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Positive solutions of the p-Laplacian with potential terms on weighted Riemannian manifolds with linear diameter growth

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Abstract. In this paper, we consider the p-Laplacian with potential terms on a connected, noncompact, complete weighted Riemannian manifold whose Ricci curvature has quadratic decay, or a lower bound. We investigate the structure and the behavior of positive solutions under the assumption that the metric spheres of the Riemannian manifold have linear diameter growth.

1. Introduction

Consider a weighted Riemannian manifold $(M, g_M, e^{-f}\, dv_M)$ of dimension m, where (M, g_M) is a Riemannian manifold of dimension m, f is a smooth function on M, and dv_M is the volume element induced by the metric g_M. In what follows, the measure $e^{-f}\, dv_M$ is denoted by μ_f.

For a vector field $X \in L^1_{\mathrm{loc}}(\Omega, TM)$ on a domain Ω, the divergence $\mathrm{div}^f X$ of X relative to the measure μ_f is defined weakly by

$$\int \psi \,\mathrm{div}^f X \, d\mu_f = -\int g_M(X, \nabla\psi)\, d\mu_f$$

for all $\psi \in C_0^\infty(\Omega)$. We simply write $\mathrm{div} X$ if the weight function f is constant. Then $\mathrm{div}^f X = \mathrm{div} X - g_M(X, \nabla f)$.

Fix $p \in (1, +\infty)$. The p-Laplacian $\Delta_{f;p}$ acts on $L^{1,p}_{\mathrm{loc}}(M)$ by

$$\Delta_{f;p} u = \mathrm{div}^f (|\nabla u|^{p-2}\, \nabla u)$$

in the weak sense, that is,

$$\int \psi\, \Delta_{f;p} u \, d\mu_f = -\int g_M(|\nabla u|^{p-2}\, \nabla u, \nabla\psi)\, d\mu_f$$

for all $\psi \in C_0^\infty(M)$.

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Fix a domain $\Omega \subset M$ and a real-valued function $W \in L^\infty_{\mathrm{loc}}(\Omega)$. The p-Laplace equation in Ω with potential W is the equation of the form

$$Q'_{p;W}(u) = -\Delta_{f;p} u + W|u|^{p-2}u = 0 \quad \text{in } \Omega.$$

This is the Euler–Lagrange equation associated with the functional

$$Q_{p;W}(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p + W|u|^p)\, d\mu_f.$$

A generalized Allegretto–Piepenbrink theorem says that $Q_{p;W}(u) \geq 0$ for all $u \in L^{1,p}_{\mathrm{loc}}(\Omega)$ if and only if $Q'_{p,W}(v) = 0$ admits a positive solution $v \in L^{1,p}_{\mathrm{loc}}(\Omega) \cap C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ (see Pinchover and Psaradakis [24] and references therein). In this paper, we are interested in the structure and the behavior of positive solutions in M. There have been extensive studies on this subject over the recent decades; see for example [1, 10, 11, 14, 20, 23–26] and references therein.

We let $B(x,t)$ (respectively, $S(x,t)$) be the open metric ball around a point x with radius t (respectively, the metric sphere centered at x of radius t). Fix $o \in M$ as a reference point and let r be the distance to o. Given $\sigma \in (0,1)$ and $t \in (0,+\infty)$, we denote by $\mathrm{dis}^{(\sigma;t)}$ the (extended) distance induced on $M \setminus B(o,(1-\sigma)t)$, and by $\mathrm{diam}^{(\sigma;t)}(S(o,t))$ the diameter of $S(o,t)$ in $M \setminus B(o,(1-\sigma)t)$ relative to the (extended) distance. We define

$$\delta^{(\sigma;\infty)}(M) = \limsup_{t\to\infty} \frac{1}{t}\,\mathrm{diam}^{(\sigma;t)}(S(o,t)) \in [0,+\infty].$$

Obviously, $\delta^{(\sigma;\infty)}(M) \leq \delta^{(\sigma';\infty)}(M)$ if $0 < \sigma' \leq \sigma < 1$. We note that M has only one end, that is, for sufficiently large compact sets $K \subset M$, the difference $M \setminus K$ has exactly one unbounded connected component if $\delta^{(\sigma;\infty)}(M) < +\infty$. Correspondingly to the case where $\sigma = 1$ in the definition of $\delta^{(\sigma;\infty)}(M)$, we let

$$\delta^{(\infty)}(M) = \limsup_{t\to\infty} \frac{1}{t}\,\mathrm{diam}(S(o,t)) \in [0,2],$$

where the diameter of the sphere $S(o,t)$ is measured in M. It is obvious that $\delta^{(\infty)}(M) \leq \delta^{(\sigma;\infty)}(M)$, and we note that if $\delta^{(\infty)}(M) < 2$, then $\delta^{(\sigma;\infty)}(M) = \delta^{(\infty)}(M)$ for $\frac{1}{2}\delta^{(\infty)}(M) < \sigma < 1$ (see Proposition 4.3 (ii)).

In order to state the main results of this paper, we need some terminology; see Li and Tam [18], Grigor'yan and Saloff-Coste [13]. Fix a constant $C_A > 1$. We say that a metric space (M, dis_M) has *relatively connected annuli* with respect to o, or satisfies condition (RCA), if for any $t \geq C_A^2$ and all $x, y \in S(o,t)$, there exists a continuous path $\gamma\colon [0,L] \to M$ with $\gamma(0) = x$, $\gamma(L) = y$ whose image is contained in $B(o, C_A t) \setminus B(o, C_A^{-1} t)$ (see [13], Definition 5.1). We observe that condition (RCA) holds for some $C_A > 1$ if $\delta^{(\sigma;\infty)}(M) < +\infty$ for some $\sigma \in (0,1)$. We say that a weighted manifold (M, g_M, μ_f) satisfies the *volume comparison condition* (VC) if there exists a positive constant C_V such that, for all $t > 0$ and all $x \in S(o,t)$, we have that $\mu_f(B(o,t)) \leq C_V \mu_f(B(x,t/2))$ (see [18] and [13], Definition 4.3).

Theorem 1.1. *Let (M, g_M, μ_f) be a connected, noncompact, complete weighted Riemannian manifold of dimension m. Suppose that the Ricci curvature Ric_M of M satisfies*

$$\inf_M (1+r)^2 \,\mathrm{Ric}_M > -\infty, \tag{1.1}$$

the weight function f satisfies

$$\sup_M (1+r)|\nabla f| < +\infty, \tag{1.2}$$

and further,

$$\delta^{(\sigma;\infty)}(M) < +\infty$$

for some $\sigma \in (0,1)$. Given $p \in (1,\infty)$, let W be a bounded function on M such that

$$\sup_M (1+r)^p |W| < +\infty,$$

and assume that $Q_{p;W} \geq 0$. Then the following assertions hold.

(i) (*Annulus Harnack inequality*) *There is a constant $C_H > 0$ such that for any $t > 0$ and for any positive solution $u \in L^{1,p}_{\mathrm{loc}}(M) \cap C^{1,\alpha}_{\mathrm{loc}}(M)$ to the equation $Q'_{p;W}(u) = 0$ in M,*

$$\sup_{S(o,t)} u \leq C_H \inf_{S(o,t)} u.$$

(ii) *In the case where $p = 2$, a positive solution to $Q'_{2;W}(u) = 0$ in M is unique up to multiple constants.*

(iii) *Let $u \in L^{1,p}_{\mathrm{loc}}(M) \cap C^{1,\alpha}_{\mathrm{loc}}(M)$ be a positive solution to $Q'_{p;W}(u) = 0$ in M. If $W \geq 0$ and u is unbounded, then $\lim_{x\in M\to\infty} u(x) = +\infty$; if $W \leq 0$ and $\inf_M u = 0$, then $\lim_{x\in M\to\infty} u(x) = 0$.*

(iv) *Let $\phi(r)$ be a nonnegative C^1 function on $[0,\infty)$ such that $\phi'(r) \leq 0$, $\sup_{t\geq 0} \phi(t) t^p < +\infty$ and*

$$\int_1^\infty (t\phi(t))^{1/(p-1)}\, dt = \infty.$$

Let $u \in L^{1,p}_{\mathrm{loc}}(M) \cap C^{1,\alpha}_{\mathrm{loc}}(M)$ be a positive solution to the equation $Q'_{p;W}(u) = 0$ in M. If

$$\phi(r(x)) \leq W(x) \leq \frac{\Lambda}{(1+r(x))^p}$$

for some positive constant Λ and all $x \in M$ (respectively,

$$-\frac{\Lambda}{(1+r(x))^p} \leq W(x) \leq -\phi(r(x))$$

for some positive constant Λ and all $x \in M$), then $\lim_{x\in M\to\infty} u(x) = +\infty$ (respectively, $\lim_{x\in M\to\infty} u(x) = 0$).

In the case where $p = 2$, we have the following.

Theorem 1.2. *Let (M, g_M, μ_f) be a connected, noncompact, complete weighted Riemannian manifold of dimension m satisfying* (1.1) *and* (1.2)*. Suppose that* (RCA) *and* (VC) *are satisfied, and that the following growth condition holds for some $\beta > 2$:*

$$C_\beta \left(\frac{t}{s}\right)^\beta \leq \frac{\mu_f(B(o,t))}{\mu_f(B(o,s))} \tag{1.3}$$

for $1 \leq s \leq t$, where C_β is a positive constant less than 1. Let W be a bounded function on M satisfying

$$|W(x)| \leq \psi(r(x))$$

for all $x \in M$, where $\psi(r)$ is a nonnegative C^1 function on $[0, \infty)$ such that $\psi'(t) \leq 0$ and

$$\int_0^\infty t\psi(t)\, dt < +\infty.$$

Then the following assertions hold.

(i) *There exists a unique solution $v \in C^{1,\alpha}_{\mathrm{loc}}(M)$ of the Poisson equation $\Delta_{f;2} v = W$ in M which tends to zero at infinity.*

(ii) *Assume that there is a positive solution $u \in L^{1,2}_{\mathrm{loc}}(M) \cap C^{1,\alpha}_{\mathrm{loc}}(M)$ of $Q'_{2;W}(u) = 0$ in M.*

  (ii-a) *If $Q'_{2;W}$ is subcritical, that is, if it admits a* (*positive minimal*) *Green function, then $u(x)$ converges to a positive constant a as $x \in M \to \infty$, and one has*

$$u(x) = a - \int_M G(x,y) W(y) u(y)\, d\mu_f(y), \quad x \in M,$$

  where $G(x, y)$ denotes the Green function of the Laplacian $\Delta_{f;2}$.

  (ii-b) *If $Q'_{2;W}$ is critical, that is, if it does not admit the Green function, then $u(x)$ converges to zero as $x \in M \to \infty$, and one has*

$$u(x) = -\int_M G(x,y) W(y) u(y)\, d\mu_f(y), \quad x \in M.$$

Now we consider a family $\mathcal{F}$ of balls in M. We say that $\mathcal{F}$ satisfies the *volume doubling property* (VD) with a constant $C_D > 1$ if, for any ball $B(x,t) \in \mathcal{F}$,

$$\mu_f(B(x,t)) \leq C_D\, \mu_f(B(x,t/2)).$$

If all balls in M satisfy (VD), then we say that (M, g_M, μ_f) satisfies (VD). It is shown that $\delta^{(\sigma;\infty)}(M) < +\infty$ for some $\sigma \in (0,1)$ if (M, g_M, μ_f) satisfies (RCA) and (VD) (see Proposition 4.3 (i)).

We say that $\mathcal{F}$ satisfies the *Poincaré inequality* (PI(p)) ($1 \leq p < +\infty$) with a constant $C_P > 0$ if, for any $B(x,t) \in \mathcal{F}$ and every $u \in C^1(B(x,t))$,

$$\int_{B(x,t/2)} |u - u_{B(x,t/2)}|^p\, d\mu_f \leq C_P\, t^p \int_{B(x,t)} |\nabla u|^p\, d\mu_f,$$

where

$$u_{B(x,t/2)} = \frac{1}{\mu_f(B(x,t/2))} \int_{B(x,t/2)} u \, d\mu_f .$$

If all balls in M satisfy (PI(p), then we say that (M, g_M, μ_f) satisfies (PI(p)).

In this paper, we call a ball $B(x,t)$ *remote* to a fixed point o if $t \leq \frac{1}{4} r(x)$ (see [13], Section 4). Then under conditions (1.1) and (1.2), a family of balls remote to o satisfies (VD) and (PI(p)) for a fixed $p \in [1, +\infty)$ (see Proposition 2.17). In fact, keeping the assumption that $\delta^{(\sigma;\infty)}(M) < +\infty$ for some $\sigma \in (0,1)$, if we replace (1.1) and (1.2) with (VD) and (PI(p)) (respectively, (VD) and (PI(2)) for all remote balls, then we obtain Theorem 1.1 (i), (iii) (respectively, Theorem 1.1 (ii) and Theorem 1.2); however, we do not know if the assertion (iv) of Theorem 1.1 must hold.

When (M, g_M, μ_f) satisfies (VD) and (PI(2)), and further the growth condition (1.3) with $\beta > 2$, a result of Ancona (see [2], Proposition 3.1), proves that

$$\int_M \frac{C_1 |u(x)|^2}{1 + r(x)^2} \, d\mu_f(x) \leq \int_M |\nabla u|^2 \, d\mu_f$$

for some positive constant C_1 and all $u \in C_0^\infty(M)$. This is a Hardy type inequality on M, and we can apply Theorem 1.1 (iv) ($p = 2$) to a positive solution to the equation $-\Delta_{f;2} u - \frac{C_1}{1+r^2} u = 0$ in M.

Let W be the bounded potential on M considered in Theorem 1.2. In order to prove the assertion (ii-a), we use another result by Ancona (see Theorem 3.3 in [2]), proving that the Green function $G^W(x,y)$ of $Q'_{2;W}$ satisfies

$$C_2^{-1} \int_{\mathrm{dis}_M(x,y)}^{\infty} \frac{t \, dt}{\mu_f(B(x,t))} \leq G^W(x,y) \leq C_2 \int_{\mathrm{dis}_M(x,y)}^{\infty} \frac{t \, dt}{\mu_f(B(x,t))}$$

for some $C_2 \geq 1$ and for all $x, y \in M$. Moreover, in view of Theorem 10.5 in [11] by Grigor'yan, and its proof, we see that in (ii-a), the heat kernel p_t^W of the operator $Q'_{2;W}$ satisfies the two-sided Gaussian estimate (or the Li–Yau estimate) as follows:

$$\frac{C_3^{-1}}{\mu_f(B(x,\sqrt{t}))} e^{-C_4 \, \mathrm{dis}_M(x,y)^2/t} \leq p_t^W(x,y) \leq \frac{C_3}{\mu_f(B(x,\sqrt{t}))} e^{-C_5 \, \mathrm{dis}_M(x,y)^2/t}$$

for all $x, y \in M$ and $t > 0$, where C_3, C_4 and C_5 are positive constants (see Remark 4.6 (ii)).

A weighted Riemannian manifold (M, g_M, μ_f) is called *p-parabolic* if every positive, continuous p-supersolution on M, that is, a positive continuous function $v \in L^{1,p}_{\mathrm{loc}}(M)$ satisfying $\Delta_{f;p} v \leq 0$ weakly on M, is constant, and *p-nonparabolic* otherwise. In Theorem 1.2, the weighted manifold M is 2-nonparabolic, since (1.3) ($\beta > 2$) is assumed (see [7], [8], Theorem 1.5), and it will be conjectured that if $\beta > p$ and the function ψ is a nonnegative C^1 function such that $\psi'(t) \leq 0$ and $\int_0^{+\infty} (t\psi(t))^{1/(p-1)} dt < +\infty$, then any positive solution u to the equation $-\Delta_{f;p} u + W|u|^{p-2} u = 0$ in M converges to a positive constant at infinity if $|W| \leq \psi(r)$ on M (see [25] and references therein for related problems). We remark that if (M, g_M, μ_f) is p-parabolic, then for any nonnegative $W \in L^\infty_{\mathrm{loc}}(M)$ which does not vanish identically, a positive solution v to equation $-\Delta_{f;p} v + W|v|^{p-2} v = 0$ in M is unbounded, because $\sup_M v - v$ is p-superharmonic if $\sup_M v < +\infty$.

Now we need some terminology to state the next result. For $n \in (-\infty, +\infty]$, the n-dimensional Bakry–Émery Ricci curvature is defined by

$$\mathrm{Ric}_f^n = \mathrm{Ric}_M + Ddf - \frac{df \otimes df}{n-m}$$

if $n \in (-\infty, +\infty) \setminus \{m\}$, and

$$\mathrm{Ric}_f^\infty = \mathrm{Ric}_M + Ddf$$

if $n = +\infty$. We assume that $n = m$ if and only if f is constant. We note that in Theorems 1.1 and 1.2, we can replace conditions (1.1) and (1.2) with the following one:

$$\inf_M (1+r)^2 \,\mathrm{Ric}_f^n > -\infty$$

for some $n > m$ (see Remark 2.14 and Corollary 2.18). Now we state:

Theorem 1.3. *Let (M, g_M, μ_f) be a connected, noncompact, complete weighted Riemannian manifold of dimension m. Suppose that for some $n \in [m, +\infty)$ and $\kappa \geq 0$,*

$$\mathrm{Ric}_f^n \geq -(n-1)\kappa \quad \text{on } M. \tag{1.4}$$

(i) *Let $u \in L^{1,p}_{\mathrm{loc}}(M) \cap C^{1,\alpha}_{\mathrm{loc}}(M)$ be a positive solution to the equation $-\Delta_{f,p} u + \Lambda |u|^{p-2} u = 0$ in M, where Λ is a positive constant. Then one has*

$$Z(p,n,\kappa,\Lambda) \leq \sup_M |\nabla \log u| \leq Y(p,n,\kappa,\Lambda). \tag{1.5}$$

Here $Z(p,n,\kappa,\Lambda)$ is the unique positive root of the equation

$$(p-1) Z^p + (n-1)\sqrt{\kappa}\, Z^{p-1} = \Lambda,$$

and $Y(p,n,\kappa,\Lambda)$ is the unique positive root of the equation

$$(p-1) Y^p - (n-1)\sqrt{\kappa}\, Y^{p-1} = \Lambda.$$

(ii) *Given $p > 1$ and $\Lambda > 0$, suppose that*

$$\delta^{(\infty)}(M) < \frac{Z(p,n,\kappa,\Lambda)}{Y(p,n,\kappa,\Lambda)} \ (\leq 1). \tag{1.6}$$

Then every positive solution u to the equation $-\Delta_{f,p} u + \Lambda |u|^{p-2} u = 0$ in M is an exhaustion function and satisfies

$$u(x) \geq u(o)\, e^{C r(x) + C'}, \quad x \in M,$$

where $C = \frac{1}{2}(Z(p,n,\kappa,\Lambda) - \delta^{(\infty)}(M) Y(p,n,\kappa,\Lambda))$ and C' is a constant independent of u.

We note that if $\kappa = 0$, then $Z(p,n,0,\Lambda) = Y(p,n,0,\Lambda) = (\Lambda/(p-1))^{1/p}$, the equalities hold in (1.5), and $C = \frac{1}{2}(\Lambda/(p-1))^{1/p}(1 - \delta^{(\infty)}(M))$ (see Example 4.7 for a simple example of Riemannian manifolds satisfying $\delta^{(\infty)}(M) < 1$).

In the case $p = 2$, applying Theorem 2.6 of Ancona [2] to Theorem 1.3, we have:

Corollary 1.4. *Let (M, g_M, μ_f) be as in Theorem 1.3 and assume (1.4). Let Λ be a positive constant and let W be a locally bounded function on M satisfying*

$$\inf \left\{ Q_{2;W}(v) \mid v \in C_0^\infty(M), \textstyle\int_M v^2 d\mu_f = 1 \right\} > 0.$$

Suppose that there exists a nonnegative, nonincreasing function $\Psi(t)$ *on* $[0,+\infty)$ *with* $\int_0^\infty \Psi(t)\,dt < +\infty$ *such that*

$$|W(x) - \Lambda| \leq \Psi(r(x)), \quad x \in M.$$

Then the following assertions hold:

(i) *A positive solution* u *to the equation* $Q'_{2,W}(u) = 0$ *in* M *satisfies*

$$u(x) \leq u(y)\, e^{Y(2,n,\kappa,\Lambda)\,\mathrm{dis}_M(x,y)+C''}$$

for all $x, y \in M$, *where* C'' *is a positive constant independent of* u.

(ii) *If* (1.6) *with* $p = 2$ *holds, then a positive solution* u *of the equation* $Q'_{2;W}(u) = 0$ *in* M *satisfies*

$$u(x) \geq u(o)\, e^{Cr(x)+C'}, \quad x \in M,$$

where $C = \frac{1}{2}(Z(2,n,\kappa,\Lambda) - \delta^{(\infty)}(M)Y(2,n,\kappa,\Lambda))$ *as in Theorem* 1.3, *and* C' *is a constant independent of* u.

We remark that in the case where $p = 2$ and $f = 0$, (1.5) is proved by Borbély [4] in a different way from ours. To get the upper bound in (1.5), we refer to the method in Sung and Wang [32], Dung and Dat [9], where positive eigenfunctions with eigenvalue λ (≥ 0), that is, solutions to the equation $\Delta_{f,p}u + \lambda|u|^{p-2}u = 0$ in M, are studied, and the gradient estimate from above by the constant $Y(p,n,\kappa,-\lambda)$ is proved. For the lower bound in (1.5), we employ the Laplacian comparison theorem derived from the assumption on a lower bound for the tensor Ric_f^n.

The outline of the paper is as follows. In Section 2, we recall first a comparison principle for the operators $Q'_{p;W}$ under consideration and then we show some Laplacian comparison results to derive volume doubling properties (VD) and scaled Poincaré inequalities (PI(p)) on metric balls. In Section 3, we derive Harnack inequalities for positive solutions to the equation $Q'_{p;W}(u) = 0$ with bounded potentials W. Based on the Harnack inequalities, we completes the proof of Theorem 1.1. Section 4 is devoted to proving Theorem 1.2 and furthermore discussing some results, remarks and examples concerning Theorems 1.1, 1.2 and 1.3; for example, we prove that (M, g_M, μ_f) fulfills (RCA) and (VC) if $\delta^{(\infty)}(M) < 1$ and some volume growth conditions are satisfied (see Proposition 4.3 (iii)). In Section 5, we study positive solutions to the equation $Q'_{p;\Lambda}(u) = 0$ in M, where λ is a positive constant, and Theorem 1.3, Corollary 1.4 and a related rigidity result are verified.

2. Laplacian comparison results

Let (M, g_M, μ_f) be a connected, complete weighted Riemannian manifold of dimension m. In this section, we first mention a comparison principle for operators $Q'_{p;W}$ on a domain of M to employ sub/supersolution techniques in our situation. We refer to Pinchover and Psaradakis [24]. Secondly, we discuss some Laplacian comparison results to derive volume doubling properties and scaled Poincaré inequalities on metric balls.

We begin with the following.

Theorem 2.1 ([24]). *Let Ω be a bounded Lipschitz domain in M. Given a function $W \in L^\infty(\Omega)$, suppose that $\inf_{u \in W_0^{1,p}(\Omega)} Q_{p;W}(u)/\|u\|^p_{L^p(\Omega)} > 0$, that is, that the principal eigenvalue of the operator $Q'_{p;W}$ is positive. Let $f, \phi, \psi \in L^{1,p}(\Omega) \cap C(\overline{\Omega})$, where $f \geq 0$ a.e. in Ω and $f > 0$ on $\partial\Omega$, and*

$$\begin{cases} Q'_{p,W}(\psi) \leq 0 \leq Q'_{p;W}(\phi) & \text{in } \Omega \text{ in the weak sense,} \\ \psi \leq f \leq \phi & \text{on } \partial\Omega, \\ 0 \leq \phi & \text{in } \Omega. \end{cases}$$

Then there exists a unique nonnegative solution $v \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ of

$$\begin{cases} Q'_{p;W}(v) = 0 & \text{in } \Omega, \\ v = f & \text{on } \partial\Omega \end{cases}$$

such that $\psi \leq v \leq \phi$ in Ω.

Proof. See Proposition 5.2 and Theorem 5.3 in [24]. ∎

To prove Theorem 1.1, we need the following.

Lemma 2.2. *Let u be a positive solution to the equation $-\Delta_{f;p}u + W|u|^{p-2}u = 0$ in a domain including $M \setminus B(o, T)$ for some $T \geq 0$. For $t \geq T$, denote $m(t) = \inf_{S(o,t)} u$ and $M(t) = \sup_{S(o,t)} u$.*

(i) *Suppose that $W \geq 0$. Then $M(t)$ is monotone for large t and converges to a number $M \in [0, +\infty]$ as $t \to +\infty$.*

(ii) *Suppose that $W \leq 0$. Then $m(t)$ is monotone for large t and converges to a number $m \in [0, +\infty]$ as $t \to +\infty$.*

Proof. For $t_1, t_2 \in (T, +\infty)$ with $t_1 < t_2$, we write $A(t_1, t_2)$ for $B(o, t_2) \setminus \overline{B(o, t_1)}$. Suppose first that $W \geq 0$. We compare u with a constant function $v = \max\{M(t_1), M(t_2)\}$, and we have $Q'_{p;W}(v) = Wv^{p-1} \geq 0 = Q'_{p;W}(u)$ in $A(t_1, t_2)$. For a connected component Ω of $A(t_1, t_2)$, we have $v \geq u$ on $\partial\Omega$, so that $v \geq u$ in Ω by Theorem 2.1. Thus $v \geq u$ in $A(t_1, t_2)$. This shows that $M(t) \leq \max\{M(t_1), M(t_2)\}$ for $t \in [t_1, t_2]$. Then it is easy to see that $M(t)$ is monotone for large t, and converges to a number $M \in [0, +\infty]$ as $t \to +\infty$. Similarly we can prove that $m(t) \geq \min\{m(t_1), m(t_2)\}$ for $t \in [t_1, t_2]$ if $W \leq 0$, which shows that $m(t)$ is also monotone for large t and hence converges to a number $m \in [0, +\infty]$ as $t \to +\infty$. This completes the proof of Lemma 2.2. ∎

Now we show some Laplacian comparison results on (M, g_M, μ_f). Take a point $x \in M$ and express the volume density in the geodesic polar coordinates centered at x as

$$dv_{g|\exp_x(r\xi)} = I(x, r, \xi)\, dr\, dv_\xi$$

for $r > 0$ and $\xi \in S_xM = \{\xi \in T_xM \mid |\xi| = 1\}$, where dv_ξ is the Riemannian volume element of the unit sphere S_xM. When we put

$$\tau_x(\xi) = \sup\{t > 0 \mid \mathrm{dis}_M(x, \exp_x t\xi) = t\} \in (0, +\infty]$$

for $\xi \in S_x M$, $I(x,t,\xi)$ is a positive smooth function on $(0, \tau_x(\xi))$ satisfying $I(x,0,\xi) = 0$ and $\lim_{t\to 0} I(x,t,\xi)/t^{m-1} = 1$. We denote by r_x the distance function to x. Then at $y = \exp_x(t\xi)$ $(0 < t < \tau_x(\xi))$, we have

$$\Delta r_x(y) = \frac{I'(x,t,\xi)}{I(x,t,\xi)} \quad \text{and} \quad \Delta_f r_x(y) = \frac{I_f'(x,t,\xi)}{I_f(x,t,\xi)},$$

where $\Delta_f = \Delta - \nabla f (= \Delta_{f;2})$, and $I_f(x,t,\xi) = e^{-f(t,\xi)} I(x,t,\xi)$ is the f-volume density in the geodesic polar coordinates (t,ξ).

We assume that there is a positive smooth function χ on $(0, R)$ $(0 < R \leq +\infty)$ such that $m - 1 \leq \limsup_{t\to 0} t\chi'(t)/\chi(t) < +\infty$ and

$$\frac{I_f'(x,t,\xi)}{I_f(x,t,\xi)} \leq \frac{\chi'(t)}{\chi(t)} \tag{2.1}$$

for $t \in (0, \tau_x(\xi) \wedge R)$.

Lemma 2.3. *Fix a point $x \in M$ and let $\chi(t)$ be as above. Then for a smooth function $\eta\colon [0,R) \to \mathbb{R}$ with $\eta' > 0$, one has*

$$\Delta_{f;p}\eta(r_x) \leq \Big(((\eta')^{p-1})' + \frac{\chi'}{\chi}(\eta')^{p-1} \Big)(r_x)$$

in the weak sense on $B(x,R)$; more precisely, for any nonnegative smooth function ϕ on $B(x,R)$ with compact support, one has

$$\int_{B(x,R)} |\nabla\eta(r_x)|^{p-2} g(\nabla\phi, \nabla\eta(r_x))\, d\mu_f \geq \int_{B(x,R)} -\phi\Big(((\eta')^{p-1})' + \frac{\chi'}{\chi}(\eta')^{p-1} \Big)(r_x)\, d\mu_f.$$

Proof. See, e.g., Proposition 3.7 in [28]. ∎

A Laplacian comparison result is stated in the following lemma.

Lemma 2.4. *Fix a point $x \in M$, and let $k(t)$ and $h(t)$ be continuous functions on $[0,R)$ such that*

$$\mathrm{Ric}_M \geq (m-1)k(r_x), \quad |\nabla f| \leq h(r_x)$$

on $B(x,R)$. Let $J(t)$ be a unique solution of the equation $J'' + kJ = 0$ in $[0,R)$, subject to the initial conditions $J(0) = 0$ and $J'(0) = 1$, and suppose that $J > 0$ on $(0,R)$. Then

$$\chi(t) = J(t)^{m-1} \exp \int_0^t h(s)\, ds$$

satisfies (2.1).

We remark that $J(t) \geq t$ for all $t \geq 0$ if $R = +\infty$ and k is nonpositive on $[0,+\infty)$.

Let κ be a nonnegative constant. In what follows, we write

$$s_\kappa(t) = \begin{cases} \frac{1}{2\sqrt{\kappa}}\,(e^{\sqrt{\kappa}t} - e^{-\sqrt{\kappa}t}) & \text{if } \kappa > 0, \\ t & \text{if } \kappa = 0. \end{cases}$$

We also let $c_\kappa(t) = s_\kappa'(t)$.

Lemma 2.5. *Let $\chi(t)$, $k(t)$, $h(t)$ and R be as in Lemma 2.4.*

(i) *Suppose that $k(t) = -\kappa$ and $h(t) = \alpha$, where κ and α are nonnegative constants. Then $\chi(t) = s_\kappa(t)^{m-1}e^{\alpha t}$ satisfies (2.1) with $R = +\infty$.*

(ii) *Suppose that $R = +\infty$ and k is nonpositive on $[0, +\infty)$, and moreover that $k(t) = -\kappa t^{-2}$ and $h(t) = \alpha t^{-1}$ for all $t \geq T$, where $\kappa \geq 0$, $\alpha \geq 0$, and $T > 0$ are some constants. Let $\beta(m, \kappa, \alpha) = \alpha + (m-1)(1 + \sqrt{1+4\kappa})/2$. Then one has*

$$\chi(t) = t^{\beta(m,\kappa,\alpha)} \left(C + C' t^{-\sqrt{1+4\kappa}}\right)^{m-1}$$

for all $t \geq T$, where $C > 0$ and C' are constants.

Proof. (i) The first assertion is obvious.

(ii) The solution J of the equation $J'' + k(t)J = 0$ in $[0, +\infty)$ is expressed as

$$J(t) = C_1\, t^{(1+\sqrt{1+4\kappa})/2} + C_2\, t^{(1-\sqrt{1+4\kappa})/2}$$

for all $t \geq T$, where $C_1 > 0$ and C_2 are some constants; moreover, we have

$$\exp \int_0^t h(s)\, ds = C_3 t^\alpha$$

for all $t \geq T$ and some constant $C_3 > 0$. These prove the assertion. ■

Now we fix a point o of M and write simply r for $r_o = \mathrm{dis}_M(o, *)$. Let W be a function in $L^\infty_{\mathrm{loc}}(M)$.

We assume first that $W \geq 0$ everywhere, and that there is a nonnegative continuous function $W_*(t)$ on $[0, \infty)$ such that

$$0 \leq W_*(r) \leq W \quad \text{on } M.$$

Lemma 2.6. *Let χ and $W_*(t)$ be as above. Suppose that for some constants a and b, with $0 \leq a < b$, $W_*(t) = 0$ for $t \in [0, a]$ and $W_*(t) > 0$ for $t \in (a, b)$. Then there exists a function $\eta \in C^1[a, +\infty) \cap C^2(a, +\infty)$ such that*

(i) $\eta(a) = 1,\ \eta'(a) = 0;$

(ii) $\eta(t) > 1,\ \eta'(t) > 0$ *for* $t > a$;

(iii) *it satisfies*

$$\left(\chi(t)\, \eta'(t)^{p-1}\right)' = W_*(t)\, \chi(t)\, \eta(t)^{p-1} \quad \text{on } (a, +\infty). \tag{2.2}$$

Proof. Let $\chi_\varepsilon(t) = \chi(t + \varepsilon)$ for $\varepsilon \in (0, 1]$. Then we can deduce from the existence and uniqueness theorems for ordinary differential equations that there are an interval $[a, R_\varepsilon)$ and a unique positive solution $\eta_\varepsilon \in C^1[a, R_\varepsilon) \cap C^2(a, R_\varepsilon)$ to the equation

$$\left(\chi_\varepsilon(t)\, |\eta_\varepsilon'(t)|^{p-2}\, \eta_\varepsilon'(t)\right)' = W_*(t)\, \chi_\varepsilon(t)\, \eta_\varepsilon(t)^{p-1},$$

subject to the initial conditions $\eta_\varepsilon(a) = 1$ and $\eta_\varepsilon'(a) = \varepsilon$. In fact, we have

$$\eta_\varepsilon'(t) = \left(\varepsilon^{p-1}\, \frac{\chi_\varepsilon(a)}{\chi_\varepsilon(t)} + \frac{1}{\chi_\varepsilon(t)} \int_a^t W_*(s)\, \chi_\varepsilon(s)\, \eta_\varepsilon(s)^{p-1}\, ds\right)^{1/(p-1)} > 0,$$

so that $1 \le \eta_{\varepsilon,\delta}(s) \le \eta_{\varepsilon,\delta}(t)$ for $a \le s \le t < R_\varepsilon$. We put here

$$\Phi_\varepsilon(t) = \Big(\varepsilon^{p-1} \frac{\chi_\varepsilon(a)}{\chi_\varepsilon(t)} + \frac{1}{\chi_\varepsilon(t)} \int_a^t W_*(s)\, \chi_\varepsilon(s)\, ds\Big)^{1/(p-1)}, \quad t \in [a, +\infty).$$

Then we get

$$\Phi_\varepsilon(t) \le \eta_\varepsilon'(t) \le \Phi_\varepsilon(t)\, \eta_\varepsilon(t)$$

for $t \in (a, R_\varepsilon)$. These show that

$$1 + \int_a^t \Phi_\varepsilon(s)\, ds \le \eta_\varepsilon(t) \le \exp \int_a^t \Phi_\varepsilon(s)\, ds,$$
$$\Phi_\varepsilon(t) \le \eta_\varepsilon'(t) \le \Phi_\varepsilon(t) \exp \int_a^t \Phi_\varepsilon(s)\, ds$$

for $t \in [a, R_\varepsilon)$. Now we put

$$\rho^*(t) = \Big(\varepsilon^{p-1} \max_{0\le\delta\le1} \frac{\chi_\delta(a)}{\chi_\delta(t)} + \max_{0\le\delta\le1} \frac{1}{\chi_\delta(t)} \int_a^t W_*(s)\, \chi_\delta(s)\, ds\Big)^{1/(p-1)},$$
$$\rho_*(t) = \Big(\min_{0\le\delta\le1} \frac{1}{\chi_\delta(t)} \int_a^t W_*(s)\, \chi_\delta(s)\, ds\Big)^{1/(p-1)},$$

for $t \in [a, +\infty)$. Then $\rho^*(t)$ and $\rho_*(t)$ are continuous functions on $[a, +\infty)$ satisfying $\rho_*(a) = 0$, $\rho_*(t) > 0$ for $t > a$, and

$$\rho_*(t) \le \Phi_\varepsilon(t) \le \rho^*(t)$$

for all $\varepsilon \in (0, 1]$ and for all $t \in [a, +\infty)$. Then we obtain

$$1 + \int_a^t \rho_*(s)\, ds \le \eta_\varepsilon(t) \le \exp \int_a^t \rho^*(s)\, ds,$$
$$\rho_*(t) \le \eta_\varepsilon'(t) \le \rho^*(t) \exp \int_a^t \rho^*(s)\, ds,$$

for all $\varepsilon \in (0, 1]$ and for all $t \in [a, R_\varepsilon)$. These estimates show that $[a, +\infty)$ is the right maximal interval of existence for the solutions η_ε, and that the above estimates hold for all $t \in [a, +\infty)$. Furthermore, as ε goes to zero, η_ε converges to a function $\eta \in C^1[a, +\infty) \cap C^2(a, +\infty)$, which is a solution to (2.2) subject to the initial conditions $\eta(a) = 1$ and $\eta'(a) = 0$. ∎

We remark that if $W_*(0) > 0$, then the same conclusions as in the above lemma with $a = 0$ hold. In what follows, we assume that the function η is defined on $[0, +\infty)$ by setting $\eta(t) = 1$ on $[0, a]$ if $a > 0$.

Proposition 2.7. *Let W and $\eta(t)$ be as above.*

(i) *Let $u \in L^{1,p}_{\mathrm{loc}}(M) \cap C(M)$ satisfy $-\Delta_{f;p} u + W|u|^{p-2}u \le 0$ on M in the weak sense. If $u(x_0) > 0$ for some $x_0 \in M$, then*

$$\max_{S(o,t)} u \ge \frac{u(x_0)}{\eta(r(x_0))}\, \eta(t)$$

for all $t \ge r(x_0)$.

(ii) *Let* $u \in L^{1,p}_{\mathrm{loc}}(M) \cap C(M)$ *satisfy* $-\Delta_{f;p}u + W|u|^{p-2}u \geq 0$ *on* M *in the weak sense. If* $u(x_0) < 0$ *for some* $x_0 \in M$*, then*

$$\min_{S(o,t)} u \leq \frac{u(x_0)}{\eta(r(x_0))}\,\eta(t)$$

for all $t \geq r(x_0)$.

Proof. By Lemma 2.3, we have

$$\Delta_{f;p}\eta(r) \leq W_*(r)\,\eta(r)^{p-1} \leq W\eta(r)^{p-1}$$

in the weak sense on M. Suppose that $u(x_0) > 0$ for some $x_0 \in M$ and that

$$\max_{S(o,t)} u < \frac{u(x_0)}{\eta(r(x_0))}\,\eta(t)$$

for some $t > r(x_0)$. We take $\varepsilon > 0$ in such a way that $\max_{S(o,t)} u < (1-\varepsilon)\frac{u(x_o)}{\eta(r(x_0))}\eta(t)$. Then it follows from Theorem 2.1 that $u \leq (1-\varepsilon)\frac{u(x_0)}{\eta(r(x_0))}\eta(r)$ in $B(o,t)$; in particular, we have $u(x_0) \leq (1-\varepsilon)u(x_0)$, so that $u(x_0) \leq 0$. But this contradicts the assumption. Thus (i) is proved. Applying the same arguments as above to $-u$, we can show the second assertion (ii). ∎

Corollary 2.8. *Let* W *and* $\eta(t)$ *be as above. Let* $u \in L^{1,p}_{\mathrm{loc}}(M) \cap C^{1,\alpha}_{\mathrm{loc}}(M)$ *be a solution to the equation* $-\Delta_{f;p}u + W|u|^{p-2}u = 0$ *on* M*. We have that*

(i) u *is positive everywhere on* M *if* $\liminf_{y\in M\to\infty} u(y)/\eta(r(y)) \geq 0$ *and* $u(x) > 0$ *for some* $x \in M$*,*

(ii) u *vanishes identically if* $\lim_{y\in M\to\infty} |u(y)|/\eta(r(y)) = 0$.

Lemma 2.9. *Let* $k(t)$*,* $h(t)$*,* $\chi(t)$*, and* R *be as in Lemma 2.4. Assume that* $R = +\infty$ *and that* k *is nonpositive on* $[0,+\infty)$*, and moreover that* $k(t) = -\kappa t^{-2}$ *and* $h(t) = \alpha t^{-1}$ *for all* $t \geq T$*, where* $\kappa \geq 0$*,* $\alpha \geq 0$ *and* $T > 0$ *are some constants.*

(i) *Suppose that* $W_*(t)$ *is nonincreasing in* $[T,+\infty)$ *and* $\int_0^\infty (W_*(s)s)^{1/(p-1)}ds = +\infty$*. Then* $\eta(t)$ *tends to infinity as* $t \to \infty$.

(ii) *Suppose that* $W_*(t) = \lambda t^{-p}$ *for all* $t \geq T$*, where* λ *is some positive constant. Let* $\gamma(p,m,\kappa,\alpha,\lambda)$ *be the positive solution of the equation*

$$x|x|^{p-2}(x(p-1) + \beta(m,\kappa,\alpha) + 1 - p) = \lambda.$$

Then η *satisfies*

$$\eta(t) \geq C(1+t)^{\gamma(p,m,\kappa,\alpha,\lambda)}$$

for some positive constant C *and all* $t \geq 0$.

Proof. (i) Since $\eta(t)$ is nondecreasing and $W_*(t)$ is nonincreasing in $[T,\infty)$, we have

$$\begin{aligned}\chi(t)(\eta'(t))^{p-1} &= \chi(T)(\eta'(T))^{p-1} + \int_T^t W_*(s)\,\chi(s)\,\eta(s)^{p-1}\,ds\\ &\geq \eta(T)^{p-1}\,W_*(t)\int_T^t \chi(s)\,ds,\end{aligned}$$

so that we get

$$\eta'(t) \geq \eta(T)\, W_*(t)^{1/(p-1)} \Big(\frac{1}{\chi(t)} \int_T^t \chi(s)\, ds \Big)^{1/(p-1)}$$

for all $t \geq T$. Since we have by Lemma 2.5 (ii), $C_4^{-1} t^\beta \leq \chi(t) \leq C_4 t^\beta$ for some constant $C_4 > 1$, where $\beta = \beta(m, \kappa, \alpha)$ in Lemma 2.5, we see that

$$\frac{1}{\chi(t)} \int_T^t \chi(s)\, ds \geq \frac{2^{\beta+1} - 1}{2^{\beta+1}(\beta+1)\, C_4^2}\, t$$

for all $t \geq 2T$, so that we obtain

$$\eta'(t) \geq \Big(\frac{2^{\beta+1} - 1}{2^{\beta+1}(\beta+1)\, C_4^2} \Big)^{1/(p-1)} \eta(T)\, (W_*(t)\, t)^{1/(p-1)}$$

for all $t \geq 2T$. This shows that

$$\eta(t) \geq \eta(2T) + \Big(\frac{2^{\beta+1} - 1}{2^{\beta+1}(\beta+1) C_4^2} \Big)^{1/(p-1)} \eta(T) \int_{2T}^t (W_*(s)s)^{1/(p-1)}\, ds$$

for all $t \geq 2T$. Thus (i) is proved.

(ii) Let $\sigma(t) = C_5 t^{\gamma(p,m,\kappa,\alpha,\lambda)}$, where C_5 is a positive constant chosen later. Then σ satisfies the same equation (2.2) as η in $[T, +\infty)$. This shows that $\eta(t) \geq \sigma(t)$ for $t \geq T$ if we choose C_5 in such a way that $\eta(T) \geq \sigma(T)$ and $\eta'(T) \geq \sigma'(T)$. ■

Now we consider a function W in $L^\infty_{\text{loc}}(M)$ such that $W \leq 0$ everywhere on M. We assume that there is a nonnegative continuous function $W_*(t)$ on $[0, \infty)$ such that

$$W \leq -W_*(r) \leq 0 \quad \text{on } M.$$

Lemma 2.10. *Let χ and W_* be as above. Suppose that for some constants a and b, with $0 \leq a < b$, $W_*(t) = 0$ for $t \in [0, a]$ and $W_*(t) > 0$ for $t \in (a, b)$. Then there exist an interval $[a, R)$ with $a < R \leq +\infty$ and a function $\omega \in C^1[a, R) \cap C^2(a, R)$ such that*

(i) $\omega(a) = 1,\ \omega'(a) = 0$;

(ii) $0 < \omega(t) < 1,\ \omega'(t) < 0$ *for* $t \in (a, R)$;

(iii) *it satisfies*

$$\big(\chi(t)(-\omega'(t))^{p-1}\big)' = W_*(t)\, \chi(t)\, \omega(t)^{p-1} \quad \textit{on } (a, R); \tag{2.3}$$

(iv) $[a, R)$ *is the right maximal interval of existence for the positive solution ω, and* $\lim_{t\to R} \omega(t) = 0$ *if* $R < +\infty$.

Proof. As in the proof of Lemma 2.6, we let $\chi_\varepsilon(t) = \chi(t + \varepsilon)$ for $\varepsilon \in (0, 1]$. Then there are an interval $[a, R_\varepsilon)$ and a unique positive solution $\omega_\varepsilon \in C^1[a, R_\varepsilon) \cap C^2(a, R_\varepsilon)$ to

$$\big(\chi_\varepsilon(t)\, |\omega_\varepsilon'(t)|^{p-2}\, \omega_\varepsilon'(t)\big)' = -W_*(t)\, \chi_\varepsilon(t)\, \omega_\varepsilon(t)^{p-1} \tag{2.4}$$

subject to the initial conditions $\omega_\varepsilon(a) = 1$, $\omega'_\varepsilon(a) = -\varepsilon$; moreover, $[a, R_\varepsilon)$ is the right maximal interval of existence for the positive solution ω_ε, and in the case where $R_\varepsilon < +\infty$, $\lim_{t\to R}\omega_\varepsilon(t) = 0$. We note here that equation (2.4) is also expressed as follows:

$$\omega''_\varepsilon(t) = -\frac{\chi'_\varepsilon(t)\,\omega'_\varepsilon(t)}{(p-1)\,\chi_\varepsilon(t)} - \frac{W_*(t)\,\omega_\varepsilon(t)^{p-1}}{(p-1)\,(-\omega'_\varepsilon(t))^{p-2}}. \tag{2.5}$$

Then we have

$$-\omega'_\varepsilon(t) = \left(\varepsilon^{p-1}\,\frac{\chi_\varepsilon(a)}{\chi_\varepsilon(t)} + \frac{1}{\chi_\varepsilon(t)}\int_a^t W_*(s)\,\chi_\varepsilon(s)\,\omega_\varepsilon(s)^{p-1}\,ds\right)^{1/(p-1)} > 0$$

so long as $\omega_\varepsilon(t)$ exists and keeps to be positive. Thus this holds on $[a, R_\varepsilon)$, and in particular we have $1 \geq \omega_{\varepsilon,\delta}(s) \geq \omega_{\varepsilon,\delta}(t)$ for $a \leq s \leq t < R_\varepsilon$. Using these inequalities, we see that

$$\Phi_\varepsilon(t)\,\omega_\varepsilon(t) \leq -\omega'_\varepsilon(t) \leq \Phi_\varepsilon(t)$$

for $t \in (a, R_\varepsilon)$, where as in the proof of Lemma 2.6, we let

$$\Phi_\varepsilon(t) = \left(\varepsilon^{p-1}\,\frac{\chi_\varepsilon(a)}{\chi_\varepsilon(t)} + \frac{1}{\chi_\varepsilon(t)}\int_a^t W_*(s)\,\chi_\varepsilon(s)\,ds\right)^{1/(p-1)}.$$

Using the last inequality, we obtain

$$\omega_\varepsilon(t) \geq 1 - \int_a^t \Phi_\varepsilon(s)\,ds,$$

from which it follows that

$$\Phi_\varepsilon(t)\left(1 - \int_a^t \Phi_\varepsilon(s)\,ds\right) \leq -\omega'_\varepsilon(t). \tag{2.6}$$

Here, in view of (2.5), we notice that

$$\omega''_\varepsilon(t) \leq -\omega'_\varepsilon(t)\,\frac{\chi'(t)}{(p-1)\chi(t)},$$

and hence we have

$$-\omega'_\varepsilon(t) \geq -\omega'_\varepsilon(s)\left(\frac{\chi_\varepsilon(s)}{\chi_\varepsilon(t)}\right)^{1/(p-1)}, \quad a \leq s \leq t < R_\varepsilon.$$

This together with (2.6) shows that

$$-\omega'_{\varepsilon,\delta}(t) \geq \left(\frac{\chi_\varepsilon(s)}{\chi_\varepsilon(t)}\right)^{1/(p-1)}\Phi_\varepsilon(s)\left(1 - \int_a^s \Phi_\varepsilon(u)\,du\right), \quad a \leq s \leq t < R_\varepsilon.$$

Now, as in the proof of Lemma 2.6, we have continuous functions $\rho^*(t)$, $\rho_*(t)$ on $[a, +\infty)$ satisfying $\rho_*(a) = 0$, $\rho_*(t) > 0$ for $t > a$, and $\rho_*(t) \leq \Phi_\varepsilon(t) \leq \rho^*(t)$ for all $\varepsilon \in (0, 1]$ and for all $t \in [a, +\infty)$. Here we fix a number $b > a$ in such a way that $\int_a^b \rho^*(s)\,ds < 1$, and then define a positive continuous function $\sigma_*(t)$ by putting $\sigma_*(t) = 1$ for $a \le t \le b$ and

$$\sigma_*(t) = \min_{0\le\delta\le1} \left(\frac{\chi_\delta(b)}{\chi_\delta(t)}\right)^{1/(p-1)}$$

for $t \ge b$. Then we obtain

$$\sigma_*(t)\,\rho_*(t\wedge b)\Big(1 - \int_a^{t\wedge b} \rho^*(s)\,ds\Big) \le -\omega_\varepsilon'(t) \le \rho^*(t);$$
$$1 - \int_a^t \rho^*(s)\,ds \le \ \omega_\varepsilon(t) \ \le 1$$

for all $\varepsilon \in (0, 1]$ and for all $t \in [a, R_\varepsilon)$. We remark that $t < R_\varepsilon$ if $\int_a^t \rho^*(s)ds < 1$. These estimates show that there are an interval $[a, R)$ and a positive function $\omega \in C^1[a, R) \cap C^2(a, R)$ which is a unique solution to equation (2.3), subject to the conditions $\omega(a) = 1$ and $\omega'(a) = 0$, such that $[a, R)$ is the maximal interval of existence for ω, and as ε goes to zero, ω_ε converges to ω. ∎

We remark that if $W_*(0) > 0$, then the same conclusions as in the above lemma with $a = 0$ hold. In what follows, we assume that the function ω is defined on $[0, R)$ by setting $\omega(t) = 1$ on $[0, a]$ if $a > 0$.

Proposition 2.11. *Let W and $\omega(t)$ be as above. Let $v \in L^{1,p}_{\mathrm{loc}}(B(o, R)) \cap C(B(o, R))$ be a positive function satisfying*

$$-\Delta_{f;p} v + W|v|^{p-2}v \ge 0 \tag{2.7}$$

on $B(o, R)$ in the weak sense. Then

$$\omega(t) \ge \frac{1}{v(o)} \min_{S(o,t)} v \tag{2.8}$$

for $t \in [0, R)$. In particular, if $R = +\infty$, that is, if (2.7) is satisfied on M, then (2.8) holds for all $t \ge 0$.

Proof. We observe that $\omega(r)$ satisfies

$$\Delta_{f;p}\omega(r) \ge -W_*(r)\,\omega(r)^{p-1} \ge W\omega(r)^{p-1}$$

in $B(o, R)$. Then we can deduce (2.8) from the same argument as in Proposition 2.7, together with Theorem 2.1. ∎

Lemma 2.12. *Let $k(t)$, $h(t)$, $\chi(t)$ and R be as in Lemma 2.4. Assume that $Q_{p;W} \ge 0$, $R = +\infty$, and k is nonpositive on $[0, +\infty)$, and moreover that $k(t) = -\kappa t^{-2}$ and $h(t) = \alpha t^{-1}$ for all $t \ge T$, where $\kappa \ge 0$, $\alpha \ge 0$, $T > 0$ are some constants.*

(i) *Suppose that $W_*(t)$ is nonincreasing in $[T, +\infty)$ and $\int_0^{+\infty} (W_*(t)t)^{1/(p-1)} dt = +\infty$. Then $\omega(t)$ tends to zero as $t \to +\infty$.*

(ii) *Suppose that* $p = 2$ *and* $W_*(t) = \lambda t^{-2}$ *for all* $t \geq T$, *where* λ *is a positive constant less than* $(\beta-1)^2/4$ *with* $\beta = \beta(m,\kappa,\alpha)$ *in Lemma* 2.5. *Then* $\omega(t)$ *satisfies*

$$\omega(t) \leq C\,(1+t)^{-\zeta}$$

for some positive constant C *and all* $t \geq 0$, *where*

$$\zeta = \min\left\{\frac{1}{2}((\beta-1) - \sqrt{(\beta-1)^2 - 4\lambda}\,), 2\sqrt{1+4\kappa} + \frac{\lambda}{\beta-1}\right\}.$$

Proof. (i) Let $t \geq 2T$. Since $\omega(t)$ and $W_*(t)$ are nonincreasing, we have by (2.3),

$$\begin{aligned} 1-\omega(t) &= 1-\omega(T) + \int_T^t \Big(\frac{1}{\chi(s)}\int_0^s W_*(x)\,\chi(x)\,\omega(x)^{p-1}\,dx\Big)^{1/(p-1)} ds \\ &\geq \int_T^t \Big(\frac{1}{\chi(s)}\int_T^s W_*(x)\,\chi(x)\,\omega(x)^{p-1}\,dx\Big)^{1/(p-1)} ds \\ &\geq \omega(t)\int_T^t \Big(\frac{W_*(s)}{\chi(s)}\int_T^s \chi(x)\,dx\Big)^{1/(p-1)} ds. \end{aligned}$$

We recall that

$$\frac{1}{\chi(s)}\int_T^s \chi(x)\,dx \geq C_6\, s$$

for some constant $C_6 > 0$ and all $s \geq 2T$ (see the proof of Lemma 2.9). Therefore we have

$$1-\omega(t) \geq \omega(t)\,C_6 \int_{2T}^t (W_*(s)s)^{1/(p-1)}\,ds,$$

and hence we obtain

$$\omega(t) \leq \frac{1}{1 + C_6\int_{2T}^t (W_*(s)s)^{1/(p-1)}\,ds}.$$

Thus $w(t)$ tends to zero if $\int_0^{+\infty}(W_*(t)t)^{1/(p-1)}\,dt = +\infty$.

(ii) Let $t \geq 2T$. Then we have

$$\chi(t)\,\omega'(t) = \chi(T)\,\omega'(T) - \int_T^t \frac{\lambda}{s^2}\,\chi(s)\,\omega(s)\,ds < -\omega(t)\int_T^t \frac{\lambda\chi(s)}{s^2}\,ds$$

and hence we get

$$\frac{\omega'(t)}{\omega(t)} \leq -\frac{\lambda}{\chi(t)}\int_T^t \frac{\chi(s)}{s^2}\,ds.$$

In view of Lemma 2.5 (ii), we see that

$$-\frac{\lambda}{\chi(t)}\int_T^t \frac{\chi(s)}{s^2}\,ds = -\frac{\lambda}{\beta-1}\,\frac{1}{t} + O(t^{-\sqrt{1+4\kappa}}),$$

so that

$$\frac{\omega'(t)}{\omega(t)} \leq -\frac{\lambda}{\beta-1}\,\frac{1}{t} + O(t^{-\sqrt{1+4\kappa}}). \tag{2.9}$$

Note here that $\beta > 1$. These show that

$$\omega(t) \le C_7 t^{-\lambda/(\beta-1)} \tag{2.10}$$

for all $t \ge 2T$ and some constant $C_7 > 0$. Then it follows from (2.9) and (2.10) that

$$\omega'(t) \le -\frac{\lambda}{\beta-1} C_7 t^{-\lambda/(\beta-1)-1}(1+o(1)). \tag{2.11}$$

We now continue the argument to improve the decay order. Let

$$\begin{aligned}
E(t) &= \frac{\chi'(t)}{\chi(t)} - \frac{\beta}{t}\Big(= (m-1)\frac{J'(t)}{J(t)} + h(t) - \frac{\beta}{t}\Big),\\
\delta_\pm &= \frac{1}{2}\Big(-(\beta-1) \pm \sqrt{(\beta-1)^2 - 4\lambda}\Big),\\
F(t) &= a t^{\delta_+} + b t^{\delta_-},\\
G(t) &= t^{\delta_+}\int_T^t s^{1-\beta-2\delta_+}\Big(\int_T^s x^{\beta+\delta_+-1}(-E(x)\omega'(x))\,dx\Big)\,ds.
\end{aligned}$$

Here, a and b are constants chosen in such a way that $F(T) = \omega(T)$ and $F'(T) = \omega'(T)$. Then F and G respectively satisfy

$$F''(t) + \frac{\beta}{t}F'(t) + \frac{\lambda}{t^2}F(t) = 0, \quad F(T) = \omega(T), \quad F'(T) = \omega'(T);$$

$$G''(t) + \frac{\beta}{t}G'(t) + \frac{\lambda}{t^2}G(t) = -E(t)\omega'(t), \quad G(T) = G'(T) = 0.$$

Therefore the uniqueness theorem for ordinary differential equations implies

$$\omega(t) = F(t) + G(t), \quad t \ge T.$$

Since $J(t) = c\,t^{(1+\sqrt{1+4\kappa})/2} + d\,t^{(1-\sqrt{1+4\kappa})/2}$ for $t \ge T$ and some constants $c > 0$ and d, we have

$$E(t) = O(t^{-2\sqrt{1+4\kappa}-1})$$

and in view of (2.9), (2.10) and (2.11), we deduce that

$$G(t) = O(t^{-2\sqrt{1+4\kappa}-\lambda/(\beta-1)}).$$

In this way, we obtain

$$\omega(t) \le C_8 t^{-\zeta}, \quad t \ge T$$

for some constant $C_8 > 0$. This completes the proof of Lemma 2.12. ∎

We have started our arguments from Lemma 2.4. Here we mention the following.

Lemma 2.13. *Let $n \in (m, +\infty)$ and fix a point $x \in M$. Let $k(t)$ be a continuous function on $[0, R)$ $(R \in (0, +\infty])$ such that*

$$\mathrm{Ric}_f^n \ge (n-1)k(r_x)$$

on $B(x, R)$*. Let* $J(t)$ *be a unique solution of the equation* $J'' + kJ = 0$ *in* $[0, R)$*, subject to the initial conditions* $J(0) = 0$ *and* $J'(0) = 1$*, and suppose that* $J > 0$ *on* $(0, R)$*. Then*

$$\chi(t) = J(t)^{n-1}$$

satisfies (2.1).

Proof. See [16] for the case where n is an integer greater than m, and [27] for $n \in (m, +\infty)$. ∎

Remark 2.14. By starting with this lemma, instead of Lemma 2.4, we have Lemmas 2.5, 2.9, and 2.12, where α and m are respectively replaced with 0 and n; furthermore, Theorems 1.1 and 1.2, and Proposition 3.7 stated at the end of Section 3, hold if we replace (1.1) and (1.2) with the condition $\inf_M (1 + r)^2 \operatorname{Ric}_f^n > -\infty$.

Now we are concerned with the volume growth and scale-invariant Poincaré inequalities on the weighted Riemannian manifold (M, g_M, μ_f). By virtue of Subsection 5.6.3 in Saloff-Coste [31], we have the following.

Lemma 2.15. *Fix* $p \in [1, +\infty)$*,* $R > 0$ *and a point* $y \in M$*. Suppose that there is a positive nondecreasing* C^1 *function* $\chi(t)$ *on* $(0, R)$ *satisfying* (2.1) *for all* $x \in B(y, R)$ *and* $t \in (0, \tau_x(\xi) \wedge 2R)$*, and furthermore that there is a constant* $F(R)$ *such that*

$$\chi(t) \le F(R)\,\chi(t/2) \tag{2.12}$$

for all $0 < t < 2R$*. Then the following volume doubling property* (VD) *and scale-invariant Poincaré inequalities* (PI(p))*, respectively, hold*:

(i) *for any ball* $B(x, 2t) \subset B(y, R)$*,*

$$\mu_f(B(x,t)) \le 4F(R)\,\mu_f(B(x,t/2));$$

(ii) *for every ball* $B(x, 2t) \subset B(y, R)$ *and any* $u \in L^{1,p}_{\mathrm{loc}}(B(x, 2t))$*,*

$$\int_{B(x,t/2)} |u - u_{B(x,t/2)}|^p \, d\mu_f \le 4F(R) t^p \int_{B(x,t)} |\nabla u|^p \, d\mu_f,$$

where

$$u_{B(x,t/2)} = \frac{1}{\mu_f(B(x,t/2))} \int_{B(x,t/2)} u \, d\mu_f.$$

Proof. Since

$$\left(\frac{I_f(x,t,\xi)}{\chi(t)}\right)' = \frac{I_f(x,t,\xi)}{\chi(t)} \left(\frac{I_f'(x,t,\xi)}{I_f(x,t,\xi)} - \frac{\chi'(t)}{\chi(t)}\right) \le 0,$$

we have

$$\frac{I_f(x,s,\xi)}{\chi(s)} \ge \frac{I_f(x,t,\xi)}{\chi(t)}, \quad 0 < s \le t < \tau_x(\xi) \wedge 2R.$$

Moreover, since

$$\left(\frac{\int_0^t I_f(x,r,\xi)\,dr}{\int_0^t \chi(r)\,dr}\right)' = \frac{\chi(t)}{\left(\int_0^t \chi(r)\,dr\right)^2}\int_0^t \left(\frac{I_f(x,t,\xi)}{\chi(t)} - \frac{I_f(x,s,\xi)}{\chi(s)}\right)\chi(s)\,ds \le 0$$

for $0 < t < \tau_x(\xi) \wedge 2R$, we get

$$\frac{\int_0^{s\wedge\tau(\xi)} I_f(x,r,\xi)\,dr}{\int_0^{s\wedge\tau(\xi)} \chi(r)\,dr} \ge \frac{\int_0^{t\wedge\tau(\xi)} I_f(x,r,\xi)\,dr}{\int_0^{t\wedge\tau(\xi)} \chi(r)\,dr}, \quad 0 < s \le t < 2R.$$

Noting that

$$\frac{\int_0^{s\wedge\tau(\xi)} \chi(r)\,dr}{\int_0^{t\wedge\tau(\xi)} \chi(r)\,dr} \ge \frac{\int_0^{s} \chi(r)\,dr}{\int_0^{t} \chi(r)\,dr},$$

we obtain

$$\frac{\int_0^{s\wedge\tau(\xi)} I_f(x,r,\xi)\,dr}{\int_0^{s} \chi(r)\,dr} \ge \frac{\int_0^{t\wedge\tau(\xi)} I_f(x,r,\xi)\,dr}{\int_0^{t} \chi(r)\,dr}.$$

This shows that

$$\frac{\mu_f(B(x,s))}{\int_0^s \chi(r)\,dr} \ge \frac{\mu_f(B(x,t))}{\int_0^t \chi(r)\,dr}, \quad 0 < s < t \le 2R. \tag{2.13}$$

Finally, if $\chi(t/2) \ge \chi(t)/F(R)$, then

$$\frac{\mu_f(B(x,s))}{\mu_f(B(x,t))} \ge \frac{1}{2F(R)}, \quad 0 < \frac{t}{2} \le s \le t < 2R, \tag{2.14}$$

and

$$\frac{I_f(x,s,\xi)}{I_f(x,t,\xi)} \ge \frac{1}{F(R)}, \quad 0 < \frac{t}{2} \le s \le t \le \tau_x(\xi) \wedge 2R. \tag{2.15}$$

Obviously, (2.14) shows the volume doubling property (VD) in (i). Moreover, in view of the proof of Theorem 5.6.5 in [31], (2.15) yields the inequalities (PI(p)) in (ii). This completes the proof of Lemma 2.15. ■

Similarly, we have the following.

Lemma 2.16. *Fix $p \in [1,\infty)$, $R > 0$ and a point $y \in M$. Suppose that*

$$\sup_{B(y,R)} f - \inf_{B(y,R)} f \le b$$

for some positive constant b, and that there is a positive nondecreasing C^1 function $\chi_*(t)$ on $(0,R)$ satisfying $m - 1 \le \limsup_{t\to 0} t\chi_*'(t)/\chi_*(t) < +\infty$,

$$\frac{I'(x,t,\xi)}{I(x,t,\xi)} \le \frac{\chi_*'(t)}{\chi_*(t)} \tag{2.16}$$

for all $x \in B(y,R)$ and $t \in (0, \tau_x(\xi) \wedge 2R)$, and furthermore

$$\chi_*(t) \le F(R)\chi_*(t/2), \quad 0 < t \le 2R, \tag{2.17}$$

for some $F(R)$.

Then the following volume doubling property (VD) *and scale-invariant Poincaré inequalities* (PI(p))*, respectively, hold*:

(i) *for any ball* $B(x, 2t) \subset B(y, R)$*,*

$$\mu_f(B(x,t)) \le 4F(R)\, e^b\, \mu_f(B(x,t/2));$$

(ii) *for every ball* $B(x, 2t) \subset B(y, R)$ *and any* $u \in L^{1,p}_{\mathrm{loc}}(B(x, 2t))$*,*

$$\int_{B(x,t/2)} |u - u_{B(x,t/2)}|^p \, d\mu_f \le 4F(R) e^b t^p \int_{B(x,t)} |\nabla u|^p \, d\mu_f.$$

Making use of Lemmas 2.15 and 2.16, we extend the Bishop–Gromov volume doubling property and (a weak form of) a theorem due to Buser [5] to weighted Riemannian manifolds in the following result.

Proposition 2.17. *Let* (M, g_M, μ_f) *be a connected, noncompact, complete weighted Riemannian manifold of dimension* m*. Fix* $p \in [1, +\infty)$*,* $R > 0$*, and a point* $y \in M$*.*

(i) *Suppose that the Ricci curvature* Ric_M *is bounded from below by* $-(m-1)\kappa$ *on* $B(y, R)$*, and that* $\sup_{B(y,R)} f - \inf_{B(y,R)} f \le b$*, where* κ *and* b *are nonnegative constants. Then* (VD) *and* (PI(p)) (*as in Lemma* 2.16) *hold with a constant* $e^{C(m)(1+b+\sqrt{\kappa}R)}$*, where* $C(m)$ *is a constant depending only on* m*.*

(ii) *Suppose that the Bakry-Émery Ricci curvature* Ric_f^n *with* $n > m$ *is bounded from below by* $-(n-1)\kappa$ *on* $B(y, R)$*, where* κ *is a nonnegative constant. Then* (VD) *and* (PI(p)) (*as in Lemma* 2.15) *hold with a constant* $e^{C(n)(1+\sqrt{\kappa}R)}$*, where* $C(n)$ *is a positive constant depending only on* n*.*

(iii) *Suppose that the Bakry–Émery Ricci curvature* Ric_f^∞ *is bounded from below by* $-(m-1)\kappa$ *on* $B(y, R)$ *and that* $\sup_{B(y,R)} f - \inf_{B(y,R)} f \le b$*, where* κ *and* b *are nonnegative constants. Then* (VD) *and* (PI(p)) (*as in Lemma* 2.15) *hold with a constant* $e^{C(m)(1+b)(1+\sqrt{\kappa}R)}$.

Proof. For the assertion (i), we let $\chi_*(t) = s_\kappa(t)^{m-1}$. Then by the assumption, we see that χ_* satisfies (2.16) and we can take $F(R) = 2^{m+1} e^{(m-1)\sqrt{\kappa}R}$ which satisfies (2.17). Hence (i) follows from Lemma 2.16.

For the assertion (ii), we let $\chi(t) = s_\kappa(t)^{n-1}$. Then by the assumption on the tensor Ric_f^n, χ satisfies (2.1) (see Lemma 2.12), and we can take $F(R) = 2^{n+1} e^{(n-1)\sqrt{\kappa}R}$, which satisfies (2.12). Hence (ii) follows from Lemma 2.15.

We consider assertion (iii). It is shown by Wei and Wylie [35] that

$$\frac{I_f'(x,t,\xi)}{I_f(x,t,\xi)} \le (m-1)\sqrt{\kappa}\coth(\sqrt{\kappa}t) + \frac{2\kappa}{\sinh^2(\sqrt{\kappa}t)} \int_0^t (f(s,\xi) - f(t,\xi)) \cosh(2\sqrt{\kappa}s)\, ds.$$

Since $\sup_{B(y,R)} f - \inf_{B(y,R)} f \le b$, we obtain

$$\begin{aligned}\frac{I_f'(x,t,\xi)}{I_f(x,t,\xi)} &\le (m-1)\sqrt{\kappa}\coth(\sqrt{\kappa}t) + \frac{2\kappa b}{\sinh^2(\sqrt{\kappa}t)} \int_0^t \cosh(2\sqrt{\kappa}s)\, ds \\ &\le (m-1+2b)\sqrt{\kappa}\coth(\sqrt{\kappa}t).\end{aligned}$$

Hence letting $\chi(t) = s_\kappa(t)^{m-1+2b}$, we have (2.1) and take $F(R) = 2^{m+2b} e^{(m-1+2b)\sqrt{\kappa}R}$, which satisfies (2.12). In this way, (iii) follows from Lemma 2.15. ∎

Corollary 2.18. *Let (M, g_M, μ_f)be as above. A family of balls remote to a fixed point o satisfies* (VD) *and* (PI(p)) *under one of the following conditions*:

(i) $\mathrm{Ric}_M \geq -\frac{(m-1)\kappa}{(1+r)^2}$ *and* $|\nabla f| \leq \frac{\alpha}{1+r}$ *on M for some constants $\kappa \geq 0$ and $\alpha \geq 0$*;

(ii) $\mathrm{Ric}_f^n \geq -\frac{(n-1)\kappa}{(1+r)^2}$ $(n > m)$ *on M for some constant $\kappa \geq 0$*;

(iii) $\mathrm{Ric}_f^\infty \geq -\frac{(m-1)\kappa}{(1+r)^2}$ *on M and*

$$\sup\left\{ \sup_{B(o,2^{k+2})\setminus B(o,2^k)} f - \inf_{B(o,2^{k+2})\setminus B(o,2^k)} f \;\middle|\; k = 1, 2, \ldots \right\} \leq b < +\infty$$

for some constants $\kappa \geq 0$ and $b \geq 0$.

Proposition 2.19. *Let (M, g_M, μ_f) be a connected, noncompact, complete weighted Riemannian manifold of dimension m and assume that* (VC) *holds.*

(i) *Suppose that* $\mathrm{Ric}_M \geq -\frac{(m-1)\kappa}{(1+r)^2}$ *and* $|\nabla f| \leq \frac{\alpha}{1+r}$ *on M for some constants $\kappa \geq 0$ and $\alpha \geq 0$. Then one has*

$$\mu_f(B(o,t)) \leq C(1+t)^{m+\alpha}$$

for some constant $C > 0$ and all $t > 0$.

(ii) *Suppose that* $\mathrm{Ric}_f^n \geq -\frac{(n-1)\kappa}{(1+r)^2}$ $(n > m)$ *on M for some constant $\kappa \geq 0$. Then one has*

$$\mu_f(B(o,t)) \leq C'(1+t)^n$$

for some constant $C' > 0$ and all $t > 0$.

(iii) *Suppose that for some constants $\kappa \geq 0$ and $b \geq 0$,* $\mathrm{Ric}_f^\infty \geq -\frac{(m-1)\kappa}{(1+r)^2}$ *on M and* $\sup\{\sup_{B(o,2^{k+2})\setminus B(o,2^k)} f - \inf_{B(o,2^{k+2})\setminus B(o,2^k)} f \mid k = 1, 2, \ldots\} \leq b < +\infty$. *Then one has*

$$\mu_f(B(o,t)) \leq C''(1+t)^{m+4b}$$

for some constant $C'' > 0$ and all $t > 0$.

Proof. Since (VC) is assumed, we have for $x \in S(o,t)$,

$$\mu_f(B(o,t)) \leq C_V \mu_f(B(x,t/2)),$$

so it is enough to show that $\mu_f(B(x,t/2)) \leq C(1+t)^{m+\alpha}$, $\mu_f(B(x,t/2)) \leq C'(1+t)^n$, and $\mu_f(B(x,t/2)) \leq C''(1+t)^{m+4b}$, respectively, under the assumptions in (i), (ii) and (iii).

We consider assertion (i). It follows from the assumption on f that $|f(x) - f(o)| \leq \alpha \int_0^{r(x)} (1+s)^{-1} ds = \log(1+r(x))^\alpha$ for $x \in M$. Hence we get

$$e^{-f} \leq e^{|f(o)|}(1+r)^\alpha \quad \text{on } M.$$

Now we fix a point $x \in S(o,t)$. Since $\mathrm{Ric}_M \geq -\kappa(1+t/2)^{-2} \geq -4\kappa t^{-2}$ on $B(x,t/2)$, we have by (2.13) (after letting s go to 0 and letting ω_m stand for the volume of the unit sphere of Euclidean space $\mathbb{R}^m$),

$$\mu_f(B(x,t/2)) \leq \sup_{B(x,t/2)} e^{-f} \cdot \mu_0(B(x,t/2)) \leq e^{|f(o)|}(1+2t)^{\alpha}\,\omega_m \int_0^{t/2} (s_{2\sqrt{\kappa}/t}(\tau))^{m-1} d\tau$$
$$\leq e^{|f(o)|}(1+2t)^{\alpha}\,\omega_m\,(s_{2\sqrt{\kappa}/t}(t/2))^{m-1} \int_0^{t/2} d\tau \leq e^{|f(o)|}\,C(m,\kappa)\,(1+t)^{m+\alpha},$$

where $C(m,\kappa)$ is a positive constant depending only on m and κ.

For the remaining assertions, the same arguments as above are valid, and we omit the proofs of (ii) and (iii). ■

Remark 2.20 ([13], subsection 2.2; [12], (15.68)). For any subset U of M and $R > 0$, we consider a family of balls $\mathcal{F} = \{B(x,t) | x \in U, t \leq R\}$. Assume that the family $\mathcal{F}$ satisfies (VD) with constant C_D. Set $\gamma = \log_2 C_D$. Then, for all $0 < s < t \leq R$, we have

$$\frac{\mu_f(B(x,t))}{\mu_f(B(x,s))} \leq C_D \left(\frac{t}{s}\right)^{\gamma}.$$

For any $B(x,t) \in \mathcal{F}$ with $t < R/2$, assume that $S(x,3t/4) \cap U \neq \emptyset$. Let y be a point of $S(x,3t/4) \cap U$. Then we obtain

$$\mu_f(B(x,t)) \geq \mu_f(B(x,t/2)) + \mu_f(B(y,t/4))$$
$$\geq \mu_f(B(x,t/2)) + C_D^{-3}\,\mu_f(B(y,2t)) \geq \mu_f(B(x,t/2)(1 + C_D^{-3}).$$

We say that a family $\mathcal{F}$ of balls in M as above satisfies the *reverse volume doubling property* (RVD) with a constant $C_{\mathrm{RD}} > 1$ if, for any ball $B(x,t) \in \mathcal{F}$ with $t < R$,

$$\mu_f(B(x,t)) \geq C_{\mathrm{RD}}\,\mu_f(B(x,t/2)).$$

Then, for all $0 < s < t \leq R/2$,

$$\frac{\mu_f(B(x,t))}{\mu_f(B(x,s))} \geq C_{\mathrm{RD}} \left(\frac{t}{s}\right)^{\beta},$$

where $\beta = \log_2 C_{\mathrm{RD}}$.

Now we let Λ be a positive constant and consider the equation $Q'_{p;\Lambda}(u) = 0$ in M. We denote by $\eta_{p,\Lambda}$ the solution of (2.2) with $\chi(t) = s_\kappa(t)^{n-1}$ and $W_* = \Lambda$ subject to the initial conditions $\eta_{p,\Lambda}(0) = 1$ and $\eta'_{p,\Lambda}(0) = 0$. Since $\Lambda > 0$, it is easy to see that $\eta'_{p,\Lambda} > 0$ on $(0,+\infty)$, so $\eta_{p,\Lambda}(t) > 1$. Moreover, it follows from Lemma 2.3 that $\eta_{p,\Lambda}(r)$ satisfies $-\Delta_{p,f}\,\eta_{p,\Lambda}(r) + \Lambda\eta_{p,\Lambda}(r)^{p-1} \geq 0$ on M in the weak sense.

To prove Theorem 1.3 (i), we need the following.

Lemma 2.21. *Let $Z(p,n,\kappa,\Lambda)$ be the unique positive root of the equation $(p-1)Z^p + (n-1)\sqrt{\kappa}Z^{p-1} = \Lambda$. Then one has*

$$\lim_{t\to\infty} \frac{1}{t} \log \eta_{p,\Lambda}(t) = Z(p,n,\kappa,\Lambda).$$

Proof. For a positive constant a, let $\rho_a(t) = c_\kappa(t)^a$ $(t \in [0,\infty))$. Then ρ_a satisfies

$$(\rho_a'(t)^{p-1})' + (m-1)\,\frac{c_\kappa(\rho)}{s_\kappa(\rho)}\,\rho_a'(t)^{p-1} \le \lambda(a,t)\,\rho_a(t)^{p-1},$$

where we put

$$\lambda(a,t) = a^{p-1}\kappa^{p-1}\Big((a-1)(p-1)\kappa\Big(\frac{s_\kappa(t)}{c_\kappa(t)}\Big)^p + (n+p-2)\Big(\frac{s_\kappa(t)}{c_\kappa(t)}\Big)^{p-2}\Big).$$

We observe that

$$\lim_{t\to\infty}\lambda(a,t) = (p-1)(\sqrt{\kappa}a)^p + (n-1)\sqrt{\kappa}\,(\sqrt{\kappa}\,a)^{p-1},$$

so that for $a = \kappa^{-1/2}Z(p,n,\kappa,\Lambda)$,

$$\lim_{t\to\infty}\lambda(\kappa^{-1/2}Z(p,n,\kappa,\Lambda),t) = \Lambda.$$

Let a be less than $\kappa^{-1/2}Z(p,n,\kappa,\Lambda)$. Then there exists a positive number τ such that $\lambda(a,t) < \Lambda$ for all $t \ge \tau$. We take a positive number b in such a way that $b\rho_a(\tau) < \eta_{p,\Lambda}(\tau)$ and $b\rho_a'(\tau) < \eta_{p,\Lambda}'(\tau)$. Then it holds that $b\rho_a'(t) < \eta_{p,\Lambda}'(t)$ for all $t \ge \tau$. In fact, we suppose contrarily that for some $t_* > \tau$, $b\rho_a'(t) < \eta_{p,\Lambda}'(t)$ for all $t \in [\tau, t_*)$ and $b\rho_a'(t_*) = \eta_{p,\Lambda}'(t_*)$. Since $b\rho_a(s) < \eta_{p,\Lambda}(s)$ for $s \in [\tau, t_*]$, we obtain

$$\begin{aligned}
s_\kappa(t_*)^{n-1}\eta_{p,\Lambda}'(t_*)^{p-1} &= s_\kappa(\tau)^{n-1}\eta_{p,\Lambda}'(\tau)^{p-1} + \int_\tau^{t_*}\Lambda s_\kappa(s)^{n-1}\eta_{p,\Lambda}'(s)^{p-1}\,ds\\
&> s_\kappa(\tau)^{n-1}b^{p-1}\rho_a'(\tau)^{p-1} + \int_\tau^{t_*}\lambda(a,s)s_\kappa(s)^{n-1}b^{p-1}\rho_a'(s)^{p-1}\,ds\\
&\ge b^{p-1}s_\kappa(\tau)^{n-1}\rho_a'(\tau)^{p-1} + b^{p-1}\int_\tau^{t_*}(s_\kappa(s)^{n-1}\rho_a'(s)^{p-1})'\,ds\\
&= b^{p-1}s_\kappa(t_*)^{n-1}\rho_a'(t_*)^{p-1} = s_\kappa(t_*)^{n-1}\eta_{p,\Lambda}'(t_*)^{p-1}.
\end{aligned}$$

This is absurd. Thus we see that $b\rho_a'(t) < \eta_{p,\Lambda}'(t)$ for all $t \ge \tau$, and hence $b\rho_a(t) < \eta_{p,\Lambda}(t)$ for all $t \ge \tau$. This shows that

$$\sqrt{\kappa}a = \lim_{t\to\infty}\frac{1}{t}\log b\rho_a(t) \le \liminf_{t\to\infty}\frac{1}{t}\log\eta_{p,\Lambda}(t).$$

This holds for any $a < \kappa^{-1/2}Z(p,n,\kappa,\Lambda)$. Thus we get

$$Z(p,n,\kappa,\Lambda) \le \liminf_{t\to\infty}\frac{1}{t}\log\eta_{p,\Lambda}(t).$$

Similarly, we can deduce that

$$\limsup_{t\to\infty}\frac{1}{t}\log\eta_{p,\Lambda}(t) \le Z(p,n,\kappa,\Lambda).$$

In this way, we obtain $\lim_{t\to\infty}\frac{1}{t}\log\eta_{p,\Lambda}(t) = Z(p,n,\kappa,\Lambda)$. This completes the proof of Lemma 2.21. ∎

3. Harnack inequalities and proof of Theorem 1.1

Let (M, g_M, μ_f) be a connected, noncompact, complete weighted Riemannian manifold of dimension m. Let Ω be an open subset of M. In this section, we assume the volume doubling property (VD) and the family of (weak) scaled Poincaré inequalities (PI(p)) ($p \in (1, +\infty)$) as follows:

(i) there exists a positive constant C_D such that, for any ball $B(x, 2t) \subset \Omega$,

$$\mu_f(B(x,t)) \le C_D\, \mu_f(B(x,t/2));$$

(ii) there exists a positive constant C_P such that for every ball $B(x, 2t) \subset \Omega$ and any $u \in L^{1,p}_{\mathrm{loc}}(B(x, 2t))$,

$$\int_{B(x,t/2)} |u - u_{B(x,t/2)}|^p \, d\mu_f \le C_P\, t^p \int_{B(x,t)} |\nabla u|^p \, d\mu_f,$$

where

$$u_{B(x,t/2)} = \frac{1}{\mu_f(B(x,t/2))} \int_{B(x,t/2)} u \, d\mu_f.$$

Then it is known that the family (SI(p)) of Sobolev inequalities holds in such a way that for some constants $k > 1$ and $C_S > 0$, and for every ball $B(x, 2t) \subset \Omega$ and any $v \in L^{1,p}_0(B(x,t))$,

$$\left(\int_{B(x,t)} |v|^{pk} \, d\mu_f \right)^{1/k} \le \frac{C_S\, t^p}{\mu_f(B(x,t))^{p/\nu}} \int_{B(x,t)} |\nabla v|^p + t^{-p} |v|^p \, d\mu_f,$$

where we can take $k = \nu/(\nu - p)$ with $\nu = \max\{p+1, \log_2 C_D\}$, and C_S depends only on C_D and C_P (See [7], Lemma 4.3; [29], [30], [31] and references therein.)

A Harnack inequality for positive p-harmonic functions is obtained in Coulhon, Holopainen and Saloff-Coste [7] by running the Moser iteration as in [30] under the assumption that volume doubling property and suitable Poincaré inequalities hold. In fact, the result is established in a natural framework including the usual p-Laplacians. Along the line of [7], we extends the Harnack inequality for positive solutions of equation $-\Delta_{f;p} u + W|u|^{p-2}u = 0$, where W is a locally bounded potential function. We refer also to [26].

The main result of this section is the following.

Theorem 3.1. *Let (M, g_M, μ_f) be a noncompact, connected, complete weighted Riemannian manifold of dimension m. The volume doubling property* (VD) *and the family* (PI(p)) *of Poincaré inequalities with constants C_D and C_P respectively are satisfied in an open subset Ω. Then for any nonnegative function $u \in L^{1,p}_{\mathrm{loc}}(B(x,2t))$, $B(x,2t) \subset \Omega$, satisfying*

$$-\lambda\, |u|^{p-2} u \le \Delta_{f;p} u \le \Lambda\, |u|^{p-2} u$$

in the weak sense on $B(x, 2t)$, where λ and Λ are positive constants, one has

$$\sup_{B(x,t)} u \le C \inf_{B(x,t)} u.$$

Here C is a positive constant depending only on C_D, C_P, p, $t^p\lambda$, and $t^p\Lambda$.

We start with:

Theorem 3.2. *Assume* (SI(p)) *is satisfied on* Ω *and let* $B(x,2t) \subset \Omega$. *Let* $0 < \sigma < \sigma' \le 1$ *and* $0 < \alpha < +\infty$. *For a nonnegative function* u *in* $L^{1,p}_{\mathrm{loc}}(B(x,2t))$ *satisfying* $-\lambda|u|^{p-2}u \le \Delta_{f;p}u$ *in the weak sense on* $B(x,2t)$, *where* λ *is a positive constant, one has*

$$\sup_{B(x,\sigma t)} u \le C {C_S}^{\nu/p}\Big(\frac{1}{\mu_f(B(x,t))}\int_{B(x,\sigma' t)} u^\alpha \, d\mu\Big)^{1/\alpha},$$

where C *is a positive constant depending only on* p, k, σ, σ', α *and* $t^p\lambda$.

Proof. For the case where $\lambda = 0$, the theorem is shown in [7], Theorems 4.4 and 4.5, and we can adapt the proof for our case. ∎

Now we are concerned with a positive function $u \in W^{1,p}_{\mathrm{loc}}(B(x,2t))$ satisfying $\Delta_{f,p}u \le \lambda|u|^{p-2}u$, where λ is a positive constant. We begin with:

Lemma 3.3. *Suppose that* (SI(p)) *is satisfied and* $B(x,2t) \subset \Omega$. *Let* $0 < \sigma < \sigma' \le 1$, $0 < s' < k^{-1}s < s < k(p-1)$, *and* $0 < q < +\infty$. *For a positive function* u *in* $L^{1,p}_{\mathrm{loc}}(B(x,2t))$ *satisfying* $\Delta_{f,p}u \le \Lambda|u|^{p-2}u$ *in the weak sense, one has*

$$\Big(\frac{1}{\mu_f(B(x,t))}\int_{B(x,\sigma t)} u^s \, d\mu\Big)^{1/s} \le \big[C {C_S}^{\nu^2/p^2}(\sigma'-\sigma)^{-\nu^2/p}\big]^{1/s'-1/s}\Big(\frac{1}{\mu_f(B(x,t))}\int_{B(x,\sigma' t)} u^{s'} \, d\mu\Big)^{1/s'}$$

and

$$\sup_{B(x,\sigma t)} u^{-q} \le C C_S^{\nu/p}(\sigma'-\sigma)^{-1/\nu}\frac{1}{\mu_f(B(x,t))}\int_{B(x,\sigma' t))} u^{-q} \, d\mu,$$

where C *is a positive constant depending only on* p, ν *and* $t^p\Lambda$.

Proof. For the case where $\Lambda = 0$, the theorem is shown in [7], Theorems 4.6. 4.7, and we can adapt the proof for our case. See also [26], Chapter 7. ∎

Now by referring to the proof of Theorem 3.1 in [7], we prove the following.

Lemma 3.4. *Suppose that* (PI(p)) *is satisfied on* Ω *and let* $B(x,2t) \subset \Omega$. *Let* $0 < \delta < 1$ *and let* u *be a positive function in* $L^{1,p}_{\mathrm{loc}}(B(x,2t))$ *satisfying* $\Delta_{f,p}u \le \Lambda|u|^{p-2}u$ *in the weak sense. Then*

$$\int_{B(x,\delta t)} |\nabla \log u|^p \, d\mu_f \le \frac{2^p(1+t^p\Lambda)}{(p-1)(1-\delta)^p t^p}\mu_f(B(x,t)).$$

Proof. In this proof, we write B, $B(s)$ ($0 < s < t$) and $V(s)$ respectively for $B(x,t)$, $B(x,s)$ and $\mu_f(B(x,s))$. For any function $w \in L^{1,p}_{\mathrm{loc}}(B)$, $w \ge \varepsilon > 0$, we have

$$-\Delta_{f;p}\log w + \frac{\Delta_{f;p}w}{w^{p-1}} = (p-1)|\nabla \log w|^p$$

in the weak sense, that is, for any nonnegative function $\psi \in L_0^{1,p}(B)$ with compact support in B, we have

$$\int g(\nabla\psi, \nabla \log w)|\nabla \log w|^{p-2}\, d\mu_f - \int g\Big(\nabla w, \nabla\Big(\frac{\psi}{w^{p-1}}\Big)\Big)|\nabla w|^{p-2}\, d\mu_f = (p-1)\int \psi|\nabla \log w|^p\, d\mu_f.$$

This shows that $\log u$ satisfies

$$(p-1)\int \psi|\nabla \log u|^p\, d\mu_f - \Lambda\int \psi\, d\mu_f \le \int \langle\nabla\psi, \nabla\log u\rangle|\nabla \log u|^{p-2}\, d\mu_f$$

for any nonnegative $\psi \in L_0^{1,p}(B)$. Taking

$$\psi(y) = \begin{cases} 1 & \text{if } y \in B(s), \\ 1 - \frac{1}{\varepsilon}(\mathrm{dis}_M(x,y) - 1) & \text{if } y \in B(s+\varepsilon) \setminus B(s), \\ 0 & \text{otherwise,} \end{cases}$$

yields

$$(p-1)\int_{B(s)} |\nabla\log u|^p\, d\mu_f \le \frac{1}{\varepsilon}\int_{B(s+\varepsilon)\setminus B(s)} |\nabla \log u|^{p-1}\, d\mu_f + \lambda\int_{B(t+\varepsilon)} d\mu_f.$$

Since

$$\frac{1}{\varepsilon}\int_{B(s+\varepsilon)\setminus B(t)} |\nabla\log u|^{p-1}\, d\mu_f \le \Big(\frac{V(s+\varepsilon)-V(t)}{\varepsilon}\Big)^{1/p}\Big(\frac{1}{\varepsilon}\int_{B(s+\varepsilon)\setminus B(s)} |\nabla \log u|^p\, d\mu_f\Big)^{1/p'},$$

where $p' = \frac{p}{p-1}$, we get

$$(p-1)\int_{B(s)} |\nabla\log u|^p\, d\mu_f \le \Big(\frac{V(s+\varepsilon)-V(s)}{\varepsilon}\Big)^{1/p}\Big(\frac{1}{\varepsilon}\int_{B(s+\varepsilon)\setminus B(s)} |\nabla \log u|^p\, d\mu_f\Big)^{p'} + \Lambda V(s+\varepsilon).$$

Thus putting $H(s) = (p-1)\int_{B(s)} |\nabla\log u|^p\, d\mu_f$ and letting ε tend to 0 yield

$$H(s) \le \Big(\frac{H'(s)}{p-1}\Big)^{1/p'} V'(s)^{1/p} + \Lambda V(s),$$

and hence

$$\frac{1}{V'(s)^{1/p}} \le \frac{1}{(p-1)^{1/p'}}\Big(\frac{H'(s)}{H(s)^{p'}}\Big)^{1/p'} + \frac{\Lambda V(s)}{H(s)}\,\frac{1}{V'(s)^{1/p}}.$$

Suppose that $2\Lambda V(t) = H(s_0)$ for some $s_0 \in (0, t)$. Since

$$\frac{\Lambda V(s)}{H(s)} \le \frac{\Lambda V(t)}{H(s_0)} = \frac{1}{2},$$

we have

$$\frac{1}{2^{p'}} \frac{1}{V'(s)^{p/p'}} \le \frac{1}{p-1} \frac{H'(s)}{H(s)^{p'}}, \quad s_0 \le s \le t.$$

Integrating both sides from s' to s for $s_0 \le s' < s \le t$, we obtain

$$\frac{1}{2^{p'}} \int_{s'}^{s} \frac{d\sigma}{V'(\sigma)^{p'/p}} \le \frac{1}{H(s')^{p'/p}} - \frac{1}{H(s)^{p'/p}}. \tag{3.1}$$

The left-hand side can be bounded from below by $((s-s')^p/(V(s)-V(s')))^{1/(p-1)}$, because

$$(s-s')^p = \Big(\int_{s'}^{s} d\sigma\Big)^p \le \Big(\int_{s'}^{s} V'(\sigma)\, d\sigma \int_{s'}^{t} \frac{1}{V'(\sigma)^{p'/p}}\, d\sigma\Big)^{p/p'}$$
$$= (V(s)-V(s'))\Big(\int_{s'}^{s} \frac{d\sigma}{V'(\sigma)^{p'/p}}\Big)^{p/p'}.$$

Hence, by (3.1), we have

$$\frac{1}{2^{p'}}\Big(\frac{(s-s')^p}{V(s)-V(s')}\Big)^{1/(p-1)} \le \frac{1}{H(s')^{p'/p}} - \frac{1}{H(s)^{p'/p}} \le \frac{1}{H(s)^{p'/p}}$$

and thus

$$H(s') \le 2^p \frac{V(s)-V(s')}{(s-s')^p}, \quad s_0 \le s' < s \le t.$$

This shows that if $\delta t \ge s_0$, then

$$H(\delta t) \le 2^p \frac{V(t)-V(\delta t)}{t^p(1-\delta)^p} \le 2^p \frac{V(t)}{t^p(1-\delta)^p},$$

and if $\delta t < s_0$, then

$$H(\delta t) \le H(s_0) = 2\Lambda V(t).$$

In this way, we obtain

$$H(\delta t) \le 2^p\Big(\frac{1}{t^p(1-\delta)^p} + \Lambda\Big)V(t) < \frac{2^p(1+t^p\Lambda)}{(1-\delta)^p t^p} V(t).$$

If $H(s) < 2\Lambda V(t)$ for any $s \in (0,t)$, then we have

$$H(\delta t) < 2\Lambda V(t) < \frac{2^p(1+t^p\Lambda)}{(1-\delta)^p t^p} V(t).$$

This completes the proof of Lemma 3.4. ∎

In order to arrive at Theorem 3.1, we need an abstract lemma due to Bombieri and Giusti [3], which simplifies considerably Moser's original proof of the Harnack inequality.

Consider a collection of measurable subsets U_σ, $0 < \sigma \le 1$, of a fixed measure space endowed with a measure μ, such that $U_\sigma \subset U_{\sigma'}$ if $\sigma \le \sigma'$. In our application, U_σ will be $B(x, \sigma t)$ for some fixed metric ball $B(x,t) \subset M$.

Lemma 3.5 ([3]; [31], Subsection 2.2.3). *Fix $0 < \delta < 1$. Let γ and C be positive constants and let $0 < \alpha_0 \le +\infty$. Let g be a positive measurable function on $U_1 = U$ which satisfies*

$$\Big(\int_{U_\sigma} g^{\alpha_0}\, d\mu\Big)^{1/\alpha_0} \le \big(\sigma'-\sigma)^{-\gamma}\mu(U)^{-1}\big)^{1/\alpha - 1/\alpha_0}\Big(\int_{U_{\sigma'}} g^{\alpha}\, d\mu\Big)^{1/\alpha}$$

for all σ, σ', α such that $0 < \delta \le \sigma < \sigma' \le 1$ and $0 < \alpha \le \min\{1, \alpha_o/2\}$. Assume further that g satisfies

$$\mu(\log g > t) \le \mu(U)\, t^{-1}$$

for all $t > 0$. Then

$$\Big(\int_{U_\delta} g^{\alpha_0}\, d\mu\Big)^{1/\alpha_0} \le A\, \mu(U)^{1/\alpha_0},$$

where A depends only on δ, γ, C and a lower bound on α_0.

Theorem 3.6. *Assume the volume doubling property* (VD) *and the family of Poincaré inequalities* (PI(p)) *with constants C_D and C_P ($p \in (1, +\infty)$), respectively, are satisfied in an open subset Ω. Let $\nu = \max\{p+1, \log_2 C_D\}$, $0 < s < \nu(p-1)/(\nu - p)$, and $0 < \delta < 1$. Then a positive function $u \in L^{1,p}_{\mathrm{loc}}(B(x,2t))$, $B(x,2t) \subset \Omega$, satisfying*

$$\Delta_{f,p} u \le \Lambda u^{p-1}$$

in $B(x,2t)$ fulfills

$$\Big(\frac{1}{\mu_f(B(x,\delta t))}\int_{B(x,\delta t)} u^s\, d\mu_f\Big)^{1/s} \le C \inf_{B_x(\delta t)} u.$$

Here C is a positive constant depending only on δ, p, C_D, C_P, and $t^p\Lambda$.

Proof. Let

$$c = \frac{1}{\mu_f(B(x,\delta t))}\int_{B(x,\delta t)} \log u\, d\mu.$$

In view of Lemma 3.3, we can apply Lemma 3.5 to $e^{-c}u$ and $e^c u^{-1}$. First it follows from (PI(p)) and Lemma 3.4 that

$$\int_{B(x,\delta t)} |\log u - c|\, d\mu \le \mu_f(B(x,\delta t))^{1-1/p}\Big(\int_{B(x,\delta t)} |\log u - c|^p\, d\mu\Big)^{1/p}$$
$$\le C_1\, \mu_f(B(x,\delta\rho)),$$

where we put $C_1 = 2(1 + t^p\lambda)^{1/p}(p-1)^{-1/p}(1-\delta)^{-1}C_P$. This shows that for any $\tau > 0$,

$$\tau\mu(\{x \in \delta B \mid \log e^{-c}u \ge \tau\}) \le \int_{\delta B} |\log u - c|\, d\mu \le C_1\, \mu_f(B(x,\delta t)).$$

Similarly, we have

$$\tau\mu(\{x \in \delta B \mid \log e^{c}u^{-1} \ge \tau\}) \le C_1\, \mu_f(B(x,\delta t)).$$

Then it follows from Lemma 3.5 that

$$\Big(\int_{B(\delta t)} u^s\, d\mu\Big)^{1/s} \le A\mu_f(B(x,\delta t))^{1/s} e^c, \quad 0 < s < \frac{\nu(p-1)}{\nu - p}$$

and also

$$e^c \sup_{\delta B} u^{-1} \le A.$$

These show the required inequality. ■

It is clear that Theorem 3.1 is derived from Theorems 3.2 and 3.6.

Proof of Theorem 1.1. (i) By the assumptions, we assume that for some positive constants κ, α and α',

$$\mathrm{Ric}_M \ge -\kappa(m-1)(1+r)^{-2}, \quad |\nabla f| \le \alpha(1+r)^{-1} \quad \text{and} \quad |W| \le \alpha'(1+r)^{-p}$$

on M. Let $b = \sup_{t>0} t^{-1}\mathrm{diam}^{(\sigma;\infty)}(S(o,t))$ and it is assumed that b is finite. We fix a positive integer k in such a way that $\sigma \le 2^{k-2}(1-\sigma)$. For any $x, y \in S(o,t)$, let $\gamma_{xy}\colon [0,L] \to M \setminus B(o,(1-\sigma)t)$ ($L = \mathrm{dis}^{(\sigma;t))}(x,y)$) be a curve parametrized by arc-length joining $x = \gamma_{xy}(0)$ to $y = \gamma_{xy}(L)$. We choose a nonnegative integer j in such a way that

$$\frac{\sigma j}{2^{k+1}} \le \frac{L}{t} < \frac{\sigma(j+1)}{2^{k+1}}.$$

Note that $j \le 2^{k+1}\sigma^{-1}b$, since $L \le t\ b$. Let $x_i = \gamma_{xy}(2^{-k-1}\sigma t i)$ ($i = 0, 1, \dots, j$) and $x_{j+1} = y$. Note also that $B(x_i, 2^{-k}\sigma t)$ ($i = 0, \dots, j$) are all remote balls, and on $M \setminus B(o,(1-(1+2^{-k})\sigma)t)$ which includes $\bigcup_{i=0}^{j+1} B(x_i, 2^{-k}\sigma t)$, we have the Ricci curvature bounded from below by $-(m-1)\kappa(1+(1-(1+2^{-k})\sigma)t)^{-2}$, $|\nabla f|$ bounded from above by $\alpha(1+(1-(1+2^{-k})\sigma)t)^{-1}$ and $|W|$ bounded from above by $\alpha'(1+(1-(1+2^{-k})\sigma)t)^{-p}$. Since $2^{-k}\sigma t < 1 + (1-(1+2^{-k})\sigma)t$, it follows from Theorem 3.1 that $u(x_i) \le C_2 u(x_{i+1})$ ($i = 0, \dots, j$), and hence we have $u(x) \le {C_2}^{j+1}u(y)$, where C_2 is a positive constant independent of u and t. This completes the proof of assertion (i).

(ii) Based on the annulus Harnack inequalities in the first assertion and using the same arguments as in Theorem 7.1 in [23], we can verify the second one. We omit the details of the proof.

(iii) Let $M(t) = \sup_{S(o,t)} u$ and $m(t) = \inf_{S(o,t)} u$. If $W \ge 0$ and u is unbounded, then Lemma 2.2 shows that $M(t)$ diverges to infinity as $t \to \infty$. By the annulus Harnack inequality, $M(t) \le C_H m(t)$ for all $t \ge 0$. This implies that $u(x) \to +\infty$ as $x \in M \to \infty$. When $W \le 0$ and $\inf_M u = 0$, we see from Lemma 2.2 that $m(t)$ tends to zero as $t \to \infty$. Thus the annulus Harnack inequality shows that $u(x)$ goes to zero as $x \in M \to \infty$.

(iv) Let $\eta(t)$ be the solution of equation (2.2) with $W_*(t) = \phi(t)$. Then by Proposition 2.7 (i), we have $\sup_{S(o,t)} u \ge u(o)\eta(t)$, and by Lemma 2.9 (i), $\lim_{t\to\infty} \eta(t) = +\infty$, so that $\sup_M u = +\infty$. This proves that $\lim_{x\in M\to\infty} u(x) = +\infty$.

Now let $\omega(t)$ be the solution of (2.3) with $W_*(t) = \phi(t)$. Then by Proposition 2.11 and Lemma 2.12 (i), we have $\omega(t) \ge u(o)^{-1}\inf_{S(o,t)} u$ and $\lim_{t\to\infty} w(t) = 0$. These show that $\inf_M u = 0$, and hence $\lim_{x\in M\to\infty} u(x) = 0$. ■

Before ending this section, we have by Lemma 2.9(ii) and Lemma 2.12(ii), the following.

Proposition 3.7. *Let (M, g_M, μ_f) be a connected, noncompact, and complete weighted Riemannian manifold of dimension m satisfying (1.1), (1.2) and $\delta^{(\sigma;\infty)}(M) < +\infty$ for some $\sigma \in (0,1)$. Given a bounded function W on M, assume that $Q_{p;W} \geq 0$. Let $u \in L^{1,p}_{\mathrm{loc}}(M) \cap C^{1,\alpha}_{\mathrm{loc}}(M)$ be a positive solution to the equation $Q'_{p;W}(u) = 0$ in M.*

(i) *Suppose that*

$$\frac{\lambda}{(1+r)^p} \leq W \leq \frac{\Lambda}{(1+r)^p}$$

for some positive constants λ, Λ with $\lambda < \Lambda$. Then one has

$$u \geq C(1+r)^{\gamma(p,m,\kappa,\alpha,\lambda)} \quad \text{in } M,$$

where C is a positive constant and $\gamma(p,m,\kappa,\alpha,\lambda)$ is the positive solution of

$$x|x|^{p-2}(x(p-1) + \beta(m,\kappa,\alpha) + 1 - p) = \lambda$$

with $\beta(m,\kappa,\alpha) = \alpha + (m-1)(1+\sqrt{1+4\kappa})/2$.

(ii) *Suppose that $p = 2$ and*

$$-\frac{\Lambda}{(1+r)^2} \leq W \leq -\frac{\lambda}{(1+r)^2}$$

for some positive constants λ, Λ with $\lambda < \Lambda$. Then one has

$$u \leq \frac{C'}{(1+r)^{\zeta(m,\kappa,\alpha,\lambda)}} \quad \text{in } M,$$

where C' is a positive constant and

$$\zeta(m,\kappa,\alpha,\lambda) = \min\left\{\tfrac{1}{2}\left((\beta-1) - \sqrt{(\beta-1)^2 - 4\lambda}\,\right), 2\sqrt{1+4\kappa} + \tfrac{\lambda}{\beta-1}\right\}.$$

with $\beta = \beta(m,\kappa,\alpha)$.

4. Proof of Theorem 1.2

Under the conditions (1.1), (1.2), (VC) and (RCA), we can conclude from Theorem 5.2, Corollary 5.4 and Theorem 2.7 in [13] that (M, g_M, μ_f) satisfies (VD) with a constant $C_D > 1$ and (PI(2)), or equivalently, that the following two-sided estimate for the heat kernel $p(t,x,y)$ of the Laplacian $\Delta_{f;2}$ holds:

$$\begin{aligned}\frac{C_0}{V(x,\sqrt{t})}\exp\Big(-C_0'\frac{\mathrm{dis}_M(x,y)^2}{t}\Big) &\leq p(t,x,y) \\ &\leq \frac{C_0'}{V(x,\sqrt{t})}\exp\Big(-C_0\frac{\mathrm{dis}_M(x,y)^2}{t}\Big)\end{aligned} \tag{4.1}$$

for all $x, y \in M$, $t > 0$ and some $C_0' > C_0 > 0$, where we put $V(x,t) = \mu_f(B(x,t))$.

Moreover, in view of (1.3) ($\beta > 2$), the Green function

$$G(x, y) = \int_0^\infty p(t, x, y)\, dt$$

exists and satisfies

$$C_1^{-1} \int_{\mathrm{dis}_M(x,y)}^\infty \frac{t}{V(x,t)}\, dt \le G(x, y) \le C_1 \int_{\mathrm{dis}_M(x,y)}^\infty \frac{t}{V(x,t)}\, dt \tag{4.2}$$

for all $x, y \in M$ and some $C_1 > 1$.

Let ψ be a positive nonincreasing C^1 function on $[0, \infty)$ such that

$$\int_0^\infty t\psi(t)\, dt < +\infty. \tag{4.3}$$

We first observe that for $0 \le a < b < +\infty$,

$$\begin{aligned} &\int_{\{a \le r \le b\}} \psi(r)\, d\mu_f \\ &\qquad \le C_\beta^{-1}\, \psi(a)\, a^\beta\, V(o, b)\, b^{-\beta} + \beta\, C_\beta^{-1}\, V(o, b)\, b^{-\beta} \int_a^b \psi(r)\, r^{\beta-1}\, dr. \end{aligned} \tag{4.4}$$

In fact, using the growth condition (1.3), we have

$$\begin{aligned} \int_{\{a \le r \le b\}} \psi(r)\, d\mu_f &= \int_a^b \psi(r)\, V'(o, r)\, dr \\ &= \psi(b)\, V(o, b) - \psi(a)\, V(o, a) - \int_a^b \psi'(r)\, V(o, r)\, dr \\ &\le \psi(b)\, V(o, b) + C_\beta^{-1}\, V(o, b)\, b^{-\beta} \int_a^b -\psi'(r)\, r^\beta\, dr \\ &\le \psi(b)\, V(o, b) + C_\beta^{-1}\, V(o, b)\, b^{-\beta} \Big(\psi(a)\, a^\beta - \psi(b)\, b^\beta + \beta \int_a^b \psi(r)\, r^{\beta-1}\, dr\Big) \\ &\le C_\beta^{-1}\, \psi(a)\, a^\beta\, V(o, b)\, b^{-\beta} + \beta\, C_\beta^{-1}\, V(o, b)\, b^{-\beta} \int_a^b \psi(r) r^{\beta-1}\, dr. \end{aligned}$$

Thus we obtain (4.4).

Now for a nonnegative number k, we let

$$G_k(x) = \int_{M \setminus B(o,k)} G(x, z)\, \psi(r(z))\, d\mu_f(z), \quad x \in M.$$

Then we have:

Lemma 4.1. *The following assertions hold*:

(i) $\lim_{x \in M \to \infty} G_0(x) = 0$,

(ii) $\lim_{k \to \infty} \sup_{x \in M} G_k(x) = 0$.

Proof. To estimate $G_k(x)$, we put

$$G_{k;1}(x) = \int_{\{k\le r(z)\le 2r(x);\, d(x,z)\ge r(x)/2\}} G(x,z)\,\psi(r(z))\,d\mu_f(z),$$
$$G_{k;2}(x) = \int_{\{r(z)\ge k;\, d(x,z)\le r(x)/2\}} G(x,z)\,\psi(r(z))\,d\mu_f(z),$$
$$G_{k;3}(z) = \int_{\{r(z)\ge 2r(x)\}} G(x,z)\,\psi(r(z))\,d\mu_f(z).$$

In view of (4.2) and the volume doubling property (VD) with a constant $C_D > 1$ (see Remark 2.20), we see that

$$\begin{aligned} G(x,z) &\le C_1 \int_{r(x)/2}^{\infty} \frac{t\,dt}{V(x,t)} = C_1 \int_{r(x)/2}^{\infty} \frac{V(o,t)}{V(x,t)}\,\frac{t\,dt}{V(o,t)} \\ &\le C_1 \int_{r(x)/2}^{\infty} \frac{V(o,t)}{V(x,t+r(x))}\,\frac{V(x,t+r(x))}{V(x,t)}\,\frac{t\,dt}{V(o,t)} \\ &\le C_1\,C_D \left(\frac{t+r(x)}{t}\right)^{\gamma} \int_{r(x)/2}^{\infty} \frac{t\,dt}{V(o,t)} \le 3^{\gamma} C_1\,C_D \int_{r(x)}^{\infty} \frac{t\,dt}{V(o,t)} \end{aligned}$$

if $d(x,z) \ge r(x)/2$. Putting $C_2 = 3^{\gamma} C_1 C_D$, we have

$$G_{k;1}(x) \le C_2 \int_{r(x)/2}^{\infty} \frac{t\,dt}{V(o,t)} \int_{\{k\le r(z)\le 2r(x); d(x,z)\ge r(x)/2\}} \psi(r(z))\,d\mu_f(z).$$

Since we assume the volume growth (1.3), we get

$$\int_{r(x)/2}^{\infty} \frac{t\,dt}{V(o,t)} \le 2^{\beta} \frac{C_{\beta}^{-1}}{V(o,r(x)/2)} \int_{r(x)/2}^{\infty} t\left(\frac{r(x)}{t}\right)^{\beta} dt = \frac{C_{\beta}^{-1}\,r(x)^2}{2^2(\beta-2)\,V(o,r(x)/2)},$$

and we have by (4.4),

$$\begin{aligned} &\int_{\{k\le r(z)\le 2r(x); d(x,z)\ge r(x)/2\}} \psi(r(z))\,d\mu_f(z) \\ &\quad\le \frac{C_{\beta}^{-1}\,\psi(k)\,k^{\beta}\,V(o,2r(x))}{2^{\beta}\,r(x)^{\beta}} + \frac{\beta\,C_{\beta}^{-1}\,V(o,2r(x))}{2^{\beta}\,r(x)^{\beta}} \int_k^{2r(x)} \psi(r)\,r^{\beta-1}\,dr \\ &\quad\le \frac{C_{\beta}^{-1}\,\psi(k)\,k^{\beta}\,V(o,2r(x))}{2^{\beta}\,r(x)^{\beta}} + \frac{\beta}{2^{\beta}}\,C_{\beta}^{-1}\,V(o,2r(x)) \int_{k/r(x)}^{2} \psi(r(x)t)\,t^{\beta-1}\,dt. \end{aligned}$$

In this way, we obtain

$$G_{k;1}(x) \le C_3 \frac{\psi(k)k^{\beta}}{r(x)^{\beta-2}} + C_3\,r(x)^2 \int_0^2 \psi(r(x)t)\,t^{\beta-1}\,dt,$$

where we put $C_3 = C_1\,C_D^2\,C_{\beta}^{-2}\,16^{\gamma}\,\beta(\beta-2)^{-1}$.

For $G_{k;2}(x)$, we have

$$
\begin{aligned}
G_{k;2}(x) &\le \psi((r(x)/2)\vee k)\int_{\{d(x,z)\le r(x)/2\}} G(x,z)\,d\mu_f(z)\\
&\le C_1\,\psi((r(x)/2)\vee k)\int_0^{r(x)/2}\int_r^\infty \frac{t\,dt}{V(x,t)}\,V'(x,r)\,dr\\
&= C_1\,\psi((r(x)/2)\vee k)\Big(\int_{r(x)/2}^\infty \frac{t\,dt}{V(x,t)}\,V(x,r(x)/2)+\int_0^{r(x)/2} r\,dr\Big)\\
&\le C_1\,\psi((r(x)/2)\vee k)\Big(\int_{r(x)/2}^{3r(x)} \frac{tV(x,r(x)/2)}{V(x,t)}\,dt+\int_{3r(x)}^\infty \frac{t\,V(x,r(x)/2)}{V(x,t)}\,dt+r(x)^2\Big)\\
&\le C_1\,\psi((r(x)/2)\vee k)\Big(6r(x)^2+\int_{3r(x)}^\infty \frac{t\,V(o,3r(x)/2)}{V(o,t-r(x))}dt\Big)\\
&\le C_1\,\psi((r(x)/2)\vee k)\Big(6r(x)^2+C_\beta\int_{3r(x)}^\infty t\Big(\frac{3r(x)/2}{t-r(x)}\Big)^\beta\,dt\Big)\\
&\le 6C_1\Big(1+\frac{C_\beta}{\beta-2}\Big)\,\psi((r(x)/2)\vee k)\,r(x)^2.
\end{aligned}
$$

Finally, we consider $G_{k;3}(x)$. Since we have for $t\ge 2r(x)$,

$$
V(x,t)\ge V(o,t-r(x))\ge C_D\Big(\frac{t-r(x)}{t}\Big)^\gamma V(o,t)\ge \frac{C_D}{2^\gamma}\,V(o,t),
$$

we get

$$
\begin{aligned}
G_{k;3}(x) &\le C_1\int_{\{r(z)\ge 2r(x)\}}\int_{d(x,z)}^\infty \frac{t\,dt}{V(x,t)}\,\psi(r)\,d\mu_f(z)\\
&\le C_1\int_{\{r(z)\ge 2r(x)\}}\int_{r(z)-r(x)}^\infty \frac{t\,dt}{V(x,t)}\,\psi(r)\,d\mu_f(z)\\
&\le C_1C_D\int_{r(x)}^\infty\Big(\frac{t}{V(o,t)}\int_{\{2r(x)\le r(z)\le t+r(x)\}}\psi(r)\,d\mu_f(z)\Big)\,dt.
\end{aligned}
$$

Since we have by (4.4),

$$
\begin{aligned}
&\int_{\{2r(x)\le r\le t+r(x)\}}\psi(r)\,d\mu_f\\
&\qquad\le C_\beta^{-1}\,\frac{V(o,t+r(x))}{(t+r(x))^\beta}\Big(\psi(2r(x))\,(2r(x))^\beta+\beta\int_{2r(x)}^{t+r(x)}\psi(r)\,r^{\nu-1}\,dr\Big)
\end{aligned}
$$

and

$$
\frac{V(o,t+r(x))}{V(o,t)}\le C_D\Big(\frac{t+r(x)}{t}\Big)^\gamma\le C_D\,2^\gamma,
$$

putting $C_4 = 2^\gamma \beta C_1 C_D^2 C_\beta^{-1}$, we get

$$
\begin{aligned}
G_{k;3}(x) &\le C_4 \int_{r(x)}^{\infty} \frac{t}{(t+r(x))^\beta} \Big(\psi(2r(x))\,(2r(x))^\beta + \int_{2r(x)}^{t+r(x)} \psi(r)\,r^{\beta-1}\Big)\,dt \\
&\le C_4\,\psi(2r(x))\,(2r(x))^\beta \int_{r(x)}^{\infty} \frac{t\,dt}{(t+r(x))^\beta} \\
&\quad + C_4 \int_{r(x)}^{\infty} \frac{t}{(t+r(x))^\beta} \int_{2r(x)}^{t+r(x)} \psi(r)\,r^{\beta-1}\,dr\,dt \\
&\le C_4(\beta-2)^{-1}\,\psi(2r(x))(2r(x))^2 + C_4 \int_{2r(x)}^{\infty} \psi(r)\,r^{\beta-1} \int_{r-r(x)}^{\infty} \frac{t\,dt}{(t+r(x))^\beta}\,dr \\
&\le C_4\,(\beta-2)^{-1}\,\psi(2r(x))\,(2r(x))^2 + C_4 \int_{2r(x)}^{\infty} \psi(r)\,r\,dr.
\end{aligned}
$$

In this way, we obtain

$$
\begin{aligned}
G_k(x) \le C_5\Big(\frac{\psi(k)\,k^\beta}{r(x)^{\beta-2}} &+ \int_0^2 \psi(r(x)t)\,(r(x)t)^2\,t^{\beta-3}\,dt + \psi((r(x)/2)\vee k)\,r(x)^2 \\
&+ \psi(2r(x))\,(2r(x))^2 + \int_{r(x)}^{\infty} t\,\psi(t)\,dt\Big)
\end{aligned}
$$

for some positive constant C_5 and for all $k \ge 0$ and $x \in M$. This shows the assertions in the lemma. The proof of Lemma 4.1 is completed. ∎

Now we will finish the proof of Theorem 1.2.

(i) It follows from Lemma 4.1 (i) that $v(x) = -\int_M G(x,y)W(y)d\mu_f(y)$ is a unique solution of equation $\Delta_{f;2} v = W$ in M which tends to zero at infinity.

(ii-a) By the assumption that $\int_0^\infty t\psi(t)dt$ converges, we are able to apply a result by Ancona (see Theorem 3.3 in [2]) to assert that the Green functions $G^W(x,y) = \int_0^\infty p_t^W(x,y)\,dt$ of $Q'_{2;W}$ and $G(x,y)$ are equivalent in the sense that

$$
C_6^{-1}\,G(x,y) \le G^W(x,y) \le C_6\,G(x,y), \quad x,y \in M
$$

for some $C_6 \ge 1$, which implies that

$$
C_6^{-2}\,\frac{G(x,y)}{G(o,y)} \le \frac{G^W(x,y)}{G^W(o,y)} \le C_6^2\,\frac{G(x,y)}{G(o,y)}, \quad x,y \in M.
$$

Since

$$
\lim_{y\in M\to\infty} \frac{G(x,y)}{G(o,y)} = 1 \quad \text{and} \quad \lim_{y\in M\to\infty} \frac{G^W(x,y)}{G^W(o,y)} = \frac{u(x)}{u(o)}
$$

by Theorem 1.1 (ii), we get

$$
C_6^{-2}\,u(o) \le u(x) \le C_6^2\,u(o), \quad x \in M.
$$

Then Lemma 4.1 (i) shows that $\hat{v}(x) = \int_M G(x,y)\,W(y)\,u(y)\,d\mu_f(y)$ converges for all $x \in M$, and $\hat{v}(x)$ tends to zero as $x \in M \to \infty$. Thus $u + \hat{v}$ is harmonic and bounded in M, so that it must be a constant, say a. In this way, we conclude that $u(x) = a - \int_M G(x,y)W(y)u(y)\,d\mu_f(y)$ for all $x \in M$.

(ii-b) We assume here that $Q'_{2;W}$ is critical, that is, $Q'_{2;W}$ does not admit the Green function. Then following a result due to Pinchiover (Theorem 4.2 in [21]), we are able to take a function V of class $C^{0,\alpha}(M)$ with compact support in such a way that $Q'_{2;V+W}$ is subcritical and

$$u(x) = \int_M G^{V+W}(x,y)\,V(y)\,u(y)\,d\mu_f(y), \quad x \in M.$$

Since $|V + W| \le C_7\,\psi(r)$ for some constant $C_7 > 0$, G^{V+W} is equivalent to G and hence it turns out that $u(x)$ tends to zero as $x \in M \to \infty$. This shows that $u + \hat{v}$ is a harmonic function on M tending to zero at infinity. Thus we conclude that $u + \hat{v} = 0$, namely, $u(x) = -\int_M G(x,y)W(y)u(y)\,d\mu_f(y)$. ∎

Now we end this section with some results, remarks, and examples related to Theorems 1.1, 1.2 and 1.3. We begin with the following.

Proposition 4.2 ([13]). *Let (M, g_M, μ_f) be a connected, noncompact, complete weighted Riemannian manifold. Condition* (VD) *for remote balls relative to a fixed point is satisfied, and* (VC) *holds true if and only if* (VD) *for all balls is satisfied.*

Proof. See Lemma 4.4 and Proposition 4.7 in [13]. ∎

Proposition 4.3. *Let (M, g_M, μ_f) be a connected, noncompact and complete weighted Riemannian manifold.*

(i) *Suppose that* (RCA) *holds true and* (VD) *for all balls is satisfied. Then $\delta^{(\sigma;\,\infty)}(M) < +\infty$ for some $\sigma \in (0,1)$.*

(ii) *Suppose that $\delta^{(\infty)}(M) < 2$. Then $\delta^{(\sigma;\infty)}(M) = \delta^{(\infty)}(M)$ for all $\sigma \in (\frac{1}{2}\delta^{(\infty)}(M), 1)$.*

(iii) *Let o be a point of M. Suppose that there are constants β, γ, C_β, C_γ such that $0 < \beta \le \gamma$, $C_\beta \le 1 \le C_\gamma$, and*

$$C_\beta\Big(\frac{t}{s}\Big)^\beta \le \frac{\mu_f(B(o,t))}{\mu_f(B(o,s))} \le C_\gamma\Big(\frac{t}{s}\Big)^\gamma \tag{4.5}$$

for all $1 \le s \le t$, and suppose that $\delta^{(\infty)}(M) < 1$. Then (M, g_M, μ_f) satisfies (RCA) *and* (VC).

Proof. (i) Let C_A be a constant greater than 1 in condition (RCA). We take constants σ, $\delta \in (0,1)$ in such a way that $0 < \delta < 4^{-1}C_A^{-1}$, and $1 - \sigma < C_A^{-1} - 2\delta$. By (RCA), for any two points on $S(o,t)$, there is a path connecting these points in $B(o, C_A t) \setminus B(o, C_A^{-1} t)$. Set $A^*(t)$ to be the union of $B(o, C_A t) \setminus B(o,t)$ and the δt-neighborhoods of all such paths. This construction ensures that $A^*(t)$ is a connected set which contains $S(o,t)$ and is included in $M \setminus B(o, (1-\sigma)t)$ (see [13], Subsection 5.1). We consider a maximal set $\{x_i \,|\, i = 1, 2, \dots, N\}$ of points in $A^*(t)$ at distance at least δt from each other (i.e., an δt-net in $A^*(t)$). Then $\{B(x_i, \delta t/2)\,|\, i = 1, \dots, N\}$ is a set of pairwise disjoint balls and the union of $\{B(x_i, \delta t)\,|\, i = 1, \dots, N\}$ covers $A^*(t)$. Associated with the covering is a graph consisting of the set of vertices $\boldsymbol{V}$ and the set of edges $\boldsymbol{E}$ by setting

$$\boldsymbol{V} = \{x_i \,|\, i = 1, 2, \dots, N\} \quad \text{and} \quad \boldsymbol{E} = \{(x_i, x_j) \in \boldsymbol{V} \times \boldsymbol{V} \mid \mathrm{dis}_M(x_i, x_j) < 2\delta t\}$$

(see [13], Subsection 3.1). Since $A^*(t)$ is connected, it follows that the associated graph $(\boldsymbol{V}, \boldsymbol{E})$ is connected. Moreover, in view of (VD) and (VC), we see that the cardinality N of $\boldsymbol{V}$ is bounded from above by a constant N^* which is independent of t. In fact, since

$$(C_A^{-1} - \delta)t < r(x_i) < (C_A + \delta)t,$$

we have

$$\begin{aligned}\mu_f(B(o,(C_A^{-1} - \delta)t)) &\le \mu_f(B(o, r(x_i))) \le C_V\, \mu_f(B(x_i, r(x_i)/2) \\ &\le C_V\, \mu_f(B(x_i, (C_A + \delta)t/2)) \le C_V\, C_D \Big(\frac{C_A + \delta}{\delta}\Big)^{\gamma} \mu_f(B(x_i, \delta t/2)),\end{aligned}$$

and hence

$$\begin{aligned}N\, \mu_f(B(o,(C_A^{-1} - \delta)t)) &\le C_V\, C_D \Big(\frac{C_A + \delta}{\delta}\Big)^{\gamma} \sum_{i=1}^{N} \mu_f(B(x_i, \delta t/2)) \\ &= C_V\, C_D \Big(\frac{C_A + \delta}{\delta}\Big)^{\gamma} \mu_f\Big(\bigcup_{i=1}^{N} B(x_i, \delta t/2)\Big) \\ &\le C_V\, C_D \Big(\frac{C_A + \delta}{\delta}\Big)^{\gamma} \mu_f(B(o,(C_A + \delta)t)) \\ &\le C_V\, C_D^2 \Big(\frac{C_A + \delta}{\delta}\Big)^{\gamma} \Big(\frac{C_A + \delta}{C_A^{-1} - \delta}\Big)^{\gamma} \mu_f(B(o,(C_A^{-1} - \delta)t).\end{aligned}$$

In this way, we obtain

$$N \le C_V\, C_D^2 \Big(\frac{C_A + \delta}{\delta}\Big)^{\gamma} \Big(\frac{C_A + \delta}{C_A^{-1} - \delta}\Big)^{\gamma} =: N^*.$$

Then for any pair of points of $A^*(t)$, there is a path in $M \setminus B(o,(1-\sigma)t)$ joining these points whose length is at most $2\delta(N^* + 2)t$. This shows that the diameter of $A^*(r)$ in $M \setminus B(o,(1-\sigma)t)$ is bounded from above by $2\delta(N^* + 2)t$. In this way, we can deduce that

$$\delta^{(\sigma;\infty)}(M) = \limsup_{t\to\infty} \frac{1}{t} \operatorname{diam}^{(\sigma;\infty)}(S(o,t)) \le 2\delta(N^* + 2).$$

(ii) We take positive numbers $\varepsilon, t_\varepsilon$ so that $\delta^{(\infty)}(M) + \varepsilon < 2\sigma$ and $t^{-1}\operatorname{diam}(S(o,t)) < \delta^{(\infty)}(M) + \varepsilon$ for all $t \ge t_\varepsilon$. For $x, y \in S(o,t)$ $(t \ge t_\varepsilon)$, let $\gamma_{xy}\colon [0, L] \to M$ $(L := \operatorname{dis}_M(x,y))$ be a distance minimizing curve joining $x = \gamma_{xy}(0)$ to $y = \gamma_{xy}(L)$. Since $t^{-1}\operatorname{dis}_M(x,y) < \delta^{(\infty)}(M) + \varepsilon < 2\sigma$, we see that γ_{xy} is included in $M \setminus B(o,(1-\sigma)t)$. This implies that $t^{-1}\operatorname{dis}^{(\sigma;t)}(x,y) = t^{-1}L < \delta^{(\infty)}(M) + \varepsilon$, and hence

$$\frac{1}{t} \operatorname{diam}^{(\sigma;t)}(S(o,t)) < \delta^{(\infty)}(M) + \varepsilon$$

so that we have

$$\delta^{(\sigma;\infty)}(M) \le \delta^{(\infty)}(M) + \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain

$$\delta^{(\sigma;\infty)}(M) \le \delta^{(\infty)}(M) < 2\sigma.$$

Since $\delta^{(\infty)}(M) \leq \delta^{(\sigma;\infty)}(M)$, we thus have

$$\delta^{(\sigma;\infty)}(M) = \delta^{(\infty)}(M) < 2\sigma$$

for $\sigma \in (\frac{1}{2}\delta^{(\infty)}(M), 1)$.

(iii) We first fix a constant $b > 2$ large enough so that $C_\beta b^\beta > 2$. Take a number $\sigma \in (\delta^{(\infty)}(M), 1)$. Then there exists $t_\sigma > 0$ such that $t^{-1}\mathrm{diam}(S(o,t)) \leq \sigma$ for all $t \geq t_\sigma$. Let $t \in [bt_\sigma, +\infty)$ and $a \in [b^{-1}, 1]$. For any $x \in S(o,t)$ and $y \in S(o,at)$, we take a point $z \in S(o,at)$ in such a way that $\mathrm{dis}_M(x,z) = (1-a)t$. Then we get

$$\mathrm{dis}_M(x,y) \leq \mathrm{dis}_M(x,z) + \mathrm{dis}_M(z,y) \leq (1-a)t + \sigma at \leq (1-(1-\sigma)b^{-1})t.$$

This shows that $S(o,at) \subset B(x,(1-(1-\sigma)b^{-1})t)$, and hence

$$B(o,t) \setminus B(o,b^{-1}t) = \bigcup_{b^{-1} \leq a < 1} S(o,at) \subset B(x,(1-(1-\sigma)b^{-1})t).$$

Therefore using (4.5), we have

$$\begin{aligned}\mu_f(B(x,(1-(1-\sigma)b^{-1})t) &\geq \mu_f(B(o,t)) - \mu_f(B(o,b^{-1}t)) \\ &\geq (C_\beta b^\beta - 1)\,\mu_f(B(o,b^{-1}t)) \geq \mu_f(B(o,b^{-1}t)\end{aligned}$$

for all $t \geq bt_\sigma$. Since $1/2 > 1-(1-\sigma)b^{-1}$, we have $\mu_f(B(x,t/2)) > \mu_f(B(x,(1-(1-\sigma)b^{-1})t)$, and by (4.5), we get $\mu_f(B(o,b^{-1}t)) \geq C_\gamma^{-1} b^\gamma \mu_f(B(o,t))$. These prove that

$$\mu_f(B(x,t/2)) \geq C_\gamma^{-1} b^\gamma \mu_f(B(o,t)).$$

In this way, we see that (VC) holds. ∎

Corollary 4.4. *Let (M, g_M, μ_f) be as above. Assume that* (VD) *and* (PI(2)) *for remote balls to a fixed point hold true. Then (M, g_M, μ_f) satisfies* (RCA)*,* (VD) *and* (PI(2)) *if* (4.5) *is satisfied and $\delta^{(\infty)}(M) < 1$,*

Proof. By Proposition 4.3 (iii), we see that (RCA) and (VC) hold, so that the corollary follows from Theorem 5.2 in [13]. ∎

Fix $p \in (1,+\infty)$. A function $u \in L^{1,p}_{\mathrm{loc}}(M) \cap C^{1,\alpha}_{\mathrm{loc}}(M)$ satisfying $\Delta_{f;p} u = 0$ in M is said to be p-harmonic. Now as an application of the annulus Harnack inequality in Theorem 1.1 (i) to p-harmonic functions, we prove the following.

Theorem 4.5. *Let (M, g_M, μ_f) be a connected, noncompact, complete weighted Remannian manifold. Assume that* (VD) *and* (PI(p)) *hold for all remote balls with respect to a reference point $o \in M$, and $\delta^{(\sigma;\infty)}(M)$ is finite for some $\sigma \in (0,1)$.*

(i) *A positive p-harmonic function on M is constant.*

(ii) *There is a positive number ρ such that if a p-harmonic function h on M satisfies*

$$|h(x)| \leq C(1+r(x))^{\rho/\log(1+\delta^{(\infty)}(M))}$$

for some positive constants C and all $x \in M$, then h is constant. In particular, if $\delta^{(\infty)}(M) = 0$, then any p-harmonic function h on M with polynomial growth is constant.

Proof. (i) This is a consequence from the annulus Harnack inequality and the maximum principle for p-harmonic functions.

(ii) For a nonconstant p-harmonic function h on M, let $m(t) = \inf_{S(o,t)} h$ and $M(t) = \sup_{S(o,t)} h$, and let $v(t) = M(t) - m(t)$. Let δ be a positive number. Then $h - m((1 + \delta + 3\delta^{(\infty)}(M)/4)t)$ is p-harmonic and positive on $B(o, (1 + \delta + 3\delta^{(\infty)}(M)/4)t)$, and moreover for sufficiently large $t \geq t_0$, we can apply the argument in the proof of Theorem 1.1 (i) to the function $h - m((1 + \delta + 3\delta^{(\infty)}(M)/4)t)$ by noting that the curve γ_{xy} there stays in $B(o, (1 + \delta/2 + 3\delta^{(\infty)}(M)/4)t) \setminus B(o, (1 - \sigma)t)$, and obtain

$$h(x) - m((1 + \delta + 3\delta^{(\infty)}(M)/4)t) \leq C_H \big(h(y) - m((1 + \delta + 3\delta^{(\infty)}(M)/4)t)\big)$$

for all $x, y \in S(o,t)$ and all $t \geq t_0$, where C_H is a constant independent of h and t. This shows that

$$M(t) - m((1 + \delta + 3\delta^{(\infty)}(M)/4)t) \leq C_H \big(m(t) - m((1 + \delta + 3\delta^{(\infty)}(M)/4)t)\big). \tag{4.6}$$

Since $M(1 + \delta + 3\delta^{(\infty)}(M)/4)t) - h$ is also p-harmonic and positive on $B(o, (1 + \delta + 3\delta^{(\infty)}(M)/4)t)$, we get

$$M((1 + \delta + 3\delta^{(\infty)}(M)/4)t) - m(t) \leq C_H \big(M((1 + \delta + 3\delta^{(\infty)}(M)/4)t) - M(t)\big). \tag{4.7}$$

Then it follows from (4.6) and (4.7) that

$$v((1 + \delta + 3\delta^{(\infty)}(M)/4)t) + v(t) \leq C_H \big(v((1 + \delta + 3\delta^{(\infty)}(M)/4)t) - v(t)\big),$$

and hence

$$\frac{C_H + 1}{C_H - 1} v(t) \leq v((1 + \delta + 3\delta^{(\infty)}(M)/4)t)$$

for all $t \geq t_0$. Thus letting $D = \frac{C_H+1}{C_H-1}$, we have

$$D^q v(t) \leq v((1 + \delta + 3\delta^{(\infty)}(M)/4)^q t)$$

for all $t \geq t_0$ and positive integers q. This shows that

$$\begin{aligned}\frac{\log D(\log t - \log t_0)}{\log(1 + \delta + 3\delta^{(\infty)}(M)/4)} &\leq \log v(t) - \log v(t_0) + \log D \\ &\leq \sup_{B(o,t)} \log 2|h| - \log v(t_0) + \log D.\end{aligned}$$

Suppose that $\delta^{(\infty)}(M) > 0$ and $|h| \leq C_8(1 + r)^{\rho/\log(1+\delta^{(\infty)}(M))}$. Then we have

$$\frac{\log D}{\log(1 + \delta + 3\delta^{(\infty)}(M)/4)} (\log t - \log t_0) \leq \frac{\rho}{\log(1 + \delta^{(\infty)}(M))} \log(1 + t) + C_9$$

for all $t \geq t_0$ and some positive constant C_9. Now taking $\delta < \delta^{(\infty)}(M)/4$, we see that h must be constant if $\rho \leq \log D$.

Suppose that $\delta^{(\infty)}(M) = 0$ and $|h| \le C_{10}(1 + r)^\nu$ for some positive constants C_{10} and ν. Then we have

$$\frac{\log D}{\log(1+\delta)}(\log t - \log t_0) \le \nu \log(1 + t) + C_{11}$$

for all $t \ge t_0$ and some positive constant C_{11}. Taking δ so small that $\log D > \nu \log(1 + \delta)$, we conclude that h must be constant. ∎

Remark 4.6. (i) In the case where $p = 2$, Theorem 4.5 generalizes some result in [15], and moreover the last statement in Theorem 4.5 (ii) is extended by Carron [6]. But it is not clear whether such an extension in [6] is possible for the case where p is different from 2.

(ii) In Theorem 1.2, the potential W under consideration satisfies the following conditions:

$$\sup_{x\in M} \int_M G(x, y) W_+(y)\, d\mu_f(y) < +\infty;$$

$$\lim_{k\to\infty} \sup_{x\in M} \int_{M\setminus B(o,k)} G(x, y) W_-(y)\, d\mu_f(y) < 1,$$

as shown in Lemma 4.1. According to Theorem 4.1 in Devyver [8], these are sufficient conditions for the heat kernel of $Q'_{2;W}$ to satisfy the Li–Yau estimate, under the conditions that (M, g_M, μ_f) satisfies [VD] and [PI(2)], and further W is subcritical.

Nonnegative Schrödinger operators, and their heat semigroups, have been studied intensively by many authors (see, e.g., [11, 14] and references therein).

(iii) In Theorem 1.2, the Ricci curvature of the Riemannian manifold M possesses a lower bound as in (1.1), and a Hardy type inequality holds on M as mentioned in the introduction. Then a recent result due to Munteanu, Sung and Wang (see Theorem 1.5 in [19]), is also applicable to deduce existence and sharp estimates for solutions to the Poisson equation on M as in the first assertion of Theorem 1.2 with $f = 0$. The method in [19] is different from ours and more effective in dealing with a wider class of Riemannian manifolds.

Example 4.7. Let g be a Riemannian metric on $\mathbb{R}^m$ such that g can be represented in the polar coordinates (r, θ) in $\mathbb{R}^m$ as follows: $g = dr^2 + \rho(r)^2 d\theta^2$, where $\rho(r)$ is a positive smooth function on $(0, +\infty)$ such that $\rho(0) = 0$ and $\rho'(0) = 1$. We assume that $\rho(r) = Cr^d$ for $r \ge 1$, where C is a positive constant and d is a constant less than or equal to 1. Then in this model space $M = (\mathbb{R}^m, g)$, $\delta^{(\sigma;\infty)}(M) < +\infty$ for any $\sigma \in (0, 1)$, $\delta^{(\infty)}(M) = \delta^{(\sigma;\infty)}(M) = 0$ if $d < 1$, and $\delta^{(\infty)}(M) = \sqrt{2(1 - \cos(\min\{\pi, C\pi\})}$ if $d = 1$; in particular $\delta^{(\infty)}(M) < 2$ for $C < 1$ and $\delta^{(\infty)}(M) < 1$ for $C < 1/3$. The Riemannian volume element of M is given by $dv_M = \rho(r)^{m-1} dr dv_\theta$, where dv_θ is the Riemannian volume element of the unit sphere $S^{m-1}(1)$ of dimension $m - 1$. Given $\gamma \in \mathbb{R}$, let $f(x) = -\log(1 + r^2)^{\gamma/2}$ and define a new measure by $\mu_f = (1 + r^2)^{\gamma/2} dv_M$. Obviously (M, g_M, μ_f) satisfies conditions (1.1) and (1.2).

(i) The following conditions are mutually equivalent:

(a-1) $\gamma + (m - 1)d + 1 > 0$,

(a-2) (VC) holds,

(a-3) M satisfies (VD) and (PI(2)).

(ii) The following are mutually equivalent (see [7], Proposition 3.4):

(b-1) $\gamma + (m-1)d + 1 > p$,

(b-2) The growth conditions (1.3) holds and the power $\beta > p$,

(b-3) (M, g_M, μ_f) is p-nonparabolic.

(iii) (See Example 9.1 in [23]) Let $u(x) = 2 + \sin(\log \sqrt{1 + r(x)^2})$ and define $W = \Delta_{f;2} u/u$. Clearly, $1 \le u(x) \le 3$, $|W(x)| \le C/(1 + r(x)^2)$, and u is the unique (up to a constant multiple) positive solution of the equation $Q'_{2;W} v = 0$ in M. But $\lim_{x\to\infty} u(x)$ does not exist.

(iv) (See Example 9.2 in [23]) Let $d = 1$ and $\vartheta \colon S^{m-1}(1) \to [-1, 1]$ a nonconstant smooth function. For $x \in M$ with $r(x) \ge 1$, let $u(x) = 2 + \vartheta(x/r(x))$, and extend the function u as a smooth positive function on M. Let $W = \Delta_{f;2} u/u$. Then $|W| \le C/(1 + r(x)^2)$ and $u(x)$ is a bounded positive function which is bounded away from zero. Moreover, $u(x)$ is the unique (up to a constant multiple) positive solution of the equation $Q'_{2;W} v = 0$ in M. But $\lim_{x\to\infty} u(x)$ does not exist.

Example 4.8. Li and Tam [18] shows that property (VC) is satisfied in the following two classes of connected, complete, noncompact Riemannian manifolds.

(i) (M, g_M) has asymptotically nonnegative sectional curvature, that is, there exists a point $o \in M$ and a continuous nonincreasing function $k \colon (0, +\infty) \to (0, +\infty)$ satisfying $\int_0^\infty s k(s)\, ds < \infty$ such that the sectional curvature $\mathrm{Sect}(x)$ of M at a point x is greater than or equal to $-k(\mathrm{dis}_M(o, x))$.

(ii) M has nonnegative Ricci curvature outside a compact set and the first Betti number is finite.

We notice that M has asymptotically nonnegative sectional curvature if, for some $C > 0$ and $\varepsilon > 0$, $\mathrm{Sect}(x) \ge -C\,\mathrm{dis}_M(o, x)^{-2-\varepsilon}$; on the other hand, we have $\mathrm{Sect}(x) \ge -C\,\mathrm{dis}_M(o, x)^{-2}$ for some $C > 0$ if M has asymptotically nonnegative sectional curvature.

5. Proof of Theorem 1.3

We first demonstrate that estimate (1.5) in Theorem 1.3 is optimal.

Example 5.1. Let $(L, g_L, e^{-\eta} dv_L)$ be a connected, complete weighted Riemannian manifold of dimension $m - 1$, and let $M = \mathbb{R} \times L$ with a warped product metric

$$g_M = dt^2 + e^{2\sqrt{\kappa}t} g_L,$$

where κ is a nonnegative constant. Suppose that for some $n \ge m$, $\mathrm{Ric}_\eta^{n-1} \ge 0$ on L. Define a weight function on M by $f(t, x) = (n - m)\sqrt{\kappa}\, t + \eta(x)$. Then it can be directly verified that $\mathrm{Ric}_f^n \ge -(n-1)\kappa\, g_M$. We let $\hat{u}(t, x) = e^{at}$ for a positive constant a, and we put $\Lambda = (p-1)a^p + (n-1)\sqrt{\kappa}\, a^{p-1}$. Then $\hat{u}$ satisfies

$$\Delta_{f;p} \hat{u} = \Lambda\, \hat{u}^{p-1} \quad \text{and} \quad |\nabla \log \hat{u}| = a = Z(p, n, \kappa, \Lambda)$$

on M. Now take a number $b > (n-1)/(p-1)$ in such a way that $(p-1)b^p - (n-1)\sqrt{\kappa}\, b^{p-1} = \Lambda$, and let $\check{u}(t, x) = e^{-bt}$. Then $\check{u}$ satisfies

$$\Delta_{f;p} \check{u} = \Lambda\, \check{u}^{p-1} \quad \text{and} \quad |\nabla \log \check{u}| = b = Y(p, n, \kappa, \Lambda) \quad \text{on } M.$$

In fact, we have a rigidity result as follows:

Theorem 5.2. *Let (M, g_M, μ_f) be a connected, noncompact complete weighted Riemannian manifold of dimension m such that $\mathrm{Ric}_f^n \geq -(n-1)\kappa\, g_M$ for some constants $\kappa \geq 0$ and $n \geq m$. Let u be a positive solution to the equation $-\Delta_{f;p}u + \Lambda |u|^{p-2}u = 0$ in M, where Λ is a positive constant. Suppose that there is a point $y \in M$ such that $|\nabla \log u(y)| = \sup_M |\nabla \log u| = Y(p,n,\kappa,\Lambda)$ or $|\nabla \log u(y)| = \sup_M |\nabla \log u| = Z(p,n,\kappa,\Lambda)$. Then M is isometric to a warped product $\mathbb{R} \times_{e^{\sqrt{\kappa}t}} L$ as in Example 5.1; in the case where $|\nabla \log u(y)| = \sup_M |\nabla \log u| = Y(p,n,\kappa,\Lambda)$, $u(t,x) = e^{-Y(p,n,\kappa,\Lambda)t}$, $f(t,x) = (n-m)\sqrt{\kappa}\, t + \eta(x)$ for some $\eta \in C^\infty(L)$ satisfying $\mathrm{Ric}_\eta^{n-1} \geq 0$ on L, and in the case where $|\nabla \log u(y)| = \sup_M |\nabla \log u| = Z(p,n,\kappa,\Lambda)$, $u(t,x) = e^{Z(p,n,\kappa,\Lambda)t}$, $f(t,x) = -(n-m)\sqrt{\kappa} t + \eta(x)$ for some $\eta \in C^\infty(L)$ satisfying $\mathrm{Ric}_\eta^{n-1} \geq 0$ on L.*

Theorem 5.2 will be verified at the end of the present section. We remark that in the case where $p = 2$ and $f = 0$, Theorem 5.2 is proved by Borbély [4] in a different way from ours.

Now we need some preliminary results to prove the upper estimate in (1.5) of Theorem 1.3.

Consider a positive solution u to the equation $-\Delta_{p,f}u + \Lambda |u|^{p-2}u = 0$ in the metric ball $B(o,R)$ of radius R around a fixed point o of M. In what follows, we write simply $B(R)$ and $V(R)$ respectively for $B(o,R)$ and $\mu_f(B(o,R))$. We set

$$v = -(p-1)\log u, \quad h = |\nabla v|^2 \quad \text{and} \quad K = \{x \in B(R) \mid h(x) = 0\}.$$

We note that u is smooth on $B(R) \setminus K$. We consider the following linear operator $\mathcal{L}_f$ on $B(R) \setminus K$:

$$\mathcal{L}_f \psi = e^f \operatorname{div}\left(e^{-f} h^{p/2-1} A(\nabla \psi)\right) - p\, h^{p/2-1} \langle \nabla v, \nabla \psi \rangle,$$

where

$$A = \mathrm{id} + (p-2)\,\frac{\nabla v \otimes \nabla v}{|\nabla v|^2}\cdot$$

Then we have

$$\mathcal{L}_f h = 2h^{p/2-1}\Big(|Ddv|^2 + \mathrm{Ric}_M(\nabla v, \nabla v) + Ddf(\nabla v, \nabla v)\Big) + \left(\frac{p}{2} - 1\right) h^{p/2-2} |\nabla h|^2 \tag{5.1}$$

(see Lemma 2.1 in [9], Lemma 2.1 in [17]). We also observe that v satisfies

$$\Delta_{f;p} v = |\nabla v|^p - \Lambda (p-1)^{p-1},$$

which is rewritten as follows:

$$\Delta v - \langle \nabla f, \nabla v \rangle + \left(\frac{p}{2} - 1\right) h^{-1} \langle \nabla h, \nabla v \rangle - h + \Lambda (p-1)^{p-1} h^{1-p/2} = 0. \tag{5.2}$$

Let $\{e_1, \ldots, e_m\}$ be a local orthonormal frame of TM with $e_1 = \nabla v / |\nabla v|$ in an open set Ω of $B(R) \setminus K$, and let $\{e_1^*, \ldots, e_m^*\}$ be the dual frame. We write

$$Dd\, v = \sum_{i,j=1}^m v_{ij} e_i^* \otimes e_j^*.$$

Then the following identities hold:

$$v_{11} = \frac{1}{2} h^{-1} \langle \nabla v, \nabla h \rangle, \tag{5.3}$$

$$\sum_{i=1}^{m} {v_{1i}}^2 = \frac{1}{4} h^{-1} |\nabla h|^2, \tag{5.4}$$

$$\sum_{i=2}^{m} v_{ii} = h - \Lambda(p-1)^{p-1} h^{1-p/2} - (p-1) v_{11} + \langle \nabla f, \nabla v \rangle \quad \text{(by (5.2))}. \tag{5.5}$$

Then using the inequality

$$(a+b)^2 \geq \frac{a^2}{1+\delta} - \frac{b^2}{\delta}, \quad \delta = \frac{n-m}{m-1},$$

we can derive from (5.5) that

$$\frac{1}{m-1}\Big(\sum_{i=2}^{m} v_{ii}\Big)^2 \geq \frac{1}{n-1} (h - \Lambda(p-1)^{p-1} h^{1-p/2} - (p-1) v_{11})^2 - \frac{1}{n-m} \langle \nabla f, \nabla v \rangle^2, \tag{5.6}$$

where the equality holds if and only if

$$(n-m)(h - \Lambda(p-1)^{p-1} h^{1-p/2} - (p-1) v_{11}) = -(n-1) \langle \nabla f, \nabla v \rangle. \tag{5.7}$$

We note that

$$\begin{aligned}(h - \Lambda(p-1)^{p-1} h^{1-p/2} - (p-1) v_{11})^2 \\ \geq (h - \Lambda(p-1)^{p-1} h^{1-p/2})^2 - 2(p-1)(h - \Lambda(p-1)^{p-1} h^{1-p/2}) v_{11},\end{aligned} \tag{5.8}$$

where the equality holds if and only if

$$v_{11} = 0. \tag{5.9}$$

Furthermore, we observe that

$$\sum_{i,j=2}^{m} v_{ij}^2 \geq \frac{1}{m-1} \Big(\sum_{i=2}^{m} v_{ii}\Big)^2, \tag{5.10}$$

where the equality holds at a point $x \in \Omega$ if and only if for some $\tau(x) \in \mathbb{R}$,

$$\sum_{i,j=2}^{m} v_{ij}(x)\, e_i^* \otimes e_j^* = \tau(x) \sum_{i=2}^{m} e_i^* \otimes e_i^*. \tag{5.11}$$

Then we obtain

$$\begin{aligned}
|Dd\,v|^2 &\geq \sum_{i=1}^{m} v_{1i}^2 + \sum_{i,j=2}^{m} v_{ij}^2 \\
&\geq \frac{1}{4}h^{-1}|\nabla h|^2 + \frac{1}{m-1}\Big(\sum_{i=2}^{m} v_{ii}\Big)^2 \quad \text{(by (5.4), (5.10))} \\
&\geq \frac{1}{4}h^{-1}|\nabla h|^2 + \frac{1}{n-1}(h - \Lambda(p-1)^{p-1}h^{1-p/2} - (p-1)v_{11})^2 \\
&\quad - \frac{1}{n-m}\langle \nabla f, \nabla v\rangle^2 \quad \text{(by (5.6))} \\
&\geq \frac{1}{4}h^{-1}|\nabla h|^2 + \frac{1}{n-1}(h - \Lambda(p-1)^{p-1}h^{1-p/2})^2 - \frac{p-1}{n-1}\langle \nabla v, \nabla h\rangle \\
&\quad + \frac{\Lambda(p-1)^{p-1}}{n-1}h^{-1}|\nabla v|^{p-2}\langle \nabla v, \nabla h\rangle - \frac{\langle \nabla v, \nabla h\rangle^2}{n-m} \quad \text{(by (5.8), (5.3))}.
\end{aligned} \tag{5.12}$$

By (5.12) and the assumption that $\mathrm{Ric}_f^n \geq -(n-1)\kappa\, g$, we get

$$\begin{aligned}
\mathcal{L}_f h &= 2h^{p/2-1}(|Dd\,v|^2 + \mathrm{Ric}_M(\nabla v, \nabla v)) + \frac{p-2}{2}h^{p/2-2}|\nabla h|^2 \\
&= 2h^{p/2-1}(|Dd\,v|^2 + \mathrm{Ric}_f^n(\nabla v, \nabla v)) + \frac{2}{n-m}h^{p/2-1}\langle \nabla v, \nabla f\rangle^2 \\
&\quad + \frac{p-2}{2}h^{p/2-2}|\nabla h|^2 \\
&\geq 2h^{p/2-1}\Big(\frac{1}{2}h^{-1}|\nabla h|^2 + \frac{1}{n-1}(h - \Lambda(p-1)^{p-1}|\nabla v|^{2-p})^2 - \frac{p-1}{n-1}\langle \nabla v, \nabla h\rangle \\
&\quad + \frac{\Lambda(p-1)^p}{n-1}h^{-1}|\nabla v|^{2-p}\langle \nabla v, \nabla h\rangle - (n-1)\kappa|\nabla v|^2\Big) + \frac{p-2}{2}h^{p/2-2}|\nabla h|^2 \\
&= \frac{2}{n-1}h^{p/2-1}\Big((h - \Lambda(p-1)^{p-1}h^{1-p/2})^2 - ((n-1)\sqrt{\kappa}\,h^{1/2})^2\Big) \\
&\quad - \frac{2(p-1)}{n-1}h^{p/2-1}\langle \nabla v, \nabla h\rangle + \frac{p}{2}h^{p/2-2}|\nabla h|^2 + \frac{2\Lambda(p-1)^p}{n-1}h^{-1}\langle \nabla v, \nabla h\rangle.
\end{aligned}$$

Thus we have the following.

Lemma 5.3. *One has*

$$\begin{aligned}
\mathcal{L}_f h \geq{}& \frac{2}{n-1}h^{p/2-1}\big((h - \Lambda(p-1)^{p-1}h^{1-p/2})^2 - ((n-1)\sqrt{\kappa}\,h^{1/2})^2\big) \\
&- \frac{2(p-1)}{n-1}h^{p/2-1}\langle \nabla v, \nabla h\rangle + \frac{p}{2}h^{p/2-2}|\nabla h|^2 + \frac{2\Lambda(p-1)^p}{n-1}h^{-1}\langle \nabla v, \nabla h\rangle
\end{aligned}$$

in $B(R)\setminus K$, where the equality holds if and only if there hold (5.7), (5.9), (5.11) and

$$\mathrm{Ric}_f^n(\nabla v, \nabla v) = -(n-1)\kappa\,|\nabla v|^2. \tag{5.13}$$

Lemma 5.4. *Let u be a positive solution of the equation $-\Delta_{f,p}u + \Lambda|u|^{p-2}u = 0$ in M. Then $|\nabla \log u|$ is bounded.*

Proof. Since

$$\big(h - \Lambda(p-1)^{p-1} h^{1-p/2}\big)^2 - \big((n-1)\sqrt{\kappa}\, h^{1/2}\big)^2 \\ > h^2 - 2\Lambda(p-1)^{p-1} h^{2-p/2} - (n-1)^2 \kappa h,$$

it follows from Lemma 5.3 that

$$\mathcal{L}_f h \geq -2(n-1)\kappa h^{p/2} + \frac{2}{n-1} h^{p/2+1} - \frac{4\Lambda(p-1)^{p-1}}{n-1} h + \frac{p-1}{2} h^{p/2-2} |\nabla h|^2 \\ - \frac{2(p-1)}{n-1} h^{p/2-1} \langle \nabla h, \nabla v\rangle + \frac{2\lambda(p-1)^p}{n-1} h^{-1} \langle \nabla h, \nabla v\rangle$$

in $B(R) \setminus K$.

Then for a nonnegative function ψ with compact support in $B(R) \setminus K$, we have

$$\begin{aligned} &\int_{B(R)} \langle h^{p/2-1} \nabla h + (p-2) h^{p/2-2} \langle \nabla v, \nabla h\rangle \nabla v, \nabla \psi\rangle \, d\mu_f \\ &\quad + p \int_{B(R)} h^{p/2-1} \langle \nabla h, \nabla v\rangle \psi \, d\mu_f + \frac{2}{n-1} \int_{B(R)} h^{p/2+1} \psi \, d\mu_f \\ &\leq 2(n-1)\kappa \int_{B(R)} h^{p/2} \psi \, d\mu_f + \frac{2(p-1)}{n-1} \int_{B(R)} h^{p/2-1} \langle \nabla h, \nabla v\rangle \psi \, d\mu_f \\ &\quad - \frac{2\Lambda(p-1)^p}{n-1} \int_{B(R)} h^{-1} \langle \nabla h, \nabla v\rangle \psi \, d\mu_f + \frac{4\Lambda(p-1)^{p-1}}{n-1} \int_{B(R)} h\psi \, d\mu_f \end{aligned} \tag{5.14}$$

(see (2.4) in [9]).

For constants $\varepsilon > 0$ and $b > 2$, we choose

$$\psi = h_\varepsilon^b \eta^2,$$

where $h_\varepsilon = (h-\varepsilon)^+$, $\eta \in C_0^\infty(B(R))$ is nonnegative and less than or equal to 1, and b is to be determined later. Then a direct calculation shows that

$$\nabla \psi = b h_\varepsilon^{b-1} \eta^2 \nabla h + 2 h_\varepsilon^b \eta \nabla \eta.$$

Insert this identity into (5.14), we obtain

$$\begin{aligned} & b \int_{B(R)} \Big(h^{p/2-1} h_\varepsilon^{b-1} |\nabla h|^2 + (p-2) h^{p/2-2} h_\varepsilon^{b-1} \langle \nabla v, \nabla h\rangle^2\Big) \eta^2 \, d\mu_f \\ &+ 2 \int_{B(R)} h^{p/2-1} h_\varepsilon^b \eta \langle \nabla h, \nabla \eta\rangle \, d\mu_f + p \int_{B(R)} h^{p/2-1} h_\varepsilon^b \eta^2 \langle \nabla h, \nabla v\rangle d\mu_f \\ &+ 2(p-2) \int_{B(R)} h^{p/2-2} h_\varepsilon^b \eta \langle \nabla h, \nabla v\rangle \langle \nabla v, \nabla \eta\rangle \, d\mu_f \\ &+ \frac{2}{n-1} \int_{B(R)} h^{p/2+1} h_\varepsilon^b \eta^2 \, d\mu_f \\ \leq\; & 2(n-1)\kappa \int_{B(R)} h^{p/2} h_\varepsilon^b \eta^2 \, d\mu_f + \frac{2(p-1)}{n-1} \int_{B(R)} h^{p/2-1} \langle \nabla h, \nabla v\rangle h_\varepsilon^b \eta^2 \, d\mu_f \\ &- \frac{2\Lambda(p-1)^p}{n-1} \int_{B(R)} \langle \nabla h, \nabla v\rangle h^{-1} h_\varepsilon^b \eta^2 \, d\mu_f + \frac{4\Lambda(p-1)^{p-1}}{n-1} \int_{B(R)} h h_\varepsilon^b \eta^2 \, d\mu_f. \end{aligned} \tag{5.15}$$

Since we have

$$h^{p/2-1} h_\varepsilon^{b-1} |\nabla h|^2 + (p-2) h^{p/2-2} h_\varepsilon^{b-1} \langle \nabla v, \nabla h \rangle^2 \geq a_0 h^{p/2-1} h_\varepsilon^{b-1} |\nabla h|^2,$$

where $a_0 = 1$ if $p \geq 2$ and $a_0 = (p-1)$ if $p \in (1,2)$, by replacing the integrand of the first term of the left side in (5.15) with the right side of the just above inequality and passing ε to 0, we obtain

$$\begin{aligned}
& a_0 b \int_{B(R)} h^{p/2+b-2} |\nabla h|^2 \eta^2 \, d\mu_f \\
& \quad + 2 \int_{B(R)} h^{p/2+b-1} \langle \nabla h, \nabla \eta \rangle \eta \, d\mu_f \\
& \quad + 2(p-2) \int_{B(R)} h^{p/2+b-2} \langle \nabla v, \nabla h \rangle \langle \nabla v, \nabla \eta \rangle \eta \, d\mu_f \\
& \quad + p \int_{B(R)} h^{p/2+b-1} \langle \nabla v, \nabla h \rangle \eta^2 \, d\mu_f + \frac{2}{n-1} \int_{B(R)} h^{p/2+b+1} \eta^2 \, d\mu_f \\
& \leq 2(n-1)\kappa \int_{B(R)} h^{p/2+b} \eta^2 \, d\mu_f \\
& \quad + \frac{2(p-1)}{n-1} \int_{B(R)} h^{p/2+b-1} \langle \nabla h, \nabla v \rangle^2 \eta^2 \, d\mu_f \\
& \quad - \frac{2\Lambda (p-1)^p}{n-1} \int_{B(R)} h^{b-1} \langle \nabla h, \nabla v \rangle \eta^2 \, d\mu_f \\
& \quad + \frac{4\Lambda (p-1)^{p-1}}{n-1} \int_{B(R)} h^{b+1} \eta^2 \, d\mu_f
\end{aligned} \tag{5.16}$$

(see (2.5) in [9], (2.5) in [34]). Using (5.16), we see that

$$\begin{aligned}
& a_0 b \int_{B(R)} h^{p/2+b-2} |\nabla h|^2 \eta^2 \, d\mu_f + \frac{2}{n-1} \int_{B(R)} h^{p/2+b+1} \eta^2 \, d\mu_f \\
& \quad \leq 2(n-1)\kappa \int_{B(R)} h^{p/2+b} \eta^2 \, d\mu_f + \frac{4\Lambda (p-1)^{p-1}}{n-1} \int_{B(R)} h^{b+1} \eta^2 \, d\mu_f \\
& \quad + I_1 + I_2 + I_3
\end{aligned} \tag{5.17}$$

(see (2.6) in [34]), where we put

$$\begin{aligned}
I_1 &= \frac{p(n+1)-2}{n-1} \int_{B(R)} h^{p/2+b-1/2} |\nabla h| \eta^2 \, d\mu_f, \\
I_2 &= \frac{2\Lambda (p-1)^p}{n-1} \int_{B(R)} h^{b-1/2} |\nabla h| \eta^2 \, d\mu_f, \\
I_3 &= 2(1 + |p-2|) \int_{B(R)} h^{p/2+b-1} |\nabla h| |\nabla \eta| \eta \, d\mu_f.
\end{aligned}$$

Now applying Young's inequality to I_1, I_2, and I_3 respectively, we obtain

$$
\begin{aligned}
|I_1| &= 2\int_{B(R)} \frac{\sqrt{a_0 b}}{2}\, h^{p/2+b-2)/2}|\nabla h|\,\eta\cdot\frac{p(n+1)-2}{\sqrt{a_0 b}(n-1)}\, h^{(p/2+b+1)/2}\,\eta\, d\mu_f \\
&\le \frac{a_0 b}{4}\int_{B(R)} h^{p/2+b-2}|\nabla h|^2\eta^2\, d\mu_f + \frac{(p(n+1)-2)^2}{a_0 b(n-1)^2}\int_{B(R)} h^{p/2+b+1}\eta^2\, d\mu_f, \\
|I_2| &= 2\int_{B(R)} \frac{\sqrt{a_0 b}}{2}\, h^{(p/2+b-2)/2}|\nabla h|\,\eta\cdot\frac{2|\Lambda|(p-1)^p}{\sqrt{a_0 b}(n-1)}\, h^{(b-p/2+1)/2}\,\eta\, d\mu_f \\
&\le \frac{a_0 b}{4}\int_{B(R)} h^{p/2+b-2}|\nabla h|^2\eta^2\, d\mu_f + \frac{4\Lambda^2(p-1)^{2p}}{a_0 b(n-1)^2}\int_{B(R)} h^{b-p/2+1}\eta^2\, d\mu_f, \\
|I_3| &= 2\int_{B(R)} \frac{\sqrt{a_0 b}}{2}\, h^{(p/2+b-2)/2}|\nabla h|\,\eta\cdot\frac{2(1+|p-2|)}{\sqrt{a_0 b}}\, h^{(p/2+b)/2}|\nabla\eta|\, d\mu_f \\
&\le \frac{a_0 b}{4}\int_{B(R)} h^{p/2+b-2}|\nabla h|^2\eta^2\, d\mu_f + \frac{4(1+|p-2|)^2}{a_0 b}\int_{B(R)} h^{p/2+b}|\nabla\eta|^2\, d\mu_f.
\end{aligned}
$$

In what follows, b is chosen in such a way that

$$
\frac{(p(n+1)-2)^2}{a_0 b} < \frac{1}{n-1}, \tag{5.18}
$$

and a_i $(i=1,2,3,\ldots)$ stand for positive constants depending only on n and p.

Now it follows from (5.17) and (5.18) that

$$
\begin{aligned}
& b\int_{B(R)} h^{p/2+b-2}|\nabla h|^2\eta^2\, d\mu_f + \frac{1}{n-1}\int_{B(R)} h^{p/2+b+1}\eta^2\, d\mu_f \\
&\qquad \le a_1\kappa\int_{B(R)} h^{p/2+b}\eta^2\, d\mu_f + \frac{a_2}{b}\int_{B(R)} h^{p/2+b}|\nabla\eta|^2\, d\mu_f \\
&\qquad\quad + \frac{a_3\Lambda^2}{b}\int_{B(R)} h^{b-p/2+1}\eta^2\, d\mu_f + a_4\int_{B(R)} \Lambda h^{b+1}\eta^2\, d\mu_f.
\end{aligned} \tag{5.19}
$$

Using

$$
|\nabla(h^{p/4+b/2}\eta)|^2 \le \frac{(p/2+b)^2}{2}\, h^{p/2+b-2}|\nabla h|^2\eta^2 + 2h^{p/2+b}|\nabla\eta|^2,
$$

we have by (5.19),

$$
\begin{aligned}
& \int_{B(R)} |\nabla(h^{p/4+b/2}\eta)|^2\, d\mu_f + a_4 b\int_{B(R)} h^{p/2+b+1}\eta^2\, d\mu_f \\
&\qquad \le a_5 b\kappa\int_{B(R)} h^{p/2+b}\eta^2\, d\mu_f + a_6\int_{B(R)} h^{p/2+b}|\nabla\eta|^2\, d\mu_f \\
&\qquad\quad + a_7\Lambda^2\int_{B(R)} h^{b-p/2+1}\eta^2\, d\mu_f \quad + a_8 b\Lambda\int_{B(R)} h^{b+1}\eta^2\, d\mu_f.
\end{aligned} \tag{5.20}
$$

We recall here the following Sobolev embedding theorem of Saloff-Coste [29, 30]:

$$\Big(\int_{B(R)} |\phi|^{2n/(n-2)}\, d\mu_f\Big)^{(n-2)/n} \le e^{C(n)(1+\sqrt{\kappa}R)} V(R)^{-2/n} \int_{B(R)} (R^2|\nabla\phi|^2 + \phi^2)\, d\mu_f$$

for any $\phi \in C_0^\infty(B(R))$, where $C(n)$ is some positive constant depending only on n, and $V(R)$ stands for $\mu_f(B(R))$.

Now letting $\phi = h^{p/4+b/2}\eta$, we have

$$\Big(\int_{B(R)} h^{\frac{(p/2+b)n}{n-2}} \eta^{\frac{2n}{n-2}}\, d\mu_f\Big)^{(n-2)/n} \le e^{C(n)(1+\sqrt{\kappa}R)}\, V(R)^{-2/n}\Big(R^2 \int_{B(R)} |\nabla(h^{p/4+b/2}\eta)|^2\, d\mu_f + \int_{B(R)} h^{p/2+b}\eta^2\, d\mu_f\Big) \tag{5.21}$$

(see (2.9) in [9]). Let $b_0 = a_9 + \sqrt{\kappa}R$, where we assume that b_0 satisfies (5.18). We put

$$\begin{aligned}
I_4 &= a_4 e^{C(n)\, b_0}\, b\, R^2\, V(R)^{-2/n} \int_{B(R)} h^{p/2+b+1}\, \eta^2\, d\mu_f,\\
I_5 &= a_5\, \kappa\, e^{C(n)\, b_0}\, b\, R^2\, V(R)^{-2/n} \int_{B(R)} h^{p/2+b}\, \eta^2\, d\mu_f,\\
I_6 &= a_6 e^{C(n)\, b_0}\, R^2\, V(R)^{-2/n} \int_{B(R)} h^{p/2+b}\, |\nabla\eta|^2\, d\mu_f,\\
I_7 &= a_7\, \Lambda^2\, e^{C(n)\, b_0}\, R^2\, V(R)^{-2/n} \int_{B(R)} h^{b-p/2+1}\, \eta^2\, d\mu_f,\\
I_8 &= a_8\, \Lambda\, e^{C(n)\, b_0}\, b\, R^2\, V(R)^{-2/n} \int_{B(R)} h^{b+1}\, \eta^2\, d\mu_f,\\
I_9 &= e^{C(n)\, b_0}\, V(R)^{-2/n} \int_{B(R)} h^{p/2+b}\, \eta^2\, d\mu_f.
\end{aligned}$$

Then (5.20) and (5.21) combined give

$$\begin{aligned}
&\Big(\int_{B(R)} h^{\frac{(p/2+b)n}{n-2}} \eta^{\frac{2n}{n-2}}\, d\mu_f\Big)^{(n-2)/n} + I_4 \qquad (5.22)\\
&\le e^{C(n)\, b_0}\, V(R)^{-2/n} R^2 \Big(\int_{B(R)} |\nabla(h^{p/4+b/2}\eta)|^2\, d\mu_f + a_4 b \int_{B(R)} h^{p/2+b+1}\, \eta^2\, d\mu_f\Big)\\
&\quad + e^{C(n)\, b_0}\, V(R)^{-2/n} \int_{B(R)} h^{p/2+b}\, \eta^2\, d\mu_f\\
&\le e^{C(n)\, b_0}\, V(R)^{-2/n}\, R^2 \Big(a_5\, b\, \kappa \int_{B(R)} h^{p/2+b}\, \eta^2\, d\mu_f + a_6 \int_{B(R)} h^{p/2+b}\, |\nabla\eta|^2\, d\mu_f\\
&\quad + a_7\, \Lambda^2 \int_{B(R)} h^{b-p/2+1}\, \eta^2\, d\mu_f + a_8\, b\, \Lambda \int_{B(R)} h^{b+1}\, \eta^2\, d\mu_f\Big)\\
&\quad + e^{C(n)\, b_0}\, V(R)^{-2/n} \int_{B(R)} h^{p/2+b}\, \eta^2\, d\mu_f\\
&\le I_5 + I_6 + I_7 + I_8 + I_9
\end{aligned}$$

(see (2.10) in [9]).

Now we let $D = \{x \in B(R) \mid h(x) \geq 10\kappa a_5/a_4\}$. Since

$$a_5 \kappa e^{C(n) b_0} b R^2 V(R)^{-2/n} \int_D h^{p/2+b} \eta^2 \, d\mu_f \leq \frac{1}{10} I_4,$$

we obtain

$$\begin{aligned} I_5 &< \frac{1}{10} I_4 + a_5 \kappa e^{C(n) b_0} b R^2 V(R)^{-2/n} \int_{B(R)\setminus D} h^{p/2+b} \eta^2 \, d\mu_f \\ &< \frac{1}{10} I_4 + a_5 \kappa e^{C(n) b_0} b R^2 \Big(\frac{10 a_5 \kappa}{a_4}\Big)^{p/2+b} V(R)^{1-2/n} \\ &< \frac{1}{10} I_4 + a_{10}{}^{p/2+b} e^{C(n) b_0} \kappa^{p/2+b+1} b R^2 V(R)^{1-2/n}. \qquad (5.23) \end{aligned}$$

Similarly, we get

$$\begin{aligned} I_7 &< \frac{1}{10} I_4 + a_7 \Lambda^2 \Big(\frac{10 a_7 \Lambda^2}{a_4}\Big)^{(b-p/2+1)/p} e^{C(n) b_0} R^2 V(R)^{1-2/n} \qquad (5.24) \\ &< \frac{1}{10} I_4 + a_{10}^{p/2+b} e^{C(n) b_0} \Lambda^{2b/p+2/p+1} R^2 V(R)^{1-2/n}, \\ I_8 &< \frac{1}{10} I_4 + a_8 \Lambda \Big(\frac{10 a_5 \Lambda}{a_4}\Big)^{2(b+1)/p} e^{C(n) b_0} b R^2 V(R)^{1-2/n} \qquad (5.25) \\ &< \frac{1}{10} I_4 + a_{10}^{p/2+b} e^{C(n) b_0} \Lambda^{2b/p+2/p+1} b R^2 V(R)^{1-2/n}, \\ I_9 &< \frac{1}{10} I_4 + \Big(\frac{10}{a_4 b R^2}\Big)^{p/2+b} e^{C(n) b_0} V(R)^{1-2/n} \qquad (5.26) \\ &< \frac{1}{10} I_4 + a_{10}^{p/2+b} \Big(\frac{1}{bR^2}\Big)^{p/2+b}. \end{aligned}$$

So far as I_6 is concerned, we let $\eta_1 \in C_0^\infty(B(R))$ satisfy $0 \leq \eta_1 \leq 1$ in $B(R)$, $\eta_1 = 1$ in $B(3R/4)$, $|\nabla \eta_1| \leq 10/R$, and choose $\eta = \eta_1^{p/2+b+1}$. Then we have

$$R^2 |\nabla \eta|^2 \leq 10^2 (p/2 + b + 1)^2 \eta^{\frac{p/2+b}{p/2+b+1}}.$$

Employing the Hölder and the Young inequalities, we then obtain

$$\begin{aligned} R^2 & \int_{B(R)} h^{p/2+b} |\nabla \eta|^2 \, d\mu_f \\ &\leq 10^2 (p/2 + b + 1)^2 \int_{B(R)} h^{p/2+b} \eta^{\frac{p+2b}{p/2+b+1}} \, d\mu_f \\ &\leq 10^2 (p/2 + b + 1)^2 V(R))^{\frac{1}{p/2+b+1}} \Big(\int_{B(R)} h^{p/2+b+1} \eta^2 \, d\mu_f\Big)^{\frac{p/2+b}{p/2+b+1}} \\ &\leq \frac{a_4 b R^2}{2 a_6} \int_{B(R)} h^{p/2+b+1} \eta^2 \, d\mu_f \\ &\quad + a_{10} (p/2 + b)^{p/2+b} (p/2 + b + 1)^{p/2+b+1} \Big(\frac{2 a_6}{a_4 b R^2}\Big)^{p/2+b} V(R), \end{aligned}$$

so that we get

$$
\begin{aligned}
(5.27)\quad I_6 &\le \frac{1}{2} I_4 \\
&\quad + a_{10}\, e^{C(n)\, b_0}\, (p/2+b)^{p/2+b} (p/2+b+1)^{p/2+b+1} \Big(\frac{2a_6}{a_4 b R^2}\Big)^{p/2+b} V(R)^{1-2/n} \\
&< \frac{1}{2} I_4 + a_{11}^{p/2+b}\, e^{C(n)\, b_0} \Big(\frac{1}{bR^2}\Big)^{p/2+b} (p/2+b)^{p/2+b} (p/2+b+1)^{p/2+b+1}.
\end{aligned}
$$

Thus it follows from (5.22) through (5.27) that

$$
\begin{aligned}
(5.28)\quad &\Big(\int_{B(3R/4)} h^{\frac{(p/2+b)n}{n-2}}\, d\mu_f\Big)^{\frac{n-2}{n(p/2+b)}} \\
&\le a_{11}\, e^{C(n) b_0/(p/2+b)} V(R)^{(n-2)/n(p/2+b)} \Big(\kappa^{p/2+b+1} bR^2 + \Lambda^{2b/p+2/p+1}(1+b)R^2 \\
&\quad + \Big(\frac{1}{bR^2}\Big)^{p/2+b} + \Big(\frac{1}{bR^2}\Big)^{p/2+b} (p/2+b)^{p/2+b} (p/2+b+1)^{p/2+b+1}\Big)^{1/(p/2+b)}.
\end{aligned}
$$

Now we write $G(R,b)$ for the right-hand side of (5.28). We fix $r > 1$ and take $R > 2r$. Then

$$
\Big(\int_{B(r)} h^{\frac{(p/2+b)n}{n-2}}\, d\mu_f\Big)^{\frac{n-2}{n(p/2+b)}} \le \Big(\int_{B(3R/4)} h^{\frac{(p/2+b)n}{n-2}}\, d\mu_f\Big)^{\frac{n-2}{n(p/2+b)}} \le G(R,b).
$$

We let $b = a_9 + R$ keep to satisfy (5.18), and observe that $V(R) \le a_{12}\, e^{(n-1)R}$. Then we see that $G(R,b)$ is bounded as $R \to \infty$. Therefore we have

$$
\sup_{B(r)} h = \lim_{R\to\infty} \Big(\int_{B(r)} h^{\frac{(p/2+b)n}{n-2}}\, d\mu_f\Big)^{\frac{n-2}{n(p/2+b)}} \le \sup_{R\ge 2} G(R,b) < +\infty.
$$

Finally, letting $r \to \infty$, we conclude that h is bounded in M. ∎

Lemma 5.5. *Suppose there is a point $y \in M$ such that*

$$
h(y) = \sup_M h = (p-1)^2\, Y(p,n,\kappa,\lambda)^2
$$

or

$$
h(y) = \sup_M h = (p-1)^2\, Z(p,n,\kappa,\Lambda)^2.
$$

Then h is constant on M.

Proof. Let $(x^1, \ldots, x^m)$ be local coordinates on a neighborhood Ω of y in $M \setminus K$. We write

$$
g_M = \sum_{i,j=1}^{m} g_{ij}\, dx^i \otimes dx^j
$$

and let $G = \det(g_{ij})$. We define functions A, B_1 and B_2 on $\Omega \times \mathbb{R} \times \mathbb{R}^m$ respectively by

$$\begin{aligned} A(x,s,\xi) &= e^{-f(x)}\sqrt{G(x)}\,h(x)^{p/2-1}(\xi + (p-2)\,h(x)^{-2}\langle \nabla v(x), \xi\rangle \nabla v(x)),\\ B_1(x,s,\xi) &= -\frac{2}{n-1}\,e^{f(x)}\sqrt{G(x)}\,h(x)^{1-p/2}\\ &\quad + (s^{p/2} + (n-1)\sqrt{\kappa}\,s^{(p-1)/2} - (p-1)^{p-1}\Lambda)\\ &\quad\ \cdot (s^{p/2} - (n-1)\sqrt{\kappa}\,s^{(p-1)/2} - (p-1)^{p-1}\Lambda),\\ B_2(x,s,\xi) &= e^{f(x)}\sqrt{G(x)}\Big(\frac{2(p-1)}{n-1}\,h(x)^{p/2-1}\langle \nabla v(x), \xi\rangle\\ &\quad - \frac{2\Lambda(p-1)^p}{(n-1)}\,h(x)^{-1}\langle \nabla v(x), \xi\rangle - \frac{(p-1)}{2}\,h(x)^{p/2-1}\langle \nabla v(x), \xi\rangle\\ &\quad - p\,h(x)^{p/2-1}\langle \nabla h(x), \xi\rangle\Big). \end{aligned}$$

Then Lemma 5.3 shows that

$$\operatorname{div}(A(x,h,\nabla h)) + B_1(x,h,\nabla h) + B_2(x,h,\nabla h) \geq 0$$

on Ω. Moreover, the constant functions

$$c_1 = (p-1)^2\,Y(p,n,\kappa,\Lambda)^2 \quad \text{and} \quad c_2 = (p-1)^2\,Z(p,n,\kappa,\Lambda)^2$$

satisfy

$$\begin{aligned} \operatorname{div}(A(x,c_i,\nabla c_i)) &= 0,\\ B_1(x,c_i,\nabla c_i) &= 0,\\ B_2(x,c_i,\nabla c_i) &= 0 \quad (i=1,2). \end{aligned}$$

Therefore, letting $w = c_1 - h$ in case $h(y) = \sup_M h = (p-1)^2 Y(p,n,\kappa,\Lambda)^2$, or $w = c_2 - h$ in case $h(y) = \sup_M h = (p-1)^2 Z(p,n,\kappa,\Lambda)^2$, we see that w satisfies $w(y) = 0$, $w \geq 0$ in M and

$$\begin{aligned} \text{(5.29)}\quad \operatorname{div}A(x,w,\nabla w) &+ B_1(x,c_i - w, \nabla(c_i - w)) + B_2(x, c_i - w, \nabla(c_i - w))\\ &= \operatorname{div}A(x,h,\nabla h) - B_1(x,h,\nabla h) - B_2(x,h,\nabla h) \leq 0. \end{aligned}$$

Then we can apply the weak Harnack inequality for supersolutions due to Trudinger [33] to get

$$\int_{B(y,t)} w\,dx \leq C \inf_{B(y,t)} w$$

for a sufficiently small number t, where C is a positive constant. This shows that $w \equiv 0$ in $B(y,t)$ and hence in $B(R)$, since $w(y) = 0$ (see [26], Theorem 2.5.1). Since M is connected, we can conclude that $w = 0$ everywhere in M. This proves Lemma 5.5. ∎

Lemma 5.6. *One has*

$$\text{(5.30)}\qquad \langle \nabla f, \nabla v\rangle = -(n-m)(p-1)\sqrt{\kappa}\,Y(p,n,\kappa,\Lambda),$$

$$\text{(5.31)}\qquad Ddv = (p-1)\sqrt{\kappa}\,Y(p,n,\kappa,\Lambda)\Big(g_M - \frac{1}{h}\,dv \otimes dv\Big),$$

if $h \equiv (p-1)^2 Y(p,n,\kappa,\Lambda)^2$, *and*

$$\langle \nabla f, \nabla v\rangle = (n-m)(p-1)\sqrt{\kappa}\, Z(p,n,\kappa,\Lambda), \tag{5.32}$$

$$Ddv = -(p-1)\sqrt{\kappa}\, Z(p,n,\kappa,\Lambda)(g_M - \frac{1}{h} dv \otimes dv), \tag{5.33}$$

if $h \equiv (p-1)^2 Z(p,n,\kappa,\Lambda)^2$.

Proof. We consider the case where $h \equiv (p-1)^2 Y(p,n,\kappa,\Lambda)^2$. We note first that $v_{11} = 0$ by (5.3), and hence it follows from (5.7) that

$$\langle \nabla f, \nabla v\rangle = -(n-m)(p-1)\sqrt{\kappa}\, Y(p,n,\kappa,\Lambda).$$

Since $\Delta v = (m-1)\tau$ in (5.11), making use of (5.5), we get

$$Ddv = (p-1)\sqrt{\kappa}\, Y(p,n,\kappa,\Lambda) \times (g_M - h^{-1} dv \otimes dv).$$

Similarly, we see that

$$\langle \nabla f, \nabla v\rangle = (n-m)(p-1)\sqrt{\kappa}\, Z(p,n,\kappa,\Lambda)$$

and

$$Ddv = -(p-1)\sqrt{\kappa}\, Z(p,n,\kappa,\Lambda)(g_M - h^{-1} dv \otimes dv)$$

if $h \equiv (p-1)^2 Z(p,n,\kappa,\Lambda)^2$. ∎

Proof of Theorem 1.3. Let u be a positive solution of $-\Delta_{p;f} u + \Lambda |u|^{p-2} u = 0$ in M. So far, as the upper estimate of $|\nabla \log u|$ is concerned, since $\sup_M |\nabla \log u| < +\infty$ by Lemma 5.3, we are able to apply the same arguments as in [32] and [9] to prove that $|\nabla \log u| \le Y(p,n,\kappa,\Lambda)$.

Suppose now that $\sup_M |\nabla \log u| \le (1-\varepsilon) Z(p,n,\kappa,\Lambda)$ for some $\varepsilon \in (0,1)$. Then it follows that

$$|\log u(x) - \log u(y)| \le (1-\varepsilon)\, Z(p,n,\kappa,\Lambda)\, \mathrm{dis}_M(x,y)$$

for all $x, y \in M$. On the other hand, in view of Lemma 2.21, we can take a large r_ε so that $\log \eta_{p,\Lambda}(r) \ge (1-\varepsilon/2) Z(p,n,\kappa,\Lambda)\, r$ for all $r \ge r_\varepsilon$, and by Proposition 2.7, we find points x_r of $S(o,r)$ such that

$$\log u(x_r) \ge \log u(o) + \log \omega_{p,n,\Lambda}(r) \ge \log u(o) + (1-\varepsilon/2)\, Z(p,n,\kappa,\Lambda)\, r$$

for all $r \ge r_\varepsilon$. But this is absurd, because we have

$$\log u(x_r) \le \log u(o) + (1-\varepsilon) Z(p,n,\kappa,\Lambda) r.$$

Thus we have proved that $Z(p,n,\kappa,\Lambda) \le \sup_M |\nabla \log u|$. This completes the proof of the first assertion of Theorem 1.3.

Now we prove the second one. We first observe from (1.6) that

$$\log u(x) \ge \log u(y) - Y(p,n,\kappa,\Lambda)\, \mathrm{dis}_M(x,y)$$

for all $x, y \in M$.

Now we take positive numbers ε and r_ε in such a way that

$$\varepsilon\Big(\frac{1}{Y(p,n,\kappa,\Lambda)}+1\Big) \le \frac{1}{2}\Big(\frac{Z(p,n,\kappa,\Lambda)}{Y(p,n,\kappa,\Lambda)} - \delta_\infty(M)\Big),$$
$$\frac{\log \eta_{p,\lambda}(r)}{r} \ge Z(p,n,\kappa,\Lambda) - \varepsilon,$$
$$\frac{\operatorname{diam}(S(o,r))}{r} \le \delta_\infty(M) + \varepsilon$$

for all $r \ge r_\varepsilon$. For such r, we let x_r be a point of $S(o,r)$ such that $u(x_r) = \max_{S(o,r)} u$. Then for any $x \in S(o,r)$, we have

$$\begin{aligned}
\log u(x) &\ge \log u(x_r) - Y(p,n,\kappa,\Lambda)\operatorname{dis}_M(x,x_r)\\
&\ge \log u(o) + \log \eta_{p,\Lambda}(r) - Y(p,n,\kappa,\Lambda)\operatorname{diam}(S(o,r))\\
&= \log u(o) + r\Big(\frac{\log \eta_{p,\Lambda}(r)}{r} - Y(p,n,\kappa,\Lambda)\,\frac{\operatorname{diam}(S(o,r))}{r}\Big)\\
&\ge \log u(o) + r\,Y(p,n,\kappa,\Lambda)\Big(\frac{Z(p,n,\kappa,\Lambda)}{Y(p,n,\kappa,\Lambda)} - \delta_\infty(M) - \frac{\varepsilon}{Y(p,n,\kappa,\Lambda)} - \varepsilon\Big)\\
&\ge \log u(o) + \frac{1}{2}\,(Z(p,n,\kappa,\Lambda) - \delta_\infty(M)\,Y(p,n,\kappa,\Lambda))\,r.
\end{aligned}$$

Applying the Harnack inequality to u in $B(o,2r_\varepsilon)$, we have

$$\log u(x) \ge \log u(o) - C_1$$

for some positive constant C_1 and all $x \in B(o,r_\varepsilon)$. These show that

$$\log u(x) \ge \log u(o) + \frac{1}{2}\,(Z(p,n,\kappa,\Lambda) - \delta_\infty(M)Y(p,n,\kappa,\Lambda))\operatorname{dis}_M(o,x) - C_2$$

for some positive constant C_2 and all $x \in M$. This completes the proof of Theorem 1.3. ■

Proof of Corollary 1.4. Let $G^\Lambda(x,y)$ and $G^W(x,y)$ be respectively the Green functions of $Q'_{2,\Lambda}$ and $Q'_{2,W}$. Then by the assumptions, we can apply Theorem 2.6 of Ancona [2] to show that there is a constant $C_3 > 1$ such that

$$C_3{}^{-1}\,G^\Lambda(x,y) \le G^W(x,y) \le C_3\,G^\Lambda(x,y), \quad x,y \in M.$$

Let

$$K^\Lambda(x,y) = \frac{G^\Lambda(x,y)}{G^\Lambda(o,y)} \quad \text{and} \quad K^W(x,y) = \frac{G^W(x,y)}{G^W(o,y)}.$$

Let ξ be a point of the Martin boundary $\partial\mathcal{M}$ of the operator $Q'_{2,W}$ and $\{y_k\}$ a sequence of points of M which converges to ξ. By taking a subsequence if necessary, denoted by the same letters, $\{y_k\}$, we may assume that $K^\Lambda(x,y_k)$ converges, as $k \to \infty$, to a function $u_\xi(x)$ on M which is a positive solution of $Q'_{2,\Lambda}(u) = 0$. Then we have

$$C_3^{-2}\,u_\xi(x) \le K^W(x,\xi) \le C_3{}^2\,u_\xi(x)$$

for all $x \in M$. Since we have, by (1.6),

$$u_\xi(x) \leq u_\xi(y)\, e^{Y(2,n,\kappa,\Lambda)\, \mathrm{dis}_M(x,y)}$$

for all $y \in M$, we get

$$K^W(x,\xi)\,(\leq u_\xi(y)\, C_3^2\, e^{Y(2,n,\kappa,\Lambda)\, \mathrm{dis}_M(x,y)}) \leq K^W(y,\xi)\, C_3^4\, e^{Y(2,n,\kappa,\Lambda)\, \mathrm{dis}_M(x,y)}$$

for $\xi \in \partial\mathcal{M}$. Integrating both sides with respect to a Radon measure ν on the Martin boundary $\partial\mathcal{M}$ with $\int_{\partial\mathcal{M}} d\nu(\xi) = 1$, we obtain

$$\int_{\partial\mathcal{M}} K^W(x,\xi)\, d\nu(\xi) \leq \int_{\partial\mathcal{M}} K^W(y,\xi)\, d\nu(\xi)\, {C_3}^4\, e^{Y(2,n,\kappa,\Lambda)\, \mathrm{dis}_M(x,y)}.$$

Since a positive solution u of $Q'_{2,W}(u) = 0$ is represented by

$$u(x) = u(o) \int_{\partial\mathcal{M}} K^W(x,\xi)\, d\nu(\xi), \quad x \in M$$

for some Radon measure ν as above on the Martin boundary, we have

$$u(x) \leq u(y)\, {C_3}^4\, e^{Y(2,n,\kappa,\Lambda)\, \mathrm{dis}_M(x,y)}$$

for all $x, y \in M$.

Now we assume (1.6) ($p = 2$). Then it follows from the second assertion of Theorem 1.3 that

$$e^{Cr(x)-C'} \leq u_\xi(x) \leq {C_3}^2\, K^W(x,\xi)$$

for all $x \in M$, and hence we get

$$e^{Cr(x)-C'} \leq {C_3}^2 \int_{\partial\mathcal{M}} K^W(x,\xi)\, d\nu(\xi) = {C_3}^2\, u(x)$$

for all $x \in M$. This completes the proof of Corollary 1.4. ■

Proof of Theorem 5.2. Suppose that there exists a point y of M such that $h(y) = \sup_M h = (p-1)^2\, Y(p,n,\kappa,\Lambda)^2$. Then it follows from Lemma 5.4 that h is constant and equal to $(p-1)^2\, Y(p,n,\kappa,\Lambda)^2$. Let

$$B = \frac{v}{|\nabla v|} = \frac{v}{(p-1)\, Y(p,n,\kappa,\Lambda)}.$$

Then we can deduce from (5.4) and (5.30) that B is a smooth function on M satisfying $|\nabla B| = 1$ and

$$DdB = \sqrt{\kappa}(g - dB \otimes dB). \tag{5.34}$$

Put $L = B^{-1}(0)$ and let $\{\Omega_t\}$ be the complete flow generated by the gradient ∇B of B. We observe that Ω_t induces a diffeomorphism between L and $B^{-1}(t)$ by sending $z \in L$ to $\Omega_t(z) \in B^{-1}(t)$. Then it follows from (5.34) that

$$|d\Omega_t(\boldsymbol{v})| = e^{\sqrt{\kappa}t}\, |\boldsymbol{v}| \tag{5.35}$$

for all $t > 0$ and $\boldsymbol{v} \in T_z L$. We define a diffeomorphism $\Theta: \mathbb{R} \times L \to M$ by

$$\Theta(t, z) = \Omega_t(z).$$

Then (5.35) implies that

$$\Theta^* g_M = dt^2 + e^{2\sqrt{\kappa}t} g_L.$$

Therefore, (M, g_M) is the warped product of $\mathbb{R}$ and L with the warping function $e^{\sqrt{\kappa}t}$. This shows, in particular, that $\mathrm{Ric}_M(\nabla B, \nabla B) = -(m-1)\kappa$. Since $\mathrm{Ric}_f^n(\nabla B, \nabla B) = -(n-1)\kappa$ by (5.13) and $\langle \nabla f, \nabla B \rangle^2 = (n-m)^2 \kappa$ by (5.30), we get $Ddf(\nabla B, \nabla \beta) = 0$, which implies that $\frac{d^2}{dt^2} f(\Omega_t(z)) = 0$ for all $t \in \mathbb{R}$ and $z \in L$. Thus we have

$$f(t, z) = \langle \nabla f, \nabla B \rangle\, t + \eta(z) = -(n-m)\sqrt{\kappa}\, t + \eta(z),$$

where we set $\eta(z) = f(0, z)$. The $(n-1)$-dimensional Bakry–Émery Ricci tensor Ric_L^{n-1} of the weighted Riemannian manifold $(L, g_L, e^{-\eta} dv_L)$ with weight $e^{-\eta}$ satisfies

$$\mathrm{Ric}_L^{n-1} = \mathrm{Ric}_M^n + (2n - 3m + 1)\,\kappa\, e^{2\sqrt{\kappa}t} g_L \geq -3(n-m)\kappa\, e^{2\sqrt{\kappa}t} g_L$$

on $T_{(t,z)}(\{t\} \times L)$, where $T_z L$ is identified with $T_{(t,z)}(\{t\} \times L)$. Thus letting $t \to -\infty$, we get $\mathrm{Ric}_L^{n-1} \geq 0$ on L.

When there exists a point o of M such that $h(o) = \sup_M h = (p-1)^2 Z(p, n, \kappa, \Lambda)^2$, we let

$$B = -\frac{v}{|\nabla v|} = -\frac{v}{(p-1)\, Z(p, n, \kappa, \Lambda)}.$$

Then we use (5.32) and (5.33), and repeat the same argument as above to get the conclusion. This completes the proof of Theorem 5.2. ∎

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