's quantum field theory (cf. [10]). Due to the vast development of analytic techniques in the rough path theory, the study of the signature transform has been enhanced to a new level of maturity over the last decade by

many authors. A landmark result was the uniqueness theorem proved by Hambly–Lyons in their well-known work [13] in 2010. The uniqueness theorem asserts that the signature of a bounded variation path uniquely determines the underlying path up to tree-like pieces (heuristically, a tree-like piece is a portion of the path in which it travels out and reverses back along itself). This result was later generalised to the rough path context in [3]. The uniqueness theorem has stimulated a stream of exciting problems related to reconstructing paths from their signatures and studying paths through functions on the signature space (see [5, 6, 12, 17]). One reason to work with the signature transform is that it has a nice intrinsic algebraic structure (linearisation of non-linear path functions) that is concealed at the level of paths (cf. [18]).

As a consequence of the uniqueness theorem, one naturally expects that many quantitative properties of a path can be recovered from its signature. In the bounded variation case, there is an elegant and important question along this line. A simple application of the triangle inequality shows that the signature of a continuous path γ with finite length satisfies the following estimate:

$$\left\| \int_{0 < t_1 < \dots < t_n < L} d\gamma_{t_1} \otimes \dots \otimes d\gamma_{t_n} \right\| \leq \int_{0 < t_1 < \dots < t_n < L} |d\gamma_{t_1}| \cdots |d\gamma_{t_n}| = \frac{\text{Length}(\gamma)^n}{n!}$$

for every $n \geq 1$. What is non-trivial and surprising is that this estimate becomes asymptotically sharp as $n \rightarrow \infty$. It was conjectured implicitly in [13] and later made explicit by Chang–Lyons–Ni [7] that, for any continuous, *tree-reduced* (i.e., not containing tree-like pieces) path with finite length, after normalisation one expects that

$$(1.1) \quad \text{Length}(\gamma) = \lim_{n \rightarrow \infty} \left\| n! \left(\int_{0 < t_1 < \dots < t_n < L} d\gamma_{t_1} \otimes \dots \otimes d\gamma_{t_n} \right) \right\|^{1/n}.$$

This conjectural property is deep and surprising, as it suggests that the tree-reduced property is eliminating the fine-scale interactions and cancellations of the path increments in the n -th order iterated integral as n increases. Understanding such a phenomenon is an important step towards obtaining effective signature lower bounds, which is in turn essential for establishing convergence of signature inversion schemes. This point is particularly relevant in the work of Chang–Lyons [6], which is also a key missing ingredient to theorise their proposed inversion algorithm in the general bounded variation context.

The signature asymptotics formula (1.1) was established in [13, 16] for \mathcal{C}^1 -paths (i.e., continuously differentiable) parametrised at unit speed. There is an important reason for pushing our understanding towards the general bounded variation case. The conjectural formula (1.1), as well as other similar signature inversion properties, if proven to be true, should be a pure consequence of *tree-reducedness* as suggested by the uniqueness theorem. As a result, identifying a suitable condition to quantify the “degree of being tree-reduced” is an essential step towards establishing general signature inversion properties. If a path γ is \mathcal{C}^1 at unit speed, one can see that γ cannot produce a tree-like piece. Indeed, if γ' is a continuous function on S^n , it immediately rules out the possibility that γ makes an π -turn at some point. However, this viewpoint embeds regularity assumptions of the path into the detection of tree-reducedness, making the latter property opaque.

For a deeper understanding about the essence of the underlying phenomenon, it is important to develop an approach that separates the non-degeneracy property of tree-reducedness from regularity properties of the path and reveals a property like (1.1) as a

consequence of tree-reducedness. The main contribution of the present article is to provide an attempt along this philosophy. Our intuition behind capturing tree-reducedness is very simple: we require that the path does not make a π -turn locally, and if it does it makes it in a way avoiding the creation of a tree-like piece (cf. Definition 2.5 for a more precise formulation). Our main result is stated in Theorem 2.6 below, which confirms the conjectural formula (1.1) for planar paths with bounded variation that satisfy such a condition. At the moment, extending the current analysis to higher dimensions is a challenging task (see Section 7 for a brief discussion). Nonetheless, the two dimensional situation already appears to be highly non-trivial and contains several essential ideas. Our methodology, which is partly inspired by the series [4, 9, 13, 15, 16] of works, is based on lifting the underlying path onto the special linear group $\mathrm{SL}_2(\mathbb{R})$ and developing fine analysis on the behaviour of the associated angle dynamics. It has a similar nature as the method developed in [4], but the underlying difficulties are in different directions. The method in [4] deals with multi-level interactions of different signature components (which is only relevant if the path is rough), while the current work deals with fine-scale interactions of different time periods due to rapid oscillations of the path. It is reasonable to expect that suitable combination of the two viewpoints may lead to deeper understanding towards the more general rough path situation. We mention that $\mathrm{SL}_2(\mathbb{R})$ -developments were also used by Lyons–Sidorova [15] to establish a decay property of the logarithmic signature. In their work, the $\mathrm{SL}_2(\mathbb{R})$ -structure was mainly used for the geometry of its exponential map. In our approach, the $\mathrm{SL}_2(\mathbb{R})$ -structure surprisingly simplifies the ODE dynamics by producing a decoupled ODE system for the $\mathrm{SL}_2(\mathbb{R})$ -action which may not be the case under other types of developments.

Organisation. In Section 2, we recall a few notation and state our main theorem. In Section 3, we recall some basic notions on Cartan developments of paths and derive the core equations in the context of $\mathrm{SL}_2(\mathbb{R})$ -developments. In Section 4, we establish several preliminary lemmas on the angle dynamics that are critical for later analysis. In Section 5, we develop the proof of the main theorem. In Section 6, we discuss how our method can be adapted to deal with a more singular type of paths. In Section 7, we discuss a few natural questions to be further investigated.

2. Statement of the main result

In this section, we provide the main set-up of the present article and state our main result.

We start by recalling some standard notation about paths and their signatures. Let V be a Banach space. For each $n \geq 1$, we denote $V^{\otimes n}$ as the completion of the algebraic tensor product $V^{\otimes_a n}$ under the *projective tensor norm*, which is defined by (cf. [19])

$$(2.1) \quad \|\xi\|_\pi \triangleq \inf \left\{ \sum_i |v_1^i| \cdots |v_n^i| : \xi = \sum_i v_1^i \otimes \cdots \otimes v_n^i \right\}, \quad \xi \in V^{\otimes_a n}.$$

The projective tensor norm is the largest among all admissible tensor norms (cf. [19]).

Definition 2.1. Let $\gamma: [0, L] \rightarrow V$ be a continuous path with finite length. The *signature* of γ is the formal tensor series of global iterated path integrals against γ defined by

$$S(\gamma) \triangleq \left(1, \gamma_L - \gamma_0, \dots, \int_{0 < t_1 < \dots < t_n < L} d\gamma_{t_1} \otimes \dots \otimes d\gamma_{t_n}, \dots \right) \in \prod_{n=0}^{\infty} V^{\otimes n}.$$

Let γ be a given continuous path with finite length. We define the following normalised signature asymptotics functional:

$$(2.2) \quad L_1(\gamma) \triangleq \lim_{n \rightarrow \infty} \left\| n! \left(\int_{0 < t_1 < \dots < t_n < L} d\gamma_{t_1} \otimes \dots \otimes d\gamma_{t_n} \right) \right\|_{\pi}^{1/n}.$$

It is known (cf. [1, 7]) that the limit in (2.2) is well defined and that the quantity $L_1(\gamma)$ remains the same if the limit is replaced by the supremum over $n \geq 1$. Using the triangle inequality, it is immediate to see that $L_1(\gamma) \leq \|\gamma\|_{1\text{-var}}$. The reverse inequality (for tree-reduced paths) is the main challenging question.

In the present article, we restrict ourselves to two dimensional paths. We assume that \mathbb{R}^2 is equipped with the Euclidean norm, and that the tensor products $(\mathbb{R}^2)^{\otimes n}$, $n \geq 1$, are equipped with the associated projective tensor norm (cf. (2.1)). We consider continuous paths in \mathbb{R}^2 with finite lengths, parametrised at unit speed. Mathematically, these paths are defined by

$$(2.3) \quad \gamma : [0, L] \rightarrow \mathbb{R}^2, \quad \gamma_t = (x_t, y_t) = \left(x_0 + \int_0^t \cos \beta_s ds, y_0 + \int_0^t \sin \beta_s ds \right),$$

where $\beta: [0, L] \rightarrow \mathbb{R}$ is a (Lebesgue) measurable function. For the purpose of this paper, readers may take (2.3) as the definition of γ . In the arXiv version of this article [2], we outline an argument that every non-constant continuous path with finite 1-variation can be reparametrised into the form (2.3).

We are going to propose a natural sufficient condition that captures the tree-reduced property, and to establish the asymptotics formula (1.1) for paths satisfying such condition. For the sake of preciseness, we recapture the definition of tree-reducedness as follows (cf. [3]). Recall that a *real tree* is a metric space in which any two distinct points can be joined by a unique non-self-intersecting path up to reparametrisation, and such a path is a geodesic in the metric sense.

Definition 2.2. Let $\gamma: [s, t] \rightarrow E$ be a continuous path in some topological space E . We say that γ is *tree-like* if there exists a real tree \mathcal{T} and two continuous maps

$$\xi : [s, t] \rightarrow \mathcal{T} \quad \text{and} \quad \Phi : \mathcal{T} \rightarrow E,$$

such that $\xi_s = \xi_t$ and $\gamma = \Phi \circ \xi$. A *tree-like piece* of a path $\gamma: [s, t] \rightarrow E$ is a portion $[u, v] \subseteq [s, t]$ such that $\gamma|_{[u, v]}$ is tree-like. A path is said to be *tree-reduced* if it does not contain any tree-like pieces.

Heuristically, being tree-reduced means that there is no portion of the path γ along which it reverses back to cancel itself right away. In the figure below, the first path is tree-reduced, while the second one contains a tree-like piece.

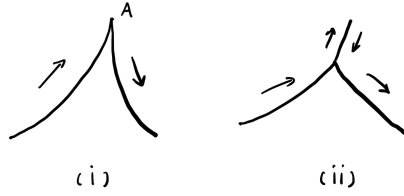


Figure 1. Tree-reduced and non-tree-reduced paths.

From now on, we consider a path $\gamma: [0, L] \rightarrow \mathbb{R}^2$ given by (2.3), where $\beta: [0, L] \rightarrow \mathbb{R}$ is a given measurable function. It is clear that γ is parametrised at unit speed, and L is the length of γ .

A natural idea to capture the tree-reduced property in terms of the angular path β is to require that β_t locally takes values in an interval of length less than π . This rules out the possibility that γ turns around to cancel itself. On the other hand, making a π -turn does not necessarily produce a tree-like piece, as illustrated by the cusp path in Figure 1 (i). In order to include this possibility, a natural extension is to make the above requirement on β hold outside an arbitrarily small interval that contains the cusp singularity.

To make the above idea precise, we first introduce the following definition.

Definition 2.3. Let $\gamma: [0, L] \rightarrow \mathbb{R}^2$ be a path given by (2.3). We say that γ is a *regular cusp*, if the following two conditions hold:

- (i) there is a real number $a \in \mathbb{R}$ such that

$$\beta_t \in [a, a + \pi] \quad \text{for a.a. } t \in [0, L];$$

- (ii) for any $\delta > 0$, there is a closed subset $F \subseteq L$ which is a finite disjoint union of closed intervals, as well as two real numbers $a_\delta > a$, $b_\delta < a + \pi$, such that

$$\mu(F^c) < \delta \quad \text{and} \quad \beta_t \in [a_\delta, b_\delta] \quad \text{for a.a. } t \in F.$$

Here μ denotes the Lebesgue measure.

Since the definition is only concerned with the angular path β , we sometimes simply say that β is a regular cusp. The typical shape of a regular cusp is illustrated by Figure 1 (i). We mention that there is another type of cusps that is more singular in terms of detecting the tree-reduced property and is thus harder to deal with. We discuss this case in Section 6 (see Theorem 6.1 below).

Example 2.4. A special situation of Definition 2.3 is when

$$(2.4) \quad \beta_t \in [a, b] \quad \text{for a.a. } t \in [0, L]$$

for some $a, b \in \mathbb{R}$ satisfying $b - a < \pi$. In this case, there is no cusp singularities and the conditions in Definition 2.3 are satisfied trivially.

Note that Definition 2.3 is global, while producing a tree-like piece or not is a local issue. To capture the tree-reduced property, it is more natural to localise Definition 2.3. This leads to the following definition, which will be assumed throughout the rest of the present article.

Definition 2.5. We say that γ is *strongly tree-reduced* if, for any $t \in (0, L)$, there exists a neighbourhood (u_t, v_t) of t such that $\gamma|_{[u_t, v_t] \cap [0, L]}$ is a regular cusp.

Our main result is stated as follows.

Theorem 2.6. *Let $\gamma: [0, L] \rightarrow \mathbb{R}^2$ be a path given by (2.3). Suppose that γ is strongly tree-reduced. Then the signature asymptotics formula (1.1) holds.*

Theorem 2.6 contains the case of \mathcal{C}^1 -paths at unit speed as a particular example. Indeed, if the angular path β_t is continuous, for each given t the path β_s takes values in an interval of length less than π when s is near t . The result also contains the case of piecewise \mathcal{C}^1 -paths whose intersection angles are strictly less than π . For the situation where the intersection angle is π , Theorem 2.6 still applies if the cusp singularity has the nature of Definition 2.3. Also see Section 6 below for adapting the analysis to the case of even more singular cusps.

On the other hand, the property stated in Definition 2.5 is a condition that captures the tree-reduced property and has no implication on the regularity of the path γ . Our analysis relies purely on such a non-degeneracy condition. In contrast to the analysis developed in [13, 16], regularity assumptions on β are not relevant here and measurability is sufficient for our purpose.

Remark 2.7. It is also interesting to point out that, although heuristically convincing, it is not at all obvious to directly show that strong tree-reducedness implies tree-reducedness in the sense of Definition 2.2. This is however an immediate consequence of Theorem 2.6 and the uniqueness theorem in [3].

To summarise the main idea in our strategy, we consider the development of γ into the special linear group $\mathrm{SL}_2(\mathbb{R})$ which acts on the plane \mathbb{R}^2 in the canonical way. The core of our approach is to look at the action of Γ_L (where $(\Gamma_t)_{0 \leq t \leq L}$ is the development of γ) from a dynamical viewpoint, and to carefully examine the behaviour of the associated angle dynamics at a microscopic level. We will elaborate this point more precisely as we develop the analysis in the following sections.

3. $\mathrm{SL}_2(\mathbb{R})$ -developments and the associated ODE dynamics

Our starting point of proving Theorem 2.6 is to develop the path γ onto a suitably chosen Lie group from Cartan's perspective. In this section, we first recall the general construction of path developments under the framework of [4]. We then specialise the development to a particular Lie group and establish the associated ODE dynamics. The analysis of this ODE dynamics is the core ingredient in our approach, which will be performed in later sections.

3.1. The Cartan development of rough paths and the intermediate lower estimate

Let V be a finite dimensional normed vector space. Let G be a finite dimensional Lie group with Lie algebra \mathfrak{g} . Suppose that $F: V \rightarrow \mathfrak{g}$ is a given linear map, and $\rho: \mathfrak{g} \rightarrow \mathrm{End}(W)$ is a given representation (i.e., a Lie algebra homomorphism) of \mathfrak{g} over a finite dimensional

normed vector space W . Set

$$\Phi \triangleq \rho \circ F : V \rightarrow \text{End}(W).$$

Let $\mathbf{X} = (\mathbf{X}_t)_{0 \leq t \leq L}$ be a geometric rough path over V (cf. [14]).

Definition 3.1. The *Cartan development* of \mathbf{X} onto G under F is the solution to the differential equation

$$\begin{cases} dG_t = G_t \cdot F(d\mathbf{X}_t), & 0 \leq t \leq L, \\ G_0 = e, \end{cases}$$

where e is the identity of G . Respectively, the *Cartan development of \mathbf{X} onto $\text{Aut}(W)$* under Φ is the solution to the linear differential equation

$$\begin{cases} d\Gamma_t = \Gamma_t \cdot \Phi(d\mathbf{X}_t), & 0 \leq t \leq L, \\ \Gamma_0 = \text{Id}. \end{cases}$$

Remark 3.2. There are two important reasons for considering path developments. The first one is that the end point Γ_L of the development, when one varies \mathfrak{g} and the representation ρ in a suitably chosen class, should encode essentially all information about the original rough path \mathbf{X} (up to tree-like pieces). The second one is that the development is defined by a “linear” equation, hence linearising the analysis of nonlinear functionals on path space. These two points are similar to the philosophy of working with the signature of \mathbf{X} , but the path development allows much richer algebraic structures through choosing the Lie algebras and representations properly. This philosophy has not yet been fully explored in the literature, and could be of potential interest for further applications in the study of rough paths and stochastic processes.

We now recall a general lower estimate proved in [4]. We only state the version for bounded variation paths. Let $\gamma: [0, L] \rightarrow V$ be a continuous path with finite length. Recall that $L_1(\gamma)$ is the functional defined by the normalised signature asymptotics (2.2). Under the above set-up, for each $\lambda > 0$, let $(\Gamma_t^\lambda)_{0 \leq t \leq L}$ be the Cartan development of $\lambda \cdot \gamma$ onto $\text{Aut}(W)$ under Φ .

Proposition 3.3. *The quantity $L_1(\gamma)$ admits the following lower estimate:*

$$L_1(\gamma) \geq \overline{\lim}_{\lambda \rightarrow \infty} \frac{\log \|\Gamma_L^\lambda\|_{W \rightarrow W}}{\lambda \|\Phi\|_{V \rightarrow \text{End}(W)}},$$

where $\|\cdot\|_{X \rightarrow Y}$ denotes the operator norm between Banach spaces X and Y .

3.2. $\text{SL}_2(\mathbb{R})$ -developments

We now specialise our study to the context of Theorem 2.6. In particular, let $V = \mathbb{R}^2$ whose canonical basis is denoted as $\{e_1, e_2\}$. Let $\gamma: [0, L] \rightarrow V$ be a continuous path with finite length L , parametrised at unit speed. More explicitly, the path γ is defined by the integral (2.3), where the angular path $\beta: [0, L] \rightarrow \mathbb{R}^2$ is a given measurable function.

Under the setting of Section 3.1, we choose $G = \text{SL}_2(\mathbb{R})$, the space of real 2×2 matrices with determinant one. The group $\text{SL}_2(\mathbb{R})$ is a three dimensional Lie group, whose

Lie algebra is given by $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$, the space of real 2×2 matrices with zero trace. The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ admits a Lorentzian metric defined by

$$\langle A, B \rangle \triangleq \frac{1}{2} \text{Tr}(AB), \quad A, B \in \mathfrak{sl}_2(\mathbb{R}),$$

which has signature $(+, +, -)$. An element $A \in \mathfrak{sl}_2(\mathbb{R})$ is *hyperbolic/elliptic/parabolic* if $\langle A, A \rangle$ is positive/negative/zero. An orthonormal basis of $\mathfrak{sl}_2(\mathbb{R})$ under the Lorentzian metric $\langle \cdot, \cdot \rangle$ is given by

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that E_1 and E_2 are hyperbolic elements, and that E_3 is elliptic.

We define the linear map $F: V \rightarrow \mathfrak{sl}_2(\mathbb{R})$ explicitly by mapping the basis $\{e_1, e_2\}$ to the hyperbolic elements $\{E_1, E_2\}$ respectively, i.e., $F(e_i) \triangleq E_i$, for $i = 1, 2$.

Finally, the representation $\rho: \mathfrak{sl}_2(\mathbb{R}) \rightarrow \text{End}(W)$ is taken to be the canonical matrix representation, i.e., the action of \mathfrak{g} on $W = \mathbb{R}^2$ via matrix multiplication. We assume that W is also equipped with the Euclidean norm. Note that the group $\text{SL}_2(\mathbb{R})$ also acts on W via matrix multiplication. If we view $\text{SL}_2(\mathbb{R})$ as a subspace of the matrix algebra $\text{Mat}_2(\mathbb{R}) \cong \text{End}(W)$, the Cartan developments onto $\text{SL}_2(\mathbb{R})$ and $\text{Aut}(W)$ are identical. For each $\lambda > 0$, the Cartan development of $\lambda \cdot \gamma$ onto $\text{SL}_2(\mathbb{R})$ is denoted as $\Gamma^\lambda = (\Gamma_t^\lambda)_{0 \leq t \leq L}$. More explicitly, Γ^λ satisfies the differential equation

$$\frac{d\Gamma_t^\lambda}{dt} = \lambda \Gamma_t^\lambda \cdot \begin{pmatrix} \cos \beta_t & \sin \beta_t \\ \sin \beta_t & -\cos \beta_t \end{pmatrix}, \quad \Gamma_0^\lambda = \text{Id}.$$

Note that such equation needs to be understood in the integral sense or in the t -a.e. sense.

Under the above specific choice, Proposition 3.3 yields the following lower estimate for the quantity $L_1(\gamma)$:

$$(3.1) \quad L_1(\gamma) \geq \overline{\lim}_{\lambda \rightarrow \infty} \frac{\log \|\Gamma_L^\lambda\|_{\mathbb{R}^2 \rightarrow \mathbb{R}^2}}{\lambda}.$$

To see this, we only need to check that the operator $\Phi = \rho \circ F: \mathbb{R}^2 \rightarrow \text{End}(\mathbb{R}^2)$ has norm one. But this follows from the relation

$$|\Phi(v)(w)| = |v| \cdot |w|, \quad \forall v, w \in \mathbb{R}^2,$$

which can be verified explicitly.

Remark 3.4. From the geometric viewpoint, it is also natural to consider the action of $\text{SL}_2(\mathbb{R})$ on the upper half plane \mathbb{H} via Möbius transformation, since $\text{SL}_2(\mathbb{R})$ is the isometry group of \mathbb{H} when \mathbb{H} is equipped with the Lobachevsky hyperbolic metric. In this case, the action of Γ_t^λ on \mathbb{H} gives the hyperbolic development of γ (see [13] for an equivalent hyperbolic framework). However, we do not take this geometric viewpoint and work with linear actions instead.

3.3. The decoupled ODE system and the associated angle dynamics

In view of (3.1), in order to obtain a sharp lower bound for $L_1(\gamma)$, we need to estimate $\|\Gamma_L^\lambda\|_{\mathbb{R}^2 \rightarrow \mathbb{R}^2}$ effectively. For this purpose, we look at the action of Γ_L^λ from a dynamical perspective which we now describe.

We denote by $\Gamma_{s,t}^\lambda$, $t \in [s, L]$ the Cartan development of $\lambda \cdot \gamma|_{[s,L]}$ evaluated at time t . A simple calculation shows that

$$(3.2) \quad \Gamma_{s,u}^\lambda = \Gamma_{s,t}^\lambda \cdot \Gamma_{t,u}^\lambda, \quad \forall s \leq t \leq u.$$

As a result, for a given initial vector $\xi^\lambda \in \mathbb{R}^2$, the action $\Gamma_L^\lambda \xi^\lambda$ can be studied through the following dynamical perspective:

$$(3.3) \quad \Gamma_L^\lambda \xi^\lambda = \Gamma_{t_0,t_1}^\lambda \cdot \Gamma_{t_1,t_2}^\lambda \cdots \Gamma_{t_{n-1},t_n}^\lambda \xi^\lambda,$$

where $\mathcal{P} = \{t_i\}_{0 \leq i \leq n}$ is an arbitrarily fine partition of $[0, L]$. The dynamics (3.3) reduces to the following simple equation when we take $\text{mesh}(\mathcal{P}) \rightarrow 0$.

Lemma 3.5. *Let $w_t^\lambda \triangleq \Gamma_{L-t,L}^\lambda \xi^\lambda$. Then $w_L^\lambda = \Gamma_L^\lambda \xi^\lambda$, and $(w_t^\lambda)_{0 \leq t \leq L}$ is the unique solution to the differential equation*

$$(3.4) \quad \begin{cases} \frac{dw_t^\lambda}{dt} = \lambda \begin{pmatrix} \cos \beta_{L-t} & \sin \beta_{L-t} \\ \sin \beta_{L-t} & -\cos \beta_{L-t} \end{pmatrix} \cdot w_t^\lambda, & 0 \leq t \leq L, \\ w_0^\lambda = \xi^\lambda. \end{cases}$$

Proof. Let t be given, and let $h > 0$. According to the relation (3.2), we have

$$w_{t+h}^\lambda = \Gamma_{L-t-h,L-t}^\lambda \cdot w_t^\lambda.$$

It follows from the equation of the Cartan development that

$$\begin{aligned} \frac{w_{t+h}^\lambda - w_t^\lambda}{h} &= \frac{\Gamma_{L-t-h,L-t}^\lambda - \text{Id}}{h} \cdot w_t^\lambda \\ &= \frac{\lambda}{h} \int_{L-t-h}^{L-t} \Gamma_{L-t-h,u}^\lambda \begin{pmatrix} \cos \beta_u & \sin \beta_u \\ \sin \beta_u & -\cos \beta_u \end{pmatrix} \cdot w_t^\lambda du, \end{aligned}$$

The result follows by letting $h \rightarrow 0^+$. ■

Notation. From now on, we denote $\alpha_t \triangleq \beta_{L-t}$. It is obvious that α satisfies Definition 2.5 if and only if β does.

Our next step is to rewrite the equation (3.4) using polar coordinates. Let

$$w_t^\lambda = \rho_t^\lambda e^{i\phi_t^\lambda}, \quad t \in [0, L].$$

We also write the initial vector as $\xi^\lambda = \rho_0^\lambda e^{i\phi_0^\lambda}$, where $\rho_0^\lambda = 1$ and the angle ϕ_0^λ is given fixed.

Lemma 3.6. *The pair $(\rho_t^\lambda, \phi_t^\lambda)$ satisfies the following ODE system:*

$$(3.5) \quad \begin{cases} \frac{d\rho_t^\lambda}{dt} = \lambda \rho_t^\lambda \cos(\alpha_t - 2\phi_t^\lambda), \\ \frac{d\phi_t^\lambda}{dt} = \lambda \sin(\alpha_t - 2\phi_t^\lambda). \end{cases}$$

Proof. Firstly, note that we have

$$(3.7) \quad \begin{aligned} d w_t^\lambda &= d\rho_t^\lambda \cdot \begin{pmatrix} \cos \phi_t^\lambda \\ \sin \phi_t^\lambda \end{pmatrix} + \rho_t^\lambda d\phi_t^\lambda \cdot \begin{pmatrix} -\sin \phi_t^\lambda \\ \cos \phi_t^\lambda \end{pmatrix} \\ &= \begin{pmatrix} \cos \phi_t^\lambda & -\sin \phi_t^\lambda \\ \sin \phi_t^\lambda & \cos \phi_t^\lambda \end{pmatrix} \cdot \begin{pmatrix} d\rho_t^\lambda \\ \rho_t^\lambda d\phi_t^\lambda \end{pmatrix}. \end{aligned}$$

In addition, according to the equation (3.4), we have

$$(3.8) \quad d w_t^\lambda = \lambda \rho_t^\lambda dt \cdot \begin{pmatrix} \cos \alpha_t & \sin \alpha_t \\ \sin \alpha_t & -\cos \alpha_t \end{pmatrix} \cdot \begin{pmatrix} \cos \phi_t^\lambda \\ \sin \phi_t^\lambda \end{pmatrix}.$$

By comparing (3.7) and (3.8), we arrive at

$$\begin{pmatrix} d\rho_t^\lambda \\ \rho_t^\lambda d\phi_t^\lambda \end{pmatrix} = \lambda \rho_t^\lambda dt \cdot \begin{pmatrix} \cos(\alpha_t - 2\phi_t^\lambda) \\ \sin(\alpha_t - 2\phi_t^\lambda) \end{pmatrix},$$

which yields the desired ODE system. ■

It is clear that the angular path ϕ_t^λ is absolutely continuous. Observe that the ODE system for $(\rho_t^\lambda, \phi_t^\lambda)$ is decoupled, in the sense that the angular equation (3.6) does not depend on ρ_t^λ . In addition, by linearity the radial component ρ_t^λ can be easily solved from the radial equation (3.5) (recall $\rho_0^\lambda = 1$) as

$$(3.9) \quad \rho_t^\lambda = \exp\left(\lambda \int_0^t \cos(\alpha_s - 2\phi_s^\lambda) ds\right), \quad t \in [0, L].$$

Remark 3.7. The following viewpoint can be taken to avoid ambiguity when choosing the angular component ϕ_t^λ . Given the initial angle ϕ_0^λ , the angular equation (3.6) admits a unique solution ϕ_t^λ . Define ρ_t^λ by the formula (3.9) accordingly. Then $w_t^\lambda \triangleq \rho_t^\lambda e^{i\phi_t^\lambda}$ gives the solution to the equation (3.4) which clearly coincides with $(\Gamma_{L-t, L}^\lambda \xi^\lambda)_{0 \leq t \leq L}$ by uniqueness.

Since $\rho_L^\lambda = |\Gamma_L^\lambda \xi^\lambda|$ and ξ^λ is assumed to be a unit vector, we have the following important lemma, which is a direct consequence of the estimate (3.1).

Lemma 3.8. *The quantity $L_1(\gamma)$ satisfies*

$$(3.10) \quad L_1(\gamma) \geq \overline{\lim}_{\lambda \rightarrow \infty} \int_0^L \cos(\alpha_t - 2\phi_t^\lambda) dt,$$

where the initial angle $\phi_0^\lambda \in \mathbb{R}$ is arbitrarily given.

Lemma 3.8 provides a clearer picture of proving Theorem 2.6. In order to produce the lower bound of $L_1(\gamma)$ given by the length L , we are led to showing that *the angular path $2\phi_t^\lambda$ is close to α_t for most of the time when λ is large*. At a heuristic level, this phenomenon is reasonable, since the angular equation (3.6) is suggesting a strong mean-reversing behaviour of $2\phi_t^\lambda$ towards the path α_t when λ is large. However, making this phenomenon mathematically precise is a non-trivial challenging task, since no regularity assumptions are made on the path α_t and one has to rely on fine measure-theoretic arguments. The analysis simplifies substantially if α_t is assumed to be a continuous function (see Appendix A for a discussion on this case).

4. Some important lemmas on the global behaviour of the angle dynamics

The core of our approach is to analyse the angle dynamics for ϕ_t^λ . In this section, we derive several key lemmas on the behaviour of ϕ_t^λ relative to the path α_t under global assumptions on α_t . These results will be used in a localised situation when we prove the main theorem in the next section.

The first lemma tells us that $2\phi_t^\lambda$ remains in the same range as α_t 's provided that the initial angle $2\phi_0^\lambda$ does.

Lemma 4.1. *Let $a \in \mathbb{R}$, and suppose that $\alpha_t \in [a, a + \pi]$ for a.a. $t \in [0, L]$. If $2\phi_0^\lambda \in (a, a + \pi)$, then $2\phi_t^\lambda \in (a, a + \pi)$ for every $t \in [0, L]$ and $\lambda > 0$.*

Proof. Recall that $2\phi_t^\lambda$ is absolutely continuous. Set $b \triangleq a + \pi$. Suppose on the contrary that $2\phi_t^\lambda$ leaves $(a, a + \pi)$ at some time, and let us assume that $2\phi_t^\lambda$ hits the end point b before hitting a . Define

$$t_2 \triangleq \inf \left\{ t : 2\phi_t^\lambda = b \right\} \quad \text{and} \quad t_1 \triangleq \sup \left\{ t < t_2 : 2\phi_t^\lambda < \frac{a+b}{2} \right\}.$$

We take $t_1 = 0$ if $2\phi_t^\lambda \geq (a+b)/2$ for all $t < t_2$. Apparently, we have $2\phi_{t_1}^\lambda < b$, $2\phi_{t_2}^\lambda = b$ and $2\phi_t^\lambda \in [(a+b)/2, b]$ for all $t \in [t_1, t_2]$. By using the equation (3.6) of ϕ_t^λ , for a.a. $t \in [t_1, t_2]$ we have

$$\begin{aligned} \frac{d\phi_t^\lambda}{dt} &= \lambda \sin(\alpha_t - 2\phi_t^\lambda) \\ &= \lambda \sin(\alpha_t - 2\phi_t^\lambda) \mathbf{1}_{\{a \leq \alpha_t < (a+b)/2\}} + \lambda \sin(\alpha_t - 2\phi_t^\lambda) \mathbf{1}_{\{(a+b)/2 \leq \alpha_t \leq b\}}. \end{aligned}$$

In the first region, we know that $\alpha_t - 2\phi_t^\lambda \in [-\pi, 0]$ and thus the sine function is non-positive. In the second region, note that both of $\alpha_t - 2\phi_t^\lambda$ and $b - 2\phi_t^\lambda$ lie in $[-\pi/2, \pi/2]$. As a result, we have

$$\sin(\alpha_t - 2\phi_t^\lambda) \leq \sin(b - 2\phi_t^\lambda)$$

in this region. Note moreover that $b - 2\phi_t^\lambda \in [0, \pi/2]$ for $t \in [t_1, t_2]$. It follows that

$$\frac{d\phi_t^\lambda}{dt} \leq \lambda \sin(b - 2\phi_t^\lambda) \mathbf{1}_{\{(a+b)/2 \leq \alpha_t \leq b\}} \leq \lambda (b - 2\phi_t^\lambda).$$

Equivalently, we have

$$\frac{d}{dt}(2e^{2\lambda t}\phi_t^\lambda) \leq 2\lambda b e^{2\lambda t}.$$

By integrating the above inequality over $[t_1, t_2]$, we arrive at

$$2\phi_{t_2}^\lambda \leq b - (b - 2\phi_{t_1}^\lambda) e^{-2\lambda(t_2-t_1)} < b,$$

which is a contradiction. A similar argument also leads to a contradiction in the case when $2\phi_t^\lambda$ hits a before b . Therefore, we conclude that $2\phi_t^\lambda \in (a, b)$ for all time. ■

The next lemma quantifies how much $2\phi_t^\lambda$ can deviate from an interval $[a, b]$ if the path α_t does not always stay in this interval.

Lemma 4.2. *Let $a, b \in \mathbb{R}$ be such that $0 < b - a < \pi$. Define*

$$r \triangleq 2\lambda \mu(\{t : \alpha_t \notin [a, b]\}),$$

Suppose that $b - a + r < \pi$ and that $2\phi_0^\lambda \in [a, b]$. Then

$$2\phi_t^\lambda \in [a - r, b + r], \quad \forall t \in [0, L].$$

Proof. Suppose on the contrary that $2\phi_t^\lambda$ exits the interval $[a - r, b + r]$ at some time, and assume that it exits at the end point $b + r$. We can then find a time τ such that

$$b + r < 2\phi_\tau^\lambda < a + \pi.$$

Define

$$t_2 \triangleq \inf\{t < \tau : 2\phi_t^\lambda = 2\phi_\tau^\lambda\} \quad \text{and} \quad t_1 \triangleq \sup\{t < t_2 : 2\phi_t^\lambda \in [a, b]\},$$

and write

$$A \triangleq \{t : \alpha_t \in [a, b]\}.$$

Note that $2\phi_{t_1}^\lambda = b$. It follows from the angular equation (3.6) that

$$\begin{aligned} 2\phi_{t_2}^\lambda - b &= -2\lambda \int_{t_1}^{t_2} \sin(2\phi_t^\lambda - \alpha_t) dt \\ &= -2\lambda \int_{[t_1, t_2] \cap A} \sin(2\phi_t^\lambda - \alpha_t) dt - 2\lambda \int_{[t_1, t_2] \cap A^c} \sin(2\phi_t^\lambda - \alpha_t) dt \\ &\leq -2\lambda \int_{[t_1, t_2] \cap A^c} \sin(2\phi_t^\lambda - \alpha_t) dt \leq 2\lambda \mu([t_1, t_2] \cap A^c). \end{aligned}$$

Therefore, we have

$$b + r < 2\phi_\tau^\lambda = 2\phi_{t_2}^\lambda \leq b + 2\lambda \mu([t_1, t_2] \cap A^c) \leq b + r,$$

which is a contradiction. The case when $2\phi_t^\lambda$ exits $[a - r, b + r]$ through the end point $a - r$ can be treated in a similar way. Consequently, we conclude that $2\phi_t^\lambda \in [a - r, b + r]$ for all time. ■

Remark 4.3. Heuristically, Lemma 4.2 tells us that the longer α_t stays in $[a, b]$ (equivalently, the smaller r is), the less will $2\phi_t^\lambda$ deviate from $[a, b]$. In the special case when $\alpha_t \in [a, b]$ a.a. $t \in [0, L]$ (i.e., when $r = 0$), we have $2\phi_t^\lambda \in [a, b]$ for all time. But this conclusion is slightly weaker than Lemma 4.1 since we have assumed $b < a + \pi$ here.

The final lemma quantifies how fast $2\phi_t^\lambda$ gets attracted to the region where α_t stays for most of the time, if initially $2\phi_0^\lambda$ is far away from this region.

Lemma 4.4. *Let $a < b$ be such that $b - a < \pi$. Let $c < d$ and $\varepsilon > 0$ be such that*

$$[c - \varepsilon, d + \varepsilon] \subseteq (a, b) \quad \text{and} \quad \varepsilon < \pi - (b - a).$$

Suppose that $\alpha_t \in [a, b]$ for a.a. $t \in [0, L]$, and

$$(4.1) \quad 2\phi_0^\lambda \in (a, b) \setminus [c - \varepsilon, d + \varepsilon].$$

Define $B \triangleq \{t : \alpha_t \in [c, d]\}$ and

$$\tau \triangleq \inf\{t : 2\phi_t^\lambda \in [c - \varepsilon, d + \varepsilon]\}.$$

Then

$$(4.2) \quad \tau \leq \frac{b - a}{2\lambda \sin \varepsilon} + \frac{1 + \sin \varepsilon}{\sin \varepsilon} \mu(B^c).$$

Proof. First of all, we know from Lemma 4.1 that $2\phi_t^\lambda \in [a, b]$ for all t . In view of the assumption (4.1), suppose that $2\phi_0^\lambda \in (d + \varepsilon, b)$. Then for a.a. $t \in [0, \tau]$, we have

$$\frac{d}{dt}(2\phi_t^\lambda) = -2\lambda \sin(2\phi_t^\lambda - \alpha_t) = -2\lambda \sin(2\phi_t^\lambda - \alpha_t) \mathbf{1}_B - 2\lambda \sin(2\phi_t^\lambda - \alpha_t) \mathbf{1}_{B^c}.$$

Note that for $t \in B \cap [0, \tau]$ we have

$$\varepsilon \leq 2\phi_t^\lambda - \alpha_t \leq b - a < \pi.$$

Since $\varepsilon < \pi - (b - a)$, it follows that

$$\sin(2\phi_t^\lambda - \alpha_t) \geq \sin \varepsilon \quad \text{on } B \cap [0, \tau].$$

Therefore,

$$\frac{d}{dt}(2\phi_t^\lambda) \leq -2\lambda(\sin \varepsilon) \mathbf{1}_B + 2\lambda \mathbf{1}_{B^c} \quad \text{a.a. } t \in [0, \tau].$$

By integrating the above inequality, we obtain

$$\begin{aligned} 2\phi_\tau^\lambda &\leq 2\phi_0^\lambda - 2\lambda \sin \varepsilon \cdot \mu(B \cap [0, \tau]) + 2\lambda \mu(B^c \cap [0, \tau]) \\ &= 2\phi_0^\lambda - 2\lambda \sin \varepsilon \cdot (\tau - \mu(B^c \cap [0, \tau])) + 2\lambda \mu(B^c \cap [0, \tau]) \\ &= 2\phi_0^\lambda - 2\lambda \tau \sin \varepsilon + 2\lambda(1 + \sin \varepsilon) \mu(B^c \cap [0, \tau]). \end{aligned}$$

Therefore,

$$\begin{aligned} 2\lambda \tau \sin \varepsilon &\leq 2\phi_0^\lambda - 2\phi_\tau^\lambda + 2\lambda(1 + \sin \varepsilon) \mu(B^c \cap [0, \tau]) \\ &\leq b - a + 2\lambda(1 + \sin \varepsilon) \mu(B^c \cap [0, \tau]). \end{aligned}$$

Rearranging the terms gives the estimate (4.2). A similar argument gives the same conclusion for the case $2\phi_0^\lambda \in (a, c - \varepsilon)$. Note that the situation when $\tau = L$ (i.e., when $2\phi_t^\lambda$ never enters $[c - \varepsilon, d + \varepsilon]$) is included in the above argument. ■

Remark 4.5. Heuristically, Lemma 4.4 tells us that, if α_t stays in $[c, d]$ for most of the time (i.e., if $\mu(B^c)$ is small) and if λ is large, it takes a short period of time for $2\phi_t^\lambda$ to enter the interval $[c - \varepsilon, d + \varepsilon]$ (i.e., τ is small).

Remark 4.6. The fact that $b - a < \pi$ is critical to make use of the monotonicity property of the sine function in the proof of Lemma 4.4. The precise use of this lemma in the proof of the main theorem requires a minor technical modification (cf. Lemma 5.3 below).

5. Proof of the main theorem

In this section, we develop the proof of Theorem 2.6. To make our strategy more transparent, we first prove the theorem under the global assumption of regular cusps (cf. Definition 2.3). This part contains the essential idea of the proof. After that, we localise the result to the context of strongly tree-reduced paths (cf. Definition 2.5).

5.1. Proof of Theorem 2.6: the global case

Suppose that $\gamma: [0, L] \rightarrow \mathbb{R}^2$ is a path defined by (2.3), where $\beta: [0, L] \rightarrow \mathbb{R}$, the angular path, is a given measurable function. In this subsection, we aim at proving the following result.

Theorem 5.1. *Suppose that γ is a regular cusp in the sense of Definition 2.3. Then the signature asymptotics formula (1.1) holds.*

Vaguely speaking, our strategy is to analyse the local behaviour of the angle dynamics (3.6) on each sub-interval of suitable partitions of $[0, L]$, and then to examine how these microscopic effects accumulate on the global scale. The analysis for the former point is based on suitable localisation of the results obtained in Section 4.

We now develop the precise details of the proof of Theorem 5.1. To better convey the logic and reasoning, we shall divide the argument into several major steps. Recall that $\alpha_t \triangleq \beta_{L-t}$. From Definition 2.3 (i), we know that the angular path α_t satisfies

$$\alpha_t \in [a, a + \pi] \quad \text{for a.a. } t \in [0, L],$$

where $a \in \mathbb{R}$ is given fixed. If one does not want to bother with cusps, the argument below appears to be simpler under the assumption that $\alpha_t \in [a, b]$ for a.a. t , where $b - a < \pi$ (cf. Example 2.4).

5.1.1. Step one: localising the path α_t . Fix $\delta > 0$. According to Definition 2.3 (ii), there is a closed subset $F_1 \subseteq [0, L]$, which is a finite disjoint union of closed intervals, as well as two real numbers $a_\delta > a, b_\delta < b \triangleq a + \pi$, such that $\mu(F_1^c) < \delta$ and

$$\alpha_t \in [a_\delta, b_\delta] \quad \text{for a.a. } t \in F_1.$$

To proceed further, we first recall the classical Lusin theorem (cf. Folland [11]) as follows.

Theorem 5.2. *Let $f: [p, q] \rightarrow \mathbb{C}$ be a Lebesgue measurable function. Then for any $\eta > 0$, there exists a compact set $E \subseteq [p, q]$, such that $\mu(E^c) < \eta$ and $f|_E$ is continuous.*

Let $\eta > 0$ be another given number, and let $\varepsilon > 0$ be such that

$$(5.1) \quad \varepsilon < \min \left\{ a_\delta - a, b - b_\delta, \frac{\pi}{4} \right\}.$$

Note that ε is independent of η . According to the above Lusin theorem, we can choose a compact subset $F_2 \subseteq F_1$, such that $\mu(F_1 \setminus F_2) < \eta$ and $\alpha|_{F_2}$ is (uniformly) continuous. As a result, there exists $\rho > 0$ such that

$$s, t \in F_2, \quad |t - s| < \rho \implies |\alpha_t - \alpha_s| < \varepsilon.$$

Since $\alpha_t \in [a_\delta, b_\delta]$ a.e. on F_1 , by further reducing F_2 if necessary, we may assume that

$$\alpha_t \in [a_\delta, b_\delta] \quad \text{for every } t \in F_2.$$

Recall that F_1 is a finite disjoint union of, say, N_δ closed intervals. Given $\lambda > 0$, we set

$$(5.2) \quad n \triangleq [c \cdot \lambda] + 1, \quad \text{where } c \triangleq \varepsilon \sin \varepsilon.$$

The reason for choosing this c will be clear later on. We consider the partition $\mathcal{P}_n = \{t_i^n\}_{0 \leq i \leq n}$ of F_1 which divide each closed interval in F_1 into small sub-intervals of equal length. When λ (and thus n) is large enough, we can ensure that

$$(5.3) \quad \text{mesh}(\mathcal{P}_n) = \frac{\mu(F_1)}{n} < \rho.$$

Note. We have introduced several parameters $\delta, \varepsilon, \eta$ and λ . At some point later on, we will introduce one more independent parameter M . All these parameters need to pass to the limit in the last step. It may be helpful to keep in mind that the following order of taking limits will be implemented eventually:

$$(5.4) \quad \lambda \rightarrow \infty, \quad \eta \rightarrow 0^+, \quad M \rightarrow \infty, \quad \varepsilon \rightarrow 0^+, \quad \delta \rightarrow 0^+.$$

In what follows, we work with any given $\lambda > 0$ that satisfies (5.3). This is legal in the spirit of (5.4), since the first limiting procedure we will take is sending $\lambda \rightarrow \infty$. For each $1 \leq i \leq n$, we write $I_i^n \triangleq [t_{i-1}^n, t_i^n]$ and define

$$\alpha_i^n \triangleq \inf \{ \alpha_t : t \in F_2 \cap I_i^n \} \quad \text{and} \quad \beta_i^n \triangleq \sup \{ \alpha_t : t \in F_2 \cap I_i^n \}.$$

Note that

$$(5.5) \quad 0 \leq \beta_i^n - \alpha_i^n < \varepsilon \quad \text{and} \quad \alpha_i^n, \beta_i^n \in [a_\delta, b_\delta].$$

5.1.2. Step two: the local behaviour of the angle dynamics. Now we consider the $\mathrm{SL}_2(\mathbb{R})$ -development of γ_t constructed in Section 3.2. Recall that the function ϕ_t^λ satisfies the angular equation (3.6). We assume that $2\phi_0^\lambda \in (a, b)$. The core of our argument concerns with understanding the local behaviour of $2\phi_t^\lambda$ on each sub-interval I_i^n and its accumulated effect on the global scale. In particular, there are two key points that we shall establish in a precise way:

(i) The time it takes $2\phi_t^\lambda$ ($t \in I_i^n$) to enter the interval $[\alpha_i^n - \varepsilon, \beta_i^n + \varepsilon]$ adds up (over i) to a negligible quantity.

(ii) Once $2\phi_t^\lambda \in [\alpha_i^n - \varepsilon, \beta_i^n + \varepsilon]$ at some $t \in I_i^n$, the portion of $2\phi_u^\lambda$ on $[t, t_i^n]$ provides a main contribution in the lower estimate of the radial function ρ_t^λ defined by (3.9) (or equivalently, the integral appearing in (3.10)).

We quantify these two points precisely in Step three below. The main ingredient in the current step is the following localised version of Lemma 4.4. Let us introduce

$$\tau_i^n \triangleq \inf\{t \in I_i^n : 2\phi_t^\lambda \in [\alpha_i^n - \varepsilon, \beta_i^n + \varepsilon]\} \quad \text{and} \quad \sigma_i^n \triangleq \tau_i^n - t_{i-1}^n.$$

The quantity σ_i^n gives the amount of time within I_i^n before $2\phi_t^\lambda$ enters the “good” region $[\alpha_i^n - \varepsilon, \beta_i^n + \varepsilon]$. If $2\phi_{t_{i-1}^n}^\lambda \in [\alpha_i^n - \varepsilon, \beta_i^n + \varepsilon]$, we trivially have $\sigma_i^n = 0$. Otherwise, we have the following estimate for the time period σ_i^n , which is a minor adaptation of Lemma 4.4.

Lemma 5.3. *The quantity σ_i^n satisfies the following estimate:*

$$\sigma_i^n \leq \frac{\pi}{2\lambda \sin \varepsilon} + \frac{1 + \sin \varepsilon}{\sin \varepsilon} \mu((F_1 \setminus F_2) \cap I_i^n).$$

Proof. We expect to apply Lemma 4.4 to the context of

$$[a, b] = [a_\delta, b_\delta], \quad [c, d] = [\alpha_i^n, \beta_i^n] \quad \text{and} \quad [0, L] = I_i^n.$$

However, the application is not entirely obvious, since we do not know if $2\phi_{t_{i-1}^n}^\lambda \in [a_\delta, b_\delta]$.

The point is, we do know that $2\phi_t^\lambda \in [a, b]$ for all t (cf. Lemma 4.1), and the previous proof of Lemma 4.4 remains valid under the requirement (5.1). To elaborate this, we only consider the case when $2\phi_{t_{i-1}^n}^\lambda > \beta_i^n + \varepsilon$, as the other scenario is similar. We set

$$B_i^n \triangleq \{t \in I_i^n : \alpha_t \in [\alpha_i^n, \beta_i^n]\}.$$

In the same way as in that proof, we have

$$\frac{d}{dt}(2\phi_t^\lambda) \leq -2\lambda \sin(2\phi_t^\lambda - \alpha_t) \mathbf{1}_{B_i^n} + 2\lambda \mathbf{1}_{(B_i^n)^c}$$

for a.a. $t \in [t_{i-1}^n, t_i^n]$. Note that $2\phi_t^\lambda \in [\beta_i^n + \varepsilon, b]$ on $[t_{i-1}^n, t_i^n]$. Therefore, we have

$$\varepsilon \leq 2\phi_t^\lambda - \alpha_t \leq b - \alpha_i^n \leq b - a_\delta \quad \text{on } B_i^n \cap [t_{i-1}^n, t_i^n].$$

By using the choice (5.1) of ε , we see that

$$\sin(2\phi_t^\lambda - \alpha_t) \geq \sin \varepsilon \quad \text{on } B_i^n \cap [t_{i-1}^n, t_i^n].$$

The rest of the argument is identical to the proof of Lemma 4.4, yielding the estimate

$$2\lambda \sigma_i^n \sin \varepsilon \leq 2\phi_{i-1}^\lambda - 2\phi_{\tau_i}^\lambda + 2\lambda(1 + \sin \varepsilon) \mu((B_i^n)^c \cap I_i^n).$$

Since $F_2 \cap I_i^n \subseteq B_i^n$, we have

$$\begin{aligned} 2\lambda \sigma_i^n \sin \varepsilon &\leq 2\phi_{i-1}^\lambda - 2\phi_{\tau_i}^\lambda + 2\lambda(1 + \sin \varepsilon) \mu((F_1 \setminus F_2) \cap I_i^n) \\ &\leq b - a + 2\lambda(1 + \sin \varepsilon) \mu((F_1 \setminus F_2) \cap I_i^n) \\ &= \pi + 2\lambda(1 + \sin \varepsilon) \mu((F_1 \setminus F_2) \cap I_i^n). \end{aligned}$$

Rearranging the inequality gives the desired estimate. ■

5.1.3. Step three: the global estimate. According to the intermediate lower estimate given by Lemma 3.8, our task is to estimate the integral

$$I_\lambda \triangleq \int_0^L \cos(2\phi_t^\lambda - \alpha_t) dt$$

from below when λ is large. For this purpose, we first write

$$\begin{aligned} I_\lambda &= \int_{F_2^c} \cos(2\phi_t^\lambda - \alpha_t) dt + \int_{F_2} \cos(2\phi_t^\lambda - \alpha_t) dt \\ &\geq -\mu(F_1^c) - \mu(F_1 \setminus F_2) + \sum_{i=1}^n \int_{F_2 \cap I_i^n} \cos(2\phi_t^\lambda - \alpha_t) dt \\ &\geq -\delta - \eta + \sum_{i=1}^n \int_{F_2 \cap I_i^n} \cos(2\phi_t^\lambda - \alpha_t) dt. \end{aligned}$$

To analyse the summation on the right-hand side, we decompose it as

$$\sum_{i=1}^n \int_{F_2 \cap I_i^n} \cos(2\phi_t^\lambda - \alpha_t) dt = J_n + K_n,$$

where

$$J_n \triangleq \sum_{i=1}^n \int_{F_2 \cap [t_{i-1}^n, \tau_i^n]} \cos(2\phi_t^\lambda - \alpha_t) dt \quad \text{and} \quad K_n \triangleq \sum_{i=1}^n \int_{F_2 \cap [\tau_i^n, t_i^n]} \cos(2\phi_t^\lambda - \alpha_t) dt,$$

respectively.

For the term J_n , according to Lemma 5.3 and the choice (5.2) of n , we have

$$\begin{aligned} J_n &\geq -\sum_{i=1}^n \mu(F_2 \cap [t_{i-1}^n, \tau_i^n]) \geq -\sum_{i=1}^n \sigma_i^n \\ (5.6) \quad &\geq -\frac{\pi n}{2\lambda \sin \varepsilon} - \frac{1 + \sin \varepsilon}{\sin \varepsilon} \sum_{i=1}^n \mu((F_1 \setminus F_2) \cap I_i^n) \geq -\frac{\pi \varepsilon}{2} - \frac{(1 + \sin \varepsilon)\eta}{\sin \varepsilon}. \end{aligned}$$

For the term K_n , we introduce an extra independent parameter $M > 0$. Define \mathcal{B}_n to be the collection of those i 's such that

$$\mu((F_1 \setminus F_2) \cap I_i^n) > \frac{M}{n} \eta,$$

and set $\mathcal{G}_n \triangleq \mathcal{B}_n^c$. Then we have

$$\eta > \mu(F_1 \setminus F_2) = \sum_{i=1}^n \mu(((F_1 \setminus F_2)) \cap I_i^n) \geq |\mathcal{B}_n| \times \frac{M\eta}{n},$$

where $|\mathcal{B}_n|$ denotes the number of elements in \mathcal{B}_n . In particular, $|\mathcal{B}_n| \leq n/M$. It follows that

$$\begin{aligned} K_n &= \left(\sum_{i \in \mathcal{B}_n} + \sum_{i \in \mathcal{G}_n} \right) \int_{F_2 \cap [\tau_i^n, t_i^n]} \cos(2\phi_t^\lambda - \alpha_t) dt \\ &\geq - \sum_{i \in \mathcal{B}_n} \mu(F_2 \cap [\tau_i^n, t_i^n]) + \sum_{i \in \mathcal{G}_n} \int_{F_2 \cap [\tau_i^n, t_i^n]} \cos(2\phi_t^\lambda - \alpha_t) dt \\ &\geq -\frac{L}{n} \times |\mathcal{B}_n| + \sum_{i \in \mathcal{G}_n} \int_{F_2 \cap [\tau_i^n, t_i^n]} \cos(2\phi_t^\lambda - \alpha_t) dt \\ (5.7) \quad &\geq -\frac{L}{M} + \sum_{i \in \mathcal{G}_n} \int_{F_2 \cap [\tau_i^n, t_i^n]} \cos(2\phi_t^\lambda - \alpha_t) dt. \end{aligned}$$

We now estimate the last term on the right-hand side of (5.7). The point is to apply Lemma 4.2 to the context where

$$[0, L] = [\tau_i^n, t_i^n], \quad [a, b] = [\alpha_i^n - \varepsilon, \beta_i^n + \varepsilon],$$

and

$$r = r_i^n \triangleq 2\lambda \cdot \mu(\{t : \alpha_t \notin [\alpha_i^n - \varepsilon, \beta_i^n + \varepsilon]\} \cap [\tau_i^n, t_i^n]).$$

As one assumption in the lemma, we already have $2\phi_{\tau_i^n}^\lambda \in [\alpha_i^n - \varepsilon, \beta_i^n + \varepsilon]$. We must also verify the other standing assumption that

$$(5.8) \quad (\beta_i^n + \varepsilon) - (\alpha_i^n - \varepsilon) + r_i^n < \pi.$$

To this end, first note that

$$r_i^n \leq 2\lambda \mu((F_1 \setminus F_2) \cap I_i^n).$$

Furthermore, for those $i \in \mathcal{G}_n$, we have

$$\mu((F_1 \setminus F_2) \cap I_i^n) \leq \frac{M\eta}{n}.$$

As a result,

$$r_i^n \leq \frac{2\lambda M\eta}{n} = \frac{2M\eta}{\varepsilon \sin \varepsilon}.$$

It follows from (5.5) that

$$(\beta_i^n + \varepsilon) - (\alpha_i^n - \varepsilon) + r_i^n < 3\varepsilon + \frac{2M\eta}{\varepsilon \sin \varepsilon}$$

for each $i \in \mathcal{G}_n$. For fixed ε and M , when η is small we can ensure that the condition (5.8) is met. We emphasise that such a requirement is legal in view of the limiting order (5.4) that will be implemented eventually. Now we can apply Lemma 4.2 to conclude that

$$2\phi_t^\lambda \in [\alpha_i^n - \varepsilon - r_i^n, \beta_i^n + \varepsilon + r_i^n], \quad \forall t \in [\tau_i^n, t_i^n].$$

Consequently, for each $i \in \mathcal{G}_n$ and $t \in F_2 \cap [\tau_i^n, t_i^n]$, we have

$$(5.9) \quad |2\phi_t^\lambda - \alpha_t| \leq \beta_i^n - \alpha_i^n + \varepsilon + r_i^n < 2\varepsilon + \frac{2M\eta}{\varepsilon \sin \varepsilon}.$$

For fixed ε and M , we further require η to be small enough so that $2\varepsilon + \frac{2M\eta}{\varepsilon \sin \varepsilon} < \frac{\pi}{2}$. As a consequence, we obtain

$$\begin{aligned} \sum_{i \in \mathcal{G}_n} \int_{F_2 \cap [\tau_i^n, t_i^n]} \cos(2\phi_t^\lambda - \alpha_t) dt &\geq \cos\left(2\varepsilon + \frac{2M\eta}{\varepsilon \sin \varepsilon}\right) \cdot \sum_{i \in \mathcal{G}_n} \mu(F_2 \cap [\tau_i^n, t_i^n]) \\ &= \cos\left(2\varepsilon + \frac{2M\eta}{\varepsilon \sin \varepsilon}\right) \cdot \sum_{i \in \mathcal{G}_n} (\mu(F_2 \cap I_i^n) - \mu(F_2 \cap [t_{i-1}^n, \tau_i^n])) \\ &= \cos\left(2\varepsilon + \frac{2M\eta}{\varepsilon \sin \varepsilon}\right) \cdot \mu(F_2) - \cos\left(2\varepsilon + \frac{2M\eta}{\varepsilon \sin \varepsilon}\right) \cdot \sum_{i \in \mathcal{B}_n} \mu(F_2 \cap I_i^n) \\ &\quad - \cos\left(2\varepsilon + \frac{2M\eta}{\varepsilon \sin \varepsilon}\right) \cdot \sum_{i=1}^n \mu(F_2 \cap [t_{i-1}^n, \tau_i^n]) \\ &\geq \cos\left(2\varepsilon + \frac{2M\eta}{\varepsilon \sin \varepsilon}\right) \cdot (L - \delta - \eta) - \frac{L}{M} \cos\left(2\varepsilon + \frac{2M\eta}{\varepsilon \sin \varepsilon}\right) \\ &\quad - \cos\left(2\varepsilon + \frac{2M\eta}{\varepsilon \sin \varepsilon}\right) \cdot \left(\frac{\pi\varepsilon}{2} + \frac{(1 + \sin \varepsilon)\eta}{\sin \varepsilon}\right). \end{aligned}$$

To reach the second term in the last inequality, we have used the fact that

$$(5.10) \quad \sum_{i \in \mathcal{B}_n} \mu(F_2 \cap I_i^n) \leq \frac{L}{n} \times |\mathcal{B}_n| \leq \frac{L}{M},$$

and to reach the third term we have used the estimate (5.6).

Gathering all the above estimates we have obtained so far, we arrive at

$$(5.11) \quad \begin{aligned} I_\lambda &\geq -\delta - \eta - \frac{\pi\varepsilon}{2} - \frac{(1 + \sin \varepsilon)\eta}{\sin \varepsilon} - \frac{L}{M} + \cos\left(2\varepsilon + \frac{2M\eta}{\varepsilon \sin \varepsilon}\right) \cdot (L - \delta - \eta) \\ &\quad - \frac{L}{M} \cos\left(2\varepsilon + \frac{2M\eta}{\varepsilon \sin \varepsilon}\right) - \cos\left(2\varepsilon + \frac{2M\eta}{\varepsilon \sin \varepsilon}\right) \cdot \left(\frac{\pi\varepsilon}{2} + \frac{(1 + \sin \varepsilon)\eta}{\sin \varepsilon}\right). \end{aligned}$$

The proof of Theorem 5.1 is thus completed by passing to the limit in the order specified by (5.4).

The following estimate is a direct consequence of the above argument. It plays an essential role for proving Theorem 2.6 in the more general context in the next subsection. We continue to use the same notation as before and to make all the standing requirements for the parameters δ , η , ε and M . However, we do not take limit for these parameters.

Corollary 5.4. *There exists $\Lambda = \Lambda(\delta, \eta, \varepsilon, M)$ such that, whenever $\lambda > \Lambda$ and $s \in [0, L]$ satisfies $2\phi_s^\lambda \in (a, b)$, we have*

$$(5.12) \quad \mu\left(\left\{t \in F_2 \cap [s, L] : |2\phi_t^\lambda - \alpha_t| > 2\varepsilon + \frac{2M\eta}{\varepsilon \sin \varepsilon}\right\}\right) \leq \frac{\pi\varepsilon}{2} + \frac{1 + \sin \varepsilon}{\sin \varepsilon} \eta + \frac{L}{M}.$$

Proof. The number Λ is chosen so that for any $\lambda > \Lambda$, we have $\beta_i^n - \alpha_i^n < \varepsilon$ (see the discussion in Section 5.1.1 leading to the property (5.5)). Recall that ε depends on δ , and that η is small depending on ε and M . Hence Λ depends on all these parameters. Suppose that $2\phi_s^\lambda \in (a, b)$. We treat $2\phi_s^\lambda$ as the initial condition and restrict the previous analysis to the interval $[s, L]$. The argument leading to (5.9) implies that

$$\left\{t \in F_2 \cap [s, L] : |2\phi_t^\lambda - \alpha_t| > 2\varepsilon + \frac{2M\eta}{\varepsilon \sin \varepsilon}\right\} \subseteq \left(\bigcup_{i \in \mathcal{B}_n} (F_2 \cap I_i^n)\right) \cup \left(\bigcup_{i \in \mathcal{B}_n} [t_{i-1}^n, \tau_i^n]\right).$$

The inequality (5.12) then follows from (5.10) and (5.6). ■

5.2. Proof of Theorem 2.6: the local case

We now proceed to develop the proof of Theorem 2.6 in general. Let us suppose that $\gamma: [0, L] \rightarrow \mathbb{R}^2$ is strongly tree-reduced in the sense of Definition 2.5. Our strategy is to cover the path by small intervals, so that on each local interval the corresponding estimate (5.12) holds. There is a key missing ingredient in order to patch the estimates (5.12) over different intervals. We must make sure that the initial condition for $2\phi_s^\lambda$ on each of the covering intervals falls in an appropriate region (a, b) to trigger the relevant estimate. Obtaining such consistency property is non-trivial, since the initial condition for the current covering is the terminal condition for the previous covering.

To reduce technical considerations, let us first make one simplification by assuming that, in Definition 2.5 the open interval $(0, L)$ is replaced by the closed interval $[0, L]$. Namely, we assume that for each $t \in [0, L]$, there exists a neighbourhood (u_t, v_t) of t such that $\gamma|_{[u_t, v_t] \cap [0, L]}$ is a regular cusp. This simplification allows us to make use of compactness and finite covers. At the end of this subsection, we discuss how to remove this restriction (cf. Section 5.2.4).

5.2.1. Step one: a covering lemma. To make the intuition clearer, it is important to choose a nice covering of $[0, L]$ out of the above assumption. This is the content of the following lemma.

Lemma 5.5. *There exist points u_1, \dots, u_{k-1} and v_1, \dots, v_k , such that*

- (i) $[0, L] = [v_0, v_1] \cup [v_1, v_2] \cup \dots \cup [v_{k-1}, v_k]$, where $v_0 \triangleq 0$;
- (ii) $u_i \in (v_{i-1}, v_i)$ for each $1 \leq i \leq k-1$;
- (iii) $\alpha|_{[u_{i-1}, v_i]}$ is a regular cusp for each $1 \leq i \leq k$, where $u_0 \triangleq 0$.

Proof. By compactness, we can find a finite family $\mathcal{A} = \{I_i : i = 1, \dots, l\}$ of distinct intervals that cover $[0, L]$, where each interval I_i is relatively open in $[0, L]$ and $\alpha|_{\overline{I_i} \cap [0, L]}$ is a regular cusp for each i . The point $t = 0$ is covered by some member in \mathcal{A} , say $I_1 = (0, v_1)$. If $v_1 \geq L$, we set $v_1 \triangleq L$ and we are done. Otherwise, the point $t = v_1$ is covered by some member in $\mathcal{A} \setminus \{I_1\}$, say $I_2 = (u'_1, v_2)$. If $v_2 \geq L$, we are done by setting $v_2 \triangleq L$ and choosing any point $u_1 \in (u'_1, v_1)$. If $v_2 < L$, we continue the process. Inductively, v_i is covered by some member in $\mathcal{A} \setminus \{I_1, \dots, I_i\}$, say $I_{i+1} = (u'_i, v_{i+1})$. We choose $u_i \in (\max\{u'_i, v_{i-1}\}, v_i)$ and proceed further. The process terminates after finitely many steps since \mathcal{A} is finite. ■

The figure below illustrates the covering specified by Lemma 5.5 when $k = 4$.

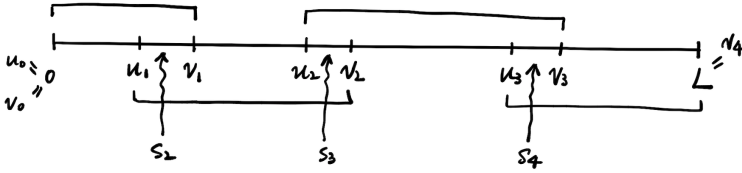


Figure 2. The covering structure when $k = 4$.

In what follows, we always work with a fixed covering structure given by Lemma 5.5.

5.2.2. Step two: consistency of initial conditions. From Lemma 5.5, we know that $\alpha|_{[u_{i-1}, v_i]}$ is a regular cusp. In particular, we know by assumption that

$$\alpha_t \in [a_i, a_i + \pi] \quad \text{for a.a. } t \in [u_{i-1}, v_i]$$

with some $a_i \in \mathbb{R}$. The main issue here is that, we cannot directly apply the results from Section 5.1, since we do not know whether $2\phi_{u_{i-1}}^\lambda \in (a_i, a_i + \pi)$. Such a requirement on the initial condition is critical in the previous argument. Nonetheless, the following lemma tells us that we can find $s_i \in [u_{i-1}, v_{i-1}]$ (which may depend on λ) such that $2\phi_{s_i}^\lambda \in (a_i, a_i + \pi)$. As a result, s_i can be treated as the initial time over the portion of $[s_i, v_i]$.

Lemma 5.6. *There exists $\Lambda > 0$ such that, for any $\lambda > \Lambda$ and $1 \leq i \leq k - 1$, we can always find $s_i \in [u_{i-1}, v_{i-1}]$ satisfying*

$$2\phi_{s_i}^\lambda \in (a_i, a_i + \pi).$$

Proof. We continue with the notation in Section 5.1 but applied to the context of $\alpha|_{[u_{i-1}, v_i]}$ for each i . Recall that $\delta, \varepsilon, \eta$ and M are given parameters. We are then able to define two compact subsets $F_i \supseteq F'_i$ of $[u_{i-1}, v_i]$ playing the roles of F_1 and F_2 , and two numbers $a'_i > a_i, b'_i < b_i \triangleq a_i + \pi$ playing the roles of a_δ and b_δ in that section. We further require that these parameters satisfy the following constraints:

$$(5.13) \quad 2\varepsilon + \frac{2\eta M}{\varepsilon \sin \varepsilon} < \min_{1 \leq i \leq k} \min\{a'_i - a_i, b_i - b'_i\},$$

and

$$(5.14) \quad \left(\frac{\pi \varepsilon}{2} + \frac{1 + \sin \varepsilon}{\sin \varepsilon} \eta + \frac{L}{M} \right) + 2(\delta + \eta) < \min_{1 \leq i \leq k-1} (v_i - u_i).$$

More quantitatively, we first choose δ and ε to be small, then M to be large, and finally η to be small. This is consistent with the limiting order (5.4). Now we are in a position to apply the quantitative estimate given by Corollary 5.4 to each portion $\alpha|_{[u_{i-1}, v_i]}$. Note that we do not take limits for the parameters δ , ε , η and M here. The constant Λ appearing in Corollary 5.4 depends on these parameters as well as on the fixed covering structure given by the $[u_{i-1}, v_i]$'s.

For any given $\lambda > \Lambda$, we are going to choose $s_i \in [u_{i-1}, v_{i-1})$ inductively on i , such that $2\phi_{s_i}^\lambda \in (a_i, b_i)$. We start by fixing $2\phi_0^\lambda \in (a_1, b_1)$ and choosing $s_1 \triangleq 0$. Suppose that $s_i \in [u_{i-1}, v_{i-1})$ is already selected with the desired property, and we want to define s_{i+1} properly. According to Corollary 5.4 and the requirement (5.14), we have

$$\mu \left(\left\{ t \in F'_i \cap [s_i, v_i] : |2\phi_t^\lambda - \alpha_t| > 2\varepsilon + \frac{2\eta M}{\varepsilon \sin \varepsilon} \right\} \right) < v_i - u_i - 2\delta - 2\eta.$$

Since $\mu((F'_i)^c) < \delta + \eta$ from Section 5.1, it follows that

$$\begin{aligned} & \mu \left(\left\{ t \in F'_i \cap [s_i, v_i] : |2\phi_t^\lambda - \alpha_t| \leq 2\varepsilon + \frac{2\eta M}{\varepsilon \sin \varepsilon} \right\} \right) \\ & > \mu(F'_i \cap [s_i, v_i]) - (v_i - u_i - 2\delta - 2\eta) \\ & \geq (v_i - s_i) - (\delta + \eta) - (v_i - u_i - 2\delta - 2\eta) = u_i - s_i + \delta + \eta. \end{aligned}$$

As a result, we have

$$\mu(C_i \cap [u_i, v_i]) > \delta + \eta,$$

where

$$C_i \triangleq \left\{ t \in [s_i, v_i] : |2\phi_t^\lambda - \alpha_t| \leq 2\varepsilon + \frac{2\eta M}{\varepsilon \sin \varepsilon} \right\}.$$

Since $\mu((F'_{i+1})^c) < \delta + \eta$, we conclude that

$$C_i \cap F'_{i+1} \cap [u_i, v_i] \neq \emptyset.$$

Pick any point in the above set and define it as s_{i+1} . Note that from Section 5.1 we also have $\alpha_{s_{i+1}} \in [a'_{i+1}, b'_{i+1})$ (since $s_{i+1} \in F'_{i+1}$). Therefore, the requirement (5.13) further implies that $2\phi_{s_{i+1}}^\lambda \in (a_{i+1}, b_{i+1})$. This gives the desired construction of s_{i+1} . ■

5.2.3. Step three: patching up the estimates. We now proceed to establish the global lower estimate. We continue to work in the previous set-up. For each given $\lambda > \Lambda$, the previous choice of s_i allows us to apply the estimate (5.12) to $\alpha|_{[s_i, v_i]}$. In particular, for each i we have

$$\mu \left(\left\{ t \in F'_i \cap [s_i, v_i] : |2\phi_t^\lambda - \alpha_t| > 2\varepsilon + \frac{2\eta M}{\varepsilon \sin \varepsilon} \right\} \right) \leq \frac{\pi \varepsilon}{2} + \frac{1 + \sin \varepsilon}{\sin \varepsilon} \eta + \frac{L}{M}.$$

Let us define

$$D \triangleq \left\{ t \in [0, L] : |2\phi_t^\lambda - \alpha_t| > 2\varepsilon + \frac{2\eta M}{\varepsilon \sin \varepsilon} \right\}.$$

Then we have

$$(5.15) \quad \begin{aligned} \mu(D) &\leq \sum_{i=1}^k \mu\left(\left\{t \in [s_i, v_i] : |2\phi_t^\lambda - \alpha_t| > 2\varepsilon + \frac{2\eta M}{\varepsilon \sin \varepsilon}\right\}\right) \\ &\leq k\left(\frac{\pi\varepsilon}{2} + \frac{1 + \sin \varepsilon}{\sin \varepsilon}\eta + \frac{2L}{M} + \eta + \delta\right), \end{aligned}$$

and

$$\begin{aligned} \int_0^L \cos(\alpha_t - 2\phi_t^\lambda) dt &\geq -\mu(D) + \cos\left(2\varepsilon + \frac{2\eta M}{\varepsilon \sin \varepsilon}\right) \mu(D^c) \\ &\geq L \cos\left(2\varepsilon + \frac{2\eta M}{\varepsilon \sin \varepsilon}\right) - \left(1 + \cos\left(2\varepsilon + \frac{2\eta M}{\varepsilon \sin \varepsilon}\right)\right) \mu(D). \end{aligned}$$

By substituting the estimate (5.15) and taking limit in the order (5.4), we conclude that

$$\overline{\lim}_{\lambda \rightarrow \infty} \int_0^L \cos(\alpha_t - 2\phi_t^\lambda) dt \geq L.$$

5.2.4. Step four: removing the assumption at the end points. Finally, we come back to relax the requirement on the endpoints $t = 0, L$. More precisely, we now assume that for each $t \in (0, L)$ (not including the endpoints), there is a neighbourhood (u_t, v_t) of t on which γ is a regular cusp. Having all the previous analysis at hand, dealing with this situation only requires minor technical effort.

To elaborate this, let $\kappa > 0$ be a given number. Then we can write

$$\int_0^L \cos(\alpha_t - 2\phi_t^\lambda) dt \geq -2\kappa + \int_\kappa^{L-\kappa} \cos(\alpha_t - 2\phi_t^\lambda) dt.$$

On the other hand, we know that $\gamma|_{[\kappa, L-\kappa]}$ satisfies Definition 2.5 up to the end points κ and $L - \kappa$. In order to apply the previous results to $\gamma|_{[\kappa, L-\kappa]}$, the only requirement is a suitable initial condition for $2\phi_\kappa^\lambda$. But we know that (cf. Lemma 3.4)

$$\rho_\kappa^\lambda e^{i\phi_\kappa^\lambda} = w_\kappa^\lambda = \Gamma_{L-\kappa, L}^\lambda \xi^\lambda.$$

Since $\Gamma_{L-\kappa, L}^\lambda$ is invertible, by choosing ξ^λ properly we can certainly guarantee that $2\phi_\kappa^\lambda$ satisfies a desired condition (i.e., $2\phi_\kappa^\lambda \in (a_1, a_1 + \pi)$) using the notation from the previous discussion). As a result, we conclude that

$$\overline{\lim}_{\lambda \rightarrow \infty} \int_0^L \cos(\alpha_t - 2\phi_t^\lambda) dt \geq -2\kappa + (L - 2\kappa) = L - 4\kappa.$$

By letting $\kappa \rightarrow 0^+$, we obtain the desired estimate.

Up to this point, the proof of Theorem 2.6 is complete.

6. An extension: singular cusps

We have mentioned at the beginning that there is another type of cusps which is more singular in terms of detecting the tree-reduced property and is thus harder to deal with. In this section, we discuss how the previous analysis, when combined with a suitable comparison lemma, can be adapted to treat this more singular case. For the sake of conciseness and for conveying the essential idea better, we only consider a typical example instead of trying to write down an abstract condition capturing such type of cusps. We remark at the end of this section on how the argument can be adapted to a more general situation.

We first illustrate this singular type of cusps in the figure below.

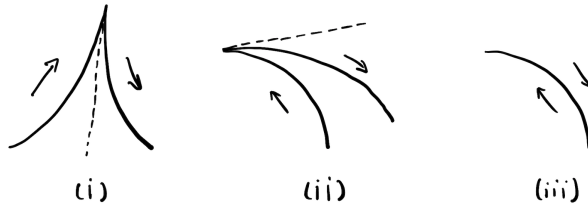


Figure 3. Regular cusp, singular cusp and tree-like cusp.

The leftmost path represents a typical regular cusp in the sense of Definition 2.3. The middle path represents the singular cusp that we are considering here. The rightmost path represents a tree-like cusp that has trivial signature. Note that these paths are all local, i.e., we are only zooming in the part near the cusp singularity.

It is not hard to describe why these two types of cusps are different in terms of the “degree of tree-reducedness”. For the moment let us assume that the paths are C^2 near the cusp singularity point. For the regular cusp in Figure 3 (i), one detects its tree-reducedness directly from the fact that the second derivative of the path (more precisely, the first derivative of the angular path β_t) does not change sign when passing through the singularity. This also accounts for the property that β_t takes values in an interval of length strictly less than π after removing an arbitrarily small neighbourhood of the singularity (cf. Definition 2.3). However, for the singular cusp in Figure 3 (ii), one *cannot* detect whether it is tree-reduced or not by examining the sign of the second derivative. Instead, the way to distinguish it from the tree-like cusp in Figure 3 (iii) is through looking at the precise magnitude of the second derivative on both sides of the singularity. To put it in another way, it is the speed of change for β_t rather than the direction of change that distinguishes it from being tree-like. Such information is finer than what is contained in Definition 2.3. As a result, dealing with this case requires more delicate analysis.

In what follows, we consider one typical example of singular cusps. To be more convenient for using the equation (3.4), we directly specify α_t instead of β_t (recall that they are related by $\alpha_t = \beta_{L-t}$). Let $L, r > 0$ and $a \in \mathbb{R}$ be given fixed. Consider a given *continuous, strictly increasing* function

$$\theta : [0, L/2] \rightarrow [a - r, a]$$

with $\theta_0 = a - r$ and $\theta_{L/2} = a$. The simplest example of θ is the linear function. We define the angular path $\alpha: [0, L] \rightarrow \mathbb{R}$ by

$$(6.1) \quad \alpha_t \triangleq \begin{cases} \theta_t, & t \in [0, L/2), \\ c \cdot (\theta_{L-t} - a) + (a - \pi), & t \in (L/2, L], \end{cases}$$

where $c > 0$ is a given fixed number. Note that $\alpha_{L/2-} = a$ and $\alpha_{L/2+} = a - \pi$. The resulting path γ_t is given by the equation (2.3) where $\beta_t \triangleq \alpha_{L-t}$.

When $c = 1$, γ is tree-like. When $c \neq 1$, γ has the shape of Figure 3 (ii). By considering the reversal path if necessary, we may assume without loss of generality that $0 < c < 1$. In addition, since only the local behaviour of γ near the singularity is relevant, we may also assume that r is small, say $r < \pi/4$.

Our main result of this section is the following.

Theorem 6.1. *The signature asymptotics formula holds for the path γ defined as above.*

To prove this result, we first state a lemma which also directly solves the case when $\alpha: [0, L] \rightarrow \mathbb{R}$ is a continuous function (i.e., the \mathcal{C}^1 -case). We defer its proof to the appendix so as not to distract the reader from the main discussion. In the \mathcal{C}^1 -case, Lyons and Xu [16] also had a similar result for the equations of the hyperbolic development.

Lemma 6.2. *Let $\alpha: [0, L] \rightarrow \mathbb{R}$ be a continuous function. Recall that the function ϕ_t^λ is defined by the angular equation (3.6). Suppose that there exists $\kappa \in (0, \pi)$ such that*

$$|2\phi_0^\lambda - \alpha_0| \leq \kappa \quad \text{for all } \lambda > 0.$$

Then for any $t_0 > 0$, $2\phi_t^\lambda$ converges uniformly to α_t on $[t_0, L]$ as $\lambda \rightarrow \infty$. In addition, if $2\phi_0^\lambda = \alpha_0$ for all λ , then the uniform convergence holds on $[0, L]$.

The key point for proving Theorem 6.1 is to compare the angle dynamics $2\phi_t^\lambda$ associated with the cusp path α_t to the one corresponding to the tree-like case. Let us begin with the trivial observation that the angular equations (3.6) on $[0, L/2]$ for the two cases ($c \neq 1$ vs $c = 1$) are identical. In addition, if we take $2\phi_0^\lambda = \alpha_0$, according to Lemma 6.2 and Lemma 4.1, we have

$$2\phi_t^\lambda \in R_1 \triangleq [a - r, a], \quad \forall t \in [0, L/2],$$

and $\|2\phi_t^\lambda - \alpha_t\|_{\infty; [0, L/2]}$ can be made arbitrarily small when λ is large.

To understand the portion of $[L/2, L]$, we first look at the tree-like situation $c = 1$. Let us use ψ_t^λ to denote the corresponding solution to the angular equation (3.6) on $[L/2, L]$ for this case. By the definition (6.1) with $c = 1$, we know that, for all $t \in [L/2, L]$,

$$(6.2) \quad \frac{d}{dt} \psi_t^\lambda = -\lambda \sin(2\psi_t^\lambda - (\theta(L-t) - \pi)) = \lambda \sin(2\psi_t^\lambda - \theta(L-t)).$$

As a result of uniqueness, we have

$$\psi_t^\lambda = \psi_{L-t}^\lambda = \phi_{L-t}^\lambda, \quad \forall t \in [L/2, L].$$

In other words, on the second half $[L/2, L]$, the path ψ_t^λ is just the reversal of the first half $[0, L/2]$.

Now we return to the cusp situation with $0 < c < 1$ given fixed. The angular equation (3.6) can be rewritten as

$$(6.3) \quad \frac{d\phi_t^\lambda}{dt} = \lambda \sin(2\phi_t^\lambda - \theta_{L-t} - \varepsilon \cdot (a - \theta_{L-t})), \quad \forall t \in [L/2, L],$$

where $\varepsilon \triangleq 1 - c$. Recall that

$$\psi_t^\lambda \triangleq \phi_{L-t}^\lambda \quad (t \in [L/2, L])$$

gives the angular solution in the tree-like case on $[L/2, L]$.

The following comparison lemma is the key step towards the proof of Theorem 6.1.

Lemma 6.3. *For each $\lambda > 0$, we have*

$$\phi_t^\lambda \leq \psi_t^\lambda, \quad \forall t \in [L/2, L].$$

Proof. Let

$$\zeta_t \triangleq \psi_t^\lambda - \phi_t^\lambda, \quad t \in [L/2, L].$$

Then $\zeta_{L/2} = 0$, and using the equations (6.2), (6.3) for the two functions, we see that ζ_t satisfies the equation

$$\frac{d\zeta_t}{dt} = 2\lambda \cos \frac{(2\psi_t^\lambda - \theta_{L-t}) + (2\phi_t^\lambda - \theta_{L-t}) - \varepsilon(a - \theta_{L-t})}{2} \sin \left(\zeta_t + \frac{\varepsilon(a - \theta_{L-t})}{2} \right).$$

Firstly, we claim that $\zeta_t \geq 0$ when t is small. To this end, we set

$$\eta_t \triangleq \varepsilon(a - \theta_{L-t})/2.$$

By the assumption on θ_t , the function η_t is non-negative, and strictly increases from 0 to $\varepsilon r/2$. Therefore, we have

$$|\zeta_t| \leq 2\lambda \int_{L/2}^t |\sin(\zeta_s + \eta_s)| ds \leq 2\lambda \int_{L/2}^t |\zeta_s| ds + 2\lambda \left(t - \frac{L}{2}\right) \eta_t.$$

It follows from Grönwall's inequality that

$$|\zeta_t| \leq 2\lambda \left(t - \frac{L}{2}\right) \eta_t \times e^{2\lambda(t-L/2)}, \quad t \in [L/2, L].$$

As a consequence, we have

$$\zeta_t + \eta_t \geq \left(1 - 2\lambda \left(t - \frac{L}{2}\right) e^{2\lambda(t-L/2)}\right) \eta_t.$$

In particular, there exists $\tau_1 \in (L/2, L)$ (depending on λ) such that

$$\zeta_t + \eta_t \geq \frac{1}{2} \eta_t \geq 0, \quad \forall t \in [L/2, \tau_1].$$

We also require that $\zeta_t + \eta_t \leq \pi$ by shrinking τ_1 if necessary. As a result, we have

$$\sin(\zeta_t + \eta_t) \geq 0, \quad \forall t \in [L/2, \tau_1].$$

On the other hand, observe that

$$|2\psi_t^\lambda - \theta_{L-t}| \leq r$$

since they both stay in the region R_1 . By the relation $\phi_{L/2}^\lambda = \psi_{L/2}^\lambda$ and the continuity of ϕ_t^λ at $t = L/2$, there exists $\tau_2 \in (L/2, L)$ (depending on λ) satisfying

$$\cos \frac{(2\psi_t^\lambda - \theta_{L-t}) + (2\phi_t^\lambda - \theta_{L-t}) - \varepsilon(a - \theta_{L-t})}{2} \geq \cos \frac{(3 + \varepsilon)r}{2} =: \kappa_r > 0$$

for $t \in [L/2, \tau_2]$ (recall we have presumed that $c \in (0, 1)$ and $r < \pi/4$). By taking

$$\tau \triangleq \tau_1 \wedge \tau_2,$$

we obtain

$$\zeta_t \geq 2\lambda \kappa_r \int_{L/2}^t \sin(\zeta_s + \eta_s) ds \geq 0, \quad \forall t \in [L/2, \tau].$$

Next, we claim that $\zeta_t \geq 0$ for all $t \in [L/2, L]$. Suppose on the contrary that, there is some t such that $\zeta_t < 0$. This t must be in the interval $(\tau, L]$. A standard argument allows us to find $t_1, t_2 \in [\tau, L]$ such that $\zeta_{t_1} = 0$ and $\zeta_t < 0$ for $t \in (t_1, t_2]$. However, since η_t is strictly increasing, we know that $\eta_{t_1} \in (0, \varepsilon r/2)$. As a result,

$$\zeta'_{t_1} = 2\lambda \cos \left(2\psi_{t_1}^\lambda - \theta_{L-t_1} - \frac{\varepsilon(a - \theta_{L-t_1})}{2} \right) \sin \eta_{t_1} \geq 2\lambda \cos \left(\frac{(2 + \varepsilon)r}{2} \right) \sin \eta_{t_1} > 0,$$

which is clearly a contradiction. Therefore, $\zeta_t \geq 0$ on $[L/2, L]$. \blacksquare

Now we are in a position to give the proof of Theorem 6.1. The idea is that, at any fixed time $t^* > L/2$, when λ is large, $2\phi_{t^*}^\lambda$ gets pushed into the interval

$$R_2 \triangleq [a - \pi - cr, a - \pi]$$

(the range of α on $[L/2, L]$). As a consequence, we can then apply results obtained in Section 5 to the portion of $[t^*, L]$. The theorem then follows as $t^* - L/2$ can be made arbitrarily small.

Proof of Theorem 6.1. Let $t_1 < t_2$ be two fixed times in $(L/2, L)$. We claim that there exists $\Lambda > 0$ such that

$$(6.4) \quad 2\phi_{t_2}^\lambda \in R_2, \quad \forall \lambda > \Lambda.$$

If this is true, the argument developed in Section 5 applied to the portion of $[t_2, L]$ implies that

$$\begin{aligned} \overline{\lim}_{\lambda \rightarrow \infty} \int_0^L \cos(\alpha_t - 2\phi_t^\lambda) dt &= \overline{\lim}_{\lambda \rightarrow \infty} \left(\int_0^{L/2} + \int_{L/2}^{t_2} + \int_{t_2}^L \right) \cos(\alpha_t - 2\phi_t^\lambda) dt \\ &\geq \frac{L}{2} + (L - t_2) - \left(\frac{L}{2} - t_2 \right). \end{aligned}$$

The desired lower bound follows by letting $t_2 \rightarrow L/2$.

The proof of the claim (6.4) contains the following three observations.

(i) From Lemma 6.3, we know that

$$2\phi_{t_1}^\lambda \leq 2\psi_{t_1}^\lambda = 2\phi_{L-t_1}^\lambda.$$

(ii) According to Lemma 6.2, $2\psi_{t_1}^\lambda$ and θ_{L-t_1} can be made arbitrarily close when λ is large.

(iii) The distance between α_{t_1} and θ_{L-t_1} is strictly less than π . Indeed,

$$\theta_{L-t_1} - \alpha_{t_1} = \theta_{L-t_1} - c \cdot (\theta_{L-t_1} - a) - a + \pi = \pi - (1-c)(a - \theta_{L-t_1}),$$

which is less than π since $c \in (0, 1)$ and $\theta_{L-t_1} < a$.

To prove the claim (6.4) precisely, first observe from Lemma 4.1 that, if $2\phi_t^\lambda$ ever enters the region R_2 during $(L/2, t_2)$, it will remain in R_2 afterwards since $\alpha_t \in R_2$ for $t \in [L/2, L]$. In particular, we have $2\phi_{t_2}^\lambda \in R_2$ in this case. Let us now assume the other case that $2\phi_t^\lambda \in [a - \pi, a]$ for $t \in [L/2, t_2]$. According to the above points (i)–(iii), in this case we see that the distance between $2\phi_{t_1}^\lambda$ and α_{t_1} is uniformly less than π for all large λ . As a consequence of Lemma 6.2, we conclude that

$$\lim_{\lambda \rightarrow \infty} |2\phi_{t_2}^\lambda - \alpha_{t_2}| = 0.$$

In particular, $2\phi_{t_2}^\lambda \in R_2$ when λ is large. This proves the desired claim. \blacksquare

We give some further comments to conclude the discussion for this section. Although we are only considering a particular type of examples here, the above argument can be adapted to deal with the more general situation where α_t is C^2 near the singularity $t_* \in (0, L)$ and $|\alpha'(t_* -)| \neq |\alpha'(t_* +)|$. For simplicity, suppose that on the portion of $[0, t_*]$ we are in the setting of Section 5, so that we have the estimate

$$(6.5) \quad \int_0^{t_*} \cos(\alpha_t - 2\phi_t^\lambda) dt \gtrsim t_*$$

when λ is large. To deal with the portion after t_* , the point is that (6.1) provides a good approximation of the actual path α_t in a small neighbourhood $(t_* - h, t_* + h)$ of t_* , where the parameter $c \neq 1$ captures the difference between the magnitudes of the left and right derivatives of α at the singularity t_* . Using a suitable comparison lemma, one can then show that, after passing through the singularity t_* , when λ is large the angular path $2\phi_t^\lambda$ gets pushed into the region where $\alpha|_{(t_*, t_* + h)}$ belongs. As a result, the initial condition of $2\phi_t^\lambda$ on the portion of $[t_* + h, L]$ is favorable (relative to $\alpha(t_* + h)$), and the analysis developed in Section 5 again leads to the estimate

$$(6.6) \quad \int_{t_* + h}^L \cos(\alpha_t - 2\phi_t^\lambda) dt \gtrsim L - t_* - h$$

when λ is large. By adding up the two estimates (6.5), (6.6) and letting $h \rightarrow 0^+$, we arrive at the formula (1.1).

7. Some further questions

One may wonder how much further one needs to go towards a complete solution to the length conjecture (1.1). In our modest opinion, a critical step is to identify a suitable way to capture the *degree* of tree-reducedness at a *quantitative* level, which is at the same time convenient to be utilised in the signature analysis. This is a quite challenging part of the problem as the tree-reduced property is essentially topological. Although Theorem 2.6 does not resolve the problem completely, we believe that it provides an interesting and important attempt along this philosophy. For instance, in the simplified context of Example 2.4, the condition that β_t stays in an interval of length $r < \pi$ reflects the extent to which the path cannot turn around immediately (thus being tree-reducedness). As r gets smaller, the path tends to be “more tree-reduced”. While as r gets closer to π , the condition becomes less sensitive for detecting the tree-reduced property, and the extreme case of $r = \pi$ allows the possibility of creating tree-like pieces.

To go deeper into the study, among others there are at least two interesting questions one can raise and investigate.

Question 7.1. *Can we extend the current approach to higher dimensional paths?*

At the moment, this is not entirely straight forward. For higher dimensional paths, one may resort to Cartan developments onto higher dimensional Lie groups, e.g., $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SO}(p, q)$ etc. If one designs the development in a clever way, it may not be too surprising to end up with an ODE system in which the angular component for the group action is decoupled from the radial component just like the equations (3.5) and (3.6). However, one faces another level of challenge (which is harder to overcome) due to the lack of monotonicity properties for the angular equation, since the angle dynamics is now taking values in the n -sphere S^n rather than in \mathbb{R} (or S^1).

On the other hand, we have made use of the intuition that the tree-reduced property is reflected by the incapability of making a π -turn locally. There is a weaker type of conditions that captures this property in a more direct way, which is expressed in terms of turning angles:

$$(7.1) \quad |\beta_t - \beta_s| \leq \kappa, \quad \text{for a.a. } s, t,$$

with some given constant $\kappa \in (0, \pi)$.

Question 7.2. *Is it possible to prove the signature asymptotics formula (1.1) under the condition (7.1), or more generally under a suitably localised version of (7.1)?*

Clearly, the condition (2.4) implies the condition (7.1). However, it is possible to construct a bounded variation path that satisfies (7.1) but is not strongly tree-reduced in the sense of Definition 2.5. For instance, let $\{A, B, C\}$ be a Lebesgue measurable partition of $[0, 1]$, such that for every open subset U of $[0, 1]$ one has

$$\mu(U \cap A) > 0, \quad \mu(U \cap B) > 0 \quad \text{and} \quad \mu(U \cap C) > 0.$$

The existence of $\{A, B, C\}$ is a (non-trivial) exercise in real analysis. Define $\beta: [0, 1] \rightarrow S^1$ by

$$\beta_t = 0 \cdot \mathbf{1}_A(t) + e^{2\pi i/3} \cdot \mathbf{1}_B(t) + e^{4\pi i/3} \cdot \mathbf{1}_C(t).$$

Then β_t satisfies (7.1) with $\kappa = 2\pi/3$, where the distance $|\beta_t - \beta_s|$ is understood as the S^1 -distance. The resulting path $\gamma_t \triangleq \int_0^t \beta_s ds$ is tree-reduced but not strongly tree-reduced. At this point, it is not so clear if the current strategy can be adapted to deal with this weaker type of conditions.

A. Proof of Lemma 6.2 and the \mathcal{C}^1 -case

In this section, we give the proof of Lemma 6.2, which also directly implies the signature asymptotics formula (1.1) for planar \mathcal{C}^1 -paths. Let $\alpha: [0, L] \rightarrow \mathbb{R}$ be a given continuous function.

We first consider the case when the initial condition of $2\phi_t^\lambda$ coincides with α_0 .

Lemma A.1. *For each $\lambda > 0$, let $(\phi_t^\lambda)_{0 \leq t \leq L}$ be the solution to the differential equation*

$$(A.1) \quad \begin{cases} d\phi_t^\lambda = \lambda \sin(\alpha_t - 2\phi_t^\lambda) dt, & 0 \leq t \leq L, \\ \phi_0^\lambda = \alpha_0/2. \end{cases}$$

Then $2\phi_t^\lambda$ converges uniformly to α_t on $[0, L]$ as $\lambda \rightarrow \infty$.

Proof. Let

$$\omega(h) \triangleq \sup_{|s-t|<h} |\alpha_t - \alpha_s|$$

be the modulus of continuity of α . Given $\varepsilon \in (0, \pi/4)$, choose $h = h_\varepsilon$ so that $\omega(h) < \varepsilon$. Let

$$\lambda > \frac{\|\alpha\|_\infty}{h \sin \varepsilon}.$$

We claim that $|2\phi_t^\lambda - \alpha_t| < 2\varepsilon$ for all $t \in [0, L]$. Suppose on the contrary that $|2\phi_t^\lambda - \alpha_t| \geq 2\varepsilon$ for some t . Define

$$\begin{aligned} t_2 &\triangleq \inf\{s \in [0, L] : |2\phi_s^\lambda - \alpha_s| \geq 2\varepsilon\}, \\ t_1 &\triangleq \sup\{0 \leq s \leq t_2 : |2\phi_s^\lambda - \alpha_s| \leq \varepsilon\}. \end{aligned}$$

Apparently, $0 < t_1 < t_2 \leq L$. Moreover, $|2\phi_{t_1}^\lambda - \alpha_{t_1}| = \varepsilon$ and

$$\varepsilon \leq |2\phi_s^\lambda - \alpha_s| \leq 2\varepsilon, \quad \forall s \in [t_1, t_2].$$

Using the differential equation (A.1), we also have

$$(A.2) \quad 2\phi_{t_2}^\lambda - \alpha_{t_2} = 2\phi_{t_1}^\lambda - \alpha_{t_1} + 2\lambda \int_{t_1}^{t_2} \sin(\alpha_s - 2\phi_s^\lambda) ds + (\alpha_{t_1} - \alpha_{t_2}).$$

If $2\phi_{t_2}^\lambda - \alpha_{t_2} = 2\varepsilon$, by the definition of t_1 we must have $2\phi_{t_1}^\lambda - \alpha_{t_1} = \varepsilon$ and thus

$$\varepsilon \leq 2\phi_s^\lambda - \alpha_s \leq 2\varepsilon, \quad \forall s \in [t_1, t_2].$$

Therefore, (A.2) implies that

$$2\varepsilon \leq \varepsilon - 2\lambda(t_2 - t_1) \sin \varepsilon + |\alpha_{t_1} - \alpha_{t_2}| \\ \leq \begin{cases} \varepsilon + \omega(h) < 2\varepsilon, & \text{if } t_2 - t_1 < h, \\ \varepsilon + 2\|\alpha\|_\infty - 2\lambda h \sin \varepsilon < \varepsilon, & \text{if } t_2 - t_1 \geq h. \end{cases}$$

This is clearly a contradiction. The case when $2\phi_{t_2}^\lambda - \alpha_{t_2} = -2\varepsilon$ is treated in a similar way. Consequently, we conclude that $|2\phi_t^\lambda - \alpha_t| < 2\varepsilon$ for all $t \in [0, L]$, provided that $\lambda > \|\alpha\|_\infty / (h \sin \varepsilon)$. ■

Now we extend the above argument to give a proof of Lemma 6.2. Namely, under the assumption that

$$(A.3) \quad |2\phi_0^\lambda - \alpha_0| \leq \kappa < \pi \quad \forall \lambda > 0,$$

we want to establish the uniform convergence of $2\phi_t^\lambda$ towards α_t on $[t_0, L]$ where $t_0 > 0$ is a given fixed time. In order to use the previous proof, the crucial point is to see that, when λ is large, the quantity $2\phi_t^\lambda - \alpha_t$ can be brought down to the region $(-\varepsilon, \varepsilon)$ at some time in $[0, t_0]$. The heuristic reason for such a property is simple to describe. Since $|2\phi_0^\lambda - \alpha_0|$ is uniformly less than π , at the initial stage of the dynamics (i.e., when t is small), the quantity $\sin(2\phi_t^\lambda - \alpha_t)$ is uniformly away from zero. Therefore, when λ is large, the mean-reversing property gets rather significant and is thus pushing $2\phi_t^\lambda$ to be close to α_t very quickly. Let us now make the heuristics precise.

Lemma A.2. *Suppose that (A.3) holds for some given constant $\kappa \in (0, \pi)$. Let $t_0 > 0$ be fixed. Then for any $\varepsilon > 0$, there exists $\Lambda = \Lambda_{\varepsilon, t_0} > 0$ such that for each $\lambda > \Lambda$ we have*

$$(A.4) \quad |2\phi_s^\lambda - \alpha_s| < \varepsilon \quad \text{for some } s \in [0, t_0].$$

Proof. Let $\kappa' \in (\kappa, \pi)$ be fixed. By the continuity of α_t at the origin, there exists $\delta \in (0, t_0)$ such that

$$t \in [0, \delta] \implies |\alpha_t - \alpha_0| < \kappa' - \kappa.$$

Given $\varepsilon > 0$, we define

$$\Lambda \triangleq \frac{2\|\alpha\|_\infty + \kappa}{2\delta \sin \varepsilon}.$$

For each given $\lambda > \Lambda$, we claim that (A.4) holds. Suppose on the contrary that

$$|2\phi_s^\lambda - \alpha_s| \geq \varepsilon, \quad \forall s \in [0, t_0].$$

By continuity, we have either

$$(i) \ 2\phi_s^\lambda - \alpha_s \geq \varepsilon, \quad \forall s \in [0, t_0] \quad \text{or} \quad (ii) \ 2\phi_s^\lambda - \alpha_s \leq -\varepsilon, \quad \forall s \in [0, t_0].$$

Suppose that case (i) holds. Define

$$s_1 \triangleq \inf \{s \in [0, t_0] : 2\phi_s^\lambda - \alpha_s = \kappa'\}.$$

Then we must have $s_1 > \delta$. Indeed, consider the equation

$$(A.5) \quad 2\phi_{s_1}^\lambda - \alpha_{s_1} = 2\phi_0^\lambda - \alpha_0 - (\alpha_{s_1} - \alpha_0) - 2\lambda \int_0^{s_1} \sin(2\phi_s^\lambda - \alpha_s) ds.$$

Note that

$$\varepsilon \leq 2\psi_s^\lambda - \alpha_s \leq \kappa' < \pi, \quad \forall s \in [0, s_1],$$

and thus $\sin(2\phi_s^\lambda - \alpha_s)$ is positive on $[0, s_1]$. If $s_1 \leq \delta$, the left-hand side of (A.5) equals κ' , while the right-hand side is strictly less than

$$\kappa + (\kappa' - \kappa) - 2\lambda \int_0^{s_1} \sin(2\phi_u^\lambda - \alpha_u) du \leq \kappa'.$$

This is clearly a contradiction. Therefore, $s_1 > \delta$. Now using the same equation (A.5), we see that the left-hand side is bounded below by ε while the right-hand side is bounded above by

$$\kappa + 2\|\alpha\|_\infty - 2\lambda s_1 \sin \varepsilon \leq \kappa + 2\|\alpha\|_\infty - 2\lambda \delta \sin \varepsilon.$$

This leads to a contradiction, since the quantity on the right hand side of the above inequality is negative when $\lambda > \Lambda$ by the definition of Λ . The discussion of case (ii) is similar. ■

Now we are able to complete the proof of Lemma 6.2.

Proof of Lemma 6.2. Given $\varepsilon > 0$, define

$$\Lambda_1 \triangleq \frac{\|\alpha\|_\infty}{h \sin \varepsilon} \quad \text{and} \quad \Lambda_2 \triangleq \frac{2\|\alpha\|_\infty + \kappa}{2\delta \sin \varepsilon},$$

which are the two constants appearing in the proofs of Lemma A.1 and Lemma A.2, respectively. Define $\Lambda \triangleq \max\{\Lambda_1, \Lambda_2\}$. For each $\lambda > \Lambda$, we know from Lemma A.2 that there is $s \in [0, t_0]$ (which may depend on λ) such that (A.4) holds. In addition, exactly the same argument as in the proof of Lemma A.1 allows us to conclude that

$$|2\phi_t^\lambda - \alpha_t| \leq 2\varepsilon, \quad \forall t \in [s, L]$$

In particular, we have

$$\sup_{t \in [t_0, L]} |2\phi_t^\lambda - \alpha_t| \leq 2\varepsilon.$$

This gives the desired uniform convergence. ■

Remark A.3. It is not hard to see why $2\phi_0^\lambda = \alpha_0 \pm \pi$ are “bad” initial conditions. Consider the extreme example where $\alpha_t \equiv \alpha_0$. If $2\phi_0^\lambda = \alpha_0 \pm \pi$, then we have $\phi_t^\lambda \equiv \phi_0^\lambda$, which is never close to $\alpha_t/2$. If we perform an explicit calculation for the tree-like path $v \sqcup (-v)$ ($v \in \mathbb{R}^2$), this is exactly what happens in the $(-v)$ -part.

Corollary A.4. Let $\gamma: [0, L] \rightarrow \mathbb{R}^2$ be a path defined by the equation (2.3), where the angular path $\beta: [0, L] \rightarrow \mathbb{R}^2$ is a continuous function. Then the signature asymptotics formula (1.1) holds for γ .

Proof. With $\alpha_t \triangleq \beta_{L-t}$ and $2\phi_0^\lambda \triangleq \alpha_0$, Lemma 6.2 shows that the angular path $2\phi_t^\lambda$ converges uniformly to α_t as $\lambda \rightarrow \infty$. The result then follows from the lower estimate (3.10). ■

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