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Real Kaehler submanifolds in codimension up to four

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Abstract. Let $f\colon M^{2n} \to \mathbb{R}^{2n+4}$ be an isometric immersion of a Kaehler manifold of complex dimension $n \geq 5$ into Euclidean space with complex rank at least 5 everywhere. Our main result is that, along each connected component of an open dense subset of M^{2n}, either f is holomorphic in $\mathbb{R}^{2n+4} \cong \mathbb{C}^{n+2}$, or it is in a unique way a composition $f = F \circ h$ of isometric immersions. In the latter case, we have that $h\colon M^{2n} \to N^{2n+2}$ is holomorphic and $F\colon N^{2n+2} \to \mathbb{R}^{2n+4}$ belongs to the class, by now quite well understood, of non-holomorphic Kaehler submanifolds in codimension two. Moreover, the submanifold F is minimal if and only if f is minimal.

1. Introduction

By a *real Kaehler submanifold* $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ we mean an isometric immersion of a connected Kaehler manifold (M^{2n}, J) of complex dimension $n \geq 2$ into Euclidean space with codimension p. Throughout this paper, it is assumed that f is locally substantial, which means that the image of f restricted to any open subset of M^{2n} does not lie inside a proper affine subspace of $\mathbb{R}^{2n+p}$. Moreover, if the codimension is even then f, when restricted to any open subset of M^{2n}, is not allowed to be holomorphic in $\mathbb{R}^{2n+2q} \cong \mathbb{C}^{n+q}$. Notice that conditions that yield that f is holomorphic have been given in [5], and that f is just minimal, in [2].

The study of real Kaehler submanifolds has acquired increasing relevance since the pioneering work by Dajczer and Gromoll in [8]. Clearly, the main motivation for their study is that when these submanifolds are minimal, they enjoy many of the feature properties of minimal surfaces. For instance, if simply-connected, they admit an associated one-parameter family of non-congruent isometric minimal submanifolds with the same Gauss map, and can be realized as the real part of its holomorphic representative. Moreover, they are pluriharmonic (sometimes called pluriminimal) submanifolds and, in particular, they are austere submanifolds. Furthermore, as seen in Appendix A, there are several cases when a classification is reached through a Weierstrass type representation. For a partial account of results and references on this subject of research, we refer to [13].

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It is well known that the second fundamental form $\alpha\colon TM \times TM \to N_f M$ of a real Kaehler submanifold $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ with codimension $p = 1$ or $p = 2$ has necessarily a large kernel. The complex dimension of that kernel is measured by way of the notion of (complex) rank.

The *rank* $\varrho_f(x)$ of f at $x \in M^{2n}$ is given by

$$2\varrho_f(x) = 2n - \dim \Delta(x) \cap J\Delta(x),$$

where

$$\Delta(x) = \{Y \in T_x M : \alpha(X, Y) = 0 \text{ for any } X \in T_x M\}$$

is known as the relative nullity subspace of f at $x \in M^{2n}$.

As recalled below, outside a flat point, the rank is $\varrho_f = 1$ if the codimension is $p = 1$, and that it is $\varrho_f \leq 2$ at any point if $p = 2$. The situation is quite different for submanifolds in higher codimension, in part due to the presence of compositions of isometric immersions. For instance, already for $p = 3$ we have that if $F\colon N^{2n+2} \to \mathbb{R}^{2n+3}$ is a real Kaehler hypersurface and $g\colon M^{2n} \to N^{2n+2}$ an holomorphic submanifold, then the composition of isometric immersions $f = F \circ g\colon M^{2n} \to \mathbb{R}^{2n+3}$ may have any rank since there is no bound under holomorphicity. Thus, in the search of local classifications of real Kaehler submanifolds in higher codimension than two, but still low, a necessary step is to provide conditions that impose the existence of a composition.

At this time there is substantial knowledge about the local real Kaehler submanifolds that are free of flat points and lie in codimension of at most four. On one hand, the ones in codimension one or two are quite well understood. On the other hand, for the higher codimensions three and four, it turns out that under a proper rank assumption, the submanifold has to be a composition as the one for $p = 3$ discussed above.

The non-flat real Kaehler hypersurfaces $f\colon M^{2n} \to \mathbb{R}^{2n+1}$, $n \geq 2$, have been locally classified by Dajczer and Gromoll [8] by way of the so called Gauss parametrization in terms of a pseudoholomorphic surface in the $2n$-dimensional round sphere and any smooth function on the surface. This was made possible because in this case the rank is $\varrho_f = 1$; see Theorem 15.14 in [13] for a more detailed proof of this classification. It turns out that the hypersurface is minimal if and only if the function on the surface is an eigenvector of the Laplacian for the eigenvalue 2. For this special case, there is a Weierstrass type parametrization given by Hennes [16].

For real Kaehler submanifolds $f\colon M^{2n} \to \mathbb{R}^{2n+2}$, $n \geq 3$, Dajczer [6] showed that we have $\dim \Delta(x) \geq 2n - 4$ at any $x \in M^{2n}$. For a discussion of the classification for the real Kaehler submanifolds that lie in codimension two in terms of its rank, see Appendix A.

We say that real Kaehler submanifold $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ admits a *Kaehler extension* if there exist a (maybe flat) real Kaehler submanifold $F\colon N^{2n+2\ell} \to \mathbb{R}^{2n+p}$, $\ell \geq 1$, and a holomorphic isometric embedding $j\colon M^{2n} \to N^{2n+2\ell}$, such that $f = F \circ j$.

For $f\colon M^{2n} \to \mathbb{R}^{2n+3}$, $n \geq 4$, it was stated by Dajczer and Gromoll [10] that if $\dim \Delta < 2n - 6$ everywhere, then there exists an open dense subset of M^{2n} such that, along each connected component, the submanifold admits a unique Kaehler extension to a real Kaehler hypersurface. Unfortunately, the existence part of the proof in [10] depends on an algebraic lemma that went unproven. But that result is correct for the specific non-symmetric bilinear form given by (1) in that paper, as follows from Lemma 3.6 or from

Lemma 1 in [19]. A nice observation due to Yan and Zheng [19] is that this result should hold under a weaker assumption on the rank of the submanifold. In this paper, we provide a proof of this.

The case of real Kaehler submanifolds $f: M^{2n} \to \mathbb{R}^{2n+4}$, $n \geq 5$, is treated by Yan and Zheng in [19] and [18]. It is proved in [19] that, if the rank of f satisfies $\varrho_f > 4$ everywhere, there exists an open dense subset of M^{2n} such that f restricted to each connected component admits a Kaehler extension $F: N^{2n+2} \to \mathbb{R}^{2n+4}$. In the present paper, by way of an alternative approach, we are able to complement in some directions the Main Theorem in [19]. In particular, we show that their result is correct in spite of a rather minor inaccuracy in the proof. In fact, their statement that the vanishing of a shape operator on one normal direction amounts to a reduction of the codimension of the submanifold does not hold. It is just elementarily false that in this case there is a reduction of the codimension and, of course, there is no statement in Spivak [17] that proves such a claim. Nevertheless, it turns out that even in this case their theorem is correct since, in this situation, the submanifold is holomorphic inside a flat submanifold as the one given in the non-minimal case (i)-(a) in Appendix A.

The main achievement of this paper is to prove that the Kaehler extension is *unique*, up to reparametrizations, in opposition to a assertion in [19]. Consequently, the submanifold f is minimal if and only if its extension F is also minimal. What makes our proof of uniqueness possible is that the way we construct the extensions is somehow more restricted than in [19]; see Remark 2.6.

Theorem 1.1. *Let $f: M^{2n} \to \mathbb{R}^{2n+p}$, $3 \leq p \leq 4$ and $n > p$, be a real Kaehler submanifold whose rank satisfies $\varrho_f > p$ everywhere. Then there exists an open dense subset of M^{2n} such that the restriction of f to each connected components admits a unique Kaehler extension $F: N^{2n+2} \to \mathbb{R}^{2n+p}$. Moreover, the rank of F is constant $\varrho_F \leq p-2$ and F is a minimal submanifold if and only if f is minimal.*

Under the rank assumption required in the above result, the manifold M^{2n} is free of points where all the sectional curvatures vanish. In fact, by a classical result going back to Cartan, at such a point we have $\dim \Delta \geq 2n - p$ and hence $\varrho_f \leq p$.

Theorem 1.1 is sharp even if the submanifold is asked to be isometrically complete. For instance, the extrinsic product immersion of two complete minimal ruled submanifolds lying in codimension two classified in [9] has rank four.

From Appendix A, it follows that the geometric options for the extension F, and hence for f, in Theorem 1.1 not to be minimal are quite limited.

We observe that Yan and Zheng in [19] made a very interesting and rather challenging conjecture: if the codimension is $p \leq 11$, then Kaehler extensions always exist for real Kaehler submanifolds $f: M^{2n} \to \mathbb{R}^{2n+p}$, $n > p$, if the rank satisfies $\varrho_f > p$ everywhere. In this respect, the results in the first section of this paper, that hold for any codimension, should be useful. Finally, in [3] it is shown why in the Yang and Zheng conjecture, limiting the codimension to 11 is essential. This was done proving that the structure of the second fundamental form until that codimension is the one expected, and this by an argument that fails beyond that codimension. As for the full conjecture, we intend to give an answer in a forthcoming paper that makes use of some results from the present one.

2. A class of Kaehler extensions

The goal of this section is to establish a set of conditions for a real Kaehler submanifold to admit a Kaehler extension of a certain type. The result achieved holds regardless of the size the codimension, and should be of use for further applications.

We first introduce some notations and definitions. Let $\gamma\colon V \times V \to W$ be a bilinear form between real vector spaces. The image of γ is the vector subspace of W given by

$$\mathcal{S}(\gamma) = \operatorname{span}\{\gamma(X,Y) \text{ for all } X, Y \in V\},$$

whereas the (right) nullity of γ is the vector subspace of V defined by

$$\mathcal{N}(\gamma) = \{Y \in V : \gamma(X,Y) = 0 \text{ for all } X \in V\}.$$

Henceforth $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ stands for a real Kaehler submanifold, and

$$\alpha : TM \times TM \to N_f M$$

its second fundamental form. If $P \subset N_f M(x)$ is a vector subspace and if α_P is the P-component of the second fundamental form α at $x \in M^{2n}$, then the *complex kernel* of α_P is the tangent vector subspace

$$\mathcal{N}_c(\alpha_P) = \mathcal{N}(\alpha_P) \cap J\mathcal{N}(\alpha_P).$$

Throughout this section, L denotes a normal vector subbundle of (real) rank $2\ell > 0$ that satisfies $L(x) \subset \mathcal{S}(\alpha(x))$ everywhere and is endowed with the induced metric and vector bundle connection. Moreover, it is assumed that L carries an isometric complex structure $\mathcal{J} \in \Gamma(\operatorname{Aut}(L))$, that is, a vector bundle isometry that satisfies $\mathcal{J}^2 = -I$. Furthermore, it is required the complex tangent vector subspaces $D(x) = \mathcal{N}_c(\alpha_{L^\perp}(x))$ to have constant (even) dimension and thus form a holomorphic tangent subbundle denoted by D.

In the sequel, the pair $(L, \mathcal{J})$ is required to satisfy the following conditions:

($\mathcal{C}_1$) The complex structure $\mathcal{J} \in \Gamma(\operatorname{Aut}(L))$ is parallel, that is,

$$(\nabla^\perp_X \mathcal{J}\eta)_L = \mathcal{J}(\nabla^\perp_X \eta)_L \quad \text{for any } X \in \mathfrak{X}(M) \text{ and } \eta \in \Gamma(L),$$

and the second fundamental form of f satisfies

$$\mathcal{J}\alpha_L(X,Y) = \alpha_L(X,JY) \quad \text{for any } X, Y \in \mathfrak{X}(M), \tag{2.1}$$

or, equivalently, the shape operators satisfy

$$A_{\mathcal{J}\eta} = J \circ A_\eta = -A_\eta \circ J \quad \text{for } \eta \in \Gamma(L).$$

($\mathcal{C}_2$) The subbundle L is parallel along D in the normal connection of f, that is,

$$\nabla^\perp_Y \eta \in \Gamma(L) \quad \text{for any } Y \in \Gamma(D) \text{ and } \eta \in \Gamma(L).$$

We observe that the subbundle L is necessarily proper in $N_1 = \mathcal{S}(\alpha)$ along any open subset $U \subset M^{2n}$ where the latter has constant rank since, otherwise we would have from the condition $(\mathcal{C}_2)$ that $N_1 = N_f M$, and then f would be holomorphic along U as established by Proposition 3.1 below.

Let the vector bundle $TM \oplus L$ over M^{2n} be endowed with the complex structure $\hat{\mathcal{J}} \in \Gamma(\mathrm{Aut}(TM \oplus L))$ defined by

$$\hat{\mathcal{J}}(X + \eta) = JX + \mathcal{J}\eta. \tag{2.2}$$

Condition $(\mathcal{C}_1)$ easily gives that $\hat{\mathcal{J}}$ is parallel in the induced vector bundle connection defined by

$$\hat{\nabla}_X(Y + \eta) = (\tilde{\nabla}_X(Y + \eta))_{TM\oplus L},$$

where $\tilde{\nabla}$ denotes the Euclidean connection in the ambient space $\mathbb{R}^{2n+p}$. That is, we have

$$(\tilde{\nabla}_X \hat{\mathcal{J}}(Y + \eta))_{TM\oplus L} = \hat{\mathcal{J}}((\tilde{\nabla}_X(Y + \eta))_{TM\oplus L}) \tag{2.3}$$

for any $X, Y \in \mathfrak{X}(M)$ and $\eta \in \Gamma(L)$.

Proposition 2.1. *The distribution $D = \mathcal{N}_c(\alpha_{L^\perp})$ is integrable.*

Proof. The Codazzi equation $(\nabla^\perp_X \alpha)(S, T) = (\nabla^\perp_S \alpha)(X, T)$ and the condition $(\mathcal{C}_2)$ give

$$(\nabla^\perp_X \alpha(S, T))_{L^\perp} + \alpha_{L^\perp}(X, \nabla_S T) = 0 \tag{2.4}$$

for any $X \in \mathfrak{X}(M)$ and $S, T \in \Gamma(D)$.

From (2.4) it follows that $[S, T] \in \mathcal{N}(\alpha_{L^\perp})$ for any $S, T \in \Gamma(D)$. On the other hand, from (2.1) we have $\alpha(S, JT) = \alpha(JS, T)$ for any $S, T \in \Gamma(D)$. This and (2.4) give

$$\alpha_{L^\perp}(X, J[S, T]) = \alpha_{L^\perp}(X, \nabla_S JT - \nabla_T JS) = (\nabla^\perp_X(\alpha(T, JS) - \alpha(S, JT)))_{L^\perp} = 0$$

for any $X \in \mathfrak{X}(M)$ and $S, T \in \Gamma(D)$. Thus also $J[S, T] \in \mathcal{N}(\alpha_{L^\perp})$ and hence $[S, T] \in D$ for any $S, T \in \Gamma(D)$. ∎

Proposition 2.1 gives that M^{2n} carries an holomorphic foliation. Let $i: \Sigma \to M^{2n}$ denote the inclusion of the leaf Σ through $x \in M^{2n}$, and let $g: \Sigma \to \mathbb{R}^{2n+p}$ be the isometric immersion $g = f \circ i$.

We have that

$$\tilde{\nabla}_S T = f_*(\nabla_S T)_D + f_*(\nabla_S T)_{D^\perp} + \alpha^f(S, T)$$

for any $S, T \in \Gamma(D)$. Hence

$$\alpha^g(y)(S, T) = f_*(i(y))(\nabla_{i_*S} i_*T)_{D^\perp} + \alpha^f(i(y))(i_*S, i_*T) \tag{2.5}$$

for any $S, T \in \mathfrak{X}(\Sigma)$.

We have that $g = f \circ i: \Sigma \to \mathbb{R}^{2n+p}$ satisfies $g(\Sigma) \subset f_*T_xM \oplus L(x)$. To see this, observe that the normal bundle of g splits orthogonally as $N_g\Sigma = i^*(f_*D^\perp \oplus L \oplus L^\perp)$, and that the condition $(\mathcal{C}_2)$ gives that the vector subbundle $i^*L^\perp$ of $N_g\Sigma$ is constant in $\mathbb{R}^{2n+p}$. Thus, we may also see $g(\Sigma)$ as a submanifold of $\mathbb{R}^{2n+2\ell} = f_*T_xM \oplus L(x) \subset \mathbb{R}^{2n+p}$ when it is convenient.

Proposition 2.2. *The submanifolds $g\colon \Sigma \to \mathbb{R}^{2n+2\ell}$ are holomorphic. Moreover, for any given g, the map $\psi\colon \mathcal{S}(\alpha^g) \to \mathcal{S}(\alpha^f|_{D\times D})$ defined by*

$$\psi(\alpha^g(S,T)) = \alpha^f(i_*S, i_*T)$$

is an isomorphism.

Proof. Let $\bar{\mathcal{J}}$ be the complex structure on $\mathbb{R}^{2n+2\ell}$ induced by $\hat{\mathcal{J}}$. Then

$$\bar{\mathcal{J}} g_*T = \hat{\mathcal{J}} f_* i_* T = f_* J i_* T = f_* i_* J|_{T\Sigma} T = g_* J|_{T\Sigma} T$$

for any $T \in \mathfrak{X}(\Sigma)$, and hence g is holomorphic.

From (2.5), the map ψ is surjective. To prove injectivity, let $\delta = \sum_{j=1}^k \alpha^g(S_j, T_j)$ for $S_j, T_j \in \mathfrak{X}(\Sigma)$ satisfy

$$\psi(\delta) = \sum_{j=1}^k \alpha^f(i_*S_j, i_*T_j) = 0.$$

From (2.1) we have

$$\sum_{j=1}^k \alpha^f(i_*S_j, i_*JT_j) = \mathcal{J} \sum_{j=1}^k \alpha^f(i_*S_j, i_*T_j) = 0.$$

We obtain from (2.4) that

$$\sum_{j=1}^k \alpha_{L^\perp}(X, \nabla_{S_j} i_*T_j) = -\sum_{j=1}^k (\nabla_X^\perp \alpha^f(i_*S_j, i_*T_j))_{L^\perp} = 0,$$

and, similarly, that

$$\sum_{j=1}^k \alpha_{L^\perp}(X, J\nabla_{S_j} i_*T_j) = 0.$$

Hence

$$\sum_{j=1}^k \nabla_{S_j} i_*T_j \in \Gamma(D),$$

and we conclude from (2.5) that $\delta = 0$. ■

In the sequel, the pair $(L, \mathcal{J})$ is assumed to satisfy the additional condition:

($\mathcal{C}_3$) $L = \mathcal{S}(\alpha|_{D\times D})$ at any point of M^{2n}.

Lemma 2.3. *Let $\pi\colon \Lambda \to M^{2n}$ be the $\hat{\mathcal{J}}$-invariant vector subbundle of $TM \oplus L$ defined by*

(2.6) $$\Lambda = \operatorname{span}\{(\nabla_S T)_{D^\perp} + \alpha^f(S,T)\colon S, T \in \Gamma(D)\}.$$

Then rank $\Lambda = 2\ell$ *and* $\Lambda \cap TM = 0$. *Moreover, we have that*

(2.7) $$\tilde{\nabla}_X \lambda \in f_*TM \oplus L$$

for any $\lambda \in \Gamma(\Lambda)$ and $X \in \mathfrak{X}(M)$.

Proof. From Proposition 2.2, it follows that rank $\mathcal{S}(\alpha^g) = \text{rank}\, i^*L$. If Σ is the leaf of D that contains $x \in M^{2n}$, we have from (2.5) that $\Lambda(i(x)) = \mathcal{S}(\alpha^g)(x)$. Thus Λ has constant dimension at each $x \in M^{2n}$ and hence is a subbundle. If $\lambda \in \Lambda \cap TM$, it follows from Proposition 2.2 and (2.5) that $\lambda = 0$ and hence $\Lambda \cap TM = 0$.

We obtain from (2.4) that

$$(\tilde{\nabla}_X(f_*(\nabla_S T)_{D^\perp} + \alpha^f(S,T)))_{L^\perp} = 0$$

for any $S, T \in \Gamma(D)$, as we wished. ∎

Lemma 2.4. *A real Kaehler submanifold $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ is minimal if and only if it is pluriharmonic, that is, if*

$$\alpha(JX, Y) = \alpha(X, JY) \quad \textit{for any } X, Y \in \mathfrak{X}(M) \tag{2.8}$$

or, equivalently, if the shape operators satisfy $J \circ A_\xi = -A_\xi \circ J$ for any $\xi \in N_f M$.

Proof. This is Theorem 1.2 in [11] or Theorem 15.7 in [13]. ∎

Theorem 2.5. *Let $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ be an embedding and let $\pi\colon \Lambda^{2\ell} \to M^{2n}$ be the vector subbundle of $TM \oplus L^{2\ell}$ defined by (2.6). Let $N^{2n+2\ell}$ be an open neighborhood of the 0-section $j\colon M^{2n} \to N^{2n+2\ell}$ of $\Lambda^{2\ell}$ such that the map $F\colon N^{2n+2\ell} \to \mathbb{R}^{2n+p}$ defined by*

$$F(\lambda) = f(\pi(\lambda)) + \lambda$$

is an embedding. Then F is a Kaehler extension of $f = F \circ j$, and its second fundamental form satisfies $\mathcal{N}_c(\alpha^F) = D \oplus \Lambda$ at any point and thus, in particular, its rank is $2\varrho_F = 2n - \text{rank}\, D$. Moreover, the submanifold F is minimal if and only if f is minimal.

Proof. If $\lambda_0 \in N^{2n+2\ell}$ and $x_0 = \pi(\lambda_0)$, let $\xi \in \Gamma(\Lambda)$ be such that $\xi(x_0) = \lambda_0$. Then

$$\tilde{\nabla}_X F(\lambda_0) = f_*(x_0)X + \tilde{\nabla}_X \xi(x_0) \quad \text{if } X \in T_{x_0}M.$$

Thus $T_\lambda N = T_{\pi(\lambda)}M \oplus L(\pi(\lambda))$ and $N_F N(\lambda) = L^\perp(\pi(\lambda))$ for any $\lambda \in N^{2n+2\ell}$.

We first show that F is a real Kaehler submanifold. Let $J^N \in \Gamma(\text{Aut}(TN))$ be the complex structure defined by $J^N(\lambda) = \hat{\mathcal{J}}(\pi(\lambda))$, where $\hat{\mathcal{J}}$ is given by (2.2). Thus J^N is constant along the fibers of Λ. Then J^N is parallel with respect to the Levi-Civita connection $\hat{\nabla}$ on $N^{2n+2\ell}$, since by (2.3) we have that

$$\hat{\nabla}_X J^N(Y + \xi) = (\tilde{\nabla}_X \hat{\mathcal{J}}(Y + \xi))_{TM \oplus L} = \hat{\mathcal{J}}(\tilde{\nabla}_X(Y + \xi))_{TM \oplus L} = J^N \hat{\nabla}_X(Y + \xi)$$

for any $X, Y \in \mathfrak{X}(M)$ and $\xi \in \Gamma(L)$.

We show next that $\mathcal{N}_c(\alpha^F) = D \oplus \Lambda$. The condition $(\mathcal{C}_2)$ gives

$$\tilde{\nabla}_S \eta = -A^f_\eta S + \nabla^\perp_S \eta = \nabla^\perp_S \eta \in \Gamma(L^\perp)$$

for any $S \in \Gamma(D)$ and $\eta \in \Gamma(L^\perp)$. We obtain $D \oplus \Lambda \subset \mathcal{N}(\alpha^F)$. Since D is J-invariant and Λ is $\hat{\mathcal{J}}$-invariant, then $D \oplus \Lambda$ is J^N-invariant. Thus $D \oplus \Lambda \subset \mathcal{N}_c(\alpha^F)$. For the other inclusion, we have to verify that if $Z \in \mathfrak{X}(M) \cap \mathcal{N}_c(\alpha^F)$, then $Z \in \Gamma(D)$. Since

$${}^F\nabla^\perp_Z \eta = \tilde{\nabla}_Z \eta = -A^f_\eta Z + {}^f\nabla^\perp_Z \eta \quad \text{and} \quad {}^F\nabla^\perp_{JZ} \eta = \tilde{\nabla}_{JZ} \eta = -A^f_\eta JZ + {}^f\nabla^\perp_{JZ} \eta$$

for any $\eta \in \Gamma(L^\perp)$, then $A^f_\eta Z = A^f_\eta JZ = 0$.

Assume that f is minimal. We first prove the following fact:

$$\mathcal{J}(\nabla^\perp_X \eta)_L + (\nabla^\perp_{JX}\eta)_L = 0 \quad \text{for any } X \in \mathfrak{X}(M) \text{ and } \eta \in \Gamma(L^\perp). \tag{2.9}$$

Let $\lambda \in \Gamma(\Lambda)$ be such that $(\lambda)_L = \mathcal{J}(\nabla^\perp_X \eta)_L + (\nabla^\perp_{JX}\eta)_L$. Using first that (2.8) holds and then (2.7) at the end of the argument, we obtain

$$\begin{aligned}\|(\lambda)_L\|^2 &= \langle \lambda, \mathcal{J}(\nabla^\perp_X \eta)_L + (\nabla^\perp_{JX}\eta)_L\rangle = \langle \lambda, -JA_\eta X + \mathcal{J}(\nabla^\perp_X \eta)_L - A_\eta JX + (\nabla^\perp_{JX}\eta)_L\rangle \\ &= \langle \lambda, \hat{\mathcal{J}}(\tilde\nabla_X \eta)_{TM\oplus L} + (\tilde\nabla_{JX}\eta)_{TM\oplus L}\rangle = \langle \tilde\nabla_X \hat{\mathcal{J}}\lambda, \eta\rangle - \langle \tilde\nabla_{JX}\lambda, \eta\rangle = 0.\end{aligned}$$

Since $\tilde\nabla_X \eta = -A^f_\eta X + \nabla^\perp_X \eta$, then $A^F_\eta X = A^f_\eta X - (\nabla^\perp_X \eta)_L$. Using (2.9), we have

$$\begin{aligned}A^F_\eta J^N X &= A^f_\eta JX - (\nabla^\perp_{JX}\eta)_L = -JA^f_\eta X + \mathcal{J}(\nabla^\perp_X \eta)_L \\ &= -\hat{\mathcal{J}}(A^f_\eta X - (\nabla^\perp_X \eta)_L) = -J^N A^F_\eta X\end{aligned}$$

for any $\eta \in \Gamma(L^\perp)$ and $X \in \mathfrak{X}(M)$. Since we have seen that $\mathcal{N}_c(\alpha^F) = D \oplus \Lambda$, then $\alpha^F(\delta, \xi) = \alpha^F(\delta, J^N \xi) = 0$ holds for any $\delta \in TN$ and $\xi \in \Lambda$, and hence F is a minimal immersion.

Assume that F is minimal. Since J is holomorphic, then $\alpha^j(JS, T) = \alpha^j(S, JT)$ for any $S, T \in \mathfrak{X}(\Sigma)$. Now since $f = F \circ j$, then

$$\begin{aligned}\alpha^f(S, JT) &= F_*\alpha^j(S, JT) + \alpha^F(j_*S, j_*JT) = F_*\alpha^j(JS, T) + \alpha^F(j_*S, J^N j_*T) \\ &= F_*\alpha^j(JS, T) + \alpha^F(J^N j_*S, j_*T) = F_*\alpha^j(JS, T) + \alpha^F(j_*JS, j_*T) \\ &= \alpha^f(JS, T),\end{aligned}$$

and hence f is a minimal submanifold. ∎

Remark 2.6. In Proposition 2 in [19], the extension is obtained by means of a developable ruling L, whereas here Theorem 2.5 uses the subbundle Λ given by (2.6). This is more restricted since Λ is a special case of a canonical developable ruling as defined in [19]. In fact, our Λ is J-invariant, and that may not be the case of L.

3. The proof of Theorem 1.1

The proof of Theorem 1.1 requires several results. The first one holds for any codimension and is of independent interest.

Proposition 3.1. *Let $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ be an isometric immersion of a Kaehler manifold. Assume that $N_1(x) = \mathcal{S}(\alpha(x))$ satisfies $N_1(x) = N_f M(x)$ at any $x \in M^{2n}$, and that there is an isometry $\mathcal{J} \in \Gamma(\mathrm{Aut}(N_f M))$ such that*

$$\mathcal{J}\alpha(X, Y) = \alpha(X, JY) \quad \textit{for any } X, Y \in \mathfrak{X}(M). \tag{3.1}$$

Then p is even and f is holomorphic.

Proof. From (3.1), we have that the $\mathcal{J}$ is a complex structure which we claim to be parallel in the normal connection. If we apply $\mathcal{J}$ to the Codazzi equation $(\nabla^{\perp}_X \alpha)(Y,Z) = (\nabla^{\perp}_Y \alpha)(X,Z)$, subtract $(\nabla^{\perp}_X \alpha)(Y,JZ) = (\nabla^{\perp}_Y \alpha)(X,JZ)$, and then use (3.1), we obtain

$$\mathcal{K}(X)\alpha(Y,Z) = \mathcal{K}(Y)\alpha(X,Z), \tag{3.2}$$

where $\mathcal{K}(X) \in \Gamma(\mathrm{End}(N_f M))$ is the skew-symmetric tensor defined by

$$\mathcal{K}(X)\eta = \mathcal{J}\nabla^{\perp}_X \eta - \nabla^{\perp}_X \mathcal{J}\eta.$$

If we denote

$$\langle \mathcal{K}(X)\alpha(Y,Z), \alpha(S,T)\rangle = (X,Y,Z,S,T),$$

then, by (3.2) and since $\mathcal{K}(X)$ is skew-symmetric, we obtain

$$\begin{aligned}(X,Y,Z,S,T) &= -(X,S,T,Y,Z) = -(S,X,T,Y,Z) = (S,Y,Z,X,T)\\ &= (Y,S,Z,X,T) = -(Y,X,T,S,Z) = -(T,X,Y,S,Z)\\ &= (T,S,Z,X,Y) = (Z,S,T,X,Y) = -(Z,X,Y,S,T)\\ &= -(X,Y,Z,S,T) = 0\end{aligned}$$

for any $X,Y,Z,S,T \in \mathfrak{X}(M)$. Because $N_1 = N_f M$ everywhere, then $\mathcal{K}(X) = 0$ for any $X \in \mathfrak{X}(M)$, and the claim has been proved.

It follows from the claim that $\mathcal{J}$ is constant along the submanifold and hence $J \oplus \mathcal{J}$ extends to a complex structure on $\mathbb{R}^{2n+p}$, still denoted by $\mathcal{J}$, such that $\mathcal{J} \circ f_* = f_* \circ \mathcal{J}$, and therefore f is holomorphic. ■

Let $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ be a real Kaehler submanifold and let $N_f M(x) \oplus N_f M(x)$ be endowed with the inner product of signature (p,p) given by

$$\langle\langle (\xi_1,\xi_2),(\eta_1,\eta_2)\rangle\rangle = \langle \xi_1,\eta_1\rangle - \langle \xi_2,\eta_2\rangle.$$

We call a bilinear form $\varphi\colon T_xM \times T_xM \to N_f M(x) \oplus N_f M(x)$ *flat* if

$$\langle\langle \varphi(X,Y),\varphi(Z,T)\rangle\rangle - \langle\langle \varphi(X,T),\varphi(Z,Y)\rangle\rangle = 0$$

for any $X,Y,Z,T \in T_xM$.

Lemma 3.2. *The bilinear form $\gamma\colon T_xM \times T_xM \to N_1(x) \oplus N_1(x)$ defined by*

$$\gamma(X,Y) = (\alpha(X,Y), \alpha(X,JY)) \tag{3.3}$$

is flat.

Proof. It is well known that the curvature tensor of a Kaehler manifold M^{2n} satisfies $R(X,Y) = R(JX,JY)$ and $R(X,Y)JZ = JR(X,Y)Z$ for any $X,Y,Z \in T_xM$. Then a roughly short straightforward computation making use of this as well as the Gauss equation of f gives the flatness. ■

A vector subspace $V \subset N_f M(x) \oplus N_f M(x)$ is called a degenerate space if $V \cap V^{\perp} \neq 0$, and nondegenerate otherwise.

Lemma 3.3. *Let $f\colon M^{2n}\to\mathbb{R}^{2n+p}$, $p<n$, be an isometric immersion of a Kaehler manifold and let $P\subset N_1(x)$ be a vector subspace of dimension* $\dim P\leq 5$. *Assume that the bilinear form $\gamma_P\colon T_xM\times T_xM\to P\oplus P$ defined by*

$$\gamma_P(X,Y)=(\alpha_P(X,Y),\alpha_P(X,JY))$$

is flat and that the vector space $\mathcal{S}(\gamma_P)$ is nondegenerate. Then

$$\dim\mathcal{N}_c(\alpha_P(x))\geq 2n-2\dim\,.$$

In particular, if $p\leq 5$ and the flat bilinear form γ in (3.3) satisfies that $\mathcal{S}(\gamma)$ is a nondegenerate vector space, then $\varrho_f(x)\leq p$.

Proof. The proof follows from Proposition 5 in [4]. ∎

Lemma 3.4. *Let $f\colon M^{2n}\to\mathbb{R}^{2n+p}$, $p<n$ and $2\leq p\leq 5$, be an isometric immersion of a Kaehler manifold with rank $\varrho_f>p$ everywhere.*

(i) *At any $x\in M^{2n}$, there are a subspace $L(x)\subset N_1(x)$ of* $\dim L(x)=2\ell(x)>0$ *and an isometry $\mathcal{J}(x)\in\mathrm{Aut}(L(x))$ such that*

$$\mathcal{J}(x)\,\alpha_{L(x)}(X,Y)=\alpha_{L(x)}(X,JY)\quad\text{for any } X,Y\in T_xM, \tag{3.4}$$

the subspace $\mathcal{S}(\gamma_{L^\perp(x)})$ is nondegenerate, and $\dim\mathcal{N}_c(\alpha_{L(x)^\perp})\geq 2n-2p+4\ell(x)$.

(ii) *On each connected component of an open dense subset $U\subset M^{2n}$, we have that $\ell(x)=\ell$ and* $\dim\mathcal{N}_c(\alpha_{L(x)^\perp})$ *are constant, the subspaces $L(x)$ form a normal vector subbundle $L\subset N_1$ of* rank $L=2\ell$, *there is an isometry $\mathcal{J}\in\Gamma(\mathrm{Aut}(L))$ that satisfies (2.1), and there is a tangent subbundle $\mathcal{N}_c(\alpha_{L^\perp})$ such that* rank $\mathcal{N}_c(\alpha_{L^\perp})\geq 2n-2p+4\ell$.

Proof. By Lemma 3.3, the bilinear form γ given by (3.3) satisfies that $\mathcal{S}(\gamma)$ is a degenerate subspace at any $x\in M^{2n}$, that is, we have that $\mathcal{U}(x)=\mathcal{S}(\gamma)\cap(\mathcal{S}(\gamma))^\perp\neq 0$. If $(\xi,\eta)\in\mathcal{U}(x)$, then also $(\eta,-\xi)\in\mathcal{U}(x)$, since

$$\langle\langle\gamma(X,Y),(\eta,-\xi)\rangle\rangle=\langle\langle\gamma(X,JY),(\xi,\eta)\rangle\rangle$$

for any $X,Y\in T_xM$. Then $\dim\mathcal{U}(x)=2\ell(x)>0$.

We have that $\pi_1(\mathcal{U}(x))=\pi_2(\mathcal{U}(x))$, where $\pi_j\colon N_1(x)\oplus N_1(x)\to N_1(x)$, $j=1,2$, is the projection onto the j-th component. Since $\pi_j|_{\mathcal{U}(x)}$, $j=1,2$, is injective, then $L(x)=\pi_j(\mathcal{U}(x))$ satisfies $\dim L(x)=2\ell(x)$. We have $\mathcal{U}(X)\subset\mathcal{S}(\gamma_{L(x)})\subset L(x)\oplus L(x)$ and thus, by dimension reasons, we obtain $\mathcal{U}(X)=\mathcal{S}(\gamma_{L(x)})$. Then

$$\langle\alpha_{L(x)}(X,Y),\alpha_{L(x)}(Z,T)\rangle=\langle\alpha_{L(x)}(X,JY),\alpha_{L(x)}(Z,JT)\rangle \tag{3.5}$$

for any $X,Y,Z,T\in T_xM$. Thus, there is an isometry $\mathcal{J}(x)\in\mathrm{Aut}(L(x))$ such that (3.4) holds. We have that γ is flat, and (3.5) just says that also

$$\gamma_{L(x)}(X,Y)=(\alpha_{L(x)}(X,Y),\alpha_{L(x)}(X,JY))$$

is flat. Since $\gamma(x) = \gamma_{L(x)} + \gamma_{L^\perp(x)}$, then also $\gamma_{L^\perp(x)}$ is flat. Having that the subspace $\mathcal{S}(\gamma_{L^\perp(x)})$ is nondegenerate, then Lemma 3.3 gives

$$\dim \mathcal{N}_c(\alpha_{L^\perp(x)}) \geq 2n - 2p + 4\ell(x).$$

Let $U \subset M^{2n}$ be an open sense subset such that $\ell(x)$ and $\dim \mathcal{N}_c(\alpha_{L^\perp(x)})$ are constant on each connected component. Along any component, the subspaces $\mathcal{U}(x)$ form a vector bundle and thus also the subspaces $L(x)$ do, since they possess equal dimension. Finally, that $\mathcal{J} \in \Gamma(\mathrm{Aut}(L))$ is smooth follows from (3.4). ∎

For codimension $p = 2$, the following result generalizes the one in [6]. A proof should also follow from the arguments in [19].

Theorem 3.5. *Let $f: M^{2n} \to \mathbb{R}^{2n+2}$, $n \geq 3$, be an isometric immersion of a Kaehler manifold. If the rank is $\varrho_f > 2$ everywhere, then f is an holomorphic submanifold.*

Proof. Lemma 3.4 gives that $N_1 = N_f M$ and an isometry $\mathcal{J} \in \Gamma(\mathrm{Aut}(N_f M))$ satisfying

$$\mathcal{J}\alpha(X, Y) = \alpha(X, JY)$$

for any $X, Y \in \mathfrak{X}(M)$. Then Proposition 3.1 yields that f is holomorphic. ∎

Lemma 3.6. *Let $f: M^{2n} \to \mathbb{R}^{2n+p}$, $3 \leq p \leq 4$ and $n > p$, be a real Kaehler submanifold with rank $\varrho_f > p$ everywhere. Then along each connected component, say U_0, of an open dense subset of M^{2n}, there exists a pair $(L, \mathcal{J})$ such that $L \subset N_1|_{U_0}$ is a vector subbundle of rank 2 and $\mathcal{J} \in \Gamma(\mathrm{Aut}(L))$ an isometric complex structure that satisfies (2.1).*

Moreover, the subspaces $D(x) = \mathcal{N}_c(\alpha_{L^\perp}(x))$ form a tangent vector subbundle such that rank $D \geq 2n - 2p + 4$. *Furthermore, if $(L', \mathcal{J}')$ is a pair along U_0 such that $L' \subset N_1$ is a vector subbundle and the isometry $\mathcal{J}' \in \Gamma(\mathrm{Aut}(L'))$ satisfies (2.1), then $(L, \mathcal{J}) = (L', \mathcal{J}')$.*

Proof. By Lemma 3.4, on each connected component of an open dense subset U of M^{2n} there are a normal vector subbundle $L \subset N_1$ and an isometry $\mathcal{J} \in \Gamma(\mathrm{Aut}(L))$ such that rank $L = 2$ if $p = 3$, and either rank $L = 2$ or rank $L = 4$ if $p = 4$. Moreover, $D = \mathcal{N}_c(\alpha_{L^\perp})$ satisfies $\dim D \geq 2n - 2p + 2\,\mathrm{rank}\, L$. If rank $L = 4$, then $p = 4$ and $L = N_1 = N_f M$. Hence (2.1) holds for $L = N_f M$, and Proposition 3.1 yields that f is holomorphic, which has been excluded. Therefore, rank $L = 2$ on each connected component of U, and rank $D \geq 2n - 2p + 4$.

We prove the uniqueness part of the statement. From Proposition 3.1, we obtain that rank $L' = 2$. By assumption, we have that $\gamma_{L'}(X, Y) = (\alpha_{L'}(X, Y), \alpha_{L'}(X, JY))$ satisfies

$$\langle\langle \gamma_{L'}(X, Y), \gamma_{L'}(Z, T) \rangle\rangle = 0$$

for any $X, Y, Z, T \in \mathfrak{X}(M)$, and since $\gamma = \gamma_{L'} + \gamma_{L'^\perp}$, thus $\gamma_{L'^\perp}$ is flat. We claim that $D' = \mathcal{N}_c(\alpha_{L'^\perp})$ satisfies $\dim D' \geq 2n - 2p + 4$. If $\mathcal{S}(\gamma_{L'^\perp})$ is nondegenerate, the claim follows from Lemma 3.3. Thus, it suffices to show that

$$\mathcal{U}' = \mathcal{S}(\gamma_{L'^\perp}) \cap (\mathcal{S}(\gamma_{L'^\perp}))^\perp \neq 0$$

leads to a contradiction. If $(\xi, \bar{\xi}) \in \mathcal{U}'$, then also $(\bar{\xi}, -\xi) \in \mathcal{U}'$, since

$$\langle\!\langle \gamma_{L'^\perp}(X, Y), (\bar{\xi}, -\xi) \rangle\!\rangle = \langle\!\langle \gamma_{L'^\perp}(X, JY), (\xi, \bar{\xi}) \rangle\!\rangle = 0.$$

Hence $\dim \mathcal{U}' = 2$ and $p = 4$. Since $\mathcal{U}' \subset \mathcal{S}(\gamma_{L'^\perp}) \subset (L'^\perp \cap N_1) \oplus (L'^\perp \cap N_1)$, then

$$\mathcal{U}' = \mathcal{S}(\gamma_{L'^\perp}) \tag{3.6}$$

and thus

$$\langle\!\langle \gamma_{L'^\perp}(X, Y), \gamma_{L'^\perp}(S, JT) \rangle\!\rangle = 0.$$

Then there is a complex structure of the form $(\mathcal{J}' \oplus \bar{\mathcal{J}}) \in \Gamma(\mathrm{Aut}(N_f M))$ such that

$$(\mathcal{J}' \oplus \bar{\mathcal{J}})\alpha(X, Y) = \alpha(X, JY).$$

Being $\pi_j|_{\mathcal{U}'}$ injective, then $\pi_j(\mathcal{U}') = N_1 \cap L'^\perp = L'^\perp$, and hence $L'^\perp \subset N_1$ by (3.6). Therefore $N_1 = N_f M$ and thus f is holomorphic by Proposition 3.1, which is not allowed and proves the claim.

Suppose that we have $L \neq L'$. If $p = 3$, then $\dim L^\perp \oplus L'^\perp = 2$. Since we have $\dim D, \dim D' \geq 2n - 2$ then $\dim \mathcal{N}_c(\alpha_{L^\perp \oplus L'^\perp}) \geq \dim D \cap D' \geq 2n - 4$. We have that $L \cap (L^\perp \oplus L'^\perp) \neq 0$ and thus (2.1) gives that $\varrho_f \leq 2$, a contradiction. Hence $p = 4$.

If $L^\perp + L'^\perp = N_f M$, we have that $D \cap D' \subset \mathcal{N}_c(\alpha)$, which is not possible since it yields $\varrho_f \leq 4$. Hence $\dim(L^\perp + L'^\perp) = 3$ and then $L \cap (L^\perp + L'^\perp) \neq 0$. Since (2.1) is equivalent to $A_{\mathcal{J}\eta} = J \circ A_\eta$ for any $\eta \in \Gamma(L)$, it follows that $D \cap D' \subset \mathcal{N}_c(\alpha_L)$. Hence if $S \in \Gamma(D \cap D')$, then

$$\alpha(X, S) = \alpha_L(X, S) + \alpha_{L^\perp}(X, S) = 0,$$

where the first term on the right-hand side vanishes since $S \in \Gamma(D \cap D')$, and the second since $S \in \Gamma(D)$. Thus we obtain again that $D \cap D' \subset \mathcal{N}_c(\alpha)$. ■

Lemma 3.7. *Let $f : M^{2n} \to \mathbb{R}^{2n+p}$, $3 \leq p \leq 4$ and $n > p$, be a real Kaehler submanifold with rank $\varrho_f > p$ everywhere. Then any pair $(L, \mathcal{J})$ given by Lemma 3.6 satisfies the conditions $(\mathcal{C}_1)$, $(\mathcal{C}_2)$ and $(\mathcal{C}_3)$.*

Proof. The condition $(\mathcal{C}_1)$ is trivially satisfied.

For the proof of the condition $(\mathcal{C}_2)$, we first show that the distribution D is integrable. Given $\delta \in \Gamma(L)$, from the Codazzi equation $(\nabla_Z A)(\xi; Y) = (\nabla_Y A)(\xi; Z)$ for $\xi = \mathcal{J}\delta$ and since $A_{\mathcal{J}\delta} = J \circ A_\delta$, we have

$$J(\nabla_Z A_\delta Y - A_\delta \nabla_Z Y) - A_{\nabla^\perp_Z \mathcal{J}\delta} Y = J(\nabla_Y A_\delta Z - A_\delta \nabla_Y Z) - A_{\nabla^\perp_Y \mathcal{J}\delta} Z$$

for any $Y, Z \in \mathfrak{X}(M)$. From the above Codazzi equation for $\xi = \delta$ and the condition $(\mathcal{C}_1)$, we obtain

$$JA_{(\nabla^\perp_Z \delta)_{L^\perp}} Y - A_{(\nabla^\perp_Z \mathcal{J}\delta)_{L^\perp}} Y = JA_{(\nabla^\perp_Y \delta)_{L^\perp}} Z - A_{(\nabla^\perp_Y \mathcal{J}\delta)_{L^\perp}} Z$$

for any $Y, Z \in \mathfrak{X}(M)$. Hence

$$\langle \alpha(Y, X), (\nabla^\perp_Z \delta)_{L^\perp} \rangle - \langle \alpha(Y, JX), (\nabla^\perp_Z \mathcal{J}\delta)_{L^\perp} \rangle = 0 \tag{3.7}$$

for any $\delta \in \Gamma(L)$, $Z \in \Gamma(D)$ and $X, Y \in \mathfrak{X}(M)$.

On one hand, the Codazzi equation $(\nabla^\perp_{Z_1}\alpha)(Z,Z_2)=(\nabla^\perp_{Z_2}\alpha)(Z,Z_1)$ gives

$$\alpha_{L^\perp}(Z,[Z_1,Z_2])=(\nabla^\perp_{Z_1}\alpha(Z,Z_2)-\nabla^\perp_{Z_2}\alpha(Z,Z_1))_{L^\perp} \tag{3.8}$$

for any $Z_1,Z_2\in\Gamma(D)$ and $Z\in\mathfrak{X}(M)$. On the other hand, from the Codazzi equation $(\nabla^\perp_Z\alpha)(Z_i,JZ_j)=(\nabla^\perp_{Z_i}\alpha)(Z,JZ_j)$, we obtain

$$(\nabla^\perp_Z\alpha(Z_i,JZ_j))_{L^\perp}=(\nabla^\perp_{Z_i}\alpha(Z,JZ_j))_{L^\perp}-\alpha_{L^\perp}(Z,\nabla_{Z_i}JZ_j)$$

for any $Z_i,Z_j\in\Gamma(D)$ and $Z\in\mathfrak{X}(M)$. Since $\alpha(Z_1,JZ_2)=\alpha(JZ_1,Z_2)$ by (2.1) then

$$\alpha_{L^\perp}(Z,J[Z_1,Z_2])=(\nabla^\perp_{Z_1}\alpha(Z,JZ_2)-\nabla^\perp_{Z_2}\alpha(Z,JZ_1))_{L^\perp} \tag{3.9}$$

for any $Z_1,Z_2\in\Gamma(D)$ and $Z\in\mathfrak{X}(M)$.

Using first (3.8) and (3.9), and then (3.7) for $\delta=\alpha(Z,Z_i)$, we obtain

$$\begin{aligned}\langle\alpha_{L^\perp}(Y,X),\alpha_{L^\perp}(Z,[Z_1,Z_2])\rangle-\langle\alpha_{L^\perp}(Y,JX),\alpha_{L^\perp}(Z,J[Z_1,Z_2])\rangle&\\ =\langle\alpha_{L^\perp}(Y,X),\nabla^\perp_{Z_1}\alpha(Z,Z_2)\rangle-\langle\alpha_{L^\perp}(Y,X),\nabla^\perp_{Z_2}\alpha(Z,Z_1)\rangle&\\ -\langle\alpha_{L^\perp}(Y,JX),\nabla^\perp_{Z_1}\alpha(Z,JZ_2)\rangle+\langle\alpha_{L^\perp}(Y,JX),\nabla^\perp_{Z_2}\alpha(Z,JZ_1)\rangle&\\ =0&\end{aligned}$$

for any $Z_1,Z_2\in\Gamma(D)$ and $X,Y,Z\in\mathfrak{X}(M)$. Thus we have shown that

$$\langle\!\langle\gamma_{L^\perp}(Y,X),\gamma_{L^\perp}(Z,[Z_1,Z_2])\rangle\!\rangle=0$$

for any $Z_1,Z_2\in\Gamma(D)$ and $X,Y,Z\in\mathfrak{X}(M)$. Since the subspace $\mathcal{S}(\gamma_{L^\perp})$ is nondegenerate by part (i) of Lemma 3.4, we have $\gamma_{L^\perp}(Z,[Z_1,Z_2])=0$ for any $Z_1,Z_2\in\Gamma(D)$ and $Z\in\mathfrak{X}(M)$. Hence $[Z_1,Z_2]\in\Gamma(D)$, as we wished.

Since D is integrable, then the Codazzi equation $(\nabla_S A)(\eta,T)=(\nabla_T A)(\eta,S)$ yields

$$A_{(\nabla^\perp_S\eta)_L}T=A_{(\nabla^\perp_T\eta)_L}S \tag{3.10}$$

for any $S,T\in\Gamma(D)$ and $\eta\in\Gamma(L^\perp)$. If the condition $(\mathcal{C}_2)$ does not hold, there is $S\in\Gamma(D)$ and $\eta\in\Gamma(L^\perp)$ such that $\mu=(\nabla^\perp_S\eta)_L\neq 0$. It follows from (3.10) and $\dim D\geq 2n-2p+4$ that $\dim\ker A_\mu\geq 2n-2p+2$, and hence we have by (2.1) that $\varrho_f\leq p-1$, which is a contradiction.

From (2.1) we have that $\mathcal{S}(\alpha^f|_{D\times D})$ is of even dimension. Then L satisfies condition $(\mathcal{C}_3)$ since, otherwise, we have $\alpha|_{D\times D}=0$ and hence $A_\xi D\subset D^\perp$ if $\xi\in\Gamma(L)$. But then $\dim\ker A_\xi|_D\geq 2n-4p+8$, and (2.1) gives that $\varrho_f\leq 2p-4$, which is a contradiction. ■

Finally, we are in the condition to prove our main result.

Proof of Theorem 1.1. By Lemma 3.6 and Lemma 3.7, there is an open dense subset of M^{2n} such that along any connected component, say U, the submanifold $f|_U$ is an embedding and there is a unique pair $(L,\mathcal{J})$, where $L\subset N_1|_U$ has rank $L=2$, and a J-invariant vector subbundle $D=\mathcal{N}_c(\alpha_{L^\perp})$ with rank $D\geq 2n-2p+4$ such that conditions $(\mathcal{C}_1)$ to $(\mathcal{C}_3)$ are satisfied. Then it follows from Theorem 2.5 that $f|_U$ admits a Kaehler extension F as in the statement.

We now argue for the uniqueness of the Kaehler extension. Let $F\colon N^{2n+2} \to \mathbb{R}^{2n+p}$, $n \geq p+1$ and $3 \leq p \leq 4$, be a real Kaehler submanifold such that the tangent vector subspaces $\Delta^c(z) = \mathcal{N}_c(\alpha^F(z))$ satisfy that $\dim \Delta^c(z)$ is constant and thus form a tangent vector subbundle. In fact, it is easy to verify that the distribution Δ^c is integrable and that its leaves are totally geodesic submanifolds in M^{2n} as well as in $\mathbb{R}^{2n+p}$.

From Lemma 3.3 if $p=3$, and Theorem 3.5 if $p=4$, we have rank $\Delta^c \geq 2n-2p+6$. Then let $j\colon M^{2n} \to N^{2n+2}$ be an holomorphic submanifold of N^{2n+2} such that the real Kaehler submanifold $f = F \circ j\colon M^{2n} \to \mathbb{R}^{2n+p}$ is substantial and satisfies that $\varrho_f(x) > p$ at any $x \in M^{2n}$. To conclude the proof, we have to show that F is the unique Kaehler extension of f up to a reparametrization.

From Lemmas 3.6 and 3.7, along each connected component, say U, of an open dense subset of M^{2n} there is a unique pair $(L, \mathcal{J})$ where $L \subset N_1|_U$ is a vector subbundle of rank two and the isometric complex structure $\mathcal{J} \in \Gamma(\mathrm{Aut}(L))$ satisfies

$$\mathcal{J}\alpha_L(X,Y) = \alpha_L(X,JY) \quad \text{for any } X,Y \in \mathfrak{X}(U).$$

Moreover, there is a tangent vector subbundle $D = \mathcal{N}_c(\alpha^f_{L^\perp})$ with rank $D \geq 2n-2p+4$ and the conditions $(\mathcal{C}_1)$ to $(\mathcal{C}_3)$ hold.

Since $f = F \circ j$, then $N_fM = F_*N_jM \oplus j^*N_FN$, and hence

$$\alpha^f(X,Y) = F_*\alpha^j(X,Y) + \alpha^F(j_*X, j_*Y) \tag{3.11}$$

for any $X,Y \in \mathfrak{X}(M)$. The vector subspace $E \subset T_xM$ given by

$$j_*E = \Delta^c(j(x)) \cap j_*T_xM$$

satisfies $\dim E \geq 2n-2p+4$ since the codimension of j_*T_xM in $T_{j(x)}N$ is two. Being j holomorphic, then $\bar{J}\alpha^j(X,Y) = \alpha^j(X,JY)$ for any $X,Y \in \mathfrak{X}(M)$, where $\bar{J}$ is the complex structure of N^{2n+2}. Since $\Delta^c(j(x)) \cap j_*T_xM$ is $\bar{J}$-invariant, then E is J-invariant, and therefore either $\alpha^j|_{E\times E} = 0$ or $\mathcal{S}(\alpha^j|_{E\times E}) = N_jM$.

Suppose that $\alpha^j|_{E\times E} = 0$. If $0 \neq \xi \in \Gamma(N_jM)$ and A^j_ξ is the shape operator of j, then $\ker A^j_\xi|_E \subset E \cap \mathcal{N}_c(\alpha^f)$. In fact, if $S \in \ker A^j_\xi|_E$, then rank $N_jM = 2$ gives that

$$S, JS \in \ker A^j_\xi|_E \cap \ker A^j_{\bar{J}\xi}|_E \subset \mathcal{N}_c(\alpha^j) \cap E.$$

Since $j_*E \subset \Delta^c$, we have $\alpha^F(j_*X, j_*S) = 0$ if $S \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$. Now (3.11) yields $\ker A^j_\xi|_E \subset \mathcal{N}_c(\alpha^f)$ and thus $\dim \mathcal{N}_c(\alpha^f) \geq 2n-4p+8$. Then $\varrho_f \leq 2p-4$, which is a contradiction. It follows that $\mathcal{S}(\alpha^j|_{E\times E}) = N_jM$, and we obtain from (3.11) that $\mathcal{S}(\alpha^f|_{E\times E}) = F_*N_jM \subset N_1$.

The vector subbundle $F_*N_jM \subset N_1$ is endowed with the complex structure $F_* \circ \bar{J}|_{N_jM}$, and $F_*\bar{J}|_{N_jM}\alpha^j(X,Y) = F_*\alpha^j(X,JY)$ holds. Now, the uniqueness part of Lemma 3.6 yields that $L = F_*N_jM$ and $\mathcal{J} = F_* \circ \bar{J}|_{F_*N_jM}$.

We claim that $E = D$. Since from (3.11) we have $\alpha^f_{L^\perp}(X,Y) = \alpha^F(j_*X, j_*Y)$, then $j_*D = \mathcal{N}_c(\alpha^F|_{j_*TM\times j_*TM})$, and hence $\Delta^c \cap j_*TM \subset j_*D$. Given $S \in \Gamma(D)$, the condition $(\mathcal{C}_2)$ yields $\nabla^\perp_S F_*\xi \in \Gamma(L)$ for any $\xi \in \Gamma(N_jM)$. Thus $\alpha^F(j_*S, \xi) = 0$ for any $\xi \in \Gamma(N_jM)$. Since $\alpha^F(j_*S, j_*X) = \alpha^f_{L^\perp}(S,X) = 0$ for any $X \in \mathfrak{X}(M)$, hence $j_*S \in \Delta^c$, giving the other inclusion and hence the claim.

Let $\pi\colon \Lambda \to U$ be the vector subbundle of $f_*TU \oplus L$ given by (2.6). Then Lemma 2.3 and Theorem 2.5 yield the Kaehler extension $\bar{F}\colon \bar{N}^{2n+2} \subset \Lambda \to \mathbb{R}^{2n+p}$ of f defined by $\bar{F}(\lambda) = f(\pi(\lambda)) + \lambda$. To conclude the proof, it remains to show that $\bar{N}^{2n+2}$ can be chosen sufficiently small so that $\bar{F}(\lambda) \in F(N)$ if $\lambda = f_*(\nabla_S T)_{D^\perp} + \alpha^f(S,T) \in \bar{N}^{2n+2}$. On one hand, from (3.11) and since $j_*D \subset \Delta^c$ from the claim, we obtain

$$\begin{aligned}\lambda &= F_*j_*(\nabla_S T)_{D^\perp} + F_*\alpha^j(S,T) = F_*j_*(\nabla_S T)_{D^\perp} + F_*({}^N\nabla_S j_*T)_{N_jM}\\ &= F_*({}^N\nabla_S j_*T)_{j_*D^\perp \oplus N_jM}.\end{aligned}$$

On the other hand, we have

$${}^N\nabla_S j_*T = ({}^N\nabla_S j_*T)_{j_*D} + ({}^N\nabla_S j_*T)_{j_*D^\perp \oplus N_jM}.$$

Since the distribution Δ^c is totally geodesic and $j_*D \subset \Delta^c$, it follows that

$$R = ({}^N\nabla_S j_*T)_{j_*D^\perp \oplus N_jM} \in \Gamma(j^*\Delta^c).$$

The restriction of F to the leaf of Δ^c that contains $j(\pi(\lambda))$ is an open subset of the affine subspace $F(j(\pi(\lambda))) + F_*\Delta^c(j(\pi(\lambda)))$ of $\mathbb{R}^{2n+p}$. Then

$$\bar{F}(\lambda) = F(j(\pi(\lambda))) + F_*R \in F(N),$$

as we wished. ∎

The following result generalizes Theorem 2 in [10] since our rank hypothesis is weaker than the corresponding assumption there.

Theorem 3.8. *Let $f\colon M^{2n} \to \mathbb{R}^{2n+3}$, $n \geq 4$, be a real Kaehler submanifold with rank $\varrho_f > 3$ everywhere. Then f is locally isometrically rigid unless there exists an open subset $U \subset M^{2n}$ such that the Kaehler extension F of $f|_U$ is either a flat or a minimal hypersurface. In this case, any isometric deformation of $f|_U$ is the restriction of an isometric deformation of F.*

Proof. By Lemma 3.6 and Lemma 3.7, let $V \subset M^{2n}$ be an open subset of M^{2n} on which there are a unique pair $(L, \mathcal{J})$, with $L \subset N_1$ of rank 2, and a tangent subbundle $D = \mathcal{N}_c(\alpha_{L^\perp})$ with rank $D \geq 2n-2$, such that the conditions $(\mathcal{C}_1)$ to $(\mathcal{C}_3)$ are satisfied.

The subset $U \subset V$ of points where $\dim D = 2n-2$ is open and dense since otherwise f would not be locally substantial. We argue for points in U. Since $\mathcal{N}_c(\alpha) = \mathcal{N}_c(\alpha_L) \cap D$ and $\dim \mathcal{N}_c(\alpha) \leq 2n-8$ by the rank assumption, then

$$\dim(\mathcal{N}_c(\alpha_L) + D) = \dim \mathcal{N}_c(\alpha_L) + \dim D - \dim \mathcal{N}_c(\alpha) \geq \dim \mathcal{N}_c(\alpha_L) + 6. \tag{3.12}$$

Thus $\dim \mathcal{N}_c(\alpha_L) \leq 2n-6$. Since $\mathcal{N}(\alpha_L) = \mathcal{N}_c(\alpha_L)$ by (2.1), hence $\dim \mathcal{N}(\alpha_L) \leq 2n-6$.

We claim that $\dim \mathcal{N}(\alpha) < 2n-6$ on U. Since $\mathcal{N}(\alpha) \subset \mathcal{N}(\alpha_L)$, we may suppose that $\dim \mathcal{N}(\alpha_L) = 2n-6$. Since the subspace $\mathcal{N}_c(\alpha_L) + D$ is J-invariant, then either $\dim \mathcal{N}(\alpha_L) + D = 2n-2$, or $\mathcal{N}(\alpha_L) + D = TM$. In the former case, the equality part in (3.12) gives $\varrho_f = 3$, a contradiction. Hence $\mathcal{N}(\alpha_L) + D = TM$. In that case, and since

$\dim D = 2n-2$, then $\dim \mathcal{N}(\alpha_{L^\perp}) \leq 2n-1$. Hence, from $\mathcal{N}(\alpha) = \mathcal{N}(\alpha_L) \cap \mathcal{N}(\alpha_{L^\perp})$ we get

$$2n = \dim(\mathcal{N}(\alpha_L) + \mathcal{N}(\alpha_{L^\perp})) = \dim \mathcal{N}(\alpha_L) + \dim \mathcal{N}(\alpha_{L^\perp}) - \dim \mathcal{N}(\alpha)$$

and thus $\dim \mathcal{N}(\alpha) < 2n-6$ as claimed. Now the proof follows from Theorem 2 in [10]. ∎

A. Appendix

The local structures of the substantial real Kaehler submanifolds $F\colon M^{2n} \to \mathbb{R}^{2n+2}$ for $n \geq 3$ are discussed next with separation in the cases were F is a minimal submanifold or is free of points where it is minimal.

The non-minimal case.

We assume further that either M^{2n} is flat or nowhere flat. In the non-flat case, the classifications given below was obtained from [15].

(i) If $M^{2n} = U \subset \mathbb{R}^{2n+2}$ is an open subset where either $N_1 = \mathcal{S}(\alpha^F)$ has rank one and it is not parallel in the normal connection at any point, or it satisfies that rank $N_1 = 2$ everywhere.

(a) If rank $N_1 = 1$, then Theorem 1 in [12] gives that $F = H \circ i$, where $i\colon U \to V$ is a totally geodesic inclusion with $V \subset \mathbb{R}^{2n+1}$ an open subset, and $H\colon V \to \mathbb{R}^{2n+2}$ is an isometric immersion free of totally geodesic points. Moreover, if $Z(i(y)) \in T_{i(y)}V$ is an eigenvector corresponding to the unique nonzero principal curvature of H, then the conditions $Z(i(y)) \notin i_*T_yU$ and $Z(i(y)) \notin N_{i(y)}U$ hold at any $y \in U$.

(b) If rank $N_1 = 2$, the nicest local parametric classification is given by Corollary 18 in [14].

(ii) An open subset of a cylinder $h \times \mathrm{id}\colon L^2 \times \mathbb{C}^{n-1} \to \mathbb{R}^{2n+2}$ over a non-flat and nowhere minimal surface $h\colon L^2 \to \mathbb{R}^4$.

(iii) A composition of isometric immersions $F = h \circ f$, where $f\colon M^{2n} \to V \subset \mathbb{R}^{2n+1}$ is a non-flat real Kaehler hypersurface, and $h\colon V \to \mathbb{R}^{2n+2}$ is not totally geodesic.

(iv) An open subset of an extrinsic product of two Euclidean real Kaehler hypersurfaces where at least one is neither flat nor minimal.

The minimal case.

For minimal real Kaehler submanifolds $F\colon M^{2n} \to \mathbb{R}^N$ in any codimension, there is the representation given in [1] and discussed in the Appendix of Chapter 15 in [13]. Very roughly, the holomorphic representative of the submanifold is determined by a set of n independent holomorphic functions which span an isotropic subspace of $\mathbb{C}^N$ and have to satisfy certain integrability conditions, thus this cannot be seen as a classification.

(i) If $\varrho_F = 1$, we have:

(a) An open subset of a cylinder $h \times \mathrm{id}\colon L^2 \times \mathbb{C}^{n-} \to \mathbb{R}^{2n+2}$ over a substantial minimal surface $h\colon L^2 \to \mathbb{R}^4$.

(b) A cylinder over a submanifold parametrically classified by Theorem 27 in [7] by means of a Weierstrass type representation given in terms of $(m-1)$-isotropic surface. These surfaces have been completely described in [7].

(ii) If $\varrho_F = 2$, we have:

(a) An open subset of the extrinsic product of two minimal Euclidean real Kaehler hypersurfaces.

(b) Examples can be constructed by the use of the representation (8) in [9] as explained by part (i) of the Remark given there. If M^{2n} is complete, there is the parametric classification provided in [9]. A classification in the local case remains an open problem unless $n = 2$, for which there is the classification obtained in [16]. Finally, for complete examples for $n = 2$ see [10].

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Note. In a previous version of the paper published online-first, there was a mistake in the statement of Theorem 2.5. The sentence "[...] satisfies $\mathcal{N}_c(\alpha^F) = D \oplus \Lambda$ at any point of M^{2n} and thus [...]" was changed to the correct "[...] satisfies $\mathcal{N}_c(\alpha^F) = D \oplus \Lambda$ at any point and thus [...]" on August 31, 2023.

References

[1] Arezzo, C., Pirola, G. P. and Solci, M.: The Weierstrass representation for pluriminimal submanifolds. *Hokkaido Math. J.* **33** (2004), no. 2, 357–367.

[2] Chion, S. and Dajczer, M.: Minimal real Kaehler submanifolds. *Mat. Contemp.* **49** (2022), 236–250.

[3] Chion, S. and Dajczer, M.: The second fundamental form of the real Kaehler submanifolds. Preprint 2023, arXiv: 2302.12038.

[4] de Carvalho, A., Chion, S. and Dajczer, M.: Conformal Kaehler Euclidean submanifolds. *Differential Geom. Appl.* **82** (2022), article no. 101893, 12 pp.

[5] de Carvalho, A., Chion, S. and Dajczer, M.: Holomorphicity of real Kaehler submanifolds. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **24** (2023), no. 2, 821–837.

[6] Dajczer, M.: A characterization of complex hypersurfaces in $\mathbb{C}^m$. *Proc. Amer. Math. Soc.* **105** (1989), no. 2, 425–428.

[7] Dajczer, M. and Florit, L. A.: A class of austere submanifolds. *Illinois J. Math.* **45** (2001), no. 3, 735–755.

[8] Dajczer, M. and Gromoll, D.: Real Kaehler submanifolds and uniqueness of the Gauss map. *J. Differential Geom.* **22** (1985), no. 1, 13–28.

[9] Dajczer, M. and Gromoll, D.: The Weierstrass representation for complete minimal real Kaehler submanifolds of codimension two. *Invent. Math.* **119** (1995), no. 2, 235–242.

[10] Dajczer, M. and Gromoll, D.: Real Kaehler submanifolds in low codimension. *Differential Geom. Appl.* **7** (1997), no. 4, 389–395.

[11] Dajczer, M. and Rodríguez, L.: Rigidity of real Kaehler submanifolds. *Duke Math. J.* **53** (1986), no. 1, 211–220.

[12] Dajczer, M. and Tojeiro, R.: Submanifolds with nonparallel first normal bundle. *Canad. Math. Bull.* **37** (1994), no. 3, 330–337.

[13] Dajczer, M. and Tojeiro, R.: *Submanifold theory. Beyond an introduction*. Universitext, Springer, New York, 2019.

[14] Florit, L. A. and de Freitas, G. M.: Classification of codimension two deformations of rank two Riemannian manifolds. *Comm. Anal. Geom.* **25** (2017), no. 4, 751–797.

[15] Florit, L. A. and Zheng, F.: Complete real Kähler submanifolds in codimension two. *Math. Z.* **258** (2008), no. 2, 291–299.

[16] Hennes, P.: *Weierstrass representations of minimal real Kahler submanifolds*. Ph.D. Thesis, State University of New York, Stony Brook. 2001.

[17] Spivak, M.: *A comprehensive introduction to differential geometry, Vol 4*. Second edition. Publish or Perish, Wilmington, DE, 1979.

[18] Yan, J. and Zheng, F.: A Dajczer–Rodríguez type cylinder theorem for real Kähler submanifolds. *Pure Appl. Math. Q.* **9** (2013), no. 3, 563–577.

[19] Yan, J. and Zheng, F.: An extension theorem for real Kähler submanifolds in codimension 4. *Michigan Math. J.* **62** (2013), no. 2, 421–441.

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