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Noncommutative determinants, Cauchy–Binet formulæ and Capelli-type identities
II. Grassmann and quantum oscillator algebra representation

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Abstract. We prove that, for X, Y, A and B matrices with entries in a non-commutative ring such that

$$[X_{ij}, Y_{k\ell}] = -A_{i\ell} B_{kj},$$

satisfying suitable commutation relations (in particular, X is a Manin matrix), the following identity holds:

$$\text{col-det}\, X \;\text{col-det}\, Y = \langle 0 \mid \text{col-det}(aA + X(I - a^\dagger B)^{-1} Y) \mid 0 \rangle.$$

Furthermore, if also Y is a Manin matrix,

$$\text{col-det}\, X \;\text{col-det}\, Y = \int \mathcal{D}(\psi, \bar{\psi}) \exp\Big(\sum_{k \geq 0} \frac{(\bar{\psi} A \psi)^k}{k+1} (\bar{\psi} X B^k Y \psi) \Big).$$

Here $\langle 0 \mid$ and $\mid 0 \rangle$, are respectively the bra and the ket of the ground state, $a^\dagger$ and a the creation and annihilation operators of a quantum harmonic oscillator, while $\bar{\psi}_i$ and ψ_i are Grassmann variables in a Berezin integral. These results should be seen as a generalization of the classical Cauchy–Binet formula, in which A and B are null matrices, and of the non-commutative generalization, the Capelli identity, in which A and B are identity matrices and $[X_{ij}, X_{k\ell}] = [Y_{ij}, Y_{k\ell}] = 0$.

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Contents

1. Introduction

1.1. The Cauchy–Binet theorem. Let R be a commutative ring, and let $M = (M_{ij})_{i,j=1}^n$ be a $n \times n$ matrix with elements in R. The determinant of the matrix M can be defined as

$$\det M \overset{\text{def}}{=} \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) M_{\sigma(1)1} M_{\sigma(2)2} \dots M_{\sigma(n)n}, \tag{1}$$

where $\mathcal{S}_n$ is the permutation group of the set $[n] = \{1, 2, \dots, n\}$ and $\operatorname{sgn}(\sigma)$ is the sign of the permutation σ.

Let X be a $n \times m$ matrix and Y a $m \times n$ matrix with elements in the commutative ring R. For each subset $I \subseteq [m]$ let be $X_{[n],I}$ the minor of X with columns in I and similarly $Y_{I,[n]}$ the minor of Y with rows in I. The classical Cauchy–Binet formula relates the product of the determinant of these matrices to the determinant of the product. More precisely

$$\sum_{\substack{L \subseteq [m] \\ |L| = n}} \det X_{[n],L} \det Y_{L,[n]} = \det(XY). \tag{2}$$

In order to generalize the definition (1) to matrices with elements in a *noncommutative* ring R, the first problem encountered is that it is ambiguous without an ordering prescription for the product. Rather, numerous alternative "determinants" can be defined: for instance, the *column-determinant*

$$\text{col-det}\, M \overset{\text{def}}{=} \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} M_{\sigma(i)\,i} \tag{3}$$

and the *row-determinant*

$$\text{row-det}\, M \overset{\text{def}}{=} \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} M_{i\sigma(i)}. \tag{4}$$

(Note that col-det M = row-det M^{T}.) It is intended above that, when dealing with non-commuting quantities having indices depending on a single integer, the product symbol Π denotes an "ordered product", i.e.

$$\prod_{i=k}^{k+\ell} f_i \overset{\text{def}}{=} f_k f_{k+1} \dots f_{k+\ell}.$$

In [1] we have proven, in collaboration with A. D. Sokal, non-commutative generalizations of the Cauchy–Binet formula. In order to express our result, we called the matrix M *column-pseudo-commutative* in the case

$$[M_{ij}, M_{k\ell}] = [M_{i\ell}, M_{kj}], \quad \text{for all } i, j, k, \ell \tag{5}$$

and

$$[M_{ij}, M_{i\ell}] = 0, \quad \text{for all } i, j, \ell. \tag{6}$$

(Similarly, we said a matrix M to be *row-pseudo-commutative* in case M^{T} is column-pseudo-commutative).[1] Furthermore, we said that M has *weakly column-symmetric* (*and row-antisymmetric*) *commutators* if (5) holds for $i \neq k$ (and (6) not necessarily holds).

We proved (see [1], Proposition 1.2) the following result.[2]

Proposition 1.1 (noncommutative Cauchy–Binet). *Let R be a ring, and let X be a $n \times m$ matrix and Y a $m \times n$ matrix with elements in R. Suppose that*

$$[X_{ij}, Y_{k\ell}] = -A_{i\ell}\delta_{kj}, \quad \textit{for all } i, j, k, \ell,$$

with A a $n \times n$ matrix. Then

(a) *if X is row-pseudo-commutative, then*

$$\sum_{\substack{L \subseteq [m] \\ |L| = n}} \text{col-det}\, X_{[n],L}\ \text{col-det}\, Y_{L,[n]} = \text{col-det}(XY + Q^{\text{col}}) \tag{7}$$

where

$$Q^{\text{col}}_{ij} \overset{\text{def}}{=} A_{ij}(n - j);$$

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[1] Note that (5) implies $2[M_{ij}, M_{i\ell}] = 0$, i.e. *twice* equation (6), a subtlety, of relevance only when the field K over which the ring R is defined is of characteristic 2, that will appear several times along the paper.

[2] Here we perform a change of notation for future convenience $(A^{\mathrm{T}} \to X, B \to Y, h \to A)$ and consider only the case $r = n$.

(b) *if Y is column-pseudo-commutative, then*

$$\sum_{\substack{L\subseteq[m]\\|L|=n}} \text{row-det}\, X_{[n],L}\ \text{row-det}\, Y_{L,[n]} = \text{row-det}(XY + Q^{\text{row}})$$

where

$$Q^{\text{row}}_{ij} \stackrel{\text{def}}{=} A_{ij}(i-1);$$

(c) *in particular, if* $[X_{ji}, X_{\ell k}] = 0$ *and* $[Y_{ij}, Y_{k\ell}] = 0$ *whenever* $j \neq \ell$, *then*

$$\begin{aligned}\sum_{\substack{L\subseteq[m]\\|L|=n}} \det X_{[n],L} \det Y_{L,[n]} &= \text{col-det}(XY + Q^{\text{col}})\\ &= \text{row-det}(XY + Q^{\text{row}}).\end{aligned}$$

With respect to the commutative case (2), the determinants are replaced by one of its non-commutative generalizations, but the left-hand side keeps the same form, while on the right-hand side the product XY requires an additive correction.

An example of a non-commutative ring R is the Weyl algebra $A_{m\times n}(K)$ over some field K of characteristic 0 (e.g. $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$) generated by a $m \times n$ collection $Z = (z_{ij})$ of commuting indeterminates (“positions”) and the corresponding collection $\partial = (\partial/\partial z_{ij})$ of differential operators (proportional to “momenta”); so that

$$\left[z_{ij}, \frac{\partial}{\partial z_{k\ell}}\right] = -\delta_{ik}\delta_{j\ell} \tag{8a}$$

and

$$[z_{ij}, z_{k\ell}] = \left[\frac{\partial}{\partial z_{ij}}, \frac{\partial}{\partial z_{k\ell}}\right] = 0. \tag{8b}$$

If we set $m = n$, $X = Z^{\mathrm{T}}$ and $Y = \partial$, we soon get $A_{ij} = \delta_{ij}$ for each $i, j \in [n]$ and

$$\begin{aligned}\det X \det \partial &= \text{col-det}[X^{\mathrm{T}}\partial + \text{diag}(n-1, n-2, \dots, 0)]\\ &= \text{row-det}[X^{\mathrm{T}}\partial + \text{diag}(0, 1, \dots, n-1)]\end{aligned}$$

which are the Capelli identities [2], [3], [4], and [5] of classical invariant theory [6], [7], and [8], a field of research that, in more than a century, has remained active up to recent days (a forcedly incomplete selection of papers on the subject includes [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], and [24]). Because of this example, the correction term due to the presence of the matrix Q which appears in the non-commutative case is sometimes called the “quantum” correction with respect to the formula in the commutative case (2).

Chervov, Falqui, and Rubtsov give in [29] an extremely interesting survey of the algebraic properties of row-pseudo-commutative matrices (which they call "Manin matrices", because a similar notion has proven fruitful in the context of quantum groups, where it arose already two decades ago in Manin's work [25], [26], [27], and [28]), when the ring R is an associative algebra over a field of characteristic $\neq 2$. In particular, [29], Section 6, contains an interesting generalization of our result. Another recent interesting survey, on combinatorial methods in the study of non-commutative determinants, is the Ph.D. Thesis of M. Konvalinka [30].

In this paper we will investigate a stronger version of Proposition 1.1. In particular we relax the condition that for all i, j, k, ℓ

$$[X_{ij}, Y_{k\ell}] = -A_{i\ell}\delta_{kj} \tag{9}$$

to

$$[X_{ij}, Y_{k\ell}] = -A_{i\ell}B_{kj} \tag{10}$$

where B is a $m \times m$ matrix whose elements are supposed to commute with everything.

Remark that, whenever B is invertible,[3] from (10) by multiplication of B^{-1}_{js} and sum over j we get

$$[(XB^{-1})_{is}, Y_{k\ell}] = -A_{i\ell}\delta_{ks}$$

which is of the form (9), and similarly by multiplication of B^{-1}_{sk} and sum over k

$$[X_{ij}, (B^{-1}Y)_{s\ell}] = -A_{i\ell}\delta_{sj}$$

and, as if X is row-pseudo-commutative also XB^{-1} is such, while if Y is column-pseudo-commutative also $B^{-1}Y$ is such. Thus, quite trivially, Proposition 1.1 can be used to express, for example in the case (a)

$$\sum_{\substack{I,L\subseteq[m]\\|L|=|I|=n}} \text{col-det}\, X_{[n],I}\ \det B^{-1}_{IL}\ \text{col-det}\, Y_{L,[n]} = \text{col-det}(XB^{-1}Y + Q^{\text{col}}).$$

In agreement with the philosophy of the original Capelli identity, our goal in this paper is in another direction: we want to find generalizations of Proposition 1.1, under the more general (10), in which the left-hand side of (7) (and variants) is kept *exactly in this form* (with no dependence from B whatsoever), and investigate for a generalized "quantum correction" on the right-hand side.

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[3] Recall that, in our case, this is not just a matter of the matrix being non-singular: as the entries B_{ij} are valued in a ring, not even the single entries, even when non-zero, are guaranteed to have a multiplicative inverse, i.e. not even the case $n = 1$ is easy.

We have not been able to reach an expression as simple as we got previously in Proposition 1.1, (not even in the case when B is invertible). However, we have found closed formulas with the help of the algebra and the Hilbert space of a single "bosonic quantum oscillator" (also known as *Weyl–Heisenberg algebra*), and, also, as a Berezin integral in Grassmann algebra, corresponding to "fermionic quantum oscillators" (see respectively the following Propositions 1.2 and 1.4, which are the main results of the paper).

We point out here a possible source of confusion. While, at the foundations of invariant theory, Capelli identities have been discovered within their explicit realization in Weyl algebra (the example of (8)), it is nowadays clear, and along the lines e.g. of [1], [29], and several other papers, that the appropriate context of this family of identities is the identification of sufficient conditions on the commutation rules for the elements of the involved matrices, regardless from the presentation of rings R, and matrices valued in R, realizing these rules. To characterize and classify these realizations (or, even, to determine their existence) is a problem that we find important, but of separate interest, and we do not treat it here. The role of the Weyl–Heisenberg and Grassmann algebras mentioned above is *not* at the level of the explicit realization of the matrices. It consists instead of an auxiliary structure, implementing certain combinatorial relations at the level of manipulation of commutators, that arise along the lines of the proof.

We annotate here an interesting paper, by Blasiak and Flajolet [31], presenting a collection of classical and new facts on the role of the Weyl–Heisenberg algebra in combinatorics, in the spirit of the discussion above.

1.2. The bosonic quantum oscillator. Following the classical treatment of the quantum oscillator by Dirac [33], Chapter 6, let us introduce the operator a and its adjoint $a^\dagger$, called respectively *annihilation* and *creation* operator, and the Hermitian number operator $N = a^\dagger a$.

They satisfy the commutation relations of the Weyl–Heisenberg algebra

$$[a, a^\dagger] = 1, \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger. \tag{11}$$

Let $\mid n\rangle$ with $n \in \mathbb{N}$ be the eigenstate of N corresponding to the eigenvalue n, that is

$$N \mid n\rangle = n \mid n\rangle.$$

In particular the lowest eigenstate of N, $\mid 0\rangle$, is annihilated by a

$$a \mid 0\rangle = 0.$$

Without loss of generality, we assume it to be of unit norm, $\langle 0 \mid 0\rangle = 1$. Our first generalization of the Capelli identity is stated within this framework.

Proposition 1.2. *Let R be a ring, and let X be a $n \times m$ matrix and Y a $m \times n$ matrix with elements in R. Suppose that*

$$[X_{ij}, Y_{k\ell}] = -A_{i\ell} B_{kj} \quad \textit{for all } i, j, k, \ell$$

with A a $n \times n$, and B a $m \times m$ matrix whose elements commute with everything. Then

(a) *if X is row-pseudo-commutative, and*

$$[X_{ij}, A_{k\ell}] - [X_{kj}, A_{i\ell}] = 0 \quad \textit{for all } i, j, k, \ell \tag{12}$$

then

$$\sum_{\substack{L \subseteq [m] \\ |L|=n}} \text{col-det}\, X_{[n],L}\, \text{col-det}\, Y_{L,[n]} = \langle 0 \mid \text{col-det}[aA + X(1 - a^{\dagger} B)^{-1} Y] \mid 0 \rangle;$$

(b) *if Y is column-pseudo-commutative, and*

$$[Y_{ij}, A_{k\ell}] - [Y_{i\ell}, A_{kj}] = 0 \quad \textit{for all } i, j, k, \ell \tag{13}$$

then

$$\sum_{\substack{L \subseteq [m] \\ |L|=n}} \text{row-det}\, X_{[n],L}\, \text{row-det}\, Y_{L,[n]} = \langle 0 \mid \text{row-det}[a^{\dagger} A + X(1 - a B)^{-1} Y] \mid 0 \rangle;$$

(c) *in particular, if $[X_{ji}, X_{\ell k}] = 0$ and $[Y_{ij}, Y_{k\ell}] = 0$ whenever $j \neq \ell$, then*

$$\sum_{\substack{L \subseteq [m] \\ |L|=n}} \det X_{[n],L} \det Y_{L,[n]} = \langle 0 \mid \text{col-det}[a A + X(1 - a^{\dagger} B)^{-1} Y] \mid 0 \rangle = \langle 0 \mid \text{row-det}[a^{\dagger} A + X(1 - a B)^{-1} Y] \mid 0 \rangle.$$

The further commutation condition (12) (and the counterpart (13) for case (b)) appears as a subtle technicality, that we did not succeed to avoid. Note however that, as shown in Lemmata 3.6 and 3.7 through an analysis of the consequences of the Jacobi Identity, it is implied by a very mild condition on B, (informally, that two vectors $\vec{u}, \vec{v} \in R^m$ exist such that the scalar product $(\vec{u}, B\vec{v})$ is a regular element of the ring, i.e., it is not zero, and not a divisor of zero). In particular, this is obviously the case under the circumstances originally treated in [1], where $B = I$.

As an example, let the non-commutative ring R be the Weyl algebra $A_{m\times s}(K)$ over some field K of characteristic 0 (e.g. $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$) generated by an $m \times s$ collection $Z = (z_i^a)$ with $i \in [n]$ and $a \in [s]$ of commuting indeterminates and the corresponding collection $\partial = (\partial/\partial z_i^a)$ of differential operators; so that

$$\left[z_i^a, \frac{\partial}{\partial z_j^b}\right] = -\delta_{ij}\delta^{ab};$$

and

$$[z_i^a, z_j^b] = \left[\frac{\partial}{\partial z_i^a}, \frac{\partial}{\partial z_j^b}\right] = 0.$$

Let

$$X_{ij} = \sum_{a=1}^{s} z_i^a \alpha_j^a \quad \text{and} \quad Y_{k\ell} = \sum_{a=1}^{s} \beta_k^a \frac{\partial}{\partial z_\ell^a},$$

with α_j^a, β_k^a commuting with everything, so that for all $i, \ell \in [n]$ and $j, k \in [m]$

$$[X_{ij}, X_{k\ell}] = [Y_{ij}, Y_{k\ell}] = 0$$

and

$$[X_{ij}, Y_{k\ell}] = -\delta_{i\ell} \sum_{a=1}^{s} \beta_k^a \alpha_j^a$$

which, in our notation means that

$$A_{i\ell} = \delta_{i\ell} \quad \text{and} \quad B_{kj} = \sum_{a=1}^{s} \beta_k^a \alpha_j^a .$$

Remark that the rank of the $m \times m$ matrix B is $\min(m, s)$, in particular, when $s < m$, B is not invertible.

In the particular case in which $B_{ij} = \delta_{ij}$ for each $i, j \in [m]$, both Proposition 1.1 and 1.2 apply. As a consequence, the right hand sides must be equal and, for example, if X is row-pseudo-commutative, then

$$\text{col-det}(XY + Q^{\text{col}}) = \langle 0 \mid \text{col-det}[a\, A + (1 - a^\dagger)^{-1} XY] \mid 0\rangle,$$

while, if Y is column-pseudo-commutative, then

$$\text{row-det}(XY + Q^{\text{row}}) = \langle 0 \mid \text{row-det}[a^\dagger A + (1 - a)^{-1} XY] \mid 0\rangle.$$

These relations are indeed valid regardless from the fact that A is related to the commutator of X and Y, i.e. they are a consequence of a stronger fact

Proposition 1.3. *Let R be a ring and U and V be two $n \times n$ matrices with elements in R. Then*

$$\text{col-det}(U + Q^{\text{col}}) = \langle 0 \mid \text{col-det}(aV + (1-a^\dagger)^{-1}U) \mid 0 \rangle$$

where

$$Q^{\text{col}}_{ij} \overset{\text{def}}{=} V_{ij}(n-j),$$

and

$$\text{row-det}(U + Q^{\text{row}}) = \langle 0 \mid \text{row-det}(a^\dagger V + (1-a)^{-1}U) \mid 0 \rangle \tag{14}$$

where

$$Q^{\text{row}}_{ij} \overset{\text{def}}{=} V_{ij}(i-1).$$

This fact, together with a generalization, is proven in Section 2.

1.3. The Grassmann algebra. The determinant of a $n \times n$ matrix M with elements in a commutative ring can be represented as a Berezin integral over the Grassmann algebra generated by the $2n$ anti-commuting variables $\{\psi_i, \bar{\psi}_i\}_{i\in[n]}$ (for an introduction to such a topic we invite the interested reader to refer to [34], Appendix B). More precisely:

$$\det M = \int \mathcal{D}(\psi, \bar{\psi}) \exp(\bar{\psi} M \psi), \tag{15}$$

where

$$\mathcal{D}(\psi, \bar{\psi}) \overset{\text{def}}{=} \prod_{i=1}^{n} d\psi_i \, d\bar{\psi}_i .$$

Therefore the Cauchy–Binet theorem can also be written as the identity

$$\sum_{\substack{L\subseteq[m]\\ |L|=n}} \det X_{[n],L} \det Y_{L,[n]} = \int \mathcal{D}(\psi, \bar{\psi}) \exp(\bar{\psi} X Y \psi).$$

We have obtained the following generalization.

Proposition 1.4. *Let R be a ring containing the rationals, and let X be a $n \times m$ matrix and Y a $m \times n$ matrix with elements in R. Suppose that*

$$[X_{ij}, Y_{k\ell}] = -A_{i\ell} B_{kj} \quad \textit{for all } i, j, k, \ell$$

with A a $n \times n$, and B a $m \times m$ matrix whose elements commute with everything. Let I_m the $m \times m$ identity matrix. Assume that

$$[X_{ij}, A_{k\ell}] - [X_{kj}, A_{i\ell}] = 0 \quad \textit{for all } i, j, k, \ell \tag{16}$$

and

$$[Y_{ij}, A_{k\ell}] - [Y_{i\ell}, A_{kj}] = 0 \quad \textit{for all } i, j, k, \ell. \tag{17}$$

Then

(a) *if X and Y are row-pseudo-commutative, then*

$$\begin{aligned}
&\sum_{\substack{L\subseteq[m]\\|L|=n}} \text{col-det}\, X_{[n],L}\, \text{col-det}\, Y_{L,[n]} \\
&\quad = \int \mathcal{D}(\psi,\bar{\psi}) \exp\Big(\sum_{k\ge 0} \frac{(\bar{\psi}A\psi)^k}{k+1} (\bar{\psi}XB^kY\psi) \Big) \\
&\quad = \int \mathcal{D}(\psi,\bar{\psi}) \exp\Big(-\bar{\psi}X \frac{\ln(1-(\bar{\psi}A\psi)B)}{(\bar{\psi}A\psi)B} Y\psi \Big);
\end{aligned} \tag{18}$$

(b) *if X and Y are column-pseudo-commutative, then*

$$\begin{aligned}
&\sum_{\substack{L\subseteq[m]\\|L|=n}} \text{row-det}\, X_{[n],L}\, \text{row-det}\, Y_{L,[n]} \\
&\quad = \int \mathcal{D}(\psi,\bar{\psi}) \exp\Big(\sum_{k\ge 0} (\bar{\psi}XB^kY\psi) \frac{(\bar{\psi}A\psi)^k}{k+1} \Big) \\
&\quad = \int \mathcal{D}(\psi,\bar{\psi}) \exp\Big(-\bar{\psi}X \frac{\ln(1-(\bar{\psi}A\psi)B)}{(\bar{\psi}A\psi)B} Y\psi \Big).
\end{aligned}$$

The commutation condition (16) in the hypotheses above is identical to the condition (12) in Proposition 1.2. Thus, as stated earlier, the following Lemmata 3.6 and 3.7 discuss mild conditions on B that would imply it.

However we are not aware of equally satisfactory conditions under which the hypothesis (17) holds. In particular, the hypothesis that Y is row-commutative would have rather suggested to interchange indices i and k in the second summand, instead of j and ℓ. A sufficient condition would be that Y is both row- and column-pseudo-commutative, i.e., that it is *tout-court* commutative, as in this situation the column-analogue of Lemmata 3.6 and 3.7 would apply (note, with the hypotheses of the lemmas now being on B^{T}). We are not aware of any set of matrices realizing the hypotheses of the proposition above and in which Y is not commutative, nor we have a proof that such a realization cannot exist (see the discussion at the end of Section 3).

We will prove Proposition 1.3 in Section 2. Then in Section 3 we recall some basic facts which were useful in our proof of Proposition 1.1, and will also be needed in the following. This section includes also a discussion on the conditions on the commutation of X and A. Section 4 is of combinatorial nature. It presents a lemma on the weighted enumeration of a family of lattice paths (of *Łukasiewicz type*), that is used later on in our proofs of Capelli-like identities. Section 5 presents the proof of Proposition 1.2, the non-commutative Cauchy–Binet formula in Quantum oscillator algebra representation. Section 6 presents a small variant of this formula, in which coherent states of the quantum oscillator are used. In Section 7 we derive a useful specialization of the Campbell–Baker–Hausdorff formula, which we use in Section 8 to give a proof of Proposition 1.4, the non-commutative Cauchy–Binet formula in Grassmann algebra representation. In Section 9 we give a short proof of Proposition 1.4, for the case $B = I$.

Acknowledgments. This work is a continuation of a previous work, done in collaboration with A. D. Sokal. As always in these cases, it is hard to "trace a boundary" on authorship. It is clear that the many discussions together had a prominent role in the genesis of the present paper.

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2. The bosonic oscillator and multilinear non-commutative functions

At the beginning of Section 1.2, we set some notations for the bosonic oscillator. Among other things, we fixed the normalization of the state $\mid 0\rangle$. There exists a residual freedom in choosing the relative norm of states $\mid n\rangle$, that we fix here, by setting for each $m, n \in \mathbb{N}$

$$(a^\dagger)^n \mid m\rangle = \mid m+n\rangle \tag{19a}$$

and

$$\langle m \mid a^n = \langle m+n \mid, \tag{19b}$$

from which it follows

$$a^n \mid m\rangle = \frac{m!}{(m-n)!} \mid m-n\rangle, \tag{20a}$$

and

$$\langle m \mid (a^\dagger)^n = \langle m-n \mid \frac{m!}{(m-n)!}, \tag{20b}$$

and

$$\langle n \mid m\rangle = n!\,\delta_{nm}.$$

As, for $m \in \mathbb{N}$, the states $|m\rangle$ form a complete set, we have

$$1 = \sum_{m\geq 0} \mid m\rangle \frac{1}{m!}\langle m \mid, \tag{21}$$

as operators acting on the Hilbert space.

In this section we prove Proposition 1.3. The two cases are analogous, and we study the 'row' case, that is we choose to prove identity (14). We shall in fact prove a more general result, for a family of multilinear non-commutative functions. Both results are statements on the fact that, taking scalar products, implement substitutional rules on suitable polynomials in the algebra of the quantum oscillator, in a way non dissimilar to the content of 'modern' umbral calculus *à la* Rota.

Proposition 2.1. *Let R be a ring, k, n and $\{m(i)\}_{1\leq i\leq n}$ integers, and $\{x_{ij}^{(h)}\}$ a collection of expressions in R, for $0 \leq h \leq k$, $i \in [n]$ and $j \in [m(i)]$. Consider also a Weyl–Heisenberg algebra as in (11), with operators commuting with the x's. Take $f(a)$ a formal power series in a, such that $f(0) = 1$ and $f'(0) \neq 0$, so that both $f(a)$ and $f'(a)$ are invertible. Consider a further indeterminate s, and let $g(a,s)$ be the formal power series in a and s defined as*

$$g(a,s) \overset{\text{def}}{=} s\Big[\frac{\partial}{\partial a} f(a)^{-s}\Big]^{-1} = -[f'(a)]^{-1} f(a)^{s+1}. \tag{22}$$

Then, introduce the operators

$$\chi_h(a,a^\dagger) \overset{\text{def}}{=} \frac{1}{h!}(a^\dagger g(a,s))^h f(a)^{-sh-1}.$$

Let

$$y_{ij} \overset{\text{def}}{=} \sum_{h=0}^{k} \binom{i-1}{h}_s x_{ij}^{(h)},$$

with

$$\binom{\ell}{h}_s \overset{\text{def}}{=} \frac{1}{h!}\ell(\ell-s)\dots(\ell-(h-1)s) = \begin{cases} s^h\dbinom{\ell/s}{h}, & s\neq 0;\\[2ex] \dfrac{\ell^h}{h!}, & s=0.\end{cases}$$

Define

$$z_{ij}(a,a^\dagger) \overset{\text{def}}{=} \sum_{h=0}^{k} \chi_h(a,a^\dagger) x_{ij}^{(h)}.$$

Then, for any polynomial ϕ *of the* $\mathcal{N}$ *variables* $\{y_{ij}\}$ *in the ring* R*, homogeneous of degree* n*, and with monomials of the form* $\prod_{i=1}^{n} y_{ij(i)}$ *(with the product in order),*[4] *the following representation holds*

$$\phi(\{y_{ij}\}) = \langle 0 \mid \phi(\{z_{ij}(a, a^{\dagger})\}) \mid 0\rangle. \tag{23}$$

We recognize the identity (14) as a special case, with $k = 1$, $x_{ij}^{(0)} = U_{ij}$, $x_{ij}^{(1)} = V_{ij}$, and $f(a) = 1 - a$. (Thus in particular, $\chi_0 = (1-a)^{-1}$ and $\chi_1 = a^{\dagger}$.) The polynomial ϕ is chosen to be $\phi(y) = \text{row-det}\, Y$, for Y the matrix with entries $y_{ij} = U_{ij} + (i-1)V_{ij}$. This correspondence is valid regardless of s, as s appears explicitly only for $k \geq 2$.

Towards the end of the proof of this theorem we will need a lemma in quantum oscillator algebra, which we prove immediately.

Lemma 2.2. *For any indeterminates* ℓ *and* s*,* $f(a)$ *and* $g(a,s)$ *as above, and any* h *and* m *in* $\mathbb{N}$*,*

$$\begin{aligned} C_{\ell,h,m} &\overset{\text{def}}{=} \frac{1}{h!}\langle 0 \mid f(a)^{-\ell}(a^{\dagger} g(a,s))^{h} f(a)^{\ell-hs} \mid m\rangle \\ &= \binom{\ell}{h}_{s} \delta_{m,0}. \end{aligned}$$

Proof. Indeed, if $h = 0$ we trivially have

$$C_{\ell,0,m} = \langle 0 \mid m\rangle = \delta_{m,0},$$

while if $h > 0$ we can write

$$\begin{aligned} C_{\ell,h,m} &= \frac{1}{h!}\langle 0 \mid f(a)^{-\ell} a^{\dagger} g(a,s)(a^{\dagger} g(a,s))^{h-1} f(a)^{\ell-hs} \mid m\rangle \\ &= \frac{1}{h!}\langle 0 \mid (a^{\dagger} f(a)^{-\ell} + [f(a)^{-\ell}, a^{\dagger}]) g(a,s)(a^{\dagger} g(a,s))^{h-1} f(a)^{\ell-hs} \mid m\rangle \\ &= \frac{\ell}{h!}\langle 0 \mid f(a)^{-\ell-1}(-f'(a) g(a,s))(a^{\dagger} g(a,s))^{h-1} f(a)^{\ell-hs} \mid m\rangle \\ &= \frac{\ell}{h!}\langle 0 \mid f(a)^{-(\ell-s)}(a^{\dagger} g(a,s))^{h-1} f(a)^{(\ell-s)-(h-1)s} \mid m\rangle \\ &= \frac{\ell}{h} C_{\ell-s,h-1,m}, \end{aligned}$$

where we used the fact that $\langle 0 \mid a^{\dagger} = 0$, and definition (22). So we get the result by induction in h. □

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[4] This means that ϕ is *multilinear* in each set $Y_i = \{y_{ij}\}_{j \in [m(i)]}$.

Proof of Proposition 2.1. A generic monomial of ϕ can be labeled by a vector $J = (j(1), \dots, j(n)) \in [m(1)] \times \cdots \times [m(n)]$, thus ϕ has the form

$$\phi(\{y_{ik}\}) = \sum_{J=(j(1),\dots,j(n))} c_J \prod_{i=1}^{n} y_{ij(i)}.$$

Both y_{ij}'s and $z_{ij}(a, a^\dagger)$'s are defined as a sum of $k+1$ terms. Perform the corresponding expansion on both sides of (23), and label each term by a vector $\mu \in \{0, \dots, k\}^n$. For the expression on the left hand side we have

$$\phi(\{y_{ik}\}) = \sum_{J,\mu} c_J \left(\prod_{i=1}^{n} \binom{i-1}{\mu(i)}_s \right) \prod_{i=1,\dots,n} x_{ij(i)}^{(\mu(i))},$$

while for the one on the right hand side we have

$$\langle 0 \mid \phi(\{z_{ik}(a, a^\dagger)\}) \mid 0 \rangle = \sum_{J,\mu} c_J \langle 0 \mid \prod_{i=1,\dots,n} \chi_{\mu(i)} \mid 0 \rangle \prod_{i=1,\dots,n} x_{ij(i)}^{(\mu(i))}.$$

As the $x_{ij}^{(h)}$ are arbitrary non-commuting indeterminates, and ϕ is arbitrary, the identity must hold separately for each summand labeled by a pair (J, μ), i.e. that for any vector μ we have to prove that

$$\prod_{\substack{\ell \in [n] \\ \mu(\ell) \neq 0}} \binom{\ell-1}{\mu(\ell)}_s = \langle 0 \mid \prod_{i=1,\dots,n} \chi_{\mu(i)} \mid 0 \rangle.$$

Let $(\ell_1, \dots, \ell_k)$ be the ordered list of indices i such that $\mu(i) \neq 0$, so that

$$\prod_{i=1,\dots,n} \chi_{\mu(i)} = \chi_0^{\ell_1-1} \chi_{\mu(\ell_1)} \chi_0^{\ell_2-\ell_1-1} \chi_{\mu(\ell_2)} \chi_0^{\ell_3-\ell_2-1} \chi_{\mu(\ell_3)} \cdots \chi_{\mu(\ell_k)} \chi_0^{n-\ell_k},$$

where all the powers are non-negative integers, and all $\mu(\ell_j)$'s are in the range $\{1, \dots, k\}$. The expression $\chi_0^{-1} = f(a)$ is defined as a formal power series, and we can write

$$\prod_{i=1,\dots,n} \chi_{\mu(i)} = \Big(\prod_{\alpha=1,\dots,k} \chi_0^{\ell_\alpha-1} \chi_{\mu(\ell_\alpha)} \chi_0^{-\ell_\alpha} \Big) \chi_0^n.$$

Let

$$\hat{O}_{\ell,h} \overset{\text{def}}{=} \chi_0^{\ell-1} \chi_h \chi_0^{-\ell}.$$

We need to prove that, for any k-uple $\ell_1 < \cdots < \ell_k$,

$$\prod_{\alpha=1}^{k} \binom{\ell_\alpha - 1}{\mu(\ell_\alpha)}_s = \langle 0 \mid \Big(\prod_{\alpha=1,\dots,k} \hat{O}_{\ell_\alpha, \mu(\ell_\alpha)} \Big) f(a)^{-n} \mid 0 \rangle.$$

First of all realize that $f(a)^{-n} \mid 0\rangle = \mid 0\rangle$. Then, because of Lemma 2.2,

$$\langle 0 \mid \widehat{O}_{\ell_1,\mu(\ell_1)} \mid m\rangle = \delta_{m,0}\binom{\ell_1 - 1}{\mu(\ell_1)}_s$$

so that by introducing a resolution of the identity, equation (21), we get a recursion in α

$$\begin{aligned}\langle 0 \mid \prod_{\alpha=1,\dots,k} \widehat{O}_{\ell_\alpha,\mu(\ell_\alpha)} \mid 0\rangle &= \sum_{m\geq 0}\langle 0 \mid \widehat{O}_{\ell_1,\mu(\ell_1)} \mid m\rangle \frac{1}{m!}\langle m \mid \prod_{\alpha=2,\dots,k} \widehat{O}_{\ell_\alpha,\mu(\ell_\alpha)} \mid 0\rangle \\ &= \sum_{m\geq 0}\frac{\delta_{m,0}}{m!}\binom{\ell_1 - 1}{\mu(\ell_1)}_s \langle m \mid \prod_{\alpha=2,\dots,k} \widehat{O}_{\ell_\alpha,\mu(\ell_\alpha)} \mid 0\rangle \\ &= \binom{\ell_1 - 1}{\mu(\ell_1)}_s \langle 0 \mid \prod_{\alpha=2,\dots,k} \widehat{O}_{\ell_\alpha,\mu(\ell_\alpha)} \mid 0\rangle,\end{aligned}$$

which proves the statement of the theorem. □

3. Some properties of commutators

Let us begin by recalling two elementary facts [1], Lemmata 2.1 and 2.2, that we used repeatedly and shall use in this paper.

Lemma 3.1 (Translation lemma). *Let $\mathcal{A}$ be an abelian group, and let $f : \mathcal{S}_n \to \mathcal{A}$. Then, for any $\tau \in \mathcal{S}_n$, we have*

$$\sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) f(\sigma) = \operatorname{sgn}(\tau) \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) f(\sigma \circ \tau).$$

Proof. Just note that both sides equal

$$\sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma \circ \tau) f(\sigma \circ \tau).$$ □

Lemma 3.2 (Involution lemma). *Let $\mathcal{A}$ be an abelian group, and let $f : \mathcal{S}_n \to \mathcal{A}$. Suppose that there exists a pair of distinct elements $i, j \in [n]$ such that*

$$f(\sigma) = f(\sigma \circ (ij))$$

for all $\sigma \in \mathcal{S}_n$ (where (ij) denotes the transposition interchanging i with j). Then

$$\sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) f(\sigma) = 0.$$

Proof. We have

$$\begin{aligned}\sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}(\sigma)f(\sigma) &= \sum_{\sigma\,:\,\sigma(i)<\sigma(j)}\operatorname{sgn}(\sigma)f(\sigma)+\sum_{\sigma\,:\,\sigma(i)>\sigma(j)}\operatorname{sgn}(\sigma)f(\sigma)\\ &= \sum_{\sigma\,:\,\sigma(i)<\sigma(j)}\operatorname{sgn}(\sigma)f(\sigma)-\sum_{\sigma'\,:\,\sigma'(i)<\sigma'(j)}\operatorname{sgn}(\sigma')f(\sigma'\circ(ij))\\ &= 0,\end{aligned}$$

where in the second line we made the change of variables $\sigma'=\sigma\circ(ij)$ and used $\operatorname{sgn}(\sigma')=-\operatorname{sgn}(\sigma)$ (or equivalently used the translation lemma). □

In the following we shall need of a less restrictive notion than the pseudo-commutative matrix. Let us begin by observing that

$$\mu_{ijkl}\overset{\text{def}}{=}[M_{ij},M_{kl}]$$

is manifestly antisymmetric under the simultaneous interchange $i\leftrightarrow k$, $j\leftrightarrow l$. So symmetry under one of these interchanges is equivalent to antisymmetry under the other. Let us therefore say that a matrix M has *row-symmetric* (and *column-antisymmetric*) *commutators* if

$$[M_{ij},M_{kl}]=[M_{kj},M_{il}]\quad\text{for all } i,j,k,l \tag{24}$$

and *column-symmetric* (and *row-antisymmetric*) *commutators* if

$$[M_{ij},M_{kl}]=[M_{il},M_{kj}]\quad\text{for all } i,j,k,l.$$

Then we shall need the following two lemmata.

Lemma 3.3. *For a n-dimensional matrix M with row-symmetric commutators, that is satisfying (24), any vector $(\ell_1,\dots,\ell_n)$, and any permutation $\pi\in\mathcal{S}_n$, we have*

$$\sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}(\sigma)\prod_{i=1}^{n}M_{\sigma(i)\,\ell_i}=\sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}(\sigma)\prod_{i=1}^{n}M_{\sigma\pi(i)\,\ell_{\pi(i)}}.$$

Proof. It suffices to prove the lemma for a single transposition of elements, consecutive after the permutation σ, namely $\pi=(\sigma(i)\,\sigma(i+1))$. We denote as L_σ and R_σ the factors on left and on the right (note that they do not depend from $\sigma(i)$ and $\sigma(i+1)$). We can write the statement as

$$\sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}(\sigma)L_\sigma M_{\sigma(i)\ell_i}M_{\sigma(i+1)\ell_{i+1}}R_\sigma=\sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}(\sigma)L_\sigma M_{\sigma(i+1)\ell_{i+1}}M_{\sigma(i)\ell_i}R_\sigma.$$

The difference of the two expressions is, by definition,

$$\sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) L_\sigma [M_{\sigma(i)\ell_i}, M_{\sigma(i+1)\ell_{i+1}}] R_\sigma$$

which vanishes because the hypothesis (24) allows the application of the involution lemma. □

Now we have a sequence of lemmas exploring the consequences of the Jacobi identity.

Lemma 3.4. *Let R be a ring, and let X and Y be matrices with elements in R.*

(a) *If X is row-pseudo-commutative then, at fixed Y_{ef}, for all a,b,c,d the antisymmetric part of $[X_{ab},[X_{cd},Y_{ef}]]$ in the exchange of a with c is symmetric in the exchange of b and d, that is*

$$[X_{ab},[X_{cb},Y_{ef}]] - [X_{cb},[X_{ab},Y_{ef}]] = 0 \tag{25}$$

and

$$\begin{aligned}&[X_{ab},[X_{cd},Y_{ef}]] - [X_{cb},[X_{ad},Y_{ef}]]\\ &\quad + [X_{ad},[X_{cb},Y_{ef}]] - [X_{cd},[X_{ab},Y_{ef}]] = 0;\end{aligned} \tag{26}$$

(b) *if Y is column-pseudo-commutative then, at fixed X_{ef}, for all a,b,c,d the antisymmetric part of $[Y_{ab},[Y_{cd},X_{ef}]]$ in the exchange of b with d is symmetric in the exchange of a and c, that is*

$$[Y_{ab},[Y_{ad},X_{ef}]] - [Y_{ad},[Y_{ab},X_{ef}]] = 0$$

and

$$\begin{aligned}&[Y_{ab},[Y_{cd},X_{ef}]] - [Y_{ad},[Y_{cb},X_{ef}]]\\ &\quad + [Y_{cb},[Y_{ad},X_{ef}]] - [Y_{cd},[Y_{ab},X_{ef}]] = 0.\end{aligned}$$

Proof. (a) Start from the Jacobi Identity applied to the triplet (X_{ab}, X_{cd}, Y_{ef}),

$$[X_{ab},[X_{cd},Y_{ef}]] + [Y_{ef},[X_{ab},X_{cd}]] + [X_{cd},[Y_{ef},X_{ab}]] = 0.$$

If we set $d = b$, as X is row-pseudo-commutative, $[X_{ab}, X_{cb}] = 0$ so that (25) follows. For (26), consider also the Jacobi identity for the triplet (X_{cb}, X_{ad}, Y_{ef}) to obtain

$$[X_{cb},[X_{ad},Y_{ef}]] + [Y_{ef},[X_{cb},X_{ad}]] + [X_{ad},[Y_{ef},X_{cb}]] = 0$$

so that, by subtraction and the hypothesis that X is row-pseudo-commutative then

$$[X_{ab},[X_{cd},Y_{ef}]] - [X_{cb},[X_{ad},Y_{ef}]] + [X_{ad},[X_{cb},Y_{ef}]] - [X_{cd},[X_{ab},Y_{ef}]] = 0.$$

The proof of (b) is similar. □

This lemma implies the following result.

Corollary 3.5. *If X and Y are as in case* (a) *of the lemma above, and furthermore they satisfy the commutation relation* (10)*, $[X_{ij}, Y_{k\ell}] = -A_{i\ell}B_{kj}$, then, for every a, b, c, e, and f,*

$$([X_{ab}, A_{cf}] - [X_{cb}, A_{af}])B_{eb} = 0 \tag{27}$$

and, for every a, b, c, d, e, and f,

$$([X_{ab}, A_{cf}] - [X_{cb}, A_{af}])B_{ed} + (b \leftrightarrow d) = 0. \tag{28}$$

We are now ready to state sufficient conditions on B, for having the commutation relation (12), $[X_{ij}, A_{k\ell}] - [X_{kj}, A_{i\ell}] = 0$.

Recall that, in a ring R, a nonzero element x is a *left zero divisor* if there exists a nonzero y such that $xy = 0$. Right zero divisors are analogously defined. A nonzero element of a ring that is not a left zero divisor is called *left-regular* (and analogously for right). Then we have the following lemma.

Lemma 3.6. *Let X, A and B as in Corollary 3.5, of sizes respectively $n \times m$, $n \times n$ and $m \times m$, and B_{ij} commuting with every other matrix element. Suppose that there exist an index $d \in [m]$, and a vector $\vec{u} \in R^m$, such that $(\vec{u}B)_d$ is left-regular. Then*

$$[X_{ij}, A_{k\ell}] - [X_{kj}, A_{i\ell}] = 0 \quad \textit{for all } i, j, k, \ell.$$

Proof. Equations (27) and (28) are valid with B written on the left or on the right, as it commutes with everything. Consider equation (27), with arbitrary a, c, f, setting $b = d$, and summing over e, after multiplying on the left by u_e. This gives

$$\Big(\sum_e u_e B_{ed}\Big)([X_{ad}, A_{cf}] - [X_{cd}, A_{af}]) = 0.$$

As $(\vec{u}B)_d$ is left-regular, we obtain $[X_{ad}, A_{cf}] - [X_{cd}, A_{af}] = 0$. Now consider any other index $b \neq d$, and equation (28), again summing over e, after multiplying on the left by u_e. We obtain

$$\Big(\sum_e u_e B_{ed}\Big)([X_{ab}, A_{cf}] - [X_{cb}, A_{af}])$$
$$= -\Big(\sum_e u_e B_{eb}\Big)([X_{ad}, A_{cf}] - [X_{cd}, A_{af}]).$$

As the right-most factor on the right hand side is zero, the whole right hand side vanishes. As the left-most factor on the left hand side is left-regular, we have that $[X_{ab}, A_{cf}] - [X_{cb}, A_{af}] = 0$, thus completing the proof. □

Furthermore, we can also state the following lemma.

Lemma 3.7. *Let X, A and B as in Corollary 3.5, of sizes respectively $n \times m$, $n \times n$ and $m \times m$, and B_{ij} commuting with every other matrix element. Suppose that there exist a vector $\vec{u} \in R^m$, and a vector $\vec{v} \in R^m$, with v_i's commuting with X, A and B elements and among themselves, such that the scalar product $2(\vec{u}, B\vec{v})$ is left-regular. Then*

$$[X_{ij}, A_{k\ell}] - [X_{kj}, A_{i\ell}] = 0 \quad \textit{for all } i, j, k, \ell.$$

Proof. Remark that, except for the annoying factor 2, this lemma is a generalization of Lemma 3.6, to which it (almost) reduces for $\vec{v}_i = \delta_{i,d}$.

Analogously to Lemma 3.6, consider equation (28), with arbitrary a, c, f, summing over e, b, d, after multiplying on the left by $u_e v_b v_d$. This gives

$$\Big(\sum_{e,d} u_e B_{ed} v_d\Big) \sum_b ([X_{ab} v_b, A_{cf}] - [X_{cb} v_b, A_{af}]) \\ + \Big(\sum_{e,b} u_e B_{eb} v_b\Big) \sum_d ([X_{ad} v_d, A_{cf}] - [X_{cd} v_d, A_{af}]) = 0.$$

Performing the sums shows that the two terms are identical. As $2(\vec{u}, B\vec{v})$ is left-regular, we obtain $[(X\vec{v})_a, A_{cf}] - [(X\vec{v})_c, A_{af}] = 0$. Now take any index b, and consider again equation (28), but summing only over e and d, after multiplying on the left by $u_e v_d$. We obtain

$$\Big(\sum_e u_e B_{ed} v_d\Big)([X_{ab}, A_{cf}] - [X_{cb}, A_{af}]) \\ = -\Big(\sum_e u_e B_{eb}\Big)([(X\vec{v})_a, A_{cf}] - [(X\vec{v})_c, A_{af}]).$$

As the right-most factor on the right hand side is zero, the whole right hand side vanishes. As the left-most factor on the left hand side is left-regular, we have that $[X_{ab}, A_{cf}] - [X_{cb}, A_{af}] = 0$, thus completing the proof. □

An analysis similar to the one of Corollary 3.5, performed on matrix Y assumed to be row-pseudo-commutative (remark that Lemma 3.4(a) exchanging X and Y is a valid starting point at this aim), gives

$$[Y_{ab}, A_{eb}]B_{cf} = [Y_{cb}, A_{eb}]B_{af}$$

and

$$[Y_{ab}, A_{ed}]B_{cf} + (b \leftrightarrow d) = [Y_{cb}, A_{ed}]B_{af} + (b \leftrightarrow d).$$

These equations are comparatively weaker w.r.t. equations (27) and (28), at the aim of establishing sufficient conditions on B for the hypothesis (17) in Proposition 1.4 to hold. Indeed, while in the previous case we have already the appropriate exchange structure, mixed to further exchanges, in this new case the exchange of indices has nothing in common with (17).

A simple sufficient condition is that Y is in fact commutative, $[Y_{ij}, Y_{k\ell}] = 0$ for all i, j, k, ℓ, as this would imply in particular that it is column-pseudo-commutative, and the validity follows from the cases (b) of the lemmas above. Another case leading to interesting simplifications is when B is the identity matrix, and $m \geq 2$. In this case, taking $f = c \neq a$ gives

$$[Y_{ab}, A_{eb}] = 0$$

and

$$[Y_{ab}, A_{ed}] + (b \leftrightarrow d) = 0.$$

Thus we see that, in this case, either the field has characteristic 2, or the only possibility for (17) to hold is that $[Y_{ij}, A_{k\ell}] = 0$ for all i, k and $j \neq \ell$.

4. A weighted enumeration of Łukasiewicz paths

Let n an integer. For $0 \leq t \leq n$, consider the “symbols”

$$\vec{\nu}_t = (\nu_1, \dots, \nu_t \mid \nu_{t+1}, \dots, \nu_n),$$

n-uples of integers with $\nu_i \geq -1$ for $i \leq t$ and $\nu_i \geq 0$ for $i > t$. These symbols are intended as formal indeterminates generating a linear space over $\mathbb{Z}$. Consider the quotient given by the relations

$$\begin{aligned}
&(\nu_1, \dots, \nu_{t-1} \mid \nu_t, \dots, \nu_n) \\
&\quad = (\nu_1, \dots, \nu_{t-1}, \nu_t \mid \nu_{t+1}, \dots, \nu_n) \\
&\qquad + \sum_{k=t+1}^{n} (\nu_1, \dots, \nu_{t-1}, -1 \mid \nu_{t+1}, \dots, \underbrace{\nu_k + \nu_t + 1}_{k\text{-th}}, \dots, \nu_n).
\end{aligned} \tag{29}$$

Remark that the sum $|\vec{\nu}_t| = \nu_1 + \cdots + \nu_n$, that we call the *norm* of the symbol, is homogeneous in all the terms of the relation, and that, if the left hand side of (29) satisfies the bounds above on the ν_i's, the bounds are satisfied also by all the summands on the right hand side.

Let us call *height* of $(\nu_1, \dots, \nu_t \mid \nu_{t+1}, \dots, \nu_n)$ the integer $H = \nu_{t+1} + \cdots + \nu_n$. Then the other combination $\nu_1 + \cdots + \nu_t$ is just the norm minus the height. We shall call t the *level* of $(\nu_1, \dots, \nu_t \mid \nu_{t+1}, \dots, \nu_n)$. We define $V_{t,s}$ as the space of all symbols with level t and norm s.

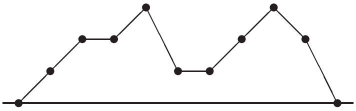
Consider any triplet (t, t', s) with $0 \leq t \leq t' \leq n$ and $s \geq -t$. Relation (29) can be seen as a recursion, allowing to write any symbol $\vec{\nu}_t \in V_{t,s}$ as a linear combination of symbols $\vec{\nu}'_{t'} \in V_{t',s}$. We will restrict our attention to the symbols with zero norm. For $t = 0$, we have a unique possible symbol in $V_{0,0}$, that is, $\vec{\nu}_0 = (\mid 0 \dots 0)$. As a consequence, and from the closure property above, for each $0 \leq t \leq n$ there exists a set of integers $c(\vec{\nu}_t)$ such that

$$\vec{\nu}_0 = \sum_{\vec{\nu}_t \in V_{t,0}} c(\vec{\nu}_t)\, \vec{\nu}_t. \tag{30}$$

In the following Lemma 4.1 we determine a formula for $c(\vec{\nu}_t)$, which is the main result of the section. Before going to the lemma, it is useful to introduce a graphical interpretation for these symbols.

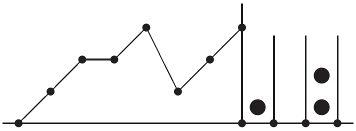
Symbols of maximal level, $\vec{\nu}_n = (\nu_1, \dots \nu_n \mid)$, are in bijection with paths γ on the half-line, that is, if represented as a 'time trajectory' in two dimensions, paths with height remaining always non-negative, starting at $(0, 0)$ and arriving at $(n, 0)$, and with steps of the form $(1, s)$. The bijection just consists in performing a jump of $-\nu_i$ at the i-th step. Thus, in our problem we have only steps $s \leq 1$. Paths with exactly this set of allowed steps are known as *Łukasiewicz paths* (see [35], p. 71, or [36], Example 3, p. 14). An example of symbol-path correspondence is

$(-1, -1, 0, -1, 2, 0, -1, -1, 1, 2 \mid)$


.

More generally, symbols of level t and height H are in bijection with pairs (γ, π), where γ is a path as above, terminating at (t, H), and π is a partition of H 'stones' into $n - t$ boxes (that we represent graphically as the columns with indices from $t + 1$ to n, following the path). For example

$(-1, -1, 0, -1, 2, -1, -1 \mid 1, 0, 2)$


.

Paths in one dimension can be described equivalently, either by the sequence of jumps $-\nu_i$, as above, or by the height profile $h_i = \sum_{j=1}^{i} (-\nu_j)$. Both notations will be useful in the following.

One easily sees that a necessary condition for $c(\vec{\nu}_t) \neq 0$ is that the corresponding path never goes below the horizontal axis. Indeed, the recursion is such that, if the left hand side of (29) has non-negative height H, then this is true also for all the summands on the right hand side. Another way of seeing this property is to realize that our graphical structures (γ, π) form a family which is stable under the recursion, and H, which is both the final height in the path and the number of stones, must remain always non-negative.

Our lemma states the following result.

Lemma 4.1. *For $\vec{\nu}_t = (\gamma, \pi)$, the function $c(\vec{\nu}_t)$ depends only on γ (and not on π), and is given by*

$$c(\gamma) = h_t! \prod_{\substack{i \in [t] \\ h_i \leq h_{i-1}}} \frac{h_{i-1}!}{h_i!}.$$

In particular, when $t = n$, the path must have $h_n = 0$ and therefore

$$c(\gamma) = \prod_{\substack{i \in [n] \\ h_i \leq h_{i-1}}} \frac{h_{i-1}!}{h_i!}. \tag{31}$$

Proof. Consider equation (29) to derive a recursion for the coefficients. For the symbol $\vec{\nu}_t = (\nu_1, \dots, \nu_t \mid \nu_{t+1}, \dots, \nu_n)$ we have

$$c(\vec{\nu}_t)$$
$$= \begin{cases} c((\nu_1, \dots, \nu_{t-1} \mid \nu_t, \dots, \nu_n)) & \text{if } \nu_t \geq 0; \\ \displaystyle\sum_{k=t+1}^{n} \sum_{\nu'=1}^{\nu_k} c((\nu_1, \dots, \nu_{t-1} \mid \nu' - 1, \nu_{t+1}, \dots, \underbrace{\nu_k - \nu'}_{k\text{-th}}, \dots, \nu_n)) & \text{if } \nu_t = -1. \end{cases}$$

We proceed by induction in t, starting from the trivial unique solution $c(\vec{\nu}_0)$ of (30) for $t = 0$. Assuming the formula for $c(\vec{\nu}_t)$ valid up to $t - 1$, we have

$$c(\vec{\nu}_t) = \begin{cases} h_{t-1}! \displaystyle\prod_{\substack{i \in [t-1] \\ h_i \leq h_{i-1}}} \frac{h_{i-1}!}{h_i!} & \text{if } \nu_t \geq 0; \\ h_{t-1}! \displaystyle\prod_{\substack{i \in [t-1] \\ h_i \leq h_{i-1}}} \frac{h_{i-1}!}{h_i!} \sum_{k=t+1}^{n} \sum_{\nu'=1}^{\nu_k} 1 & \text{if } \nu_t = -1. \end{cases}$$

In the case $\nu_t \geq 0$, we have $h_t \leq h_{t-1}$ and therefore

$$h_{t-1}! \prod_{\substack{i \in [t-1] \\ h_i \leq h_{i-1}}} \frac{h_{i-1}!}{h_i!} = h_t! \prod_{\substack{i \in [t] \\ h_i \leq h_{i-1}}} \frac{h_{i-1}!}{h_i!}$$

as required. If $\nu_t = -1$, remark that

$$\sum_{k=t+1}^{n} \sum_{\nu'=1}^{\nu_k} 1 = \sum_{k=t+1}^{n} \nu_k = h_t,$$

then, as $h_t = h_{t-1} + 1 > h_{t-1}$, we soon get that

$$h_t\, h_{t-1}! \prod_{\substack{i \in [t-1] \\ h_i \leq h_{i-1}}} \frac{h_{i-1}!}{h_i!} = h_t! \prod_{\substack{i \in [t-1] \\ h_i \leq h_{i-1}}} \frac{h_{i-1}!}{h_i!} = h_t! \prod_{\substack{i \in [t] \\ h_i \leq h_{i-1}}} \frac{h_{i-1}!}{h_i!},$$

which completes the proof. □

Now, for symbols of maximal level, $(\nu_1, \ldots \nu_n \mid)$, we give a representation in quantum oscillator algebra of the combinatorial formula for the coefficients $c(\vec{\nu}_t)$

Lemma 4.2. *For $\nu \geq -1$, define the operator in the Weyl–Heisenberg algebra*

$$\chi(\nu) = \begin{cases} (a^{\dagger})^{\nu} & \text{if } \nu \geq 0; \\ a & \text{if } \nu = -1. \end{cases}$$

Then, when the symbol $\vec{\nu}_n = (\nu_1, \ldots \nu_n \mid)$ corresponds to a path γ as described above,

$$\langle 0 \mid \chi(\nu_1) \ldots \chi(\nu_n) \mid 0 \rangle = c(\vec{\nu}_n) = \prod_{\substack{i \in [n] \\ h_i \leq h_{i-1}}} \frac{h_{i-1}!}{h_i!},$$

while otherwise

$$\langle 0 \mid \chi(\nu_1) \ldots \chi(\nu_n) \mid 0 \rangle = 0.$$

Proof. We proceed by induction. Assume that, for a sequence $\nu_1, \ldots, \nu_t$ such that the corresponding path remains positive,

$$\langle 0 \mid \chi(\nu_1) \ldots \chi(\nu_t) = \langle h_t \mid \prod_{\substack{i \in [t] \\ h_i \leq h_{i-1}}} \frac{h_{i-1}!}{h_i!}.$$

Then, we analyze the application of the operator $\chi(\nu_{t+1})$ to the right. If $\nu_{t+1} = -1$, because of (19), the application of a consistently brings $\langle h_t \mid$ to $\langle h_t + 1 \mid = \langle h_{t+1} \mid$. If $\nu_{t+1} \geq 0$, because of (20), the application of $(a^\dagger)^\nu$ brings $\langle h_t \mid$ to $\langle h_t - \nu \mid$, with an extra factor $h_t!/(h_t - \nu)!$ (which, in particular, is zero if the path goes below the horizontal axis). Taking finally the scalar product with $\mid 0\rangle$ ensures that the path ends at height zero. □

5. The Capelli identity in Weyl–Heisenberg algebra

We are now ready to prove Proposition 1.2.

Proof of Proposition 1.2. (a) As a first step, by simply using the fact that X is row-pseudo-commutative, in [1], Section 3, we get that

$$\sum_{\substack{L\subseteq[m]\\|L|=n}} \text{col-det}\, X_{[n],L}\ \text{col-det}\, Y_{L,[n]}$$
$$= \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) \sum_{l_1,\dots,l_n\in[m]} \Big(\prod_{i=1}^{n} X_{\sigma(i)l_i}\Big) \prod_{j=1}^{n} Y_{l_j\, j},$$

because only l_i's which are permutations in $\mathcal{S}_n$ have non-vanishing contribution in the sum. This remark would be already enough to set the Cauchy–Binet theorem in the simple case in which X commutes with Y [1], Proposition 3.1.

The second step of the proof comes from analyzing which terms do arise from commuting the factor $Y_{l_1 1}$ to the position between $X_{\sigma(1)l_1}$ and $X_{\sigma(2)l_2}$, and so on recursively, by using the general formula

$$x_1[x_2\dots x_r, y] = x_1 \sum_{s=2}^{r} x_2\dots x_{s-1}[x_s, y]x_{s+1}\dots x_r.$$

As an illustration, we consider the first application of this procedure:

$$\Big(\prod_{i=1}^{n} X_{\sigma(i)l_i}\Big)\Big(\prod_{j=1}^{n} Y_{l_j\, j}\Big)$$
$$= X_{\sigma(1)l_1} Y_{l_1 1} \Big(\prod_{i=2}^{n} X_{\sigma(i)l_i}\Big) \prod_{j=2}^{n} Y_{l_j\, j}$$
$$+ \sum_{k=2}^{n} \Big(\prod_{r=1}^{k-1} X_{\sigma(r)l_r}\Big)[X_{\sigma(k)l_k}, Y_{l_1 1}]\Big(\prod_{i=k+1}^{n} X_{\sigma(i)l_i}\Big) \prod_{j=2}^{n} Y_{l_j\, j}.$$

Then

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}(\sigma)\sum_{l_1,\dots,l_n\in[m]}\Big(\prod_{i=1}^{n}X_{\sigma(i)l_i}\Big)\prod_{j=1}^{n}Y_{l_j j}\\
&\quad=\sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}(\sigma)\Big[(XY)_{\sigma(1)1}\sum_{l_2,\dots,l_n\in[m]}\Big(\prod_{i=2}^{n}X_{\sigma(i)l_i}\Big)\prod_{j=2}^{n}Y_{l_j j}\\
&\qquad-\sum_{k=2}^{m}\sum_{l_1,\dots,l_n\in[m]}\Big(\prod_{r=1}^{k-1}X_{\sigma(r)l_r}\Big)A_{\sigma(k)1}B_{l_1l_k}\Big(\prod_{i=k+1}^{n}X_{\sigma(i)l_i}\Big)\prod_{j=2}^{n}Y_{l_j j}\Big].
\end{aligned}\tag{32}$$

Consider the summands for each k in the second row on the right hand side of (32). First of all, consider Lemma 3.3 applied to a matrix X', defined as $X'_{ij}=X_{ij}$ if $i\neq k$ and A_{ij} if $i=k$. We are in the hypothesis of the lemma because X is row-pseudo-commutative and satisfies the condition (12). One can then write those summands as

$$-\sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}(\sigma)\sum_{l_1,\dots,l_n\in[m]}A_{\sigma(k)1}\Big(\prod_{r=2}^{k-1}X_{\sigma(r)l_r}\Big)X_{\sigma(1)l_1}B_{l_1l_k}\Big(\prod_{i=k+1}^{n}X_{\sigma(i)l_i}\Big)\prod_{j=2}^{n}Y_{l_j j}.$$

Then, using the translation lemma for $\sigma\to\sigma\circ(1\,k)$, and performing the sum over l_1

$$+\sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}(\sigma)\sum_{l_2,\dots,l_n\in[m]}A_{\sigma(1)1}\Big(\prod_{r=2}^{k-1}X_{\sigma(r)l_r}\Big)(XB)_{\sigma(k)l_k}\Big(\prod_{i=k+1}^{n}X_{\sigma(i)l_i}\Big)\prod_{j=2}^{n}Y_{l_j j}.$$

When $B_{ij}=\delta_{ij}$ the product of matrices X becomes of the same form of the first term of the right hand side of (32). This procedure can be repeated iteratively and, ultimately, was enough to prove Proposition 1.1.

However, as the commutation of X's and Y's now produces extra matrices B, we have to deal with an induction expression of a more general form. One easily sees that, at all steps, matrices B will only act on X's from the right, so, in order to deal with the generic step t of the procedure (beside $t=1$ seen in detail above), we will consider expressions of the form

$$\sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}(\sigma)L(\sigma)\sum_{l_t\in[m]}\Big(\prod_{i=t}^{n}(XB^{\nu(i)})_{\sigma(i)l_i}\Big)\prod_{j=t}^{n}Y_{l_j j}$$

where $L(\sigma)$ depend only from $\sigma_1, \dots \sigma_{t-1}$ and $\nu(i)$ are non-negative integers. This form includes the initial situation at $t = 0$, and, as we see in a moment, is stable when t is increased. Indeed we have

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) L(\sigma) \sum_{l_t\in[m]} \Big(\prod_{i=t}^{n} (XB^{\nu(i)})_{\sigma(i) l_i} \Big) \prod_{j=t}^{n} Y_{l_j j} \\
&\quad = \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) L(\sigma) (XB^{\nu(t)} Y)_{\sigma(t) t} \Big(\prod_{i=t+1}^{n} (XB^{\nu(i)})_{\sigma(i) l_i} \Big) \prod_{j=t+1}^{n} Y_{l_j j} \\
&\qquad + \sum_{k=t+1}^{n} \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) L(\sigma) \sum_{l_t\in[m]} \Big(\prod_{r=t}^{k-1} (XB^{\nu(r)})_{\sigma(r) l_r} \Big) \\
&\qquad\quad \times A_{\sigma(k) t} (B^{\nu(k)+1})_{l_t l_k} \Big(\prod_{i=k+1}^{n} (XB^{\nu(i)})_{\sigma(i) l_i} \Big) \prod_{j=t+1}^{n} Y_{l_j j}.
\end{aligned}$$

In the last summands, we would like to commute the term $A_{\sigma(k)t}$ in front of all X's, as it carries the smallest column-index. This is indeed possible, at the light of Lemma 3.3. Consider this lemma applied to a matrix X', defined as

$$X'_{ij} = \begin{cases} (XB^{\nu(j)})_{ij} & \text{if } i \neq k; \\ A_{ij} & \text{if } i = k. \end{cases}$$

We are in the hypothesis of the lemma because X is row-pseudo-commutative and satisfies the condition (12), and therefore the same is true when replacing X with $XB^{\nu(j)}$ because $B^{\nu(j)}$ acts on the column indices. Then apply the involution lemma with $(t\ k)$, and sum over l_t where appropriate. We can thus write

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) L(\sigma) \sum_{l_t\in[m]} \Big(\prod_{i=t}^{n} (XB^{\nu(i)})_{\sigma(i) l_i} \Big) \prod_{j=t}^{n} Y_{l_j j} \\
&\quad = \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) L(\sigma)\, (XB^{\nu(t)} Y)_{\sigma(t) t} \Big(\prod_{i=t+1}^{n} (XB^{\nu(i)})_{\sigma(i) l_i} \Big) \prod_{j=t+1}^{n} Y_{l_j j} \\
&\qquad + \sum_{k=t+1}^{n} \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) L(\sigma)\, A_{\sigma(t) t} \Big(\prod_{r=t+1}^{k-1} (XB^{\nu(r)})_{\sigma(r) l_r} \Big) \\
&\qquad\quad \times (XB^{\nu(k)+\nu(t)+1})_{\sigma(k) k} \Big(\prod_{i=k+1}^{n} (XB^{\nu(i)})_{\sigma(i) l_i} \Big) \prod_{j=t+1}^{n} Y_{l_j j}.
\end{aligned} \tag{33}$$

The relevant point in this expression is that all of the $n-t+1$ summands are of the same form of the original left hand side, with one less matrix Y to be reordered. However, while in the simpler case $B_{ij}=\delta_{ij}$ the various terms were *identical* up to the prefactor, and could be collected together in a simple induction, here they differ in the set of exponents $\{\nu(i)\}$. Not accidentally, the combinatorics of these lists of exponents has already been discussed in Section 4. Indeed we can identify

$$\begin{aligned}
&(\nu_1,\dots,\nu_{t-1}\mid \nu_t,\dots,\nu_n)\\
&\quad\overset{\text{def}}{=} \sum_{\sigma\in\mathcal{S}_n} \mathrm{sgn}(\sigma)\Big(\prod_{i=1}^{t-1} M^{(\nu_i)}_{\sigma(i)i}\Big) \sum_{l_t,\dots,l_n\in[m]} \Big(\prod_{j=t}^{n}(XB^{\nu_j})_{\sigma(j)l_j}\Big)\prod_{r=t}^{n} Y_{l_r r},
\end{aligned}$$

where parameters ν_i have to be integers, and $\nu_i \geq -1$ for $i=1,\dots,t-1$, while $\nu_i\geq 0$ for $i=t,\dots,n$. The matrix elements $M^{(\nu_j)}_{ij}$ are A_{ij} if $\nu_j=-1$ and $(XB^{\nu_j}Y)_{ij}$ if ν_j is non-negative. In particular

$$\vec{\nu}_0 = (\,|\underbrace{0\dots0}_{n}) = \sum_{\sigma\in\mathcal{S}_n}\mathrm{sgn}(\sigma)\sum_{l_1,\dots,l_n\in[m]}\Big(\prod_{i=1}^{n} X_{\sigma(i)l_i}\Big)\prod_{j=1}^{n} Y_{l_j j}.$$

Our rule (33) coincides with (29) under this identification. We can apply Lemma 4.1 to get

$$\sum_{\substack{L\subseteq[m]\\|L|=n}} \text{col-det}\, X_{[n],L}\,\text{col-det}\, Y_{L,[n]} = \sum_{\gamma} c(\gamma)\sum_{\sigma\in\mathcal{S}_n}\mathrm{sgn}(\sigma)\Big(\prod_{i=1}^{n} M^{(\nu_i(\gamma))}_{\sigma(i)i}\Big),$$

where notations are as in Section 4, i.e. γ is a directed path in the upper half-plane starting from the origin, the heights $(h_0,\dots,h_{t-1})$, are given by $h_{i+1}-h_i=-\nu_i$, each ν_i is in the set $\{-1,0,1,2,\dots\}$, and the coefficients $c(\gamma)$ are given by (31).

Now we can use Lemma 4.2 to obtain

$$\begin{aligned}
&\sum_{\substack{L\subseteq[m]\\|L|=n}} \text{col-det}\, X_{[n],L}\,\text{col-det}\, Y_{L,[n]}\\
&\qquad= \sum_{\vec{\nu}_n}\langle 0\mid \sum_{\sigma\in\mathcal{S}_n}\mathrm{sgn}(\sigma)\Big(\prod_{i=1}^{n}\chi(\nu_i)\,M^{(\nu_i)}_{\sigma(i)i}\Big)\mid 0\rangle\\
&\qquad= \langle 0\mid \sum_{\sigma\in\mathcal{S}_n}\mathrm{sgn}(\sigma)\prod_{i=1}^{n}\Big(\sum_{\nu_i=-1}^{\infty}\chi(\nu_i)\,M^{(\nu_i)}\Big)_{\sigma(i)i}\mid 0\rangle\\
&\qquad= \langle 0\mid \text{col-det}\Big(\sum_{\nu=-1}^{\infty}\chi(\nu)\,M^{(\nu)}\Big)\mid 0\rangle,
\end{aligned}\tag{34}$$

but

$$\begin{aligned}\sum_{\nu=-1}^{\infty} \chi(\nu) M^{(\nu)} &= a M^{(-1)} + \sum_{\nu=0}^{\infty} (a^\dagger)^\nu M^{(\nu)} \\ &= aA + X \sum_{\nu=0}^{\infty} (a^\dagger B)^\nu Y \\ &= aA + X(I - a^\dagger B)^{-1} Y,\end{aligned}$$

so we got our thesis. □

6. Holomorphic representation

The results of Proposition 1.2 can also be expressed as a multiple integral in the complex plane, a structure that, within the language of the quantum oscillator, is called a *holomorphic representation*. We shall use the *coherent states* of the quantum oscillator, which are the states $\mid z\rangle$ defined as

$$\mid z\rangle \overset{\text{def}}{=} \exp(z a^\dagger) \mid 0\rangle$$

with $z \in \mathbb{C}$ a complex number. From the commutation relations (11) it soon follows the fundamental property of these states

$$a \mid z\rangle = z \mid z\rangle$$

that is, it is an eigenstate of the annihilation operator. And, of course

$$\langle z \mid \overset{\text{def}}{=} \langle 0 \mid \exp(\bar{z} a) \quad \text{and} \quad \langle z \mid a^\dagger = \langle z \mid \bar{z},$$

where $\bar{z}$ is the complex-conjugate of z. One easily verifies that two different coherent states are not orthogonal

$$\langle z \mid z' \rangle = \exp(\bar{z} z').$$

However, since coherent states obey a closure relation, any state can be decomposed on the set of coherent states. They hence form an overcomplete basis. This closure relation can be expressed by the resolution of the identity

$$\int \frac{dz\, d\bar{z}}{i\pi} \exp(-|z|^2) \mid z\rangle\langle z \mid = 1.$$

Let us consider the evaluation of

$$\langle 0 \mid (f_1(a^\dagger) + g_1(a)) \dots (f_n(a^\dagger) + g_n(a)) \mid 0\rangle,$$

where $\{f_\alpha, g_\alpha\}_{1\le\alpha\le n}$ are $2n$ generic expressions in a ring R, for which we have *a priori* no knowledge on the commutators.[5] We are ultimately interested in the case, corresponding to Proposition 1.2,

$$\begin{aligned}&\langle 0 \mid \text{col-det}(F(a^\dagger) + G(a)) \mid 0\rangle\\ &\quad = \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma)\langle 0 \mid \prod_{j=1}^{n} (F_{\sigma(j)j}(a^\dagger) + G_{\sigma(j)j}(a)) \mid 0\rangle\end{aligned}$$

(the product is ordered), with

$$F(a^\dagger) = X(1 - a^\dagger B)^{-1} Y \quad \text{and} \quad G(a) = aA.$$

Let $z_0 = z_n = 0$, and introduce $n-1$ intermediate coherent states, with parameters $z_1, \dots, z_{n-1}$, to get (with no more need of ordered products on the right hand side)

$$\begin{aligned}&\langle 0 \mid \prod_{j=1}^{n} (f_j(a^\dagger) + g_j(a)) \mid 0\rangle\\ &\quad = \int \prod_{j=1}^{n-1} \frac{dz_j\, d\bar{z}_j}{i\pi} e^{-|z_j|^2} \prod_{j=1}^{n} \langle z_{j-1} \mid f_j(a^\dagger) + g_j(a) \mid z_j\rangle.\end{aligned}$$

Each scalar product is easily evaluated according to

$$\langle u \mid f(a^\dagger) + g(a) \mid v\rangle = (f(\bar{u}) + g(v))e^{\bar{u}v},$$

so that

$$\begin{aligned}&\langle 0 \mid \prod_{j=1}^{n} (f_j(a^\dagger) + g_j(a)) \mid 0\rangle\\ &\quad = \int \prod_{j=1}^{n-1} \frac{dz_j d\bar{z}_j}{i\pi}\, e^{-\sum_{j=1}^{n} \bar{z}_j(z_j - z_{j+1})} \prod_{j=1}^{n} (f_j(\bar{z}_{j-1}) + g_j(z_j)),\end{aligned}$$

---

[5] We mean here that, for $f(a^\dagger) = \sum_i (a^\dagger)^i f_i$, $g(a) = \sum_j a^j g_j$, with f_i's and g_j's in a *commutative* ring, $[f(a^\dagger), g(a)] = \sum_{i,j} f_i g_j [(a^\dagger)^i, a^j]$, and the commutators are known, although complicated in general. However, if the coefficients f_i's and g_j's are valued in a generic *non-commutative* ring, even if commuting with the Weyl–Heisenberg algebra, we have unknown extra terms of type $[f_i, g_j]$, namely: $[f(a^\dagger), g(a)] = \sum_{i,j} (g_j f_i [(a^\dagger)^i, a^j] + (a^\dagger)^i a^j [f_i, g_j])$.

and in particular

$$\langle 0 \mid \text{col-det}(aA + X(1-a^\dagger B)^{-1}Y) \mid 0\rangle$$

$$= \int \prod_{j=1}^{n-1} \frac{dz_j\, d\bar{z}_j}{i\pi}\, e^{-\sum_{j=1}^{n} \bar{z}_j(z_j - z_{j+1})}\, \text{col-det}\, M(z)$$

and

$$M_{ij}(z) = A_{ij} z_j + (X(1-\bar{z}_{j-1}B)^{-1}Y)_{ij}.$$

The equation above, jointly with Proposition 1.2, provides a representation of the non-commutative Cauchy–Binet expression in terms of an integral over n (commuting) complex variables. This result is somewhat implicit in Proposition 1.2, and the standard general facts on the holomorphic representation of the quantum oscillator.

Let us however observe that, in Section 5, we could have derived *directly* the holomorphic representation, from the Cauchy–Binet left hand side, instead of the representation in terms of creation and annihilation operators. We only need to follow a different track at the very final step of the proof, where, in equation (34), we use the combinatorial Lemma 4.2.

The equivalent lemma for coherent states is based on the formula[6]

$$\int \frac{dz\, d\bar{z}}{i\pi}\, z^p \bar{z}^q \exp(-\bar{z}(z-\eta)) = \frac{p!}{(p-q)!}\, \eta^{p-q}, \tag{35}$$

and reads (using notations as described in Section 4 for paths γ, symbols $\vec{\nu}_n$, coefficients $c(\vec{\nu}_n)$, and conversion between ν_i's and h_i's)

Lemma 6.1. *For $\nu \geq -1$, define the monomials*

$$\chi_i(\nu) = \begin{cases} \bar{z}_{i-1}^{\nu}, & \nu \geq 0; \\ z_i, & \nu = -1. \end{cases}$$

---

[6] Which is easily proven, e.g. in generating function,

$$\sum_{p,q} \frac{\zeta^p \xi^q}{p!q!} \int \frac{dz\, d\bar{z}}{i\pi} z^p \bar{z}^q \exp(-\bar{z}(z-\eta))$$

$$= \int \frac{dz\, d\bar{z}}{i\pi} \exp(-\bar{z}(z-\eta) + \bar{z}\xi + \zeta z) = \exp(\zeta(\eta+\xi)),$$

while

$$\sum_{p,q} \frac{\zeta^p \xi^q}{p!q!} \frac{p!}{(p-q)!}\, \eta^{p-q} = \sum_{p,q} \frac{\zeta^p}{p!} (\eta+\xi)^p = \exp(\zeta(\eta+\xi)).$$

Then, when the symbol $\vec{\nu}_n = (\nu_1, \dots \nu_n|)$ *corresponds to a path* γ, *setting* $z_0 = z_n = 0$,

$$\int \prod_{j=1}^{n-1} \frac{dz_j\, d\bar{z}_j}{i\pi} e^{-\sum_{j=1}^{n-1} \bar{z}_j (z_j - z_{j+1})} \chi_1(\nu_1) \dots \chi_n(\nu_n) = c(\vec{\nu}_n) = \prod_{\substack{i \in [n] \\ h_i \le h_{i-1}}} \frac{h_{i-1}!}{h_i!},$$

while otherwise the integral above is zero.

Proof. We try to follow as closely as possible the reasoning in the proof of Lemma 4.2. We proceed by induction. Assume that, for a sequence $\nu_1, \dots, \nu_t$ such that the corresponding path remains positive,

$$\int \prod_{j=1}^{t-1} \frac{dz_j d\bar{z}_j}{i\pi} e^{-\sum_{j=1}^{t-1} \bar{z}_j (z_j - z_{j+1})} \chi_1(\nu_1) \dots \chi_t(\nu_t) = z_t^{h_t} \prod_{\substack{i \in [t] \\ h_i \le h_{i-1}}} \frac{h_{i-1}!}{h_i!}.$$

This is indeed the case for $t = 0$ (where, as customary for products over empty sets, we have $1 = 1$), and in the more convincing case $t = 1$ (where we have no integrations to perform, and, as $z_0 = 0$, $\chi_1(\nu_1) = z_1$, 1 and 0 respectively if $\nu_1 = -1, 0$, or strictly positive).

Then, we analyze the consequence of increasing t on both sides of the equation. On the left hand side, we should multiply by $e^{-\bar{z}_t (z_t - z_{t+1})} \chi_{t+1}(\nu_{t+1})$, and then integrate over $dz_t\, d\bar{z}_t$. If $\nu_{t+1} = -1$, $\chi_{t+1}(\nu_{t+1}) = z_{t+1}$ and $h_{t+1} = h_t + 1$, while if $\nu_{t+1} \ge 0$, $\chi_{t+1}(\nu_{t+1}) = \bar{z}_t^{\nu_{t+1}}$ and $h_{t+1} = h_t - \nu_{t+1}$. In both cases, the integral is of the form (35), and we get

$$\int \frac{dz_t d\bar{z}_t}{i\pi} e^{-\bar{z}_t (z_t - z_{t+1})} z_t^{h_t} z_{t+1} = z_{t+1}^{h_t + 1} = z_{t+1}^{h_{t+1}}$$

and

$$\int \frac{dz_t d\bar{z}_t}{i\pi} e^{-\bar{z}_t (z_t - z_{t+1})} z_t^{h_t} \bar{z}_t^{\nu_{t+1}} = \frac{h_t!}{(h_t - \nu_{t+1})!} z_{t+1}^{h_t - \nu_{t+1}} = \frac{h_t!}{h_{t+1}!} z_{t+1}^{h_{t+1}}.$$

In the two cases, the integration produces the appropriate relative factor, which, in particular, is zero if the path goes below the horizontal axis (because of a $1/k!$ factor, with $k < 0$). At the last step, we remain with a factor $z_n^{h_n}$. As $z_n = 0$, we select only the paths terminating at height zero. ☐

7. A lemma on the Campbell–Baker–Hausdorff formula

The goal of this section is to prove the following relation, which is a preparatory lemma to our Capelli identity in Grassmann representation, proven in the next section.

Proposition 7.1. *Let a and $a^\dagger$ be the generators of a Weyl–Heisenberg algebra, i.e. $[a, a^\dagger] = 1$, and $f(x)$ a formal power series. Then, at the level of formal power series, we have*

$$\begin{aligned} \exp(a^\dagger + f(a)) &= \exp(a^\dagger) \exp\Big(\sum_{k \geq 0} \frac{1}{(k+1)!} (\partial^k f)(a) \Big) \\ &= \exp(a^\dagger) \exp\Big(\frac{\exp(\partial) - 1}{\partial} f(a) \Big). \end{aligned} \tag{36}$$

The proposition above is a special case of the *Campbell–Baker–Hausdorff* (CBH) formula [37], [38], [39], and [40]. We give here a proof that makes use only of the existence of a CBH formula (and not the explicit expressions known in the literature). Furthermore, an additional argument provides a slightly longer variant, which instead is completely self-contained.

We recall that, given two elements x and y in a non-commutative ring, the Campbell–Baker–Hausdorff formula is an expression for $\ln(\exp(x)\exp(y))$ as a formal infinite sum of elements of the Lie algebra generated by x and y:

$$\exp(x)\exp(y) = \exp(x + y + z) \quad \text{and} \quad z = S(x, y).$$

The first few terms read

$$S(y; x) = \frac{1}{2}[x, y] + \frac{1}{12}[x - y, [x, y]] + \cdots,$$

and the generic summand in this series has the form

$$[z_{s(1)}, [z_{s(2)}, \dots [z_{s(k-1)}, z_{s(k)}] \dots]]$$

for some integer $k \geq 2$, $(s(1), \dots, s(k)) \in \{0, 1\}^k$, and the identification $z_0 = x$, $z_1 = y$. Of course, terms with $s(k) = s(k-1)$ vanish in any Lie algebra, and many other strings are redundant, e.g., besides the trivial $[\dots, [x, y] \dots] = -[\dots, [y, x] \dots]$, a first non-trivial relation is $[x, [y, [x, y]]] = [y, [x, [x, y]]]$.

The existence statement is relatively easy to obtain. The full expression at all orders with coefficients in closed form is complicated, but redundant forms (in the sense above) are well-known in the literature; see e.g. [32], pp. 134 and 135.

Formal inversion (that is, solving with respect to y, leaving z as an indeterminate) is easily achieved. Define the inverse problem as

$$\exp(x+z) = \exp(x)\exp(z+y) \quad \text{and} \quad y = \widetilde{S}(x;z); \tag{37}$$

then, multiplying both sides by e^{-x} from the left, one obtains

$$\widetilde{S}(x;z) = S(-x, x+z).$$

The existence result for $\widetilde{S}$ follows from existence for S and the relation above.

Proof of Proposition 7.1. Our proposition corresponds to the solution of the inverse problem (37), finding an expression for $\widetilde{S}(x;z)$, in the special case of $x = a^\dagger$ and $z = f(a)$.

In this case many commutators vanish. We have

$$[\underbrace{a^\dagger, [a^\dagger, \ldots [a^\dagger}_{k}, f(a)] \ldots]] = (-\partial)^k f(a)$$

where $\partial^k f$ denotes the k-th derivative of f (as a power series). So, all the expressions above do commute with $f(a)$ and we see that in our case all non-vanishing strings are the ones of the form $(0, 0, \ldots, 0, 1)$ (the ones $(0, 0, \ldots, 0, 1, 0)$ are also non-vanishing but clearly redundant). In other terms, writing for a generic Lie algebra

$$\widetilde{S}(x;z) = \sum_{k \geq 1} c_k \, [\underbrace{x, [x, \ldots [x}_{k}, z] \ldots]] \; + \mathcal{O}(z^2)$$

(where $\mathcal{O}(\cdot)$ is in the sense of polynomials in the enveloping algebra), we get in our case

$$\widetilde{S}(a^\dagger; f(a)) = \sum_{k \geq 1} c_k \, [\underbrace{a^\dagger, [a^\dagger, \ldots [a^\dagger}_{k}, f(a)] \ldots]] = \sum_{k} c_k \, (-\partial)^k f(a). \tag{38}$$

Observe that, again in the enveloping algebra,

$$[\underbrace{x, [x, \ldots [x}_{k}, z] \ldots]] = \sum_{h=0}^{k} (-1)^h \binom{k}{h} x^{k-h} z x^h$$

and that

$$\exp(x+z) = \exp(x) + \sum_{k \geq 0} \sum_{h=0}^{k} \frac{1}{(k+1)!} x^{k-h} z x^h + \mathcal{O}(z^2).$$

Appealing to the existence of a solution, we can determine the c_k's by matching the coefficient of zx^k on the two sides of (37), using (38) and (7), obtaining

$$c_k = \frac{1}{(k+1)!}$$

that, with the fact $\sum_{k\geq 0} x^k/(k+1)! = (e^x - 1)/x$ (used here at the level of formal power series), gives our statement.

Avoiding to appeal to the existence statement requires to match all possible other linear monomials, of the kind $x^h z x^{k-h}$. Then, the consistency of the assignment of c_k's boils down to the following relation: for each k and h positive integers,

$$\sum_{i=0}^{h} (-1)^{h-i} \binom{k+1}{i} \binom{k-i}{h-i} = 1.$$

This is proven by observing that

$$\binom{k-i}{h-i} = (-1)^{h-i} \binom{h-k-1}{h-i},$$

and, using Chu–Vandermonde convolution,

$$\sum_i \binom{n}{i} \binom{m}{k-i} = \binom{n+m}{k}. \qquad \square$$

If instead of $a^\dagger$ we have $ca^\dagger$, with c some commuting quantity, the same reasoning can be done, and a simple scaling applies to all formulas. The corresponding generalization of (36) is

$$\begin{aligned} \exp\Big(ca^\dagger + f(a)\Big) &= \exp(c\,a^\dagger) \exp\Big(\sum_{k\geq 0} \frac{c^k}{(k+1)!} (\partial^k f)(a)\Big) \\ &= \exp(ca^\dagger) \exp\Big(\frac{\exp(c\partial) - 1}{c\partial} f(a)\Big). \end{aligned} \tag{39}$$

We shall need also the identity obtained by Hermitian conjugation

$$\begin{aligned} \exp\Big(ca + f(a^\dagger)\Big) &= \exp\Big(\sum_{k\geq 0} \frac{c^k}{(k+1)!} (\partial^k f)(a^\dagger)\Big) \exp(ca) \\ &= \exp\Big(\frac{\exp(c\partial) - 1}{c\,\partial} f(a^\dagger)\Big) \exp(ca). \end{aligned} \tag{40}$$

8. The Capelli identity in Grassmann algebra

Besides column- and row-determinants, defined in (3) and (4) respectively, another possible non-commutative generalization of the determinant is the *symmetric-determinant*:

$$\text{sym-det}\, M \overset{\text{def}}{=} \frac{1}{n!} \sum_{\sigma,\tau\in\mathcal{S}} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \prod_{i=1}^{n} M_{\sigma(i)\tau(i)}. \tag{41}$$

In contrast to the cases of the column- and row-determinant, the definition (41) demands in general the inclusion of rational numbers in the field K over which the ring R is defined.

For any permutation $\tau \in \mathcal{S}_n$ let us denote M^τ the matrix with entries $(M^\tau)_{ij} = M_{i\tau(j)}$, and ${}^\tau M$ the matrix with entries $({}^\tau M)_{ij} = M_{\tau(i)j}$. We clearly have, for any matrix M,

$$\text{col-det}\, {}^\tau M = \operatorname{sgn}(\tau) \,\text{col-det}\, M \quad \text{and} \quad \text{row-det}\, M^\tau = \operatorname{sgn}(\tau)\, \text{row-det}\, M,$$

while in general the action of the symmetric group on columns and rows, respectively for the two cases, is not simple.

Indeed, the symmetric-determinant reads

$$\text{sym-det}\, M = \frac{1}{n!} \sum_{\tau\in\mathcal{S}_n} \operatorname{sgn}(\tau)\, \text{col-det}\, M^\tau = \frac{1}{n!} \sum_{\tau\in\mathcal{S}_n} \operatorname{sgn}(\tau)\, \text{row-det}\, {}^\tau M \tag{42}$$

and no relevant further simplifications are possible in general.

However, for a n-dimensional matrix M with weakly row-symmetric commutators, (and thus in particular if M is row-pseudo-commutative), in [1], Lemma 2.6(a), we proved that *both* actions of the symmetric group are simple, i.e. also

$$\text{col-det}\, M^\tau = \operatorname{sgn}(\tau)\, \text{col-det}\, M \tag{43}$$

(and similarly for the row-determinant, if M has weakly column-symmetric commutators), and therefore for such a matrix the expression (42) simplifies (in particular, rationals are not necessary)

Corollary 8.1. *For a n-dimensional matrix M with weakly row-symmetric commutators*

$$\text{sym-det}\, M = \text{col-det}\, M.$$

Our interest in the symmetric-determinant follows from the remark that it provides the generalization of the Berezin integral representation (15) for the determinant of a matrix with commuting elements. Indeed, for M a $n\times n$ matrix with elements in a non-commutative ring R, if R contain the rationals (or M is row-pseudo-commutative), and $\{\bar{\psi}_i, \psi_i\}_{i\in[n]}$ a set of $2n$ Grassmann variables commuting with the entries M_{ij}, we have

$$\int \mathcal{D}(\psi, \bar{\psi}) \exp(\bar{\psi} M \psi) = \text{sym-det}\, M. \tag{44}$$

Comparatively, the Grassmann formulas for the column- and row-determinant are more cumbersome, as they require an ordering of the n factors

$$\int d\psi_n \dots d\psi_1\ (\psi M)_1 \dots (\psi M)_n = \text{col-det}\, M \tag{45}$$

and

$$\int d\psi_n \dots d\psi_1 (M\psi)_1 \dots (M\psi)_n = \text{row-det}\, M.$$

Grassmann indeterminates present the advantage of encoding our commutation relations in a simple way. For example, we have the following result.

Lemma 8.2. *Let R be a ring, and A a $n\times n$ matrix with elements in R. Let the $\{\psi_i\}_{i\in[n]}$ be nilpotent Grassmann indeterminates, that is $\psi_i^2 = 0$ and their anti-commutators $\{\psi_i, \psi_j\} = 0$ vanish.*

(a) *Let X be a $n \times m$ matrix with elements in R such that*

$$[X_{ij}, A_{k\ell}] - [X_{kj}, A_{i\ell}] = 0 \quad \textit{for all } i, j, k, \ell. \tag{46}$$

Then

$$\{(\psi X)_j, (\psi A)_\ell\} \overset{\text{def}}{=} \sum_{i\in[n]} \sum_{k\in[n]} \{\psi_i X_{ij}, \psi_k A_{k\ell}\} = 0.$$

(b) *Let Y be a $m \times n$ matrix with elements in R such that*

$$[Y_{ij}, A_{k\ell}] - [Y_{i\ell}, A_{kj}] = 0 \quad \textit{for all } i, j, k, \ell.$$

Then

$$\{(Y\psi)_i, (A\psi)_k\} \overset{\text{def}}{=} \sum_{j\in[n]} \sum_{\ell\in[n]} \{Y_{ij}\psi_j, A_{k\ell}\psi_\ell\} = 0.$$

Proof. (a) We have that

$$\begin{aligned}
\{(\psi X)_j, (\psi A)_\ell\} &= \sum_{i,k\in[n]} (\psi_i X_{ij} \psi_k A_{k\ell} + \psi_k A_{k\ell} \psi_i X_{ij}) \\
&= \sum_{i,k\in[n]} \psi_i \psi_k [X_{ij}, A_{k\ell}] \\
&= \sum_{1\le i<k\le n} \psi_i \psi_k ([X_{ij}, A_{k\ell}] - [X_{kj}, A_{i\ell}]),
\end{aligned}$$

where we have taken into account that $i \neq k$ because the ψ's are nilpotent and we have put together the terms in which both ψ_i and ψ_k appears. But now each term in the sum vanish by the hypothesis (46). The case (b) is identical. ☐

This result is used to prove the following lemma.

Lemma 8.3. *Let R be a ring, and X a $n \times m$, Y a $m \times n$, A a $n \times n$ and B a $m \times m$ matrix with elements in R. Let the $\{\bar{\psi}_i, \psi_i\}_{i \in [n]}$ be nilpotent Grassmann indeterminates commuting with R, that is $\bar{\psi}_i^2 = \psi_i^2 = 0$ and their anti-commutators $\{\bar{\psi}_i, \bar{\psi}_j\} = \{\bar{\psi}_i, \psi_j\} = \{\psi_i, \psi_j\} = 0$ vanish. If*

$$[X_{ij}, A_{k\ell}] - [X_{kj}, A_{i\ell}] = [Y_{ij}, A_{k\ell}] - [Y_{i\ell}, A_{kj}] = 0 \quad \text{for all } i, j, k, \ell$$

and the elements of B commute with the ones of A, then for each integer s

$$[\bar{\psi} X B^s Y \psi, \bar{\psi} A \psi] = 0.$$

Proof. Indeed, as B_{ij}'s and $A_{k\ell}$'s do commute, we can write the commutator as

$$\begin{aligned} &[\bar{\psi} X B^s Y \psi, \bar{\psi} A \psi] \\ &\quad = \sum_{r \in [n]} (\bar{\psi} X B^s)_r [(Y\psi)_r, \bar{\psi} A \psi] + [(\bar{\psi} X)_r, \bar{\psi} A \psi]\, (B^s Y \psi)_r . \end{aligned}$$

Consider separately each of the resulting commutators:

$$[(\bar{\psi} X)_r, \bar{\psi} A \psi] = \sum_{k \in [n]} \{(\bar{\psi} X)_r, (\bar{\psi} A)_k\}\, \psi_k = 0 \tag{47a}$$

and

$$[(Y\psi)_r, \bar{\psi} A \psi] = \sum_{k \in [n]} \bar{\psi}_k \{(Y\psi)_r, (A\psi)_k\} = 0, \tag{47b}$$

where we used Lemma 8.2. ☐

We have now all the ingredients to prove Proposition 1.4.

Proof of Proposition 1.4. (a) As Y is row-pseudo-commutative, and we assumed that our ring contains the rationals, using (43), we can rewrite the left hand side of (18) as

$$\begin{aligned} &\sum_{\substack{L \subseteq [m] \\ |L| = n}} \text{col-det}\, X_{[n],L}\, \text{col-det}\, Y_{L,[n]} \\ &\quad = \frac{1}{n!} \sum_{\tau \in \mathcal{S}_n} \text{sgn}(\tau) \sum_{\substack{L \subseteq [m] \\ |L| = n}} \text{col-det}\, X_{[n],L}\, \text{col-det}(Y^{\tau})_{L,[n]} . \end{aligned} \tag{48}$$

From the hypotheses we soon have that, for any permutation $\tau \in \mathcal{S}_n$, the matrices X, Y^τ, A^τ, B satisfy the hypothesis of Proposition 1.2(a) and, therefore, as X is row-pseudo-commutative, we have that

$$\sum_{\substack{L \subseteq [m] \\ |L|=n}} \text{col-det}\, X_{[n],L}\, \text{col-det}(Y^\tau)_{L,[n]} = \langle 0 \mid \text{col-det}(aA^\tau + X(1-a^\dagger B)^{-1} Y^\tau) \mid 0 \rangle .$$

Note that, on the right hand side, the permutation τ has exactly the action from the right on the matrix $M = aA + X(1-a^\dagger B)^{-1} Y$. Thus, the combination in (48) corresponds to the definition (42) of the symmetric-determinant,

$$\frac{1}{n!} \sum_{\tau \in \mathcal{S}_n} \text{sgn}(\tau)\, \text{col-det}(aA^\tau + X(1-a^\dagger B)^{-1} Y^\tau) = \text{sym-det}(aA + X(1-a^\dagger B)^{-1} Y).$$

We can use the Grassmann representation, (44), for the expression above, to conclude that

$$\sum_{\substack{L \subseteq [m] \\ |L|=n}} \text{col-det}\, X_{[n],L}\, \text{col-det}\, Y_{L,[n]} \\ = \int \mathcal{D}(\psi, \bar{\psi}) \langle 0 \mid \exp(\bar{\psi} A \psi\, a + \bar{\psi} X (1-a^\dagger B)^{-1} Y \psi) \mid 0 \rangle .$$

Now we use the result in (40) by posing $c = \bar{\psi} A \psi$ and $f(a^\dagger) = \bar{\psi} X (1-a^\dagger B)^{-1} Y \psi$. Using the hypotheses (16) and (17) of the proposition, we can verify the hypothesis of Lemma 8.3, therefore our quantities c and $f(a^\dagger)$ commute (as required for (40) to apply), and we get

$$\exp(\bar{\psi} A \psi\, a + \bar{\psi} X (1-a^\dagger B)^{-1} Y \psi) = \exp(g(a^\dagger)) \exp(\bar{\psi} A \psi\, a) \tag{49}$$

with $g(a^\dagger)$ determined according to (40),[7]

$$g(a^\dagger) = \sum_{k \geq 0} \frac{(\bar{\psi} A \psi)^k}{k+1} (\bar{\psi} X B^k (1-a^\dagger B)^{-k-1} Y \psi).$$

Note that, in the sum, k cannot become larger that $n-1$, because of the nilpotency of the Grassmann indeterminates.

---

[7] Note at this aim that, if $[M_{ij}, M_{k\ell}] = 0$, $\frac{\partial}{\partial \xi} (\vec{u}, (I - \xi M)^{-s} \vec{v}) = s(\vec{u} M, (I - \xi M)^{-s-1} \vec{v})$.

In (49) the creation and annihilation operators are ordered into a polynomial with monomials of the form $(a^\dagger)^k a^h$ (i.e., they are *antinormal–*, or *anti-Wick–ordered*), and the whole expression is drastically simplified because

$$\exp(a\bar{\psi}A\psi) \mid 0\rangle = \mid 0\rangle$$

and

$$\langle 0 \mid \exp(g(a^\dagger)) = \langle 0 \mid \exp(g(0)) = \langle 0 \mid \exp\Big(\sum_{k\geq 0} \frac{(\bar{\psi}A\psi)^k}{k+1}(\bar{\psi}XB^kY\psi)\Big).$$

As there are no more creation and annihilation operators, we can just drop the factor $\langle 0 \mid 0\rangle = 1$, to obtain the purely fermionic representation

$$\sum_{\substack{L\subseteq[m]\\|L|=n}} \text{col-det}\, X_{[n],L}\, \text{col-det}\, Y_{L,[n]} = \int \mathcal{D}(\psi,\bar{\psi}) \exp\Big(\sum_{k\geq 0} \frac{(\bar{\psi}A\psi)^k}{k+1}(\bar{\psi}XB^kY\psi)\Big),$$

or, by summing over k, intending $\ln(I-M) = \sum_{k\geq 1}\frac{1}{k}M^k$ as a polynomial, truncated by the nilpotence of $\bar{\psi}A\psi$, and using $[(\bar{\psi}X)_r, \bar{\psi}A\psi] = 0$ for every r (valid because of Lemma 8.2, see equation (47)),

$$\sum_{\substack{L\subseteq[m]\\|L|=n}} \text{col-det}\, X_{[n],L}\, \text{col-det}\, Y_{L,[n]} = \int \mathcal{D}(\psi,\bar{\psi}) \exp\Big(-\bar{\psi}X\frac{\ln(1-(\bar{\psi}A\psi)B)}{(\bar{\psi}A\psi)B}Y\psi\Big),$$

as announced.

For the case (b), consider now the matrices ${}^\tau X, Y, {}^\tau A, B$ which satisfy the hypothesis of Proposition 1.2(b) and therefore, as X and Y are column-pseudo-commutative, following the procedure above,

$$\sum_{\substack{L\subseteq[m]\\|L|=n}} \text{row-det}\, X_{[n],L}\, \text{row-det}\, Y_{L,[n]} = \int \mathcal{D}(\psi,\bar{\psi})\langle 0 \mid \exp(a^\dagger\bar{\psi}A\psi + \bar{\psi}X(1-aB)^{-1}Y\psi) \mid 0\rangle$$

and, to conclude, we proceed as in the previous case, except that we use the identity (39) instead of (40). ☐

9. Direct proof of the Grassmann representation for $B = I$

We have proven a Grassmann version of the non-commutative Cauchy–Binet formula as a consequence of the Weyl–Heisenberg version. Considering also the necessary analysis of combinatorics of Łukasiewicz paths, for the latter, and of Campbell–Baker–Hausdorff formula, for the former, the proof is quite composite. It is conceivable that a more direct proof may exist.

In this section we give such a proof, in the simplified situation in which, besides the hypotheses in Proposition 1.4, we have that B is the identity matrix. Indeed, in this case, the version of non-commutative Cauchy–Binet formula obtained in [1] (and reported here as Proposition 1.1(a)), and the Grassmann algebra representation of Proposition 1.4(a), hold simultaneously. We produce here a short proof of the specialized Proposition 1.4(a), taking Proposition 1.1(a) as the starting point.

Actually, just like in Proposition 1.3, we will end up proving that this relation between the right hand sides of (7) and (18) is in fact valid regardless from the fact that A is related to the commutator of X and Y, i.e. they are a consequence of the following stronger fact.

Proposition 9.1. *Let R be a ring containing the rationals, and U and V be two $n \times n$ matrices with elements in R. Let $\bar{\psi}_i$, ψ_i, with $1 \leq i \leq n$, be Grassmann indeterminates. Define*

$$(Q^{\mathrm{col}}(V))_{ij} \stackrel{\mathrm{def}}{=} V_{ij}(n-j). \tag{50}$$

Assume that

$$[\bar{\psi} U \psi, \bar{\psi} V \psi] = 0 \tag{51}$$

and that, for any permutation τ,

$$\operatorname{sgn}(\tau)\, \text{col-det}(U^{\tau} + Q^{\mathrm{col}}(V^{\tau})) = \text{col-det}(U + Q^{\mathrm{col}}(V)). \tag{52}$$

Then

$$\text{col-det}(U + Q^{\mathrm{col}}(V)) = \int \mathcal{D}(\psi, \bar{\psi}) \exp\Big(\sum_{k \geq 0} \frac{(\bar{\psi} V \psi)^k}{k+1} (\bar{\psi} U \psi) \Big). \tag{53}$$

Proof. Remark that, for s and t commuting indeterminates, at the level of power series,

$$\exp\Big(s \sum_{k \geq 0} \frac{t^{k+1}}{k+1} \Big) = (1-t)^{-s} = \sum_{n \geq 0} \frac{t^n}{n!} (s + (n-1))(s + (n-2)) \ldots s.$$

With the choice $t \to tv$ and $s \to u/(tv)$, with u, v and t commuting, we get that

$$\exp\Big(t\, u \sum_{k \geq 0} \frac{(t\, v)^k}{k+1} \Big) = \sum_{n \geq 0} \frac{t^n}{n!} (u + (n-1)v)(u + (n-2)v) \ldots u.$$

We apply this formula to the right hand side of (53), with $u = \bar\psi U \psi$, $v = \bar\psi V \psi$, and t a formal indeterminate that counts the degree in Grassmann variables (the coefficient of order t^k has k factors $\bar\psi_i$'s and k ψ_j's). In particular, Grassmann integration selects only the term t^n, and we get

$$\int \mathcal{D}(\psi, \bar\psi) \exp\Big(\sum_{k \geq 0} \frac{(\bar\psi V \psi)^k}{k+1} (\bar\psi U \psi) \Big) \\ = \frac{1}{n!} \int \mathcal{D}(\psi, \bar\psi) (\bar\psi (U + V(n-1))\psi)(\bar\psi(U + V(n-2))\psi) \ldots (\bar\psi U \psi). \tag{54}$$

The left hand side of (53), using (45), reads

$$\int d\bar\psi_n \ldots d\bar\psi_1 (\bar\psi(U + Q^{\mathrm{col}}))_1 \ldots (\bar\psi(U + Q^{\mathrm{col}}))_n,$$

that is, given the expression (50) for Q^{col},

$$\int d\bar\psi_n \ldots d\bar\psi_1 (\bar\psi(U + V(n-1)))_1 (\bar\psi(U + V(n-2)))_2 \ldots (\bar\psi U)_n. \tag{55}$$

We can introduce a trivial factor $1 = \int d\psi_n \ldots d\psi_1 \psi_1 \ldots \psi_n$, and reorder the Grassmann variables, and terms in the integration measure, to rewrite (55) as

$$\int \mathcal{D}(\psi, \bar\psi) (\bar\psi(U + A(n-1)))_1 (\bar\psi(U + A(n-2)))_2 \ldots (\bar\psi U)_n \psi_n \ldots \psi_1.$$

We can exploit the invariance in the hypothesis (52), and the fact that our ring contains the rationals, to replace the expression above by its symmetrization

$$\frac{1}{n!} \sum_\tau \mathrm{sgn}(\tau) \int \mathcal{D}(\psi, \bar\psi) (\bar\psi(U^\tau + A^\tau(n-1)))_1 \ldots (\bar\psi U^\tau)_n \psi_n \ldots \psi_1.$$

As $(M^\tau)_{ij} = M_{i\tau(j)}$, we just have

$$\frac{1}{n!} \sum_\tau \mathrm{sgn}(\tau) \int \mathcal{D}(\psi, \bar\psi) (\bar\psi(U + A(n-1)))_{\tau(1)} \ldots (\bar\psi U)_{\tau(n)} \psi_n \ldots \psi_1.$$

Note that the factors $(n - j)$, multiplying the matrix entries of A, remain unchanged in their ordering, and in particular the values of j are distinct from the indices, now $\tau(j)$, in the corresponding product. Reorder the factors ψ_i's so to compensate for the signature of the permutation

$$\frac{1}{n!} \sum_\tau \int \mathcal{D}(\psi, \bar\psi) (\bar\psi(U + A(n-1)))_{\tau(1)} \ldots (\bar\psi U)_{\tau(n)} \psi_{\tau(n)} \ldots \psi_{\tau(1)},$$

and extend the sum to all n-uples of integers

$$\frac{1}{n!} \sum_{i_1,\dots,i_n \in [n]} \int \mathcal{D}(\psi, \bar{\psi}) (\bar{\psi}(U + A(n-1)))_{i_1} \\ (\bar{\psi}(U + A(n-2)))_{i_2} \dots (\bar{\psi} U)_{i_n} \psi_{i_n} \dots \psi_{i_1}$$

(this is possible because repeated indices give zero, from the nilpotence of ψ_i variables). Reordering the ψ_i's next to the factors with the corresponding indices, and performing the sum over indices i_α's, gives

$$\frac{1}{n!} \int \mathcal{D}(\psi, \bar{\psi}) (\bar{\psi}(U + A(n-1))\psi)(\bar{\psi}(U + A(n-2))\psi) \dots (\bar{\psi} U \psi),$$

which coincides with (54), as was to be proven. □

Our case of interest is recovered by setting $U = XY$ and $V = A$. The hypothesis (51) holds, as a consequence of Lemma 8.3 specialized to $B = I$ (of which, because of Lemma 3.6, the hypotheses are satisfied), while the hypothesis (52) is verified by observing that, for any permutation τ, the three matrices X, Y^τ and A^τ satisfy the hypotheses of Proposition 1.1(a), and by applying (43) to the *left* hand side of the proposition statement (we use at this aim the fact that Y has weakly row-symmetric commutators, as implied by the hypotheses of Proposition 1.4(a)). Conversely, equations (43) and (52) are *not* immediately related, as, because of the factors $n - j$ in Q^{col}, the matrix on the left hand side of (52) does not correspond to the action of τ from the right.

Remark that, with respect to Proposition 1.3, the level of generality of this proposition in comparison to the specialization pertinent to Capelli-like identities is less pronounced. This is mainly due to the fact that the hypothesis (52) is in fact very demanding. Indeed, it implies in particular that, for any permutation τ and any transposition $(j\ j+1)$ of consecutive elements,

$$\text{col-det}(U^\tau + Q^{\mathrm{col}}(V^\tau)) + \text{col-det}(U^{\tau\circ(j\,j+1)} + Q^{\mathrm{col}}(V^{\tau\circ(j\,j+1)})) = 0.$$

Using the representation (45) of column-determinants, gives

$$\int d\psi_n \dots d\psi_1 L[(\psi(U + V(n-j)))_r (\psi(U + V(n-j-1)))_s + (r \leftrightarrow s)]R = 0,$$

where L and R are appropriate factors, corresponding, $i \neq j, j+1$, to the product of $(\psi(U + Q^{\mathrm{col}}))_i$. A sufficient condition for the integral to vanish is that the combination in square brackets is zero. Strictly speaking, this is not also necessary, but it is hard to imagine a different mechanism for the quantity above to vanish, and still the original column-determinant being non-trivial. So we keep on investigating under which conditions on U and V we have, for every r, s and j,

$$(\psi(U + V(n-j)))_r (\psi(U + V(n-j-1)))_s + (r \leftrightarrow s) = 0.$$

Matching the terms with different degree in j gives

$$\begin{aligned}&\{(\psi V)_r, (\psi V)_s\} = 0,\\ &\{(\psi U)_r, (\psi V)_s\} = -\{(\psi U)_s, (\psi V)_r\}\end{aligned} \tag{56}$$

and

$$\{(\psi U)_r, (\psi U)_s\} = (\psi U)_r(\psi V)_s + (\psi U)_s(\psi V)_r.$$

Incidentally, equation (56) implies (51), thus the three equations above are sufficient for Proposition 9.1 to apply.

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