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Carleson perturbations for the regularity problem

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Abstract. We prove that the solvability of the regularity problem in $L^q(\partial\Omega)$ is stable under Carleson perturbations. If the perturbation is small, then the solvability is preserved in the same L^q, and if the perturbation is large, the regularity problem is solvable in L^r for some other $r \in (1,\infty)$. We extend an earlier result from Kenig and Pipher to very general unbounded domains, possibly with lower dimensional boundaries as in the theory developed by Guy David and the last two authors. To be precise, we only need the domain to have non-tangential access to its Ahlfors regular boundary, together with a notion of gradient on the boundary.

1. Introduction

1.1. History and motivation

In the last 40 years, and even more in the last 10 years, there have been impressive developments at the intersection of harmonic analysis, elliptic PDEs, and geometric measure theory. Their main goal is to understand as much as possible the interaction between geometry of (the boundary of) a domain and bounds on solutions of boundary value problems.

The first important result beyond the complex plane is due to Dahlberg in [13, 14], and it states that the Dirichlet problem is solvable in L^2 whenever the domain is Lipschitz. Since then, considerable efforts have been devoted to weakening the conditions on domains Ω and theirs boundaries, and to replacing the harmonic functions (that are solutions to $-\Delta u = 0$) by solutions of elliptic operators in the form $L = -\operatorname{div} A\nabla$. These two directions are not independent from each other, because with the help of changes of variables, we can make $\partial\Omega$ smoother, and the price to pay is rougher coefficients for the matrix A.

As far as the Dirichlet boundary value problem is concerned, mathematicians in the area have a pretty clear picture. When the operator is the Laplacian, the solvability of the Dirichlet problem in L^p for some large $p \in (1,\infty)$ is equivalent to the fact that the boundary of the domain $\partial\Omega$ is uniformly rectifiable of dimension $n-1$ (see [23,24] for the definition) and the domain has sufficient access to the boundary. A non-exhaustive list of works that helped to arrive to this conclusion includes [4,5,21,43,57]. One cannot replace

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the Laplacian by a general uniformly elliptic operator and still preserve the L^p-solvability of the Dirichlet problem (see [6, 54]). The uniformly elliptic operators $L = -\operatorname{div} A\nabla$ that preserve the L^p solvability of the Dirichlet problem fall into two classes. The first one is the t-independent operators (see for instance [42, 45, 48]), and the second one is related to Carleson measures, either via perturbations (e.g., [15,32–34], and more recently, for domains satisfying the capacity condition, [1, 8]), or via the oscillations of A (also known as Dahlberg–Kenig–Pipher operators, see [28, 51]). Many of the results have been extended to complex valued elliptic operators and elliptic systems ([26,29,30,41]). For an interested reader, who is new to this area, a nice and detailed discussion on those topics can be found in the introduction of [37].

A natural question to ask is whether those results for the Dirichlet boundary value problem have analogues for other boundary value problems, such as the Neumann problem and the regularity problem. However, those problems appear to be considerably more complicated; some results are shown in [2, 3, 31, 41, 46, 49, 50, 60], but they do not go as far as one would expect, for instance they do not go beyond Lipschitz domains.

In the recent impressive breakthrough [56], Mourgoglou and Tolsa have shown the solvability of the regularity problem in some Sobolev spaces for the Laplacian on open bounded domains satisfying the corkscrew condition and with uniformly rectifiable boundaries. The key point is the use of an alternative to the classical boundary Sobolev space (called the Hajłasz–Sobolev spaces) to bypass the lack of connectedness of the boundary of the domains. The importance of the Hajłasz–Sobolev spaces is supported by a counterexample from the authors, that shows that the result is false when one uses the classical Sobolev spaces. Mourgoglou and Tolsa complete their article by giving additional geometric conditions (that we interpret as connectedness on the boundary – like the validity of a Poincaré inequality on boundary balls) for which the classical Sobolev spaces and the Hajłasz–Sobolev spaces are the same, which ultimately give the existence of some non-Lipschitz domains where the regularity problem is solvable for the Laplacian in the classical Sobolev spaces. After the submission of our article, the two new manuscripts [25] and [55] successfully extended the solvability of the regularity problem to all the Dahlberg–Kenig–Pipher operators, hence generalizing some results from [31] and [56].

In our article, we look at the stability of the regularity problem under Carleson perturbations [50] on a ball, and we prove that we can extend it in several directions: first we consider operators which are not necessarily symmetric; second, we extend the geometric setting to uniform domains (which are domains with non-tangential access and Ahlfors regular boundaries, using as Mourgoglou and Tolsa the Hajłasz–Sobolev spaces); and third, we allow low dimensional boundaries, which were studied for the Dirichlet problem by Guy David, Zihui Zhao, Bruno Poggi, and the two last authors (see [18–20, 22, 35–37, 52, 53]). Combined with another paper under preparation ([16]), we ultimately prove the solvability of the regularity problem on the complement of a Lipschitz graph of lower dimension.

The purpose of this article is to adapt the method used by Kenig and Pipher in [50] to a more general setting, by relying on the elliptic theory developed in [22].

1.2. Introduction to the setting

The aim of this subsection is to introduce results from [19] and [20] and to give basic definitions adapted to the setting at hand.

As mentioned in the previous subsection, we understand now that we can characterize the uniformly rectifiable sets $\Gamma \subset \mathbb{R}^n$ of dimension $n-1$ via some bounds on the oscillations of the bounded harmonic functions on Ω (or the solvability of the Dirichlet problem), where $\Gamma = \partial\Omega$ and Ω has enough access to its boundary. Guy David and the two last authors launched a program to extend this characterization of uniform rectifiability to uniformly rectifiable sets of lower dimension $d \le n-2$. In this case, the domain $\Omega = \mathbb{R}^n \setminus \Gamma$ has plenty of access to its boundary (see Proposition 2.4). However, a bounded harmonic function in Ω is also a bounded harmonic function in $\mathbb{R}^n$, and thus does not "see" the boundary Γ. For that reason, the authors developed in [19] an elliptic theory that is adapted to low-dimensional boundaries by using some operators whose coefficients are elliptic and bounded with respect to a weight. Let us give a quick presentation of this theory.

Consider a domain $\Omega \subset \mathbb{R}^n$ whose boundary is d-dimensional Ahlfors regular, that is, there exist a measure σ supported on $\partial\Omega$ and $C_\sigma >$ such that

$$C_\sigma^{-1} r^d \le \sigma(\Delta(x,r)) \le C_\sigma r^d \quad \text{for } x \in \partial\Omega,\ r > 0, \tag{1.1}$$

where $\Delta(x,r) := B(x,r) \cap \partial\Omega$ is a boundary ball. If (1.1) holds for some measure σ, then it works also with $\sigma' := \mathcal{H}^d_{|\partial\Omega}$, the d-dimensional Hausdorff on $\partial\Omega$. The incoming results would also be true for bounded domains when we ask (1.1) only when $r \le \operatorname{diam}(\Omega)$, but the proof would require splitting cases (even though the two cases are fairly similar) and we do not tackle it here.

Observe that when $d < n-1$, we necessarily have that $\Omega = \mathbb{R}^n \setminus \partial\Omega$, and the domain Ω automatically has access to its boundary (see Proposition 2.4). When $d \ge n-1$, we assume that Ω satisfies the interior corkscrew point condition and the interior Harnack chain condition (see Definitions 2.1 and 2.2), which means that Ω is 1-sided NTA and hence uniform.

Consider a class of operators $\mathcal{L} = -\operatorname{div} A\nabla$ on Ω, where the coefficients are elliptic and bounded with respect to the weight $w(X) := \operatorname{dist}(X,\partial\Omega)^{d+1-n}$. To be more precise, we assume that there exists $\lambda > 0$ such that

$$\lambda|\xi|^2 w(X) \le A(X)\xi\cdot\xi \quad \text{and} \quad |A(X)\xi\cdot\zeta| \le \lambda^{-1} w(X)|\xi||\zeta|, \quad \xi,\zeta \in \mathbb{R}^n,\ X \in \Omega. \tag{1.2}$$

If we write $\mathcal{A}$ for the rescaled matrix $w^{-1}A$, then the operators that we consider are in the form $\mathcal{L} := -\operatorname{div}[w\mathcal{A}\nabla]$, where $\mathcal{A}$ satisfies the classical elliptic condition

$$\lambda|\xi|^2 \le \mathcal{A}(X)\xi\cdot\xi \quad \text{and} \quad |\mathcal{A}(X)\xi\cdot\zeta| \le \lambda^{-1}|\xi||\zeta|, \quad \xi,\zeta \in \mathbb{R}^n,\ X \in \Omega. \tag{1.3}$$

A weak solution to $\mathcal{L}u = 0$ lies in $W^{1,2}_{\mathrm{loc}}(\Omega)$ and satisfies

$$\int_\Omega \mathcal{A}\nabla u \cdot \nabla\varphi \, dm = 0 \quad \text{for } \varphi \in C_0^\infty(\Omega),$$

where $dm(Y) = w(Y)\,dY$.

The weak solutions to $\mathcal{L}u = 0$ satisfy De Giorgi–Nash–Moser estimates (interior and at the boundary). We can also construct a Green function for $\mathcal{L}$, an elliptic measure on $\partial\Omega$, and derive the comparison principle, also known as CFMS estimates. The full elliptic theory is presented in Subsections 2.3 to 2.6.

There are two fairly standard ways to construct weak solutions. The first one is using the Lax–Milgram theorem in an appropriate weighted Sobolev space (see Lemma 2.8). The second one, that will be the one used in the present article, is via the elliptic measure, which is a collection of probability measures $\{\omega^X\}_{X\in\Omega}$ such that, for every compactly supported continuous function f on $\partial\Omega$, the function defined as

$$u_f(X) := \int_{\partial\Omega} f(x)\, d\omega^X(x) \tag{1.4}$$

belongs to $C^0(\overline{\Omega})$, and is a weak solution to $\mathcal{L}u = 0$, and satisfies $u \equiv f$ on $\partial\Omega$. Note that (1.4) will be used to provide a formal solution to

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

We are ready to introduce the Dirichlet boundary value problem.

Definition 1.1 (Dirichlet problem). The Dirichlet problem is solvable in L^p if there exists $C > 0$ such that for every $f \in C_c(\partial\Omega)$, the solution u_f constructed by (1.4) satisfies

$$\|N(u_f)\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{L^p(\partial\Omega,\sigma)}, \tag{1.5}$$

where N is the non-tangential maximal function defined as

$$N(v)(x) := \sup_{\gamma(x)} |v| \tag{1.6}$$

and $\gamma(x) := \{X \in \Omega,\ |X - x| \leq 2\delta(X)\}$ is a cone with vertex at $x \in \partial\Omega$.

In Definition 1.1, the data f lies in $C_c(\partial\Omega)$ instead of $L^p(\partial\Omega,\sigma)$, so that we have a way to construct u_f *a priori* using the harmonic measure. Once we know that (1.5) holds for any $f \in C_c(\partial\Omega)$, we can construct *a posteriori* the solutions u_f for any $f \in L^p(\partial\Omega,\sigma)$ by density, and those solutions will satisfy (1.4) and (1.5).

1.3. Main results

We shall use $\delta(X)$ for $\operatorname{dist}(X,\partial\Omega)$, $w(X)$ for $\delta(X)^{d+1-n}$, $dm(X)$ for $\delta(X)^{d+1-n}dX$, and B_X for $B(X,\delta(X)/4)$. In this section, we consider two elliptic operators $\mathcal{L}_0$ and $\mathcal{L}_1$ in the form $\mathcal{L}_i = -\operatorname{div}[w\mathcal{A}_i\nabla]$, where $\mathcal{A}_i$ is real, not necessarily symmetric, and uniformly elliptic (1.3).

We define the disagreement between $\mathcal{A}_0$ and $\mathcal{A}_1$ as:

$$a(X) := \sup_{Y\in B_X} |\mathcal{E}(Y)|, \quad \mathcal{E}(Y) := \mathcal{A}_0(Y) - \mathcal{A}_1(Y). \tag{1.7}$$

Assume that $\delta(X)^{d-n}|a(X)|^2 dX$ is a Carleson measure, that is, there exists $M > 0$ such that

$$\int_{B(x,r)\cap\Omega} |a(X)|^2 \frac{dX}{\delta(X)^{n-d}} \leq M r^d \quad \text{for } x \in \partial\Omega,\ r > 0. \tag{1.8}$$

The stability of the solvability of the Dirichlet problem under Carleson perturbations was established in [34] (when Ω is a ball), [9] and [10] (when Ω is a uniform domain in with $n-1$-dimensional boundary), [1] and [8] (for domains satistying the capacity condition), [52] (when Ω is uniform with lower dimensional boundary), and [37] (1-sided NTA domains and enough basic bounds on the harmonic measure, a setting that includes all the previous ones and more). These results are as follows:

Theorem 1.2 ([10, 34, 37, 52]). *Let Ω be a uniform domain, and let $\mathcal{L}_0$ and $\mathcal{L}_1$ be two elliptic operators whose coefficients are real, non necessarily symmetric, and uniformly elliptic in the sense* (1.3). *Assume that the Dirichlet problem for the operator $\mathcal{L}_0$ is solvable in L^{p_0}* (*see Definition* 1.1). *If the disagreement* (1.7) *satisfies the Carleson measure condition* (1.8)*, then there exists $p_1 \in (1,\infty)$ such that the Dirichlet problem for $\mathcal{L}_1$ is solvable in L^{p_1}.*

Moreover, there exists $\varepsilon_0 > 0$ (*that depends on p_0 and $\mathcal{L}_0$*) *such that if the Carleson norm M in* (1.8) *is smaller then ε_0, then the Dirichlet problem for $\mathcal{L}_1$ is solvable in the same L^{p_0}.*

We know that if the Dirichlet problem is solvable in L^p, then it is solvable in L^r for all $r \in (p,\infty)$. In this sense, p_1 is *a priori* larger than p_0. The second part of the theorem says that if the disagreement is small enough, then we can preserve the solvability of the Dirichlet problem in the same L^{p_0} space.

Our main theorem addresses this type of perturbations for the regularity problem. The only forerunner in this case is [50]. However, contrary to [50], here we treat domains with rough and/or low dimensional boundaries, and operators whose coefficients are not necessarily symmetric. Let us state exactly what we prove.

Theorem 1.3. *Let Ω be a uniform domain* (*see Definition* 2.3)*, and let $\mathcal{L}_0$ and $\mathcal{L}_1$ be two elliptic operators whose coefficients are real, non necessarily symmetric, and uniformly elliptic in the sense* (1.3). *Assume that the Dirichlet problem for the adjoint operator $\mathcal{L}_1^*$ is solvable in $L^{q'}$* (*see Definition* 1.1).

If the disagreement (1.7) *satisfies the Carleson measure condition* (1.8)*, then for any $f \in C_c(\partial\Omega)$, the two solutions $u_{0,f}$ and $u_{1,f}$ to $\mathcal{L}_0 u_{0,f} = 0$ and $\mathcal{L}_1 u_{1,f} = 0$ constructed by* (1.4) *satisfy*

$$\|\widetilde{N}(\nabla u_{1,f})\|_{L^q(\partial\Omega,\sigma)} \leq CM \|\widetilde{N}(\nabla u_{0,f})\|_{L^q(\partial\Omega,\sigma)}, \tag{1.9}$$

where $\widetilde{N}$ is the averaged non-tangential maximal function defined as

$$\widetilde{N}(v)(x) := \sup_{X\in\gamma(x)} \left(\fint_{B_X} |v|^2 \, dX \right)^{1/2}. \tag{1.10}$$

The two quantities in (1.9) *can be infinite, but the left-hand side has to be finite as soon as the right-hand side is finite. The constant $C > 0$ depends only on n, the uniform constants of Ω, the ellipticity constant λ, the parameter q, and the constant in* (1.5).

The above theorem assumes the solvability of the Dirichlet problem for $\mathcal{L}_1^*$. It is not very surprising, because it can be seen as a partial converse of Theorem 1.5 below. That is, if the Dirichlet problem for $\mathcal{L}_1^*$ is solvable in $L^{q'}$, and if $\mathcal{L}_1$ satisfies extra conditions, then the regularity problem for $\mathcal{L}_1$ is solvable in L^q.

We wanted to state the above theorem independently of the notion of regularity problem. We remark that it can also be used for the Neumann problem, although not directly, and we leave this question for a future article.

We turn now to the definition of the regularity problem, which is long overdue.

1.4. The regularity problem

Informally speaking, the regularity problem in L^q reduces to a bound on $\|\widetilde{N}(\nabla u_f)\|_{L^q(\partial\Omega)}$ in terms of the tangential derivatives of u_f on the boundary (i.e., in terms of the derivatives of f). If $\partial\Omega = \mathbb{R}^d$ is a plane, then we want to show that

$$\|\widetilde{N}(\nabla u_f)\|_{L^q(\mathbb{R}^d)} \leq C\|\nabla_{\mathbb{R}^d} f\|_{L^q(\mathbb{R}^d)}, \tag{1.11}$$

where here $\nabla_{\mathbb{R}^d}$ is the classical gradient in $\mathbb{R}^d$. Similarly, on a more complicated boundary, the regularity problem would correspond to the estimate

$$\|\widetilde{N}(\nabla u_f)\|_{L^q(\partial\Omega,\sigma)} \leq C\|\nabla_{\partial\Omega,q} f\|_{L^q(\partial\Omega,\sigma)}, \tag{1.12}$$

where $\nabla_{\partial\Omega,q}$ is a notion of (tangential) gradient that may depend on $\partial\Omega$ and q. If $\partial\Omega = \Gamma$ is the graph of a Lipschitz function, then we can extend the notion of gradient from $\mathbb{R}^d$ to Γ, for instance by finding a bi-Lipschitz map $\rho\colon \Gamma \to \mathbb{R}^d$ and define

$$\nabla_\Gamma f(x) := \nabla_{\mathbb{R}^d}[f \circ \rho] \circ \rho^{-1}(x) \quad \text{for almost every } x \in \partial\Omega.$$

On the other hand, let us take $\partial\Omega := P_1 \cup P_2$ to be the union of two low dimensional planes that do not intersect (and then $\Omega = \mathbb{R}^n \setminus \partial\Omega$). We have a well-defined gradient $\nabla_{P_1\cup P_2}$ on $\partial\Omega$ (because we have a gradient on planes), and we also have elliptic operators and solutions thanks to the elliptic theory from [19]. However, since $P_1 \cup P_2$ is not connected, $\nabla_{P_1\cup P_2} f = 0$ does not necessarily imply that f is constant on $P_1 \cup P_2$, and thus we can never have

$$\|\widetilde{N}(\nabla u_f)\|_{L^q(P_1\cup P_2)} \leq C\|\nabla_{P_1\cup P_2} f\|_{L^q(P_1\cup P_2)}. \tag{1.13}$$

Recall that Theorem 1.3 only requires Ω to be uniform, and so nothing can stop Ω from being very rough (even purely unrectifiable) and not connected. So if we do not want to lose too much from Theorem 1.3, we would prefer a notion of gradient that exists for any set, and for any function obtained by restricting the ones from $C_0^\infty(\mathbb{R}^n)$ to $\partial\Omega$.

Fortunately for us, Lipschitz functions exist on every metric space, and a notion of gradient was derived from it. For a Borel function $f\colon \partial\Omega \to \mathbb{R}$, we say that a non-negative Borel function g is *a generalized gradient of* f if

$$|f(x) - f(y)| \leq |x-y|(g(x)+g(y)) \quad \text{for } \sigma\text{-a.e. } x, y \in \partial\Omega. \tag{1.14}$$

The collection of all generalized gradients is denoted by $D(f)$. Then for any $p \geq 1$, the space $\dot W^{1,p}(\partial\Omega,\sigma)$ is the space of Borel functions that have a generalized gradient in L^p, and we equip it with the semi-norm

$$\|f\|_{\dot W^{1,p}(\partial\Omega,\sigma)} := \inf_{g\in D(f)} \|g\|_{L^p(\partial\Omega,\sigma)}. \tag{1.15}$$

Hajłasz introduced those spaces in [39], that is why they are often called Hajłasz–Sobolev spaces. We refer an interested reader to [40] for more information on Sobolev spaces on general metric spaces.

It should not a big surprise to bring up Hajłasz–Sobolev spaces here, since they have already been used to study boundary values problems in [44] and recently in [56]. In the latter article, the authors proved that in bounded domains with $(n-1)$-dimensional uniformly rectifiable boundaries, the solvability in L^p of the Dirichlet problem for the Laplacian is equivalent to the solvability of the regularity problem, defined below with the Hajłasz–Sobolev space.

Note that any function f which lies in a Hajłasz–Sobolev space supports a Poincaré inequality, that is, for any $p \in [1,\infty]$ and for any boundary ball $\Delta = \Delta(x,r)$, we have

$$\|f - f_\Delta\|_{L^p(\Delta,\sigma)} \leq C_p\, r \inf_{g\in D(f)} \|g\|_{L^p(\Delta,\sigma)}, \tag{1.16}$$

where

$$f_\Delta = \fint_\Delta f\, d\sigma.$$

The proof of this fact is immediate. Indeed, if $g \in D(f)$, we have

$$\begin{aligned}\fint_\Delta |f - f_\Delta|^p\, d\sigma &\leq \fint_\Delta \fint_\Delta |f(x) - f(y)|^p\, d\sigma(x)\, d\sigma(y)\\ &\leq \fint_\Delta \fint_\Delta |x-y|^p (g(x)+g(y))^p\, d\sigma(x)\, d\sigma(y)\\ &\leq C_p \fint_\Delta g(x)^p \fint_\Delta |x-y|^p\, d\sigma(y)\, d\sigma(x) \leq C_p r^p \fint_\Delta g(x)^p\, d\sigma(x),\end{aligned}$$

where we used the symmetry of the roles of x and y between the first and the second line.

Definition 1.4 (Regularity problem). The regularity problem is solvable in L^p if there exists $C > 0$ such that for every compactly supported Lipschitz function f, the solution u_f constructed by (1.4) satisfies

$$\|\widetilde N(\nabla u_f)\|_{L^p(\partial\Omega,\sigma)} \leq C \|f\|_{\dot W^{1,p}(\partial\Omega,\sigma)}, \tag{1.17}$$

where $\dot W^{1,p}(\partial\Omega,\sigma)$ is the Hajłasz–Sobolev space defined above, and the maximal function $\widetilde N$ is defined in (1.10).

The regularity problem targets the question whether the oscillations of u can be controlled by the oscillations of its trace, in a similar way that in the Dirichlet problem, u is controlled by its trace. We replace N by $\widetilde N$ because, contrary to u_f which lies in $L^\infty_{\rm loc}(\Omega)$ thanks to the Moser estimate (2.7), we can only be certain of the fact that ∇u_f lies in $L^2_{\rm loc}(\Omega)$.

It would be reassuring to know that the Hajłasz–Sobolev spaces are the classical Sobolev spaces in the basic settings, which is not obvious at first glance. We have indeed:

$$\text{when } \partial\Omega \text{ is a plane or the graph of a Lipschitz function, } \dot{W}^{1,p}(\partial\Omega,\sigma) \text{ is the classical homogeneous Sobolev space.} \tag{1.18}$$

The proof of the equivalence[1] is a consequence of Lemma 6.5 in [56].

Note also that the Hajłasz–Sobolev spaces contain the compactly supported Lipschitz functions, so they cannot be too small. Besides, we have the following duality result between regularity and Dirichlet problems, proven in the appendix.

Theorem 1.5. *Let Ω be a uniform domain, and let $\mathcal{L} = -\operatorname{div}[w\mathcal{A}\nabla]$ be an elliptic operator whose coefficients satisfy* (1.3).

If the regularity problem (defined with the Hajłasz–Sobolev spaces) for $\mathcal{L}$ is solvable in L^q for some $q \in (1,\infty)$, then the Dirichlet problem for the adjoint $\mathcal{L}^*$ is solvable in $L^{q'}$, where $1/q + 1/q' = 1$.

The combination of Theorems 1.2, 1.3, and 1.5 gives the following corollary.

Corollary 1.6. *Let Ω be a uniform domain, and let $\mathcal{L}_0$ and $\mathcal{L}_1$ be two elliptic operators whose coefficients are uniformly elliptic in the sense of* (1.3). *Assume that the disagreement* (1.7) *satisfies the Carleson measure condition* (1.8). *Then the following holds.*

(1) *If there exists $q_0 \in (1,\infty)$ such that the regularity problem for the operator $\mathcal{L}_0$ is solvable in L^q for any $q \in (1,q_0]$, then there exists $q_1 \in (1,\infty)$ such that the regularity problem for $\mathcal{L}_1$ is solvable in L^q for any $q \in (1,q_1]$.*

(2) *If there exists $q_0 \in (1,\infty)$ such that the regularity problem for the operator $\mathcal{L}_0$ is solvable in L^{q_0}, and if the Carleson norm M in* (1.8) *is smaller than ε_0 (depending only on q_0 and $\mathcal{L}_0$), then the regularity problem for $\mathcal{L}_1$ is solvable in the same L^{q_0}.*

Remark 1.7. When the boundary is smooth (flat or Lipschitz), then the Hajłasz–Sobolev spaces and the regular Sobolev spaces are the same (1.18), and the solvability of the regularity problem for an operator $\mathcal{L}$ in L^q immediately implies the solvability of the regularity problem in L^p, $p \in (1,q]$. The proof of this result is the same as that of Theorem 5.2 in [49] (see also [27]), which treats the case of bounded domains with smooth boundary.

However, the proof cannot be directly adapted to our context, because the proof relies on the properties of Hardy–Sobolev spaces on the boundary, which are not constructed yet in our setting that uses the generalized gradient.

Proof of Corollary 1.6. Let $f \in C_c(\mathbb{R}^n)$ and let $u_{0,f}$ and $u_{1,f}$ be the two solutions to $\mathcal{L}_0 u_{0,f} = 0$ and $\mathcal{L}_1 u_{1,f} = 0$ constructed by (1.4). Theorem 1.5 shows that the Dirichlet problem for $\mathcal{L}_0^*$ is solvable in $L^{q_0'}$, and then Theorem 1.2 implies that the Dirichlet problem for $\mathcal{L}_1^*$ is solvable in $L^{q_1'}$ for some $q_1' \in [q_0',\infty)$. Theorem 1.3 further provides the estimate

$$\|\widetilde{N}(\nabla u_{1,f})\|_{L^{q_1}(\partial\Omega,\sigma)} \leq C\|\widetilde{N}(\nabla u_{0,f})\|_{L^{q_1}(\partial\Omega,\sigma)}, \tag{1.19}$$

where $1/q_1 + 1/q_1' = 1$.

[1] The equivalence is established in a much more general situation, involving uniformly rectifiable sets and Poincaré inequalities.

Since $q_1' \geq q_0'$, we have $q_0 \leq q_1$ and hence the regularity problem for $\mathcal{L}_0$ is solvable in L^{q_1}, that is,

$$\|\widetilde{N}(\nabla u_{0,f})\|_{L^{q_1}(\partial\Omega,\sigma)} \leq C\|\nabla f\|_{L^{q_1}(\partial\Omega,\sigma)}. \tag{1.20}$$

We combine (1.19) and (1.20) to get that $\|\widetilde{N}(\nabla u_{1,f})\|_{q_1} \leq C\|\nabla f\|_{q_1}$, which concludes the first part of the corollary.

When M is small, Theorem 1.2 says that we can take $q_1' = q_0'$ (hence $q_1 = q_0$) in the above reasoning. The second part of the corollary follows. ■

1.5. Conditions on the operator implying the solvability in L^p of the regularity problem

1.5.1. Combination with [16]. In another article [16], we use the perturbation result from the present article to show that the regularity problem is solvable in L^2 for a certain class of elliptic operators on $\mathbb{R}^n \setminus \mathbb{R}^d$, $d < n-1$.

Theorem 1.8 (Theorem 1.1 in [16]). *Let $d < n-1$ and $\Omega = \mathbb{R}^n \setminus \mathbb{R}^d := \{(x,t) \in \mathbb{R}^d \times \mathbb{R}^{n-d},\ t \neq 0\}$. Assume that the operator $\mathcal{L} = -\operatorname{div}[|t|^{d+1-n}\mathcal{A}\nabla]$ satisfies (1.3) and is such that the matrix $\mathcal{A}$ can be written*

$$\mathcal{A} = \mathcal{B} + \mathcal{C}, \quad \text{with} \quad \mathcal{B} = \begin{pmatrix} \mathcal{B}_{||} & 0 \\ 0 & b\,\mathrm{Id}_{n-d} \end{pmatrix}, \tag{1.21}$$

and

$$\int_{B(x,r)} \int_{|t|<r} \big(|t||\nabla \mathcal{B}| + |\mathcal{C}|\big)^2 \frac{dt\,dx}{|t|^{n-d}} \leq Mr^d \quad \text{for } x \in \mathbb{R}^d,\ r > 0. \tag{1.22}$$

There exists $\varepsilon_0 > 0$ depending only on n, d, and the ellipticity constant λ such that if the constant M in (1.22) is smaller than ε_0, then the regularity problem for $\mathcal{L}$ is solvable in L^2.

Thanks to (1.18), the solvability of the regularity problem above means that for any compactly supported Lipschitz function f on $\partial\Omega$, the solution u_f constructed by (1.4) satisfies

$$\|\widetilde{N}(\nabla u_f)\|_{L^2(\mathbb{R}^d)} \leq C\|\nabla_t f\|_{L^2(\mathbb{R}^d)},$$

where the constant $C > 0$ depends only on d, n, and λ, and where ∇_t is the (tangential) gradient on $\mathbb{R}^d$.

If the matrix $\mathcal{C}$ is not included, Theorem 1.8 can be seen as the higher co-dimensional analogue of Theorem 5.1 in [31]. The perturbation theory that we developed here allows us to add such term $\mathcal{C}$ to the coefficients of the operator.

The objective of the project that includes both the present article and [16] is to prove the solvability of the regularity problem when Ω and $\mathcal{L}$ are like those in [18], that is, when Ω is the complement of a Lipschitz graph of low dimension.

In domains with codimension 1 boundary, such results are classically obtained by using a change of variables that turns the Lipschitz domain into $\mathbb{R}^n_+$ (if the domain is unbounded) or a ball (if the domain is bounded). Such gain in regularity on the boundary is paid for by less regularity on the coefficients of the elliptic operators $\mathcal{L}$. In the case of Lipschitz domains with codimension 1 boundary, the change of variable used to flatten Lipschitz boundaries turns smooth operators like the Laplacian to operators in the form $-\operatorname{div} \mathcal{B}\nabla$ with $\mathcal{B}$ like in (1.22), see [51]. That is, the perturbation theory is not needed in this case.

However, the change of variable from [51] is not suitable to flatten Lipschitz graphs of low dimension, and another change of variable is needed, like the one in [18]. This second change of variable is almost isometric in the non-tangential direction, and the conjugate elliptic operator will have coefficients in the form (1.21) and (1.22). Thus we can deduce from Theorem 1.8 the solvability of the regularity problem on the complement of a small Lipschitz graph of low dimension.

Corollary 1.9. *Let $d < n-1$. Let Γ be the graph of a Lipschitz function φ. Consider the domain $\Omega := \mathbb{R}^n \setminus \Gamma$ and the operator $L_\alpha = -\operatorname{div} D_\alpha^{d+1-n}\nabla$, where D_α is the regularized distance*

$$D_\alpha(X) := \Big(\int_\Gamma |X-y|^{-d-\alpha} \, d\mathcal{H}^d(y) \Big)^{-1/\alpha},$$

$\mathcal{H}^d$ is the d-dimensional Hausdorff measure, and $\alpha > 0$.

There exists $\varepsilon_0 > 0$ that depends only on α and n such that if the Lipschitz constant $\|\nabla\varphi\|_\infty$ is smaller than ε_0, then the regularity problem for $\mathcal{L}$ is solvable in L^2, which means that, for any compactly supported Lipschitz function f on $\partial\Omega$, the solution u_f constructed by (1.4) satisfies

$$\|\widetilde{N}(\nabla u_f)\|_{L^2(\Gamma)} \leq C \|\nabla_t f\|_{L^2(\Gamma)},$$

where the constant $C > 0$ depends only on α and n, and where ∇_t is the (tangential) gradient on Γ.

Proof. We use the bi-Lipschitz change of variable ρ constructed in [18]. So the solvability of the regularity problem for L_α (defined on $\mathbb{R}^n \setminus \Gamma$) in L^q is equivalent to the solvability regularity problem in L^q for an operator $\mathcal{L}_\rho = -\operatorname{div}[|t|^{d+1-n}\mathcal{A}_\rho\nabla]$ (defined on $\mathbb{R}^n \setminus \mathbb{R}^d$), where $\mathcal{A}_\rho$ satisfies

$$\mathcal{A}_\rho = \begin{pmatrix} \mathcal{B}_{||} & 0 \\ 0 & b\,\mathrm{Id}_{n-d} \end{pmatrix} + \mathcal{C} = \mathcal{B} + \mathcal{C}$$

and

$$\int_{B(x,r)} \int_{|t|<r} \big(|t||\nabla\mathcal{B}| + |\mathcal{C}|\big)^2 \frac{dt\,dx}{|t|^{n-d}} \leq C_\alpha \|\nabla\varphi\|_\infty r^d \quad \text{for } x \in \mathbb{R}^d,\ r > 0. \tag{1.23}$$

The corollary follows now from Theorem 1.8. ■

1.5.2. Combination with [56]. In uniform domains, Corollary 1.6 and part (a) of Corollary 1.7 in [56][2] give:

[2]The results in [56] are stronger because they do not require the existence of Harnack chains inside the domain, like we do.

Theorem 1.10 ([56]). *If Ω be a bounded uniform domain with a uniformly rectifiable boundary, then the regularity problem (defined with the Hajłasz–Sobolev spaces) for the Laplacian is solvable in L^q for any $q \in (1, q_0]$, where $q_0 > 1$ sufficiently small.*

Thanks to Corollary 1.6, we know that the above result can be extended to perturbations of the Laplacian as well.

1.6. Plan of the article

A brief summary of this article is as follows. In Section 2, we state the precise statement of the assumptions on our domain, and we recall the elliptic theory that shall be needed for the proof of Theorem 1.3. In particular, we construct the elliptic measure and we link it to the solvability of the Dirichlet problem. In Section 3, we construct the elliptic measure and Green function with pole at infinity, which are very convenient tools to deal with unbounded domains. Section 4 is devoted to the proof of Theorem 1.3, where we shall follow the strategy of [50] to our more general setting.

In the rest of the article, we shall use $A \lesssim B$ when there exists a constant C such that $A \leq C\,B$, where the dependence of $C > 0$ into the parameters will be either obvious from context or recalled. We shall also write $A \approx B$ when $A \lesssim B$ and $B \lesssim A$.

2. Our assumptions and the elliptic theory

2.1. Our assumptions on the domain

In addition to the fact that the boundary $\partial\Omega$ is d-dimensional Ahlfors regular, see (1.1), we assume two extra hypotheses on the domain: the interior corkscrew point condition (quantitative openness) and the interior Harnack chain condition (quantitative connectedness).

Definition 2.1 (Corkscrew point condition). We say that Ω satisfies the corkscrew point condition if there exists $c_0 > 0$ such that, for any $x \in \partial\Omega$ and any $r > 0$, there exists $X \in B(x, r) \cap \Omega$ such that $B(X, c_0 r) \subset \Omega$.

Such a point X is called the corkscrew point associated to x and r. Sometimes, the pair (x, r) will be given by a boundary ball Δ, that is, we say that X is a corkscrew point associated to a boundary ball Δ if X is a corkscrew point associated to x and r where $\Delta = \Delta(x, r) := B(x, r) \cap \partial\Omega$.

Definition 2.2 (Harnack chain condition). We say that Ω satisfies the Harnack chain condition if the following holds. For any $K > 1$, there exists an integer N_K such that for any $X, Y \in \Omega$ that satisfies $|X - Y| \leq K \min\{\delta(X), \delta(Y)\}$, there exist at most N_K balls $B_1, \dots B_{N_K}$ such that

(1) $X \in B_1$ and $Y \in B_{N_K}$,

(2) $2B_i \in \Omega$ for $1 \leq i \leq N_K$,

(3) $B_i \cap B_{i+1} \neq \emptyset$ for $1 \leq i \leq N_K - 1$.

Definition 2.3 (Uniform domain). We say that Ω is uniform if Ω satisfies the corkscrew point condition and the Harnack chain condition, and if $\partial\Omega$ is d-Ahlfors regular.

The constants in (1.1), Definition 2.1, and Definition 2.2 are referred as "the uniform constants of Ω".

Lemmas 11.7 and 2.1 of [19] show that:

Proposition 2.4. *Let $d < n-1$, and let $\Omega = \mathbb{R}^n \setminus \Gamma$. If $\Gamma = \partial\Omega$ is d-dimensional Ahlfors regular, then Ω is uniform.*

2.2. Quantitative version of absolutely continuity

In this article, we focus on doubling measures on $\partial\Omega$, which are non-negative Borel measures μ that satisfy

$$\mu(2\Delta) \leq C_\mu \mu(\Delta) \quad \text{for any boundary ball } \Delta \subset \partial\Omega. \tag{2.1}$$

Two measures that will be considered in this paper are the Ahlfors regular (hence doubling) measure σ and the elliptic measure with pole at infinity ω that will be constructed in Section 3 and is doubling according to Lemma 3.4. These two measures will be comparable, more precisely, A^∞-absolutely continuous with each other; the definition is given below.

Definition 2.5 (A_∞-absolute continuity). Let ν, μ be two doubling measures on $\partial\Omega$. We say that μ is A_∞-absolute continuous with respect to ν (or $\mu \in A_\infty(\nu)$, in short) if for each $\varepsilon > 0$, there exists $\xi = \xi(\varepsilon) > 0$ such that for every surface ball Δ, and every Borel set $E \subset \Delta$, we have that

$$\frac{\nu(E)}{\nu(\Delta)} < \xi \implies \frac{\mu(E)}{\mu(\Delta)} < \varepsilon. \tag{2.2}$$

The A^∞-absolute continuity is related to the following stronger property.

Definition 2.6 (The reverse Hölder class RH_p). Let ν and μ be two doubling measures on $\partial\Omega$ that are absolutely continuous with respect to each other. We say that $\nu \in \mathrm{RH}_p(\mu)$ if there exists a constant C_p such that for every surface ball Δ, the Radon–Nikodym derivative $k = d\nu/d\mu$ satisfies

$$\Big(\fint_\Delta |k|^p \, d\mu \Big)^{1/p} \leq C_p \fint_\Delta k \, d\mu = C_p \, \frac{\nu(\Delta)}{\mu(\Delta)}. \tag{2.3}$$

The A_∞ and RH_p classes satisfy several important properties, which are recalled here.

Theorem 2.7 (Properties of A_∞ measures; Theorem 1.4.13 of [47], [58]). *Let μ and ν be two doubling measures on $\partial\Omega$, and let Δ be a surface ball. The following statements hold.*

(i) *If $\mu \in A_\infty(\nu)$, then ν is absolutely continuous with respect to ν on Δ.*

(ii) *The class A_∞ is an equivalence relationship, that is, $\mu \in A_\infty(\nu)$ implies $\nu \in A_\infty(\mu)$.*

(iii) *We have that $\mu \in A_\infty(\nu)$ if and only if there exist a constant $C > 0$ and $\theta > 0$ such that for each surface ball Δ and each Borel set $E \subset \Delta$, we have that*

$$\frac{\mu(E)}{\mu(\Delta)} \leq C \Big(\frac{\nu(E)}{\nu(\Delta)} \Big)^\theta.$$

(iv) *$\mu \in A_\infty(\nu)$ if and only if we can find $p > 1$ such that $\mu \in \mathrm{RH}_p(\nu)$, i.e.,*

$$A_\infty(\nu) = \bigcup_{p>1} \mathrm{RH}_p(\nu).$$

(v) *$\mu \in \mathrm{RH}_p(\nu)$ if and only if the uncentered Hardy–Littlewood maximal function with the measure μ, defined as*

$$(\mathcal{M}_\mu f)(x) := \sup_{\Delta \ni x} \fint_{\Delta'} |f| \, d\mu,$$

satisfies the estimate

$$\|\mathcal{M}_\mu f\|_{L^{p'}(\partial\Omega,\nu)} \leq C \|f\|_{L^{p'}(\partial\Omega,\nu)} \quad \text{for } f \in L^{p'}(\partial\Omega,\nu),$$

where p' is the Hölder conjugate of p, that is, $1/p + 1/p' = 1$.

2.3. The basic elliptic theory

To lighten the notations, in the rest of the article, we shall write $\delta(X)$ for $\operatorname{dist}(X, \partial\Omega)$, $w(X)$ for $\delta(X)^{d+1-n}$, $dm(X)$ for $w(X)dX$, and B_X for $B(X, \delta(X)/4)$. The measure m is doubling, as shown in Lemma 2.3 of [19], but more importantly, m satisfies some boundary and interior Poincaré inequalities (see Lemma 4.2 in [19] when $d < n-1$, and Theorem 7.1 in [20] for the statement in any dimension).

The correct function spaces to study our elliptic equations are the weighted Sobolev space,

$$W := \{u \in L^1_{\mathrm{loc}}(\Omega) : \|u\|_W := \|\nabla u\|_{L^2(\Omega,dm)} < +\infty\}, \tag{2.4}$$

and the space of traces

$$H := \left\{ f : \|f\|_H := \int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(x) - f(y)|}{|x-y|^{d+1}} \, d\sigma(x) d\sigma(y) < \infty \right\}. \tag{2.5}$$

For these spaces, we can construct a bounded trace operator Tr: $W \to H$. By trace operator we mean that $\operatorname{Tr}(u) = u$ whenever $u \in W \cap C^0(\overline{\Omega})$, which is uniquely defined by the density of $W \cap C^0(\overline{\Omega})$ in W (see Lemma 9.19 in [20]). We shall also need

$$W_0 := \{u \in W, \ \operatorname{Tr}(u) = 0\},$$

which is also the completion of $C_0^\infty(\Omega)$ with the norm $\|.\|_W$, and the local versions of W defined for any open set $E \subset \mathbb{R}^n$ as

$$W_r(E) := \{u \in L^1_{\mathrm{loc}}(E \cap \Omega), \varphi u \in W \text{ for all } \varphi \in C_0^\infty(E)\}.$$

Note that E is not necessarily a subset of Ω, and that $W_r(\mathbb{R}^n) = W^{1,2}_{\mathrm{loc}}(\overline{\Omega}, dm) \subsetneq W$.

We are now ready to talk about weak solutions to $\mathcal{L}u = 0$. Recall that $\mathcal{L} = -\operatorname{div}(w\mathcal{A}\nabla)$ for a matrix $\mathcal{A}$ that satisfies (1.3). Let $F \subset \Omega$ be an open set. We say that u is a weak solution of $\mathcal{L}u = f$ in F if $u \in W_r(F)$ and for any $\varphi \in C_0^\infty(F)$,

$$\int_\Omega \mathcal{A}\nabla u \cdot \nabla\varphi \, dm = 0.$$

We can always construct a unique weak solution via the Lax–Milgram theorem.

Lemma 2.8 (Lemma 9.3 of [19]). *For any* $h \in W^{-1} := (W_0)^*$ *and* $f \in H$*, there exists a unique* $u \in W$ *such that* $\mathrm{Tr}(u) = f$ *and*

$$\int_\Omega \mathcal{A}\nabla u \cdot \nabla\varphi \, dm = \langle h, \varphi\rangle_{W^{-1},W_0} \qquad \text{for } \varphi \in W_0.$$

Moreover, there exists $C > 0$*, independent of* h *and* f*, such that*

$$\|u\|_W \leq C(\|f\|_H + \|h\|_{W^{-1}}).$$

Let us now recall several classical results (Caccioppoli's inequality, Moser's estimate, and Harnack's inequality inside the domain) that will be useful later. Since they are interior results, that is, where the weight w has no degeneracy, they are direct consequences of the classical theory. The precise statements can be found in [19] and [20].

Lemma 2.9 (Interior Caccioppoli inequality and Moser estimate). *Let* B *be a ball of radius* $r > 0$ *such that* $2B \subset \Omega$*, and let* $u \in W_r(2B)$ *be a weak solution to* $\mathcal{L}u = 0$ *in* $2B$*. Then*

$$\int_B |\nabla u|^2 \, dm \leq C r^{-2} \int_{2B} u^2 \, dm, \tag{2.6}$$

and

$$\sup_B |u| \leq C \fint_{2B} |u| \, dm, \tag{2.7}$$

where $C > 0$ *depends on the dimensions* d *and* n*, and on the elliptic constant* λ*.*

The interior Caccioppoli inequality (and the Moser estimate) holds if we replace $2B$ by αB in (2.6), and the constant C will then depend on $\alpha > 1$ too. Note also that we can very well replace a ball by a (Whitney) cube, that is, a cube $I \in \mathbb{R}^n$ for which $2I \subset \Omega$, and that we can replace dm by dX, since the weight w is non-degenerated on B.

Corollary 2.10. *Let* B *be a ball of radius* $r > 0$ *such that* $4B \subset \Omega$ *and let* $u \in W_r(2B)$ *be a weak solution to* $\mathcal{L}u = 0$ *in* $2B$*. Then*

$$\left(\fint_B |\nabla u|^2 \, dX\right)^{1/2} \leq C \fint_{2B} |\nabla u| \, dX,$$

where $C > 0$ *depends on the dimensions* d *and* n*, and on the elliptic constant* λ*.*

Proof. First, observe that $w(X) \approx w(Y)$ for $X, Y \in 2B$, that is, $\fint_E v \, dm \approx \fint_E v \, dX$ whenever v is nonnegative and $E \subset 2B$. Therefore, if $u_{2B} = \fint_{2B} u \, dX$, then

$$\begin{aligned}\left(\fint_B |\nabla u|^2 \, dX\right)^{1/2} &\approx \left(\fint_B |\nabla(u - u_{2B})|^2 \, dX\right)^{1/2} \lesssim \frac{1}{r}\left(\fint_{\frac{3}{2}B} |u - u_{2B}|^2 \, dm\right)^{1/2} \\ &\lesssim \frac{1}{r} \fint_{2B} |u - u_{2B}| \, dm \lesssim \frac{1}{r} \fint_{2B} |u - u_{2B}| \, dX,\end{aligned}$$

where we invoke successively (2.6) and (2.7) and use the fact that we can replace $2B$ by αB in those inequalities. The lemma follows then from the L^1-Poincaré inequality. ∎

Lemma 2.11 (Harnack inequality). *Let B be a ball such that $2B \subset \Omega$, and let $u \in W_r(2B)$ be a non-negative solution to $\mathcal{L}u = 0$ in $2B$. Then*

$$\sup_B u \leq C \inf_B u,$$

where $C > 0$ depends on the dimensions d and n, and on the elliptic constant λ.

We also have a version of Lemma 2.9 for a ball B centered at the boundary, and in this case, the solution $u \in W_r(2B)$ has to satisfy $\operatorname{Tr} u = 0$ on $2B \cap \partial\Omega$. In order to keep our article short, we will not present the result explicitly, but it is worthwhile to mention the Hölder continuity of solutions at the boundary.

Lemma 2.12 (Hölder continuity at the boundary; Lemmas 11.32 and 15.14 in [20]). *Let $B := B(x, r)$ be a ball with a center $x \in \partial\Omega$ and radius $r > 0$, and let X be a corkscrew point associated to $(x, r/2)$. For any non-negative solution $u \in W_r(B)$ to $\mathcal{L}u = 0$ in $B \cap \Omega$ such that $\operatorname{Tr} u = 0$ on $B \cap \partial\Omega$, there exists $\alpha > 0$ such that for $0 < s < r$,*

$$\sup_{B(x,s)} u \leq C \left(\frac{s}{r}\right)^{\alpha} u(X),$$

where the constants α and C depend on the dimension n, the uniform constants of Ω, and the elliptic constant λ.

We shall mention quickly that weak solutions are also Hölder continuous inside the domain, and so the solutions u that satisfy the assumptions of Lemma 2.12 are Hölder continuous in $\frac{1}{2}B \cap \overline{\Omega}$.

2.4. Elliptic measure

For solutions $u \in W$ to $\mathcal{L}u = 0$, we have a maximum principle that states

$$\sup_\Omega u \leq \sup_{\partial\Omega} \operatorname{Tr}(u) \quad \text{and} \quad \inf_\Omega u \geq \inf_{\partial\Omega} \operatorname{Tr}(u), \tag{2.8}$$

see Lemma 12.8 in [20]. The maximum principle and the Riesz representation theorem can be used to construct a family of positive regular Borel measures ω^X on $\partial\Omega$, which is called the elliptic measure.

Proposition 2.13 (Elliptic measure, Lemmas 12.13 and 12.15 in [20]). *There exists a unique collection of Borel regular probability measures $\{\omega^X\}_{X\in\Omega}$ on $\partial\Omega$ such that, for any continuous compactly supported function $f \in H$, the solution u_f constructed as*

$$u_f(X) := \int_{\partial\Omega} f(x) d\omega^X(x) \tag{2.9}$$

is the solution to $\mathcal{L}u = 0$ and $\operatorname{Tr} u_f = f$ given by Lemma 2.8.

Moreover, the construction (2.9) can be extended to all bounded functions on $\partial\Omega$ and provides a weak solution to $\mathcal{L}u = 0$.

Since the elliptic measure is a family of measures, the classical definitions of A_∞ and RH_p should be adapted to fit the scenario of elliptic measure.

Definition 2.14 (A_∞ and RH_p for elliptic measure). We say that $\{\omega^X\}_{X\in\Omega}$ is of class A_∞ with respect to the measure σ, or simply $\{\omega^X\}_{X\in\Omega} \in A_\infty(\sigma)$, if for every $\varepsilon > 0$, there exists $\xi = \xi(\varepsilon) > 0$ such that for any boundary ball $\Delta = \Delta(x,r)$ and any $E \subset \Delta$, we have

$$\frac{\sigma(E)}{\sigma(\Delta')} < \xi \implies \omega^{X_0}(E) < \varepsilon,$$

where X_0 is a corkscrew point associated to Δ.

We say that $\{\omega^X\}_{X\in\Omega} \in \mathrm{RH}_p(\sigma)$, for some $p \in (1,\infty)$, if there exists a constant $C \geq 1$ such that for each surface ball Δ with corkscrew point $X_0 \in \Omega$, we have

$$\Big(\frac{1}{\sigma(\Delta)}\int_\Delta (k^{X_0})^p\, d\sigma\Big)^{1/p} \leq C\,\frac{1}{\sigma(\Delta)}\int_\Delta k^{X_0}\, d\sigma. \tag{2.10}$$

Let us recall a result from [52] showing that in higher co-dimension, the solvability of the Dirichlet problem in $L^{p'}$ is equivalent to the fact that $\omega^X \in \mathrm{RH}_p(\sigma)$. It is an analogue of Theorem 1.7.3 of [47].

Theorem 2.15. *Let ω^X be the elliptic measure associated to $\mathcal{L}$, and let $p, p' \in (1,\infty)$ be such that $1/p + 1/p' = 1$. Then, the following statements are equivalent:*

(i) *The Dirichlet problem is solvable in L^p, that is, for each $f \in C_c(\partial\Omega)$, the solution u_f constructed by (2.9) satisfies*

$$\|N(u)\|_{L^{p'}(\partial\Omega,\sigma)} \leq C\|f\|_{L^{p'}(\partial\Omega,\sigma)},$$

where $N(u)$ is the non-tangential maximal function – see (1.6) –, and the constant C is independent of f.

(ii) *We have that $\omega \ll \sigma$ and $\omega^X \in \mathrm{RH}_p(\sigma)$ (see Definition 2.14).*

2.5. Green functions

The Green function is a function defined on $\Omega \times \Omega$ which is morally the solution to $\mathcal{L}u = \delta_Y$, where δ_Y is the Dirac distribution, with zero trace. Its properties are given below.

Theorem 2.16 (Lemma 14.60 and 14.91 in [20]). *There exists a unique function $G\colon \Omega \times \Omega \to \mathbb{R} \cup \{+\infty\}$ such that $G(X,.)$ is continuous on $\Omega \setminus \{X\}$, locally integrable in Ω for any $X \in \Omega$, and such that for any $f \in C_0^\infty(\Omega)$, the function defined by*

$$u_f(X) := \int_\Omega G(X,Y)f(Y)\,dY \tag{2.11}$$

belongs to W_0 and is a solution to $\mathcal{L}u = f$ in the sense that

$$\int_\Omega \mathcal{A}\nabla u_f \cdot \nabla\varphi\, dm = \int_\Omega f\varphi\, dm \quad \text{for } \varphi \in W_0.$$

Moreover,

(i) *for any $Y \in \Omega$, $G(.,Y) \in W_r(\mathbb{R}^n \setminus \{Y\})$ and* $\mathrm{Tr}[G(.,Y)] = 0$.

(ii) *For $Y \in \Omega$ and $\varphi \in C_0^\infty(\Omega)$,*

$$\int_\Omega \mathcal{A}\nabla_X G(X,Y)\cdot\nabla\varphi(X)\,dm(X) = \varphi(Y).$$

In particular, $G(.,Y)$ is a solution to $\mathcal{L}u = 0$ in $\Omega \setminus \{y\}$.

(iii) *For every $Y \in \Omega$, $G(.,Y) \in W^{1,2}(\Omega \setminus B_Y, dm)$, and*

$$\int_{\Omega\setminus B_Y} |\nabla_X G(X,Y)|^2\,dm(X) \le C\,\delta(Y)^{-d},$$

(iv) *For $Y \in \Omega$ and $q \in [1, n/(n-1))$, $G(.,Y) \in W^{1,q}(2B_Y)$, and*

$$\Big(\fint_{2B_Y} |\nabla_X G(X,Y)|^q\,dm(X)\Big)^{1/q} \le C_q\,\delta(Y)^{-d}.$$

In the inequalities above, $C > 0$ depends on n, the uniform constants of Ω, and the ellipticity constant λ, while C_q depends on the same parameters and q.

We only provide a condensed version of Theorem 14.60 from [20]. Indeed, we also have explicit pointwise bounds on G, but it turns out they are not useful in our article. So we omit them here.

Lemma 2.17 (Lemma 10.101 of [19]). *Let G_* be the Green function associated with the operator $\mathcal{L}^*$ (defined from the matrix $\mathcal{A}^T$). For any $X, Y \in \Omega$, $X \neq Y$, we have $G(X,Y) = G_*(Y,X)$. In particular, the function $Y \mapsto G(X,Y)$ satisfies the estimates given in Theorem 2.16.*

We need a last technical lemma.

Lemma 2.18. *Let $X \in \Omega$ and $\varphi \in C_0^\infty(\mathbb{R}^n \setminus \{X\})$. Then*

$$\int_\Omega \mathcal{A}^T\nabla G(X,Y)\cdot\nabla\varphi(Y)\,dm(Y) = -\int_{\partial\Omega} \varphi(y)\,d\omega^X(y).$$

Proof. Take $\rho < \delta(X)/2$ such that $B(X,\rho) \cap \operatorname{supp}\varphi = \emptyset$. Construct $G_*^\rho(.,X)$ to be the function in W_0 that satisfies

$$\int_\Omega \mathcal{A}^T\nabla G_*^\rho(Y,X)\cdot\nabla\phi(Y)\,dm(Y) = \fint_{B(X,\rho)} \phi\,dm \quad \text{for } \phi \in W_0 \tag{2.12}$$

as given by Lemma 2.8 and which was constructed in Section 14 of [20]. As shown in the proof of Theorem 14.60 from [20], we have that $G_*^{\rho_\eta}(.,X)$ converges to $G_*(.,X) = G(X,.)$ uniformly on compact sets of $\overline{\Omega} \setminus \{X\}$ for a subsequence $\rho_\eta \to 0$, and by the Caccioppoli inequality, we also have that $\nabla G_*^{\rho_\eta}(.,X)$ converges to $\nabla G_*(.,X) = \nabla G(X,.)$ in $L^2_{\text{loc}}(\overline{\Omega} \setminus \{X\})$.

Let now u_φ be the weak solution in W to $\mathcal{L}u_\varphi = 0$ in Ω with $\operatorname{Tr} u_\varphi = \varphi$ given by Lemma 2.8 . By Proposition 2.13, we have that

$$\int_{\partial\Omega} \varphi(y)\,d\omega^X(y) = u_\varphi(X).$$

Since $u_\varphi \in W_0$ is a weak solution to $\mathcal{L}u_\varphi = 0$, we have

$$\begin{aligned}\int_\Omega \mathcal{A}^T \nabla G_*^\rho(.,X)\cdot\nabla\varphi\, dm &= \int_\Omega \mathcal{A}^T\nabla G_*^\rho(.,X)\cdot\nabla[\varphi - u_\varphi]\, dm \\ &= \fint_{B(X,\rho)} [\varphi - u_\varphi]\, dm = -\fint_{B(X,\rho)} u_\varphi\, dm \end{aligned} \tag{2.13}$$

by (2.12) and the fact that $\varphi \equiv 0$ on $B(X,\rho)$. As previously mentioned, we have the convergence $\nabla G_*^{\rho_\eta}(.,X) \to \nabla G(X,.)$ in $L^2(\operatorname{supp}\varphi, dm)$, but we also have $\fint_{B(X,\rho)} u_\varphi\, dm \to u_\varphi(X)$ because u_φ is a solution, hence is continuous. The lemma follows from taking the convergence $\rho \to 0$ in (2.13). ∎

2.6. The comparison principle

Theorem 2.19 (Lemma 15.28 of [20]). *Let $x \in \partial\Omega$ and $r > 0$, and let X be a corkscrew point associated to x and r. There exists a constant $C > 1$ depending on n, d, the uniform constants of Ω, and the elliptic constant λ such that, for $Y \in \Omega \setminus B(x,2r)$,*

$$C^{-1} r^{d-1} G(Y,X) \le \omega^Y(\Delta(x,r)) \le C\, r^{d-1} G(Y,X). \tag{2.14}$$

The next result in line should be the fact that the elliptic measure ω^X is doubling. We need the doubling property for the elliptic measure with pole at infinity constructed in Section 3, but we shall prove this result without going through the fact that the elliptic measure is itself doubling.

At this point, it is time to introduce the comparison principle. There are two different versions of it.

Lemma 2.20 (Change of poles, Lemma 15.61 of [20]). *Let $\Delta := \Delta(x,r)$ be a boundary ball, and let X be a corkscrew point associated to Δ. If $E \subset \Delta$ is a Borel set, then for $Y \in \Omega \setminus B(x,2r)$,*

$$C^{-1}\omega^X(E) \le \frac{\omega^Y(E)}{\omega^Y(\Delta)} \le C\,\omega^X(E), \tag{2.15}$$

where $C > 0$ depends on n, the uniform constants of Ω, and the ellipticity constant λ.

Theorem 2.21 (Comparison principle, Lemma 15.64 of [20]). *Let $x \in \partial\Omega$ and $r > 0$ be given, and take X a corkscrew point associated to x and r. Let $u, v \in W_r(B(x,2r))$ be two non-negative, not identically zero, solutions of $\mathcal{L}u = \mathcal{L}v = 0$ in $B(x,2r)\cap\Omega$ such that $\operatorname{Tr} u = \operatorname{Tr} v = 0$ on $\Delta(x,2r)$. For any $Y \in \Omega \cap B(x,r)$, one has*

$$C^{-1}\frac{u(X)}{v(X)} \le \frac{u(Y)}{v(Y)} \le C\frac{u(X)}{v(X)},$$

where $C > 0$ depends only on n, the uniform constants of Ω, and the ellipticity constant λ.

The next corollary is a generalization of Corollary 6.4 of [17]. Even though Corollary 6.4 in [17] is proved for a specific operator $\mathcal{L}$, the proof of it can be adapted to any uniformly elliptic operator, because it is a direct consequence of Theorem 2.21.

Corollary 2.22 (Corollary 6.4 of [17]). *Under the same assumptions on u and v as in Theorem 2.21, for all $X, Y \in B(x, \rho) \cap \Omega$ and $0 < \rho < r/4$,*

$$\left|\frac{u(X)}{u(Y)}\frac{v(Y)}{v(X)} - 1\right| \leq C\left(\frac{\rho}{r}\right)^{\alpha},$$

where $\alpha > 0$ and $C > 0$ depend also only on n, the uniform constants of Ω, and the ellipticity constant λ.

3. The Green function and elliptic measure with pole at infinity

The elliptic measure, contrary to what its name suggests, is a collection of measures. This is pretty inconvenient: every time when we consider the elliptic measure and its properties, we have to pick a right one from the collection. It would be way more practical to have a single measure ω that will capture the (interesting) behavior of all the measures $\{\omega^X\}_{X\in\Omega}$. As we can see in (2.15), taking a pole Y further away from the boundary set E will not really matter, as long as we rescale accordingly. For bounded domains Ω, it suffices to pick a pole X_Ω which is roughly at the middle of the domain in order to have a measure $\omega := \omega^{X_\Omega}$ from which we can recover many properties that the collection $\{\omega^X\}_{X\in\Omega}$ possess. For unbounded domains, we want to morally take "$X_\Omega = \infty$". This section is devoted to the construction of the measure ω^∞ – called the elliptic measure with pole at infinity – and its Green counterpart G_*^∞, which satisfies (2.14) with "$Y = \infty$".

Definition 3.1. We say G_*^∞ and ω^∞ are the Green function and the elliptic measure with pole at infinity[3] if $G_*^\infty \in W_r(\mathbb{R}^n)$[4] is a positive solution to $\mathcal{L}^* G_*^\infty = 0$ in Ω with zero trace, and ω^∞ satisfies

$$\int_\Omega \mathcal{A}^T \nabla G_*^\infty \cdot \nabla\varphi \, dm = -\int_{\partial\Omega} \varphi \, d\omega^\infty, \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^n).$$

Lemma 3.2 (Existence and uniqueness of G^∞ and ω^∞). *Let Ω be a uniform domain and let $\mathcal{L}$ be an operator that satisfies (1.3). There exist a Green function and an elliptic measure with pole at infinity, and they are both unique up to multiplication by a positive scalar.*

Proof. The proof of the following lemma is adapted from Lemma 6.5 of [17]. One key difference is that we consider a general operator $\mathcal{L}$, which is not necessarily symmetric.

Choose a boundary point $x \in \partial\Omega$ (the choice is not important). Pick $X_0 \in B(x, 1) \cap \Omega$ a corkscrew point associated to x and 1, and then for $i \geq 1$, pick $X_i \in \Omega \setminus B(x, 2^i)$ to be a corkscrew point for x and $C2^i$. For $i \geq 1$, we define $G_*^i(X) := G(X_i, X)/G(X_i, X_0)$, where $G(., X)$ is the Green function of $\mathcal{L}$. Thanks to Harnack's inequality (Lemma 2.11), $G(X_i, X_1) > 0$ for $i > 1$. So $G^i(X)$ is well defined.

[3] If we want to be accurate, the elliptic measure with pole at infinity is for $\mathcal{L}$ while the Green function with pole at infinity is for its adjoint $\mathcal{L}^*$. Indeed, we want to take $Y = \infty$ in $G(Y, X) = G_*(X, Y)$, that is, when the pole of the Green function associated to $\mathcal{L}^*$ is ∞.

[4] $W_r(\mathbb{R}^n)$ is the set $W^{1,2}_{\text{loc}}(\overline{\Omega}, dm)$.

First, we show the existence of the Green function with pole at infinity. Let $B_j := B(x, 2^j)$. Observe that for $j < i$, one has $X_i \notin 2B_j$, so in particular G_*^i is a solution in $2B_j \cap \Omega$ and hence is Hölder continuous on $B_j \cap \overline{\Omega}$ (see Lemma 2.12). Using the Hölder continuity, the Harnack inequality (Lemma 2.11), the existence of Harnack chains (by assumption on the domain), and the fact that $G_*^i(X_0) = 1$ for all i, we also deduce that the G_*^i are uniformly bounded on $B_j \cap \Omega$. Thus, the sequence $\{G_*^i\}_{i>j}$ is uniformly bounded and uniformly equicontinuous (follows from the Hölder continuity), and by the Arzelà–Ascoli theorem, there exists a subsequence $\{G_*^{i_\eta}\}$ that converges uniformly on $B_j \cap \Omega$. By a diagonal process, we conclude that $G_*^{i_\eta}$ converges uniformly on all compact sets of $\overline{\Omega}$ to a non-negative continuous function $G_*^\infty(X)$ satisfying $G_*^\infty(X_0) = 1$.

Using the boundary Caccioppoli inequality (see Lemma 11.15 in [20], analogous of Lemma 2.9 but at the boundary), we can see that ∇G^{i_η} is a Cauchy sequence in $L^2(K)$ for all compact set $K \Subset \Omega$, and thus $\nabla G_*^{i_\eta}$ converges to a function $V \in L^2_{\text{loc}}(\overline{\Omega}, dm)$. Since $\nabla G_*^{i_\eta}$ converges to both ∇G_*^∞ and V in the sense of distributions, we deduce that $\nabla G_*^\infty = V$, hence $G_*^\infty \in W_r(\mathbb{R}^n)$.

From the previous paragraph, $\nabla G_*^{i_\eta}$ converges strongly (hence weakly) to ∇G_*^∞ in $L^2_{\text{loc}}(\Omega)$, so we easily have

$$\int_\Omega \mathcal{A}^T \nabla G_*^\infty \cdot \nabla\varphi \, dm = -\lim_{i\to\infty} \int_\Omega \mathcal{A}^T \nabla G_*^i \cdot \nabla\varphi \, dm = 0 \tag{3.1}$$

whenever $\varphi \in C_0^\infty(\Omega)$ and i large enough so that X_i is outside of $\operatorname{supp}\varphi$. We deduce that G_*^∞ is a weak solution to $\mathcal{L}^* G_*^\infty = 0$. We can now invoke the Harnack inequality (together with the existence of Harnack chains and the fact that $G_*^\infty(X_0) = 1$) to obtain that G_*^∞ is positive in Ω.

We claim that G^∞ is the unique positive solution to the operator $\mathcal{L}^*$ with zero trace (up to a positive scalar multiplication). Take another weak solution $v \in W(\mathbb{R}^n)$ to $\mathcal{L}^* v = 0$ in Ω with zero trace and $v(X_0) = 1$. Applying Corollary 2.22 with $Y = X_0$, one has

$$\left|\frac{G_*^\infty(X)}{v(X)} - 1\right| \leq C\left(\frac{\rho}{r}\right)^\alpha \tag{3.2}$$

whenever $X \in \Omega$, $B(x,\rho) \ni X$, and $r > 4\rho$. There is no limitation to take ρ/r as small as we want, hence (3.2) implies that $G_*^\infty \equiv v$. The uniqueness also proves that, in the Arzelà–Ascoli theorem, G_*^∞ is the only adherent point of the relatively compact sequence $\{G_*^i\}$. So we actually have that

$$G_*^i := \frac{G(X_i, .)}{G(X_i, X_0)} \text{ converges to } G_*^\infty \text{ uniformly on compact subsets of } \overline{\Omega} \text{ and in } W_r(\mathbb{R}^n). \tag{3.3}$$

Now, we deal with the elliptic measure ω^∞ with pole at infinity. Let us set $\omega^i = \omega^{X_i}/G(X_i, X_0)$. Theorem 2.19 entails, for $i > j$, that $\omega^{X_i}(B_j \cap \partial\Omega) \lesssim 2^{j(d-1)} G(X_i, X^j)$, hence

$$\omega^i(B_j \cap \partial\Omega) \leq C\, 2^{j(d-1)}\, G_*^i(X_j) \leq C_j,$$

because G_*^i converges to G_*^∞ uniformly on compacts. Thus, there exists a measure ω^∞ such that a subsequence ω^{i_η} converges weakly-* to ω^∞. Lemma 2.18, the convergence of ∇G_*^i to ∇G in $L^2(\overline{\Omega}, dm)$, and $\omega^{i_\eta} \overset{*}{\rightharpoonup} \omega^\infty$ all together imply that

$$\int_\Omega \mathcal{A}^T \nabla G^\infty \cdot \nabla \varphi \, dm = - \int_{\partial\Omega} \varphi \, d\omega^\infty. \tag{3.4}$$

The uniqueness of ω^∞ follows from the uniqueness of G^∞ and (3.4). Moreover, the uniqueness also shows the convergence of ω^i (instead of a subsequence), that is,

$$\omega^i \overset{*}{\rightharpoonup} \omega^\infty \quad \text{and} \quad \omega^i(E) \to \omega^\infty(E) \quad \text{for any Borel set } E \subset \partial\Omega. \tag{3.5}$$

The lemma follows. ■

The Green function and elliptic measure with pole at infinity satisfy the following CFMS-type estimates (see [7]).

Lemma 3.3. *Let $X \in \Omega$ be a corkscrew point associated to a boundary ball $\Delta_X := \Delta(x, r)$. Then*

$$C^{-1} r^{d-1} G_*^\infty(X) \leq \omega^\infty(\Delta_X) \leq C\, r^{d-1} G_*^\infty(X).$$

If moreover $E \subset \Delta_X$ is a Borel set, then

$$C^{-1} \omega^X(E) \leq \frac{\omega^\infty(E)}{\omega^\infty(\Delta_X)} \leq C\, \omega^X(E),$$

At last, when $Y \in \Omega \setminus B(x, 2r)$, we have

$$C^{-1} G(Y, X) \leq \frac{G_*^\infty(Y)}{\omega^\infty(\Delta_X)} \leq C\, G(Y, X).$$

In each case, $C > 0$ depends only on n, the uniform constants of Ω, and the elliptic constant λ.

Proof. Thanks to the convergences (3.3) and (3.5), the first two results follow directly from the estimates of Theorems 2.19 and 2.20 respectively. The third one is an easy consequence of Theorem 2.19 and the first two estimates. ■

Let us show now that the elliptic measure with pole at infinity is doubling.

Lemma 3.4 (Doubling property of ω^∞). *There exists $C > 0$ depending only on n, the uniform constants of Ω, and the elliptic constant λ, such that*

$$\omega^\infty(2\Delta) \leq C \omega^\infty(\Delta) \quad \textit{for any boundary ball } \Delta.$$

Proof. By Lemma 3.3, if r_Δ is the radius of Δ, then

$$\omega^\infty(2\Delta) \approx (2r_\Delta)^{d-1} G^\infty(X_{2\Delta}) \quad \text{and} \quad \omega^\infty(\Delta) \approx (r_\Delta)^{d-1} G^\infty(X_\Delta),$$

where X_Δ and $X_{2\Delta}$ are corkscrew points for Δ and 2Δ respectively. The lemma follows from the existence of Harnack chains (since Ω is uniform) and the Harnack inequality (Lemma 2.11). ■

The measure ω^∞ is convenient, because it allows to capture the A_∞-absolute continuity and the reverse Hölder estimates for a collection of measures (see Definition 2.14) with a single measure (Definitions 2.5 and 2.6).

Lemma 3.5. *We have*

$$\{\omega^X\}_{X\in\Omega} \in A_\infty(\sigma) \iff \omega^\infty \in A_\infty(\sigma),$$

and

$$\{\omega^X\}_{X\in\Omega} \in \mathrm{RH}_p(\sigma) \iff \omega^\infty \in \mathrm{RH}_p(\sigma).$$

Proof. The change of pole estimate (the second one) in Lemma 3.3, Definition 2.14, and Theorem 2.15 easily give the results. ■

Corollary 3.6. *Let Ω be a uniform domain, let σ be as in (1.1), and let $\mathcal{L}$ be the elliptic operator that satisfies (1.3). Write ω^∞ for the elliptic measure with pole at infinity of $\mathcal{L}$. For any fixed $p \in (1,\infty)$, the following two statements are equivalent:*

- *the Dirichlet problem of operator $\mathcal{L}$ is solvable in $L^{p'}$, that is, for each $f \in C_c(\partial\Omega)$, the solution u_f constructed by (2.9) satisfies*

$$\|N(u_f)\|_{L^{p'}(\partial\Omega,\sigma)} \leq C\|f\|_{L^{p'}(\partial\Omega,\sigma)},$$

  where C is independent of f;
- *$\omega^\infty \ll \sigma$ and $\omega^\infty \in \mathrm{RH}_p(\sigma)$.*

4. The proof of Theorem 1.3

We recall that we write $\delta(X)$ for $\mathrm{dist}(X,\partial\Omega)$, $w(X)$ for $\delta(X)^{d+1-n}$, $dm(X) = w(X)dX$, and B_X for $B(X,\delta(X)/4)$.

For the proof of Theorem 1.3, we will follow the method developed by Kenig and Pipher in [50]. In this section, $\mathcal{L}_0$ and $\mathcal{L}_1$ are two operators in the form $-\mathrm{div}[w\mathcal{A}_i\nabla]$ that satisfy (1.3). Since we assume that the Dirichlet problem for $\mathcal{L}_1^*$ is solvable in $L^{q'}(\sigma)$, by Corollary 3.6, the elliptic measure $\omega^\infty_{1,*}$ with pole at infinity satisfies the reverse Hölder bounds

$$\left(\fint_\Delta \left|\frac{d\omega^\infty_{1,*}}{d\sigma}\right|^q d\sigma\right)^{1/q} \leq C_q \frac{\omega^\infty_{1,*}(\Delta)}{\sigma(\Delta)} \quad \text{for any boundary ball } \Delta. \tag{4.1}$$

The notations u_0 and u_1 are reserved for solutions to $\mathcal{L}_0u_0 = 0$ and $\mathcal{L}_1u_1 = 1$ that satisfy the same trace condition $\mathrm{Tr}\,u_0 = \mathrm{Tr}\,u_1 = f \in C_c(\partial\Omega)\cap H$. We shall use the quantity $F(X)$ defined as

$$F(X) = \int_\Omega \nabla_Y G_1(X,Y)\cdot\mathcal{E}(Y)\nabla u_0(Y)\,dm(Y) = u_1(X) - u_0(X), \tag{4.2}$$

where G_1 is the Green function associated to the operator $\mathcal{L}_1$ and $\mathcal{E} := \mathcal{A}_0 - \mathcal{A}_1$ is the disagreement between $\mathcal{L}_0$ and $\mathcal{L}_1$. One important fact is that F is the difference of u_1 and u_0, that is,

$$u_1(X) - u_0(X) = F(X) \quad \text{for almost every } X \in \Omega. \tag{4.3}$$

Indeed, we “morally” have that

$$\mathcal{L}_1(u_1 - u_0) = (\mathcal{L}_0 - \mathcal{L}_1)u_0 = -\operatorname{div}(w\mathcal{E}\nabla u_0),$$

and so, using the properties of the Green function and the fact that $u_1 - u_0$ has zero trace,

$$\begin{aligned} u_1(X) - u_0(X) &= -\int_\Omega G_1(X,Y)\operatorname{div}(w\mathcal{E}\nabla u_0)(Y)\,dY \\ &= \int_\Omega \nabla_Y G_1(X,Y)\cdot\mathcal{E}(Y)\nabla u_0(Y)\,w(Y)\,dY = F(X). \end{aligned}$$

The actual proof of (4.3) can be found in Lemma 3.18 from [9] (for codimension 1) and Lemma 7.13 from [52] (for higher codimension).

We assume that the disagreement satisfies the Carleson measure condition

$$\int_{B(x,r)\cap\Omega} \sup_{Y\in B_X} |\mathcal{E}(Y)|^2 \frac{dX}{\delta(X)^{n-d}} \leq M r^d \quad \text{for any } x\in\partial\Omega \text{ and } r>0. \tag{4.4}$$

The condition (4.4) is well adapted to the non-tangential maximal function $\widetilde{N}$ because of the Carleson inequality

$$\int_{\partial\Omega}\Big(\int_{\gamma(x)} |\mathcal{E}(Y)|^2|\phi(Y)|^2\frac{dY}{\delta(Y)^n}\Big)^{q/2} d\sigma(x) \lesssim M\|\widetilde{N}(\phi)\|_{L^q}^q, \tag{4.5}$$

which is proved as Lemma 2.1 in [11] in the case where the boundary is flat (but the proof easily extends to our setting).

Similarly to the method found in [50], the plan of the proof is as follows:

(1) Lemma 4.1:

$$\|\widetilde{N}(F)\|_q \lesssim \int_\Omega \nabla F(Z)\cdot\vec{h}(Z)\,dZ,$$

where $\vec{h}$ is constructed by duality to have the above estimate (and so depends on F and q).

(2) Lemma 4.6:

$$\int_\Omega \nabla F(Z)\cdot\vec{h}(Z)\,dZ \lesssim M\|\widetilde{N}(\nabla u_0)\|_q\|S(v)\|_{q'},$$

where v is the solution to $\mathcal{L}_1^* v = \operatorname{div}\vec{h}$ with $\operatorname{Tr}(v) = 0$.

(3) Lemma 4.7 and Corollary 4.8:

$$\|S(v)\|_{q'} \lesssim \|N(v)\|_{q'} + \|\widetilde{N}(\delta\nabla v)\|_{q'} + \|\mathcal{M}_\omega(T(\vec{h}))\|_{L^{q'}(\partial\Omega,\sigma)},$$

where $\mathcal{M}_\omega$ is the Hardy–Littlewood maximal function with respect to the measure $\omega := \omega_{1,*}^\infty$, and where $T(\vec{h})$ is defined in (4.13) and looks a bit like a square functional.

(4) Lemma 4.10:

$$\|N(v)\|_{q'} + \|\widetilde{N}(\delta\nabla v)\|_{q'} \lesssim \|\mathcal{M}_\omega(T(\vec{h}))\|_{L^{q'}(\partial\Omega,\sigma)}.$$

(5) By the property (v) of Theorem 2.7,

$$\|\mathcal{M}_\omega(T(\vec{h}))\|_{L^{q'}(\partial\Omega,\sigma)} \lesssim \|T(\vec{h})\|_{L^{q'}(\partial\Omega,\sigma)}.$$

(6) Lemma 4.4:

$$\|T(\vec{h})\|_{q'} \lesssim 1.$$

(7) Items (1) to (6) prove that, for $f \in C_c(\partial\Omega) \cap H$,

$$\|\widetilde{N}(\nabla F)\|_{L^q(\partial\Omega,\sigma)} \leq C\|\widetilde{N}(\nabla u_0)\|_{L^q(\partial\Omega,\sigma)}, \tag{4.6}$$

that is, by (4.3),

$$\|\widetilde{N}(\nabla u_1)\|_{L^q(\partial\Omega,\sigma)} \leq C\|\widetilde{N}(\nabla u_0)\|_{L^q(\partial\Omega,\sigma)}, \tag{4.7}$$

which is the desired result.

The constants in this section are independent of f and depend on $\mathcal{L}_0$ and $\mathcal{L}_1$ (and hence on u_0 and u_1) only via the ellipticity constant λ and the reverse Hölder constants q and C_q from (4.1). The dependence in M will only appear in Lemma 4.6 and will be explicitly written.

4.1. Notation

We start this section by giving the definition of cones that we shall use. The basic cones are simply $\gamma(x) := \{X \in \Omega,\ |X - x| < 2\delta(X)\}$, but it will be also convenient for us to use cones constructed from Whitney cubes.

So we construct a family of Whitney cubes $\mathcal{W}$. We use the following convention: if $Q \in \mathbb{D}$ is a dyadic cube in $\mathbb{R}^n$, then $\ell(Q)$ denotes its diameter and

$$\kappa Q := \{X \in \mathbb{R}^n,\ \operatorname{dist}(X, Q) \leq (\kappa - 1)\ell(Q)\} \quad \text{for } \kappa \geq 1.$$

In particular, if Q^* is the dyadic parent of Q, then $Q^* \subset 2Q$. We say that the dyadic $I \in \mathbb{D}$ in $\mathbb{R}^n$ belongs to $\mathcal{W}$ if I is a maximal dyadic cube with the property that $10I \subset \Omega$. As such, a cube $I \in \mathcal{J}$ satisfies

$$10I \subset \Omega \quad \text{and} \quad 20I \cap \partial\Omega \neq \emptyset. \tag{4.8}$$

We define then $\gamma_d(x)$ as the union of the Whitney cubes that intersect $\{X \in \Omega,\ |X - x| < 3\delta(X)\}$, that is, if

$$\mathcal{W}_x := \{I \in \mathcal{W},\ |X - x| < 3\delta(X) \text{ for one } X \in I\},$$

then

$$\gamma_d(x) := \bigcup_{I \in \mathcal{W}_x} I. \tag{4.9}$$

4.2. Duality and the function *h*

The first step is to use duality to write $\|\widetilde{N}(\nabla F)\|_q$ as an integral against a function. Since we do not know *a priori* that $\|\widetilde{N}(\nabla F)\|_q$ is finite, for the rest of the section, we choose a compact subset K of Ω and we define the truncated (localized) function $\widetilde{N}$ as

$$\widetilde{N}_K(\nabla F) := \sup_{X\in\gamma(x)} \mathbb{1}_K(X) \Big(\fint_{B_X} |\nabla F|^2 \, dY \Big)^{1/2}.$$

The quantities $\widetilde{N}_K(\nabla F)(x)$ –for $x \in \partial\Omega$– and $\|\widetilde{N}_K(\nabla F)\|_q$ are all finite, and this is only a consequence of the fact that $F = u_1 - u_0 \in W^{1,2}_{\rm loc}(\Omega)$. We shall obtain bounds on $\|\widetilde{N}_K(\nabla F)\|_q$ that are independent of K, hence a bound on $\|\widetilde{N}(\nabla F)\|_q$ thanks to the monotone convergence theorem.

Lemma 4.1. *Let $q > 1$ and let $K \Subset \Omega$. There exist a compact set $K' \Subset \partial\Omega$, a bounded vector function $\vec{\alpha} \in L^\infty(\Omega, \mathbb{R}^n)$ with $\|\vec{\alpha}\|_\infty = 1$, a non-negative function $\beta(.,x) \in L^1(\Omega)$ with $\int_\Omega \beta(X,x) dX = 1$ for all $x \in \partial\Omega$, and a non-negative function $g \in L^{q'}(\partial\Omega)$ with $\|g\|_{L^{q'}(\partial\Omega)} = 1$, such that*

$$\|\widetilde{N}_K(\nabla F)\|_{L^q} \leq C \int_\Omega \nabla F(Z) \cdot \vec{h}(Z) \, dZ, \tag{4.10}$$

where C depends only on n and λ, and where $\vec{h}$ is defined as

$$\vec{h}(Z) := \int_K \vec{\alpha}(Z) \, \mathbb{1}_{2B_X}(Z) \int_{K'\cap 8B_X} \beta(X,x) \, g(x) \, d\sigma(x) \, \frac{dX}{\delta(X)^n}. \tag{4.11}$$

Remark 4.2. The function $\vec{h}$, as well as the functions g and β, depend on the compact K. It is necessary to guarantee the *a priori* finiteness of all $\|\widetilde{N}_K(\nabla F)\|_{L^q}$, and so of all the quantities we shall manipulate in the future. Moreover, the function $\vec{h}$ is compactly supported and bounded by a constant that depends on K (see Lemma 4.3), which will make future manipulations of $\vec{h}$ easier. However, *the constants in the core results of this section* (Lemmas 4.1, 4.4, 4.6, 4.7, 4.9) *shall never depend on K*, so that the bound that we obtain on $\|\widetilde{N}_K(\nabla F)\|_{L^q}$ will be transmitted to $\|\widetilde{N}(\nabla F)\|_{L^q}$.

Proof. First, recall that F is just $u_1 - u_0$, see (4.3), so we can use the reverse Hölder inequality for the gradient (see Corollary 2.10) to obtain that

$$\widetilde{N}_K(\nabla F)(x) \leq C_\Lambda \, \widetilde{N}^1_K(\nabla F)(x) \quad \text{for } x \in \partial\Omega,$$

where

$$\widetilde{N}^1_K(\nabla F) := \sup_{X\in\gamma(x)} \mathbb{1}_K(X) \fint_{2B_X} |\nabla F| \, dY.$$

Of course, this also gives that

$$\|\widetilde{N}_K(\nabla F)\|_q \lesssim \|\widetilde{N}^1_K(\nabla F)\|_q.$$

The rest of the proof relies on duality. Since $L^{q'}(\partial\Omega,\sigma)$ is the dual space of $L^q(\partial\Omega,\sigma)$ and $\widetilde{N}^1_K(\nabla F)$ is non-negative, we have

$$\|\widetilde{N}^1_K(\nabla F)\|_q = \sup_{\substack{0\leq g\in L^{q'}\\ \|g\|_{q'}=1}} \int_{\partial\Omega} \widetilde{N}^1_K(\nabla F) g\, d\sigma.$$

We are able to select a $g \in L^{q'}$ with $g \geq 0$ and $\|g\|_{q'} = 1$ such that

$$\|\widetilde{N}_K(\nabla F)\|_q \lesssim \|\widetilde{N}^1_K(\nabla F)\|_q \leq 2\int_{\partial\Omega} \widetilde{N}^1_K(\nabla F)\, g\, d\sigma.$$

By density, we can even take g to be continuous and compactly supported. We set K' to be the support of g and we obtain

$$\|\widetilde{N}_K(\nabla F)\|_q \lesssim \int_{K'} \widetilde{N}^1_K(\nabla F)\, g\, d\sigma.$$

Since L^∞ is the dual of L^1, for each $x \in \partial\Omega$, we have

$$\widetilde{N}^1_K(\nabla F)(x) = \sup_{\|\beta(.,x)\|_{L^1(\Omega)}=1} \int_K \Big(⨍_{2B_X} |\nabla F|\, dZ\Big) \beta(X,x)\, \mathbb{1}_{\gamma(x)}(X)\, dX. \tag{4.12}$$

It also means that we can find a function $\beta \geq 0$ which satisfies $\int_\Omega \beta(X,x)dX = 1$ for all $x \in \partial\Omega$, and such that

$$\begin{aligned}
\|\widetilde{N}_K(\nabla F)\|_q &\lesssim \int_{K'} \widetilde{N}^1_K(\nabla F)\, g\, d\sigma \\
&\lesssim \int_{K'} g(x) \int_K \Big(⨍_{2B_X} |\nabla F|\, dZ\Big) \beta(X,x)\, \mathbb{1}_{\gamma(x)}(X)\, dX\, d\sigma(x) \\
&\lesssim \int_K \Big(\int_{2B_X} |\nabla F|\, dZ\Big)\Big(\int_{K'\cap 8B_X} \beta(X,x)\, g(x)\, d\sigma(x)\Big) \frac{dX}{\delta(X)^n};
\end{aligned}$$

the last line is due to Fubini's theorem, since $X \in \gamma(x)$ is equivalent to $x \in \partial\Omega \cap 8B_X$.

We now take $\vec{\alpha} = \nabla F/|\nabla F|$ and, by Fubini's theorem again, we have

$$\begin{aligned}
&\|\widetilde{N}_K(\nabla F)\|_q \\
&\quad\lesssim \int_K \Big(\int_\Omega \nabla F(Z)\cdot \vec{\alpha}(Z)\, \mathbb{1}_{2B_X}(Z)\, dZ\Big)\Big(\int_{K'\cap 8B_X} \beta(X,x)\, g(x)\, d\sigma(x)\Big) \frac{dX}{\delta(X)^n} \\
&\quad= \int_\Omega \nabla F(Z)\cdot \Big(\int_K \int_{K'\cap 8B_X} \vec{\alpha}(Z)\, \mathbb{1}_{2B_X}(Z)\, \beta(X,x)\, g(x)\, d\sigma(x)\, \frac{dX}{\delta(X)^n}\Big) dZ \\
&\quad= \int_\Omega \nabla F(Z)\cdot \vec{h}(Z)\, dZ.
\end{aligned}$$

The lemma follows. ■

In the previous construction, we made sure that $\vec{h}$ is nice enough, that is, bounded and compactly supported, as shown in the next result.

Lemma 4.3. *The function $\vec{h}$ defined in (4.11) is bounded and compactly supported.*

Proof. Since $Z \in 2B_X$, we have $\delta(Z)/2 \le \delta(X) \le 2\delta(Z)$. Combined with $|\vec{\alpha}| \le 1$, we deduce

$$|\vec{h}| \lesssim \delta(Z)^{-n} \int_{K'} g(x)\, d\sigma(x) \int_K \beta(X,x)\, dX \le C_{K'}.$$

So the function $\vec{h}$ is indeed bounded.

It is also compactly supported because, in order for $\vec{h}$ to be non-zero, we need $Z \in 2B_X$ with $X \in K$. And that is possible only when Z is in a compact set that is slightly bigger than K. ∎

We want now to bound the integral $\int_\Omega \nabla F \cdot \vec{h}\, dZ$. However, as one can expect, a lot of information is hidden in $\vec{h}$. Why do we use the quantity $\vec{h}$? Because, even if $\vec{h}$ depends on F (and K), we are able to bound it independently of F (and K), as shown in the lemma below. We define first $T(\vec{h})$ as

$$T(\vec{h})(x) := \sum_{I \in \mathcal{W}_x} \ell(I)^{n-d} \sup_I |\vec{h}|, \tag{4.13}$$

where $\mathcal{W}_x$ is the collection of Whitney cubes that intersect $\gamma_3(x) := \{x \in \Omega,\ |X - x| < 3\delta(X)\}$ and $\ell(I)$ is the side-length of I, which is equivalent to $\mathrm{dist}(I, \partial\Omega)$. To build intuition, we observe that if the supremum was replaced by a L^1-average, then $T(\vec{h})(x)$ would be essentially $\int_{\gamma_d(x)} |\vec{h}|\, dX/\delta(X)^d$, that is, the integration of $|\vec{h}|$ over the radial direction(s).

Lemma 4.4. *We have*

$$\|T(\vec{h})\|_{L^{q'}(\partial\Omega,\sigma)} \le C_{q'},$$

where q is the one of Lemma 4.1 and is used to construct $\vec{h}$.

Proof. We first remove $\vec{\alpha}$ from the estimate on $\vec{h}$, because we will not be able to do anything with it, so we have

$$\begin{aligned} |\vec{h}(Z)| &\le \int_\Omega \mathbb{1}_{2B_X}(Z) \int_{8B_X} \beta(X,x) g(x)\, d\sigma(x)\, \frac{dX}{\delta(X)^n} \\ &= \int_{\partial\Omega} g(x) \Big(\int_\Omega \mathbb{1}_{2B_X}(Z)\, \beta(X,x)\, \mathbb{1}_{\gamma(x)}(X)\, \frac{dX}{\delta(X)^n} \Big)\, d\sigma(x). \end{aligned} \tag{4.14}$$

Pick a Whitney cube $I \in \mathcal{W}$. Construct I^* as

$$I^* := \{X \in \Omega, \text{ there exists } Z \in I \text{ such that } Z \in 2B_X\}.$$

Check that I^* is a Whitney region larger than I, but still has a finite overlapping. So if $b_I(x)$ denotes $\int_{I^*} \beta(X,x) dX$, we have the nice control

$$\sum_{I \in \mathcal{W}} b_I(x) \lesssim 1 \quad \text{for any } x \in \partial\Omega \tag{4.15}$$

because, by definition, $\int_\Omega \beta(X,x)\,dX = 1$ for any $x \in \partial\Omega$. We take now $Z \in I$, in this case, any X that satisfies $Z \in 2B_X$ lies in the Whitney region I^*, which implies that $\delta(X) \approx \ell(I)$. So (4.14) becomes

$$\sup_{2I} |\vec{h}| \leq \ell(I)^{-n} \int_{\partial\Omega\cap 10^3 I} g(y)\, b_I(y)\, d\sigma(y).$$

We inject this bound in the expression of $T(\vec{h})$ to obtain

$$T(\vec{h})(x) \leq \sum_{I\in\mathcal{W}_x} \ell(I)^{-d} \int_{\partial\Omega\cap 10^3 I} g(y)\, b_I(y)\, d\sigma(y). \tag{4.16}$$

We compute then the $L^{q'}$-norm of $T(\vec{h})$ by duality. Let $\phi \in L^q(\partial\Omega,\sigma)$ be any non-negative function such that $\|\phi\|_q = 1$. We claim that

$$\int_{\partial\Omega} T(\vec{h})(x)\, \phi(x)\, d\sigma(x) \lesssim 1, \tag{4.17}$$

which is exactly what we need to conclude the lemma. We use the bound (4.16) and then Fubini's theorem to write

$$\begin{aligned}\int_{\partial\Omega} T(\vec{h})(x)\, \phi(x)\, d\sigma(x) &\lesssim \int_{\partial\Omega} \phi(x) \sum_{I\in\mathcal{W}_x} \ell(I)^{-d} \int_{\partial\Omega\cap 10^3 I} g(y)\, b_I(y)\, d\sigma(y)\, d\sigma(x)\\ &\lesssim \int_{\partial\Omega} g(y) \sum_{I\in\mathcal{W}_x} b_I(y)\, \mathbb{1}_{10^3 I}(y)\, \ell(I)^{-d} \int_{\partial\Omega\cap 10^3 I} \phi(x)\, d\sigma(x)\, d\sigma(y),\end{aligned}$$

where the last line holds because $I \in \mathcal{W}_x$ implies $x \in 1000I$ (we are not trying to be optimal here). Let $\mathcal{M}_\sigma$ denote the Hardy–Littlewood maximal function with respect to the d-dimensional Ahlfors measure σ. Since $y \in 10^3 I$, we easily have

$$\mathbb{1}_{y\in 10^3}\, \ell(I)^{-d} \int_{\partial\Omega\cap 10^3 I} \phi(x)\, d\sigma(x) \lesssim \mathcal{M}_\sigma(\phi)(y)$$

and hence

$$\begin{aligned}\int_{\partial\Omega} T(\vec{h})(x)\, \phi(x)\, d\sigma(x) &\lesssim \int_{\partial\Omega} g(y) \sum_{I\in\mathcal{W}} b_I(y)\, \mathcal{M}_\sigma(\phi)(y)\, d\sigma(y)\\ &\lesssim \int_{\partial\Omega} g(y)\, \mathcal{M}_\sigma(\phi)(y)\, d\sigma(y),\end{aligned}$$

by (4.15). We invoke now Hölder's inequality and the L^q-boundedness of the operator $\mathcal{M}_\sigma$ to deduce

$$\begin{aligned}\int_{\partial\Omega} T(\vec{h})(x)\, \phi(x)\, d\sigma(x) &\lesssim \|g\|_{L^{q'}(\partial\Omega,\sigma)}\, \|\mathcal{M}_\sigma(\phi)\|_{L^q(\partial\Omega,\sigma)}\\ &\lesssim \|g\|_{L^{q'}(\partial\Omega,\sigma)}\, \|\phi\|_{L^q(\partial\Omega,\sigma)} = 1\end{aligned}$$

because, by definition, $\|g\|_{q'} = 1$ and $\|\phi\|_q = 1$. The claim (4.17) and the lemma follow.

■

4.3. The solution v

Our goal is to bound the expression $\int_\Omega \nabla F \cdot \vec{h}\, dZ$ from (4.10). However, this expression is lacking derivatives. Indeed, the techniques employed here rely on integration by parts, that is, moving gradients and derivatives from one term to another, with errors that can be controlled. So the more terms with derivatives we have, the more possibilities we get. That is why we introduce v, which is essentially the solution to the inhomogeneous Dirichlet problem

$$\begin{cases} \mathcal{L}_1^* v = \operatorname{div} \vec{h} & \text{in } \Omega, \\ \operatorname{Tr}(v) = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.18}$$

where $\vec{h}$ is the one constructed in Lemma 4.1.

We shall ultimately use two distinct representations of v. So we need to prove that those two definitions of v coincide, which is very classical in the bounded codimension 1 case but more delicate in our context (which allows higher codimensional boundaries, and the elliptic theory is not as developed).

We write G_1 for the Green function associated to $\mathcal{L}_1$ as defined in Theorem 2.16. We define v on Ω as

$$v(X) := -\int_\Omega \nabla_Z G_1(Z,X) \cdot \vec{h}(Z)\, dZ, \tag{4.19}$$

which is well defined because $\vec{h}(Z)$ is bounded and compactly supported (see Lemma 4.3) and $\nabla_Z G_1^*(X,.) = \nabla_Z G_1(.,X) \in L^r_{\mathrm{loc}}(\Omega)$ for r sufficiently close to 1 (see items (iii)-(iv) of Theorem 2.16).

Lemma 4.5. *The function $v(X)$ constructed in (4.19) lies in W_0 and satisfies*

$$\int_\Omega \mathcal{A}_1^T \nabla v \cdot \nabla \varphi \, dm = -\int_\Omega \vec{h} \cdot \nabla \varphi \, dX \quad \textit{for } \varphi \in W_0.$$

Proof. The idea of the proof is: if $\vec{h}$ were smooth, there would be no difficulty. Thus, as expected, we mollify $\vec{h}$ and we check that we can take all the desired limits.

We construct the mollifier by using a non-negative radial function $\rho \in C_0^\infty(\mathbb{R}^n)$ supported in $B(0,1)$ and satisfying $\int_{\mathbb{R}^n} \rho = 1$, and then we define $\rho_\varepsilon(Z) := \varepsilon^{-n}\rho(\varepsilon^{-1}Z)$ for $\varepsilon > 0$ and $Z \in \Omega$. We set $\vec{h}_\varepsilon := \vec{h} * \rho_\varepsilon \in C_0^\infty(\Omega)$. The fact that $\vec{h}_\varepsilon$ is compactly supported in Ω is true only for small $\varepsilon > 0$ (but it does not matter since we intend to take limits) because $\vec{h}$ is already compactly supported in the first place.

We fix $p \in (n, \infty)$, so that $\nabla_Z G_1(Z,X)$ is locally in $L^{p'}$ (see item (iv) in Theorem 2.16). Note for later that

$$\vec{h}_\varepsilon \to \vec{h} \quad \text{in } L^p(\Omega), \tag{4.20}$$

which is a classical result and essentially equivalent to the density of smooth functions in L^p.

We define

$$v_\varepsilon(X) := -\int_\Omega \nabla_Z G_1(Z,X)\cdot \vec{h}_\varepsilon(Z)\,dZ = \int_\Omega G_1(Z,X)\,\mathrm{div}(\vec{h}_\varepsilon)(Z)\,dZ. \tag{4.21}$$

Since now $\mathrm{div}(\vec{h}_\varepsilon) \in C_0^\infty(\Omega)$, by (2.11) (and Lemma 2.17), we have that v_ε is the function of W_0 that satisfies

$$\int_\Omega \mathcal{A}_1^T \nabla v_\varepsilon \cdot \nabla\varphi\, dm = \int_\Omega \mathrm{div}(\vec{h}_\varepsilon)\varphi\, dX = -\int_\Omega \vec{h}_\varepsilon \cdot \nabla\varphi\, dX \quad \text{for } \varphi \in W_0. \tag{4.22}$$

In addition, we also know the following convergences.

(1) We can pass the limit as $\varepsilon \to 0$ in the expression $\int_\Omega \nabla G_1(.,X)\cdot \vec{h}_\varepsilon\,dZ$, because $\nabla G_1(.,Z) \in L^{p'}_{\mathrm{loc}}(\Omega)$, all the h_ε are supported in the same compact subset of Ω, and $\vec{h}_\varepsilon$ converges to $\vec{h}$ in $L^p(\Omega)$ (see (4.20)). So we deduce $v_\varepsilon \to v$ pointwise (and thus in the distribution sense).

(2) The functions $\mathrm{div}(\vec{h}_\varepsilon)$ converge to $\mathrm{div}(\vec{h})$ in W^{-1}. Indeed, if $K \Subset \Omega$ is a compact set that contains the support of all the h_ε and $p \in (2,\infty)$, then

$$\begin{aligned} \|\mathrm{div}(\vec{h}_\varepsilon) - \mathrm{div}(\vec{h})\|_{W^{-1}} &= \sup_{\|\varphi\|_{W_0}\leq 1} \Big| \int_\Omega (\vec{h}-\vec{h}_\varepsilon)\cdot\nabla\varphi\,dX \Big| \\ &\leq \|\vec{h}-\vec{h}_\varepsilon\|_{L^p}\,\|\nabla\varphi\|_{L^{p'}(K)} \leq C_K\,\|\vec{h}-\vec{h}_\varepsilon\|_{L^p} \to 0 \quad \text{as } \varepsilon\to 0. \end{aligned}$$

(3) The previous convergences combined with the Lax–Milgram theorem (Lemma 2.8) imply that v_ε converges in W_0.

The combination of three convergences shows that

$$v_\varepsilon \to v \quad \text{in } W_0, \tag{4.23}$$

so in particular, $v \in W_0$ and we also have

$$\int_\Omega \mathcal{A}_1^T \nabla v\cdot\nabla\varphi\,dm = -\int_\Omega \vec{h}\cdot\nabla\varphi\,dX \quad \text{for } \varphi\in W_0$$

by taking the limit in (4.22). The lemma follows. ■

We return to the estimate of the non-tangential maximal function. Recall that at this point, we want to bound $\int_\Omega \nabla F\cdot\vec{h}\,dZ$. The next step will involve the square function of v, which is defined as

$$S(v)(x) := \Big(\int_{\gamma(x)} |\nabla v|^2 \frac{dY}{\delta(Y)^{n-2}}\Big)^{1/2}. \tag{4.24}$$

Even though the next lemma is an analogue of Corollary 2.9 in [50], we provide here an alternative proof which is self contained (up to some basic results on the Green functions) and does not rely on taking the limit of a sequence of elliptic operators with smooth coefficients.

Lemma 4.6. *Recall that M is the constant in* (4.4)*. We have*

$$\int_\Omega \nabla F \cdot \vec{h}\, dZ \leq CM \|\widetilde{N}(\nabla u_0)\|_{L^q(\partial\Omega)} \|S(v)\|_{L^{q'}(\partial\Omega)},$$

where the constant depends only on the constant in (4.5)*. Hence, thanks to Lemma* 4.1*,*

$$\|\widetilde{N}_K(\nabla F)\|_{L^q(\partial\Omega)} \lesssim M \|\widetilde{N}(\nabla u_0)\|_{L^q(\partial\Omega)} \|S(v)\|_{L^{q'}(\partial\Omega)}.$$

Proof. We claim that

$$\int_\Omega \nabla F \cdot \vec{h}\, dZ = \int_\Omega \mathcal{E} \nabla u_0 \cdot \nabla v\, dm. \tag{4.25}$$

To see how (4.25) proves our lemma, we first observe that for any positive function ϕ on Ω, by Fubini's theorem,

$$\begin{aligned}\int_{\partial\Omega} \Big(\int_{\gamma(x)} \phi(X) \frac{dX}{\delta(X)^{n-1}} \Big) d\sigma(x) &\gtrsim \int_\Omega \phi(X) \delta(X)^{-d} \sigma(8B_X \cap \partial\Omega)\, dm(X) \\ &\gtrsim \int_\Omega \phi(X)\, dm(X)\end{aligned}$$

because, if $\hat{x}$ is such that $|X - \hat{x}| = \delta(X)$, then $\sigma(8B_X) \geq \sigma(\Delta(\hat{x}, \delta(X))) \gtrsim \delta(X)^d$ by (1.1). As a consequence, by successively applying the Cauchy–Schwarz inequality and the Hölder inequality, the claim (4.25) implies

$$\begin{aligned}\int_\Omega \nabla F \cdot \vec{h}\, dZ \leq \int_\Omega |\mathcal{E}||\nabla u_0||\nabla v|\, dm &\lesssim \int_{\partial\Omega} \Big(\int_{\gamma(x)} |\mathcal{E}||\nabla u_0||\nabla v| \frac{dX}{\delta(X)^{1-n}} \Big) d\sigma(x) \\ &\lesssim \Big(\int_{\partial\Omega} \Big(\int_{\gamma(x)} |\mathcal{E}|^2 |\nabla u_0|^2 \frac{dX}{\delta(X)^n} \Big)^{q/2} d\sigma(x) \Big)^{1/q} \|S(v)\|_{L^{q'}(\partial\Omega}.\end{aligned}$$

The lemma follows then from the Carleson inequality (4.5).

So it remains to show the claim (4.25). Formally, the claim is just a permutation of integrals, that is, by using the definition (4.2) of $F(X)$ and (4.19), one has

$$\begin{aligned}\int_\Omega \nabla F \cdot \vec{h}\, dX &= \int_\Omega F \operatorname{div}(\vec{h})\, dX \\ &= \int_\Omega \int_\Omega \nabla_Y G_1(X,Y) \cdot \mathcal{E}(Y) \nabla u_0(Y) \operatorname{div}(\vec{h})(X)\, dm(Y)\, dX \\ &= \int_\Omega \mathcal{E}(Y) \nabla u_0(Y) \cdot \nabla \Big(\int_\Omega G_1(X,Y) \operatorname{div}(\vec{h})(X)\, dX \Big) dm(Y) \\ &= \int_\Omega \mathcal{E} \nabla u_0 \cdot \nabla v\, dm.\end{aligned}$$

However, the assumptions of Fubini's theorem are not satisfied, so the justification will end up being way more delicate.

The issue mainly comes from the Green function G_1, which has a degeneracy when $Z = Y$ that we cannot control very well. So instead, we shall use approximation of the

Green function. We use the same mollifier as the previous lemma. Let $\rho \in C_0^\infty(\mathbb{R}^n)$ supported in $B(0,1)$ and satisfying $\int_{\mathbb{R}^n} \rho = 1$, and then define $\rho_\eta(Z) := \eta^{-n}\rho(\eta^{-1}Z)$ for $\eta > 0$ and $Z \in \Omega$. We also construct a cut-off function $\varphi_\eta \in C_0^\infty(\Omega)$ such that $\varphi_\eta(Z) = 0$ if $\delta(Z) < 2\eta$, $\varphi_\eta(Z) = 1$ if $\delta(Z) > 4\eta$, and $|\nabla\varphi_\eta| \leq \eta^{-1}$. Check that the map $\phi \in W_0 \to \rho_\eta * (\varphi_\eta \phi)(Z)$ lies in $W^{-1} = (W_0)^*$ for all $\eta > 0$ and all $Z \in \Omega$. By the Lax–Milgram theorem (see Lemma 2.8), for each $\eta > 0$ and $X, Y \in \Omega$, we can construct[5] $G_\eta(., Y)$ and $G_\eta^*(., X)$ as the only functions in W_0 such that

$$\int_\Omega \mathcal{A}_1 \nabla_Z G_\eta(Z,Y) \cdot \nabla\phi(Z)\, dm(Z) = \rho_\eta * (\varphi_\eta \phi)(Y) \quad \text{for } \phi \in W_0, \tag{4.26}$$

and similarly,

$$\int_\Omega \mathcal{A}_1 \nabla\phi(Z) \cdot \nabla_Z G_\eta^*(Z,X)\, dm(Z) = \rho_\eta * (\varphi_\eta \phi)(X) \quad \text{for } \phi \in W_0. \tag{4.27}$$

The combination of the two above lines easily gives the nice identity

$$\begin{aligned} \mathcal{G}_\eta(X,Y) &:= [\rho_\eta * (\varphi_\eta G_\eta^*)(., X)](Y) = [\rho_\eta * (\varphi_\eta G_\eta)(., Y)](X) \\ &= \int_\Omega \mathcal{A}_1 \nabla G_\eta(Z,Y) \cdot \nabla G_\eta^*(Z,X)\, dm(Y). \end{aligned} \tag{4.28}$$

Note that the identity implies that the function $\mathcal{G}_\eta$ lies in W_0 and is smooth both in X and Y, which will make $\mathcal{G}_\eta$ a nice tool for the next lines. We define v_ε as in (4.21) and

$$v_{\varepsilon,\eta}(Y) := -\int_\Omega \nabla_X G_\eta(X,Y) \cdot \vec{h}_\varepsilon(X)\, dX.$$

We plug in $G_\eta(., Y)$ as the test function in (4.22) to get

$$v_{\varepsilon,\eta}(Y) = \int_\Omega \mathcal{A}_1^T \nabla v_\varepsilon \cdot \nabla G_\eta(., Y)\, dm = \rho_\eta * (\varphi_\eta v_\varepsilon)(Y)$$

by (4.26). By (4.23), we have that $v_\varepsilon \to v$ in W_0, and a classical convolution result yields that $v_{\varepsilon,\eta}$ converges to $v_{0,\eta} := \rho_\eta * (\varphi_\eta v)$ in W_0 (as $\varepsilon \to 0$ and uniformly in η). The fact that $\nabla v_{0,\eta} \to \nabla v$ in L^2 is also well known. So by a diagonal argument, the function

$$v_{\varepsilon,\varepsilon}(Y) = -\int_\Omega \nabla_X G_\varepsilon(X,Y) \cdot \vec{h}_\varepsilon(X)\, dX = \langle \operatorname{div}(\vec{h}), \mathcal{G}_\varepsilon(., Y)\rangle_{W^{-1},W_0}$$

converges to v in W_0 as $\varepsilon \to 0$. The same proof gives that

$$\begin{aligned} F_{\varepsilon,\varepsilon}(X) &:= \int_\Omega G_\varepsilon^*(Y,X)\, \varphi_\varepsilon(Y) \operatorname{div}(\rho_\varepsilon * (\mathcal{E}\nabla u_0))(Y)\, dm(Y) \\ &= \langle \operatorname{div}(\mathcal{E}\nabla u_0), \mathcal{G}_\varepsilon(X, .)\rangle_{W^{-1},W_0} \end{aligned}$$

converges to F in W_0.

[5] from here and forward, we drop the index 1 on G, since any Green function will always be associated to $\mathcal{L}_1$ or $\mathcal{L}_1^*$.

All these convergences show that the claim (4.25) would be proven once we establish that

$$\int_\Omega \nabla F_{\varepsilon,\varepsilon} \cdot \vec{h}\, dX = \int_\Omega \mathcal{E} \nabla u_0 \cdot \nabla v_{\varepsilon,\varepsilon}\, dm \tag{4.29}$$

for any $\varepsilon > 0$. Observe that if we replace ρ_η by $\rho_\eta * \rho_\eta$ in (4.26) and (4.27), then we replace $G_\eta(Z,Y)$ by $(\rho_\eta * G_\eta(Z,.))(Y)$ and $G^*_\eta(Z,X)$ by $(\rho_\eta * G^*_\eta(Z,.))(X)$. We deduce that we can make $G_\eta(Z,Y)$ and $G^*_\eta(Z,X)$ as smooth as we want in the second variable, and hence quantities like $\nabla_Y(\nabla_X G_\eta)(X,Y)$ make perfect sense and lie in $L^\infty_Y(L^2_X)$. From these remarks and (4.28), the identity (4.29) is just a permutation of integrals and two differentiations under the integral symbol. ∎

4.4. The $S < N$ estimate

The aim of this subsection is to bound $\|S(v)\|_q$ by the non-tangential maximal function of v, $\delta\nabla v$, and a term that depends on $T(\vec{h})$ defined in (4.13).

For the first time, we shall use (4.1), but in a weaker form (see Theorem 2.7) which says that there exist $C, \theta > 0$ such that

$$\frac{\omega^\infty_{1,*}(E)}{\omega^\infty_{1,*}(\Delta)} \le C\Big(\frac{\sigma(E)}{\sigma(\Delta)}\Big)^\theta \quad \text{and} \quad \frac{\sigma(E)}{\sigma(\Delta)} \le C\Big(\frac{\omega^\infty_{1,*}(E)}{\omega^\infty_{1,*}(\Delta)}\Big)^\theta \tag{4.30}$$

for any boundary ball $\Delta \subset \partial\Omega$ and any Borel set $E \subset \Delta$.

In the next lemma, $\mathcal{M}_\omega$ and $\mathcal{M}_\sigma$ are the Hardy–Littlewood maximal functions for the elliptic measure $\omega := \omega^\infty_{1,*}$ and for the Ahlfors regular measure σ. Moreover, S_* and $\widetilde{N}_*$ are respectively the square function and the averaged non-tangential maximal function, but defined with a wider cone $\gamma^*(x) = \{X \in \Omega,\ |X - x| < C_*\delta(X)\}$. The value C_* of the aperture does not matter much, and will be chosen to match our purpose in the next proof. The important and well-known facts are

$$\|N_*(u)\|_p \lesssim \|N(u)\|_p, \quad \|\widetilde{N}_*(u)\|_p \lesssim \|\widetilde{N}(u)\|_p, \quad \text{and} \quad \|S_*(u)\|_p \lesssim \|S(u)\|_p, \tag{4.31}$$

for every $p \in (0,\infty)$ and every u for which the considered quantities make sense (the constants depends on p but not u). The proof of (4.31) – in the case $\partial\Omega = \mathbb{R}^n$ – can be found in Chapter II, equation (25) of [59] for the non-tangential square function, and in Proposition 4 of [12] for the square function. Although the proof is written when $\partial\Omega = \mathbb{R}^n$, it can be easily extended to all doubling metric spaces.

Lemma 4.7. *For the function v constructed in (4.19), define the set*

$$E_{\beta\alpha} := \{N_*(v) + \widetilde{N}_*(\delta|\nabla v|) + \mathcal{M}_\omega(T(\vec{h})) > \beta\alpha\} \subset \partial\Omega. \tag{4.32}$$

There exist $\eta, \beta_0 > 0$ such that for all $\alpha > 0$ and $\beta \in (0,\beta_0)$, we have

$$\sigma\{x \in \partial\Omega,\ S(v)(x) > 2\alpha \quad \text{and} \quad \mathcal{M}_\sigma(\mathbb{1}_{E_{\beta\alpha}})(x) \le 1/2\} \le C\beta^\eta \sigma\{S_*(v) > \alpha\}, \tag{4.33}$$

where C is independent of α and β.

The above "good-λ" argument entails the following L^p bounds.

Corollary 4.8. *For any $p > 0$, we have*

$$\int_{\partial\Omega} |S(v)|^p \, d\sigma \leq C \int_{\partial\Omega} |N(v) + \widetilde{N}(\delta|\nabla v|) + \mathcal{M}_\omega(T(\vec{h}))|^p \, d\sigma.$$

Proof of Corollary 4.8. Let $E_{\beta\alpha}$ be the set defined in Lemma 4.7. Recall that the Hardy–Littlewood maximal operator $\mathcal{M}_\sigma$ is bounded from L^1 to weak-L^1. Then

$$\sigma\{\mathcal{M}_\sigma(\mathbb{1}_{E_{\beta\alpha}}) > 1/2\} \lesssim \int_{\partial\Omega} \mathbb{1}_{E_{\beta\alpha}} \, d\sigma = \sigma(E_{\beta\alpha}). \tag{4.34}$$

According to Lemma 4.7, we have, for $\beta \leq \beta_0$, that

$$\begin{aligned} \sigma\{S(v) > 2\alpha\} &\leq \sigma\{\mathcal{M}_\sigma(\mathbb{1}_{E_{\beta\alpha}}) > 1/2\} + \sigma\{S(v) > 2\alpha, \mathcal{M}_\sigma(\mathbb{1}_{E_{\beta\alpha}}) \leq 1/2\} \\ &\leq \sigma\{\mathcal{M}_\sigma(\mathbb{1}_{E_{\beta\alpha}}) > 1/2\} + C\beta^\eta \sigma\{S_*(v) > \alpha\}. \end{aligned} \tag{4.35}$$

The last two computations imply, for any $p > 0$, that

$$\begin{aligned} \int_{\partial\Omega} |S(v)|^p \, d\sigma &= c \int_0^\infty \alpha^{p-1} \sigma\{S(v) > 2\alpha\} d\alpha \\ &\leq C\beta^\eta \int_0^\infty \alpha^{p-1} \sigma\{S_*(v) > \alpha\} \, d\alpha + C \int_0^\infty \alpha^{p-1} \sigma(E_{\beta\alpha}) \, d\alpha \\ &\leq C\beta^\eta \int_{\partial\Omega} |S_*(v)|^p \, d\sigma + \frac{C}{\beta^p} \Big\{ \int_{\partial\Omega} |\widetilde{N}_*(v) + \widetilde{N}_*(\delta|\nabla v|) + \mathcal{M}_\omega(T_p(\vec{h}))|^p \, d\sigma \Big\} \\ &\leq C'\beta^\eta \int_{\partial\Omega} |S(v)|^p \, d\sigma + \frac{C'}{\beta^p} \Big\{ \int_{\partial\Omega} |\widetilde{N}(v) + \widetilde{N}(\delta|\nabla v|) + \mathcal{M}_\omega(T_p(\vec{h}))|^p \, d\sigma \Big\} \end{aligned} \tag{4.36}$$

by (4.31). Choose $\beta \leq \beta_0$ small enough so that $C'\beta^\eta \leq 1/2$. Hence, we can hide the square function of the last inequality of (4.36) to the left-hand side. The corollary follows. ∎

Proof of Lemma 4.7. Fix $\alpha > 0$. Define

$$\mathcal{S} := \{S_*(v) > \alpha\} \quad \text{and} \quad \mathcal{S}' := \{S(v) > 2\alpha, \mathcal{M}_\sigma(\mathbb{1}_{E_{\beta\alpha}}) \leq 1/2\}.$$

Take any surface ball Δ of radius r. It suffices to show that there exists a constant C such that for any surface ball Δ that intersects $\partial\Omega \setminus \mathcal{S}$, we have

$$\omega^\infty_{1,\infty}(F) \leq C\beta^2 \, \omega^\infty_{1,*}(\Delta), \tag{4.37}$$

where $F := \mathcal{S}' \cap \Delta$. Indeed, the bound (4.30) - which comes from the $L^{q'}$-solvability of the Dirichlet problem for $\mathcal{L}_1^*$ - immediately turns (4.37) into

$$\sigma(F) \leq C\beta^\eta \sigma(\Delta). \tag{4.38}$$

Why is (4.38) enough? Because we can construct a Whitney decomposition of $\mathcal{S}$ in the following way. For any $x \in \mathcal{S}$, we can build the boundary ball $\Delta_x := \Delta(x, \operatorname{dist}(x, \mathcal{S}^c)/40)$. Since the radius of a ball Δ_x that intersects a compact subset of $\mathcal{S}$ is uniformly bounded (depending on the compact), the Vitali covering lemma allows us to obtain a non-overlapping sub-collection $\{\Delta_{x_j}\}$ of $\{\Delta_x\}$ for which $\bigcup_j 5\Delta_{x_j} = \mathcal{S}$. Each ball $50\Delta_{x_j}$ intersects $\partial\Omega \setminus \mathcal{S}$ and so, if $F_j = \mathcal{S}' \cap 50\Delta_{x_j}$, we have by (4.38),

$$\sigma(\mathcal{S}') \leq \sum_j \sigma(F_j) \leq C\beta^\eta \sigma(50\Delta_j) \leq C'\beta^\eta \sigma(\Delta_j) \leq C'\beta^\eta \sigma(\mathcal{S}),$$

which is the desired bound (4.33).

Step 1. Let Δ be a surface ball of radius r that contains a point $x_\Delta \in \partial\Omega \setminus \mathcal{S}$, i.e., a point satisfying $S_*(v)(x_\Delta) \leq \alpha$. We write $F := \Delta \cap \mathcal{S}'$.

Observe that for any $x \in \Delta$ and $X \in \gamma(x) \setminus B(x,r)$, we have

$$|X - x_\Delta| \leq |X - x| + |x - x_\Delta| < 2\delta(X) + 2r < 2\delta(X) + 2|X - x| < 6\delta(X).$$

Consequently, $\gamma(x) \setminus B(x,r) \subset \gamma^*(x)$ as long as the aperture $C_* \geq 6$ (which we choose as such). So if $S^r(v)(x)$ is a truncated square function defined as

$$S^r(v)(x) := \Big(\int_{\gamma(x)\cap B(x,r)} |\nabla v|^2 \frac{dY}{\delta(Y)^{n-2}} \Big)^{1/2},$$

then we easily have

$$|S^r(v)(x)|^2 \geq |S(v)(x)|^2 - |S_*(v)(x_\Delta)|^2 \geq \alpha^2, \quad \text{for } x \in F,$$

that is,

$$S^r(v)(x) \geq \alpha, \quad \text{for } x \in F. \tag{4.39}$$

Step 2. In the sequel, to lighten the notation, we shall write ω for $\omega^\infty_{1,*}$, the elliptic measure with pole at infinity associated to $\mathcal{L}_1^*$. In a similar way, $G(Y)$ will denote the Green function with pole at infinity associated to $\mathcal{L}_1$. Both of them are linked together by Lemma 3.3. Let Ω_F be the saw-tooth region over F defined as $\Omega_F := \bigcup_{x\in F} \gamma(x)$. Then

$$\begin{aligned} \omega(F) &\leq \frac{1}{\alpha^2} \int_F |S^r(v)|^2 \, d\omega \leq \frac{1}{\alpha^2} \int_F \int_{\gamma(x)\cap B(x,r)} |\nabla v|^2 \frac{dY}{\delta(Y)^{n-2}} \, d\omega(x) \\ &\leq \frac{1}{\alpha^2} \int_{\Omega_F \cap \{\delta(Y)\leq r\}} |\nabla v|^2 \, \omega(B(Y, 2\delta(Y)) \cap \partial\Omega) \frac{dY}{\delta(Y)^{n-2}}. \end{aligned} \tag{4.40}$$

If $y \in \partial\Omega$ is a point such that $|Y - y| = \delta(Y)$, then

$$\omega(B(Y, 2\delta(Y)) \cap \partial\Omega) \approx \omega(\Delta(y, \delta(Y)) \approx \delta(Y)^{d-1} G(Y) \tag{4.41}$$

by the doubling property of ω (Lemma 3.4) and then Lemma 3.3. We use the above estimate in (4.40) to obtain

$$\omega(F) \lesssim \frac{1}{\alpha^2} \int_{\Omega_F \cap \{\delta(Y)\leq r\}} |\nabla v|^2 G \, \frac{dY}{\delta(Y)^{n-d-1}}.$$

Let us recall that $dm(Y) = \delta(Y)^{d+1-n}dY$. Together with the ellipticity of matrix $\mathcal{A}_1^T$, we have

$$\omega(F) \lesssim \frac{1}{\alpha^2}\int_{\Omega_F\cap\{\delta(Y)\le r\}} \mathcal{A}_1^T\nabla v\cdot\nabla v\, G\, dm. \tag{4.42}$$

Choose a cut-off function $\phi_F \in C^\infty(\mathbb{R}^n)$ such that, $0 \le \phi_F \le 1$, $\phi_F \equiv 1$ on Ω_F, and ϕ_F is supported on a larger saw-tooth region $\Omega_F^3 := \bigcup_{x\in F}\gamma_3(x)$, with $\gamma_3(x) := \{X\in\Omega,\ |X-x| < 3\delta(X)\}$. In addition, we can always pick the cut-off function ϕ_F so that $|\nabla\phi_F(Y)| \lesssim 1/\delta(Y)$. Pick another smooth function ϕ_r such that $0\le\phi_r\le 1$, $\phi_r\equiv 1$ when $\delta(Y)\le r$ and $\phi_r\equiv 0$ when $\delta(Y)\ge 2r$ and $|\nabla\phi_r|\le 2/r$. Define $\Psi = \phi_F\phi_r$. Then we have

$$|\nabla\Psi(Y)| \lesssim \frac{\mathbb{1}_{D_1}(Y)}{\delta(Y)} + \frac{\mathbb{1}_{D_2}(Y)}{\delta(Y)}, \tag{4.43}$$

where $D_1 := \{Y\in\Omega_F^3\setminus\Omega_F, \delta(Y)\le 2r\}$ and $D_2 := \{Y\in\Omega_F^3, r\le\delta(Y)\le 2r\}$. By the product rule, the term (4.42) can be rewritten as

$$\begin{aligned}\omega(F) &\lesssim \frac{1}{\alpha^2}\int_\Omega \mathcal{A}_1^T\nabla v\cdot\nabla v\,(G\Psi)\,dm\\ &= \frac{1}{\alpha^2}\Big(\int_\Omega \mathcal{A}_1^T\nabla v\cdot\nabla[vG\Psi]\,dm - \int_\Omega \mathcal{A}_1^T\nabla v\cdot\nabla G(v\Psi)\,dm - \int_\Omega \mathcal{A}_1^T\nabla v\cdot\nabla\Psi(Gv)\,dm\Big)\\ &=: \frac{1}{\alpha^2}(\mathrm{I}+\mathrm{II}+\mathrm{III}).\end{aligned}$$

The lemma will be proven once we show that I, II and III are all bounded by $C(\alpha\beta)^2\omega(\Delta)$.

Step 3. *The term* I. As we want to use Lemma 4.5, we need to check that $vG\Psi\in W_0$. We have that $v\in W_0$ (also by Lemma 4.5). Thanks to the elliptic theory recalled in Section 2, we also have that v is Hölder continuous close to the boundary (when we are outside the support of $\vec{h}$) and $G\Psi\in W_0\cap L^\infty(\Omega)$. So in order to get that $vG\Psi\in W_0$, we only need to explain why $v\in L^\infty_{\mathrm{loc}}(\Omega)$. The control of solutions for inhomogeneous Dirichlet problem was not done in [19], but that is fine, because we only require local boundedness inside the domain, so we can use the result from the classical (unweighted) theory, which can be found in Theorem 8.17 of [38]. Now, we apply Lemma 4.5, which entails that

$$\begin{aligned}\mathrm{I} &= -\int_\Omega \vec{h}\cdot\nabla(vG\Psi)\,dY\\ &= -\int_\Omega \vec{h}\cdot\nabla v\,(G\Psi)\,dY - \int_\Omega \vec{h}\cdot\nabla G\,(v\Psi)\,dY - \int_\Omega \vec{h}\cdot\nabla\Psi\,(vG)\,dY := \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.\end{aligned}$$

By the definition of $F\subset\mathcal{S}'$, for any $x\in F$, we have $\mathcal{M}_\sigma(\mathbb{1}_{E_{\beta\alpha}})(x)\le 1/2$. Thus, for any surface ball $\Delta'\subset\partial\Omega$ which contains such a point $x\in F$, we necessary have $\sigma(\Delta'\cap E_{\beta\alpha})/\sigma(\Delta')\le 1/2$. This implies that $\sigma(\Delta'\cap E_{\beta\alpha}^c)/\sigma(\Delta') > 1/2$. The A_∞-absolute continuity (4.30) yields then

$$\omega(\Delta'\cap E_{\beta\alpha}^c)/\omega_*(\Delta') \ge c > 0. \tag{4.44}$$

The comparison (4.41) now entails that

$$|\mathrm{I}_1| \leq \int_{\Omega_F^3} |\vec{h}||\nabla v| G \, dY \lesssim \int_{\Omega_F^3} |\vec{h}||\nabla v| \, \omega(\Delta(y, \delta(Y)))\, \delta(Y)^{1-d} \, dY, \tag{4.45}$$

where we recall that y is a point on the boundary such that $|Y-y| = \delta(Y)$. Since Ω_F^3 is a sawtooth region over F, there exists a constant C_0 ($C_0 = 4$) such that for all $Y \in \Omega_F^3$,

$$F \cap \Delta(y, C_0\,\delta(Y)) \neq \emptyset.$$

Thus, by (4.44),

$$\omega(\Delta(y,\delta(Y))) \leq \omega(\Delta(y, C_0\,\delta(Y))) \lesssim \omega(E_{\beta\alpha}^c \cap \Delta(y, C_0\,\delta(Y))). \tag{4.46}$$

Together with (4.46), (4.45) becomes

$$\begin{aligned} |\mathrm{I}_1| &\lesssim \int_{\Omega_F^3} |\vec{h}||\nabla v| \, \frac{\omega(E_{\beta\alpha}^c \cap \Delta(y, C_0\,\delta(Y)))}{\delta(Y)^{d-1}} \, dY \\ &\lesssim \int_{E_{\beta\alpha}^c \cap C_0'\Delta} \Big(\int_{\gamma_3(x)} \frac{|\nabla v||\vec{h}|}{\delta(Y)^{d-1}} \, dY \Big) \, d\omega(x). \end{aligned} \tag{4.47}$$

Recall that $\gamma_3(x) \subset \gamma_d(x)$, which is used in the construction of $T(\vec{h})$ in (4.13). By Hölder's inequality,

$$\begin{aligned} \int_{\gamma_3(x)} \frac{|\nabla v||\vec{h}|}{\delta(Y)^{d-1}} \, dY &\lesssim \sum_{I \in \mathcal{W}_x} \Big(\fint_I \delta |\nabla v||\vec{h}| \, dY \Big) \ell(I)^{n-d} \\ &\lesssim \sum_{I \in \mathcal{W}_x} \Big(\fint_I \delta^2 |\nabla v|^2 \, dY \Big)^{1/2} \Big(\fint_I |\vec{h}|^2 \, dY \Big)^{1/2} \ell(I)^{n-d} \lesssim \widetilde{N}_*(\delta|\nabla v|)(x) \, T(\vec{h})(x) \end{aligned} \tag{4.48}$$

if we choose the aperture C_* of the cone $\gamma_*(x)$ that defines $\widetilde{N}_*$ big enough so that $I \subset \gamma^*(x)$ for all $I \in \mathcal{W}_x$. When $x \in E_{\beta\alpha}^c$, we have $\widetilde{N}_*(\delta|\nabla v|)(x) \leq \beta\alpha$. Therefore, if x_0 is any point in $E_{\beta\alpha}^c \cap C_0'\Delta$ (if the set is empty, then $\mathrm{I}_1 = 0$ and there is nothing to prove), then (4.47) can be further continued as

$$|\mathrm{I}_1| \lesssim (\beta\alpha) \int_{E_{\beta\alpha}^c \cap C_0\Delta} T(\vec{h}) \, d\omega \lesssim (\beta\alpha)\, \omega(C_0\Delta)\, \mathcal{M}_\omega(T(\vec{h}))(x_0) \lesssim (\beta\alpha)^2 \omega(\Delta), \tag{4.49}$$

thanks to the doubling property of ω (Lemma 3.4) and the fact that $\mathcal{M}_\omega(T_p(\vec{h}))(x_0) \leq \beta\alpha$ for $x_0 \in E_{\beta\alpha}^c$.

The term I_3 is very similar to I_1. Indeed, in I_1, we only use the fact that $0 \leq \Psi \leq 1$ and is supported in Ω_F^*. For I_3, we use the fact that $|\nabla \Psi| \lesssim 1/\delta$ and is supported in Ω_F^*, and we use $N_*(v)$ instead of $\widetilde{N}_*(\delta|\nabla v|)$. So with the same reasoning as of I_1, we also have

$$|\mathrm{I}_3| \lesssim (\beta\alpha)^2 \omega(\Delta). \tag{4.50}$$

The term I_2 is slightly more different from I_1 than I_3 is, so we shall write a bit more. Observe that I_2 is the same as I_1 once you replace ∇v by $v\nabla G/G$. So similarly to (4.47), we have that

$$|\mathrm{I}_2| \lesssim \int_{E^c_{\beta\alpha}\cap C_0'\Delta} \Big(\int_{\gamma_3(x)} \frac{v|\nabla G||\vec{h}|}{G\delta(Y)^{d-1}}\, dY \Big)\, d\omega(x). \tag{4.51}$$

Then analogously to (4.48), we get that

$$\begin{aligned} \int_{\gamma_3(x)} \frac{v|\nabla G||\vec{h}|}{G\delta(Y)^{d-1}}\, dY &\lesssim \sum_{I\in\mathcal{W}_x} \Big(\fint_I v\, \frac{\delta|\nabla G|}{G}\, |\vec{h}|\, dY \Big)\, \ell(I)^{n-d} \\ &\lesssim \sum_{I\in\mathcal{W}_x} \Big(\fint_I v^2\, dY \Big)^{1/2} \Big(\fint_I \frac{\delta^2|\nabla G|^2}{G^2}\, dY \Big)^{1/2} \sup_I |\vec{h}|\, \ell(I)^{n-d}. \end{aligned} \tag{4.52}$$

Since G is a positive solution to $\mathcal{L}^1$, the Harnack inequality (Lemma 2.11) and the Caccioponi inequality (Lemma 2.9) entail that

$$\fint_I \frac{\delta^2|\nabla G|^2}{G^2}\, dY \approx \frac{\delta(X_I)^2}{G(X_I)^2} \fint_I |\nabla G|^2\, dm \lesssim \frac{1}{G(X_I)^2} \fint_{2I} G^2\, dm \approx 1, \tag{4.53}$$

whenever X_I is any point in I. So the bound (4.52) becomes

$$\int_{\gamma_3(x)} \frac{v|\nabla G||\vec{h}|}{G\delta(Y)^{d-1}} dY \lesssim \sum_{I\in\mathcal{W}_x} \Big(\fint_I v^2\, dY \Big)^{1/2} \ell(I)^{n-d} \sup_I |\vec{h}| \lesssim N_*(v)(x) T(\vec{h})(x).$$

We use this last estimate in (4.51) and we conclude that

$$|\mathrm{I}_2| \lesssim (\beta\alpha)^2\, \omega(\Delta). \tag{4.54}$$

Step 4. *Carleson estimates for* $|\nabla\Psi|$. As we shall see, the terms II and III will only involve Ψ via its gradient. So it will be useful to have good estimates on $\delta|\nabla\Psi|$, or on $\mathbb{1}_{D_1\cup D_2}$ (which is bigger by (4.43)). We aim to prove that

$$M_\Delta := \int_{C_0'\Delta} \Big(\sum_{I\in\mathcal{W}_x} \sup_I (\mathbb{1}_{D_1\cup D_2}) \Big)\, d\omega(x) \leq C\omega(\Delta), \tag{4.55}$$

where C_0' is the constant on the right-hand side of (4.47).

Even if the inequality (4.55) is presented in an unusual way, the result is fairly classical. Let us sketch it. By simply switching the integral and the sum, we have

$$M_\Delta \leq \sum_{\substack{I\in\mathcal{W}\\ I\cap D_1\neq\emptyset}} \omega(\Delta_I) + \sum_{\substack{I\in\mathcal{W}\\ I\cap D_2\neq\emptyset}} \omega(\Delta_I) := M_1 + M_2,$$

where $\Delta_I := \Delta(\xi_I, 200\ell(I))$ for a point $\xi_I \in 100I \cap \partial\Omega$ that will be chosen later. If I intersects D_2, then $\ell(I) \approx \mathrm{dist}(I,\Delta) \approx r$: there is a uniformly bounded amount of those cubes, and we also have $\omega(\Delta_I) \approx \omega(\Delta)$. We deduce

$$M_2 := \sum_{\substack{I\in\mathcal{W}\\ I\cap D_2\neq\emptyset}} \omega(\Delta_I) \lesssim \omega(\Delta)$$

as desired. As for J_1, we use the fact that we have some freedom on the choice of ξ_I. If $I \cap D_1 \supset \{X_I\} \neq \emptyset$, then we choose $\xi_I \in \partial\Omega$ such that $|\xi_I - X_I| = \delta(X_I) := r_I$. Note that we necessary have $r_I \leq 60\ell(I)$, so $\xi_I \in 100I$. Recall that $X_I \in D_1$ means that there exists $x_I \in F$ such that $|X_I - x_I| < 3\delta(X_I) = 3|X_I - \xi_I|$ but $|X_I - x| \geq 2|X_I - \xi_I|$ for all $x \in F$. Consequently,

$$10\ell(I) \leq r_I \leq \operatorname{dist}(\xi_I, F) \leq 4r_I. \tag{4.56}$$

When $I \cap D_1 \neq \emptyset$, we define $\tilde{\Delta}_I = \Delta(\xi_I, \ell(I))$ using the ξ_I that we constructed and satisfies (4.56). Notice that the collection $\{\tilde{\Delta}_I\}$ is finitely overlapping, because if $x \in \tilde{\Delta}_I$, then $\ell(I) \approx \operatorname{dist}(x, F)$ and $I \subset B(x, 62\ell(I))$, and there can be only a uniformly finite Whitney cubes with this property. We conclude by writing

$$M_1 := \sum_{\substack{I \in \mathcal{W} \\ I \cap D_1 \neq \emptyset}} \omega(\Delta_I) \lesssim \sum_{\substack{I \in \mathcal{W} \\ I \cap D_1 \neq \emptyset}} \omega(\tilde{\Delta}_I) \lesssim \omega\Big(\bigcup_{\substack{I \in \mathcal{W} \\ I \cap D_1 \neq \emptyset}} \tilde{\Delta}_I \Big) \leq \omega(C_0''\Delta) \lesssim \omega(\Delta).$$

The first and the last inequalities above hold because of the doubling property of ω (Lemma 3.4), the second inequality is due to the finite overlap of $\{\tilde{\Delta}_I\}$, and the third one is a consequence of the fact that all Δ_I (and thus $\tilde{\Delta}$) are included in a dilatation of Δ when I intersects D_1.

Step 5. *The terms* II *and* III. Let us talk about III first. We can repeat the strategy developed in Step 3 for I_1. We use the fact that $\delta \nabla \Psi \leq \mathbb{1}_{D_1 \cup D_2}$ and $dm(Y) = \delta(Y)^{d+1-n} dY$, and similarly to (4.47), we have

$$|\mathrm{III}| \lesssim \int_{E^c_{\beta\alpha} \cap C_0'\Delta} \Big(\int_{\gamma_3(x)} \frac{|\nabla v|\, v\, \mathbb{1}_{D_1 \cup D_2}}{\delta(Y)^{n-1}}\, dY \Big)\, d\omega(x). \tag{4.57}$$

Yet,

$$\begin{aligned} \int_{\gamma_3(x)} \frac{|\nabla v|\, v\, \mathbb{1}_{D_1 \cup D_2}}{\delta(Y)^{n-1}}\, dY &\leq \sum_{I \in \mathcal{W}_x} \int_I \frac{|\nabla v|\, v\, \mathbb{1}_{D_1 \cup D_2}}{\delta(Y)^{n-1}}\, dY \\ &\lesssim \sum_{I \in \mathcal{W}_x} \Big(\fint_I \delta^2 |\nabla v|^2\, dY \Big)^{1/2} \Big(\fint_I v^2\, dY \Big)^{1/2} \sup_I (\mathbb{1}_{D_1 \cup D_2}) \\ &\lesssim \widetilde{N}_*(\delta|\nabla v|)(x)\, N_*(v)(x) \sum_{I \in \mathcal{W}_x} \sup_I (\mathbb{1}_{D_1 \cup D_2}) \\ &\leq (\alpha\beta)^2 \sum_{I \in \mathcal{W}_x} \sup_I (\mathbb{1}_{D_1 \cup D_2}) \end{aligned}$$

when $x \in E^c_{\beta\alpha}$. We conclude that

$$|\mathrm{III}| \lesssim (\alpha\beta)^2 \int_{C_0'\Delta} \Big(\sum_{I \in \mathcal{W}_x} \sup_I (\mathbb{1}_{D_1 \cup D_2}) \Big)\, d\omega(x) \lesssim (\alpha\beta)^2\, \omega(\Delta)$$

by (4.55) and the doubling property of ω (Lemma 3.4).

For II, we want to use the fact that G is a solution to $\mathcal{L}_1$, so we write

$$\begin{aligned}\mathrm{II} &= -\int_\Omega \nabla v \cdot \mathcal{A}_1 \nabla G(v\Psi)\, dm \\ &= -\frac{1}{2}\int_\Omega \nabla(v^2\Psi)\cdot \mathcal{A}_1\nabla G\, dm + \int_\Omega v^2 \mathcal{A}_1 \nabla G \cdot \nabla\Psi\, dm =: \mathrm{II}_1 + \mathrm{II}_2.\end{aligned}$$

The discussion at the beginning of Step 3 shows that $v \in W_0 \cap L^\infty(\operatorname{supp}\Psi)$. So $v^2\Psi$ lies in W_0 and it is compactly supported in $\mathbb{R}^n$. Consequently, $v^2\Psi$ is a valid test function for G, and thus $\mathrm{II}_1 = 0$. Hence it remains to bound II_2, which is actually similar to III. Following again the same strategy, replacing $|\nabla v|$ by $v|\nabla G|/G$ in the argument of III, we have

$$|\mathrm{II}_2| \lesssim \int_{E^c_{\beta\alpha}\cap C_0'\Delta} \Big(\int_{\gamma_3(x)} \frac{v^2\,|\nabla G|\,\mathbb{1}_{D_1\cup D_2}}{G\delta(Y)^{n-1}}\, dY\Big)\, d\omega(x),$$

and when $x \in E^c_{\alpha\beta}$,

$$\begin{aligned}\int_{\gamma_3(x)} \frac{v^2|\nabla G|\mathbb{1}_{D_1\cup D_2}}{G\delta(Y)^{n-1}} dY &\lesssim \sum_{I\in\mathcal{W}_x}\Big(⨍_I \frac{\delta^2|\nabla G|^2}{G^2}\, dY\Big)^{1/2}\Big(⨍_I v^4 dY\Big)^{1/2} \sup_I(\mathbb{1}_{D_1\cup D_2}) \\ &\lesssim |N_*(v)(x)|^2 \sum_{I\in\mathcal{W}_x}\sup_I(\mathbb{1}_{D_1\cup D_2}) \leq (\alpha\beta)^2 \sum_{I\in\mathcal{W}_x}\sup_I(\mathbb{1}_{D_1\cup D_2}),\end{aligned}$$

where we used (4.53) for the second inequality. With a similar reasoning as the one used on III, we conclude that

$$|\mathrm{II}| = |\mathrm{II}_2| \lesssim (\alpha\beta)^2\omega(\Delta)$$

thanks to (4.55). The lemma follows. ■

4.5. Bounds on $N(v)$ and $\tilde{N}(\delta\nabla v)$

In order to finish the proof Theorem 1.3, we need to bound $N(v)$ and $\widetilde{N}(\delta\nabla v)$ by $T(\vec{h})$. We shall observe first that the bound on $\widetilde{N}(\delta\nabla v)$ is just a consequence of the bound on $N(v)$ because of the following Caccioppoli-type inequality.

Lemma 4.9. *For any $X \in \Omega$, we have*

$$\Big(⨍_{B_X} \delta^2|\nabla v|^2\, dX\Big)^{1/2} \lesssim \Big(\int_{2B_X} |v|^2\, dX\Big)^{1/2} + \delta(X)^{n-d}\Big(⨍_{2B_X} |\vec{h}|^2\, dY\Big)^{1/2}, \tag{4.58}$$

where v is constructed in (4.19).

Proof. Take $X \in \Omega$ and construct a cut-off function $\Psi \in C_0^\infty(\Omega)$ such that $0 \leq \Psi \leq 1$, $\Psi \equiv 1$ on B_X, $\Psi \equiv 0$ outside $2B_X$, and $|\nabla\Psi| \lesssim 1/\delta(X)$. By the ellipticity of $\mathcal{A}^T$, we have

$$\begin{aligned}T &:= \int_\Omega |\nabla v|^2\,\Psi^2\, dm \lesssim \int_\Omega \mathcal{A}_1^T\nabla v\cdot\nabla v\,\Psi^2\, dm \\ &= \int_\Omega \mathcal{A}_1^T\nabla v\cdot\nabla[v\Psi^2]\, dm - \int_\Omega \mathcal{A}_1^T\nabla v\cdot\nabla\Psi\,\Psi v\, dm =: T_1 + T_2.\end{aligned} \tag{4.59}$$

We want to use the fact that v is a solution to $\mathcal{L}_1^* v = \operatorname{div} \vec{h}$. Observe that $v\Psi^2$ lies in W_0, hence it is a valid test function, because $v \in W_0$ lies in $L^\infty_{\rm loc}$ (we refer to the discussion at the beginning of Step 3 of the proof of Lemma 4.7) and $\Psi \in C_0^\infty$. Lemma 4.5 entails

$$\begin{aligned} T_1 &= -\int_\Omega \vec{h} \cdot \nabla(v\Psi^2)\, dY = -\int_\Omega \vec{h} \cdot \nabla v\, \Psi^2\, dY - \int_\Omega \vec{h} \cdot \nabla \Psi\, \Psi v\, dY \\ &\lesssim \Big(\delta(X)^{n-d-1} \int_\Omega |\vec{h}|^2 \Psi^2 dY\Big)^{1/2} \Big[\Big(\int_\Omega |\nabla v|^2\, \Psi^2\, dm\Big)^{1/2} + \Big(\int_\Omega |v|^2\, |\nabla \Psi|^2\, dm\Big)^{1/2}\Big] \end{aligned}$$

by the Cauchy–Schwarz inequality and $dm(Y) \approx \delta(X)^{d+1-n} dY$ when $Y \in \operatorname{supp} \Psi \subset 2B_X$. We use the fact that Ψ is supported on $2B_X$ and $|\nabla \Psi| \lesssim 1/\delta(X)$ to further have

$$|T_1| \lesssim \Big(\delta(X)^{n-d-1} \int_{2B_X} |\vec{h}|^2\, dY\Big)^{1/2} \Big[T^{1/2} + \delta(X)^{-1}\Big(\int_{2B_X} |v|^2\, dm\Big)^{1/2}\Big] \tag{4.60}$$

Similarly, T_2 is bounded using the Cauchy–Schwarz inequality and the properties of Ψ by

$$|T_2| \lesssim T^{1/2} \delta(X)^{-1} \Big(\int_{2B_X} |v|^2\, dm\Big)^{1/2}. \tag{4.61}$$

Finally, by applying the estimates (4.60) and (4.61) to (4.59), we deduce that $T \lesssim A^{1/2} T^{1/2} + A$, where

$$A := \delta(X)^{-1} \int_{2B_X} |v|^2\, dX + \delta(X)^{n-d-1} \fint_{2B_X} |\vec{h}|^2\, dY.$$

Since all the quantities that we considered are finite, this bound on T self improves to $T \lesssim A$. The lemma follows easily. ■

Lemma 4.10. *Let v be the weak solution constructed in (4.19). Then*

$$N(v) + \widetilde{N}(\delta |\nabla v|) \leq C\, \mathcal{M}_\omega(T(\vec{h})), \tag{4.62}$$

where $\mathcal{M}_\omega$ is the Hardy–Littlewood maximal function with respect to $\omega := \omega^\infty_{1,*}$, the elliptic measure with pole at infinity associated to $\mathcal{L}_1^*$.

Proof. Fix $x_0 \in \partial\Omega$ and then $X \in \gamma_*(x_0)$, where $\gamma_*(x)$ is a cone with a bigger aperture so that $\bigcup_{Y \in \gamma(x)} 2B_Y \subset \gamma_*(x)$. We want to show that

$$|\vec{h}(X)|\, \delta(X)^{n-d} \lesssim \mathcal{M}_\omega(T(\vec{h}))(x_0). \tag{4.63}$$

and

$$v(X) \lesssim \mathcal{M}_\omega(T(\vec{h}))(x_0). \tag{4.64}$$

Indeed, once these two estimates are proven, then the bound $\widetilde{N}(\delta|\nabla v|) \lesssim \mathcal{M}_\omega(T(\vec{h}))$ will follow thanks to Lemma 4.9. The bound (4.63) is also fairly immediate. Take x such that $|X - x| = \delta(X)$, and check that $X \in \gamma(y)$ for any y in a small boundary ball $\Delta(x, c\delta(X))$. Hence, we easily have $|\vec{h}(X)|\delta(X)^{n-d} \leq T(\vec{h})(y)$ for $y \in \Delta(x, c\delta(X))$ by definition of $T(\vec{h})$, and $\Delta(x, c\delta(X)) \subset \Delta(x_0, C\delta(X))$ for C large enough depending only on the aperture of $\gamma_*(x_0)$. The inequality (4.63) follows.

It remains to show (4.64). By definition,

$$v(X) := -\int_\Omega \nabla_Y G(Y,X) \cdot \vec{h}(Y)\, dY,$$

where $G(Y,X)$ is the Green function with pole at X associated to $\mathcal{L}_1$. We shall treat differently the cases where Y is close to X and far from X. We define S_X as the union of Whitney cubes $I \in \mathcal{W}$ (constructed in Subsection 4.1) for which $3I \ni X$. The function $v(X)$ can be decomposed as

$$v(X) = -\int_{\Omega\setminus S_X} \nabla_Y G(Y,X) \cdot \vec{h}(Y)\, dY - \int_{S_X} \nabla_Y G(Y,X) \cdot \vec{h}(Y)\, dY := \widetilde{v} + v_0. \tag{4.65}$$

Step 1. *Bound on* $\widetilde{v}$. By definition of S_X, we have

$$\begin{aligned} |\widetilde{v}(X)| &= \Bigg| \sum_{\substack{I\in\mathcal{W}\\ X\notin 3I}} \int_I \nabla_Y G(Y,X)\cdot \vec{h}(Y)\, dY \Bigg| \\ &\leq \sum_{\substack{I\in\mathcal{W}\\ X\notin 3I}} \Big(\int_I |\nabla_Y G(Y,X)|^2\, dY\Big)^{1/2} \Big(\int_I |\vec{h}(Y)|^2\, dY\Big)^{1/2} \\ &\lesssim \sum_{\substack{I\in\mathcal{W}\\ X\notin 3I}} \ell(I)^{-1} \Big(\int_{2I} G(Y,X)^2\, dY\Big)^{1/2} \Big(\int_I |\vec{h}(Y)|^2\, dY\Big)^{1/2} \end{aligned} \tag{4.66}$$

by Hölder's inequality, and then by Caccioppoli's inequality (see Lemma 2.9, that we can use because $G(.,X)$ is a solution on $2I$).

We want now to estimate $G(Y,X)$. Pick a point Y_I in I. By the Harnack inequality (Lemma 2.11), we have $G(Y,X) \approx G(Y_I,X)$ for all $Y \in 2I$. So (4.66) becomes

$$|\widetilde{v}(X)| \lesssim \sum_{\substack{I\in\mathcal{W}\\ X\notin 3I}} \ell(I)^{n-1} G(Y_I,X) \sup_I |\vec{h}|. \tag{4.67}$$

Our next objective is (4.68). We give the details, but a reader who is an expert in the elliptic theory may want to skip them. First, we shall introduce several notations. For $j \geq 1$, let us denote by $\Delta_j := \Delta(x_0, 2^j\delta(X))$ the boundary balls and X_j the corkscrew points associated to Δ_j. It is also fair to pick $X_1 := X$. We partition $\mathcal{W}$ into $\bigcup_{j\geq 1} \mathcal{W}_j$, where

$$\mathcal{W}_1 := \{I \in \mathcal{W},\ |Y_I - X| \leq 2\delta(X)\},$$

and for $j \geq 2$,

$$\mathcal{W}_j := \{I \in \mathcal{W},\ 2^{j-1}\delta(X) < |Y_I - X| \leq 2^j\delta(X)\}.$$

Observe that we can find an integer a that depends only on n and the aperture of $\gamma_*(x_0)$ such that we have

$$\begin{aligned} 2I &\in B(x_0, 2^{j+a}\delta(X)) \quad \text{for } I \in \mathcal{W}_j,\ j \geq 1,\\ \text{and} \quad 2I &\cap B(x_0, 2^{j-a}\delta(X)) = \emptyset \quad \text{for } I \in \mathcal{W}_j,\ j \geq 2. \end{aligned}$$

So, for each $j \geq 1$, we take j_- to be the biggest value for which X_{j_-} stays within $B(x_0, 2^{j-a-1}\delta(X))$, and $j_- = 1$ if there are none, and we take j_+ to be the smallest value for which X_{j_+} is outside $B(x_0, 2^{j+a+1})$. Note that by construction, $|j - j_-| + |j - j_+| \lesssim 1$. When $I \in \mathcal{W}_j$, the function $G(Y_I, .)$ is a solution on $B(x_0, 2^{j+a}\delta(X))$. Therefore, the Hölder continuity at the boundary (Lemma 2.12) entails that

$$G(Y_I, X) \lesssim 2^{-j\alpha} G(Y_I, X_{j_-}).$$

Our choice of X_{j_-} and X_{j_+} allows the construction of a Harnack chain of balls of (uniformly) finite length that links X_{j_-} to X_{j_+} and avoids B_{Y_I}. So by the Harnack inequality (Lemma 2.11), the above estimate is equivalent to

$$G(Y_I, X) \lesssim 2^{-j\alpha} G(Y_I, X_{j_+}).$$

Lemma 3.3 implies now that

$$G(Y_I, X_{j_+}) \approx \frac{G(Y_I)}{\omega(\Delta_j)},$$

where G is the Green function with pole at infinity associated to $\mathcal{L}_1$. If $\Delta_I := \Delta(\xi_I, \ell(I))$ (with ξ_I such that $|Y_I - \xi_I| = \delta(Y_I)$), we have by Lemma 3.3 that

$$G(Y_I) \approx \ell(I)^{1-d}\,\omega(\Delta_I).$$

Altogether, our discussion of $G(Y_I, X)$ proves that

$$G(Y_I, X) \lesssim 2^{-j\alpha}\,\ell(I)^{1-d}\,\frac{\omega(\Delta_I)}{\omega(\Delta_j)} \quad \text{when } I \in \mathcal{W}_j. \tag{4.68}$$

We inject our estimate (4.68) in (4.67) to obtain that

$$|\widetilde{v}(X)| \lesssim \sum_{j\geq 1} \frac{2^{-j\alpha}}{\omega(\Delta_j)} \sum_{I\in\mathcal{W}_j} \omega(\Delta_I)\,\ell(I)^{n-d} \sup_I |\vec{h}|.$$

Since $x \in \Delta_I$ implies that $I \in \mathcal{W}_x$, by Fubini's theorem, we have that

$$\sum_{I\in\mathcal{W}_j} \omega(\Delta_I)\ell(I)^{n-d} \sup_I |\vec{h}| \lesssim \int_{C\Delta_j} T(\vec{h})(x)\,d\omega(x)$$

and thus, thanks to the doubling property of ω (Lemma 3.4),

$$|\widetilde{v}(X)| \lesssim \sum_{j\geq 1} 2^{-j\alpha} \fint_{C\Delta_j} T(\vec{h})\,d\omega \lesssim \mathcal{M}_\omega(T(\vec{h}))(x_0),$$

which is our desired bound on $\widetilde{v}$.

Step 2. *Bound on* v_0. It remains to bound the term $\int_{S_X} \nabla_Y G(Y,X)\cdot \vec{h}(Y)dY$ in (4.65). Since $dY \approx \delta(X)^{n-d-1}\, dm(Y)$ on S_X, the bound (iv) of Lemma 2.16 shows that

$$\int_{S_X} |\nabla_Y G(Y,X)|\, dY \lesssim \delta(X)^{n-d-1} \int_{S_X} |\nabla_Y G(Y,X)|\, dm(Y) \lesssim \delta(X)^{n-d}.$$

Therefore, we have

$$|v_0(X)| \lesssim \sum_{\substack{I\in\mathcal{W}\\ 3I\ni X}} \ell(I)^{n-d} \sup_I |\vec{h}|.$$

For each $I \in \mathcal{W}$, we can pick any point $Y_I \in I$ as before and then y_I such that $|Y_I - y_I| = \delta(Y_I)$. It is fairly easy to see that $I \in \mathcal{W}_x$ for all $x \in \Delta(y_I, c\ell(I))$, with c small enough independent of I, and thus $\ell(I)^{n-d} \sup_I |\vec{h}| \leq T(\vec{h})(x)$ for $x \in \Delta(y_I, c\ell(I))$. From this we infer that

$$\ell(I)^{n-d} \sup_I |\vec{h}| \leq \fint_{\Delta(y_I, c\ell(I))} T(\vec{h})\, d\omega.$$

If $X \in 3I \cap \gamma_*(x_0)$, we necessary have $\Delta(y_I, c\ell(I)) \subset \Delta(x_0, C\ell(I))$ for C large enough. By the doubling property of ω (Lemma 3.4), we obtain

$$\ell(I)^{n-d} \sup_I |\vec{h}| \lesssim \fint_{\Delta(x_0, C\ell(I))} T(\vec{h})\, d\omega \leq \mathcal{M}_\omega(T(\vec{h}))(x_0).$$

Since the number of Whitney cubes $I \in \mathcal{W}$ for which $3I \ni X$ is (uniformly) finite, we can conclude that

$$|v_0(X)| \lesssim \mathcal{M}_\omega(T(\vec{h}))(x_0)$$

as desired. The lemma follows. ■

A. The regularity problem implies the Dirichlet problem

This section is devoted to the proof of Theorem 1.5. We shall follow closely the proof of Theorem 5.4 in [49]. Note that when the operator is the Laplacian and the domain does not have Harnack chains, this result was proved by Mourgoglou and Tolsa as Theorem 1.5 in [56]. Since the existing literature does not cover operators more general than the Laplacian, we decided to rewrite a proof in our context.

In all this section, we assume that Ω is a uniform domain (see Definition 2.3), and that $\mathcal{L} = -\operatorname{div}[w\mathcal{A}\nabla]$ is an elliptic operator satisfying (1.3).

The following Poincaré inequality will needed.

Lemma A.1. *For any* $\alpha \in [0,1)$, *any* $x \in \partial\Omega$, *any* $r > 0$, *and a function* $u \in W(B(x,2r))$ *satisfying* $\operatorname{Tr}(u) \equiv 0$ *on* $\Delta(x,2r)$, *we have*

$$\text{(A.1)} \qquad \int_{B(x,r)\cap\Omega} |u(Y)|^2 \delta(Y)^\alpha\, dm(Y) \leq C_\alpha r^2 \int_{B(x,2r)\cap\Omega} |\nabla u(Y)|^2 \delta(Y)^\alpha\, dm(Y),$$

where C_α *depends on* (*the uniform constants of*) Ω *and* α.

Proof. Define $dm'(X) := \delta(X)^\alpha \, dm(X)$ on Ω, and then check that the triple (Ω, m', σ) satisfies the assumptions (H1) to (H6) from [20]. The result is then a consequence of Theorem 7.1 in [20]. ∎

Lemma A.2. *Let $u \in W$ be a non-negative weak solution to $\mathcal{L}u = 0$ such that* $\operatorname{Tr}(u) \equiv 0$ *on $\Delta(x,r)$. Then, for each $X \in \Omega$ such that $|X - x| \approx \delta(X) \approx r$, we have*

$$\frac{u(X)}{r} \approx \Big(\fint_{B(x,r/2)\cap\Omega} |\nabla u(Y)|^2 \, dm(Y) \Big)^{1/2} \lesssim \widetilde{N}_*(\nabla u)(x). \tag{A.2}$$

Here $\widetilde{N}_*$ is defined with cones $\gamma^*(x) := \{X \in \Omega,\ |X - x| \leq C^* \delta(X)\}$ of large aperture. Besides, C^* and the implicit constants in (A.2) depend only on the uniform constants of Ω and the constants in $|X - x| \approx \delta(X) \approx r$.

Proof. *Step* 1. We have that

$$r^2 \fint_{B(x,r/2)\cap\Omega} |\nabla u(Y)|^2 \, dm(Y) \lesssim u(X)^2. \tag{A.3}$$

Indeed, since $\operatorname{Tr}(u) = 0$ on $\Delta(x, \delta(X))$, the above bound is due to two basic results from [20]: Lemma 11.15 (Caccioppoli's inequality at the boundary) and Lemma 15.14, which, used in this order, give that

$$r^2 \fint_{B(x,r/2)\cap\Omega} |\nabla u(Y)|^2 \, dm \lesssim \fint_{B(x,3r/4)\cap\Omega} |u(Y)|^2 \, dm \lesssim |u(X)|^2.$$

Step 2. We claim that for any $\alpha \in [0,1)$, we have

$$u(X)^2 \lesssim r^{2-\alpha} \fint_{B(x,r/2)\cap\Omega} |\nabla u(Y)|^2 \delta(Y)^\alpha \, dm(Y). \tag{A.4}$$

Let $X' \in \Omega \cap B(x, r/8)$ be such that $\delta(X') \approx r$; such point exists because Ω satisfies the corkscrew point condition (see Definition 2.1). Thanks to the Harnack chain condition (Definition 2.2) and the Harnack inequality (Lemma 2.11), we have $u(X) \approx u(Y)$ for any $Y \in B_{X'}$. So we obtain that

$$\begin{aligned} u(X)^2 &\lesssim \fint_{B_{X'}} |u(Y)|^2 \, dm \approx r^{-\alpha} \fint_{B_{X'}} |u(Y)|^2 \delta(Y)^\alpha \, dm \\ &\lesssim r^{-\alpha} \fint_{B(x,r/4)\cap\Omega} |u(Y)|^2 \delta(Y)^\alpha \, dm \lesssim r^{2-\alpha} \fint_{B(x,r/2)\cap\Omega} |\nabla u(Y)|^2 \delta(Y)^\alpha \, dm(Y), \end{aligned}$$

where we used the Poincaré inequality (Lemma A.1), and we can because $\operatorname{Tr}(u) = 0$ on $\Delta(x, r/2)$.

Step 3. *Conclusion.* The equivalence in (A.2) is the combination of (A.3) and (A.4) for $\alpha = 0$. It remains to prove the second bound in (A.2), that is,

$$\Big(\fint_{B(x,r/2)\cap\Omega} |\nabla u(Y)|^2 \, dm(Y) \Big)^{1/2} \lesssim \widetilde{N}_*(\nabla u)(x), \tag{A.5}$$

but this bound is an immediate consequence of

$$\int_{B(x,r/2)\cap\Omega} |\nabla u(Y)|^2\, dm(Y) \lesssim \int_{B(x,r/2)\cap\Omega,} \mathbb{1}_{\delta(Y)>\varepsilon_* r} |\nabla u(Y)|^2\, dm(Y), \tag{A.6}$$

where ε_* is a small constant that depends only on the uniform constants of Ω, because the right-hand side of (A.6) is bounded by $|\widetilde{N}_*(\nabla u)(x)|^2$ if C^* is large enough (depending on ε_*).

In order to establish (A.6), observe that (A.3) and (A.4) give that

$$\int_{B(x,r/2)\cap\Omega} |\nabla u(Y)|^2\, dm(Y) \lesssim r^{-\alpha} \int_{B(x,r/2)\cap\Omega} |\nabla u(Y)|^2 \delta(Y)^\alpha\, dm(Y)$$

and thus

$$\begin{aligned}\int_{B(x,r/2)\cap\Omega} |\nabla u(Y)|^2\, dm(Y) \leq C(\varepsilon_*)^\alpha \int_{B(x,r/2)\cap\Omega} \mathbb{1}_{\delta(Y)\leq\varepsilon_* r} |\nabla u(Y)|^2\, dm(Y) \\ + C \int_{B(x,r/2)\cap\Omega} \mathbb{1}_{\delta(Y)>\varepsilon_* r} |\nabla u(Y)|^2\, dm(Y).\end{aligned}$$

We choose $\alpha = \varepsilon_*/2 > 0$ such that $C(\varepsilon_*)^\alpha \leq 1/2$, so that we can hide the integral over $B(x,r/2)\cap\Omega\cap\{\delta(Y)\leq\varepsilon_* r\}$ in the left-hand side. The claim (A.6) and thus the lemma follow. ■

We are know ready for the proof of Theorem 1.5.

Proof of Theorem 1.5. Suppose that the regularity problem (defined using Hajłasz–Sobolev spaces) for $\mathcal{L}$ is solvable in L^q. Let ω_* be the harmonic measure with pole at infinity associated to $\mathcal{L}^*$, that is defined in Definition 3.1. By Corollary 3.6, in order to show the Dirichlet problem for $\mathcal{L}^*$ is solvable in $L^{q'}$, it suffices to show $\omega_* \ll \sigma$ and $k := d\omega_*/d\sigma$ satisfies the reverse Hölder inequality of order q.

Step 1. Thanks to the Ahlfors regularity of $\partial\Omega$, for any boundary ball Δ, there exists K (that depends only on the constant C_σ in (1.1) such that $K\Delta\setminus 3\Delta \neq \emptyset$.

Let $\Delta := \Delta(x,r)$ be a surface ball on $\partial\Omega$. We construct f on $\partial\Omega$ as

$$f(y) := \max\Big\{0, 1 - \frac{\operatorname{dist}(y, K\Delta\setminus 3\Delta)}{r}\Big\}. \tag{A.7}$$

Note that f is a non-negative function with $f\equiv 0$ on 2Δ and $\partial\Omega\setminus(K+1)\Delta$ and $f\equiv 1$ on $K\Delta\setminus 3\Delta$. The function f is Lipschitz, and if we define g on $\partial\Omega$ as $g = \frac{1}{r}\mathbb{1}_{(K+1)\Delta}$, we easily have that

$$|f(y)-f(z)| \leq |y-z|(g(y)+g(z)).$$

We deduce that g is a generalized (or Hajłasz upper) gradient of f, and thus the Hajłasz–Sobolev norm of f satisfies

$$\|f\|_{\dot W^{1,q}} \leq C\, r^{d/q-1}, \tag{A.8}$$

where C depends only on the Ahlfors regular constant C_σ.

Let u be defined from f as in (1.4), that is,

$$u(X) := \int_{\partial\Omega} f(y)\, d\omega_{\mathcal{L}}^X(y).$$

Let $X_0 \in \Omega$ be a corkscrew point for Δ, then

$$u(X_0) \approx 1. \tag{A.9}$$

Indeed, the upper bound is 1 and comes from the fact that ω^X is a probability measure. The lower bound comes from the non-degeneracy of the harmonic measure: since by definition of K, the set $K\Delta \setminus 3\Delta$ is non-empty, we can take $y \in K\Delta \setminus 3\Delta$, and then Y_0 a corkscrew point of $\Delta(y,r)$. The non-degeneracy of the harmonic measure (see for instance Lemma 15.1 in [20]) gives that $u(Y_0) \gtrsim 1$, because f is nonnegative and $f \geq 1/2$ on $\Delta(y, r/2)$. But Y_0 and X_0 can be linked by a Harnack chain, so the Harnack inequality (Lemma 2.11) entails that $u(X_0) \gtrsim 1$ as well.

Step 2. In this step, we claim that for any $y \in \Delta(x,r)$, any $0 < s < r/2$, and any $z \in \Delta(y,s)$,

$$\frac{\omega_*(\Delta(y,s))}{\sigma(\Delta(y,s))} \approx \frac{\omega_*(\Delta(x,r))}{r^{d-1}}\, \widetilde{N}_*(\nabla u)(z). \tag{A.10}$$

Let $G(.,.)$ and G^∞ be the Green function and the Green function with pole at infinity respectively; in particular, $G(.,Y)$ and G^∞ are solutions to $\mathcal{L}u = 0$. Both G^∞ and $u(.)$ are non-negative solutions for which $\mathrm{Tr}(u) = \mathrm{Tr}(G^\infty) = 0$ on 2Δ, so by the comparison principle (Theorem 2.21) and (A.9), we have,

$$\frac{u(Y)}{G^\infty(Y)} \approx \frac{u(X_0)}{G^\infty(X_0)} \approx \frac{1}{G^\infty(X_0)} \quad \text{for } Y \in B(x, 3r/2) \cap \Omega. \tag{A.11}$$

In addition, according to Lemma 3.3,

$$G^\infty(X_0) \approx r^{1-d}\, \omega_*(\Delta(x,r)). \tag{A.12}$$

Then combining (A.11), (A.12) and Lemma 3.3 again, we obtain

$$\frac{u(Y)}{\delta(Y)} \approx \frac{G^\infty(Y)}{\delta(Y)} \frac{r^{d-1}}{\omega_*(\Delta(x,r))} \approx \frac{\omega_*(\Delta(y,s))}{s^d} \frac{r^{d-1}}{\omega_*(\Delta(x,r))}, \tag{A.13}$$

where $s \approx \delta(Y)$ and $|Y - y| \lesssim s$. So if at the contrary we choose any $y \in \Delta$ and $0 < s < r/2$, we take $Y \in B(y,s) \cap \Omega$ to be such that $\delta(Y) \gtrsim s$, (A.13) and Lemma A.2 entail

$$\frac{\omega_*(\Delta(y,s))}{s^d} \approx \frac{\omega_*(\Delta(x,r))}{r^{d-1}} \frac{u(Y)}{\delta(Y)} \approx \frac{\omega_*(\Delta(x,r))}{r^{d-1}}\, N_*(\nabla u)(z). \tag{A.14}$$

The claim (A.10) follows for the Ahlfors regularity of σ.

Step 3. Assume that $E \subset \Delta$ and $\sigma(E) = 0$. Since ω_* is Borel regular, we have that $\omega_*(E) = \inf_{\substack{V \supset E \\ V \text{ open}}} \omega_*(V)$. For each open set V, we cover it by the balls $\{B_y :=$

$B(x, \operatorname{dist}(y, \partial\Omega \setminus V))\}_{y \in V}$, and using Vitali's covering lemma, we find a sequence $\{y_i\}_{i \in V}$ such that B_{y_i} are not overlapping while $5B_{y_i}$ covers V. By using (A.10) on the balls $5B_{y_i}$, we find that

$$\omega_*(5B_{y_i}) \approx \frac{\omega_*(\Delta(x,r))}{r^{d-1}} \int_{B_{y_i}} \widetilde{N}_*(\nabla u)(z)\, d\sigma(z),$$

and then

$$\omega_*(V) \approx \frac{\omega_*(\Delta(x,r))}{r^{d-1}} \int_{V} \widetilde{N}_*(\nabla u)(z)\, d\sigma(z).$$

Since we assume that the regularity problem is solvable in L^q, the function $\widetilde{N}(\nabla u)$ lies in $L^q(\partial\Omega, \sigma)$, and so by (4.31), the function $N_*(\nabla u)$ lies in $L^q(\partial\Omega, \sigma)$. We invoke the Borel regularity of σ to deduce that

$$\omega_*(E) = \inf_{\substack{V \supset E \\ V \text{ open}}} \int_V \widetilde{N}_*(\nabla u)\, d\sigma = \int_E \widetilde{N}_*(\nabla u)\, d\sigma = 0.$$

We conclude that $\omega_* \ll \sigma$.

Step 4. We have shown that $\omega_* \ll \sigma$, and therefore the Radon–Nykodym derivative $k := d\omega_*/d\sigma$ exists. Moreover, (A.10) implies for any $y \in \Delta$ that

$$k(y) := \lim_{s \to 0} \frac{\omega_*(\Delta(y,s))}{\sigma(\Delta(y,s))} \lesssim \widetilde{N}_*(\nabla u)(y).$$

As a consequence,

$$\begin{aligned}\Big(\fint_\Delta k^q\, d\sigma\Big)^{1/q} &\lesssim \frac{\omega_*(\Delta(x,r))}{r^{d-1}} \Big(\fint_\Delta |N_*(\nabla u)|^q\, d\sigma\Big)^{1/q} \\ &\lesssim \frac{\omega_*(\Delta(x,r))}{r^{d-1}} r^{-d/q} \|\widetilde{N}_*(\nabla u)\|_{L^q(\partial\Omega,\sigma)}.\end{aligned}$$

But by using successively (4.31), the solvability of the regularity problem in L^q, and (A.8), we obtain

$$\|\widetilde{N}_*(\nabla u)\|_{L^q(\partial\Omega,\sigma)} \lesssim \|\widetilde{N}(\nabla u)\|_{L^q(\partial\Omega,\sigma)} \lesssim \|f\|_{\dot{W}^{1,q}} \lesssim r^{d/q-1}.$$

The two last computations show that

$$\Big(\fint_\Delta k^q\, d\sigma\Big)^{1/q} \lesssim \frac{\omega_*(\Delta)}{r^d} \approx \frac{\omega_*(\Delta)}{\sigma(\Delta)}$$

because σ is a d-dimensional Ahlfors regular measure. We proved that $k \in \mathrm{RH}_q$, as desired, which concludes the theorem. ∎

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