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On the Sobolev quotient of three-dimensional CR manifolds

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Abstract. We exhibit examples of compact three-dimensional CR manifolds of positive Webster class, *Rossi spheres*, for which the pseudo-hermitian mass, as defined by Cheng–Malchiodi–Yang (2017), is negative, and for which the infimum of the CR-Sobolev quotient is not attained. To our knowledge, this is the first geometric context on smooth closed manifolds where this phenomenon arises, in striking contrast to the Riemannian case.

1. Introduction

The Yamabe problem consists in deforming conformally the metric of a manifold of dimension $n \geq 3$ so that its scalar curvature becomes a constant. Apart from being a natural conformal extension of the uniformization problem in two dimensions, the question was introduced in [36] for trying to attack Poincaré's conjecture. Yamabe metrics have also been applied to other contexts, such as the study of degeneration of conformal structures. For example, in [34] it is shown that the set of Yamabe Bach-flat metrics on a four-manifold is compact up to orbifold degeneration.

If one wishes to have $S_{\tilde{g}}$ constant, the following elliptic problem must be solved:

$$(Y) \qquad -\frac{4(n-1)}{n-2}\,\Delta_g u + S_g u = \overline{S}\, u^{(n+2)/(n-2)} \quad \text{on } M, \text{ for } \overline{S} \in \mathbb{R}.$$

Notice that the exponent on the right-hand side of the equation is critical with respect to the Sobolev embeddings. In [36], an attempt was made to solve (Y) by lowering the exponent by a small amount, but the possible weak convergence to zero of solutions was not excluded. Another way to attack (Y) was to view $\overline{S}$ as a Lagrange multiplier, considering the Sobolev quotient

$$(1.1) \qquad Q_{(M,g)}(u) := \frac{\int_M (c_n |\nabla_g u|^2 + S_g u^2)\, dV_g}{\big(\int_M |u|^{2^*}\, dV_g\big)^{2/2^*}} = \frac{\int_M S_{\tilde{g}}\, dV_{\tilde{g}}}{(\mathrm{Vol}_{\tilde{g}}(M))^{2/2^*}},$$

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where $c_n = 4(n-1)/(n-2)$ and $2^* = 2n/(n-2)$. If one could realise the minimum of $Q_{(M,g)}(u)$ over all non-zero u's of class $W^{1,2}(M,g)$, this would give rise to a solution of (Y): notice that it is sufficient to consider functions in $W^{1,2}(M,g)$ that are non-negative, therefore by regularity theory one would obtain a positive smooth solution. Defining then

$$Y(M,g) := \inf_{u \in W^{1,2}(M,g), u \not\equiv 0} Q_{(M,g)}(u),$$

it can be proved that this quantity is independent of the conformal representative of g, and will therefore be denoted from now on by $Y(M,[g])$. Depending on the sign of the latter quantity, $(M,[g])$ is said to be of *negative, null* or of *positive Yamabe class*.

It was proved in [35] that there exists a dimensional constant $\varepsilon_n > 0$ such that $Y(M,[g])$ is attained (and hence (Y) is solvable) provided $Y(M,[g]) \leq \varepsilon_n$. The result applies in particular to all manifolds with conformal classes of metrics of negative or null Yamabe class.

Consider the (normalized) Sobolev quotient in $\mathbb{R}^n$

$$S_n := \inf_{u \in C_c^\infty(\mathbb{R}^n), u \not\equiv 0} \frac{\int_{\mathbb{R}^n} c_n |\nabla u|^2 \, dx}{\left(\int_{\mathbb{R}^n} |u|^{2^*} \, dx\right)^{2/2^*}}. \tag{1.2}$$

Using the stereographic projection from S^n to $\mathbb{R}^n$, it can be proved that the above quantity coincides with the Yamabe quotient of the round sphere, i.e., for all $n \geq 3$ one has $S_n = Y(S^n, [g_{S^n}])$. It was shown in [2] that one always has $Y(M,[g]) \leq S_n$, and that (Y) is solvable provided the strict inequality holds. It was also shown in [2] that $Y(M,[g]) < S_n$ provided $n \geq 6$ and M is not locally conformally flat, i.e., when the Weyl tensor of (M,g) is not identically zero. It was proved then in [26] that $Y(M,[g]) < S_n$ in all complementary cases (provided (M,g) is not conformally equivalent to the round sphere), i.e., when (M,g) has dimension less or equal to 5 or when it is locally conformally flat. While the argument in [2] was based on a local energy expansion, the one in [26] relied on the *positive mass theorem* in general relativity, see [27–30], which is in turn related to the expansion of the Green function of the *conformal Laplacian* L_g near its pole, where

$$L_g u := -\frac{4(n-1)}{n-2} \Delta_g u + S_g u.$$

In both [2] and [26], the strict inequality was proved by evaluating the Yamabe–Sobolev quotient on (suitable perturbations of) highly concentrated extremals of (1.2) (classified in [2,33]), suitably glued to (M,g). Such extremals, parametrized using the Möbius group of S^n, can be chosen arbitrarily peaked near any point: these decay faster at infinity in higher dimensions and therefore the correction to the quotient due to the geometry of M is more *localized* in space for n large. In any case, we always have $Y(M,[g]) < S_n$ provided that $(M,g) \overset{\text{conf.}}{\not\simeq} (S^n, g_{S^n})$.

We consider in this paper compact three dimensional pseudo-hermitian manifolds (M,J,θ): these are CR manifolds, i.e., endowed with a contact structure ξ and a CR structure $J : \xi \to \xi$ such that $J^2 = -1$. We assume (M,J) to be *strictly pseudo-convex*, namely that it is globally defined a contact form θ which annihilates ξ, and for which $\theta \wedge d\theta$ is always non-zero (see [3]). We define the *Reeb vector field* as the unique T for which

$\theta(T) \equiv 1$ and $T \lrcorner d\theta = 0$. Given J as above, we can define locally a vector field Z_1 such that

$$JZ_1 = iZ_1 \quad \text{and} \quad JZ_{\bar{1}} = -iZ_{\bar{1}}, \quad \text{where} \quad Z_{\bar{1}} = (\overline{Z_1}).$$

We also let $(\theta, \theta^1, \theta^{\bar{1}})$ be the dual triple to $(T, Z_1, Z_{\bar{1}})$, so that

$$d\theta = ih_{1\bar{1}}\,\theta^1 \wedge \theta^{\bar{1}} \quad \text{for some } h_{1\bar{1}} > 0 \quad (\text{possibly replacing } \theta \text{ by } -\theta).$$

In the following, we will always assume that $h_{1\bar{1}} \equiv 1$.

The *connection* 1-form ω_1^1 and the *torsion* $A_{\bar{1}}^1$ are uniquely determined by the structure equations

$$\left\{ \begin{array}{l} d\theta^1 = \theta^1 \wedge \omega_1^1 + A_{\bar{1}}^1\,\theta \wedge \theta^{\bar{1}}, \\ \omega_1^1 + \omega_{\bar{1}}^{\bar{1}} = 0. \end{array} \right. \tag{1.3}$$

The *Tanaka–Webster curvature* (or *Webster curvature*) R_θ (or, simply, R) is then defined by the formula

$$d\omega_1^1 = R_\theta\,\theta^1 \wedge \theta^{\bar{1}} \pmod{\theta}.$$

A model with positive curvature is the round sphere $(S^3, J_{S^3}, \hat{\theta})$, with $S^3 \subseteq \mathbb{C}^2 = \{(z_1, z_2)\}$, and

$$\begin{aligned} \hat{\theta} &= \frac{1}{2}\,i(\bar{\partial} - \partial)(|z^1|^2 + |z^2|^2) = \frac{1}{2}\,i\sum_{k=1}^{2}(z^k dz^{\bar{k}} - z^{\bar{k}} dz^k), \\ Z_1 &= Z_1^{S^3} = \bar{z}^2\,\frac{\partial}{\partial z^1} - \bar{z}^1\,\frac{\partial}{\partial z^2}. \end{aligned} \tag{1.4}$$

Similarly to what happens with the classical stereographic projection, the CR three-sphere is CR equivalent to the *Heisenberg group* $\mathbb{H}^1 = \{(z,t), z \in \mathbb{C}, t \in \mathbb{R}\}$, see e.g. [11].

The Tanaka–Webster curvature enjoys conformal properties similar to the scalar curvature on Riemannian manifolds. More precisely, scaling the contact form θ by a positive function, one has the following law for the transformation of the Webster curvature:

$$L_b u := -4\,\Delta_b u + R_\theta\, u = R_{\tilde{\theta}}\, u^3; \quad \tilde{\theta} = u^2\theta. \tag{1.5}$$

Here, $R_{\tilde{\theta}}$ is the Tanaka–Webster curvature corresponding to the pseudo-hermitian structure $(J, \tilde{\theta})$, and Δ_b stands for the operator defined as follows:

$$\Delta_b f = f_{,1}{}^{1} + f_{,\bar{1}}{}^{\bar{1}} = f_{,1\bar{1}} + f_{,\bar{1}1},$$

where we have used $h^{1\bar{1}} = h_{1\bar{1}} = 1$ to raise or lower the indices, and where we set

$$f_1 = f_{,1} := Z_1 f, \quad f_{,1\bar{1}} = Z_{\bar{1}} Z_1 f - \omega_1^1(Z_{\bar{1}}) Z_1 f \quad \text{and} \quad f_{,0} = Tf.$$

The CR-invariant sub-Laplacian transforms covariantly as follows:

$$\hat{L}_b(\varphi) = u^{-(Q+2)/(Q-2)}\, L_b(u\varphi); \quad \hat{\theta} = u^2\theta,$$

where $Q = 4$ is the *homogeneous dimension* of the manifold. By (1.5), finding $\tilde{\theta}$ with constant Webster curvature corresponds to solving the following analogous problem to (Y):

$$L_b u = \overline{R}\, u^{(Q+2)/(Q-2)} \quad \text{on } M, \;\text{ for } \overline{R} \in \mathbb{R}, \; u > 0. \tag{W}$$

In [19], the counterpart of the result in [2] was obtained, i.e., if the infimum of the CR-Sobolev quotient satisfies

$$\mathcal{Y}(M,J) := \inf_{\hat\theta} \frac{\int_M R_{\hat\theta}\, \hat\theta \wedge d\hat\theta}{(\int_M \hat\theta \wedge d\hat\theta)^{1/2}} = \inf_{u \in C^\infty(M), u>0} \frac{\int_M (4|\nabla_b u|^2 + R_\theta u^2)\, \theta \wedge d\theta}{(\int_M u^4\, \theta \wedge d\theta)^{1/2}}$$
$$< \mathcal{Y}(S^3, J_{S^3}),$$

then it is attained and a solution of (W) exists (indeed, this holds true in any dimension). The same authors verified this condition when the dimension is greater or equal to five and (M,J) is not *spherical*, see [21] and [20].

However, in the CR setting new phenomena appear, related to the fact that most three-dimensional structures are non-embeddable, differently from the higher-dimensional case, see [4,6]. In [11], some results in the above directions were proved, assuming some global conditions related to the embeddability of the abstract CR structure.

More precisely, a notion of *pseudo-hermitian mass* was defined for three-dimensional *asymptotically-Heisenberg* manifolds (we refer to the latter paper for precise definitions and details) by setting

$$m(J,\theta) := i \oint_\infty \omega_1^1 \wedge \theta := \lim_{\Lambda \to +\infty} i \oint_{S_\Lambda} \omega_1^1 \wedge \theta,$$

where $S_\Lambda = \{\rho = \Lambda\}$, $\rho^4 = |z|^4 + t^2$ (with (z,t) coordinates on the Heisenberg group), and where ω_1^1 stands for the connection form of the structure. The above definition was introduced considering an analogue of the *Einstein–Hilbert action*.

As it happens in the Riemannian case, this mass is related to the expansion of the Green function of the conformal sub-Laplacian L_b on a compact manifold M. When $\mathcal{Y}(M,J) > 0$, the latter operator is invertible, so for any $p \in M$ there exists a Green function G_p satisfying distributionally

$$(-4\Delta_b + R)\, G_p = 64\pi\delta_p,$$

where δ_p in the right-hand side stands for the Dirac delta with respect to the volume measure $\theta \wedge d$. In *CR normal coordinates* (z,t) (introduced in [21] and discussed in Section 2), G_p writes as

$$G_p = 2\rho^{-2} + A + O(\rho), \tag{1.6}$$

for some $A \in \mathbb{R}$, and where $\rho^4(z,t)$ is as above. For the latter expansion, we refer to Proposition 5.2 in [11] (here we use an extra factor 4π in the definition of G_p), and to Subsection 2.1 for our notation $O(\rho)$. Given $(M,J,)$ compact and $p \in M$, consider a blow-up of contact form as follows:

$$N = (M \setminus \{p\}, J, G_p^2\theta).$$

As it is shown in [11], via an inversion of coordinates, the manifold N turns out to have asymptotically the geometry of the Heisenberg group, and its pseudo-hermitian mass satisfies

$$m = 12\pi A \tag{1.7}$$

(see Lemma 2.5 there, and recall the difference of 4π in our current notation), where A is as above.

Using crucially a result in [18], in the same paper it was also proved that the pseudo-hermitian mass is non-negative (and zero only when $(M, J,)$ is CR equivalent to S^3), provided that the *CR Paneitz operator* P on (M, J) is non-negative definite. The latter operator is

$$P\varphi := 4(\varphi_{\bar{1}}{}^{\bar{1}}{}_1 + iA_{11}\varphi^1)^1,$$

and it has a relation to the log-term coefficient in the Szegö kernel expansion, and it is pseudo-hermitian-covariant, namely $P_{\hat{\theta}}\varphi = e^{4f} P_\theta\varphi$ for the conformal change $\theta = e^{2f}\hat{\theta}$ (see [17]). By a result in [9], manifolds for which P is non-negative and $R > 0$ can be embedded into some $\mathbb{C}^N$ (see also [7]).

The assumption on the positivity of the Paneitz operator is not technical, as in [11] some counterexamples for the positivity of the pseudo-hermitian mass were also given for structures (arbitrarily) close to the spherical one, and hence with positive Webster curvature. In a recent work [32], the positivity of the Paneitz operator is shown to hold for embeddable (M, J).

In this paper we are concerned with *Rossi spheres*: these are a one-parameter-family of CR structures on the 3-sphere of the form $S^3_s := (S^3, J_{(s)}, \hat{\theta})$, where $\hat{\theta}$ is as in (1.4), and where $J_{(s)}$ is characterized by

$$(1.8)\quad J_{(s)}Z_{1(s)} = iZ_{1(s)}; \quad Z_{1(s)} = Z_1 + \frac{s}{\sqrt{1+s^2}}Z_{\bar{1}}, \quad Z_{\bar{1}(s)} = Z_{\bar{1}} + \frac{s}{\sqrt{1+s^2}}Z_1.$$

Rossi spheres are interesting because they are simple examples of CR structures on the three-sphere that cannot be embedded in $\mathbb{C}^N$. In [5], it was shown that all the holomorphic functions on such structures are even functions if $s \neq 0$. On the other hand, there are explicit embeddings in $\mathbb{C}^3$ of the quotient of the Rossi spheres by the antipodal map, see [10]. By the above discussion, it follows that the Paneitz operator cannot be non-negative here. In addition, this family of CR structures are homogeneous and if we take the standard contact form, it is *pseudo-Einstein*, i.e., $R_{,1} - iA_{11,\bar{1}} = 0$, see [8] as well as our notation for covariant derivatives in Section 2.1.

Our first main result in this paper is the following theorem.

Theorem 1.1. *For $|s|$ small, $s \neq 0$, the pseudo-hermitian mass of the Rossi spheres S^3_s is negative. More precisely, one has the expansion*

$$m_s = -18\pi s^2 + o(s^2) \quad \textit{for } s \simeq 0.$$

Remark 1.2. (a) We can generalize the construction of Rossi spheres in Theorem 1.1 as follows. According to Proposition 3.3 in [13], there exist deformations of the standard CR structure on S^3/Γ ($\Gamma = \mathbb{Z}_2$ for the case of Rossi spheres), whose universal covers are not embeddable. These CR structures (i.e., universal covers) are likely to have negative mass.

(b) We can embed $S^3_s/\mathbb{Z}_2$ into $\mathbb{C}^3$ (see, for instance, [10]). So according to [32], the CR Paneitz operator P on $S^3_s/\mathbb{Z}_2$ is non-negative definite. On the other hand, P on S^3_s cannot be non-negative definite by Theorem 1.1 and the positive mass theorem in [11] for $|s|$ small, $s \neq 0$, so that the Webster curvature of S^3_s is positive. Thus, for $|s|$ small, $s \neq 0$, $S^3_s/\mathbb{Z}_2$ provides an example of CR manifold having non-negative definite P while its covering space S^3_s does not have non-negative definite P, answering a question raised by Ngaiming Mok in a conference held in Hong Kong, 2014.

We saw before (in both low-dimensional Riemannian and CR cases) that positivity of the mass implies attainment of the Sobolev quotient. We also strengthen the relation between mass and quotient by means of the following result, which is in striking contrast with the Riemannian case.

Theorem 1.3. *For $|s|$ small, $s \neq 0$, the infimum of the CR-Sobolev quotient of S^3_s coincides with $\mathcal{Y}(S^3, J_{S^3})$ and* is not attained.

Remark 1.4. (a) The phenomenon in Theorem 1.3 is typical of some critical problems in a PDE context, like the Yamabe equation on Euclidean domains with Dirichlet boundary conditions or the case of some general elliptic operators on manifolds. However, to our knowledge this is the first time this is displayed in a *purely geometric* smooth context.

(b) We recall that in [15] and [16] the CR-Yamabe problem was solved for every three dimensional CR manifolds, but there solutions were found via variational arguments and they are not of minimal type. Theorem 1.3 shows that the use of such methods is in some cases somehow necessary.

Determining or estimating the *mass* of a manifold is in general a hard problem, since this is deeply related to the Green function of the conformal (sub-)Laplacian, which is a *global* object. The mass also appears as its zero-th order coefficient after a proper choice of conformal representative and local coordinates. After recalling some preliminary facts in Section 2 on CR normal coordinates (introduced in [21] and suited for the above expansion) and on Rossi spheres, we specialize in Section 3 to the latter manifolds. For doing this we need first to derive a suitable conformal factor satisfying a list of conditions, and then express pseudo-hermitian coordinates depending on s. By the special expression of the Green function in these coordinates, we are able to determine it quite precisely near the north pole, up to the constant term A appearing in (1.6). However, as we remarked before, also some global features of the Green function have to be understood.

For doing this, by a Taylor expansion in s worked-out at the beginning of Section 4 it is possible to characterize formally the Green function for the conformal sub-Laplacian on Rossi spheres up to an order $O(s^3)$. One problem with this expansion is that it generates singular terms, with a particularly bad behavior near the pole, if expressed with respect to the standard complex coordinates of $\mathbb{C}^2$, where S^3 embeds. Also in this case non local terms appear, which we are able to evaluate at the pole via some integral formula.

Via a careful analysis of all terms of order 1, s and s^2, we verify then in the second part of the section that the global singular expansion on S^3 matches with the one done in CR normal coordinates up to an order $O(s^3)$. This allows us to prove Theorem 1.1.

In Section 5, arguing by contradiction, we analyse the possible behaviours of minimizers for the CR Sobolev quotient. Due to a non-degeneracy result from [24], the analysis of minimizers can be reduced to a finite-dimensional one, and we show that the CR-Sobolev quotient of all candidate minimizers is strictly above the spherical one, i.e., $\mathcal{Y}(S^3, J_{S^3})$. With negative mass, this is expected for highly concentrated profiles, reversing the expansion in [26]: however such a property has to be obtained in *all cases*, i.e., even for non-concentrated profiles, in order to guarantee that the infimum of the CR-Sobolev quotient is not attained. In Proposition 5.5 this is proved for s small in a *fixed* compact set of the CR maps of S^3. This is done starting with the expansion of the quotient on Rossi spheres over the extremals of the quotient on the standard S^3, adding to them a correction term that improves their accuracy as approximate critical points for s non zero. One needs then to analyze the quotient in a regime with loss of compactness, which is particularly delicate due to the following reason. It is known from [26] that the mass of a (given) manifold plays a role in the expansion for Sobolev quotients of highly concentrated functions. In our case this must be done *uniformly in* s, and the problem could be that the *principal term* coming from the mass could become negligible as $s \to 0$. To solve this issue we exploit a symmetry $s \to -s$ for Rossi spheres, discussed in Section 2, which implies that all variational expansions are indeed *even* in s and hence the mass, which vanishes with s, gives still a dominant sign to the asymptotic expansion of the CR-Sobolev quotient. Two appendices are devoted to the estimates of the latter quantity in two different scaling regimes. To make the above arguments rigorous, we employ a finite-dimensional reduction of the problem, via a fixed point argument, which allows to solve for the CR-Yamabe equation on Rossi spheres up to a Lagrange multiplier. We obtain in this way a manifold of approximate solutions containing by construction all possible minimizers: our expansion shown then that on this manifold the CR-Sobolev quotient is strictly higher than $\mathcal{Y}(S^3, J_{S^3})$, yielding our result.

2. Background material

In this section we recall some useful facts about CR manifolds and the properties of CR normal coordinates, constructed in [21]. We then describe some general features of Rossi spheres.

2.1. Preliminary facts on CR manifolds

Let us begin by recalling the following commutation relations on tensors, see Lemma 2.3 in [23] (we also refer to this paper for our tensorial notation):

$$\begin{cases} c_{,1\bar{1}} - c_{,\bar{1}1} = i\, c_{,0} + k\, c R, \\ c_{,01} - c_{,10} = c_{,\bar{1}}\, A_{11} - k\, c A_{11,\bar{1}}\,, \\ c_{,0\bar{1}} - c_{,\bar{1}0} = c_{,1}\, A_{\bar{1}\bar{1}} + k\, c A_{\bar{1}\bar{1},1}\,. \end{cases}$$

Here, c is a tensor with 1 or $\bar{1}$ as sub-indices, k is the number of 1-sub-indices of c minus the number of $\bar{1}$-sub-indices of c and where, we recall, we are assuming that $h_{1\bar{1}} = 1$ (so $A_{\bar{1}\bar{1}} = A_{\bar{1}}^{1}$ and A_{11} is the complex conjugate of $A_{\bar{1}\bar{1}}$).

In the system of coordinates we will describe below, for $(z,t) \in \mathbb{H}^1$ near zero we will set

$$\rho^4 = |z|^4 + t^2.$$

For $k \in \mathbb{Z}$ we denote by $\tilde{O}(\rho^k)$ a function $f(z,\bar{z},t)$ for which $|f| \leq C\rho^k$ for some $C > 0$; we use instead the symbol $\tilde{O}'(\rho^k)$ for a function $f(z,\bar{z},t)$ such that

$$|f| \leq C\rho^k, \quad |\partial_z f| \leq C\rho^{k-1}\,|\partial_z \rho|, \quad |\partial_{\bar{z}} f| \leq C\rho^{k-1}\,|\partial_{\bar{z}}\rho|, \quad |\partial_t f| \leq C\rho^{k-1}\,|\partial_t \rho|.$$

One can define similarly the symbols $\tilde{O}''(\rho^k)$, $\tilde{O}'''(\rho^k)$, etc. We will use $O(\rho^k)$ for a function which is of the form $\tilde{O}^{(j)}(\rho^k)$ for every integer j, or for j large enough for our purposes.

Large positive constants are always denoted by C, and the value of C is allowed to vary from one formula to another and also within the same line. When we want to stress the dependence of the constants on some parameter (or parameters), we add subscripts to C, as C_δ, etc. Also constants with this kind of subscripts are allowed to vary.

Let us recall the notions of *pseudo-hermitian geometry* from [37] and [22]. We would need the following result in [21] (Proposition 2.5 in p. 313). For a differential form η, let us denote by $\eta_{(m)}$ the part of its Taylor series that is homogeneous of degree m in terms of parabolic dilations (see [21] for more details).

Proposition 2.1. *Let $\tilde{Z}_1$ be a special frame dual to $\tilde{\theta}^1$ (with $\tilde{h}_{1\bar{1}} = 2$), and let $\theta^1 = \sqrt{2}\tilde{\theta}^1$ be a unitary coframe ($h_{1\bar{1}} = 1$). Then in pseudo-hermitian normal coordinates (z, t) with respect to $\tilde{Z}_1$ and $\tilde{\theta}^1$, we have the following.*

(a) *For $m \geq 4$,*

$$\theta_{(2)} = \mathring{\theta}, \quad \theta_{(3)} = 0, \quad \theta_{(m)} = \frac{1}{m}\sqrt{2}\,(iz\theta^{\bar{1}} - i\bar{z}\theta^1)_{(m)}.$$

(b) *For $m \geq 3$,*

$$\theta^1_{(1)} = \sqrt{2}\,dz, \quad \theta^1_{(2)} = 0, \quad \theta^1_{(m)} = \frac{1}{m}(\sqrt{2}\,z\,\omega_1{}^1 + 2tA_{\bar{1}\bar{1}}\theta^{\bar{1}} - \sqrt{2}\,\bar{z}\,A_{\bar{1}\bar{1}}\theta)_{(m)}.$$

(c1) $(\omega_1{}^1)_{(1)} = 0$.

(c2) *For $m \geq 2$,*

$$(\omega_1{}^1)_{(m)} = \frac{1}{m}\left(\sqrt{2}R(z\theta^{\bar{1}} - \bar{z}\theta^1) + A_{11,\bar{1}}(\sqrt{2}z\theta - 2t\theta^1) - A_{\bar{1}\bar{1},1}(\sqrt{2}\bar{z}\theta - 2t\theta^{\bar{1}})\right)_{(m)}.$$

Here, $\mathring{\theta} = dt + iz\,d\bar{z} - i\bar{z}\,dz$.

Definition 2.2. Given a three dimensional pseudo-hermitian manifold (M, θ), we define a real symmetric tensor Q as

$$Q = Q_{jk}\,\theta^j \otimes \theta^k, \quad j, k \in \{0, 1, \bar{1}\},$$

with $\theta^0 := \theta$, whose components with respect to any admissible coframe are given by

$$Q_{11} = \overline{Q_{\bar{1}\bar{1}}} = 3iA_{11}, \quad Q_{1\bar{1}} = Q_{\bar{1}1} = h_{1\bar{1}}R,$$

$$Q_{01} = Q_{10} = \overline{Q_{0\bar{1}}} = \overline{Q_{\bar{1}0}} = 4A_{11,}{}^1 + iR_{,1}, \quad Q_{00} = 16\,\mathrm{Im}\,A_{11,}{}^{11} - 2\Delta_b R.$$

We have then the following result, see Theorem 3.1 in p. 315 of [21].

Proposition 2.3. *Suppose M is a strictly pseudo-convex pseudo-hermitian manifold of dimension three, and let $q \in M$. Then for any integer $N \geq 2$, there exists a choice of contact form θ such that all symmetrized covariant derivatives of Q with total order less or equal than N vanish at q, that is,*

$$Q_{\langle jk,l\rangle} = 0 \quad \text{at } q \text{ if } \mathbb{O}(jkl) \leq N.$$

By *CR normal coordinates* of order N, we mean the pseudo-hermitian normal coordinates with θ chosen as in Proposition 2.3. We recall ([21]) that for a multi index $l = (l_1, \dots, l_s)$, we count its order as

$$\mathbb{O}(l) = \mathbb{O}(l_1) + \cdots + \mathbb{O}(l_s),$$

where $\mathbb{O}(1) = \mathbb{O}(\bar{1}) = 1$ and where $\mathbb{O}(0) = 2$. The symmetrized covariant derivatives are defined by

$$Q_{\langle l\rangle} = \frac{1}{s!} \sum_{\sigma \in \mathbb{S}_s} Q_{\sigma l}, \quad \sigma l = (l_{\sigma(1)}, \dots, l_{\sigma(s)}).$$

In Proposition A.5 of [11], the following result was proved.

Proposition 2.4. *In CR normal coordinates of order $N = 4$, we have a contact form θ such that*

$$\begin{aligned}
\theta &= (1 + O(\rho^4))\mathring{\theta} + O(\rho^5)\, dz + O(\rho^5)\, d\bar{z},\\
\theta^1 &= (1 + O(\rho^4))\sqrt{2}\, dz + O(\rho^4)\, d\bar{z} + O(\rho^3)\, \mathring{\theta},\\
\omega_1^1 &= O(\rho^3)\, dz + O(\rho^3)\, d\bar{z} + O(\rho^2)\, \mathring{\theta},\\
Z_1 &= (1 + O(\rho^4))\, \mathring{Z}_1 + O(\rho^4)\, \mathring{Z}_{\bar{1}} + O(\rho^5)\, \frac{\partial}{\partial t},\\
T &= (1 + O(\rho^4))\, \frac{\partial}{\partial t} + O(\rho^3)\, \mathring{Z}_1 + O(\rho^3)\, \mathring{Z}_{\bar{1}},
\end{aligned}$$

where we recall that

$$\mathring{\theta} = dt + iz\, d\bar{z} - i\bar{z}\, dz, \quad \mathring{Z}_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z} + i\bar{z}\, \frac{\partial}{\partial t}\right) \quad \textit{and} \quad \rho^4 = t^2 + |z|^4. \tag{2.1}$$

2.2. Rossi spheres

We recall here some properties of *Rossi spheres*, introduced in [25] as a non-embeddable example of CR manifold (see also [5]). These are families of CR structures on S^3, containing the standard one, obtained in the following way.

Considering the complex vector field Z_1 as in (1.4) and its conjugate $Z_{\bar{1}}$, one defines the CR structure $J_{(s)}$ by setting $J_{(s)} Z_{1(s)} = i Z_{1(s)}$, where

$$Z_{1(s)} = Z_1 + \frac{s}{\sqrt{1+s^2}}\, Z_{\bar{1}} \quad \text{and} \quad Z_{\bar{1}(s)} = Z_{\bar{1}} + \frac{s}{\sqrt{1+s^2}}\, Z_1. \tag{2.2}$$

Corresponding to these vector fields, we have the dual forms

$$\theta^1_{(s)} = (1+s^2)\theta^1 - s\sqrt{1+s^2}\, \theta^{\bar{1}} \quad \text{and} \quad \theta^{\bar{1}}_{(s)} = (1+s^2)\theta^{\bar{1}} - s\sqrt{1+s^2}\, \theta^1,$$

where $\theta^1 = z^2\, dz^1 - z^1\, dz^2$. Compute

$$i\, \theta^1_{(s)} \wedge \theta^{\bar{1}}_{(s)} = (1+s^2)\, i\theta^1 \wedge \theta^{\bar{1}} = (1+s^2)\, d\hat{\theta}, \tag{2.3}$$

where $d\hat{\theta} = i\theta^1 \wedge \theta^{\bar{1}}$, i.e., $h_{1\bar{1}} = 1$. Hence, from (2.3) we get

$$h^{(s)}_{1\bar{1}} = \frac{1}{1+s^2} \quad \text{and} \quad h^{1\bar{1}}_{(s)} := (h^{(s)}_{1\bar{1}})^{-1} = 1 + s^2.$$

By taking

$$\tilde{\theta}^1_{(s)} = \frac{1}{\sqrt{2(1+s^2)}}\,\theta^1_{(s)},$$

we have $\tilde{h}^{(s)}_{1\bar{1}} = 2$. The Webster curvature R of $(J, \hat{\theta})$ is identically equal to 2. Then we should take $\omega_1^1 = -2i\hat{\theta}$ in the structure equation (1.3), such that $d\omega_1^1 = 2\theta^1 \wedge \theta^{\bar{1}}$. We can then determine, from the structure equation for $(J_{(s)}, \hat{\theta})$, that

$$\omega_{1(s)}^1 = -2i(1+2s^2)\,\hat{\theta}, \quad h^{1\bar{1}}_{(s)} A_{\bar{1}\bar{1}(s)} = 4is\sqrt{1+s^2} \quad \text{and} \quad R_{(s)} = 2(1+2s^2).$$

Dual to $\theta^1 = z^2 dz^1 - z^1 dz^2$, we have

$$Z_1 = Z_1^{S^3} = z^{\bar{2}}\frac{\partial}{\partial z^1} - z^{\bar{1}}\frac{\partial}{\partial z^2}.$$

The sub-Laplacian associated to $(J_{(s)}, \hat{\theta})$ reads

$$\triangle_b^{(s)} = h^{1\bar{1}}_{(s)}(Z_{1(s)}Z_{\bar{1}(s)} + Z_{\bar{1}(s)}Z_{1(s)}) = (1+2s^2)\triangle_b^{(0)} + 2s\sqrt{1+s^2}\,(Z_1^2 + Z_{\bar{1}}^2).$$

It follows that, at $s = 0$, the first-and second-order derivatives of $\triangle_b^{(s)}$ with respect to s are given by

$$-\dot{\Delta}_b = 2Z_{\bar{1}}Z_{\bar{1}} + \text{conj.} \quad \text{and} \quad -\ddot{\Delta}_b = -4\Delta_b. \tag{2.4}$$

Moreover, since $R_s = 2(1+2s^2)$, it follows that, still at $s = 0$,

$$\dot{R} = 0 \quad \text{and} \quad \ddot{R} = 8. \tag{2.5}$$

We next analyze a symmetry property of Rossi spheres, that will imply in particular the symmetry of the mass in s. Consider the diffeomorphism $\iota \colon S^3 \to S^3$ defined by

$$\iota(z^1, z^2) = (iz^1, z^2), \tag{2.6}$$

which fixes the point (0, 1). A direct computation shows that $\iota_* Z_1^{S^3} = iZ_1^{S^3}$, and hence $\iota_* Z_{\bar{1}}^{S^3} = (-i)Z_{\bar{1}}^{S^3}$. By (2.2), we compute

$$\iota_* Z_{1(s)} = \iota_* Z_1 + \frac{s}{\sqrt{1+s^2}}\,\iota_* Z_{\bar{1}} = iZ_1 + \frac{s}{\sqrt{1+s^2}}(-i)Z_{\bar{1}} = iZ_{1(-s)}. \tag{2.7}$$

It follows that

$$(\iota^* J_{(-s)})Z_{1(s)} = \iota_*^{-1} J_{(-s)}(\iota_* Z_{1(s)}) \overset{\text{(by (2.7))}}{=} \iota_*^{-1} J_{(-s)}(iZ_{1(-s)}) = \iota_*^{-1}(-Z_{1(-s)})$$
$$\overset{\text{(by the inverse of (2.7))}}{=} (-1)(-i)Z_{1(s)} = iZ_{1(s)} = J_{(s)}Z_{1(s)}.$$

Hence we have shown

$$J_{(s)} = \iota^* J_{(-s)}. \tag{2.8}$$

Let $v_{(s)}$ denote the conformal factor in $\check{\theta}_{(s)} = e^{2v_{(s)}}\hat{\theta}$, yielding CR normal coordinates with respect to $J_{(s)}$. It then follows that

$$v_{(s)} = \iota^* v_{(-s)} \quad \text{and} \quad \check{\theta}_{(s)} = \iota^* \check{\theta}_{(-s)}, \tag{2.9}$$

and hence $\check{G}_s = \iota^* \check{G}_{-s}$ by observing

$$\iota^* \hat{\theta} = \hat{\theta}.$$

Write

$$\check{G}_s = 2\rho_s^{-2} + A_s + O(\rho_s)$$

in s-CR normal coordinates near $(0, 1)$. Then $\rho_s = \iota^* \rho_{-s} = \rho_{-s} \circ \iota$, and

$$A_s = \iota^* A_{-s} = A_{-s} \circ \iota = A_{-s}$$

near the point $(0, 1)$. So, we have obtained

$$m(J_{(s)}, \theta_{(s)}) = 12\pi A_s = 12\pi A_{-s} = m(J_{(-s)}, \theta_{(-s)}),$$

where $\theta_{(s)} = \check{G}_{(s)}^2 \check{\theta}_{(s)}$. This property (and other related ones) will be crucial in the last section of the paper.

3. CR normal coordinates on Rossi spheres

In this section, we will find the main-order terms of CR normal coordinates on Rossi spheres. We first determine the principal term in the required conformal factor, then discuss pseudo-hermitian coordinates and finally CR normal coordinates. This will allow us to express with a good precision the Green function of the conformal sub-Laplacian near its pole.

3.1. Conformal factor in normalized contact form on Rossi spheres

Fix $p = (0, 1) \in S^3 \subseteq \mathbb{C}^2$, and consider a contact form $\check{\theta}'_{(s)} = e^{2v_{(s)}}\hat{\theta}'$, where $\hat{\theta}' = 2\hat{\theta} = i(\bar{\partial} - \partial)(|z^1|^2 + |z^2|^2)$, yielding CR normal coordinates (see Proposition 2.3) with respect to $J_{(s)}$ for $N = 4$. We are going to solve an equation for $v_{(s)}$ as in Lemma 3.11 of Jerison–Lee's paper ([21]). Write

$$v_{(s)} = v_2 + v_3 + \cdots, \tag{3.1}$$

where $v_2 \in \mathcal{R}_2 \subset \mathcal{P}_2$ and $v_3 \in \mathcal{P}_3$. Recall that, in the notation of [21], $\mathcal{P}_m$ denotes the vector space of polynomials in (z, t) that are homogeneous of degree m in terms of parabolic dilations (for which t has homogeneity 2), and $\mathcal{R}_m \subseteq \mathcal{P}_m$ denotes the subspace of polynomials independent of t.

First, write $v_2 \in \mathcal{R}_2$ as $v_2 = az^2 + bz\bar{z} + c\bar{z}^2$ ((z,t) being pseudo-hermitian normal coordinates for $\hat{\theta}'$ at p), satisfying

$$L_2 v_2 = -z^2 Q_{11} - \bar{z}^2 Q_{\bar{1}\bar{1}} - z\bar{z} Q_{1\bar{1}} - \bar{z} z Q_{\bar{1}1} \quad \text{and} \quad L_2 = -2|z|^2(\partial_z\partial_{\bar{z}} + \partial_{\bar{z}}\partial_z) - 12.$$

Here $Q_{11} = 3iA^{\mathrm{JL}}_{11(s)} = Q_{\bar{1}\bar{1}}$ and $Q_{1\bar{1}} = R^{\mathrm{JL}}_{1\bar{1}(s)} = Q_{\bar{1}1}$ are with respect to the Jerison–Lee coframe $\theta^1_{\mathrm{JL}} = \theta^1_{(s)}/\sqrt{1+s^2}$, with $h^{\mathrm{JL}}_{1\bar{1}(s)} = 2$ with respect to $\hat{\theta}'$ by the formulas for Q_{jk} on p. 315 in [21] and (2.3). We compute

$$\tilde{Q}_{11} = 3iA_{11(s)} = \frac{12s}{\sqrt{1+s^2}} = \tilde{Q}_{\bar{1}\bar{1}} \quad \text{and} \quad \tilde{Q}_{1\bar{1}} = R_{1\bar{1}(s)} = h^{(s)}_{1\bar{1}} R_{(s)} = 2\,\frac{1+2s^2}{1+s^2},$$

with respect to the co-frame $\theta^1_{(s)}$. A direct computation shows that

$$L_2 v_2 = -12a\,z^2 - 12c\,\bar{z}^2 - 16b\,|z|^2,$$

where $Q_{11} = 12a$, $Q_{\bar{1}\bar{1}} = 12c$, $Q_{1\bar{1}} = Q_{\bar{1}1} = 8b$, and

$$a = c = s\sqrt{1+s^2} \quad \text{and} \quad b = \frac{1}{4}(1+2s^2). \tag{3.2}$$

For v_3, we observe that all $Q_{jk,l}$'s for j,k,l being 1 or $\bar{1}$ vanish since the space derivatives of the constant R_0 is zero. On the other hand, Q_{0k} and Q_{k0} for $k = 1$ or $\bar{1}$ also vanish since they involve space derivatives by formulas on p. 315 in [21]. Altogether, the right-hand side of the equation in Lemma 3.11 in [21] for $m = 3$ equals zero, so we have

$$L_3 v_3 = 0.$$

By Lemma 3.9 in [21], we learn that L_3 is invertible on $\mathcal{P}_3$. It follows that

$$v_3 = 0. \tag{3.3}$$

Therefore, from (3.2) and (3.3), we get the following result.

Lemma 3.1. *In pseudo-hermitian coordinates, the conformal factor expands in homogeneous powers as*

$$v_{(s)} = s\sqrt{1+s^2}\,(z^2 + \bar{z}^2) + \frac{1}{4}(1+2s^2)\,|z|^2 + v_4 + \cdots$$

3.2. Pseudo-hermitian normal coordinates on Rossi spheres

Recall that, on Rossi spheres, we have

$$\theta^1_{(s)} = (1+s^2)\theta^1 - s\sqrt{1+s^2}\,\theta^{\bar{1}} \quad \text{and} \quad \omega_1{}^1{}_{(s)} = -i(1+2s^2)2\hat{\theta},$$

and that pseudo-hermitian coordinates near $(0,1)$ are defined by the equation

$$\nabla_{\dot{\sigma}}\dot{\sigma} = 2c\,\hat{T}', \quad \sigma(0) = (0,1), \tag{3.4}$$

where $\hat{T}'$ is the unique vector field such that $\hat{\theta}'(\hat{T}') = 1$ and $d\hat{\theta}'(\hat{T}', \cdot) = 0$. Recall also that

$$\hat{\theta}' = i\sum_{i=1}^{2}(z^i d\bar{z}^i - \bar{z}^i dz^i) \text{ and } \hat{T}' = -\mathrm{Im}\Big(z^1\frac{\partial}{\partial z_1} + z^2\frac{\partial}{\partial z^2}\Big) = \frac{1}{2}i\sum_{i=1}^{2}\Big(z^i\frac{\partial}{\partial z_i} - \bar{z}^i\frac{\partial}{\partial \bar{z}_i}\Big).$$

Setting

$$\dot{\sigma} = \alpha\, Z^{\mathrm{JL}}_{1(s)} + \beta\, Z^{\mathrm{JL}}_{\bar{1}(s)} + \gamma\, \hat{T}',$$

equation (3.4) becomes

$$\begin{aligned} 2c\hat{T}' &= \nabla_{\dot{\sigma}}\dot{\sigma} = (\dot{\alpha} + \alpha\,\omega^1_{1(s)}(\dot{\sigma}))\, Z^{\mathrm{JL}}_{1(s)} + (\dot{\beta} + \beta\,\omega^{\bar{1}}_{\bar{1}(s)}(\dot{\sigma}))\, Z^{\mathrm{JL}}_{\bar{1}(s)} + \dot{\gamma}\,\hat{T}' \\ &= (\dot{\alpha} - i\alpha(1+2s^2)\gamma)\, Z^{\mathrm{JL}}_{1(s)} + (\dot{\beta} + i\beta(1+2s^2)\gamma)\, Z^{\mathrm{JL}}_{\bar{1}(s)} + \dot{\gamma}\,\hat{T}'. \end{aligned} \tag{3.5}$$

If τ parametrizes the curve σ, the above formulas imply that

$$\gamma = 2c\tau, \quad \frac{\dot{\alpha}}{\alpha} = i(1+2s^2)\gamma \quad \text{and} \quad \frac{\dot{\beta}}{\beta} = -i(1+2s^2)\gamma,$$

which in turn yields

$$\alpha(t) = \alpha(0)\, e^{ic(1+2s^2)\tau^2} \quad \text{and} \quad \beta(t) = \beta(0)\, e^{-ic(1+2s^2)\tau^2}.$$

Therefore, we obtained

$$\dot{\sigma} = \alpha(0)\, e^{ic(1+2s^2)\tau^2}\, Z^{\mathrm{JL}}_{1(s)} + \beta(0)\, e^{-ic(1+2s^2)\tau^2}\, Z^{\mathrm{JL}}_{\bar{1}(s)} + 2c\tau\hat{T}.$$

Recall also that $Z_{1(s)} = Z_{1(0)} + \frac{s}{\sqrt{1+s^2}} Z_{\bar{1}(0)}$. Hence we need to solve for

$$\begin{aligned} \dot{z}_1(\tau) &= dz_1(\dot{\sigma}(\tau)) \\ &= \alpha(0)\, e^{ic(1+2s^2)\tau^2}\, dz_1(Z^{\mathrm{JL}}_{1(s)}) + \beta(0)\, e^{-ic(1+2s^2)\tau^2}\, dz_1(Z^{\mathrm{JL}}_{\bar{1}(s)}) + 2c\tau\, dz_1(\hat{T}') \\ &= \sqrt{1+s^2}\Big(\alpha(0)\, e^{i\delta\tau^2}\, \bar{z}_2(\tau) + \beta(0)\, e^{-i\delta\tau^2}\frac{s}{\sqrt{1+s^2}}\, \bar{z}_2(\tau)\Big) + c\tau i\, z_1(\tau), \end{aligned}$$

where $\delta = c(1+2s^2)$. Similarly, we obtain

$$\dot{z}_2(\tau) = \sqrt{1+s^2}\Big(-\alpha(0)\, e^{i\delta\tau^2}\, \bar{z}_1(\tau) - \beta(0)\, e^{-i\delta\tau^2}\frac{s}{\sqrt{1+s^2}}\, \bar{z}_1(\tau)\Big) + c\tau i\, z_2(\tau).$$

Once we will solve for this system, the pseudo-hermitian coordinates will be given by the map

$$(z, \bar{z}, t) = (\alpha(0)\tau, \beta(0)\tau, c\tau^2) \quad \longmapsto \quad (0,1) + \int_0^t \dot{\sigma}(\eta)\, d\eta. \tag{3.6}$$

Setting, for simplicity,

$$A_0 = \alpha(0), \quad B_0 = \beta(0)\frac{s}{\sqrt{1+s^2}}, \quad C_0 = 2c,$$
$$F_0(\tau) := \sqrt{1+s^2}\,(A_0\, e^{i\delta\tau^2} + B_0\, e^{-i\delta\tau^2}) = f_0(\tau) + i g_0(\tau),$$

we have then the system of ODEs

$$\dot{z}_1(\tau) = F_0(\tau)\,\bar{z}_2(\tau) + iC_0\,\tau\,z_1(\tau), \quad \dot{z}_2(\tau) = -F_0(\tau)\,\bar{z}_1(\tau) + iC_0\,\tau\,z_2(\tau),$$

which in real form becomes

$$\begin{cases} \dot{x}_1(\tau) = f_0(\tau)\,x_2(\tau) + g_0(\tau)\,y_2(\tau) - C_0\tau\,y_1(\tau), \\ \dot{y}_1(\tau) = g_0(\tau)\,x_2(\tau) - f_0(\tau)\,y_2(\tau) + C_0\tau\,x_1(\tau), \\ \dot{x}_2(\tau) = -f_0(\tau)\,x_1(\tau) - g_0(\tau)\,y_1(\tau) - C_0\tau\,y_2(\tau), \\ \dot{y}_2(\tau) = f_0(\tau)\,y_1(\tau) - g_0(\tau)\,x_1(\tau) + C_0\tau\,x_2(\tau). \end{cases}$$

We rewrite this system as

$$\dot{\mathfrak{X}}(\tau) = \mathfrak{A}(\tau)\,\mathfrak{X}(\tau),$$

where

$$\mathfrak{A}(\tau) = \begin{pmatrix} 0 & -C_0\tau & f_0(\tau) & g_0(\tau) \\ C_0\tau & 0 & g_0(\tau) & -f_0(\tau) \\ -f_0(\tau) & -g_0(\tau) & 0 & -C_0\tau \\ -g_0(\tau) & f_0(\tau) & C_0\tau & 0 \end{pmatrix}.$$

We can Taylor-expand the solution to an arbitrary order in τ. Differentiating the above ODE, we obtain

$$\begin{aligned} \ddot{\mathfrak{X}}(\tau) &= \dot{\mathfrak{A}}(\tau)\,\mathfrak{X}(\tau) + \mathfrak{A}(\tau)^2\,\mathfrak{X}(\tau), \\ \dddot{\mathfrak{X}}(\tau) &= \ddot{\mathfrak{A}}(\tau)\,\mathfrak{X}(\tau) + (\mathfrak{A}(\tau)\,\dot{\mathfrak{A}}(\tau) + 2\dot{\mathfrak{A}}(\tau)\,\mathfrak{A}(\tau))\,\mathfrak{X}(\tau) + \mathfrak{A}(\tau)^3\,\mathfrak{X}(\tau). \end{aligned}$$

We have that

$$\mathfrak{A}(0) = \sqrt{1+s^2}\begin{pmatrix} 0 & 0 & \mathrm{Re}A_0 + \mathrm{Re}B_0 & \mathrm{Im}A_0 + \mathrm{Im}B_0 \\ 0 & 0 & \mathrm{Im}A_0 + \mathrm{Im}B_0 & -\mathrm{Re}A_0 - \mathrm{Re}B_0 \\ -\mathrm{Re}A_0 - \mathrm{Re}B_0 & -\mathrm{Im}A_0 - \mathrm{Im}B_0 & 0 & 0 \\ -\mathrm{Im}A_0 - \mathrm{Im}B_0 & \mathrm{Re}A_0 + \mathrm{Re}B_0 & 0 & 0 \end{pmatrix},$$

$$\dot{\mathfrak{A}}(0) = \begin{pmatrix} 0 & -C_0 & 0 & 0 \\ C_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -C_0 \\ 0 & 0 & C_0 & 0 \end{pmatrix},$$

$$\ddot{\mathfrak{A}}(0) = \begin{pmatrix} 0 & 0 & 2d(\mathrm{Im}B_0 - \mathrm{Im}A_0) & -2d(\mathrm{Re}B_0 - \mathrm{Re}A_0) \\ 0 & 0 & -2d(\mathrm{Re}B_0 - \mathrm{Re}A_0) & 2d(\mathrm{Im}A_0 - \mathrm{Im}B_0) \\ 2d(\mathrm{Im}A_0 - \mathrm{Im}B_0) & 2d(\mathrm{Re}B_0 - \mathrm{Re}A_0) & 0 & 0 \\ 2d(\mathrm{Re}B_0 - \mathrm{Re}A_0) & 2d(\mathrm{Im}B_0 - \mathrm{Im}A_0) & 0 & 0 \end{pmatrix},$$

where $d = \delta\sqrt{1+s^2}$. In conclusion, looking at the first three terms in the Taylor expansion of $\mathfrak{X}(\tau)$ near (0, 1), we find that

$$\mathfrak{X}(\tau) = \begin{pmatrix} \tilde{\tau}(\mathrm{Re}A_0 + \mathrm{Re}B_0) \\ \tilde{\tau}(\mathrm{Im}A_0 + \mathrm{Im}B_0) \\ 1 - \frac{1}{2}\tilde{\tau}^2\left((\mathrm{Im}A_0 + \mathrm{Im}B_0)^2 + (\mathrm{Re}A_0 + \mathrm{Re}B_0)^2\right) \\ \frac{C_0\tilde{\tau}^2}{2}\,\frac{1}{2(1+s^2)} \end{pmatrix} + o(\tau^2), \quad \tilde{\tau} = \sqrt{1+s^2}\,\tau.$$

Recalling (3.6), we then obtain the following result.

Lemma 3.2. *Pseudo-hermitian normal coordinates near* $(0,1)$ *on Rossi spheres with respect to* $\hat{\theta}' = 2\hat{\theta}$ *are given by the following map*:

$$(z,\bar{z},t) \longmapsto \begin{pmatrix} \sqrt{(1+s^2)}\left(z + \frac{s}{\sqrt{1+s^2}}\,\bar{z}\right) \\ 1 - \frac{1}{2}(1+s^2)\left|z + \frac{s}{\sqrt{1+s^2}}\,\bar{z}\right|^2 + i\,\frac{t}{2} \end{pmatrix} + o(\rho^2).$$

Inverting in the first component, we have in particular that

$$z = \frac{1}{\sqrt{1+s^2}}\,(1+s^2)\left(z_1 - \frac{s}{\sqrt{1+s^2}}\,\bar{z}_1\right) + o(\rho^2). \tag{3.7}$$

3.3. CR normal coordinates

Recalling (3.1), Lemma 3.1 and using (3.7), we get

$$\begin{aligned} v_2 &= \frac{1}{4}\left(3\,(z_1^2 + \bar{z}_1^2)\,s\sqrt{s^2+1}\,(2s^2+1) - |z_1|^2\,(12s^4 + 12s^2 - 1)\right) \\ &= A_1\,(z_1^2 + \bar{z}_1^2) + B_1\,|z_1|^2, \end{aligned}$$

where

$$A_1 = \frac{1}{8}\,2\,(s^2+1)^{1/2}\,3s\,(2s^2+1) \quad \text{and} \quad B_1 = -\frac{1}{8}\,2\,(12s^4 + 12s^2 - 1).$$

Recall that also

$$\hat{\theta}^1_{(0)} = z^2\,dz^1 - z^1\,dz^2 \quad \text{and} \quad \hat{\theta}' = i\sum_{i=1}^{2}(z^i\,d\bar{z}^i - \bar{z}^i\,dz^i),$$

and that

$$\hat{\theta}^1_{(s)} = (1+s^2)\,\hat{\theta}^1_{(0)} - s\sqrt{1+s^2}\;\hat{\theta}^{\bar{1}}_{(0)}.$$

Conformally changing the contact form and recalling Appendix 1.1.1 in [11], we have that $\hat{\theta}^1_{(s)}$ transforms as

$$\hat{\theta}^1_{(s)} \;\longmapsto\; e^{v}\,(\hat{\theta}^1_{(s)} + 2i\,v^1\hat{\theta}'), \tag{3.8}$$

where $d\hat{\theta}' = 2i\,\hat{\theta}^1_{(0)} \wedge \hat{\theta}^{\bar{1}}_{(0)}$ and

$$v^1 = h'^{1\bar{1}}_{(s)}\,\hat{Z}_{\bar{1}(s)}v = \frac{(1+s^2)}{2}\,\hat{Z}_{\bar{1}(s)}v.$$

By computing explicitly, it turns out that

$$(v_2)^1 = \left(\frac{(s^2+1)}{2}\,\frac{\bar{z}_2\,s(2A_1\,z_1 + B_1\,\bar{z}_1)}{\sqrt{s^2+1}} + \frac{(s^2+1)}{2}\,z_2\,(2A_1\,\bar{z}_1 + B_1\,z_1)\right),$$

which can be written as

$$(v_2)^1 = A_2\,z_1\,z_2 + B_2\,z_1\,\bar{z}_2 + C_2\,\bar{z}_1\,z_2 + D_2\,\bar{z}_1\,\bar{z}_2, \tag{3.9}$$

with

$$A_2 = \frac{1}{2}(s^2+1)B_1, \quad B_2 = sA_1\sqrt{s^2+1}, \quad C_2 = A_1\,(s^2+1), \quad D_2 = \frac{1}{2}\,sB_1\sqrt{s^2+1}.$$

Up to higher order terms, we have that

$$(v_2)^1 = (A_2 + B_2) z_1 + (C_2 + D_2) \bar{z}_1.$$

Taylor expanding (3.8), up to higher-order terms, $\hat{\theta}^1_{(s)}$ transforms into

$$\hat{\theta}^1_{(s)} + \big[v_2\, \hat{\theta}^1_{(s)} + 2i\,(v_2)^1\, \hat{\theta}'\big].$$

We now multiply by a complex unit factor $e^{i\psi}$, and impose a closeness condition on $e^{i\psi}$ multiplied by the latter form, up to higher-order terms, since by Proposition 2.4 it should be approximately a constant multiple of $d\tilde{z}_{\mathrm{CR}}$, up to higher order terms. We then find

$$\begin{aligned} 0 &= d\big\{e^{i\psi}\big(\hat{\theta}^1_{(s)} + [v_2\, \hat{\theta}^1_{(s)} + 2i\,(v_2)^1\, \hat{\theta}']\big)\big\} \\ &= e^{i\psi}\big\{i\, d\psi \wedge \hat{\theta}^1_{(s)} + i\, d\psi \wedge [v_2\, \hat{\theta}^1_{(s)} + 2i\,(v_2)^1\, \hat{\theta}']\big\} \\ &\quad + e^{i\psi}\big\{d\hat{\theta}^1_{(s)} + dv_2 \wedge \hat{\theta}^1_{(s)} + v_2\, d\hat{\theta}^1_{(s)} + 2i\, d((v_2)^1) \wedge \hat{\theta}' + 2i\,(v_2)^1 d\hat{\theta}'\big\} + \text{h.o.t.} \end{aligned} \tag{3.10}$$

We also have

$$d\hat{\theta}^1_{(s)} = 2(1+s^2)\, dz^2 \wedge dz^1 - 2s\sqrt{1+s^2}\, d\bar{z}^2 \wedge d\bar{z}^1.$$

We expand $d\psi$ and $\hat{\theta}^1_{(s)}$ in homogeneous powers of the (z,t) coordinates (with respect to parabolic scaling, including differentials) as follows:

$$d\psi = (d\psi)_0 + (d\psi)_1 + (d\psi)_2 + \cdots.$$

Taylor-expanding the above system up to order one, we obtain the relations

$$\begin{cases} (d\psi)_0 \wedge \hat{\theta}^1_{(s)} = 0, \\ i(d\psi)_1 \wedge \hat{\theta}^1_{(s)} + d\hat{\theta}^1_{(s)} + (dv_2) \wedge \hat{\theta}^1_{(s)} + 2i(v_2)^1 d\hat{\theta}' + 2i(d(v_2)^1) \wedge \hat{\theta}' = 0. \end{cases} \tag{3.11}$$

The first component is easy to solve setting $(d\psi)_0 = \mu\, \hat{\theta}^1_{(s)}$ for some $\mu \in \mathbb{R}$.

For the second component, recall that we have

$$\hat{\omega}_{1(s)}^{1} = -i(1+2s^2)\,\hat{\theta}' \quad \text{and} \quad \hat{A}^1_{\bar{1}(s)} = 2is\sqrt{1+s^2}.$$

It then follows that

$$d\hat{\theta}^1_{(s)} = \hat{\theta}^1_{(s)} \wedge \hat{\omega}^1_{1(s)} + \hat{A}^1_{\bar{1}(s)}\, \hat{\theta}' \wedge \hat{\theta}^{\bar{1}}_{(s)} = -i(1+2s^2)\, \hat{\theta}^1_{(s)} \wedge \hat{\theta}' + 2is\sqrt{1+s^2}\, \hat{\theta}' \wedge \hat{\theta}^{\bar{1}}_{(s)}.$$

Moreover, we have that

$$d\hat{\theta}' = 2ih_{1\bar{1}(s)}\, \hat{\theta}^1_{(s)} \wedge \hat{\theta}^{\bar{1}}_{(s)} = \frac{2i}{1+s^2}\, \hat{\theta}^1_{(s)} \wedge \hat{\theta}^{\bar{1}}_{(s)},$$

and that (up to $\hat{\theta}'$)

$$d(v_2)^1 = \hat{Z}_{1(s)}(v_2)^1\, \hat{\theta}^1_{(s)} + \hat{Z}_{\bar{1}(s)}(v_2)^1\, \hat{\theta}^{\bar{1}}_{(s)}.$$

By the above expression of $(v_2)^1$ and (3.9), this becomes

$$\begin{aligned} d(v_2)^1 &= \big[(s^2+\tfrac{1}{2})\,B_1(|z_2|^2-|z_1|^2)+B_2\,\bar z_2^2-C_2\,\bar z_1^2+s\sqrt{1+s^2}\,A_1\,z_2^2-s^2\,A_1\,z_1^2\big]\,\hat\theta^1_{(s)} \\ &\quad+\big[s\sqrt{1+s^2}\,B_1(|z_2|^2-|z_1|^2)-B_2\,z_1^2+s^2\,A_1\,\bar z_2^2-s\sqrt{1+s^2}\,A_1\,\bar z_1^2 \\ &\quad+C_2\,z_2^2\big]\,\hat\theta^{\bar 1}_{(s)} \qquad \mathrm{mod}\ \hat\theta'. \end{aligned}$$

We next write

$$(d\psi)_1=(A_3z_1+B_3\bar z_1)\,\hat\theta^1_{(s)}+(\overline{A}_3\,\bar z_1+\overline{B}_3\,z_1)\,\hat\theta^{\bar 1}_{(s)}+C_3\,\hat\theta', \quad \overline{A}_3=-A_3,\ \overline{B}_3=-B_3.$$

The $(\hat\theta^{\bar 1}_{(s)}\wedge\hat\theta^1_{(s)})$-component of the second equation in (3.11) is given by

$$i\,(\overline{A}_3\,\bar z_1+\overline{B}_3\,z_1)+\frac{6}{1+s^2}\,(v_2)^1=0.$$

This determines A_3 and B_3 by

$$i\,\overline{A}_3+\frac{6}{1+s^2}(C_2+D_2)=0, \quad i\,\overline{B}_3+\frac{6}{1+s^2}(A_2+B_2)=0,$$

giving

$$A_3=-\frac{3}{8}\,\frac{2is}{\sqrt{1+s^2}}(7+6s^2) \quad \text{and} \quad B_3=-\frac{3}{8}\,2i\,(1-6s^2).$$

Next, the $(\hat\theta'\wedge\hat\theta^1_{(s)})$-component gives

$$iC_3+i(1+2s^2)+\hat T v_2-2i\,\hat Z_{1(s)}(v_2)^1=0.$$

Finally, the $(\hat\theta'\wedge\hat\theta^{\bar 1}_s)$-component of the second equation in (3.11) is given by

$$2is\sqrt{1+s^2}-2i\,\hat Z_{\bar 1(s)}(v_2)^1=0.$$

This is true because, as one can check,

$$\begin{aligned} \hat Z_{\bar 1(s)}(v_2)^1 &= (C_2+D_2)+\frac{s}{\sqrt{1+s^2}}\,(A_2+B_2), \\ \hat Z_{1(s)}(v_2)^1 &= (A_2+B_2)+\frac{s}{\sqrt{1+s^2}}\,(C_2+D_2). \end{aligned}$$

By a direct computation, it follows that

$$\hat Z_{\bar 1(s)}(v_2)^1=s(s^2+1)^{1/2} \quad \text{and} \quad \hat Z_{1(s)}(v_2)^1=\frac{1}{8}\,(2s^2+1).$$

These also imply

$$C_3=-\frac{3}{4}\,(1+2s^2)=-\frac{3}{2}\,s^2-\frac{3}{4}\cdot$$

Let us now try to integrate for the phase ψ. There holds

$$\begin{aligned} \hat\theta^1_{(s)} &= (1+s^2)\,dz_1-s\sqrt{1+s^2}\,d\bar z_1+\text{h.o.t.}, \\ \hat\theta' &= i\,[(z_1\,d\bar z_1-\bar z_1\,dz_1)+(d\bar z_2-dz_2)]+\text{h.o.t.} \end{aligned}$$

In this way, we have that $(d\psi)_1$ becomes

$$\begin{aligned}&\big[(A_3 z_1 + B_3 \bar z_1)(1+s^2) + (A_3 \bar z_1 + B_3 z_1) s\sqrt{1+s^2} - iC_3 \bar z_1\big]\, dz_1 \\ &\qquad + \text{conj.} - iC_3\,(dz_2 - d\bar z_2) \\ &\quad = \big\{\big[(1+s^2)A_3 + s\sqrt{1+s^2}\,B_3\big]z_1 + \big[(1+s^2)\,B_3 + s\sqrt{1+s^2}\,A_3 - iC_3\big]\bar z_1\big\}\, dz_1 \\ &\qquad + \text{conj.} - iC_3(dz_2 - d\bar z_2).\end{aligned}$$

Since $(1+s^2)B_3 + s\sqrt{1+s^2}A_3 - iC_3 = 0$, we get

$$(d\psi)_1 = A_4 z_1\, dz_1 + B_4\, dz_2 + \text{conj.},$$

with

$$A_4 = -6is\sqrt{1+s^2} \quad \text{and} \quad B_4 = -iC_3 = \frac{3}{4}\, i(1+2s^2).$$

Integrating, we find

$$(\psi)_2 = \frac{1}{2}\, A_4 z_1^2 + B_4 z_2 + \text{conj.}$$

Taylor-expanding, we then get

$$dz'_{\mathrm{CR}} = \hat\theta^1_{(s)}(1 + v_2 + i(\psi)_2) + 2i(v_2)^1\,\hat\theta' + \text{h.o.t.} \tag{3.12}$$

Writing the homogeneous 0-th and 2nd order terms in (z,t) of the right-hand side, we obtain

$$\begin{aligned}&\big[1 + A_1(z_1^2 + \bar z_1^2) + \tfrac{1}{2} iA_4(z_1^2 - \bar z_1^2) + B_1|z_1|^2 + iB_4(z_2 - \bar z_2)\big] \\ &\qquad \times \big\{(1+s^2) z_2\, dz_1 - s\sqrt{1+s^2}\,\bar z_2\, d\bar z_1 - (1+s^2) z_1\, dz_2 + s\sqrt{1+s^2}\,\bar z_1\, d\bar z_2\big\} \\ &\qquad - 2\big[(A_2+B_2)\, z_1 + (C_2 + D_2)\,\bar z_1\big]\big[z_1\, d\bar z_1 - \bar z_1\, dz_1 + d\bar z_2 - dz_2\big].\end{aligned}$$

Further expanding this, gives

$$\begin{aligned}&(1+s^2) z_2\, dz_1 - s\sqrt{1+s^2}\,\bar z_2\, d\bar z_1 - (1+s^2) z_1\, dz_2 + s\sqrt{1+s^2}\,\bar z_1\, d\bar z_2 \\ &\quad + [A_1(z_1^2 + \bar z_1^2) + \frac{1}{2}\, iA_4(z_1^2 - \bar z_1^2) + B_1|z_1|^2 + iB_4(z_2 - \bar z_2)] \\ &\qquad \times [(1+s^2)\, dz_1 - s\sqrt{1+s^2}\, d\bar z_1] \\ &\quad - 2[(A_2+B_2)z_1 + (C_2+D_2)\,\bar z_1][z_1\, d\bar z_1 - \bar z_1\, dz_1 + d\bar z_2 - dz_2].\end{aligned} \tag{3.13}$$

We next set

$$w = z_2 - \bar z_2,$$

and rewrite the terms involving z_2 as

$$\begin{aligned}&z_2 = 1 + \frac{1}{2}\, w - \frac{1}{2}\, |z_1|^2, \quad \bar z_2 = 1 - \frac{1}{2}\, w - \frac{1}{2}\, |z_1|^2, \\ &dz_2 = \frac{1}{2}\, dw - \frac{1}{2}\,(\bar z_1\, dz_1 + z_1\, d\bar z_1), \quad d\bar z_2 = -\frac{1}{2}\, dw - \frac{1}{2}\,(\bar z_1 dz_1 + z_1 d\bar z_1).\end{aligned} \tag{3.14}$$

Here, Lemma 3.2 has been used. Write (3.13) as

$$
\begin{aligned}
(3.15)\quad & C_6(z_2\,dz_1 - z_1\,dz_2) + D_6(\bar z_2\,d\bar z_1 - \bar z_1\,d\bar z_2) \\
& + [A_1(z_1^2+\bar z_1^2) + \frac{1}{2}iA_4(z_1^2-\bar z_1^2) + B_1|z_1|^2 + iB_4 w][C_8\,dz_1 + D_8\,d\bar z_1] \\
& - 2[(A_2+B_2)z_1 + (C_2+D_2)\bar z_1][z_1\,d\bar z_1 - \bar z_1\,dz_1 - dw],
\end{aligned}
$$

where

$$C_6 = C_8 = 1 + s^2 \quad\text{and}\quad D_6 = D_8 = -s\sqrt{1+s^2}.$$

We now substitute $C_6(z_2dz_1 - z_1dz_2) + D_6(\bar z_2 d\bar z_1 - \bar z_1 d\bar z_2) = C_6[(1+\frac{1}{2}w)dz_1 + \frac{1}{2}z_1^2 d\bar z_1 - \frac{1}{2}z_1 dw] + D_6[(1-\frac{1}{2}w)d\bar z_1 + \frac{1}{2}\bar z_1^2 dz_1 + \frac{1}{2}\bar z_1 dw]$ into (3.15) and collect terms involving w as follows:

$$
\begin{aligned}
(3.16)\quad & \Big(iB_4C_8 + \frac{1}{2}C_6\Big)w\,dz_1 + \Big[2(A_2+B_2) - \frac{1}{2}C_6\Big]z_1\,dw \\
& + \Big(iB_4D_8 - \frac{1}{2}D_6\Big)w\,d\bar z_1 + \Big[2(C_2+D_2) + \frac{1}{2}D_6\Big]\bar z_1\,dw.
\end{aligned}
$$

A direct computation shows that

$$(3.17)\quad iB_4C_8 + \frac{1}{2}C_6 = 2(A_2+B_2) - \frac{1}{2}C_6 = (1+s^2)\Big(-\frac{3}{2}s^2 - \frac{1}{4}\Big).$$

Similarly, we have

$$(3.18)\quad iB_4D_8 - \frac{1}{2}D_6 = 2(C_2+D_2) + \frac{1}{2}D_6 = s\sqrt{1+s}\Big(\frac{3}{2}s^2 + \frac{5}{4}\Big).$$

In view of (3.17) and (3.18), we can write (3.16) as

$$(3.19)\quad d\Big[(1+s^2)\Big(-\frac{3}{2}s^2 - \frac{1}{4}\Big)z_1 w + s\sqrt{1+s}\Big(\frac{3}{2}s^2 + \frac{5}{4}\Big)\bar z_1 w\Big].$$

On the other hand, we can write terms only involving z_1 and $\bar z_1$ in (3.15) as

$$(3.20)\quad (K_{11}z_1^2 + K_{\bar 1\bar 1}\bar z_1^2 + K_{1\bar 1}|z_1|^2)\,dz_1 + (N_{11}z_1^2 + N_{\bar 1\bar 1}\bar z_1^2 + N_{1\bar 1}|z_1|^2)\,d\bar z_1,$$

where

$$
\begin{aligned}
& K_{11} = \Big(A_1 + \frac{1}{2}iA_4\Big)C_8, \quad K_{\bar 1\bar 1} = \frac{1}{2}D_6 + \Big(A_1 - \frac{1}{2}iA_4\Big)C_8 + 2(C_2+D_2), \\
& K_{1\bar 1} = B_1C_8 + 2(A_2+B_2), \quad N_{11} = \frac{1}{2}C_6 + \Big(A_1 + \frac{1}{2}iA_4\Big)D_8 - 2(A_2+B_2), \\
& N_{\bar 1\bar 1} = \Big(A_1 - \frac{1}{2}iA_4\Big)D_8, \quad N_{1\bar 1} = B_1D_8 - 2(C_2+D_2).
\end{aligned}
$$

Observe that

$$
(3.21)\quad
\begin{aligned}
K_{\bar 1\bar 1} &= \frac{1}{2}N_{1\bar 1} = s\sqrt{1+s^2}\Big(\frac{3}{2}s^4 + \frac{3}{4}s^2 - 1\Big), \\
N_{11} &= \frac{1}{2}K_{1\bar 1} = (1+s^2)\Big(-\frac{3}{2}s^4 - \frac{9}{4}s^2 + \frac{1}{4}\Big),
\end{aligned}
$$

and

$$K_{11} = s(1+s^2)^{3/2}\left(\frac{3}{2}s^2 + \frac{15}{4}\right), \quad N_{\bar{1}\bar{1}} = -s^2(1+s^2)\left(\frac{3}{2}s^2 - \frac{9}{4}\right). \tag{3.22}$$

In view of (3.21) and (3.22), we can express (3.20) as

$$d\left\{\frac{1}{3}K_{11}z_1^3 + \frac{1}{3}N_{\bar{1}\bar{1}}\bar{z}_1^3 + K_{\bar{1}\bar{1}}\bar{z}_1^2 z_1 + N_{11}z_1^2\bar{z}_1\right\}. \tag{3.23}$$

Altogether, from (3.19) and (3.23) we obtain $\tilde{z}_{\mathrm{CR}}$ (see (3.12)) as follows:

$$\begin{aligned} \tilde{z}_{\mathrm{CR}} = {} & (1+s^2)z_1 - s\sqrt{1+s^2}\,\bar{z}_1 + (1+s^2)\left(-\frac{3}{2}s^2 - \frac{1}{4}\right)z_1 w \\ & + s\sqrt{1+s^2}\left(\frac{3}{2}s^2 + \frac{5}{4}\right)\bar{z}_1 w + s(1+s^2)^{3/2}\left(\frac{1}{2}s^2 + \frac{5}{4}\right)z_1^3 \\ & - s^2(1+s^2)\left(\frac{1}{2}s^2 - \frac{3}{4}\right)\bar{z}_1^3 + s\sqrt{1+s^2}\left(\frac{3}{2}s^4 + \frac{3}{4}s^2 - 1\right)\bar{z}_1^2 z_1 \\ & + (1+s^2)\left(-\frac{3}{2}s^4 - \frac{9}{4}s^2 + \frac{1}{4}\right)z_1^2\bar{z}_1 + \text{h.o.t.} \end{aligned} \tag{3.24}$$

The CR normal coordinate z'_{CR} with respect to Jerison–Lee's frame reads

$$z'_{\mathrm{CR}} = \frac{\tilde{z}_{\mathrm{CR}}}{\sqrt{1+s^2}}.$$

We want next to determine the t-component of CR normal coordinates. Recall the definition of $\hat{\theta}$ and (3.14): after some cancellations, one can check that

$$\hat{\theta}' = i\left\{z_1\,d\bar{z}_1 - \bar{z}_1\,dz_1 - dw + \frac{1}{2}|z_1|^2\,dw - \frac{1}{2}w(\bar{z}_1\,dz_1 + z_1\,d\bar{z}_1)\right\}.$$

We now need to consider the conformal change of contact form

$$\check{\theta}' = e^{2v}\hat{\theta}' = (1 + 2v_2 + \cdots)\hat{\theta}'.$$

Recalling that $v_2 = A_1(z_1^2 + \bar{z}_1^2) + B_1|z_1|^2$, we obtain that

$$\begin{aligned} \check{\theta}' = {} & i\,(1 + 2A_1(z_1^2+\bar{z}_1^2) + 2B_1|z_1|^2) \\ & \times\left\{\left(z_1 - \frac{1}{2}wz_1\right)d\bar{z}_1 - \left(\bar{z}_1 + \frac{1}{2}w\bar{z}_1\right)dz_1 - \left(1 - \frac{1}{2}|z_1|^2\right)dw\right\} + \text{h.o.t.} \end{aligned}$$

From straightforward computations, one finds

$$\begin{aligned} \check{\theta}' = {} & i\left(z_1 + 2A_1(z_1^2+\bar{z}_1^2)z_1 + 2B_1|z_1|^2 z_1 - \frac{1}{2}wz_1\right)d\bar{z}_1 \\ & - i\left(\bar{z}_1 + 2A_1(z_1^2+\bar{z}_1^2)\bar{z}_1 + 2B_1|z_1|^2\bar{z}_1 + \frac{1}{2}w\bar{z}_1\right)dz_1 \\ & - i\left(1 + 2A_1(z_1^2+\bar{z}_1^2) + 2B_1|z_1|^2 - \frac{1}{2}|z_1|^2\right)dw + \text{h.o.t.} \end{aligned}$$

Therefore, from (3.24) we deduce

$$
\begin{aligned}
d\tilde{z}_{\mathrm{CR}} &= (1+s^2)\,dz_1 - s\sqrt{1+s^2}\,d\bar{z}_1 + (1+s^2)\Big(-\frac{3}{2}s^2-\frac{1}{4}\Big)(w\,dz_1+z_1\,dw)\\
&\quad + s\sqrt{1+s^2}\Big(\frac{3}{2}s^2+\frac{5}{4}\Big)(w\,d\bar{z}_1+\bar{z}_1\,dw) + K_{11}\,z_1^2\,dz_1 + N_{\bar{1}\bar{1}}\,\bar{z}_1^2\,d\bar{z}_1\\
&\quad + K_{\bar{1}\bar{1}}\,(2\bar{z}_1 z_1\,d\bar{z}_1+\bar{z}_1^2\,dz_1) + N_{11}\,(2z_1\bar{z}_1\,dz_1+z_1^2\,d\bar{z}_1) + \text{h.o.t.}
\end{aligned}
$$

One can then expand $\check{\theta}' + i\,\bar{z}'_{\mathrm{CR}}dz'_{\mathrm{CR}} - i\,z'_{\mathrm{CR}}d\bar{z}'_{\mathrm{CR}}$ to find that

$$
t'_{\mathrm{CR}} = -iw(1+1/2|z_1|^2) + is|z_1|^2(z_1^2-\bar{z}_1^2) + is^2(\bar{z}_1^4-z_1^4) + \text{h.o.t.} \tag{3.25}
$$

We can summarize the above discussion into the following result.

Proposition 3.3. *The CR normal coordinates on Rossi spheres with respect to $\check{\theta}' = e^{2v}\hat{\theta}'$ are given by the formulas $z'_{\mathrm{CR}} = \frac{\tilde{z}_{\mathrm{CR}}}{\sqrt{1+s^2}}$, with $\tilde{z}_{\mathrm{CR}}$ as in (3.24) and t'_{CR} as in (3.25).*

We next collect some useful formulas derived from the latter proposition. Taylor-expanding z'_{CR} one finds

$$
\begin{aligned}
|z'_{\mathrm{CR}}|^2 &= |z_1|^2\Big(1+\frac{1}{2}|z_1|^2\Big) - s\,(z_1^2+\bar{z}_1^2+w(z_1^2-\bar{z}_1^2))\\
&\quad + \frac{1}{2}s^2\,(4|z_1|^2-4|z_1|^4-z_1^4-\bar{z}_1^4) + \text{h.o.t.},
\end{aligned}
$$

while taking its square we obtain

$$
\begin{aligned}
|z'_{\mathrm{CR}}|^4 &= |z_1|^4(1+|z_1|^2) - s|z_1|^2((z_1^2+\bar{z}_1^2)(2+|z_1|^2)+2w(z_1^2-\bar{z}_1^2))\\
&\quad + s^2((z_1^4+\bar{z}_1^4)(1-|z_1|^2)+2|z_1|^4(3-|z_1|^2)+2w(z_1^4-\bar{z}_1^4)) + \text{h.o.t.}
\end{aligned} \tag{3.26}
$$

The square of t'_{CR} is given by

$$
(t'_{\mathrm{CR}})^2 = -w^2(1+|z_1|^2) + 2sw|z_1|^2(z_1^2-\bar{z}_1^2) + 2s^2w(\bar{z}_1^4-z_1^4) + \text{h.o.t.}
$$

Summing the latter formula and (3.26) we obtain that, up to higher-order terms

$$
\begin{aligned}
(\rho'_{\mathrm{CR}})^4 &= (1+|z_1|^2)(|z_1|^4-w^2) - s\,|z_1|^2\,(z_1^2+\bar{z}_1^2)\,(2+|z_1|^2)\\
&\quad + s^2\left[(z_1^4+\bar{z}_1^4)\,(1-|z_1|^2) + 2\,|z_1|^4\,(3-|z_1|^2)\right].
\end{aligned}
$$

It is also useful to expand the quantity $e^{v_2}(\rho'_{\mathrm{CR}})^{-2}$, related to the conformal covariance for the Green function, which up to higher-order terms is given by

$$
\begin{aligned}
e^{v_2}(\rho'_{\mathrm{CR}})^{-2} &= \frac{|z_1|^2+4}{4\sqrt{(|z_1|^2+1)(|z_1|^4-w^2)}} + s\,\frac{(z_1^2+\bar{z}_1^2)(12|z_1|^4+8|z_1|^2-6w^2)}{8\left((|z_1|^2+1)(|z_1|^4-w^2)\right)^{3/2}}\\
&\quad + \frac{s^2}{8\left((|z_1|^2+1)\,(|z_1|^4-w^2)\right)^{5/2}}\left[(z_1^4+\bar{z}_1^4)(20|z_1|^6+8|z_1|^4\right.\\
&\quad \left. + 4w^2-5|z_1|^2w^2) - 4|z_1|^{10}+58|z_1|^6w^2+24|z_1|^4w^2-24|z_1|^2w^4\right].
\end{aligned} \tag{3.27}
$$

Note that, with respect to the contact from $\hat{\theta} = \frac{1}{2}\hat{\theta}'$, the CR normal coordinates and the *Heisenberg distance* would be $(z_{\mathrm{CR}}, t_{\mathrm{CR}}) = (z'_{\mathrm{CR}}/\sqrt{2}, t'_{\mathrm{CR}}/2)$ and $\rho_{\mathrm{CR}} = \rho'_{\mathrm{CR}}/\sqrt{2}$, respectively.

4. Proof of Theorem 1.1

In this section, we determine the Green function for the conformal sub-Laplacian on Rossi spheres, up to an error of order s^3. This allows to estimate the mass of Rossi spheres, which turns out to be negative for $s \neq 0$ small. This is done by deriving a formal expansion in s of the Green function G_s globally away from the pole with respect to the standard coordinates (z_1, z_2) of S^3, and comparing this with the expression for G_s in CR-normal coordinates.

4.1. Formal expansion of the Green function in powers of s

Let L_s denote the conformal sub-Laplacian for the $J_{(s)}$-structure on S^3. For $s = 0$, the fundamental solution of $L_0 \mathcal{G}_0 = 64\pi\,\delta_p$ with pole at $p = (0, 1)$ is given by

$$\mathcal{G}_0 = 2\,((1 - z_2)\,(1 - \bar{z}_2))^{-1/2}.$$

We next solve formally, up to an error $O(s^3)$, $L_s G_s = 0$ away from p in power series of s in the form

$$G_s = \mathcal{G}_0 + s\,\mathcal{G}_1 + \frac{1}{2}\,s^2(\mathcal{G}_2 + \alpha\,\mathcal{G}_0 - \mathcal{G}_3) + o(s^2), \tag{4.1}$$

where $\alpha \in \mathbb{R}$, $\mathcal{G}_1$ and $\mathcal{G}_2$ are suitable explicit singular functions near p, and $\mathcal{G}_3$ is a Hölder continuous function near p for which we would need to determine only $\mathcal{G}_3(p)$. We chose to expand the second-order term including separately $\alpha\mathcal{G}_0$: this will be useful later in order to fix the distributional component of the solution at the pole p. In principle, this should be done also for the first-order term, but by our choice of $\mathcal{G}_1$, this further correction will not be necessary.

For the above expansion, the following formulas will be used:

$$\begin{aligned} Z_1\,Z_1\,z_1^a\,(1 - z_2)^b(1 - \bar{z}_2)^c &= z_1^{a-2}\,(1 - z_2)^{b-2}\,(1 - \bar{z}_2)^c\,\big(b\,(|z_2|^2 - 1)\,(2a\,\bar{z}_2\,(z_2 - 1) - |z_2|^2 + 1) \\ &\quad + (a - 1)\,a\,\bar{z}_2^2\,(z_2 - 1)^2 + b^2\,(|z_2|^2 - 1)^2\big), \end{aligned} \tag{4.2}$$

$$Z_{\bar{1}}\,Z_{\bar{1}}\,z_1^a\,(1 - z_2)^b\,(1 - \bar{z}_2)^c = (c - 1)\,c\,z_1^{a+2}\,(1 - z_2)^b\,(1 - \bar{z}_2)^{c-2}, \tag{4.3}$$

$$\begin{aligned} Z_{\bar{1}}\,Z_1\,z_1^a\,(1 - z_2)^b\,(1 - \bar{z}_2)^c &= z_1^a\,(1-z_2)^{b-1}\,(-(1-\bar{z}_2)^{c-1})\,(a(z_2-1)((c+1)\bar{z}_2 - 1) + b(c(|z_2|^2-1) \\ &\quad + (\bar{z}_2 - 1)\,z_2)), \end{aligned} \tag{4.4}$$

$$\begin{aligned} Z_1\,Z_{\bar{1}}\,z_1^a\,(1 - z_2)^b\,(1 - \bar{z}_2)^c &= -c\,z_1^a\,(1 - z_2)^{b-1}\,(1 - \bar{z}_2)^{c-1}\,((a + 1)\,\bar{z}_2\,(z_2 - 1) \\ &\quad + b(|z_2|^2 - 1)), \end{aligned} \tag{4.5}$$

with similar ones for $\bar{z}_1^a(1 - z_2)^b(1 - \bar{z}_2)^c$, passing to conjugates.

To find the first-order correction $\mathcal{G}_1$, we differentiate the relation $L_s G_s = 0$ with respect to s, evaluating it for $s = 0$. Using (2.4) and (2.5), this yields

$$L_0\mathcal{G}_1 = -\dot{L}\mathcal{G}_0 = 8\,Z_1 Z_1 \mathcal{G}_0 + 8\,Z_{\bar{1}} Z_{\bar{1}} \mathcal{G}_0 \quad \text{on } S^3 \setminus \{p\},$$

where $\dot{L} = \frac{d}{ds}|_{s=0} L_s$. The right-hand side is given by

$$\frac{12\,((\bar{z}_2-1)^2\,\bar{z}_1^2 + z_1^2\,(z_2-1)^2)}{((\bar{z}_2-1)\,(z_2-1))^{5/2}}.$$

By formulas (4.2)–(4.5), the first-order correction $\mathscr{G}_1$ to G_s can be chosen as

$$\mathscr{G}_1 = \frac{1}{2}\,(z_1^2+\bar{z}_1^2)\Big[\frac{1}{1-z_2}+\frac{1}{1-\bar{z}_2}+2\Big]\frac{1}{((1-z_2)\,(1-\bar{z}_2))^{1/2}}. \tag{4.6}$$

We pass next to the second order expansion for G_s: we will find it up to a smooth function that can be determined at p, which is enough for our purposes. Differentiating the relation $L_s G_s = 0$ twice with respect to s and evaluating at $s = 0$, we obtain (with analogous notation to above for the s-derivatives)

$$L_0 \ddot{G} = -2\dot{L}\dot{G} - \ddot{L}\mathscr{G}_0.$$

Recalling from (2.4), (2.5) that $\ddot{L} = 4L_0$, we have

$$L_0(\ddot{G} + 4\mathscr{G}_0) = -2\dot{L}\dot{G} = 16\,Z_1 Z_1 \mathscr{G}_1 + 16\,Z_{\bar{1}} Z_{\bar{1}} \mathscr{G}_1.$$

It is possible to show by direct computation, again from (4.2)–(4.5), that $16\,Z_1 Z_1 \mathscr{G}_1 + 16\,Z_{\bar{1}} Z_{\bar{1}} \mathscr{G}_1$ equals

$$\begin{aligned}
&\frac{-1}{((1-z_2)(1-\bar{z}_2))^{7/2}}\big[z_1^4\,\big(30\,(z_2-1)^3 + 6\,(z_2-1)^2(\bar{z}_2-1) - 12\,(z_2-1)^3(\bar{z}_2-1)\big)\\
&+\bar{z}_1^4\,\big(30\,(\bar{z}_2-1)^3 + 6\,(\bar{z}_2-1)^2\,(z_2-1) - 12\,(\bar{z}_2-1)^3\,(z_2-1)\big)\\
&+30\,(\bar{z}_2-1)^5 + 30\,(z_2-1)^5 + 18\,(\bar{z}_2-1)^4\,(z_2-1) + 18\,(z_2-1)^4\,(\bar{z}_2-1)\\
&+6(\bar{z}_2-1)^5(z_2-1)^2 + 6(z_2-1)^5(\bar{z}_2-1)^2 - 18(\bar{z}_2-1)^4(z_2-1)^3 - 18(\bar{z}_2-1)^3(z_2-1)^4\\
&-12\,(z_2-1)^3\,(\bar{z}_2-1)^5 - 12\,(\bar{z}_2-1)^3\,(z_2-1)^5\big],
\end{aligned}$$

where we grouped the terms by homogeneity in $z_2 - 1$ and $\bar{z}_2 - 1$.

We can invert L_0 explicitly for the terms with factors z_1^4 and $\bar{z}_1^4$. The solution is given by

$$\mathscr{G}_{2,1} := \frac{(z_1^4+\bar{z}_1^4)\,g_{2,1}}{((1-z_2)\,(1-\bar{z}_2))^{5/2}},$$

where

$$\begin{aligned}
g_{2,1} :={}& \frac{3}{8}\,(\bar{z}_2-1)^2 + \frac{3}{8}\,(z_2-1)^2 + \frac{1}{4}\,(z_2-1)\,(\bar{z}_2-1) + \frac{3}{2}\,(z_2-1)^2\,(\bar{z}_2-1)^2\\
&-\frac{3}{4}\,(z_2-1)^2\,(\bar{z}_2-1) - \frac{3}{4}\,(z_2-1)\,(\bar{z}_2-1)^2.
\end{aligned}$$

For the other terms, we can only find an explicit approximate solution. We set

$$\begin{aligned}
g_{2,2} ={}& (z_2-1)^4 + (\bar{z}_2-1)^4 - \frac{4}{3}\,(z_2-1)^4\,(\bar{z}_2-1) - \frac{4}{3}\,(\bar{z}_2-1)^4\,(z_2-1)\\
&+4\,(\bar{z}_2-1)^3\,(z_2-1)^2 + 4\,(z_2-1)^3\,(\bar{z}_2-1)^2\\
&+\frac{11}{3}\,(z_2-1)^4\,(\bar{z}_2-1)^2 + \frac{11}{3}\,(\bar{z}_2-1)^4\,(z_2-1)^2 + 6\,(z_2-1)^3\,(\bar{z}_2-1)^3,
\end{aligned}$$

and

$$\mathcal{G}_{2,2} := \frac{3}{4}\,\frac{g_{2,2}}{((1-z_2)(1-\bar z_2))^{5/2}}.$$

Defining

$$\mathcal{G}_2 = \mathcal{G}_{2,1} + \mathcal{G}_{2,2},$$

still by (4.2)–(4.5), one finds that

$$\begin{aligned} L_0\mathcal{G}_2 - 16Z_1Z_1\mathcal{G}_1 - 16Z_{\bar 1}Z_{\bar 1}\mathcal{G}_1 &= -12\,\frac{(z_2-1)^2+(\bar z_2-1)^2-3(z_2-1)(\bar z_2-1)}{((1-z_2)(1-\bar z_2))^{1/2}} \\ &=: \Xi(z_2,\bar z_2), \end{aligned} \tag{4.7}$$

with the right-hand side now bounded on S^3.

It will be now sufficient to add a more regular correction (which is Hölder continuous by standard regularity theory) to solve the equation for $\mathcal{G}_2$ pointwise, away from p. From (4.7), setting $\mathcal{G}_3 = L_0^{-1}\Xi(z,w)$, we then find that

$$L_0(\mathcal{G}_2-\mathcal{G}_3) - 16Z_1Z_1\mathcal{G}_1 - 16Z_{\bar 1}Z_{\bar 1}\mathcal{G}_1 = 0 \quad \text{on } S^3\setminus\{p\},$$

which corresponds to (4.1) up to the term $s^2\alpha\mathcal{G}_0$, which will be determined later. To obtain $\mathcal{G}_3(p)$, we use the Green representation formula, convoluting $\Xi(z_2,\bar z_2)$ with $\mathcal{G}_0$:

$$\mathcal{G}_3(p) = \frac{1}{64\pi^2}\int_{S^3} -24\,\frac{(z_2-1)^2+(\bar z_2-1)^2-3(z_2-1)(\bar z_2-1)}{((1-z_2)(1-\bar z_2))}\,\hat\theta\wedge d\hat\theta.$$

The Taylor expansion of the integrand in $z_2, \bar z_2$ is

$$\begin{aligned} &(24-24\bar z_2^5-24\bar z_2^4-24\bar z_2^3-24\bar z_2^2)+(24\bar z_2^5+24\bar z_2^4+24\bar z_2^3+24\bar z_2^2+48\bar z_2)z_2 \\ &\quad+(24\bar z_2-24)z_2^2+(24\bar z_2-24)z_2^3+(24\bar z_2-24)z_2^4+(24\bar z_2-24)z_2^5+\cdots. \end{aligned}$$

Integrated, this gives

$$\int_{S^3}(24+48|z_2|^2)\,\hat\theta\wedge d\hat\theta = 48\cdot 2\pi^2 + 96\pi^2 = 192\pi^2,$$

which implies that

$$\mathcal{G}_3(p) = 3. \tag{4.8}$$

In conclusion, we found that

$$\ddot G = \mathcal{G}_2 - \mathcal{G}_3 + \alpha\mathcal{G}_0,$$

i.e., (4.1), where α is a real number to be determined later. We proved therefore the following result.

Proposition 4.1. *For every compact set K in $S^3\setminus\{p\}$, $p=(0,1)$, there exists a constant $C_K>0$ such that the function $\mathfrak{G}_s := \mathcal{G}_0 + s\mathcal{G}_1 + \frac{1}{2}s^2(\mathcal{G}_2+\alpha\mathcal{G}_0-\mathcal{G}_3)$ in (4.1) satisfies*

$$|L_s\mathfrak{G}_s| \le C_K s^3 \quad \text{on } K.$$

4.2. Rigorous estimates

We prove next that the function $\mathfrak{G}_s$ in Proposition 4.1 well matches with the expression of the Green function of L_s in CR normal coordinates. Recall from the end of Section 3 that $\rho_{\mathrm{CR}}^2 = \frac{1}{2}(\rho'_{\mathrm{CR}})^2$. Then, from (3.27) we obtain that

$$(4.9)\quad 4e^{v_2}\rho_{\mathrm{CR}}^{-2} = \frac{|z_1|^2+4}{\sqrt{(|z_1|^2+1)(|z_1|^4-\mathtt{w}^2)}} + s\,\frac{(z_1^2+\bar z_1^2)(12|z_1|^4+8|z_1|^2-6\mathtt{w}^2)}{2\left((|z_1|^2+1)(|z_1|^4-\mathtt{w}^2)\right)^{3/2}}$$
$$+ s^2\,\frac{1}{2\left((|z_1|^2+1)(|z_1|^4-\mathtt{w}^2)\right)^{5/2}}$$
$$\cdot\left[(z_1^4+\bar z_1^4)(20|z_1|^6+8|z_1|^4+4\mathtt{w}^2-5|z_1|^2\mathtt{w}^2)\right.$$
$$\left.-4|z_1|^{10}+58|z_1|^6\mathtt{w}^2+24|z_1|^4\mathtt{w}^2-24|z_1|^2\mathtt{w}^4\right]$$
$$+O(s^3\rho^{-2}),$$

where $\mathtt{w} = z_2 - \bar z_2$. Given the covariance property of the Green function (that is, $G_{(\tilde\theta)} = e^u G_{(\theta)}$ if $\tilde\theta = e^{2u}$), we aim to compare this expression to the function $\mathfrak{G}_s$ in Proposition 4.1 on a suitable small annulus centered around p. We do it term by term for the Taylor series in s, and for this purpose the following formulas will be useful. Since $z_2 - \bar z_2$ is purely imaginary, we can write

$$|z_1|^4 + |z_2-\bar z_2|^2 = |z_1|^4 - (z_2-\bar z_2)^2 = (|z_1|^2 + (z_2-\bar z_2))(|z_1|^2 - (z_2-\bar z_2)).$$

As $|z_1|^2 + |z_2|^2 = 1$, we get

$$(4.10)\qquad |z_1|^4 - \mathtt{w}^2 = |z_1|^4 + |z_2-\bar z_2|^2 = (1+z_2)(1+\bar z_2)(1-z_2)(1-\bar z_2).$$

Setting $\mathtt{v} = z_2 + \bar z_2 - 2$ (which is real), we have that $z_2 = 1 + \mathtt{v}/2 + \mathtt{w}/2$, which implies

$$|z_2|^2 = 1 + \mathtt{v} + \frac{\mathtt{v}^2}{4} - \frac{\mathtt{w}^2}{4} + o(\rho^4).$$

Squaring this relation, we obtain

$$(4.11)\qquad |z_1|^4 = \mathtt{v}^2 + \frac{\mathtt{v}^3}{2} - \frac{\mathtt{v}\,\mathtt{w}^2}{2} + o(\rho^6).$$

We also have that $|z_2|^2 = 1 + \mathtt{v}$ up to an error $O(\rho^4)$, so $\mathtt{v}^2 = |z_1|^4 + o(\rho^4)$. These imply that

$$(4.12)\quad |z_2|^2 + 1 - (z_2+\bar z_2) = \frac{1}{4}|z_1|^4 - \frac{1}{4}\mathtt{w}^2 + o(\rho^4) = \frac{1}{4}|z_1|^4 + \frac{1}{4}|\mathtt{w}|^2 + o(\rho^4).$$

Furthermore, there holds

$$(4.13)\qquad 1 + z_2 + \bar z_2 + |z_2|^2 \simeq 3 + \mathtt{v} + |z_2|^2 \simeq 2 + 2|z_2|^2 = 4 - 2|z_1|^2 + o(\rho^2).$$

Recalling our notation from Section 2, we have then the following result.

Lemma 4.2. *For* $\alpha = -3/4$*, the following estimate holds*:

$$4e^{v_2}\,\rho_{\mathrm{CR}}^{-2} = \Big(\mathcal{G}_0 + s\mathcal{G}_1 + \frac{1}{2}s^2(\mathcal{G}_2 + \alpha\,\mathcal{G}_0)\Big) + o(s^2)\,O''(\rho^{-2}) + o_\rho(1), \tag{4.14}$$

where $o_\rho(1) \to 0$ *as* $\rho \to 0$.

Proof. We analyse separately different orders in s for the left-hand side, and the first term in the right-hand side of (4.14).

Zero-th order in s. Recalling that $\mathcal{G}_0 = 2((1-z_2)(1-\bar{z}_2))^{-1/2}$, we need to compare the two quantities

$$\frac{|z_1|^2 + 4}{\sqrt{(|z_1|^2+1)\,(|z_1|^4 - \mathrm{w}^2)}} \quad \text{and} \quad \frac{2}{((1-z_2)(1-\bar{z}_2))^{1/2}}. \tag{4.15}$$

Taylor-expanding the terms involving $|z_1|^2$ in the left-hand side, we are left with comparing

$$\frac{4\,(1-\frac{1}{4}|z_1|^2)}{\sqrt{(|z_1|^4 - \mathrm{w}^2)}} \quad \text{and} \quad \frac{2}{((1-z_2)\,(1-\bar{z}_2))^{1/2}}.$$

Using (4.10) and multiplying by $((1-z_2)\,(1-\bar{z}_2))^{1/2}$, we are left with the comparison of

$$\frac{4\,(1-\frac{1}{4}|z_1|^2)}{((1+z_2)\,(1+\bar{z}_2))^{1/2}} \quad \text{and} \quad 2.$$

From (4.13), we are left with comparing

$$\frac{4\,(1-\frac{1}{4}|z_1|^2)}{(4-2|z_1|^2)^{1/2}} \quad \text{and} \quad 2,$$

which holds true up to an error of order $O(\rho^4)$. Therefore, the two quantities in (4.15) coincide up to an error of order $O(\rho^2)$.

First order in s. Recalling (4.6), we have that

$$\mathcal{G}_1 = \frac{1}{4}\,(z_1^2 + \bar{z}_1^2)\left[\frac{4 - 3z_2 - 3\bar{z}_2 + 2|z_2|^2}{(1-z_2)\,(1-\bar{z}_2)}\right]\mathcal{G}_0.$$

Considering the first-order term in s of (4.9), we need to compare the two quantities

$$\frac{(z_1^2 + \bar{z}_1^2)\,(12|z_1|^4 + 8|z_1|^2 - 6\mathrm{w}^2)}{2\,((|z_1|^2+1)\,(|z_1|^4 - \mathrm{w}^2))^{3/2}} \quad \text{and} \quad \frac{1}{4}\,(z_1^2 + \bar{z}_1^2)\left[\frac{4 - 3z_2 - 3\bar{z}_2 + 2|z_2|^2}{(1-z_2)\,(1-\bar{z}_2)}\right]\mathcal{G}_0.$$

Using the expression of $\mathcal{G}_0$, dividing by $(z_1^2 + \bar{z}_1^2)$ and multiplying by 2, we need to compare

$$\frac{(12|z_1|^4 + 8|z_1|^2 - 6\mathrm{w}^2)}{((|z_1|^2+1)\,(|z_1|^4 - \mathrm{w}^2))^{3/2}} \quad \text{and} \quad \left[\frac{4 - 3z_2 - 3\bar{z}_2 + 2|z_2|^2}{((1-z_2)\,(1-\bar{z}_2))^{3/2}}\right].$$

Using (4.10), this is equivalent to the comparison of

$$\frac{(12|z_1|^4+8|z_1|^2-6\mathtt{w}^2)}{((|z_1|^2+1)(1+z_2)(1+\bar z_2))^{3/2}} \quad \text{and} \quad 4-3z_2-3\bar z_2+2|z_2|^2.$$

Using (4.13) and Taylor-expanding the left-hand side, we arrive to comparing

$$\frac{(1-\frac{3}{4}|z_1|^2)(12|z_1|^4+8|z_1|^2-6\mathtt{w}^2)}{8} \quad \text{and} \quad 4-3z_2-3\bar z_2+2|z_2|^2.$$

Using instead (4.12), we transform the right-hand side, arriving to the comparison of

$$\frac{(1-\frac{3}{4}|z_1|^2)(12|z_1|^4+8|z_1|^2-6\mathtt{w}^2)}{8} \quad \text{and} \quad |z_1|^2+\frac{3}{4}|z_1|^4-\frac{3}{4}\mathtt{w}^2,$$

which is again true up to an error of order $O(\rho^6)$. Therefore, we get matching of the first-order terms in s in both sides of (4.14) up to an error $O(\rho^2)$.

Second order in s. Recalling again (4.9) and the fact that $\mathscr{G}_2$ comes with a factor $1/2$, let us first compare

$$\frac{(z_1^4+\bar z_1^4)(20|z_1|^6+8|z_1|^4+4\mathtt{w}^2-5|z_1|^2\mathtt{w}^2)}{2\,((|z_1|^2+1)(|z_1|^4-\mathtt{w}^2))^{5/2}} \quad \text{and} \quad \frac{1}{2}\mathscr{G}_{2,1}:=\frac{1}{2}\frac{(z_1^4+\bar z_1^4)\,g_{2,1}}{((1-z_2)(1-\bar z_2))^{5/2}},$$

where, up to order $O(\rho^8)$,

$$\begin{aligned} g_{2,1} := {} & \frac{3}{8}(\bar z_2-1)^2+\frac{3}{8}(z_2-1)^2+\frac{1}{4}(z_2-1)(\bar z_2-1)-\frac{3}{4}(z_2-1)^2(\bar z_2-1) \\ & -\frac{3}{4}(z_2-1)(\bar z_2-1)^2. \end{aligned}$$

Factoring out $(z_1^4+\bar z_1^4)$ and using (4.10), we need to compare

$$\frac{(20|z_1|^6+8|z_1|^4+4\mathtt{w}^2-5|z_1|^2\mathtt{w}^2)}{2\,((|z_1|^2+1)(1+z_2)(1+\bar z_2))^{5/2}} \quad \text{and} \quad \frac{1}{2}g_{2,1}.$$

Using then (4.13) and Taylor-expanding the denominator in the first term in $|z_1|^2$, we arrive to comparing

$$\left(1-\frac{5}{4}|z_1|^2\right)\frac{(20|z_1|^6+8|z_1|^4+4\mathtt{w}^2-5|z_1|^2\mathtt{w}^2)}{64} \quad \text{and} \quad \frac{1}{2}g_{2,1}.$$

Expanding $g_{2,1}$ and using (4.11), we come to the comparison of

$$\left(1-\frac{5}{4}|z_1|^2\right)\frac{(20|z_1|^6+8|z_1|^4+4\mathtt{w}^2-5|z_1|^2\mathtt{w}^2)}{64} \quad \text{and} \quad \frac{1}{2}\frac{1}{16}(-3\mathtt{v}^3+4\mathtt{v}^2+3\mathtt{v}\mathtt{w}^2+2\mathtt{w}^2),$$

which is correct, up to an error of order $O(\rho^{12})$.

We need next to compare

$$\left[\frac{-4|z_1|^{10}+58|z_1|^6\mathtt{w}^2+24|z_1|^4\mathtt{w}^2-24|z_1|^2\mathtt{w}^4}{2\,((|z_1|^2+1)(|z_1|^4-\mathtt{w}^2))^{5/2}}\right] \quad \text{and} \quad \frac{1}{2}\,\frac{3}{4}\,\frac{g_{2,2}}{((1-z_2)(1-\bar z_2))^{5/2}},$$

where, up to higher order terms,

$$g_{2,2} = (z_2-1)^4 + (\bar z_2-1)^4 - \frac{4}{3}(z_2-1)^4(\bar z_2-1) - \frac{4}{3}(\bar z_2-1)^4(z_2-1) + 4(\bar z_2-1)^3(z_2-1)^2 + 4(z_2-1)^3(\bar z_2-1)^2.$$

Using again (4.10), we then need to compare

$$\left[\frac{-4|z_1|^{10}+58|z_1|^6\mathtt{w}^2+24|z_1|^4\mathtt{w}^2-24|z_1|^2\mathtt{w}^4}{2((|z_1|^2+1)(1+z_2)(1+\bar z_2))^{5/2}}\right] \quad \text{and} \quad \frac{3}{8}g_{2,2}.$$

As before, we are then comparing

$$\left(1-\frac{5}{4}|z_1|^2\right)\frac{-4|z_1|^{10}+58|z_1|^6\mathtt{w}^2+24|z_1|^4\mathtt{w}^2-24|z_1|^2\mathtt{w}^4}{64} \quad \text{and} \quad \frac{3}{8}g_{2,2}.$$

In fact, we can add to $\mathscr{G}_2$ any multiple of $\mathscr{G}_0$. In the latter formula, we can then replace $g_{2,2}$ with $\tilde g_{2,2}$, where

$$\tilde g_{2,2} = g_{2,2} - 2(z_2-1)^2(\bar z_2-1)^2.$$

It turns out that

$$\frac{3}{8}\tilde g_{2,2} = \frac{1}{16}\mathtt{v}(\mathtt{v}^4-4\mathtt{v}^2\mathtt{w}^2+6\mathtt{v}\mathtt{w}^2+3\mathtt{w}^4).$$

Using (4.11) and the previous formula to expand $|z_1|^2$ as $|z_1|^2 = -\mathtt{v} - \frac{1}{4}\mathtt{v}^2 + \frac{1}{4}\mathtt{w}^2$, the left-hand side in the above formula becomes

$$\frac{\mathtt{v}^5}{16} + \frac{3\mathtt{v}^2\mathtt{w}^2}{8} - \frac{\mathtt{v}^3\mathtt{w}^2}{4} + \frac{3\mathtt{v}\mathtt{w}^4}{16} + O(\rho^{12}),$$

so it coincides with the right-hand side, i.e., with $\frac{3}{8}\tilde g_{2,2}$ up to error terms of order $O(\rho^{12})$. Therefore, also the second-order terms in s of both sides of (4.14) coincide up to an error of order $O(\rho^{-2})$.

It is standard to check that the above matching also holds up to computing first- and second-order derivatives, which then implies the conclusion. ∎

Proof of Theorem 1.1. Consider a small annulus of the form

$$A_r := \{r \le \rho \le 2r\},$$

and a smooth cut-off function χ_r satisfying

$$\begin{cases} \chi_r = 1 & \text{on } \{\rho \le r\}, \\ \chi_r = 0 & \text{on } \{\rho \ge 2r\}, \\ |\nabla_b \chi_r| \le C/r & \text{and} \quad |\nabla_b^2 \chi_r| + |\nabla_T \chi_r| \le C/r^2. \end{cases}$$

If v is the conformal factor as in Proposition 2.3 then, with obvious notation, the Green function conformally transforms as $G_\theta = e^{-v} G_{\hat\theta}$. Consider then the function

$$\tilde G_s = \chi_r\left(2\rho_{\mathrm{CR}}^{-2} - \frac{1}{2}\mathscr{G}_3(p)s^2\right) + (1-\chi_r)e^{-v}G_{\hat\theta}.$$

From the conformal covariance of L_s, Proposition 4.1 and Lemma 4.2, it follows that, applying the conformal sub-Laplacian with respect to the contact form θ,

$$|L_s^v \tilde{G}_s| \leq C_r\, o(s^2) \quad \text{pointwise on } S^3.$$

It then follows from standard regularity theory that the Green function G_θ of the conformal sub-Laplacian satisfies $\|G_\theta - \tilde{G}_s\|_{L^\infty(S^3)} = o(s^2)$. Sending s to zero and recalling that $\mathcal{G}_3(p) = 3$, we deduce

$$A = -\frac{3}{2}s^2 + o(s^2). \tag{4.16}$$

Therefore, given that $m = 12\pi A$ (see (4.8) and (1.7)), we obtain the conclusion. ∎

5. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 by an implicit function argument and some asymptotic expansions, which crucially use also Theorem 1.1.

We start by analysing the relation of the CR Sobolev quotient on Rossi spheres with the minimizers on standard spheres found in [20]. Recall that in [19] it was proved that for any three-dimensional CR manifold one has $\mathcal{Y}(M, J) \leq \mathcal{Y}(S^3, J_{S^3})$, which in particular implies

$$\mathcal{Y}(S^3, J_{(s)}) \leq \mathcal{Y}(S^3, J_{S^3} = J_{(0)}). \tag{5.1}$$

In [20], it was proven that $\mathcal{Y}(S^3, J_{S^3})$ is precisely attained by the following functions, up to composing (z_1, z_2) with elements of SU(2):

$$\varphi_\lambda = \lambda\Big(\frac{(|z_1|^2 + |z_2+1|^2)^2 - (z_2 - \bar{z}_2)^2}{(\lambda^2|z_1|^2 + |z_2+1|^2)^2 - \lambda^4(z_2 - \bar{z}_2)^2}\Big)^{1/2}, \quad \lambda > 0. \tag{5.2}$$

Recalling that $\hat{\theta} \wedge d\hat{\theta}$ is a volume form double with respect to the Euclidean one, the φ_λ's satisfy the following normalization condition:

$$\int_{S^3} \varphi_\lambda^4\, \hat{\theta} \wedge d\hat{\theta} = 4\pi^2 \quad \text{for all } \lambda > 0.$$

On the standard S^3, see [14], the *Folland–Stein space* $\mathfrak{S}^{1,2}(S^3)$ is defined as the completion of the (complex-valued) C^∞ functions on S^3 with respect to the norm

$$\|u\|_{\mathfrak{S}^{1,2}} := \Big(\int_{S^3} (u_{,1}\bar{u}_{,\bar{1}} + u_{,\bar{1}}\bar{u}_{,1})\, \hat{\theta} \wedge d\hat{\theta}\Big)^{1/2} + \Big(\int_{S^3} |u|^2\, \hat{\theta} \wedge d\hat{\theta}\Big)^{1/2}.$$

Notice that, for $|s|$ small, this defines an equivalent norm on Rossi spheres too: from now on, this will be assumed understood.

We show next that, if a minimizer for the CR-Sobolev quotient on Rossi spheres exists for $|s|$ small, it must be close in $\mathfrak{S}^{1,2}(S^3)$ to some function φ_λ as in (5.2). We have indeed the following result.

Lemma 5.1. *Fix $s \in \mathbb{R}$, $|s|$ small. Assume $u_s > 0$ attains* $\inf Q_{(s)} = \mathcal{Y}(S^3, J_{(s)})$. *Then, if u_s is normalized so that $\int_{S^3} u_s^4\, \hat{\theta} \wedge d\hat{\theta} = 4\pi^2$, up to a homogeneous action on S^3 there exists $\lambda > 0$ such that*

$$\|u_s - \varphi_\lambda\|_{\mathfrak{S}^{1,2}(S^3)} = o_s(1),$$

where $o_s(1) \to 0$ as $s \to 0$.

Proof. It is sufficient to notice that, if $Z_{1(s)}$ is as in (1.8), then for all smooth u's one has

$$\int_{S^3} (Z_{1(s)} u\, Z_{\bar{1}(s)} \bar{u} + Z_{\bar{1}(s)} u Z_{1(s)} \bar{u})\, \hat{\theta} \wedge d\hat{\theta} = (1 + o_s(1)) \int_{S^3} (Z_1 u\, Z_{\bar{1}} \bar{u} + Z_{\bar{1}} u\, Z_1 \bar{u})\, \hat{\theta} \wedge d\hat{\theta}.$$

Since we are assuming u_s to be normalized in $L^4(S^3)$ as in the statement, its $\mathfrak{S}^{1,2}(S^3)$-norm is uniformly bounded from above, and therefore

$$\int_{S^3} (Z_1 u_s\, Z_{\bar{1}} \bar{u}_s + Z_{\bar{1}} u_s\, Z_1 \bar{u}_s)\, \hat{\theta} \wedge d\hat{\theta} = \int_{S^3} (Z_{1(s)} u_s\, Z_{\bar{1}(s)} \bar{u}_s + Z_{\bar{1}(s)} u_s\, Z_{1(s)} \bar{u}_s)\, \hat{\theta} \wedge d\hat{\theta} + o_s(1).$$

This relation implies that u_s is nearly a minimizer also for the Sobolev-type quotient in (5.4) on the standard CR-sphere $(S^3, J_{S^3} = J_{(0)})$. By the *Ekeland variational principle* (see, e.g., Chapter I in [31]), u_s is close in $\mathfrak{S}^{1,2}(S^3)$ to a minimizing Palais–Smale sequence for such quotient. Palais–Smale sequences for problems involving critical Sobolev embeddings can be characterized by a well-known decomposition due to Struwe. For the subelliptic case, this feature is analyzed e.g. in Theorem 2.1 of [12] for domains of the Heisenberg group and functions vanishing at the boundary (the same arguments apply in the present case using dilations in normal coordinates as those introduced in [19]), or for CR manifolds in Proposition 8 of the later paper [16]. The minimality condition implies that u_s can only develop a single *bubble profile*, which is precisely our conclusion. ∎

5.1. Finite-dimensional reduction

Let φ_λ be as in (5.2), and define the following family of functions:

$$\mathcal{M} = \{\varphi_\lambda(U(\cdot)) \,:\, \lambda > 0, U \in \mathrm{SU}(2)\}\,. \tag{5.3}$$

Even though SU(2) is a three-dimensional Lie group, since φ_λ is invariant by a complex rotation in z_1, the result of these compositions is also a set of three dimensions. We previously saw that the functions in $\mathcal{M}$ are global minimizers of the CR-Sobolev quotient $Q_{(s)}$ on the standard S^3 when $s = 0$, where

$$Q_{(s)}(u) = \frac{\int_{S^3} u\, L_s u\, \hat{\theta} \wedge d\hat{\theta}}{\left(\int_{S^3} u^4\, \hat{\theta} \wedge d\hat{\theta}\right)^{1/2}}. \tag{5.4}$$

In Lemma 5 of [24], it was proved that the linearization of the Yamabe equation (with $s = 0$) at $\mathcal{M}$ is minimally degenerate, in the sense that its kernel coincides with the tangent space to $\mathcal{M}$.

As a consequence, one has that the CR-Sobolev quotient on the standard sphere is *non-degenerate in the sense of Bott* on $\mathcal{M}$. Thanks to this fact and to Lemma 5.1, for s small we can characterize with particular precision all the solutions of the CR-Yamabe equation lying in a fixed neighborhood (in $\mathfrak{S}^{1,2}$) of the manifold $\mathcal{M}$, and in particular the (hypothetical) minimal ones. We first show that the CR-Yamabe equation is always solvable, in a fixed neighborhood of $\mathcal{M}$, up to a Lagrange multiplier: see [1] for a general reference on this method.

Proposition 5.2. *For φ_λ as in (5.2), there exists a unique $w_\lambda \in \mathfrak{S}^{1,2}(S^3)$, depending smoothly on λ, such that $\|w_\lambda\|_{\mathfrak{S}^{1,2}(S^3)} \leq C s$, and which satisfies*

$$(5.5)\qquad \int_{S^3} \varphi_\lambda^2 \frac{\partial \varphi_\lambda}{\partial \lambda} w_\lambda \, \hat{\theta} \wedge d\hat{\theta} = 0 \quad \text{and} \quad L_s(\varphi_\lambda + w_\lambda) - 2(\varphi_\lambda + w_\lambda)^3 = \ell \, \varphi_\lambda^2 \frac{\partial \varphi_\lambda}{\partial \lambda}$$

for some $\ell \in \mathbb{R}$. Moreover, there exists $\delta > 0$ with the following property: if there exists a critical point of $Q_{(s)}$ in a δ-neighborhood of $\mathcal{M}$ (in $\mathfrak{S}^{1,2}$ norm), then it must be of the form $\varphi_\lambda + w_\lambda$ up to a homogeneous action on S^3 and up to a scalar multiple, with w_λ as above.

Proof. For $\lambda > 1$, φ_λ has a global maximum at $(z_1, z_2) = (0, 1)$. Locally near these functions, all other extremals can be obtained composing on the right with elements of SU(2). When also λ varies, the extremals can be described locally near the φ_λ's by

$$\Sigma_{\Lambda,\gamma} = \left\{\varphi_{\mathbf{a},\lambda}(z_1, z_2) := \varphi_\lambda(U_{\mathbf{a}}(z_1, z_2)) \ : \ \mathbf{a} \in (-\gamma, \gamma)^3, \lambda \in [1/2, 2\Lambda]\right\} \subseteq \mathcal{M},$$

where

$$(5.6)\qquad U_{\mathbf{a}}(z_1, z_2) = \left(\exp\begin{pmatrix} -i\, a_3 & a_1 + i\, a_2 \\ -a_1 + i\, a_2 & i\, a_3 \end{pmatrix}\right)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad \mathbf{a} = (a_1, a_2, a_3).$$

Consider next the CR-Yamabe equation on the standard sphere,

$$L_0 u = 2u^3 \quad \text{ on } S^3.$$

It was proved in [24] (see Lemma 5 there) that solutions of the linearized equation at φ_λ,

$$L_0 v = 6\varphi_\lambda^2 v \quad \text{ on } S^3,$$

are of the form

$$v = l_0 \frac{\partial \varphi_{\mathbf{a},\lambda}}{\partial \lambda} + \sum_{i=1}^{3} l_i \frac{\partial \varphi_{\mathbf{a},\lambda}}{\partial a_i},$$

where $l_i \in \mathbb{R}$ and where the latter derivatives are evaluated at $\mathbf{a} = 0$.

Define $\widetilde{W} = \widetilde{W}_\lambda$ to be the space of functions $\tilde{w}$ satisfying the four constraints

$$(5.7)\qquad \int_{S^3} \varphi_{\mathbf{a},\lambda}^2 \frac{\partial \varphi_{\mathbf{a},\lambda}}{\partial \lambda} \tilde{w} \, \hat{\theta} \wedge d\hat{\theta} = 0; \qquad \int_{S^3} \varphi_{\mathbf{a},\lambda}^2 \frac{\partial \varphi_{\mathbf{a},\lambda}}{\partial a_i} \tilde{w} \, \hat{\theta} \wedge d\hat{\theta} = 0; \quad i = 1, 2, 3,$$

where, again, the derivatives are evaluated at $\mathbf{a} = 0$.

It follows from the classification result in [24] and Fredholm's theory that the operator

$$A_{\widetilde{W}} : \tilde{w} \mapsto P_{\widetilde{W}}[L_s \tilde{w} - 6\varphi_\lambda^2 \tilde{w}],$$

where $P_{\widetilde{W}}$ denotes the projection onto $\widetilde{W}$, is invertible from $\widetilde{W}$ in itself, endowed with the $\mathfrak{S}^{1,2}(S^3)$-norm.

Setting $S_{\mathbf{a},\lambda}(\tilde{w}) := L_s(\varphi_\lambda + \tilde{w}) - 2(\varphi_{\mathbf{a},\lambda} + \tilde{w})^3$, equation (5.5) has a relation to the condition $P_{\widetilde{W}} S_{\mathbf{a},\lambda}(\tilde{w}) = 0$, as we will explain below. Since $A_{\widetilde{W}}$ is invertible (with the norm of the inverse uniformly bounded), we have that

$$P_{\widetilde{W}} S_{\mathbf{a},\lambda}(\tilde{w}) = 0 \quad \Longleftrightarrow \quad \tilde{w} = T_{\mathbf{a},\lambda}(\tilde{w}),$$

where

$$T_{\mathbf{a},\lambda}(\tilde{w}) = -(A_{\widetilde{W}})^{-1}\{S_{\mathbf{a},\lambda}(0) - 2[(\varphi_{\mathbf{a},\lambda} + \tilde{w})^3 - \varphi_{\mathbf{a},\lambda}^3 - 3\varphi_{\mathbf{a},\lambda}^2 \tilde{w}]\}.$$

From the smoothness in s of the $J_{(s)}$ structures, it follows that $\|T_{\mathbf{a},\lambda}(0)\| = O(s)$, where here and below $\|\cdot\| = \|\cdot\|_{\mathfrak{S}^{1,2}(S^3)}$. Moreover, it is quite standard that for s and δ small,

$$\|T_{\mathbf{a},\lambda}(\tilde{w}_1) - T_{\mathbf{a},\lambda}(\tilde{w}_2)\| = o(1)\|\tilde{w}_1 - \tilde{w}_2\|, \quad \|\tilde{w}_1\|, \|\tilde{w}_2\| \le \delta.$$

It follows that for s small, $T_{\mathbf{a},\lambda}$ is a contraction in a normed ball of radius Cs for $C > 0$ large and fixed, so in such a ball there exists a unique fixed point w_λ of $T_{\mathbf{a},\lambda}$.

In this way we found a (unique) solution to the problem

$$L_s(\varphi_\lambda + w_\lambda) - 2(\varphi_\lambda + w_\lambda)^3 = \ell\,\varphi_\lambda^2 \frac{\partial \varphi_\lambda}{\partial \lambda} + \sum_{i=1}^{3} \ell_i\,\varphi_\lambda^2 \left.\frac{\partial \varphi_{\mathbf{a},\lambda}}{\partial a_i}\right|_{\mathbf{a}=0}$$

for some Lagrange multipliers ℓ, ℓ_i. However, the last three vanish by *Palais' criticality principle*. In fact, let us recall that, being $(S^3, J_{(s)})$ a homogeneous space, $Q_{(s)}$ is invariant under the maps $U_{\mathbf{a}}$ as in (5.6). Therefore, with obvious notation, we have with the same Lagrange multipliers that

$$L_s(\varphi_{\lambda,\mathbf{a}} + w_{\lambda,\mathbf{a}}) - 2(\varphi_{\lambda,\mathbf{a}} + w_{\lambda,\mathbf{a}})^3 = \ell\,\varphi_{\lambda,\mathbf{a}}^2 \frac{\partial \varphi_{\lambda,\mathbf{a}}}{\partial \lambda} + \sum_{i=1}^{3} \ell_i\,\varphi_{\lambda,\mathbf{a}}^2 \frac{\partial \varphi_{\mathbf{a},\lambda}}{\partial a_i},$$

for $\mathbf{a}$ in a neighborhood of zero. Differentiating with respect to a_i and then scalar-multiplying by $\partial\varphi_{\mathbf{a},\lambda}/\partial a_j$, one obtains an invertible system for $(\ell_i)_i$, yielding that $\ell_i = 0$ for $i = 1, 2, 3$, as desired.

Let now u be a critical point of $Q_{(s)}$ in a δ-neighborhood of $\mathcal{M}$ for s small. Then it satisfies $L_s u = \mu\, u^3$ for some Lagrange multiplier μ. Since u is close to the family of φ_λ's, satisfying $L_0 \varphi_\lambda = 2\varphi_\lambda^3$, the multiplier μ must be δ-close to 2.

Defining $\tilde{u} = \mu^{-1/2} u$, this is still close of order δ to $\mathcal{M}$, and it satisfies $L_s \tilde{u} = 2\tilde{u}^3$, i.e., the second equation in (5.5) with $\ell = 0$. By uniqueness of the fixed point, we must then have $\tilde{u} = \varphi_\lambda + w_\lambda$, up to a homogeneous action on S^3. This concludes the proof. ∎

Remark 5.3. In Proposition 5.2, it is possible to replace the φ_λ's with other approximate solutions to the CR-Yamabe equation on Rossi spheres. With a better approximate solution, for example, one would then require a correction as in (5.5) of smaller norm, yielding a more precise expansion for the quotient $Q_{(s)}$. This observation will be crucially used in the next two sections.

5.2. Expansion of the CR Sobolev quotient

Recalling the second statement in Proposition 5.2, we analyze the CR Sobolev quotient on functions of the form $\varphi_\lambda + w_\lambda$, showing that it is strictly higher than the standard spherical one. We first show that the latter expansion is always even in s.

Lemma 5.4. *Let $s > 0$ be small, and let $w_\lambda^{(s)}$ and $w_\lambda^{(-s)}$ denote the counterparts of w_λ in Proposition 5.2 for s and $-s$, respectively. Then one has that*

$$Q_{(s)}(\varphi_\lambda + w_\lambda^{(s)}) = Q_{(-s)}(\varphi_\lambda + w_\lambda^{(-s)}).$$

Proof. Let $\iota\colon S^3 \to S^3$ be the diffeomorphism given in (2.6). We notice that φ_λ is invariant under ι and that, due to (2.7), (2.8) and (2.9), for any $u \in \mathfrak{S}^{1,2}(S^3)$ one has

$$Q_{(s)}(\iota^* u) = Q_{(-s)}(u).$$

From this covariance property and the uniqueness in Proposition 5.2, it follows that $w_\lambda^{(s)} = \iota^* w_\lambda^{(-s)}$, and therefore we get

$$Q_{(s)}(\varphi_\lambda + w_\lambda^{(s)}) = Q_{(s)}(\iota^*(\varphi_\lambda + w_\lambda^{(-s)})) = Q_{(-s)}(\varphi_\lambda + w_\lambda^{(-s)}),$$

which is the desired conclusion. ∎

We analyse next two situations. The first is when the parameter λ in the previous lemma tends to infinity or to zero, and the second when $\log\lambda$ remains bounded. In the latter case, we will show that the CR Sobolev quotient would be strictly higher than $\mathcal{Y}(S^3, J_{S^3})$, which would give a contradiction to (5.1). On the other hand, we can also rule out the former case using the estimates on the Green function in Section 4, and in particular the negativity of the mass of $(S^3, J_{(s)})$ for s small and non zero. The proofs of the next two results, beginning from the latter case, are given in Appendices A and B.

Proposition 5.5. *Let $\Lambda > 1$ be a fixed number. Then there exists $C_\Lambda > 0$ such that, for $\lambda \in [1/\Lambda, \Lambda]$ and for s small, one has $Q_{(s)}(\varphi_\lambda + w_\lambda) = 4\pi + s^2 \mathcal{A}_\lambda + \mathcal{B}_{\lambda,s}$, where*

$$\mathcal{A}_\lambda = \frac{16\pi\lambda^2(3 + 12\lambda^2 + 2\lambda^4 + 12\lambda^6 + 3\lambda^8)}{(1+\lambda^2)^6},$$

and where $|\mathcal{B}_{\lambda,s}| \le C_\Lambda s^3$.

Notice that the minimizers in [20] stay unchanged when we compose with the antipodal map on S^3 and replace λ by $1/\lambda$: this symmetry implies that $\mathcal{A}_\lambda = \mathcal{A}_{1/\lambda}$. Therefore, in the next proposition it is sufficient to consider large values of λ.

Proposition 5.6. *The following expansion holds true, uniformly in s (small):*

$$Q_{(s)}(\varphi_\lambda + w_\lambda) = 4\pi - \frac{8}{3}\frac{m_s}{\lambda^2} + O\Big(\frac{s^2}{\lambda^3}\Big) = 4\pi + 48\pi\frac{s^2}{\lambda^2}(1 + o_s(1)) + O\Big(\frac{s^2}{\lambda^3}\Big),$$

for λ large.

Remark 5.7. The above function $\lambda \mapsto \mathcal{A}_\lambda$ is positive and strictly decreasing for $\lambda > 1$, see Figure 1. Notice that the matching of the first-order correction terms for λ large in the above two propositions: the expansions are indeed obtained with two completely different approaches. However, while the *mass* does not appear in the expansions of Appendix A, it is somehow *hidden* in the fact that there we are using standard coordinates on S^3, and not CR normal coordinates.

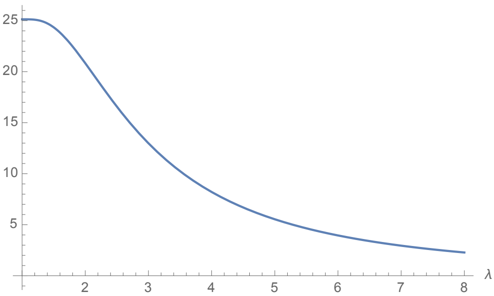

Figure 1. Graph of the function $\mathcal{A}_\lambda$.

We can finally prove our second main result.

Proof of Theorem 1.3. Assume by contradiction that u is a minimizer of the CR-Sobolev quotient $Q_{(s)}$ for $s \neq 0$ small. By Lemma 5.1, u must then lie in a δ-neighborhood of the manifold $\mathcal{M}$ defined in (5.3). From the second part of Proposition 5.2 we have also that $u = \varphi_\lambda + w_\lambda$ up to a homogeneous action on S^3, where w_λ is as in the first part of the proposition. The conclusion then follows from Proposition 5.5 and Proposition 5.6, which cover all ranges of λ for s small enough. ∎

A. Proof of Proposition 5.5

We consider the *Cayley map* from S^3 into $\mathbb{H}^1$ given by

$$\mathcal{F}(z_1, z_2) = \Big(\frac{z_1}{1+z_2}, \mathrm{Re}\Big(i\,\frac{1-z_2}{1+z_2}\Big)\Big),$$

with inverse

$$\mathcal{F}^{-1}(z,t) = \Big(\frac{2iz}{t+i(1+|z|^2)}, \frac{-t+i(1-|z|^2)}{t+i(1+|z|^2)}\Big).$$

Using $\mathcal{F}$, we can derive explicit expressions for the CR maps on S^3. Letting $\mathfrak{d}_\lambda$ denote the natural dilation in the Heisenberg group,

$$\mathfrak{d}_\lambda(z,t) = (\lambda z, \lambda^2 t), \quad \lambda > 0,$$

consider the map $\Phi_\lambda\colon S^3 \to S^3$ defined by

$$\Phi_\lambda(p) = (\mathcal{F}^{-1} \circ \mathfrak{d}_\lambda \circ \mathcal{F})(p).$$

By explicit computations, one finds that the inverse is given by

$$\Phi_\lambda^{-1}(z_1, z_2) = \Big(\frac{2\lambda(\bar z_2 + 1) z_1}{\lambda^2 |z_2+1|^2 + \bar z_2 + |z_1|^2 - z_2}, \frac{\lambda^2 |z_2+1|^2 - \bar z_2 - |z_1|^2 + z_2}{\lambda^2 |z_2+1|^2 + \bar z_2 + |z_1|^2 - z_2}\Big). \tag{A.1}$$

For later purposes, the following formula will be useful:

$$\varphi_\lambda(\Phi_\lambda^{-1}(z_1, z_2))^{-3} = \frac{1}{2}\lambda^{-1}\Big(\frac{|1+z_2|^2}{(\lambda^2|1+z_2|^2 + |z_1|^2)^2 - (z_2 - \bar z_2)^2}\Big)^{1/2}.$$

Notice also that $\varphi_{\lambda=1} \equiv 1$ on S^3.

A.1. Approximate solutions

We construct next, on every compact interval in the range of λ, approximate solutions to the CR-Yamabe equation with $s \neq 0$ up an order $O(s^2)$, improving the accuracy of the φ_λ's (approximate up to order $O(s)$) for $s \neq 0$.

Lemma A.1. *Let $\Lambda > 1$ be a fixed number. Then there exist $C_\Lambda > 0$ and regular functions $\hat w_\lambda$, depending smoothly on λ, such that, for $\lambda \in [1/\Lambda, \Lambda]$ and for s small, one has*

$$L_s(\varphi_\lambda + s\hat w_\lambda) - 2(\varphi_\lambda + s\hat w_\lambda)^3 = f_\lambda,$$

with $\|f_\lambda\|_{L^\infty(S^3)} \le C_\Lambda s^2$.

Proof. Recall that the extremals of the CR-Sobolev inequality (up to a homogeneous CR-action of S^3) have the expression in (5.2), namely

$$\begin{aligned}\varphi_\lambda &= \lambda\Big(\frac{(|z_1|^2 + |z_2+1|^2)^2 - (z_2 - \bar z_2)^2}{(\lambda^2|z_1|^2 + |z_2+1|^2)^2 - \lambda^4(z_2 - \bar z_2)^2}\Big)^{1/2} \\ &= 2\lambda\Big(\frac{|1+z_2|^2}{(\lambda^2|z_1|^2 + |z_2+1|^2)^2 - \lambda^4(z_2 - \bar z_2)^2}\Big)^{1/2},\end{aligned}$$

and for all $\lambda > 0$, they satisfy the equation

$$L_0\varphi_\lambda = -4\Delta_b\varphi_\lambda + 2\varphi_\lambda = 2\varphi_\lambda^3 \quad \text{on } S^3. \tag{A.2}$$

Our goal is to find a correction $s\hat w_\lambda$ such that $\varphi_\lambda + s\hat w_\lambda$ satisfies the CR-Yamabe equation on $(S^3, J_{(s)})$ up to an order s^2. Recalling (2.4) and (2.5), it is sufficient to solve for

$$-4\Delta_b\hat w_\lambda + 2\hat w_\lambda - 6\varphi_\lambda^2\hat w_\lambda = \mathcal{G}_\lambda := 8 Z_1 Z_1\varphi_\lambda + \text{conj}.$$

From a straightforward computation, one has that

$$\begin{aligned}&\mathcal{G}_\lambda(z_1, z_2) \\ &= \frac{192\lambda(\lambda^2-1)^2|1+z_2|^8 \operatorname{Re}[z_1^2(1+z_2+\lambda^2(z_2-1))^2]}{[(\lambda^2|z_1|^2 + |z_2+1|^2)^2 - \lambda^4(z_2-\bar z_2)^2]^{5/2}[(|z_1|^2 + |z_2+1|^2)^2 - (z_2-\bar z_2)^2]^{3/2}} \\ &= \frac{3}{4}\lambda^{-4}\frac{(\lambda^2-1)^2\operatorname{Re}[z_1^2(1+z_2+\lambda^2(z_2-1))^2]}{[(\lambda^2|z_1|^2 + |z_2+1|^2)^2 - \lambda^4(z_2-\bar z_2)^2]^{5/2}}\varphi_\lambda(z_1, z_2)^5.\end{aligned}$$

It is useful to evaluate this expression after composing with the inverse CR map defined in (A.1): by direct computation, using also (A.1), one finds that

$$\mathcal{G}_\lambda(\Phi_\lambda^{-1}(z_1,z_2)) = \frac{3}{4}\lambda^{-3}(\lambda^2-1)^2\left(\frac{|1+z_2|^2}{(\lambda^2|1+z_2|^2+|z_1|^2)^2-(z_2-\bar z_2)^2}\right)^{3/2} \times\left[z_1^2(1-\bar z_2+\lambda^2(1+\bar z_2))^4+\bar z_1^2(1-z_2+\lambda^2(1+z_2))^4\right]. \tag{A.3}$$

Let us recall the covariance of the conformal sub-Laplacian L_θ: for a conformal contact form $\tilde\theta = u^{4/(Q-2)}\,\theta$, one has

$$L_{\tilde\theta}\varphi = u^{-(Q+2)/(Q-2)}\,L_\theta(u\,\cdot).$$

Let $\mathfrak{L}_{\varphi_\lambda}$ be the linearized CR-Yamabe operator at φ_λ on $(S^3, J_{(0)})$, i.e.,

$$\mathfrak{L}_{\varphi_\lambda}v = -4\,\Delta_b v + 2v - 6\varphi_\lambda^2 v, \tag{A.4}$$

and let $\mathfrak{w}_\lambda$ denote the pull-back of $\hat w_\lambda$ via Φ_λ, namely

$$\mathfrak{w}_\lambda(z) = \varphi_\lambda^{-1}(\Phi_\lambda^{-1}(z))\,\hat w_\lambda(\Phi_\lambda^{-1}(z)). \tag{A.5}$$

Then the covariance of L_θ implies that

$$(\mathfrak{L}_{\varphi_1\equiv 1}\mathfrak{w}_\lambda)(x) = \varphi_\lambda(\Phi_\lambda^{-1}(x))^{-3}(\mathfrak{L}_{\varphi_\lambda}\hat w_\lambda)(\Phi_\lambda^{-1}(x)). \tag{A.6}$$

It follows from this formula and (A.3) that the pull-back $\mathfrak{w}_\lambda$ satisfies the following equation on S^3, which has constant coefficients on the left-hand side:

$$-4\,\Delta_b\mathfrak{w}_\lambda - 4\,\mathfrak{w}_\lambda = 12(\lambda^2-1)^2\,\mathrm{Re}\,\frac{(1-\bar z_2+\lambda^2(1+\bar z_2))\,z_1^2}{(1-z_2+\lambda^2(1+z_2))^3}. \tag{A.7}$$

This last equation can be solved explicitly in $\mathfrak{w}_\lambda$ via Fourier decomposition: in fact, the right-hand side in (A.7) is given by

$$12\,\frac{(\lambda^2-1)^2}{(\lambda^2+1)^2}\,\mathrm{Re}\,\frac{z_1^2(1-\Gamma\bar z_2)}{(1-\Gamma z_2)^3}, \quad \text{with } \Gamma = \frac{1-\lambda^2}{1+\lambda^2}.$$

Since we have the expansion

$$\frac{1}{(1-\Gamma z_2)^3} = 1 + 3\Gamma z_2 + 6\Gamma^2 z_2^2 + 10\Gamma^3 z_2^3 + 15\Gamma^4 z_2^4 + \cdots,$$

we obtain that

$$\begin{aligned}12\,\frac{(\lambda^2-1)^2}{(\lambda^2+1)^2}\,\mathrm{Re}\,\frac{z_1^2(1-\Gamma\bar z_2)}{(1-\Gamma z_2)^3} &= 12\,\frac{(\lambda^2-1)^2}{(\lambda^2+1)^2}\,\mathrm{Re}\big\{z_1^2\big[1+3\Gamma z_2+6\Gamma^2 z_2^2+10\Gamma^3 z_2^3+15\Gamma^4 z_2^4+\cdots \\ &\quad -\Gamma\bar z_2(1+3\Gamma z_2+6\Gamma^2 z_2^2+10\Gamma^3 z_2^3+15\Gamma^4 z_2^4+\cdots)\big]\big\}.\end{aligned}$$

While the set of functions of the right–hand side in the first line are spherical harmonics, i.e., satisfying

$$-\Delta_b(z_1^2 z_2^k) = (k+2)\, z_1^2 z_2^k, \tag{A.8}$$

the functions on the second line of the right-hand side are not. However, they can be easily modified in order to satisfy an eigenvalue equation. More precisely, one has that (see [21])

$$-\Delta_b\Big(z_2^k\Big(z_2\bar z_2 - \frac{k+1}{k+4}\Big)\Big) = (10+3k)\Big(z_2^k\Big(z_2\bar z_2 - \frac{k+1}{k+4}\Big)\Big). \tag{A.9}$$

Hence we rewrite the right-hand side in (A.7) in the following way:

$$\begin{aligned}
&12\,\frac{(\lambda^2-1)^2}{(\lambda^2+1)^2}\,\mathrm{Re}\,\frac{z_1^2(1-\Gamma\bar z_2)}{(1-\Gamma z_2)^3}\\
&= 12\,\frac{(\lambda^2-1)^2}{(\lambda^2+1)^2}\,\mathrm{Re}\Big\{z_1^2\Big[1+3\Gamma z_2+6\Gamma^2 z_2^2+10\Gamma^3 z_2^3+15\Gamma^4 z_2^4+\cdots\\
&\quad -\Gamma\bar z_2 - 3\Gamma^2\Big(z_2\bar z_2-\frac14\Big) - 6\Gamma^3 z_2\Big(z_2\bar z_2-\frac25\Big) - 10\Gamma^4 z_2^2\Big(z_2\bar z_2-\frac36\Big)+\cdots\\
&\quad -3\Gamma^2\frac14 - 6\Gamma^3 z_2\frac25 - 10\Gamma^4 z_2^2\frac36 - \cdots\Big]\Big\}.
\end{aligned}$$

The latter expression can in turn be rewritten as

$$\begin{aligned}
&12\Gamma^2\,\mathrm{Re}\Big\{z_1^2\Big[\sum_{k=0}^{\infty}\frac{(k+1)(k+2)}{2}(\Gamma z_2)^k\Big(1-\Gamma^2\frac{k+3}{k+4}\Big)\\
&\qquad -\Gamma^2\sum_{k=-1}^{\infty}\frac{(k+2)(k+3)}{2}(\Gamma z_2)^k\Big(z_2\bar z_2-\frac{k+1}{k+4}\Big)\Big]\Big\}.
\end{aligned} \tag{A.10}$$

Recall that by (A.7), to obtain $\mathfrak{w}_\lambda$, we need to invert the operator $-4\Delta_b - 4$ on the latter expression, so we have to divide the coefficients of the spherical harmonics respectively by (using (A.8) and (A.9)) $4(k+2)-4 = 4(k+1)$ and by $4(3k+10)-4 = 12(k+3)$. We then find

$$\begin{aligned}
\mathfrak{w}_\lambda = \frac32\Gamma^2\,\mathrm{Re}\Big\{&z_1^2\Big[\sum_{k=0}^{\infty}(k+2)\,(\Gamma z_2)^k\Big(1-\Gamma^2\frac{k+3}{k+4}\Big)\\
&-\frac13\Gamma^2\sum_{k=-1}^{\infty}(k+2)\,(\Gamma z_2)^k\Big(z_2\bar z_2-\frac{k+1}{k+4}\Big)\Big]\Big\},
\end{aligned} \tag{A.11}$$

with

$$\Gamma = (1-\lambda^2)/(1+\lambda^2).$$

Notice that since $|\Gamma|<1$ all the above series are absolutely converging on S^3. Finally, the correction $\hat w_\lambda$ to φ_λ for the CR-Yamabe equation can be obtained from (A.5). ∎

A.2. Second order expansion of the CR Sobolev quotient

We want next to analyse the order s^2 in the expansion of the CR Sobolev quotient.

Lemma A.2. *If $Q_{(s)}$ is as in (5.4), then we have that*

$$Q_{(s)}(\varphi_\lambda + s\,\hat{w}_\lambda) = 4\pi + \frac{16\pi\lambda^2(3+12\lambda^2+2\lambda^4+12\lambda^6+3\lambda^8)}{(1+\lambda^2)^6}\, s^2 + \mathcal{B}_{\lambda,s},$$

with $|\mathcal{B}_{\lambda,s}| \le C_\Lambda s^3$.

Proof. Recall that, at $s=0$, from (2.5) one has $\frac{d}{ds}R_s = 0$ and $\frac{d^2}{ds^2}R_s = 8$. We use the choice of contact form

$$\hat{\theta} = \frac{1}{2}\, i \sum_{i=1}^{2} (z_i\, d\bar{z}_i - \bar{z}_i\, dz_i)\,, \qquad \hat{\theta}\wedge d\hat{\theta} = 2\, d\sigma_{\text{Eucl}}.$$

From the expression of $-\ddot{\Delta}_b$ (in (2.4)) and of $\ddot{R}$, we have that the second derivative $\ddot{Q}(\varphi_\lambda)$ of $Q_{(s)}(\varphi_\lambda)$ at $s=0$ is given by

$$\ddot{Q}(\varphi_\lambda) = \int_{S^3} \varphi_\lambda\,(-4\,\ddot{\Delta}_b\varphi_\lambda + \ddot{R}\varphi_\lambda)\,\hat{\theta}\wedge d\hat{\theta} = \int_{S^3}\varphi_\lambda\,(-16\,\Delta_b\varphi_\lambda + 8\varphi_\lambda)\,\hat{\theta}\wedge d\hat{\theta}.$$

Using (A.2), this also becomes

$$\ddot{Q}(\varphi_\lambda) = 8\int_{S^3}\varphi_\lambda^4\,\hat{\theta}\wedge d\hat{\theta} = 32\pi^2, \tag{A.12}$$

since the integral is independent of λ and since $\varphi_{\lambda=1}\equiv 1$.

Our next goal is to expand to second order in s the quantity $Q_{(s)}(\varphi_\lambda + s\,\hat{w}_\lambda)$. We claim that

$$\begin{aligned} Q_{(s)}(\varphi_\lambda + s\,\hat{w}_\lambda) &= 4\pi + \frac{s^2}{2\pi}\left(\frac{1}{2}\,\ddot{Q}(\varphi_\lambda) - \int_{S^3}\hat{w}_\lambda\,\mathfrak{L}_{\varphi_\lambda}\,\hat{w}_\lambda\,\hat{\theta}\wedge d\hat{\theta}\right) + o(s^2) \\ &= 4\pi + \frac{s^2}{2\pi}\left(\frac{1}{2}\,\ddot{Q}(\varphi_\lambda) - \int_{S^3}\mathfrak{w}_\lambda\,\mathfrak{L}_{\varphi_1}\,\mathfrak{w}_\lambda\,\hat{\theta}\wedge d\hat{\theta}\right) + o(s^2). \end{aligned}\tag{A.13}$$

Here, $\mathfrak{w}_\lambda$ is given in (A.11) (see also (A.5)), and $\mathfrak{L}_{\varphi_\lambda}$ is given in (A.4). The latter equality follows from the covariance property (A.6). To check this claim, we want to expand $Q_{(s)}(\varphi_\lambda + s\,\hat{w}_\lambda)$, which we write as

$$\frac{\int_{S^3}(\varphi_\lambda + s\,\hat{w}_\lambda)\,(L_0 + s\,\dot{L} + \frac{1}{2}s^2\,\ddot{L})\,(\varphi_\lambda + s\,\hat{w}_\lambda)\,\hat{\theta}\wedge d\hat{\theta}}{\left(\int_{S^3}(\varphi_\lambda + s\,\hat{w}_\lambda)^4\,\hat{\theta}\wedge d\hat{\theta}\right)^{1/2}}.$$

Expanding in s, we find that this quantity is equal to

$$\begin{aligned} &Q_{(s)}(\varphi_\lambda + s\,\hat{w}_\lambda) \\ &= \frac{\int_{S^3}\left[\varphi_\lambda L_0\varphi_\lambda + s(\varphi_\lambda\dot{L}\varphi_\lambda + 2\hat{w}_\lambda L_0\varphi_\lambda) + s^2(\frac{1}{2}\varphi_\lambda\ddot{L}\varphi_\lambda + 2\hat{w}_\lambda\dot{L}\varphi_\lambda + \hat{w}_\lambda L_0\hat{w}_\lambda)\right]\hat{\theta}\wedge d\hat{\theta}}{\left(\int_{S^3}(\varphi_\lambda^4 + 4s\,\varphi_\lambda^3\,\hat{w}_\lambda + 6s^2\,\varphi_\lambda^2\,\hat{w}_\lambda^2)\,\hat{\theta}\wedge d\hat{\theta}\right)^{1/2}} \\ &\quad + o(s^2). \end{aligned}$$

The first-order term in s vanishes, as one can see using the Euler equation for φ_λ, so we will just consider the second-order term. Since $\mathfrak{w}_\lambda$ only consists of spherical harmonics of positive order, see (A.11), using (A.5) it also turns out that

$$\int_{S^3} \varphi_\lambda^3 \hat{w}_\lambda \, \hat{\theta} \wedge d\hat{\theta} = \int_{S^3} \mathfrak{w}_\lambda \, \hat{\theta} \wedge d\hat{\theta} = 0,$$

so there is no contribution to the expansion of the denominator from the first-order term (in s) in the denominator.

Since $\int_{S^3} \varphi_\lambda L_0 \varphi_\lambda \, \hat{\theta} \wedge d\hat{\theta} = 8\pi^2$ and $\int_{S^3} \varphi_\lambda^4 \, \hat{\theta} \wedge d\hat{\theta} = 4\pi^2$, we can collect these numbers in the numerator and denominator respectively to get that

$$Q_{(s)}(\varphi_\lambda + s\,\hat{w}_\lambda) = \frac{8\pi^2}{(4\pi^2)^{1/2}} \frac{1 + \frac{s^2}{8\pi^2} \int_{S^3} (\frac{1}{2} \varphi_\lambda \ddot{L} \varphi_\lambda + 2 \hat{w}_\lambda \dot{L} \varphi_\lambda + \hat{w}_\lambda L_0 \hat{w}_\lambda) \, \hat{\theta} \wedge d\hat{\theta}}{\left(1 + \frac{s^2}{4\pi^2} \int_{S^3} 6 \varphi_\lambda^2 \hat{w}_\lambda^2 \, \hat{\theta} \wedge d\hat{\theta}\right)^{1/2}} + o(s^2).$$

Taylor-expanding, one finds

$$Q_{(s)}(\varphi_\lambda + s\,\hat{w}_\lambda) = \frac{8\pi^2}{(4\pi^2)^{1/2}} \left[1 + \frac{s^2}{8\pi^2} \left(\int_{S^3} \left(\frac{1}{2} \varphi_\lambda \ddot{L} \varphi_\lambda + 2 \hat{w}_\lambda \dot{L} \varphi_\lambda + \hat{w}_\lambda L_0 \hat{w}_\lambda \right) \hat{\theta} \wedge d\hat{\theta} - \int_{S^3} 6 \varphi_\lambda^2 \hat{w}_\lambda^2 \, \hat{\theta} \wedge d\hat{\theta} \right) \right] + o(s^2).$$

We now use the fact that $\hat{w}_\lambda$ satisfies

$$\mathfrak{L}_{\varphi_\lambda} \hat{w}_\lambda := L_0 \hat{w}_\lambda - 6 \varphi_\lambda^2 \hat{w}_\lambda = -\dot{L} \varphi_\lambda$$

to deduce that

$$\begin{aligned} Q_{(s)}(\varphi_\lambda + s\,\hat{w}_\lambda) &= \frac{8\pi^2}{(4\pi^2)^{1/2}} \left[1 + \frac{s^2}{8\pi^2} \left(\int_{S^3} \left(\frac{1}{2} \varphi_\lambda \ddot{L} \varphi_\lambda - \hat{w}_\lambda \mathfrak{L}_{\varphi_\lambda} \hat{w}_\lambda \right) \hat{\theta} \wedge d\hat{\theta} \right) \right] + o(s^2) \\ &= \frac{8\pi^2}{(4\pi^2)^{1/2}} \left[1 + \frac{s^2}{8\pi^2} \left(\int_{S^3} \left(\frac{1}{2} \varphi_\lambda \ddot{L} \varphi_\lambda - \mathfrak{w}_\lambda \mathfrak{L}_{\varphi_1} \mathfrak{w}_\lambda \right) \hat{\theta} \wedge d\hat{\theta} \right) \right] + o(s^2). \qquad \text{(A.14)} \end{aligned}$$

We next compute the last integral. To explicitly integrate spherical harmonics, we need the following explicit formula (see Proposition 5.3 in [21]):

$$\int_{S^3} |z_1|^4 |z_2|^{2k} \, \hat{\theta} \wedge d\hat{\theta} = \frac{8\pi^2}{(k+1)(k+2)(k+3)}. \qquad \text{(A.15)}$$

Both $\mathfrak{w}_\lambda$ and $\mathfrak{L}_{\varphi_1} \mathfrak{w}_\lambda$ consist of two types of spherical harmonics, orthogonal to each-other. For the first series, taking real parts, we need to compute integrals of the form (notice that only products of conjugate terms contribute)

$$\frac{1}{4} \int_{S^3} (z_1^2 z_2^k + \bar{z}_1^2 \bar{z}_2^k)^2 \, \hat{\theta} \wedge d\hat{\theta} = \frac{4\pi^2}{(k+1)(k+2)(k+3)}.$$

For the second series, still taking real parts, we need to compute instead

$$\frac{1}{4}\int_{S^3}\Big[z_1^2 z_2^k\Big(z_2\bar z_2-\frac{k+1}{k+4}\Big)+\bar z_1^2\bar z_2^k\Big(z_2\bar z_2-\frac{k+1}{k+4}\Big)\Big]^2\hat\theta\wedge d\hat\theta$$
$$=\frac{1}{2}\int_{S^3}|z_1|^4|z_2|^{2k}\Big(|z_2|^4-2|z_2|^2\frac{k+1}{k+4}+\Big(\frac{k+1}{k+4}\Big)^2\Big)\hat\theta\wedge d\hat\theta.$$

Using (A.15), the expression becomes

$$\frac{12\pi^2}{(k+2)(k+3)(k+4)^2(k+5)}.$$

Therefore, from (A.10) and (A.11) we obtain

$$-\int_{S^3}\mathfrak{w}_\lambda\mathfrak{L}_{\varphi_1}\mathfrak{w}_\lambda\,\hat\theta\wedge d\hat\theta$$
$$=-24\,\frac{3}{2}\,\Gamma^4\Bigg\{\sum_{k=0}^{\infty}\frac{(k+1)(k+2)^2}{2}\Big(1-\Gamma^2\frac{k+3}{k+4}\Big)^2\Gamma^{2k}\frac{2\pi^2}{(k+1)(k+2)(k+3)}$$
$$+\sum_{k=-1}^{\infty}\frac{\Gamma^4}{6}(k+2)^2(k+3)\,\Gamma^{2k}\frac{6\pi^2}{(k+2)(k+3)(k+4)^2(k+5)}\Bigg\}.$$

After some simplification, this gives

$$-\int_{S^3}\mathfrak{w}_\lambda\mathfrak{L}_{\varphi_1}\mathfrak{w}_\lambda\,\hat\theta\wedge d\hat\theta$$
$$=-36\,\Gamma^4\pi^2\Bigg\{\sum_{k=0}^{\infty}\frac{k+2}{k+3}\Big(1-\Gamma^2\frac{k+3}{k+4}\Big)^2\Gamma^{2k}+\sum_{k=-1}^{\infty}\Gamma^4\frac{\Gamma^{2k}(k+2)}{(k+4)^2(k+5)}\Bigg\}.$$

Notice that the last series starts from $k=-1$, so after relabelling we get

$$-\int_{S^3}\mathfrak{w}_\lambda\mathfrak{L}_{\varphi_1}\mathfrak{w}_\lambda\,\hat\theta\wedge d\hat\theta$$
$$=-36\,\Gamma^4\pi^2\Bigg\{\sum_{k=0}^{\infty}\frac{k+2}{k+3}\Big(1-\Gamma^2\frac{k+3}{k+4}\Big)^2\Gamma^{2k}+\sum_{k=0}^{\infty}\Gamma^2\frac{\Gamma^{2k}(k+1)}{(k+3)^2(k+4)}\Bigg\}.$$

After some manipulation, the series reduces to a finite one, and we find

$$-\int_{S^3}\mathfrak{w}_\lambda\mathfrak{L}_{\varphi_1}\mathfrak{w}_\lambda\,\hat\theta\wedge d\hat\theta=8\pi^2\Gamma^4(\Gamma^2-3).$$

Collecting this formula and (A.14), from (A.12) and (A.13) we obtain the second order expansion

$$Q_{(s)}(\varphi_\lambda+s\,\hat w_\lambda)=4\pi+\frac{s^2}{2\pi}\Big(\frac{1}{2}\ddot Q(\varphi_\lambda)-\int_{S^3}\mathfrak{w}_\lambda\mathfrak{L}_{\varphi_1}\mathfrak{w}_\lambda\,\hat\theta\wedge d\hat\theta\Big)+o(s^2)$$
$$=4\pi+4\pi s^2(\Gamma^6-3\Gamma^4+2)+o(s^2),\quad\text{with }\Gamma=\frac{1-\lambda^2}{1+\lambda^2}.$$

This concludes the proof. ∎

We display next, see Figure 2, the graph of the function $4\pi(\Gamma^6 - 3\Gamma^4 + 2)$ in Γ. This shows that the second-order correction of the Sobolev quotient is always positive in λ, and tends to zero as $\lambda \to \infty$.

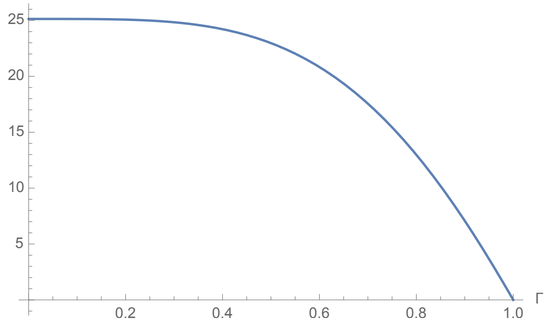

Figure 2. Graph of the function $4\pi(\Gamma^6 - 3\Gamma^4 + 2)$.

A.3. Conclusion

We can use the observation in Remark 5.3 to work out the contraction argument in Proposition 5.2 starting from $\varphi_\lambda + s\,\hat{w}_\lambda$ instead of from φ_λ only. Given the improved accuracy in Lemma A.1, the contraction can be performed in a ball of radius $O(s^2)$ in $\mathfrak{S}^{1,2}(S^3)$, yielding a corresponding correction $\check{w}_\lambda$ of that order. By Lemma A.1 and the smoothness of $Q_{(s)}$, we then have that

$$\begin{aligned} Q_{(s)}(\varphi_\lambda + s\,\hat{w}_\lambda + \check{w}_\lambda) &= Q_{(s)}(\varphi_\lambda + s\hat{w}_\lambda) + dQ_{(s)}(\varphi_\lambda + s\,\hat{w}_\lambda)[\check{w}_\lambda] + O(\|\check{w}_\lambda\|^2) \\ &= Q_{(s)}(\varphi_\lambda + s\,\hat{w}_\lambda) + O(s^4). \end{aligned}$$

By uniqueness in the fixed point of the contraction, it must be $\varphi_\lambda + s\,\hat{w}_\lambda + \check{w}_\lambda = \varphi_\lambda + w_\lambda$, so the conclusion follows from Lemma A.2.

B. Proof of Proposition 5.6

The goal of this section is to expand $Q_{(s)}$ on the functions $\varphi_\lambda + w_\lambda$ given by Proposition 5.2 for large values of λ. Since the estimates of the previous section deteriorate for λ in this range, we choose approximate solutions in terms of CR normal coordinates, better suited for highly-concentrated profiles.

Recall from the results in Section 5 of [11] that, given $p \in M$, the Green function of the conformal sub-Laplacian satisfies, in CR normal coordinates

$$G_p = 2\rho^{-2} + A + O(\rho).$$

B.1. Approximate solutions

For $p \in S^3$, fix a small number $r > 0$ and define in CR normal coordinates a function F such that

$$\begin{cases} F(z,t) = |z|^2 & \text{for } \rho \le r, \\ F \equiv 0 & \text{for } \rho \ge 2r. \end{cases}$$

In this way, F can be extended via cut-offs to all of S^3 as the zero function away from p, so F can be written as

$$F(z,t) = |z|^2 + O(\rho^5).$$

For $\lambda > 0$ large, let us consider a test function in CR normal coordinates as follows:

$$\breve{\varphi}_\lambda = \frac{\lambda}{(1 + \lambda^2 F + \lambda^4 \tilde{G})^{1/2}}, \tag{B.1}$$

where $\tilde{G} = G_p^{-2}$.

Lemma B.1. *In CR normal coordinates, one has the expansion*

$$\begin{aligned} L_b \breve{\varphi}_\lambda &= \breve{\varphi}_\lambda^3 (2 + O(\rho^3) + \lambda^{-2} O(\rho^2)) \\ &\quad + \breve{\varphi}_\lambda^5 \Big[-\frac{3}{2} |z|^2 \rho^2 (4 + \lambda^2 |z|^2) A + O(\rho^5) + O(\lambda^2 \rho^7) \Big]. \end{aligned}$$

Proof. By direct computation, we have that

$$\breve{\varphi}_{\lambda,1} = -\frac{1}{2} \frac{\lambda^3}{(1 + \lambda^2 F(z) + \lambda^4 \tilde{G})^{3/2}} [F_{,1} + \lambda^2 \tilde{G}_{,1}] = -\frac{1}{2} \breve{\varphi}_\lambda^3 [F_{,1} + \lambda^2 \tilde{G}_{,1}],$$

and similarly for its conjugate. As a consequence, we have that

$$\breve{\varphi}_{\lambda,1\bar{1}} = -\frac{1}{2} \breve{\varphi}_\lambda^3 [F_{,1\bar{1}} + \lambda^2 \tilde{G}_{,1\bar{1}}] + \frac{3}{4} \breve{\varphi}_\lambda^5 |F_{,1} + \lambda^2 \tilde{G}_{,1}|^2,$$

which implies

$$\Delta_b \breve{\varphi}_\lambda = -\frac{1}{2} \breve{\varphi}_\lambda^3 [\Delta_b F + \lambda^2 \Delta_b \tilde{G}] + \frac{3}{2} \breve{\varphi}_\lambda^5 |F_{,1} + \lambda^2 \tilde{G}_{,1}|^2.$$

By direct computation, one finds (with $G = G_p$)

$$\Delta_b \tilde{G} = -2 G^{-3} \Delta_b G + 12 G^{-4} G_{,1} G_{,\bar{1}}.$$

We then deduce

$$L_b \breve{\varphi}_\lambda = 2 \breve{\varphi}_\lambda^3 [\Delta_b F + \lambda^2 (12 G^{-4} G_{,1} G_{,\bar{1}} - 2 G^{-3} \Delta_b G)] - 6 \breve{\varphi}_\lambda^5 |F_{,1} + \lambda^2 \tilde{G}_{,1}|^2 + R \breve{\varphi}_\lambda.$$

We can next write

$$R \breve{\varphi}_\lambda = R \breve{\varphi}_\lambda^3 (\lambda^{-2} + F + \lambda^2 G^{-3} G).$$

Since G satisfies $L_b G = 0$, we get some cancellation and find that

$$L_b \breve{\varphi}_\lambda = \breve{\varphi}_\lambda^3 (2\Delta_b F + \lambda^{-2} R + R F) + 6\breve{\varphi}_\lambda^5 \big[4G^{-4} G_{,1} G_{,\bar{1}} (1 + \lambda^2 F(z) + \lambda^4 \tilde{G}) - |F_{,1} + \lambda^2 \tilde{G}_{,1}|^2\big].$$

Using some further cancellation, we then obtain

$$L_b \breve{\varphi}_\lambda = \breve{\varphi}_\lambda^3 (2\Delta_b F + \lambda^{-2} R + R F) + 6\breve{\varphi}_\lambda^5 \big[4G^{-4} G_{,1} G_{,\bar{1}} (1 + \lambda^2 F(z)) - F_{,1} F_{,\bar{1}} - \lambda^2 (F_{,1} \tilde{G}_{,\bar{1}} + \tilde{G}_{,1} F_{,\bar{1}})\big].$$

From Proposition A.5 in [11] (where a different but analogous notation is used), one has that, in CR normal coordinates,

$$Z_1 = (1 + O(\rho^4)) \mathring{Z}_1 + O(\rho^4) \mathring{Z}_{\bar{1}} + O(\rho^5) \frac{\partial}{\partial t},$$
$$\omega_1^1 = O(\rho^3)\, dz + O(\rho^3)\, d\bar{z} + O(\rho^2)\, \mathring{\theta},$$

see (2.1). By direct computation, one then has

$$G_{,1} = -\frac{i\sqrt{2}\,\bar{z}}{(t + i|z|^2)\rho^2} + O(1), \qquad F_{,1} = \frac{\bar{z}}{\sqrt{2}} + O(\rho^4),$$

$$\tilde{G}_{,1} = \frac{2\sqrt{2}\,\bar{z}\,(|z|^2 + it)}{(A\rho^2 + 2)^3} + O(\rho^6), \quad \Delta_b F = 1 + O(\rho^3).$$

Using these expressions in the above formula for $L_b \breve{\varphi}_\lambda$, one finally finds

$$L_b \breve{\varphi}_\lambda = \breve{\varphi}_\lambda^3 (2 + \lambda^{-2} O(\rho^2) + O(\rho^3)) + 6\breve{\varphi}_\lambda^5 \Big[-\frac{1}{4}|z|^2 \rho^2 (4 + \lambda^2 |z|^2) A + O(\rho^5) + O(\lambda^2 \rho^7)\Big],$$

which is the desired result. ■

If the contact form θ involved in the definition of CR normal coordinates writes as $\theta = e^{2v}\hat{\theta}$, setting

$$\bar{\varphi}_\lambda = e^{-v} \breve{\varphi}_\lambda, \tag{B.2}$$

and by the covariance property of the conformal sub-Laplacian, one has that

$$\breve{\varphi}_\lambda^4\, \theta \wedge d\theta = \bar{\varphi}_\lambda^4\, \hat{\theta} \wedge d\hat{\theta}, \quad \breve{\varphi}_\lambda L_b^{(\theta)} \breve{\varphi}_\lambda\, \theta \wedge d\theta = \bar{\varphi}_\lambda L_s \bar{\varphi}_\lambda\, \hat{\theta} \wedge d\hat{\theta}, \quad L_s = L_b^{(\hat{\theta})}. \tag{B.3}$$

These imply the invariance

$$Q_{(s)}(\bar{\varphi}_\lambda) = \frac{\int_{S^3} \bar{\varphi}_\lambda L_s \bar{\varphi}_\lambda\, \hat{\theta} \wedge d\hat{\theta}}{(\int_{S^3} \bar{\varphi}_\lambda^4\, \hat{\theta} \wedge d\hat{\theta})^{1/2}} = \frac{\int_{S^3} \breve{\varphi}_\lambda L_b^{(\theta)} \breve{\varphi}_\lambda\, \theta \wedge d\theta}{(\int_{S^3} \breve{\varphi}_\lambda^4\, \theta \wedge d\theta)^{1/2}}.$$

We then get the following consequence of Lemma B.1, concerning the differential of $Q_{(s)}$ at $\bar{\varphi}_\lambda$.

Corollary B.2. *There exists a constant $C > 0$ such that, for all s small and λ large, one has the inequality $|dQ_{(s)}(\bar{\varphi}_\lambda)[v]| \leq \frac{C}{\lambda^2}\|v\|_{\mathfrak{S}^{1,2}}$ for every $v \in \mathfrak{S}^{1,2}(S^3)$.*

Proof. By direct computation, for $v \in \mathfrak{S}^{1,2}(S^3)$, one has

$$\begin{aligned} dQ_{(s)}(\bar{\varphi}_\lambda)[v] = & \frac{2}{(\int_{S^3} \bar{\varphi}_\lambda^4 \,\hat{\theta} \wedge d\hat{\theta})^{3/2}} \int_{S^3} \Big[\Big(\int_{S^3} \bar{\varphi}_\lambda^4 \,\hat{\theta} \wedge d\hat{\theta}\Big) L_s\bar{\varphi}_\lambda \\ & - \Big(\int_{S^3} \bar{\varphi}_\lambda L_s\bar{\varphi}_\lambda \,\hat{\theta} \wedge d\hat{\theta}\Big)\bar{\varphi}_\lambda^3\Big] v \,\hat{\theta} \wedge d\hat{\theta}. \end{aligned} \tag{B.4}$$

From (B.3) and Lemma B.1, it follows that

$$\begin{aligned} &\int_{S^3} \bar{\varphi}_\lambda L_s\bar{\varphi}_\lambda \,\hat{\theta} \wedge d\hat{\theta} \\ &\quad = 2\int_{S^3} \breve{\varphi}_\lambda^4 \,\theta \wedge d\theta + \int_{S^3} [\breve{\varphi}_\lambda^4 (O(\rho^3) + \lambda^2 O(\rho^2)) + \breve{\varphi}_\lambda^6 (O(\rho^4) + \lambda^2 O(\rho^6))]\, \theta \wedge d\theta. \end{aligned}$$

Using a change of variable, it is possible then to show

$$\int_{S^3} \breve{\varphi}_\lambda^4 \,\theta \wedge d\theta = -\int_{S^3} \breve{\varphi}_\lambda L_b^\theta \breve{\varphi}_\lambda \,\theta \wedge d\theta + O(\lambda^{-2}).$$

Therefore, inserting the latter estimate and the result of Lemma B.1 into (B.4), we find that

$$|dQ_{(s)}(\bar{\varphi}_\lambda)[v]| \leq \int_{S^3} \breve{\varphi}_\lambda^3 \big[O(\rho^2) + O(\lambda^{-2}) + \breve{\varphi}_\lambda^5 \big(O(\rho^4) + \lambda^2 O(\rho^6)\big)\big]\, |v|\, \theta \wedge d\theta.$$

Applying Hölder's inequality, we get that

$$\begin{aligned} |dQ_{(s)}(\bar{\varphi}_\lambda)[v]| \leq & \Big[O(\lambda^{-2}) + \Big(\int_{S^3} \breve{\varphi}_\lambda^4 \,O(\rho^{8/3})\Big)^{3/4} + \Big(\int_{S^3} \breve{\varphi}_\lambda^{20/3} \,O(\rho^{16/3})\Big)^{3/4} \\ & + \lambda^2 \Big(\int_{S^3} \breve{\varphi}_\lambda^{20/3} \,O(\rho^8)\Big)^{3/4}\Big] \|v\|_{\mathfrak{S}^{1,2}}, \end{aligned}$$

where all integrals are computed with respect to the volume form $\theta \wedge d\theta$. By the expression of $\breve{\varphi}_\lambda$, all terms are integrable and of order λ^{-2}, which concludes the proof. ∎

B.2. Expansion of the CR Sobolev quotient

We expand next the CR Sobolev quotient $Q_{(s)}$ on the approximate solutions $\bar{\varphi}_\lambda$ in (B.2), obtaining the following result.

Lemma B.3. *Let $\breve{\varphi}_\lambda$ be defined in (B.1). Then for λ large, one has the expansion*

$$Q_{(s)}(\bar{\varphi}_\lambda) = 4\pi + 48\pi \,\frac{s^2}{\lambda^2}\,(1 + o_s(1)) + O\Big(\frac{1}{\lambda^3}\Big).$$

Proof. We use (B.3), Lemma B.1 and integrate: expanding the numerator in $Q_{(s)}$, we find that

$$\int_{S^3} \breve{\varphi}_\lambda L_b^{(\theta)} \breve{\varphi}_\lambda \, \theta \wedge d\theta$$
$$= 2 \int_{S^3} \breve{\varphi}_\lambda^4 \, \theta \wedge d\theta - \frac{3}{2} A \int_{\mathbb{H}^1} |z|^2 (4 + \lambda^2 |z|^2) \rho^2 \mathring{\varphi}_\lambda^6 \, \mathring{\theta} \wedge d\mathring{\theta}$$
$$+ \int_{S^3} \varphi^4 (O(\lambda^2 \rho^2) + O(\rho^3)) \, \theta \wedge d\theta + \int_{S^3} \varphi^6 (O(\rho^5) + O(\lambda^2 \rho^7)) \, \theta \wedge d\theta,$$

where

$$\mathring{\varphi}_\lambda = \frac{\lambda}{(1 + \lambda^2 |z|^2 + \frac{1}{4} \lambda^4 (|z|^4 + t^2))^{1/2}}, \quad (z, t) \in \mathbb{H}^1.$$

For the first term, which also appears in the above expression, we Taylor-expand $\tilde{G}$ as

$$\tilde{G} = \Big(\frac{2 + A\rho^2}{\rho^2} \Big)^{-2} = \frac{1}{4} \rho^4 (1 - A\rho^2) + O(\rho^8).$$

Therefore, $\breve{\varphi}_\lambda$ expands as

$$\breve{\varphi}_\lambda = \frac{\lambda}{\big(1 + \lambda^2 (|z|^2 + O(\rho^5)) + \frac{1}{4} \lambda^4 [\rho^4 (1 - A\rho^2) + O(\rho^8)]\big)^{1/2}}$$
$$= \Big(1 + \frac{1}{8} \frac{A \rho^6 \lambda^4}{1 + \lambda^2 |z|^2 + \frac{1}{4} \lambda^4 \rho^4} + O\Big(\frac{\rho^{12} \lambda^8}{(1 + \lambda^4 \rho^4)^2}\Big)\Big) \mathring{\varphi}_\lambda$$
$$= \mathring{\varphi}_\lambda + \frac{1}{8} A \rho^6 \lambda^2 \mathring{\varphi}_\lambda^3 + O\Big(\frac{\rho^{12} \lambda^8}{(1 + \lambda^4 \rho^4)^2}\Big) \mathring{\varphi}_\lambda.$$

Taylor-expanding the integral of the fourth power of $\breve{\varphi}_\lambda$ and using a change of variable we get that

$$\int_{S^3} \breve{\varphi}_\lambda^4 \, \theta \wedge d\theta = \int_{\mathbb{H}^1} \mathring{\varphi}_\lambda^4 \, \mathring{\theta} \wedge d\mathring{\theta} + \frac{1}{2} A \lambda^2 \int_{\mathbb{H}^1} \rho^6 \mathring{\varphi}_\lambda^6 \, \mathring{\theta} \wedge d\mathring{\theta} + O(1/\lambda^3).$$

Hence, using the fact that $\int_{\mathbb{H}^1} \mathring{\varphi}_\lambda^4 \, \mathring{\theta} \wedge d\mathring{\theta}$ is independent of λ, $Q_{(s)}(\bar{\varphi}_\lambda)$ becomes

$$\frac{2\big(\int_{\mathbb{H}^1} \mathring{\varphi}_{\sqrt{2}}^4 \, \mathring{\theta} \wedge d\mathring{\theta} + \frac{1}{2} A\lambda^2 \int_{\mathbb{H}^1} \rho^6 \mathring{\varphi}_\lambda^6 \, \mathring{\theta} \wedge d\mathring{\theta}\big) - \frac{3A}{2} \int_{\mathbb{H}^1} |z|^2 \rho^2 (4 + \lambda^2 |z|^2) \mathring{\varphi}_1^6 \, \mathring{\theta} \wedge d\mathring{\theta}}{\big(\int_{\mathbb{H}^1} \mathring{\varphi}_{\sqrt{2}}^4 \, \mathring{\theta} \wedge d\mathring{\theta} + \frac{1}{2} A \lambda^2 \int_{\mathbb{H}^1} \rho^6 \mathring{\varphi}_\lambda^6 \, \mathring{\theta} \wedge d\mathring{\theta}\big)^{1/2}}$$
$$+ O(1/\lambda^3).$$

We can expand the denominator in the latter expression as

$$
\begin{aligned}
&\Big(\int_{\mathbb{H}^1} \mathring{\varphi}^4_{\sqrt{2}}\, \mathring{\theta} \wedge d\mathring{\theta} + \frac{1}{2}\, A\lambda^2 \int_{\mathbb{H}^1} \rho^6\, \mathring{\varphi}^6_\lambda\, \mathring{\theta} \wedge d\mathring{\theta}\Big)^{-1/2} \\
&\quad = \Big(\int_{\mathbb{H}^1} \mathring{\varphi}^4_{\sqrt{2}}\, \mathring{\theta} \wedge d\mathring{\theta}\Big)^{-1/2} \left(1 + \frac{\frac{1}{2} A\lambda^2 \int_{\mathbb{H}^1} \rho^6\, \mathring{\varphi}^6_\lambda\, \mathring{\theta} \wedge d\mathring{\theta}}{\int_{\mathbb{H}^1} \mathring{\varphi}^4_1\, \mathring{\theta} \wedge d\mathring{\theta}}\, \mathring{\theta} \wedge d\mathring{\theta}\right)^{-1/2} \\
&\quad = \Big(\int_{\mathbb{H}^1} \mathring{\varphi}^4_{\sqrt{2}}\, \mathring{\theta} \wedge d\mathring{\theta}\Big)^{-1/2} \left(1 - \frac{1}{4}\, A\lambda^2\, \frac{\int_{\mathbb{H}^1} \rho^6\, \mathring{\varphi}^6_\lambda\, \mathring{\theta} \wedge d\mathring{\theta}}{\int_{\mathbb{H}^1} \mathring{\varphi}^4_1\, \mathring{\theta} \wedge d\mathring{\theta}}\right) + O(1/\lambda^3),
\end{aligned}
$$

which gives

$$
\begin{aligned}
Q_{(s)}(\bar{\varphi}_\lambda) = &\Big(\int_{\mathbb{H}^1} \mathring{\varphi}^4_{\sqrt{2}}\, \mathring{\theta} \wedge d\mathring{\theta}\Big)^{-1/2} \Big[2 \int_{\mathbb{H}^1} \mathring{\varphi}^4_{\sqrt{2}}\, \mathring{\theta} \wedge d\mathring{\theta} \\
&+ \frac{1}{2}\, A\lambda^2 \int_{\mathbb{H}^1} \rho^6\, \mathring{\varphi}^6_\lambda\, \mathring{\theta} \wedge d\mathring{\theta} - \frac{3}{2}\, A \int_{\mathbb{H}^1} |z|^2 \rho^2 (4 + \lambda^2 |z|^2)\, \mathring{\varphi}^6_1\, \mathring{\theta} \wedge d\mathring{\theta}\Big] + O\Big(\frac{1}{\lambda^3}\Big),
\end{aligned}
$$

equivalent to

$$
\begin{aligned}
\text{(B.5)}\quad Q_{(s)}(\bar{\varphi}_\lambda) = &\Big(\int_{\mathbb{H}^1} \mathring{\varphi}^4_{\sqrt{2}}\, \mathring{\theta} \wedge d\mathring{\theta}\Big)^{-1/2} \Big[2 \int_{\mathbb{H}^1} \mathring{\varphi}^4_{\sqrt{2}}\, \mathring{\theta} \wedge d\mathring{\theta} \\
&- \frac{1}{2}\, A \int_{\mathbb{H}^1} \big(3|z|^2 (4 + \lambda^2 |z|^2) - \lambda^2 \rho^4\big) \rho^2\, \mathring{\varphi}^6_\lambda\, \mathring{\theta} \wedge d\mathring{\theta}\Big] + O\Big(\frac{1}{\lambda^3}\Big).
\end{aligned}
$$

The computation on page 177 in [19] (where θ_1, in their notation, equals $2\hat{\theta}$) shows that $\mathring{\varphi}^4_{\sqrt{2}}$ is the scaling factor for the volume of the Cayley map. Recalling that $\hat{\theta} \wedge d\hat{\theta}$ is twice the (induced) Euclidean volume on S^3, this implies

$$
\text{(B.6)}\qquad \int_{\mathbb{H}^1} \mathring{\varphi}^4_{\sqrt{2}}\, \mathring{\theta} \wedge d\mathring{\theta} = \int_{S^3} \hat{\theta} \wedge d\hat{\theta} = 4\pi^2.
$$

We now make the following change of variables: $\lambda z \mapsto \sqrt{2} z'$, $\lambda^2 t \mapsto 2t'$, and notice that

$$
\mathring{\varphi}_\lambda(z,t) = \frac{\lambda}{\sqrt{2}}\, \frac{\sqrt{2}}{((1 + |z'|^2)^2 + (t')^2)^{1/2}} = \frac{\lambda}{\sqrt{2}}\, \mathring{\varphi}_{\sqrt{2}}(z', t').
$$

In this way, we have

$$
\begin{aligned}
&\int_{\mathbb{H}^1} \big(3|z|^2 (4 + \lambda^2 |z|^2) - \lambda^2 \rho^4\big) \rho^2\, \mathring{\varphi}^6_\lambda\, \mathring{\theta} \wedge d\mathring{\theta}(z,t) \\
&\qquad = \frac{4}{\lambda^2} \int_{\mathbb{H}^1} \big(3|z'|^2 (2 + |z'|^2) - (\rho')^4\big) (\rho')^2\, \mathring{\varphi}^6_{\sqrt{2}}\, \mathring{\theta} \wedge d\mathring{\theta}(z', t').
\end{aligned}
$$

One checks by direct computations that the primitive with respect to t of the integrand is

$$- \frac{3\,(|z'|^6+8\,|z'|^4+19\,|z'|^2+8)\,|z'|^6}{2\,(|z'|^2+1)^5\,(2\,|z'|^2+1)^{3/2}}$$
$$\times \log\Big(\frac{t^2\,(|z'|^4+4|z'|^2+2)-2t\,(|z'|^2+1)\,\sqrt{2|z'|^2+1}\,\sqrt{t^2+|z'|^4}+(|z'|^2+1)^2\,|z'|^4}{|z'|^4\,(t^2+(|z'|^2+1)^2)}\Big)$$
$$- \frac{t\,\sqrt{t^2+|z'|^4}}{(|z'|^2+1)^4\,(2\,|z'|^2+1)\,(t^2+(|z'|^2+1)^2)^2}$$
$$\times\big(t^2\,(3|z'|^8+4|z'|^6-17|z'|^4-4|z'|^2+2)+(|z'|^3+|z'|)^2\,(3|z'|^6-8\,|z'|^4-55\,|z'|^2-24)\big).$$

As a consequence, we deduce that

$$\int_{\mathbb{R}} (3\,|z'|^2\,(2+|z'|^2)-(\rho')^4)\,(\rho')^2\,\mathring{\varphi}^6_{\sqrt{2}}\,dt$$
$$= \frac{-3\,(|z'|^6+8|z'|^4+19|z'|^2+8)\,|z'|^6\log\Big(\frac{|z'|^4-2\,(\sqrt{2|z'|^2+1}-2)\,|z'|^2-2\,\sqrt{2|z'|^2+1}+2}{|z'|^4+2\,(\sqrt{2|z'|^2+1}+2)\,|z'|^2+2\,(\sqrt{2|z'|^2+1}+1)}\Big)}{2\,(|z'|^2+1)^5\,(2\,|z'|^2+1)^{3/2}}$$
$$- \frac{4\,(|z'|^2+1)\,\sqrt{2|z'|^2+1}\,(3\,|z'|^8+4\,|z'|^6-17\,|z'|^4-4\,|z'|^2+2)}{2\,(|z'|^2+1)^5\,(2\,|z'|^2+1)^{3/2}}.$$

Multiplying this quantity by $2\pi|z'|$, its primitive with respect to $|z'|$ is

$$- \frac{3\pi\,(|z'|^2+2)\,|z'|^8\log\Big(\frac{|z'|^4-2\,(\sqrt{2\,|z'|^2+1}-2)\,|z'|^2-2\,\sqrt{2|z'|^2+1}+2}{|z'|^4+2\,(\sqrt{2\,|z'|^2+1}+2)\,|z'|^2+2\,(\sqrt{2\,|z'|^2+1}+1)}\Big)}{2\,(|z'|^2+1)^4\,\sqrt{2|z'|^2+1}}$$
$$- \frac{4\pi\,\sqrt{2|z'|^2+1}\,(|z'|^6+6\,|z'|^4+6\,|z'|^2+1)}{2\,(|z'|^2+1)^4\,\sqrt{2|z'|^2+1}},$$

whose difference between the values $|z'| \to +\infty$ and $|z'| = 0$ is 8π. Therefore, recalling that the volume form $\mathring{\theta} \wedge d\mathring{\theta}$ is four times the Euclidean one, we obtain that

$$\int_{\mathbb{H}^1} (3\,|z|^2(4+\lambda^2\,|z|^2)-\lambda^2\,\rho^4)\,\rho^2\,\mathring{\varphi}^6_\lambda\,\mathring{\theta}\wedge d\mathring{\theta} = 32\pi.$$

Recalling (B.6) and the fact that $A = -\frac{3}{2}\,s^2(1+o_s(1))$, from (4.16) and (B.5) we deduce that

$$Q_{(s)}(\breve{\varphi}_\lambda) = 4\pi - \frac{32\,\pi A}{\lambda^2}\,s^2(1+o_s(1)) + O\Big(\frac{1}{\lambda^3}\Big) = 4\pi + \frac{48\pi}{\lambda^2}\,s^2(1+o_s(1)) + O\Big(\frac{1}{\lambda^3}\Big).$$

This concludes the proof. ∎

B.3. Conclusion

We can use the observation in Remark 5.3, to perform the contraction argument in Proposition 5.2 starting from $\breve{\varphi}_\lambda$ instead of from φ_λ only. Given the improved accuracy in

Lemma 5.5, the contraction can be performed in a ball of radius $O(1/\lambda^2)$ in $\mathfrak{S}^{1,2}(S^3)$, yielding a corresponding correction $\breve{w}_\lambda$ of that order. By Lemma 5.5 and the smoothness of $Q_{(s)}$, we then have, similarly to Subsection A.3,

$$Q_{(s)}(\breve{\varphi}_\lambda + \breve{w}_\lambda) = Q_{(s)}(\breve{\varphi}_\lambda) + O(\|\breve{w}_\lambda\|^2) = Q_{(s)}(\breve{\varphi}_\lambda) + O\Big(\frac{1}{\lambda^4}\Big).$$

By uniqueness of the fixed point, it must be $\breve{\varphi}_\lambda + \breve{w}_\lambda = \varphi_\lambda + w_\lambda$, so from Lemma B.3 we get that

$$Q_{(s)}(\varphi_\lambda + w_\lambda) = 4\pi + 48\pi \frac{s^2}{\lambda^2}(1 + o_s(1)) + O\Big(\frac{1}{\lambda^3}\Big). \tag{B.7}$$

Notice that

$$Q_{(0)}(\varphi_\lambda + w_\lambda) = Q_{(0)}(\varphi_\lambda) \equiv 4\pi,$$

and therefore the term $O(1/\lambda^3)$ appearing in (B.7) is identically zero for $s = 0$, even and smooth in s. It therefore must be of the form $O(s^2/\lambda^3)$. Hence the statement of the proposition holds true.

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