Rev. Mat. Iberoam. 39 (2023), no. 6, 2007–2016
DOI 10.4171/RMI/1411

A set with no Riesz basis of exponentials

Gady Kozma, Shahaf Nitzan and Alexander Olevskiĭ

Abstract. We show that there exists a bounded subset of $\mathbb{R}$ such that no system of exponentials can be a Riesz basis for the corresponding Hilbert space. An additional result gives a lower bound for the Riesz constant of any putative Riesz basis of the two-dimensional disk.

1. Introduction

A system of vectors $\{u_l\}$ in a separable Hilbert space H is called a basis if every vector $f \in H$ can be represented by a series

$$f = \sum c_l u_l$$

and the representation is unique. The best kind of basis is the orthonormal basis. In this paper we are interested in questions revolving around the existence of bases when u_l are taken from a specific, pre-given set. More specifically, we are interested in the case that $H = L^2(S)$ for some $S \subset \mathbb{R}$ or $\mathbb{R}^d$, bounded and of positive measure, and the u_l are exponential functions. In this case it is not always possible to find an orthogonal basis: if S is an interval, then the classic Fourier system is an orthogonal basis. However, for the union of two intervals it is easy to see that, in general, no exponential orthogonal basis exists, say for $[0,2] \cup [3,5]$. See [14] for a full treatment. So one needs some generalisation of orthogonal bases, which would possess many of their good properties, but would be more available for constructions.

Definition. The image of an orthonormal basis by a linear isomorphism of the space is called a Riesz basis. Equivalently, a system $\{u_l\}$ is Riesz basis if

(1) it is complete in H;

(2) there is a constant K such that, for any (finite) sum $P = \sum c_l u_l$, the following condition holds:

$$\frac{1}{K}\|c\|^2 \le \|P\|^2 \le K\|c\|^2, \tag{1.1}$$

where $\|c\|^2 = \sum |c_l|^2$.

2020 Mathematics Subject Classification: 42C15.
Keywords: Riesz basis, sampling, interpolation.

Remark. One may ask: is any basis in H a Riesz basis? Babenko [1] gave a counter-example. His example was a system of exponentials in a weighted L^2 space. It turned out later that the weights defined by Babenko are a specific case of the so-called Muckenhoupt weights [17] (we remark that the same condition was discovered independently and simultaneously by Krantsberg, see [13] or [19], pg. 73).

Thus the question we are interested in is as follows. Given an $S \subset \mathbb{R}$, is there a $\Lambda \subset \mathbb{R}$ such that the system $E(\Lambda) := \{e^{2\pi i \lambda x} : \lambda \in \Lambda\}$ is a Riesz basis in $L^2(S)$? It turns out that it is neither easy to construct Riesz bases, nor to prove that none exist. For the construction problem, Seip [21] constructed Riesz bases of exponentials for unions of two intervals (and some cases of unions of larger numbers of intervals). Riesz bases for arbitrary finite unions of intervals were constructed in [12]. See [3] for Riesz bases for convex, symmetric polygons, see [8, 11] for a construction of Riesz bases for multitiling sets, and [16] for exponentials with complex frequencies. See [2, 5, 15] for some recent work. Nevertheless, for many natural sets the question is still open, with famous examples being the ball and the triangle in two dimensions.

In this paper, we give an example of a set S for which no Riesz basis of exponentials exists. The set providing the example is an infinite collection of intervals with one accumulation point. The same technique allows us to show that, even if a Riesz basis of exponentials existed for a two-dimensional ball, its defining constant K (the K from (1.1)) cannot be too close to 1. In particular, this reproduces Fuglede's result that the ball has no orthogonal basis of exponentials [6], as that would correspond to $K = 1$. For different generalisations of Fuglede's result, see [7, 10].

2. Preliminaries

Throughout we use the usual notation $e(x) = e^{2\pi i x}$, and $E(\Lambda) = \{e(\lambda x) : \lambda \in \Lambda\}$ (that is, $E(\Lambda)$ is a set of functions with parameter x). Let us start by recalling the Paley–Wiener perturbation theorem.

Theorem (Paley and Wiener). *Let $S \subset \mathbb{R}^d$ be a bounded set of positive measure, and let $\Lambda = \{\lambda_n\} \subset \mathbb{R}$ be such that $E(\Lambda)$ is a Riesz basis for $L^2(S)$. Then there exists a constant $\mu = \mu(S, \Lambda)$ such that if a second sequence $\Gamma = \{\gamma_n\}$ satisfies $|\lambda_n - \gamma_n| < \mu$ for all n, then $E(\Gamma)$ is also a Riesz basis for $L^2(S)$.*

See, e.g., Section 2.3 in [12] for a proof in $d = 1$. The proof in higher dimensions is similar.

We say that a set $\Lambda \subset \mathbb{R}$ is uniformly discrete if there exists some $c > 0$ such that if $\lambda \neq \mu$ are both in Λ, then $|\lambda - \mu| > c$. It is easy to see that any Riesz basis of exponentials of $L^2(S)$ for some $S \subset \mathbb{R}^d$ is uniformly discrete. On the other hand, any set which is uniformly discrete satisfies the right inequality in (1.1), namely

$$\Big\| \sum_{\lambda \in \Lambda} c_\lambda\, e(\lambda x) \Big\|^2 \le C \sum_{\lambda \in \Lambda} |c_\lambda|^2.$$

(This inequality is called Bessel's inequality). See Proposition 2.7 in [20] – the formulation here follows from the formulation in the book by a simple duality argument.

We use c and C for constants (usually depending on the Riesz basis involved), whose value may change from formula to formula and even inside the same formula. We use c for constants which are 'small enough' and C for constants which are 'large enough'.

3. Details and proofs

We start with a result of the third named author which remained unpublished, but a version of it was included (with his permission) in the book of Heil [9], p. 296 (see also [4] for a proof in the more general context of frames, or 'overcomplete' bases). We chose to present it here as it provides the simplest demonstration of the 'translation' technique which we apply throughout this note.

Theorem 1. *Let $S \subset \mathbb{R}$ be a set of positive measure and let $w \in L^1(S)$ be a positive function. If w is not bounded away from 0 or ∞, then there is no $\Lambda \subset \mathbb{R}$ such that $E(\Lambda)$ is a Riesz basis for the weighted space $L^2(S, w)$.*

Proof. Assume that w is not bounded away from 0. In this case, one may find, for every $\varepsilon > 0$, a subset $A = A(\varepsilon) \subset S$ of positive measure and a number $t = t(\varepsilon)$ such that $w(x) < \varepsilon$ on A but $w(x) > c$ on $A + t$, where the constant c does not depend on ε. (For example, one may use the Lebesgue density theorem to find two intervals of equal length, I and J, where the relative densities of $w < \varepsilon$ and of $w > c$ are bigger than $2/3$ respectively, define t to be the distance between the centers of I and J, and take $A = \{x \in I : w(x) < \varepsilon, w(x + t) > c\}$. Then A would have relative density at least $1/3$ in I and in particular have positive measure).

Examine the function $f = \frac{1}{\sqrt{|A|}} \mathbb{1}_A$. Assume by contradiction that $E(\Lambda)$ is a Riesz basis for $L^2(S, w)$ with respect to a constant K, and develop f in this basis, i.e., write

$$f(x) = \sum_{\lambda \in \Lambda} c_\lambda \, e(\lambda x).$$

The Riesz basis property tells us that the series converges in L^2 and that

$$\sum |c_\lambda|^2 \le K \|f\|^2_{L^2(S,w)} \le K\varepsilon, \tag{3.1}$$

by the definition of A. Now perform a formal translation (by $-t$) of the series $\sum c_\lambda \, e(\lambda x)$, namely, consider a new series with coefficients $d_\lambda := c_\lambda \, e(-\lambda t)$. Since $\sum |d_\lambda|^2 = \sum |c_\lambda|^2 < \infty$, the Riesz basis property gives that the series $\sum d_\lambda \, e(\lambda x)$ converges in $L^2(S, w)$ and the limit, g, satisfies

$$\|g\|^2_{L^2(S,w)} \le K \sum_{\lambda \in \Lambda} |d_\lambda|^2 = K \sum_{\lambda \in \Lambda} |c_\lambda|^2 \overset{(3.1)}{\le} K^2 \varepsilon. \tag{3.2}$$

On the other hand, for every $x \in A + t$, we have

$$g(x) = \sum_{\lambda \in \Lambda} d_\lambda \, e(\lambda x) = \sum_{\lambda \in \Lambda} c_\lambda \, e(\lambda(x - t)) = f(x - t) = \frac{1}{\sqrt{|A|}}$$

(it is perhaps easiest to consider the equalities as holding almost everywhere and the sums converging in measure – we use here that convergence in L^2 implies convergence in measure; and that if a sum converges both in L^2 and in measure, then the two limits are almost everywhere equal). But this is a contradiction because then

$$\|g\|_{L^2(s,w)} \geq \int_{A+t} |g(x)|^2 \, w(x) \, dx \geq \int_{A+t} \frac{1}{|A|} c \, dx = c,$$

contradicting (3.2), since ε was arbitrary.

The case that w is not bounded away from ∞ is treated in a similar way. ■

Next, we turn to the main result of this note.

Theorem 2. *There exists a bounded set $S \subset \mathbb{R}$ of positive measure for which no $\Lambda \subset \mathbb{R}$ may satisfy that $E(\Lambda)$ is a Riesz basis for $L^2(S)$*

Proof. We start with $S_1 = [0,1] \cup [2,3]$. We then break the interval $[2,3]$ into $4^{4^2}+1$ intervals of equal length, and keep for each one only the left half (this is S_2). We next take the last interval of S_2, break it into $4^{4^3}+1$ intervals of equal length, and keep for each one only the left third. We continue this way, and denote $S = \bigcap S_i$.

Assume by contradiction that S has a Riesz basis of exponentials $E(\Lambda)$. By the Paley–Wiener theorem, we may assume without loss of generality that $\Lambda \subset \frac{1}{\ell}\mathbb{Z}$ for some integer $\ell \geq 4$. Let K be a Riesz constant for Λ, i.e., every $f \in L^2(S)$ can be expanded into a sum

$$f(x) = \sum_{\lambda \in \Lambda} c_\lambda \, e(\lambda x)$$

such that

$$\frac{1}{K} \sum |c_\lambda|^2 \leq \int_S |f|^2 \leq K \sum |c_\lambda|^2.$$

Assume for simplicity that $K \geq \ell$.

Fix some n. The construction of S implies that one may find 4^{4^n} intervals $I_1, \dots, I_{4^{4^n}}$ of equal length (call it ε) and distance between them $\varepsilon(n-1)$. For $i = 1, \dots, 4^{4^n}$, let

$$f_i = \frac{1}{\sqrt{\varepsilon}} \mathbb{1}_{I_i}$$

so that $\|f_i\|_{L^2(S)} = 1$. Expand $f_i(x) = \sum c_{\lambda,i} \, e(\lambda x)$. Since $\Lambda \subset \frac{1}{\ell}\mathbb{Z}$, this sum converges in $L^2[0,\ell]$, and the limit is an extension of f_i to the whole of $L^2[0,\ell]$ (to the whole of $\mathbb{R}$, if you prefer). Call these extensions $\widetilde{f_i}$ and note that the Riesz basis property gives that $\sum |c_{\lambda,i}|^2 \leq K$, and hence

$$\|\widetilde{f_i}\|^2_{L^2([0,\ell])} \leq \ell K \leq K^2. \tag{3.3}$$

Similarly, if β_i are arbitrary complex coefficients, then the expansion of $\sum \beta_i f_i$ in the Riesz basis $E(\Lambda)$ is

$$\sum_{\lambda \in \Lambda} \Big(\sum_i \beta_i \, c_{\lambda,i} \Big) e(\lambda x),$$

so the Riesz property shows that

$$\sum_{\lambda\in\Lambda}\Big|\sum_i \beta_i c_{\lambda,i}\Big|^2 \le K\Big\|\sum_i \beta_i f_i\Big\|^2_{L^2(S)} = K\sum_i |\beta_i|^2.$$

From this we conclude, as in (3.3), that

$$\Big\|\sum_i \beta_i \widetilde{f}_i\Big\|^2_{L^2([0,\ell])} \le K^2\sum|\beta_i|^2. \tag{3.4}$$

Let us further define

$$h = \frac{4}{\varepsilon}\mathbb{1}_{[-\varepsilon/8,\varepsilon/8]},$$

so that $\|h\|_1 = 1$. Let $g_i = \widetilde{f}_i * h$ (the convolution here is a periodic convolution on $[0,\ell]$). By the Cauchy–Schwarz inequality, for any coefficients β_i we have

$$\Big|\sum_i \beta_i g_i(x)\Big| \le \Big\|\sum_i \beta_i \widetilde{f}_i\Big\|_{L^2[0,\ell]}\|h\|_2 \overset{(3.4)}{\le} K\sqrt{\sum|\beta_i|^2}\cdot\frac{2}{\sqrt{\varepsilon}} \qquad \forall x\in\mathbb{R}.$$

In other words, the convolution turns the L^2 estimate for $\sum\beta_i\widetilde{f}_i$ to a pointwise estimate.

Next, enumerate all intervals comprising S which have length strictly bigger than ε as $[a_1,b_1],\dots,[a_q,b_q]$, where $q=\sum_{k<n}4^{4^k}\le 2\cdot 4^{4^{n-1}}$. Let

$$A := \bigcup_{i=1}^{q}[a_i - n\varepsilon, a_i+\varepsilon].$$

(A can be thought of as the set of left 'edges' of these intervals, as each interval $[a_i - n\varepsilon, a_i+\varepsilon]$ of A is much smaller than the corresponding $[a_i,b_i]$). Then $|A| = q(n+1)\varepsilon \le 2(n+1)\varepsilon 4^{4^{n-1}}$. By Claim 1 below (used with $N=4^{4^n}$, $M=2K/\sqrt{\varepsilon}$ and A as above), there exists an i_0 such that

$$\int_A |g_{i_0}|^2 \le \frac{M^2|A|}{\sqrt{N}} \le \frac{CK^2 n}{4^{4^{n-1}}}.$$

With i_0 selected and this crucial property of g_{i_0} proved, the theorem follows by examining a few translations of g_{i_0}.

To see this, expand f_{i_0} using the Riesz basis property and denote the coefficients by c_λ so that $f_{i_0}(x) = \sum c_\lambda e(\lambda x)$, and recall that we have extended f_{i_0} so that a similar equality holds also for $\widetilde{f}_{i_0}$ over all of $\mathbb{R}$. Hence

$$g_{i_0}(x) = \sum_{\lambda\in\Lambda} c_\lambda \widehat{h}(\lambda)\, e(\lambda x).$$

So g_{i_0} is supported spectrally on Λ, and therefore

$$\sum|c_\lambda \widehat{h}(\lambda)|^2 \ge \frac{1}{K}\int_S |g_{i_0}|^2 \ge \frac{1}{2K}, \tag{3.5}$$

where the last inequality follows from the definition of g_{i_0} as $\widetilde{f}_{i_0} * h$, which allows to calculate g_{i_0} in the middle half of the interval I_{i_0}.

On the other hand, the Plancherel formula on $[0, \ell]$ and the Riesz property imply that

$$\int_0^{\ell} |g_{i_0}|^2 = \ell \sum |c_\lambda|^2 \, |\widehat{h}(\lambda)|^2 \overset{(*)}{\leq} K \sum |c_\lambda|^2 \leq K^2 \int_S |f_{i_0}|^2 = K^2. \tag{3.6}$$

The inequality $(*)$ follows from $\|h\|_1 = 1$, which implies $|\widehat{h}(\lambda)| \leq 1$ for all λ.

Consider the following $n-1$ translations of g_{i_0}:

$$G_k(x) = g_{i_0}(x - k\varepsilon) = \sum_{\lambda \in \Lambda} c_\lambda \, \widehat{h}(\lambda) \, e(-\lambda k \varepsilon) \, e(\lambda x).$$

Each G_k is also supported spectrally on Λ, and hence

$$\int_S |G_k|^2 \geq \frac{1}{K} \sum |c_\lambda \, \widehat{h}(\lambda)|^2 \overset{(3.5)}{\geq} \frac{1}{2K^2}.$$

Consider now an interval $[a, b]$ which is a connected component of S of length larger than ε. Since $\widetilde{f}_{i_0}$ is zero on $[a, b]$, we have that g_{i_0} is zero on $[a + \frac{1}{8}\varepsilon, b - \frac{1}{8}\varepsilon]$ (simply because g_{i_0} was defined as $\widetilde{f}_{i_0} * h$). Hence G_k is zero on $[a + (k + \frac{1}{8})\varepsilon, b]$, i.e., on most of the interval, for any $k \in \{1, \ldots, n-1\}$. On the remaining part, $[a, a + (k + \frac{1}{8})\varepsilon]$, we have that G_k is a translation of g_{i_0} from the interval $[a - k\varepsilon, a + \frac{1}{8}\varepsilon] \subset A$. Therefore

$$\sum_{[a,b]: b-a > \varepsilon} \int_a^b |G_k|^2 \leq \int_A |g_{i_0}|^2 \leq \frac{CK^2 n}{4^{4^{n-1}}},$$

where the sum is over all $[a, b]$ which are connected components of S of length larger than ε. This means that, if I is the union of all components of S of length ε or less (all the I_i and then the shorter intervals), then

$$\int_I |G_k|^2 \geq \frac{1}{2K^2} - \frac{CK^2 n}{4^{4^{n-1}}},$$

and since G_k is a translation of g_{i_0}, we get

$$\int_{I - \varepsilon k} |g_{i_0}|^2 \geq \frac{1}{2K^2} - \frac{CK^2 n}{4^{4^{n-1}}}.$$

Since the sets $I - \varepsilon k$ are disjoint, for $k \in \{1, \ldots, n-1\}$, we get

$$K^2 \overset{(3.6)}{\geq} \int_0^{\ell} |g_{i_0}|^2 \geq (n-1)\Big(\frac{1}{2K^2} - \frac{CK^2 n}{4^{4^{n-1}}}\Big).$$

Since n was arbitrary, we have reached a contradiction. ∎

Claim 1. *Let $A \subset \mathbb{R}$ be a set of positive finite measure, and let $\{g_i\}_{i=1}^N \subset L^2(A)$. If for every $\beta_1, \ldots, \beta_N$ we have*

$$\Big|\sum_{i=1}^N \beta_i g_i(x)\Big| \leq M \sqrt{\sum |\beta_i|^2}, \quad \forall x \in A, \tag{3.7}$$

then there exists some i_0 such that

$$\int_A |g_{i_0}|^2 \le \frac{M^2|A|}{\sqrt{N}}.$$

Proof. Assume that $\int_A |g_i|^2 > \delta$ for some δ and all i. We get

$$\sum_i \int_A |g_i(x)|^2\,dx > \delta N.$$

Hence there exists some x_0 such that

$$\sum_i |g_i(x_0)|^2 > \frac{\delta N}{|A|}.$$

Fix some i and apply condition (3.7) with $\beta_i = 1$ and $\beta_j = 0$ for all $j \ne i$. We get that $|g_i(x_0)| \le M$. Since i was arbitrary, this holds for all i. Hence

$$\sum_i |g_i(x_0)| \ge \frac{1}{M}\sum_i |g_i(x_0)|^2 > \frac{\delta N}{|A|M}.$$

Now apply condition (3.7) with $\beta_i = g_i(x_0)/|g_i(x_0)|$. We get

$$\frac{\delta N}{|A|M} < \sum_i |g_i(x_0)| = \Big|\sum_i \beta_i g_i(x_0)\Big| \overset{(3.7)}{\le} M\sqrt{N}.$$

The claim is thus proved. ∎

Remark. It is known [12] that any finite union S of intervals has a Riesz basis of exponentials with some constant K_S. Assume $|S| = 1$ for simplicity. The result above also shows that K_S is not bounded uniformly. Indeed, stopping the construction after n steps would give a set, S_n, which is a finite union of intervals, for which $K_{S_n} > cn^{1/4}$ (the proof also needs a quantitative version of the Paley–Wiener theorem). Alternatively, the fact that K_S is unbounded can be concluded from Theorem 2 the same way the main result of [18] follows from Lemma 7 there.

This might be a good place to mention an interesting question for which we have no answer. Is K_S bounded uniformly for $S \subset [0,2]$ which are a union of two intervals? The constructions known to us [12,21] give, for $S = [0,1/2] \cap [1/2+\varepsilon, 1+\varepsilon]$ a constant that increases to ∞ as $\varepsilon \to 0$, but these are only upper bounds and we have no corresponding lower bound.

Our last result is that, even if a Riesz basis of exponentials would exist for the disk, its constant cannot be too close to 1. The way we defined the Riesz constant in (1.1), though, makes it easy to compare to orthogonal bases only when $|S| = 1$ (otherwise even an orthogonal basis of exponentials would not be normalised and would not satisfy (1.1) with $K = 1$). For simplicity, we prove the result for $D = \frac{1}{\sqrt{\pi}}\mathbb{D}$, a disk with area 1.

Theorem 3. *Any Riesz basis of exponentials for D must have $K \ge \sqrt{\frac{1+\sqrt{5}}{2}}$.*

Proof. Assume by contradiction that $\Lambda \subset \mathbb{R}^2$ satisfies that $E(\Lambda)$ is a Riesz basis for $L^2(D)$, with the Riesz constant K (the K from (1.1)) satisfying $K < \sqrt{(1+\sqrt{5})/2}$.

Fix $\varepsilon > 0$ and examine the function

$$f = \frac{1}{\sqrt{\pi}\,\varepsilon}\,\mathbb{1}_{\varepsilon\mathbb{D}}$$

i.e., a normalised indicator of a disk of radius ε around 0. Use the Riesz basis property to write

$$f(x) = \sum_{\lambda\in\Lambda} c_\lambda\, e(\langle\lambda, x\rangle),$$

with the sum converging in $L^2(D)$. As before, we use this representation to extend f to $[-2,2]^2$, though there will be two differences from the previous theorem. First, it will be more convenient to not distinguish between f and its extension, and call the extension f as well. And second, rather than using Paley–Wiener, this time we use the fact $E(\Lambda)$ is a Riesz basis in $L^2(D)$ to conclude that Λ is uniformly separated and hence $E(\Lambda)$ satisfies Bessel's inequality in $L^2([-2,2]^2)$, i.e.,

$$\|f\|_{L^2([-2,2]^2)} \le C \sum_{\lambda\in\Lambda} |c_\lambda|^2 \le CK\|f\|_D^2 = CK. \tag{3.8}$$

For a $\theta \in [0, 2\pi]$, denote $t_\theta = (1/\sqrt{\pi} - \varepsilon)(\cos\theta, \sin\theta)$ and consider the translation of f by t_θ, i.e.,

$$g_\theta(x) := \sum_{\lambda\in\Lambda} c_\lambda\, e(\langle\lambda, x - t_\theta\rangle).$$

As before, we must have

$$\|g_\theta\|^2_{L^2(D)} \le K \sum_{\lambda\in\Lambda} |c_\lambda\, e(\langle\lambda, -t_\theta\rangle)|^2 = K \sum_{\lambda\in\Lambda} |c_\lambda|^2 \le K^2.$$

However, $g_\theta(x) = \frac{1}{\sqrt{\pi}\varepsilon}$ on a disk of radius ε contained in D and also in $D + t_\theta$. Hence

$$\int_{D\setminus(D+t_\theta)} |g_\theta|^2 \le K^2 - 1.$$

Translating back we get an estimate for (the extended) f outside D, namely

$$\int_{(D-t_\theta)\setminus D} |f|^2 \le K^2 - 1. \tag{3.9}$$

This is our upper bound for f.

To get a lower bound we again examine one θ, and this time translate by $s_\theta = (\frac{1}{\sqrt{\pi}} + \varepsilon)(\cos\theta, \sin\theta)$. Denote the translated function by h_θ, namely

$$h_\theta(x) := \sum_{\lambda\in\Lambda} c_\lambda\, e(\langle\lambda, x - s_\theta\rangle),$$

and get

$$\|h_\theta\|^2_{L^2(D)} \geq \frac{1}{K} \sum_{\lambda\in\Lambda} |c_\lambda|^2 \geq \frac{1}{K^2},$$

which we again map back to f to get

$$\int_{(D-s_\theta)\setminus D} |f|^2 \geq \frac{1}{K^2}$$

(note that we used in this last step that the little disk that allowed to subtract 1 when we did the corresponding calculations for g is outside D for h_θ, so does not contribute to $\|h_\theta\|_{L^2(D)}$). With the lower bound (3.9), we get

$$\int_{(D-s_\theta)\setminus(D-t_\theta)} |f|^2 \geq \frac{1}{K^2} - (K^2 - 1).$$

When $K < \sqrt{(1+\sqrt{5})/2}$, we get that the right-hand side is at least some constant $c > 0$. Integrating over θ and using Fubini gives (omitting the details of the elementary geometry exercise involved)

$$c \leq \int_0^{2\pi} \int_{(D-s_\theta)\setminus(D-t_\theta)} |f(x)|^2 \, dx \, d\theta \leq \int_{(2/\sqrt{\pi}+\varepsilon)\mathbb{D}} C\sqrt{\varepsilon}\, |f(x)|^2 \, dx.$$

Or, in other words,

$$\int_{(2/\sqrt{\pi}+\varepsilon)\mathbb{D}} |f(x)|^2 \geq \frac{c}{\sqrt{\varepsilon}},$$

contradicting (3.8) if ε is taken to be sufficiently small. ∎

Funding. G. K. acknowledges the support of the Israel Science Foundation and of the Jesselson Foundation. S. N. is supported by NSF-CAREER grant DMS-1847796.

References

[1] Babenko, K. I.: On conjugate functions. [Russian] *Doklady Akad. Nauk SSSR* **62** (1948), 157–160.

[2] Cabrelli, C., Hare, K. E. and Molter, U.: Riesz bases of exponentials and the Bohr topology *Proc. Amer. Math. Soc.* **149** (2021), no 5, 2121–2131.

[3] Debernardi, A. and Lev, N.: Riesz bases of exponentials for convex polytopes with symmetric faces. *J. Eur. Math. Soc. (JEMS)* **24** (2022), no 8, 3017–3029.

[4] Dutkay, D. E. and Lai, C. K.: Uniformity of measures with Fourier frames. *Adv. Math.* **252** (2014), 684–707.

[5] Ferguson, S., Mayeli, A. and Sothanaphan, N.: Riesz bases of exponentials and multi-tiling in finite abelian groups. Preprint 2019, arXiv: 1904.04487.

[6] Fuglede, B.: Commuting self-adjoint partial differential operators and a group theoretic problem. *J. Funct. Anal.* **16** (1974), 101–121.

[7] Fuglede, B.: Orthogonal exponentials on the ball. *Expo. Math.* **19** (2001), no. 3, 267–272.

[8] Grepstad, S. and Lev, N.: Multi-tiling and Riesz bases. *Adv. Math.* **252** (2014), 1–6.

[9] Heil, C.: *A basis theory primer*. Expanded edition. Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, New York, 2011.

[10] Iosevich, A. and Mayeli, A.: On complete and incomplete exponential systems. Preprint 2020, arXiv: 2007.07972.

[11] Kolountzakis, M.: Multiple lattice tiles and Riesz bases of exponentials. *Proc. Amer. Math. Soc.* **143** (2015), no. 2, 741–747.

[12] Kozma, G. and Nitzan, S.: Combining Riesz bases. *Invent. Math.* **199** (2015), no. 1, 267–285.

[13] Krantsberg, A. S.: On whether the Haar system is a basis in weighted spaces [Russian]. *Moskov. Inst. Èlektron. Mašino. Trudy MIÈM* **24** (1972), 14–21.

[14] Łaba, I.: Fuglede's conjecture for a union of two intervals. *Proc. Amer. Math. Soc.* **129** (2001), no. 10, 2965–2972.

[15] Lee, D. G.: On construction of bounded sets not admitting a general type of Riesz spectrum. Preprint 2021, arXiv: 2108.07760.

[16] Lyubarskii, Y. I. and Seip, K.: Sampling and interpolating sequences for multiband-limited functions and exponential bases on disconnected sets. *J. Fourier Anal. Appl.* **3** (1997), no. 5, 597–615.

[17] Muckenhoupt, B.: Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.* **165** (1972), 207–226.

[18] Nitzan, S., Olevskiĭ, A. and Ulanovskii, A.: Exponential frames on unbounded sets. *Proc. Amer. Math. Soc.* **144** (2016), no. 1, 109–118.

[19] Olevskiĭ, A. M.: *Fourier series with respect to general orthogonal systems*. Ergebnisse der Mathematik und ihrer Grenzgebiete 86, Springer, New York-Heidelberg, 1975.

[20] Olevskiĭ, A. M. and Ulanovskii, A.: *Functions with disconnected spectrum. Sampling, interpolation, translates*. University Lecture Series, 65. American Mathematical Society, Providence, RI, 2016.

[21] Seip, K.: A simple construction of exponential bases in L^2 of the union of several intervals. *Proc. Edinburgh Math. Soc.* **38** (1995), no. 1, 171–177.

Received January 26, 2022; revised October 6, 2022. Published online February 1, 2023.

Gady Kozma
Department of Mathematics, The Weizmann Institute of Science, Rehovot 7610001, Israel;
gady.kozma@weizmann.ac.il

Shahaf Nitzan
School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA;
shahaf.nitzan@math.gatech.edu

Alexander Olevskiĭ
School of Mathematical Sciences, Tel Aviv University, Tel Aviv 6997801, Israel;
olevskii@post.tau.ac.il