Rev. Mat. Iberoam. (Online first)
DOI 10.4171/RMI/1375

Published by EMS Press

How highly connected can an orbifold be?

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Abstract. On the one hand, we provide the first examples of arbitrarily highly connected (compact) bad orbifolds. On the other hand, we show that n-connected n-orbifolds are manifolds. The latter improves the best previously known bound of Lytchak by roughly a factor of 2. For compact orbifolds and in most dimensions, we prove slightly better bounds. We obtain sharp results up to dimension 5.

1. Introduction

Orbifolds are a generalization of manifolds that incorporate local symmetries. They were introduced by Satake under the name of V-manifolds [22, 23] and later rediscovered by Thurston. Orbifolds for instance occur as quotient spaces of Lie group actions or foliations [17], as collapsed limits of manifolds under Gromov–Hausdorff convergence [6] and as moduli spaces, e.g., in Teichmüller theory.

Thurston [25] defined the notions of orbifold coverings and fundamental groups, and showed that they behave as in the classical case. More generally, *orbifold homotopy*- and (*co*)*homology groups* can be defined, after observing that every orbifold can be realized as a quotient of an almost free action of a compact Lie group on a manifold, as the usual homotopy and (co)homology groups of the associated Borel construction. In the special case of the orbifold fundamental group, one recovers Thurston's definition, because orbifold coverings are in one-to-one correspondence with coverings of a model for the Borel construction.

An orbifold is called *good* if it is covered by a manifold and *bad* otherwise. It is a natural question to ask which measurable parameters tell good and bad orbifolds apart or identify manifolds among orbifolds. Such identifiers can have also practical applications. The fact that an orbifold which admits a Riemannian metric of constant curvature is always good (see [18]) was for instance applied in [15] in the context of Alexandrov geometry. In [17], Lytchak and Wilking completed the program initiated by Ghys [7] and Gromoll–Grove [9] of classifying Riemannian foliations of spheres, and a large part of the work amounted to showing that a compact 7-connected 8-orbifold is in fact a manifold. In the context of Riemannian orbifolds all of whose geodesics are closed, the authors [2] applied

2020 Mathematics Subject Classification: Primary 57N65; Secondary 57R91.
Keywords: Orbifolds, highly connected.

a cohomology characterization of manifolds among orbifolds, see Proposition 2.1, in order to show that odd-dimensional orbifolds all of whose geodesics are closed are covered by spheres.

In terms of topological parameters, Davis asked whether a contractible orbifold is a manifold [3], p. 28. A positive answer to this question was provided by Lytchak, who actually showed that a $(2n-2)$-connected n-dimensional orbifold is a manifold [16], see also Section 2.2. In the same paper [16], Lytchak moreover asked whether arbitrarily highly connected bad orbifolds exist.

In this paper we answer Lytchak's question in the positive, and we improve his bound by roughly a factor of 2.

On improving the bound, we have the following result.

Theorem A. *An n-connected n-orbifold is a manifold for any $n \geq 1$. A $2n$-connected $(2n+1)$-orbifold is a manifold for any $n \geq 1$.*

Note that the even-dimensional part of this theorem is only a statement about non-compact orbifolds: a simply connected, compact n-orbifold has a fundamental class and can thus not be n-connected. In the compact case, we obtain a better bound by additionally exploiting Lefschetz duality and the classification of homology spheres that are covered by spheres [24].

Theorem B. *A compact $(2n-2)$-connected $2n$-orbifold is a manifold for any $n \geq 3$. For $n \geq 3$ not a power of 2, a compact $(2n-2)$-connected $(2n+1)$-orbifold is a manifold.*

In particular, our argument simplifies the tricky part (i.e., Section 4) in the work [17] of Lytchak and Wilking concerned with showing that a certain compact 7-connected 8-orbifold is a manifold. The results of Theorems A and B hinge on the fact, proved in Proposition 4.5, that m-connected orbifolds have singular strata only in codimension $> 2\lfloor (m-1)/2 \rfloor$.

On the constructive side, we answer Lytchak's question about the existence of highly connected bad orbifolds in the following way.

Theorem C. *For any $n \geq 4$, there exist compact $\lfloor n/2 - 1 \rfloor$-connected bad orbifolds in dimension n.*

Moreover, in low dimensions we present some specific constructions that yield higher connectedness than provided by Theorem C. For instance, there exists a compact, 3-connected, bad 4-orbifold. All our results are summarized in the following section.

1.1. Summary of the results

Given a dimension n, let $\kappa(n)$ be the maximum k such that there exists a bad, k-connected, n-orbifold, and let $\kappa_c(n)$ be the maximum k such that there exists a bad, k-connected, compact n-orbifold. Then the statements we prove can be written as follows.

Theorem D (Bounds for $\kappa(n)$). *The following hold:*

(1) $\lfloor n/2 \rfloor - 1 \leq \kappa(n) < 2\lfloor n/2 \rfloor$ *for* $n \geq 4$,

(2) $\kappa(3) = 1$ *and* $\kappa(4) = \kappa(5) = 3$.

Theorem E (Bounds for $\kappa_c(n)$). *The following hold*:

(1) $\lfloor n/2 \rfloor - 1 \leq \kappa_c(n) < 2\lfloor (n-1)/2 \rfloor$ *for* $n \geq 5$,

(2) $\kappa_c(2n+1) < 2n - 2$ *if* $1 < n \neq 2^k$,

(3) $\kappa_c(3) = 1$ *and* $\kappa_c(4) = 3$.

1.2. Structure of the paper

In Section 2 we recall some background about orbifolds. In particular, we recall a cohomological characterization of manifolds among orbifolds, see Proposition 2.1. In Section 3 we provide specific examples of highly connected orbifolds in low dimensions. Moreover, we prove Theorem C about the existence of arbitrarily highly connected orbifolds. In Section 4, we prove our Theorems A and B. Finally, in Section 5 we prove Theorems D and E.

Let us close the introduction with the following questions:

Question 1.1. What are the precise values of $\kappa(n)$ and $\kappa_c(n)$?

Question 1.2. (When) does $\kappa = \kappa_c$ hold?

2. Preliminaries

2.1. Orbifolds

An *n-dimensional Riemannian orbifold* is a metric length space $\mathcal{O}$ such that each point in $\mathcal{O}$ has a neighborhood that is isometric to the quotient of an n-dimensional Riemannian manifold M by an isometric action of a finite group Γ. Every such Riemannian orbifold has a canonical smooth orbifold structure in the usual sense [14]. Conversely, every smooth (effective) orbifold can be endowed with a Riemannian metric, and then the induced length metric turns it into a Riemannian orbifold in the above sense. For a point p in $\mathcal{O}$, the isotropy group of a preimage of p in a Riemannian manifold chart is uniquely determined up to conjugation in $\mathrm{O}(n)$. Its conjugacy class is called the *local group* of $\mathcal{O}$ at p and we also denote it as Γ_p. The point p is called *regular* if this group is trivial and *singular* otherwise. More precisely, an orbifold admits a stratification into manifolds, where the stratum of codimension k is given by

$$\Sigma_k = \{p \in \mathcal{O} \mid \operatorname{codimFix}(\Gamma_p) = k\}.$$

In particular, Σ_0 is the set of regular points.

Riemannian orbifolds for instance arise as a quotient of a Riemannian manifold M by an effective, isometric and almost free (i.e., isotropy groups are finite) action of a compact Lie group G. In fact, every Riemannian orbifold arises in this way. Namely, it can be obtained as the quotient of its (orthonormal) frame bundle by the natural $\mathrm{O}(n)$-action on it. The homotopy type of the corresponding Borel construction $B\mathcal{O} \overset{\mathrm{def}}{=} M \times_G EG$, which was first considered by Haefliger [11], depends only on the orbifold $\mathcal{O}$ and not on the specific representation of $\mathcal{O}$ as a quotient, see Proposition 1.51 in [1]. Hence, if $\mathcal{O}$ is a manifold, we can take $M = \mathcal{O}$, $G = \{1\}$ and see that $B\mathcal{O} = M$. The orbifold homotopy groups, cohomology groups, etc. of an orbifold $\mathcal{O}$ are defined as the corresponding invariants of $B\mathcal{O}$. In particular, an orbifold $\mathcal{O}$ is by definition k-connected, if $B\mathcal{O}$ is so.

The notion of an orbifold covering is defined in [25]. An alternative metric characterization of orbifold coverings is provided in [14].

In dimension 1, every orbifold is covered by $\mathbb{R}$. In dimension 2, the so-called football orbifolds (also known as spindle or complex weighted projective lines) provide examples of simply connected, bad, compact 2-orbifold, see Figure 1. In fact, any bad, simply connected 2-orbifold is of this type [3].

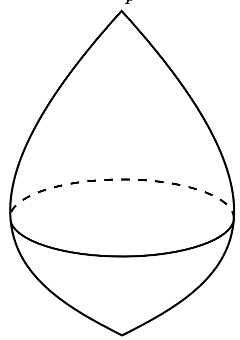

Figure 1. A football orbifold is homeomorphic to S^2 and has two cyclic singularities of order p and q. Its fundamental group is isomorphic to $\mathbb{Z}_{\gcd(p,q)}$.

An orbifold (respectively, a Riemannian orbifold) is diffeomorphic to a manifold (respectively, isometric to a Riemannian manifold) if and only if its cohomology groups are non-trivial in only finitely many degrees. This statement appears in a more general form in the work of Quillen, see Corollary 7.8 in [21]. In particular, it yields an alternative proof for Davis' question (see [3], p. 28) whether a contractible orbifold is a manifold. In the special case of an orientable orbifold, an easier argument is provided in Proposition 3.3 of [2]. For the convenience of the reader, we present this argument here in a condensed form.

Proposition 2.1. *An orientable n-orbifold $\mathcal{O}$ is a manifold, i.e., all its local groups are trivial, if and only if $H^i_{\mathrm{orb}}(\mathcal{O}) = H^i(B\mathcal{O};\mathbb{Z}) = 0$ for all $i > n$. This is the case if and only if the cohomology is nontrivial only in finitely many degrees.*

Proof. We only need to show the 'if' part. If $\mathcal{O}$ is not a manifold, then there exists some $\mathbb{Z}_p \subset \mathrm{SO}(n)$ which fixes some $x \in \mathrm{Fr}(\mathcal{O})$. In this case, the projection

$$\mathrm{Fr}(\mathcal{O})_{\mathbb{Z}_p} \overset{\mathrm{def}}{=} \mathrm{Fr}(\mathcal{O}) \times_{\mathbb{Z}_p} E\mathrm{SO}(n) \to B\mathbb{Z}_p$$

admits the section $s(b) = [x, b]$. Thus, we obtain an inclusion $H^*(B\mathbb{Z}_p) \hookrightarrow H^*(\mathrm{Fr}(\mathcal{O})_{\mathbb{Z}_p})$. Now consider the fibration

$$\mathrm{SO}(n)/\mathbb{Z}_p \to \mathrm{Fr}(\mathcal{O})_{\mathbb{Z}_p} \to B\mathcal{O} \overset{\mathrm{def}}{=} \mathrm{Fr}(\mathcal{O})_{\mathrm{SO}(n)}.$$

Since $\mathcal{O}$ is orientable, $\pi_1(B\mathcal{O})$ acts trivially on $H^*(\mathrm{SO}(n)/\mathbb{Z}_p)$ (see Lemma 3.1 in [2]), and thus the Leray–Serre spectral sequence satisfies

$$E_2^{p,q} = H^p(B\mathcal{O}; H^q(\mathrm{SO}(n)/\mathbb{Z}_p))$$

and converges to $H^{p+q}(\mathrm{Fr}(\mathcal{O})_{\mathbb{Z}_p})$. Since $\mathrm{SO}(n)/\mathbb{Z}_p$ is a finite-dimensional manifold and $H^*(\mathrm{Fr}(\mathcal{O})_{\mathbb{Z}_p})$ is nontrivial in infinitely many degrees as we have seen above, the same must be true for $H^*(B\mathcal{O})$. ∎

Remark 2.2. An orbifold can be homeomorphic to a manifold even if some of its local groups are not trivial. The question when this happens is completely answered in [13].

2.2. Lytchak's bound

We recall the following argument by Lytchak [16]. Suppose an n-orbifold $\mathcal{O}$ is k-connected for some $k \geq 1$. Then $\mathcal{O}$ is in particular orientable and so one can realize it as an almost free $G = \mathrm{SO}(n)$-quotient of a manifold M. The orbit map $o_p: G \to M$ is then k-connected as well. Thus for any $l < k$, the map $H^l(M) \to H^l(G)$ is surjective. Since the orbit map factorizes through $\pi_p: G \to G/G_p$, the map $H^l(G/G_p; \mathbb{Z}) \to H^l(G; \mathbb{Z})$ is surjective for $l < k$ as well. If $\mathcal{O}$ is not a manifold, i.e., if G_p is nontrivial for some p, then $G \to G/G_p$ is a nontrivial covering map for this p, in which case the image of $H^m(G/G_p; \mathbb{Z})$ in $H^m(G; \mathbb{Z}) = \mathbb{Z}$ is a subgroup of index $|G_p|$ for $m = \dim G = n(n-1)/2$. Since the free part of $H^*(G; \mathbb{Z})$ is generated in degree $\leq 2n-3$, it follows that $\mathcal{O}$ is a manifold if $k \geq 2n-2$.

We remark that this argument can be improved for 4-orbifolds. Indeed, in this case $H^6(G; \mathbb{Z})$ is generated in degree 3, not just in degree $\leq 2n-3 = 5$.

Therefore, the above argument shows that a 4-orbifold $\mathcal{O}$ is a manifold if it is 4-connected. In Section 4 we will see an alternative proof of this conclusion that works in any dimension.

3. Examples and constructions

In this section we provide specific examples and general constructions for highly connected orbifolds.

3.1. Low dimensional examples

An example of a compact, simply connected, bad 2-orbifold is shown in Figure 1 when p and q are coprime. We will in particular see that compact, simply connected, bad orbifolds exist in all dimensions. An example in dimension 3 is provided by the following proposition.

Proposition 3.1. *There exists a compact simply connected bad* 3*-orbifold.*

Proof. There exists a 3-orbifold with underlying space S^3 whose singular set is the trivalent graph shown in Figure 2, see [25]. The edge weights specify the orders of the corresponding cyclic local groups. The local groups at the two upper vertices are the orientation preserving symmetry group I of an icosahedron. The local groups at the two lower vertices are the orientation preserving symmetry group O of a cube. We claim that this orbifold is simply connected as an orbifold. To see this we first observe that it is the double of the dashed 3-ball. By Seifert–Van Kampen, it suffices to show that this 3-ball is simply connected as an orbifold. The latter is covered by two good orbifolds with fundamental groups I and O, respectively. Their intersection is homotopy equivalent to $\mathbb{R}^2(2,3)$, an open disk with two cyclic orbifold singularities of order 2 and 3, respectively. The fundamental group of this disk has two generators x and y represented respectively by a loop around the cyclic singularity of order 2, and a loop around the cyclic singularity of order 3. A presentation of this fundamental group is given by $\langle x, y \mid x^2, y^3 \rangle$. The fundamental groups of the two good orbifold balls isomorphic to O and I are generated by x and y as well and have presentations $\langle x, y \mid x^2, y^3, (xy)^4 \rangle$ and $\langle x, y \mid x^2, y^3, (xy)^5 \rangle$, respectively. Applying Seifert–Van Kampen again now proves the claim. ■

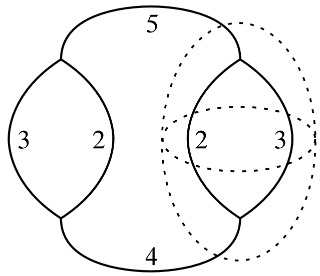

Figure 2. Compact simply connected bad 3-orbifold.

In dimension 4, higher connectedness can be achieved.

Proposition 3.2. *There exists a compact* 3*-connected bad* 4*-orbifold.*

Proof. If ρ is an irreducible representation of SU(2) on a complex even-dimensional vector space $\mathbb{C}^n$, then the induced action of SU(2) on the unit sphere $S^{2n-1} \subseteq \mathbb{C}^n$ is almost free but not free, and so the corresponding quotient $S^{2n-1}/\mathrm{SU}(2)$, which is also referred to as a *weighted quaternionic projective space*, is a $(2n-4)$-dimensional orbifold, [9], p. 154. In particular, for $n = 4$ the quotient $\mathcal{O} = S^7/\mathrm{SU}(2)$ is a compact 4-orbifold that is not a manifold.

The SU(2) action on S^7 induces a free action on the product $S^7 \times E\mathrm{SU}(2) \simeq S^7$, thus producing a principal SU(2)-bundle

$$S^7 \simeq S^7 \times E\mathrm{SU}(2) \to S^7 \times_{\mathrm{SU}(2)} E\mathrm{SU}(2) = B\mathcal{O}$$

and the corresponding long exact sequence in homotopy yields

$$\cdots \to \pi_i(S^7) \to \pi_i^{\mathrm{orb}}(\mathcal{O}) \to \pi_{i-1}(\mathrm{SU}(2)) \to \cdots$$

Since S^7 is 6-connected and $\mathrm{SU}(2) \cong S^3$ is 2-connected, the long exact sequence implies that this orbifold is 3-connected. ■

Note that by taking products with $\mathbb{R}^n$, Proposition 3.2 provides examples of (non-compact) 3-connected n-orbifolds for any $n \geq 5$. In dimension 5 we are only aware of 2-connected compact examples. In fact, the example in the following proposition can be shown to be not 3-connected.

Proposition 3.3. *There exists a compact* 2*-connected bad* 5*-orbifold.*

Proof. Consider the maps $\rho_1 \colon \mathrm{SU}(2) \to \mathrm{SU}(3)$ given by inclusion, and $\rho_2 \colon \mathrm{SU}(2) \to \mathrm{SU}(3)$ given by the composition $\mathrm{SU}(2) \to \mathrm{SO}(3) \to \mathrm{SU}(3)$. Finally, consider the action of $\mathrm{SU}(2)$ on $\mathrm{SU}(3)$ given by $g \cdot h = \rho_1(g)h(\rho_2(g))^{-1}$, which was proved to be almost free but not free by Yeroshkin [28]. In particular, the biquotient $\mathcal{O} \stackrel{\text{def}}{=} \mathrm{SU}(3)//\mathrm{SU}(2)$ is an orbifold, of dimension $\dim \mathcal{O} = \dim \mathrm{SU}(3) - \dim \mathrm{SU}(2) = 5$.

As for the example in Proposition 3.2, there is a corresponding long exact sequence

$$\cdots \to \pi_i(\mathrm{SU}(7)) \to \pi_i^{\mathrm{orb}}(\mathcal{O}) \to \pi_{i-1}(\mathrm{SU}(2)) \to \cdots$$

Since both SU(3) and SU(2) are 2-connected, the long exact sequence in homotopy implies that $\mathcal{O}$ is 2-connected as well. ∎

Remark 3.4. The examples in Propositions 3.2 and 3.3 admit metrics with non-negative sectional curvature. In addition to the examples above, we point out the existence of 4-dimensional, 3-connected orbifolds with a cohomogeneity-1 action produced by Hitchin, see [12], and by Goette–Kerin–Shankar, see Lemma 2.3 in [8]. It is an amusing fact that these orbifolds have also played an important role in finding new examples of manifolds with non-negative or positive sectional curvature. In fact, the orbifold from [12] was used by Dearricott [4] and Grove–Verdiani–Ziller [10] to produce a new 7-manifold with positive sectional curvature, while the examples in Goette–Kerin–Shankar [8] were used in the same paper to produce metrics with non-negative sectional curvature in a large class of 2-connected 7-manifolds, including all the 7-dimensional exotic spheres.

3.2. Arbitrarily highly connected bad orbifolds.

Here we present a construction of arbitrarily highly connected orbifolds. It is based on the existence of certain parallelizable lens spaces as well as a surgery construction by Milnor that allows to kill homotopy groups.

Proposition 3.5. *For any $n \geq 4$, there exist compact $\lfloor n/2 - 1 \rfloor$-connected bad orbifolds in dimension n.*

Proof. The proof is constructive. We let M be either a stably parallelizable lens space $S^{n-1}/\mathbb{Z}_p$, whose existence is guaranteed by [5], or the parallelizable product $S^{n-2}/\mathbb{Z}_p \times S^1$ of such a lens space with S^1, depending on whether n is even or odd. Take $W \stackrel{\text{def}}{=} M \times [0,1]$, a compact parallelizable n-manifold with $\partial W = M \sqcup M$. By the corollary of Theorem 2 in [19], it is possible to perform surgery and turn it into a $\lfloor n/2 - 1 \rfloor$-connected manifold W', with boundary $\partial W' = M \sqcup M$. Finally, let $\mathcal{O}$ be the orbifold obtained by gluing two copies of $X \stackrel{\text{def}}{=} \mathbb{D}^n/\mathbb{Z}_p$ to W' along their boundary. We claim that $\mathcal{O}$ is $\lfloor n/2 - 1 \rfloor$-connected as an orbifold. Since W' is simply connected, it follows that

$\pi_1^{\mathrm{orb}}(\mathcal{O}) = 1$ by Seifert–Van Kampen. Let $j\colon M \sqcup M \hookrightarrow X \sqcup X$ be the inclusion. Since for $i = 1, \ldots \lfloor n/2-1 \rfloor$ we have that $H_i^{\mathrm{orb}}(W') = H_i(W') = 0$ and that $j_*\colon H_i^{\mathrm{orb}}(M \sqcup M) \to H_i^{\mathrm{orb}}(X \sqcup X)$ is an isomorphism for $i < n-1$, cf. Proposition 3.5 in [13], Mayer–Vietoris implies that $H_i^{\mathrm{orb}}(\mathcal{O}) = 0$ for $i = 1, \ldots, \lfloor n/2-1 \rfloor$. Now Hurewicz implies that $\mathcal{O}$ is $\lfloor n/2-1 \rfloor$-connected. ∎

Remark 3.6. Killing homotopy groups via surgery becomes more delicate past degree $\lfloor n/2-1 \rfloor$ since new surgery might reintroduce homology in lower degrees, and it involves analyzing the intersection form in $H_{\lfloor n/2 \rfloor}(W)$ (when dim W is even) or the linking form in $\mathrm{Tor}(H_{\lfloor n/2 \rfloor}(W))$ (when dim W is odd), see Milnor [19] and Wall [26]. In these cases, however, one also needs vanishing conditions on ∂W such as $H_{\lfloor n/2 \rfloor}(\partial W) = H_{\lfloor n/2 \rfloor +1}(\partial W)$, which are in general not satisfied in the examples above.

4. Stratification and obstructions

The first step in proving our obstruction results is to rule out the presence of 2-torsion in sufficiently highly connected orbifolds. The following lemma generalizes a conclusion in the proof of Lemma 3.2 in [17].

Lemma 4.1. *Let $\mathcal{O}$ be a 2^{a-1}-connected n-orbifold for some $a \geq 1$. Then the local groups of $\mathcal{O}$ contained in strata of codimension $< 2^a$ have no 2-torsion.*

Proof. In any case, $\mathcal{O}$ is simply connected, and so we can realize it as an $\mathrm{SO}(n)$ quotient of its frame bundle. Suppose a local group Γ_p of $\mathcal{O}$ contains a subgroup Γ_0 of order 2 and a fixed point subspace of codimension $< 2^a$. The image of $\mathrm{SO}(n)p \times E\mathrm{SO}(n) \subset \mathrm{Fr}(\mathcal{O}) \times E\mathrm{SO}(n)$ under the projection $\mathrm{Fr}(\mathcal{O}) \times E\mathrm{SO}(n) \to B\mathcal{O}$ is a model for the classifying space of Γ_p. If we pull back the tangent bundle $T\mathcal{O}$ via the map $B\Gamma_0 \to B\Gamma_p \hookrightarrow B\mathcal{O}$, we get a bundle isomorphic to $V = E\Gamma_0 \times_{\Gamma_0} \mathbb{R}^n$.

Suppose that the non-zero element $\iota \in \Gamma_0 \subset \mathrm{SO}(n)$ has eigenvalue -1 of multiplicity $2m < 2^a$. Then we can write $2m$ "in binary" as $2m = \sum_{j=1}^{N} 2^{e_j}$, where $0 < e_1 < e_2 < \cdots e_N < a$. Furthermore, V is a bundle over $\mathbb{RP}^\infty \cong B\Gamma_0$ that decomposes as the sum of $2m$ canonical line bundles and $n-2m$ trivial line bundles. Thus the total Stiefel–Whitney class is given by

$$(1+w)^{2m} = \prod_{j=1}^{m} (1+w)^{2^{e_j}} = \prod_{j=1}^{m} (1+w^{2^{e_j}}) = 1 + w^{2^{e_1}} + R,$$

where 1 is the generator of $H^0(\mathbb{RP}^\infty; \mathbb{Z}_2)$, w is the generator of $H^1(\mathbb{RP}^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2$, and R is a multiple of $w^{2^{e_1}+1}$. This implies that $w_{2^{e_1}}(V) \neq 0$. However, since V is a pullback of $T\mathcal{O}$, and $\mathcal{O}$ is 2^{a-1}-connected, we have $w_1(V) = \cdots = w_{2^{a-1}}(V) = 0$. Moreover, since the Stiefel–Whitney classes $w_j(V)$, $j < 2^a$, are generated by $w_1(V), \ldots, w_{2^{a-1}}(V)$ via Steenrod powers (see [20], p. 94), it follows that $w_j(V) = 0$ for all $j < 2^a$. This is a contradiction with $w_{2^{e_1}}(V) \neq 0$, since by assumption $2^{e_1} < 2m < 2^a$. ∎

It turns out that the absence of 2-torsion has strong representation-theoretical implications as the following lemma shows. It for instance reflects the fact that any irreducible subgroup of $\mathrm{SO}(3)$ contains elements of order 2.

Lemma 4.2. *A real representation without trivial components of a finite group G of odd order has even dimension.*

Proof. Assume that the statement is false. Then there exists a nontrivial odd-dimensional irreducible real representation of G. Since it has odd dimension, its complexification $\rho\colon G \to U(V)$ (which we can turn to be unitary as G is finite) is irreducible as well. Let $\chi_\rho\colon G \to \mathbb{R}$ be the character of this representation: $\chi_\rho(g) = \operatorname{tr}(\rho(g))$. The fact that G has odd order implies that the map $g \mapsto g^2$ is a bijection of G. This allows us to compute the Schur-indicator of ρ:

$$\iota_\rho \overset{\text{def}}{=} \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g^2) = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) = \langle \chi_\rho, \chi_{\rho_0} \rangle = 0.$$

Here ρ_0 denotes the trivial character, and the last relation holds by the Schur orthogonality relation since ρ nontrivial. Now the claim follows by contradiction since an irreducible complex representation with trivial Schur indicator cannot be realized over the real numbers, see Theorem 4.5.6 in [27]. ∎

The conclusion of Lemma 4.2 will be essential in our proof as it allows us to skip every second dimension. This will be necessary because later on we want to apply the following proposition to turn a highly connected orbifold into a k-connected orbifold without singular strata in codimension $\geq k + 1$.

Recall that given an orbifold $\mathcal{O}$, a *strong suborbifold* is a subset $\Sigma \subset \mathcal{O}$ such that for every $p \in \Sigma$ and every chart $\phi\colon U \to V \subset \mathcal{O}$ around p (i.e., U is a Riemannian manifold with an isometric action of a finite group Γ, ϕ is Γ-invariant and induces an isometry $U/\Gamma \simeq V$), the preimage $\phi^{-1}(\Sigma \cap V)$ is a submanifold fixed set-wise by the group Γ.

Proposition 4.3. *Let $\mathcal{O}$ be a simply connected n-orbifold with a closed strong suborbifold $\Sigma \subset \mathcal{O}$ of codimension k. Then the pair $(\mathcal{O}, \mathcal{O} \setminus \Sigma)$ is $(k-1)$-connected in the orbifold topology.*

Proof. Write $\mathcal{O}$ as a quotient M/G, where $M = \operatorname{Fr}(\mathcal{O})$ is the frame bundle of $\mathcal{O}$ (cf. Section 2) and $G = \mathrm{O}(n)$. Letting $\pi\colon M \to M/G$ denote the projection, the fact that Σ is a strong suborbifold implies that $N \overset{\text{def}}{=} \pi^{-1}(\Sigma)$ is a smooth submanifold of M of codimension k, and $\mathcal{O} \setminus \Sigma$ can be globally written as a quotient $\mathcal{O} \setminus \Sigma = (M \setminus N)/G$. In particular, one has

$$\pi_i^{\text{orb}}(\mathcal{O}) = \pi_i(M \times_G EG) \quad \text{and} \quad \pi_i^{\text{orb}}(\mathcal{O} \setminus \Sigma) = \pi_i((M \setminus N) \times_G EG).$$

One has a commutative diagram

$$\begin{array}{ccccc} M \setminus N & \longrightarrow & (M \setminus N) \times_G EG & \longrightarrow & BG \\ \downarrow & & \downarrow & & \| \\ M & \longrightarrow & M \times_G EG & \longrightarrow & BG \end{array}$$

with horizontal maps being fibrations, which induces a commutative diagram with exact rows

$$\begin{array}{ccccccccccc}
\cdots \to & \pi_{i+1}(BG) & \to & \pi_i(M \setminus N) & \to & \pi_i^{\mathrm{orb}}(\mathcal{O} \setminus \Sigma) & \to & \pi_i(BG) & \to & \pi_{i-1}(M \setminus N) & \to \cdots \\
& \| & & \downarrow & & \downarrow & & \| & & \downarrow & \\
\cdots \to & \pi_{i+1}(BG) & \to & \pi_i(M) & \to & \pi_i^{\mathrm{orb}}(\mathcal{O}) & \to & \pi_i(BG) & \to & \pi_{i-1}(M) & \to \cdots
\end{array}$$

Since N has codimension k in M, by standard transversality arguments the inclusion $M \setminus N \to M$ is $(k-1)$-connected. Thus $\pi_i(M \setminus N) \to \pi_i(M)$ is an isomorphism for $i \leq k-1$. By the five lemma, $\pi_i^{\mathrm{orb}}(\mathcal{O} \setminus \Sigma) \to \pi_i^{\mathrm{orb}}(\mathcal{O})$ is an isomorphism for $i \leq k-1$ as well. This proves the result. ∎

Since the orbifold strata of $\mathcal{O}$ are special examples of strong suborbifolds, the previous proposition implies the following.

Corollary 4.4. *Let $\mathcal{O}$ be a simply connected n-orbifold. Then the pair $(\mathcal{O}, \mathcal{O} \setminus \Sigma_{\geq k})$ is $(k-1)$-connected in the orbifold topology, where $\Sigma_{\geq k}$ denotes the union of all strata of codimension at least k.*

Now we can prove our first obstruction theorem.

Proposition 4.5. *Let $\mathcal{O}$ be an m-connected orbifold for some even m. Then there are no singular strata in codimension $\leq m$.*

Proof. Let $\mathcal{O}$ be an m-connected manifold, and let a be the smallest integer such that $m < 2^a$. By Lemma 4.1, all local groups in strata of codimension $< 2^a$ have odd order. Lemma 4.2 then implies that all singular strata in codimension $\leq 2^a$ have even codimension.

Assume by contradiction that there exists a singular stratum of codimension $\leq m < 2^a$, and let $2k \geq 2$ denote the lowest codimension of a singular stratum.

Let $\Sigma_{\geq 2k+2}$ be the union of strata of codimension $\geq 2k+2$. By Corollary 4.4 and the fact that $\mathcal{O}$ is $2k$-connected, the complement $\mathcal{O}' \overset{\mathrm{def}}{=} \mathcal{O} \setminus \Sigma_{\geq 2k+2}$ is $2k$-connected as well, and it contains only singular strata of codimension $2k$, which we call Σ.

By abuse of notation, we denote by $\Sigma \subseteq \mathcal{O}'$ a connected component of Σ.

Introduce a Riemannian metric on $\mathcal{O}$, and let $D\Sigma$ denote an ϵ tubular neighbourhood of Σ, with smooth boundary $S\Sigma \overset{\mathrm{def}}{=} \partial D\Sigma$. We will need the following result.

Lemma 4.6. *$D\Sigma$ and $S\Sigma$ are strong suborbifolds of $\mathcal{O}$. Furthermore, letting $\widehat{D\Sigma}$ and $\widehat{S\Sigma}$ denote the classifying spaces of $D\Sigma$ and $S\Sigma$, respectively, the map $\widehat{S\Sigma} \to \widehat{D\Sigma}$ induced by the inclusion $S\Sigma \to D\Sigma$ is a S^{2k-1}-bundle, where $2k = \mathrm{codim}(\Sigma \subseteq \mathcal{O})$.*

Proof of Lemma 4.6. At any point $p \in \Sigma$, consider a local chart $\pi_p : \tilde{U}_p \to U_p$, where $\tilde{U}_p$ is a Riemannian manifold on which the local group Γ_p acts by fixing the (singleton) preimage $\tilde{p}$ of p. By definition of strong suborbifold, the preimage

$$\tilde{\Sigma}_p \overset{\mathrm{def}}{=} \pi_p^{-1}(\Sigma \cap U_p)$$

is a submanifold of $\tilde{U}_p$, fixed (as a set) by the local group Γ_p. In particular, $\pi_p^{-1}(D\Sigma) = D(\tilde{\Sigma}_p)$ is preserved by the action of Γ_p, as well as $\partial D(\tilde{\Sigma}_p) = \pi_p^{-1}(S\Sigma)$, and thus $D\Sigma$ and $S\Sigma$ are strong suborbifolds of $\mathcal{O}$.

Consider now the frame bundle $\phi\colon \mathrm{Fr}(\mathcal{O}) \to \mathcal{O}$, and recall that on each local chart U_p, one has $\phi^{-1}(U_p) = \tilde{U}_p \times_{\Gamma_p} \mathrm{SO}(n)$. In particular, we have

$$\begin{aligned}
\phi^{-1}(\Sigma \cap U_p) &= \tilde{\Sigma}_p \times_{\Gamma_p} \mathrm{SO}(n),\\
\phi^{-1}(D\Sigma \cap U_p) &= D(\tilde{\Sigma}_p) \times_{\Gamma_p} \mathrm{SO}(n),\\
\phi^{-1}(S\Sigma \cap U_p) &= (\partial D(\tilde{\Sigma}_p)) \times_{\Gamma_p} \mathrm{SO}(n) = \partial(D(\tilde{\Sigma}_p) \times_{\Gamma_p} \mathrm{SO}(n)).
\end{aligned}$$

Since the projection $D(\tilde{\Sigma}_p) \to \tilde{\Sigma}_p$ is a Γ_p-equivariant trivial disc bundle, with sphere bundle $\partial D(\tilde{\Sigma}_p) \to \tilde{\Sigma}_p$, there is an induced map

$$\phi^{-1}(D\Sigma \cap U_p) = D(\tilde{\Sigma}_p) \times_{\Gamma_p} \mathrm{SO}(n) \to \tilde{\Sigma}_p \times_{\Gamma_p} \mathrm{SO}(n) = \phi^{-1}(\Sigma \cap U_p)$$

which is again a disc bundle, with sphere bundle given by $\phi^{-1}(S\Sigma \cap U_p)$.

In particular, we have that $\phi^{-1}(D\Sigma)$ is a disk bundle over $\tilde{\Sigma} = \phi^{-1}(\Sigma)$, with sphere bundle $\phi^{-1}(S\Sigma)$, and therefore the inclusion $\phi^{-1}(S\Sigma) \to \phi^{-1}(D\Sigma)$ is, up to homotopy, a sphere bundle with fiber S^{2k-1}. Finally, since the inclusion $\phi^{-1}(S\Sigma) \to \phi^{-1}(D\Sigma)$ is $\mathrm{SO}(n)$-equivariant, it induces fibrations

$$\begin{array}{ccccccc}
 & & \phi^{-1}(S\Sigma) & \longrightarrow & \phi^{-1}(D\Sigma) & & \\
 & & \downarrow & & \downarrow & & \\
\widehat{S\Sigma} & = & \phi^{-1}(S\Sigma) \times_{\mathrm{SO}(n)} E\mathrm{SO}(n) & \longrightarrow & \phi^{-1}(D\Sigma) \times_{\mathrm{SO}(n)} E\mathrm{SO}(n) & = & \widehat{D\Sigma}\\
 & & \downarrow & & \downarrow & & \\
 & & B\mathrm{SO}(n) & = & B\mathrm{SO}(n) & &
\end{array}$$

where $\widehat{D\Sigma}$ and $\widehat{S\Sigma}$ are the classifying spaces of $D\Sigma$ and $S\Sigma$, respectively, and the map $\widehat{S\Sigma} \to \widehat{D\Sigma}$, being a free quotient of a sphere bundle, is again a sphere bundle. ■

We can now continue with the proof of Proposition 4.5. By Lemma 4.6, the sets $D\Sigma$ and Σ are strong suborbifolds of $\mathcal{O}$ and, letting $\widehat{S\Sigma}$ and $\widehat{D\Sigma}$ denote the classifying spaces of $S\Sigma$ and $D\Sigma$, respectively, the map $\widehat{S\Sigma} \to \widehat{D\Sigma}$ induced by the inclusion $S\Sigma \to D\Sigma$ is a S^{2k-1}-bundle. By the assumption on the structure of the singular set, $S\Sigma$ is a manifold, and in fact $\widehat{S\Sigma}(= S\Sigma)$ has finite dimensional cohomology. On the other hand, by Proposition 2.1 one has that $H^*(\widehat{D\Sigma}) = H^*_{\mathrm{orb}}(D\Sigma)$ is nonzero in infinitely many degrees.

From the Gysin sequence of $S\Sigma \to D\Sigma$, there exists an element $e \in H^{2k}_{\mathrm{orb}}(D\Sigma)$ with a long exact sequence

$$\cdots H^j_{\mathrm{orb}}(S\Sigma) \to H^{j-(2k-1)}_{\mathrm{orb}}(D\Sigma) \xrightarrow{\cup e} H^{j+1}_{\mathrm{orb}}(D\Sigma) \to H^{j+1}_{\mathrm{orb}}(S\Sigma) \to \cdots$$

By choosing an index j such that $H^{\geq j}_{\mathrm{orb}}(S\Sigma) = 0$ and $H^{j-(2k-1)}_{\mathrm{orb}}(D\Sigma) \neq 0$, it follows that $e \neq 0$. By then choosing $j = 2k - 1$, one gets

$$H^{2k-1}_{\mathrm{orb}}(S\Sigma) \to H^0_{\mathrm{orb}}(D\Sigma) \xrightarrow{\cup e} H^{2k}_{\mathrm{orb}}(D\Sigma) \to H^{2k}_{\mathrm{orb}}(S\Sigma) \to \cdots$$

i.e.,

$$0 \neq e \in \ker(j_1^* : H^{2k}_{\mathrm{orb}}(D\Sigma) \to H^{2k}_{\mathrm{orb}}(S\Sigma)).$$

Apply now the Mayer–Vietoris sequence to the partition $(D\Sigma, \mathcal{O}' \setminus \Sigma)$ of $\mathcal{O}'$:

$$0 = H^{2k}_{\mathrm{orb}}(\mathcal{O}') \to H^{2k}_{\mathrm{orb}}(D\Sigma) \oplus H^{2k}_{\mathrm{orb}}(\mathcal{O}' \setminus \Sigma) \xrightarrow{j_1^* - j_2^*} H^{2k}_{\mathrm{orb}}(S\Sigma) \to \cdots$$

But this is a contradiction, because of $0 \neq (e, 0) \in \ker(j_1^* - j_2^*)$. ■

As a corollary of Proposition 4.5, one gets:

Proof of Theorem A. In dimension 1, no bad orbifolds exist. In dimension 2, any bad orbifold is compact. Hence, we can assume that the dimension is at least 3.

By Proposition 4.5, a $2n$-connected orbifold of dimension $2n$ or $2n + 1$ does not have singular strata in codimension $\leq 2n$. But since by Lemmas 4.1 and 4.2 the codimension of singular strata must be even, this implies that there are no singular strata in either case. ■

In the compact case, we can show more by exploiting Lefschetz duality.

Proposition 4.7. *Any compact $(2n-2)$-connected $2n$-orbifold $\mathcal{O}$, $n \geq 3$, is actually a manifold.*

Proof. By Proposition 4.5, the singular strata must have codimension $\geq 2n - 1$, and by Lemmas 4.1 and 4.2, such codimension must be even. Therefore, if there are strata they must have codimension $2n$, i.e., every singular point is isolated. We denote the union of these singular points by Σ. Since Σ is finite, for sufficiently small $\epsilon > 0$ the ϵ-balls around these singular points are disjoint and every such ϵ-ball is a quotient B_ϵ/Γ_p of an ϵ-ball in a manifold chart by the action of the local group Γ_p at p. For such a small ϵ, we set $U = B_\epsilon(\Sigma)$ and $V = \mathcal{O} \setminus \Sigma$. By Proposition 4.3, $\mathcal{O} \setminus U$, which is homotopy equivalent to V, is a compact $(2n-2)$-connected manifold with boundary, and by excision and Lefschetz duality,

$$\begin{aligned} H^{\mathrm{orb}}_{2n-i}(V, V \cap U) &\cong H_{2n-i}(V, V \cap U) \cong H_{2n-i}(V, V \cap \overline{U}) \\ &\overset{\mathrm{exc}}{\cong} H_{2n-i}(\mathcal{O} \setminus U, \partial(\mathcal{O} \setminus U)) \\ &\overset{\mathrm{LD}}{\cong} H^i(\mathcal{O} \setminus U) \\ &\cong H^i(V) \cong H^i_{\mathrm{orb}}(V) \end{aligned}$$

In particular, $H_i^{\mathrm{orb}}(V, V \cap U) = 0$ for $i = 2, \ldots, 2n-1$. Via the long exact sequence in homology for the pair $(V, V \cap U)$, one gets for $i = 2, \ldots, 2n-1$ that

$$0 = H_i^{\mathrm{orb}}(V, V \cap U) \to H^{\mathrm{orb}}_{i-1}(V \cap U) \to H^{\mathrm{orb}}_{i-1}(V) = 0.$$

Hence, each component of $V \cap U$ is homotopy equivalent to a homology $(2n-1)$-sphere. On the other hand, $V \cap U \simeq S^{2n-1}/\Gamma_p$, where the local group Γ_p acts freely on S^{2n-1}. In particular,

$$H_{i-1}(S^{2n-1}/\Gamma_p) \cong H_{i-1}(V \cap U) \cong H_{i-1}^{\mathrm{orb}}(V \cap U) = 0 \quad \forall i = 2, \ldots, 2n-1.$$

Based on work by Zassenhaus [29], it was shown in [24] that there are no homology spheres in dimensions higher than 3 which are covered by a genuine sphere. Because of $(2n-1) > 3$, this implies that Γ_p is trivial and so the claim follows. ∎

A slight modification of the proof of Proposition 4.7 also shows the following statement.

Proposition 4.8. *Let $n \geq 3$ be an integer that is not a power of 2. Any compact $(2n-2)$-connected $(2n+1)$-orbifold $\mathcal{O}$ is actually a manifold.*

Proof. Let $\mathcal{O}$ be a compact $(2n-2)$-connected $(2n+1)$-orbifold. Since n is not a power of 2, $2n$ is not a power of 2 either, and there exists an integer a such that $2^{a-1}+2 \leq 2n \leq 2^a - 2$. In particular, $\mathcal{O}$ is also 2^{a-1}-connected, and by Lemma 4.1, all local groups contained in strata $\leq 2^a - 1$ have no 2-torsion. However, $2n+1 \leq 2^a - 1$, thus all local groups of $\mathcal{O}$ have no 2-torsion. By Lemma 4.2, it follows that all singular strata have even codimension and, by Proposition 4.5, the strata of codimension $\leq 2n-2$ are empty. Thus, we can assume that the singular set is a disjoint union of embedded circles. For sufficiently small $\epsilon > 0$, the ϵ-neighborhoods of these circles are disjoint, tubular neighborhoods and homotopy equivalent to $S^1 \times B^{2n}/\Gamma_p$, where Γ_p is the local group at a respective circle. For such an $\epsilon > 0$, we set $V = \mathcal{O} \setminus \Sigma$ and $U = B_\epsilon(\Sigma)$. By Proposition 4.3, the complement $\mathcal{O} \setminus B_\epsilon(\Sigma)$, which is homotopy equivalent to V, is a compact $(2n-2)$-connected manifold with boundary. The same arguments using Lefschetz duality and excision as in the proof of Proposition 4.7 give

$$H^i(V) \cong H^i_{\mathrm{orb}}(V) \cong H^{\mathrm{orb}}_{2n+1-i}(V, V \cap U).$$

In particular,

$$H_i^{\mathrm{orb}}(V, V \cap U) = 0 \quad \text{for } i = 3, \ldots, 2n.$$

Via the long exact sequence in homology for the pair $(V, V \cap U)$ one gets for $i = 3, \ldots, 2n-1$ that

$$0 = H_i^{\mathrm{orb}}(V, V \cap U) \to H_{i-1}^{\mathrm{orb}}(V \cap U) \to H_{i-1}^{\mathrm{orb}}(V) = 0,$$

i.e.,

$$H_i(V \cap U) \cong H_i^{\mathrm{orb}}(V \cap U) = 0 \quad \text{for } i = 2, \ldots, 2n-2.$$

On the other hand, we know that each component of $V \cap U$ is homotopy equivalent to $S^1 \times S^{2n-1}/\Gamma_p$, where the local group Γ_p acts freely on S^{2n-1}. The homological computation above together with the Künneth formula implies that S^{2n-1}/Γ_p is a homology $(2n-1)$-sphere. Because of $(2n-1) > 3$ this again implies that Γ_p is trivial [24]. ∎

5. Proof of Theorems D and E

The proofs of Theorems D and E amount to recollecting the results proved so far. Recall that the inequality $\kappa(n) \geq k$ is equivalent to the statement "there is a k-connected bad orbifold of dimension n", while $\kappa(n) < k$ can be restated as "every k-connected n-orbifold must be a manifold". Similar equivalences hold for κ_c in the compact case.

Proof of Theorem D. We prove each result separately.

(1) Given $n \geq 4$, the inequality $\kappa(n) \geq \lfloor n/2 \rfloor - 1$ is equivalent to Theorem C, while the inequality $\kappa(n) < 2\lfloor n/2 \rfloor$ is equivalent to Theorem A.

(2) By Proposition 3.1, $\kappa(3) \geq 1$. On the other hand, by Theorem A, $\kappa(3) < 2$. Therefore $\kappa(3) = 1$.

By Proposition 3.2, $\kappa(4) \geq 3$, while crossing the example in Proposition 3.2 with $\mathbb{R}$ shows that $\kappa(5) \geq 3$. On the other hand, Theorem A gives $\kappa(4) < 4$ and $\kappa(5) < 4$, therefore $\kappa(4) = \kappa(5) = 3$. ∎

Proof of Theorem E. We prove each result separately.

(1) Given $n \geq 4$, the inequality $\kappa_c(n) \geq \lfloor n/2 \rfloor - 1$ is equivalent to Theorem C. The inequality $\kappa_c(n) < 2\lfloor n/2 \rfloor$ can be divided in two subcases:

- $\kappa_c(2n) < 2n - 2$ follows from the first statement of Theorem B (here we need $2n \geq 6$);
- $\kappa_c(2n + 1) < 2n$ follows from Theorem A.

(2) The inequality $\kappa_c(2n + 1) < 2n - 2$ if $1 < n \neq 2^k$ is equivalent to the second statement in Theorem B.

(3) By Proposition 3.1, $\kappa_c(3) \geq 1$. On the other hand, by Theorem A, $\kappa_c(3) < 2$. Therefore $\kappa_c(3) = 1$.

By Proposition 3.2, $\kappa_c(4) \geq 3$. On the other hand, Theorem A gives $\kappa_c(4) < 4$, therefore $\kappa_c(4) = 3$. ∎

Acknowledgments. We are grateful to Alexander Lytchak for comments on a draft version of this paper, and to the referees for numerous suggestions and improvements on an earlier version of the paper. This paper was written while the first named author was visiting RUB Bochum. He would like to thank the members of the geometry and dynamics group at RUB Bochum for their hospitality.

Funding. The second named author is supported by the NSF grant DMS 1810913 and the NSF CAREER grant DMS 2042303.

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Received March 21, 2022; revised June 8, 2022. Published online August 25, 2022.

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