

Analyticity results for the cumulants in a random matrix model

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Abstract. The generating function of the cumulants in random matrix models, as well as the cumulants themselves, can be expanded as asymptotic (divergent) series indexed by maps. While at fixed genus the sums over maps converge, the sums over genera do not. In this paper we obtain alternative expansions both for the generating function and for the cumulants that cure this problem. We provide explicit and convergent expansions for the cumulants, for the remainders of their perturbative expansion (in the size of the maps) and for the remainders of their topological expansion (in the genus of the maps). We show that any cumulant is an analytic function inside a cardioid domain in the complex plane and we prove that any cumulant is Borel summable at the origin.

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1. Introduction

Random matrix theory [1, 2] studies probability laws for matrices. They were introduced more than half a century ago to model the energy spectra of large nuclei and have later proven to be ubiquitous in physics and mathematics. Applications to mathematics range from combinatorics of maps to free probability while in physics, beyond energy spectra of heavy nuclei, random matrices can be used to describe disordered systems and discretized models of random surfaces.

The application of random matrices to random surfaces and 2d quantum gravity [3] relies on the combinatorics of maps. The matrix integrals arising in random matrix theory depend on (at least) two parameters: a coupling constant λ and the size of the matrix, N . A *formal* expansion in the parameter λ of such matrix integrals yields generating functions for maps of arbitrary genus. The coupling constant λ measures the size of the map (the number of its edges), while the parameter $1/N$ turns out to measure the genus of the map. While this formal expansion is extremely successful in enumerating both maps of fixed size and arbitrary genus and maps of fixed genus and arbitrary size, it does *not* provide an estimation of the matrix integral because it *does not converge*.

This phenomenon is well understood. From a combinatorial standpoint, the divergence of this formal series is due to the proliferation of maps: while the maps of fixed genus are an exponentially bounded family, maps of arbitrary genus are not. At the analytical level, this reflects the fact that $\lambda = 0$ lies on the boundary of the analyticity domain of the generating function.

One can analyze in some depth these formal power series. Restricting to a fixed order in $1/N$ one obtains convergent series enumerating maps of fixed genus. This yields the celebrated $1/N$ expansion for random matrices [4] (see also [5, 6, 7] for rigorous mathematical results on this expansion). The series at fixed genus exhibit a critical behavior at some critical value λ_c of the coupling constant and a *formal* sum over random surfaces of arbitrary genus can be obtained by taking the so called double scaling limit $\lambda \rightarrow \lambda_c, N \rightarrow \infty$ while keeping $(\lambda_c - \lambda)N^{5/4}$ fixed. The precise status of the $1/N$ series is somewhat involved. Usually the $1/N$ series is taken as an *asymptotic series*: while each order in $1/N$ is well understood, the rest term is usually difficult to control.

Analytical control over the rest term has been achieved in the region of strictly convex potential (see for instance [8] and references therein). This region corresponds to a stable perturbation $\Re\lambda \geq 0$ or to an unstable but small perturbation (such that the perturbation potential is always dominated by the quadratic part and absolute convergence of the matrix integral is ensured). However, one would like to exert some same kind of analytic control over the rest term of the $1/N$ series

also *outside* the region of strictly convex potential. This is due to the following two facts.

- When considering the interpretation of a matrix integral as a generating function of maps, $\Re\lambda \geq 0$ corresponds to an *alternating* sum over maps. A genuine sum over maps is obtained only for $\Re\lambda < 0$. Moreover, the critical point λ_c (at which the fixed genus series become critical) lies far on the negative real axis (for bipartite quadrangulations for instance $\lambda_c = -\frac{1}{12}$). In order to study the behavior of the matrix integral in the critical regime one needs to control the rest term close to this critical point.
- In the region $\Re\lambda < 0$, instanton effects $e^{-\frac{1}{|\lambda|}}$ are expected to play a very important role. As they correspond to non trivial solutions of the classical equations of motion, they correspond precisely to the region where the perturbation potential equals the quadratic part, hence outside the strictly convex potential region.

In this paper we focus on the analyticity of the cumulants in a specific random matrix model. Building on results in random tensor theory [9, 10] and on the Loop Vertex Expansion (LVE) introduced in [11] we establish, for any cumulant, an explicit expansion which is convergent for λ in the cardioid domain:

$$\lambda \in \mathbb{C}, \quad 4|\lambda| < \cos^2\left(\frac{\arg \lambda}{2}\right).$$

We further provide explicit convergent expressions for the remainder in the expansion in λ and as a by product we prove that any cumulant is Borel summable in λ uniformly in N .

More importantly, we provide explicit expressions for the remainder in the $1/N$ expansion of any cumulant which is absolutely convergent in the cardioid domain:

$$\lambda \in \mathbb{C}, \quad 12|\lambda| < \cos^2\left(\frac{\arg \lambda}{2}\right).$$

We emphasize that this domain goes well outside the strictly convex potential region. Our paper is thus a first step towards the rigorous study of the instanton effects and of the critical regime in matrix models. However, work still remains to be done: in order to access these effects, one needs to find analytic continuations of our explicit formulae which hold all the way up to the negative real axis (and up to $-\frac{1}{12}$).

This paper is divided into four parts. In Section 2 we introduce some notation and state our main results. In Section 3 we introduce the intermediate field representation which we subsequently use for the proofs of our results which are

performed in Sections 4 and 5. Some technical details are collected in the appendices.

2. Statement of the main results

Matrix integral and normalization. In this paper, we consider the Gaussian matrix model with a quartic perturbation. The generating function of its cumulants is defined by the integral over complex $N \times N$ matrices M :

$$\mathcal{Z}[J, J^\dagger; \lambda, N] = \frac{\mathfrak{N}}{\mathfrak{D}}, \quad (1)$$

where

$$\begin{aligned} \mathfrak{N} = \int dM \exp \left\{ -\text{Tr}(MM^\dagger) - \frac{\lambda}{2N} \text{Tr}(MM^\dagger MM^\dagger) \right. \\ \left. + \sqrt{N} \text{Tr}(JM^\dagger) + \sqrt{N} \text{Tr}(MJ^\dagger) \right\} \end{aligned}$$

and

$$\mathfrak{D} = \int dM \exp \left\{ -\text{Tr}(MM^\dagger) - \frac{\lambda}{2N} \text{Tr}(MM^\dagger MM^\dagger) \right\}.$$

The source J is itself a $N \times N$ complex matrix and J^\dagger is its adjoint. The cumulants of the quartic model are obtained by taking derivatives of $\log \mathcal{Z}$ with respect to J and J^\dagger evaluated at $J = J^\dagger = 0$.

A Taylor expansion in λ , J and J^\dagger , followed by the evaluation of the Gaussian integral, expresses \mathcal{Z} as a sum over ribbon Feynman graphs (or combinatorial maps). The normalization in N has been chosen in such a way that the amplitude of a ribbon graph is $N^{\chi(G)}$, with $\chi(G)$ the Euler characteristic of the graph (i.e. the Euler characteristic of a surface of minimal number of handles in which G can be embedded). Since we are only interested in the cumulants, we divide the integral by its value at $J = J^\dagger = 0$.

The measure dM is the standard Lebesgue measure on matrices suitably normalized in such a way that $\mathcal{Z}[J, J^\dagger; \lambda, N] = 1$ for $\lambda = 0$ and $J = J^\dagger = 0$,

$$dM = \pi^N \prod_{1 \leq i, j \leq N} d\text{Re}(M_{ij}) d\text{Im}(M_{ij}).$$

In this paper, we are interested in establishing analyticity properties of the cumulants in the variable λ .

The analyticity of $\mathcal{Z}[J, J^\dagger; N, \lambda]$ is fairly easy to establish using conventional techniques. However, in order to study the analyticity of $\log \mathcal{Z}[J, J^\dagger; N, \lambda]$, these techniques have to be supplemented by a detailed study of the zeros of $\mathcal{Z}[J, J^\dagger]$ in the complex domain, which is a harder problem.

Loop Vertex Expansion (LVE) graphs and their amplitudes. The LVE is based on combinatorial maps with cilia. A *cilium* is a half edge hooked to a vertex. A combinatorial map is a graph with a distinguished cyclic ordering of the half edges incident at each vertex. Combinatorial maps are conveniently represented as *ribbon graphs* whose vertices are disks and whose edges are ribbons (allowing one to encode graphically the ordering of the half edges incident at a vertex).

Definition 1 (LVE graphs and corners). A *LVE graph* (G, T) is a connected ribbon graph G with labels on its vertices having furthermore

- a distinguished spanning tree $T \subset G$,
- a labeling of the edges of G not in T (loop edges in physics parlance),
- at most one cilium per vertex.

A *LVE tree* is a LVE graph without cycles.

A *corner* of a LVE graph (G, T) is a pair of consecutive half edges attached to the same vertex.

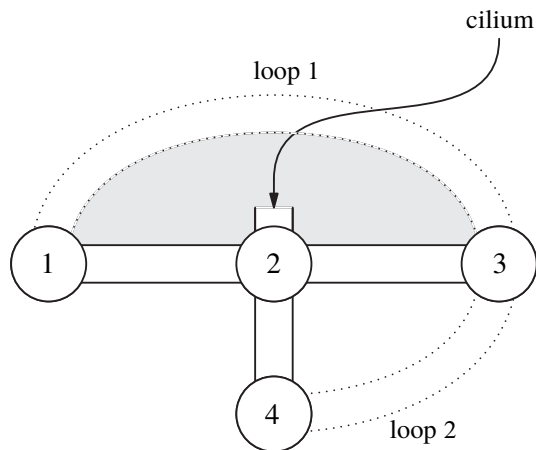


Figure 1. A LVE graph with one cilium and one broken face (coloured in grey).

We denote $K(G)$, $V(G)$, $E(G)$ and $F(G)$ the sets of cilia, vertices, edges and respectively faces of G . The edges of G not in T are called *loop edges* and we denote $L(G, T) = E(G) - E(T)$ the set of loop edges. The faces of G are partitioned between the faces which do not contain any cilium (which we sometimes call *internal faces*) and the ones which contain at least a cilium which we call *broken faces*. We denote $B(G)$ the set of broken faces of G . Each broken face corresponds to a puncture in the Riemann surface in which G is embedded, and the Euler characteristic of the graph G is

$$\chi(G) = |V(G)| - |E(G)| + |F(G)| - |B(G)| = 2 - 2g(G) - |B(G)|$$

where $|X|$ denotes the cardinality of X and $g(G)$ is the genus of the graph G .

Let us consider a LVE graph (G, T) with vertices labeled $1, \dots, |V(G)|$. We associate to every edge e of the tree T a *weakening parameter* $t_e \in [0, 1]$. For any two vertices i and j of the graph G we define

$$(C_T)_{ij} = \inf\{t_e \mid e \text{ in the unique path } \mathcal{P}_{i \leftrightarrow j}^T \text{ in } T \text{ joining } i \text{ and } j\},$$

and the infimum is 1 if $i = j$. We arrange $(C_T)_{ij}$ in a (symmetric) $V(G) \times V(G)$ matrix C_T . The matrix C_T is a *positive matrix*. This statement is non trivial and its proof can be found in [12] or [13].

To any real, positive, symmetric $n \times n$ matrix $(C_{ij})_{1 \leq i, j \leq n}$ we associate a unitary invariant normalized Gaussian measure $d\mu_C(A)$ on n random $N \times N$ Hermitian matrices $A = (A_1, \dots, A_n)$ defined by its covariance:

$$\int d\mu_C(A) A_{i|ab} A_{j|cd} = C_{ij} \delta_{ad} \delta_{bc}, \quad \int d\mu_C(A) = 1, \quad (2)$$

where $A_{i|ab}$ and $A_{j|cd}$ are the matrix elements of the matrices A_i and A_j . This Gaussian measure can be represented as a differential operator. Indeed, denoting

$$\begin{aligned} \text{Tr} \left[\frac{\partial}{\partial A_i} \frac{\partial}{\partial A_j} \right] &= \sum_{a,b} \frac{\partial}{\partial A_{i|ab}} \frac{\partial}{\partial A_{j|ba}}, \\ \frac{\partial}{\partial A_{i|ab}} &= \frac{1}{2} \left(\frac{\partial}{\partial \text{Re} A_{i|ab}} - i \frac{\partial}{\partial \text{Im} A_{i|ab}} \right), \end{aligned}$$

the Gaussian expectation of any function $F(A_1, \dots, A_n)$ is

$$\int d\mu_C(A) F(A_1, \dots, A_n) = [e^{\frac{1}{2} \sum_{ij} C_{ij} \text{Tr} \left[\frac{\partial}{\partial A_i} \frac{\partial}{\partial A_j} \right]} F(A_1, \dots, A_n)]_{A_i=0}.$$

To every loop edge $e \in L(G, T)$ we associate a parameter $s_e \in [0, 1]$. As the loop edges are labeled $1, \dots, |L(G, T)|$, we will denote $s_1, \dots, s_{|L(G, T)|}$ the parameter associated to the edge $1, \dots, |L(G, T)|$. Note that for the loop edge $e = (i, j)$ the parameter s_e and the weakening factor $(C_T)_{ij}$ are completely unrelated.

We associate to every LVE graph (G, T) the *amplitude* $\mathcal{A}_{(G,T)}[J, J^\dagger, \lambda, N]$ defined as a Gaussian integral over $|V(G)|$ Hermitian matrices $A = (A_i)_{1 \leq i \leq |V(G)|}$ (each one of size $N \times N$):

$$\begin{aligned} &\mathcal{A}_{(G,T)}[J, J^\dagger; \lambda, N] \\ &= \frac{(-\lambda)^{|E(G)|} N^{|V(G)| - |E(G)|}}{|V(G)|!} \\ &\quad \int_{1 \geq s_1 \geq \dots \geq s_{|L(G,T)|} \geq 0} \prod_{e \in L(G,T)} ds_e \\ &\quad \int_{[0,1]} \prod_{e \in E(T)} dt_e \left(\prod_{e=(i,j) \in L(G,T)} \inf_{e' \in P_{i \leftrightarrow j}^T} t_{e'} \right) \\ &\quad \int d\mu_{s_{|L(G,T)|} C_T}(A) \prod_{f \in F(G)} \text{Tr} \left\{ \overrightarrow{\prod}_{c \in \partial f} \left(1 - i \sqrt{\frac{\lambda}{N}} A_{i_c} \right)^{-1} (JJ^\dagger)^{\eta_c} \right\}, \end{aligned} \tag{3}$$

where

- $\overrightarrow{\prod}_{c \in \partial f}$ is the oriented product around the corners c on the boundary ∂f of the face f ;
- i_c is the label of the vertex the corner c belongs to;
- $\eta_c = 1, 0$ depending on whether c is followed by a cilium (1) or not (0).

We refer to Appendix C for some examples of LVE graphs and their amplitudes.

The Gaussian measure $d\mu_{s_{|L(G,T)|} C_T}(A)$ can also be written as the differential operator:

$$\begin{aligned} &\int d\mu_{s_{|L(G,T)|} C_T}(A) F(A) \\ &= \left[e^{\frac{s_{|L(G,T)|}}{2} \sum_{ij} [\inf_{(k,l) \in P_{i \leftrightarrow j}^T} t_{kl}] \text{Tr} \left[\frac{\partial}{\partial A_i} \frac{\partial}{\partial A_j} \right]} F(A) \right]_{A_i=0}. \end{aligned}$$

In the case of a LVE graph which is just a tree, $G = T$, we will use the shorthand notation $\mathcal{A}_T[J, J^\dagger; \lambda, N] \equiv \mathcal{A}_{(T,T)}[J, J^\dagger; \lambda, N]$. The amplitude simplifies drastically in this case: there are no integrals over the s parameters (and $s_{|L(G,T)|}$ is set to 1), the product over infima in the second line is empty (hence set to 1), and only one trace is obtained (as trees have only one face).

Constructive expansions of the generating function. Let \mathcal{C} be the cardioid domain in the complex plane (see Figure 2),

$$\mathcal{C} = \left\{ \lambda \in \mathbb{C} \text{ with } 4|\lambda| < \cos^2 \left(\frac{\arg \lambda}{2} \right) \right\},$$

where we choose the determination $-\pi < \arg \lambda < \pi$ of the argument (hence the argument has a cut on the negative real axis).

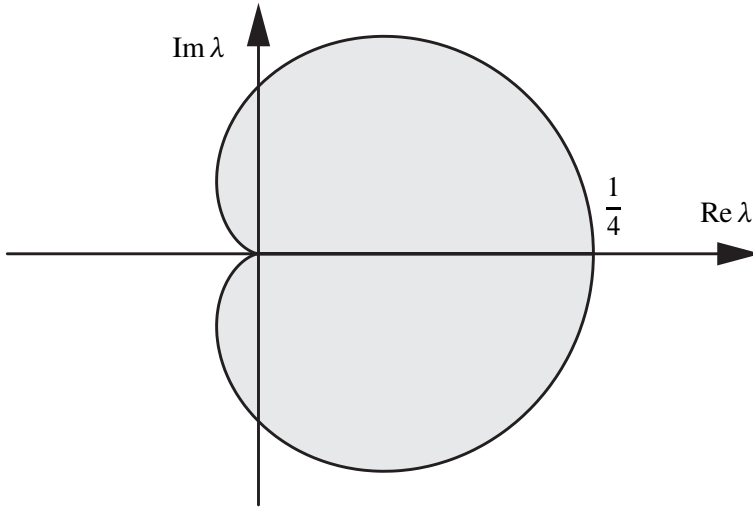


Figure 2. Analyticity domain in the complex λ plane.

Our first result is a convergent expansion of $\log \mathcal{Z}[J, J^\dagger; \lambda, N]$ as a sum over LVE trees.

Theorem 1 (Tree expansion). *For any $\lambda \in \mathcal{C}$, there exists $\epsilon_\lambda > 0$ depending on λ such that for $\|JJ^\dagger\| < \epsilon_\lambda$ the logarithm of $\mathcal{Z}[J, J^\dagger; \lambda, N]$ is given by the absolutely convergent expansion*

$$\log \mathcal{Z}[J, J^\dagger; \lambda, N] = \sum_{T \text{ LVE tree}} \mathcal{A}_T[J, J^\dagger; \lambda, N]. \quad (4)$$

In order to compare the tree expansion of Theorem 1 with the conventional perturbative expansion, it is necessary to further expand some of the loop edges. The following theorem is obtained by recursively adding loop edges to the LVE trees.

Theorem 2 (Perturbative expansion with remainder). *For any $\lambda \in \mathbb{C}$, there exists $\epsilon_\lambda > 0$ depending on λ such that, for $\|JJ^\dagger\| < \epsilon_\lambda$,*

$$\begin{aligned} & \log \mathcal{Z}[J, J^\dagger; \lambda, N] \\ &= \sum_{\substack{G \text{ ciliated ribbon graph} \\ |E(G)| \leq n}} \frac{(-\lambda)^{|E(G)|} N^{\chi(G)}}{|\text{Aut}(G)|} \prod_{f \in B(G)} \text{Tr}[(JJ^\dagger)^{c(f)}] + \mathcal{R}_n[J, J^\dagger; \lambda, N], \end{aligned} \tag{5}$$

where $c(f)$ is the number of cilia in the broken face f and the perturbative remainder at order n is a convergent sum over LVE graphs with at least $n + 1$ edges and at most $n + 1$ loop edges

$$\begin{aligned} & \mathcal{R}_n[J, J^\dagger; \lambda, N] \\ &= \sum_{\substack{(G, T) \text{ LVE graph} \\ |E(G)| = n+1}} \mathcal{A}_{(G, T)}[J, J^\dagger; \lambda, N] + \sum_{\substack{T \text{ LVE tree} \\ |E(T)| \geq n+2}} \mathcal{A}_T[J, J^\dagger; \lambda, N]. \end{aligned}$$

Note that the first term in (5) involves a sum over ribbon graphs (not LVE graphs) and reproduces the perturbative expansion over maps. The remainder is made of the more involved LVE graphs and its amplitude involves further non trivial Gaußian integrations. In particular, the ribbon graphs in the perturbative expansion do not carry labels on their vertices: this is the origin of the factor $\frac{1}{|\text{Aut}(G)|}$ (where $\text{Aut}(G)$ is the cardinal of the group of permutations of the labels on the vertices that preserves the adjacency relations of the graph). Alternatively, one could also work with labeled ribbon graphs and divide by $|V(G)|!$.

Since we are dealing with random matrices of size N , it is also possible to organize the expansion in powers of $\frac{1}{N}$. Such an expansion is governed by the genus of the ribbon graphs, as a graph with Euler characteristic $\chi(G)$ scales like $N^{\chi(G)}$. Contrary to the standard perturbative expansion, the expansion over graphs of fixed genus g has a finite ($\frac{1}{12}$) radius of convergence, as can be easily seen from their asymptotic behavior [14]. In particular, $\lambda_c = -\frac{1}{12}$ is the critical point, instrumental in constructing the double scaling limit. This motivates the introduction of the following cardioid (see Figure 3):

$$\tilde{\mathcal{C}} = \left\{ \lambda \in \mathbb{C} \text{ with } 12|\lambda| < \cos^2\left(\frac{\arg \lambda}{2}\right) \right\}. \tag{6}$$

The shift from the factor 4 for \mathcal{C} to 12 for $\tilde{\mathcal{C}}$ reflects the fact that the radius of convergence of the sum over ribbon graphs of fixed genus is $\frac{1}{12}$ while the radius of convergence of the sum over trees is $\frac{1}{4}$.

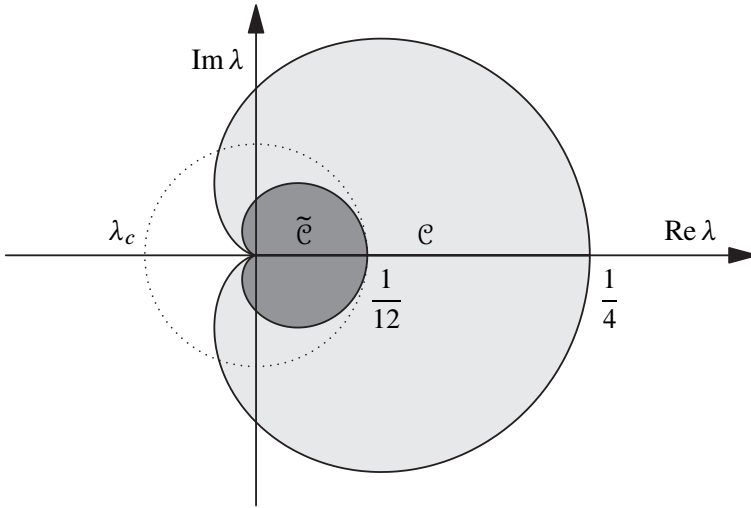


Figure 3. Analyticity domain of the topological expansion.

Theorem 3 (Topological expansion with remainder). *For any $\lambda \in \tilde{\mathcal{C}}$, there exists $\epsilon_\lambda > 0$ depending on λ such that for $\|JJ^\dagger\| < \epsilon_\lambda$ the logarithm of $\mathcal{Z}[J, J^\dagger; \lambda, N]$ is*

$$\begin{aligned} & \log \mathcal{Z}[J, J^\dagger; \lambda, N] \\ &= \left(\sum_{\substack{G \text{ ciliated ribbon graph} \\ g(G) \leq g}} \frac{(-\lambda)^{|E(G)|} N^{2-2g(G)-|B(G)|}}{|\text{Aut}(G)|} \prod_{f \in B(G)} \text{Tr}[(JJ^\dagger)^{c(f)}] \right) \quad (7) \\ &+ \tilde{\mathcal{R}}_g[J, J^\dagger; \lambda, N], \end{aligned}$$

where the order g topological remainder is given by the absolutely convergent expansion

$$\tilde{\mathcal{R}}_g[J, J^\dagger; \lambda, N] = \sum_{\substack{(G,T) \text{ LVE graph with} \\ g(G) = g + 1 \text{ and } g(G - e_{L(G,T)}) = g}} \mathcal{A}_{(G,T)}[J, J^\dagger, \lambda, N],$$

where $G - e_{L(G,T)}$ is the graph obtained by removing the loop edge with the highest label.

This expansion is also obtained by a recursive addition of loop edges to a tree, but with a stop rule which takes into account the topology: one iteratively adds loop edges as long as the genus of the LVE graph (G, T) does not exceed g .

Cumulants. The main objects of interest in this paper are the cumulants (connected correlation functions).

Definition 2 (Cumulants). The *cumulant of order $2k$* is the derivative

$$K_{a_1 b_1 c_1 d_1, \dots, a_k b_k c_k d_k}(\lambda, N) = \frac{\partial^2}{\partial J_{a_1 b_1}^* \partial J_{c_1 d_1}} \cdots \frac{\partial^2}{\partial J_{a_k b_k}^* \partial J_{c_k d_k}} \log \mathcal{Z}[J, J^\dagger; \lambda, N] \Big|_{J=J^\dagger=0}.$$

Here J_{ab}^* is the complex conjugate of J_{ab} , so that $(J^\dagger)_{ab} = J_{ba}^*$. Note that all the derivatives of $\log \mathcal{Z}$ which are not of this form vanish. For example, the order 2 cumulant is

$$K_{abcd}(\lambda, N) = \frac{\partial^2}{\partial J_{ab}^* \partial J_{cd}} \log \mathcal{Z}[J, J^\dagger] = N(\langle M_{ab} M_{cd}^* \rangle - \langle M_{ab} \rangle \langle M_{cd}^* \rangle).$$

The normalization is chosen in such a way that the contribution of a genus g graph with b broken faces scales as N^{2-2g-b} , corresponding to the Euler characteristic of a surface with punctures.

Due to the unitary invariance of the matrix model, the cumulants have a specific form. For any permutation of k elements $\sigma \in \mathfrak{S}_k$, let us write $C(\sigma)$ the integer partition of k associated to the cycle decomposition of σ and $|C(\sigma)|$ the number of cycles it contains. Let us also denote by Π_k the set of integer partitions of k (recall that a partition $\pi \in \Pi_k$ is an increasing sequence of $|\pi|$ integers $0 < k_1 \leq \dots \leq k_{|\pi|}$ such that $k_1 + \dots + k_{|\pi|} = k$). To any integer partition of k we associate a *trace invariant*:

$$\text{Tr}_\pi(X) = \text{Tr}(X^{k_1}) \dots \text{Tr}(X^{k_p}).$$

As we will see below, the cumulants are written in terms of the Weingarten functions $\text{Wg}(\tau\sigma^{-1}, N)$ [15, 16]. These functions arise when integrating over unitary matrices $U(N)$ with the invariant normalized Haar measure. Denoting U_{ab}^* the complex conjugate of U_{ab} we have [15]

$$\int dU U_{a_1 b_1} \dots U_{a_k b_k} U_{c_1 d_1}^* \dots U_{c_l d_l}^* = \delta_{kl} \sum_{\sigma, \tau \in \mathfrak{S}_k} \delta_{a_{\tau(1)} c_1} \dots \delta_{a_{\tau(k)} c_k} \delta_{b_{\sigma(1)} d_1} \dots \delta_{b_{\sigma(k)} d_k} \text{Wg}(\tau\sigma^{-1}, N).$$

The functions $\text{Wg}(\sigma, N)$ only depends on the cycle structure of σ . For low values of n , the Weingarten functions read

$$\begin{aligned} \text{Wg}((1), N) &= \frac{1}{N}, & \text{Wg}((1, 1, 1), N) &= \frac{N^2 - 2}{N(N^2 - 1)(N^2 - 4)}, \\ \text{Wg}((1, 1), N) &= \frac{-1}{N^2 - 1}, & \text{Wg}((1, 2), N) &= \frac{-1}{(N^2 - 1)(N^2 - 4)}, \\ \text{Wg}((2), N) &= \frac{-1}{N(N^2 - 1)}, & \text{Wg}((3), N) &= \frac{2}{N(N^2 - 1)(N^2 - 4)}. \end{aligned}$$

Let us choose a permutation $\zeta \in \mathfrak{S}_k$ whose cycle decomposition reproduces the contribution of the broken faces to the amplitude of a LVE graph. Specifically, if there are $b = |B(G)|$ broken faces with k_1, \dots, k_b cilia, we choose ζ to have a cycle decomposition of the form

$$\zeta = (i_1^1 \dots i_{k_1}^1) \dots (i_1^b \dots i_{k_b}^b).$$

This permutation defines a labeling of the cilia in such a way that the product of traces over the broken faces can be expressed as

$$\prod_{1 \leq m \leq b} \text{Tr} \left[J J^\dagger \prod_{1 \leq r \leq k_m} \overrightarrow{X}^{i_r^m} \right] = \sum_{1 \leq p_1, q_1 \dots \leq N} \prod_{1 \leq l \leq k} (J J^\dagger)_{p_l q_l} X_{q_l p_{\zeta(l)}}^l,$$

where X^l is the product of the resolvents

$$\left(1 - i \sqrt{\frac{\lambda}{N}} A_{i_c} \right)^{-1}$$

located on the corners separating the cilia labeled l and $\zeta(l)$. Similarly, for the $F(G) - B(G)$ unbroken faces we denote by Y^m the product of the resolvents around the unbroken face labeled m .

Proposition 1. *The amplitude of a LVE graph in eq. (3) expands in trace invariants as*

$$\mathcal{A}_{(G,T)}[J, J^\dagger, \lambda, N] = \sum_{\pi \in \Pi_k} \mathcal{A}_{(G,T)}^\pi(\lambda, N) \text{Tr}_\pi(J J^\dagger),$$

with

$$\begin{aligned}
 & A_{(G,T)}^\pi(\lambda, N) \\
 &= \frac{(-\lambda)^{|E(G)|} N^{|V(G)|-|E(G)|}}{|V(G)|!} \\
 & \int_{1 \geq s_1 \geq \dots \geq s_{|L(G,T)|} \geq 0} \prod_{e \in L(G,T)} ds_e \\
 & \int \prod_{e \in E(T)} dt_e \left(\prod_{e=(i,j) \in L(G,T)} \inf_{e' \in P_{i \leftrightarrow j}^T} t_{e'} \right) \int d\mu_{s_{|L(G,T)|}} \mathcal{C}_T(A) \\
 & \sum_{\substack{\tau, \sigma \in \mathfrak{S}_k \\ C(\sigma) = \pi}} \sum_{1 \leq p_1, \dots, p_k \leq N} Wg(\tau\sigma^{-1}, N) \prod_{1 \leq m \leq F(G)-B(G)} \text{Tr}[Y^m] \prod_{1 \leq l \leq k} X_{p_{\tau(l)} p_{\xi(l)}}^l.
 \end{aligned} \tag{8}$$

If the LVE graph (G, T) is reduced to a tree we use the shorthand notation $A_T^\pi(\lambda, N)$ instead of $A_{(T,T)}^\pi(\lambda, N)$.

Proposition 2 (Scalar cumulants). *The order $2k$ cumulants can be written as a sum over partitions of k and over two permutations of k elements:*

$$\begin{aligned}
 & K_{a_1 b_1 c_1 d_1, \dots, a_k b_k c_k d_k}(\lambda, N) \\
 &= \sum_{\pi \in \Pi_k} K_\pi(\lambda, N) \sum_{\rho, \sigma \in \mathfrak{S}_k} \prod_{1 \leq l \leq k} \delta_{c_l, a_{\rho\tau_\pi\sigma^{-1}(l)}} \delta_{d_l, b_{\rho\xi_\pi\sigma^{-1}(l)}},
 \end{aligned}$$

where τ_π and ξ_π are arbitrary permutations such that $\tau_\pi(\xi_\pi)^{-1}$ has a cycle structure corresponding to the partition π and the scalar cumulants $K_\pi(\lambda, N)$ are given by the expansion

$$K_\pi(\lambda, N) = \sum_{T \text{ LVE tree with } k \text{ cilia}} A_T^\pi(\lambda, N).$$

Choosing any other pair of permutations τ_π and ξ_π leads to an identical result, after reorganizing the sum over ρ and σ . $K_\pi(\lambda, N)$ only depends on the partition π and not on the index structure of $K_{a_1 b_1 c_1 d_1, \dots, a_k b_k c_k d_k}(\lambda, N)$ which explains why we call it *scalar cumulant*.

The main goal of this paper is to establish some analyticity results as well as bounds for the scalar cumulants $K_\pi(\lambda, N)$ regarded as functions of λ inside a cardioid with N considered as a parameter.

Constructive expansions for cumulants. Our first result states that the expansion of $K_\pi(\lambda, N)$ as a sum over trees yields an analytic function of $\lambda \in \mathbb{C}$.

Theorem 4 (Analyticity and bound for cumulants). *The series*

$$K_\pi(\lambda, N) = \sum_{T \text{ LVE tree with } k \text{ cilia}} \mathcal{A}_T^\pi(\lambda, N),$$

defines an analytic function of $\lambda \in \mathbb{C}$. Moreover, each term in this sum is bounded (for N large enough) as

$$|\mathcal{A}_T^\pi(\lambda, N)| \leq \frac{N^{2-|\pi|} |\lambda|^{|E(T)|} (k!)^2 2^{2k}}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^{2|E(T)|+k} |V(T)|!}, \tag{9}$$

where $|\pi|$ is the number of integers in the partition π of k (number of cilia).

By further expanding loop edges on each tree, we obtain a perturbative expansion with a well controlled remainder. In order to identify the graphs contributing to $K_\pi(\lambda, N)$, we say that a ciliated ribbon graph has broken faces corresponding to π if the partition of the cilia defined by the broke faces agrees with the partition π .

Theorem 5 (perturbative expansion with remainder). *The perturbative expansion of the cumulants reads:*

$$K_\pi(\lambda, N) = \sum_{\substack{G \text{ ribbon graph with } k \text{ cilia} \\ \text{broken faces corresponding to } \pi \text{ and } |E(G)| \leq n}} \frac{(-\lambda)^{|E(G)|} N^{\chi(G)}}{|\text{Aut}(G)|} + \mathcal{R}_{\pi,n}(\lambda, N).$$

The perturbative remainder $\mathcal{R}_{\pi,n}(\lambda, N)$ is a sum over LVE graphs with k cilia, at least $n + 1$ edges and at most $n + 1$ loop edges,

$$\mathcal{R}_{\pi,n}(\lambda, N) = \sum_{\substack{(G,T) \text{ LVE graphs with broken structure corresponding to } \pi \\ |E(G)| \geq n+1 \text{ and } |L(G,T)| \leq n+1}} \mathcal{A}_{(G,T)}^\pi(\lambda, N).$$

The perturbative remainder is analytic for $\lambda \in \mathbb{C}$ and for any $\lambda \in \mathbb{C}$ and N large enough it obeys the bound:

$$|\mathcal{R}_{\pi,n}(\lambda, N)| \leq N^{2-|\pi|} \left(\frac{2^{3k-1} k!}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^k} \right) (n+1)! \left(\frac{4|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2} \right)^{n+1} \left(\frac{\frac{4|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2}}{\left(1 - \frac{4|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2}\right)^{n+2}} + 2^{k+n+2} \right).$$

Borel summation for cumulants. The previous expansion defines an asymptotic expansion of the cumulants. Indeed, let us collect the contribution of all graphs of a given order in

$$a_{\pi,n}(N) = \sum_{\substack{G \text{ ribbon graph with } k \text{ cilia} \\ \text{broken faces } \pi \text{ and } |E(G)| = n}} \frac{N^{\chi(G)}}{|\text{Aut}(G)|},$$

so that for $\lambda \in \mathbb{C}$ the bound on $R_{\pi,n}(\lambda, N)$ implies

$$\lim_{\lambda \rightarrow 0} \left| \frac{K_{\pi}(\lambda, N) - \sum_{k \leq m \leq n} (-\lambda)^m a_{\pi,m}(N)}{\lambda^n} \right| = 0.$$

However, the series $\sum_n a_{\pi,n} \lambda^n$ is divergent which means that $K_{\pi}(\lambda, N)$ is not analytic at the origin. From a combinatorial point of view, the divergence of the series is due to the occurrence of too many graphs at a given order in n . Nevertheless, $\sum a_{\pi,n}(N) \lambda^n$ contains all the information required to reconstruct $K_{\pi}(\lambda, N)$ through the Borel summation procedure. The latter is based on the following theorem.

For any $R > 0$, let \mathcal{D}_R be the disc of radius R tangent at the origin (see Figure 4)

$$\mathcal{D}_R = \left\{ \lambda \in \mathbb{C} \mid \text{Re}\left(\frac{1}{\lambda}\right) > \frac{1}{R} \right\},$$

and let Σ_{σ} be the half strip (see Figure 4 for a representation of \mathcal{D}_R and Σ_R)

$$\Sigma_{\sigma} = \left\{ s \in \mathbb{C} \mid \text{distance}(s, \mathbb{R}^+) < \frac{1}{\sigma} \right\}.$$

Theorem 6 (Nevanlinna and Sokal [17]). *Let $R > 0$ and $F_{\omega}(\lambda)$ be a family of analytic functions on the disc \mathcal{D}_R depending on some parameter $\omega \in \Omega$. If there exists a sequence $a_n(\omega)$ of functions of $\omega \in \Omega$ obeying, for any $n, \lambda \in \mathcal{D}_R$ and $\omega \in \Omega$ the uniform bound:*

$$\left| F_{\omega}(\lambda) - \sum_{m=0}^n a_m(\omega) \lambda^m \right| < C \sigma^{n+1} |\lambda|^{n+1} (n+1)!, \tag{10}$$

with C and σ two positive constants that do not depend on ω , then the series

$$B_{\omega}(s) = \sum_{n=0}^{\infty} \frac{a_n(\omega)}{n!} s^n,$$

has radius of convergence σ^{-1} and can be analytically continued in the strip Σ_{σ} . Moreover, there exists a constant B such that, for any $s \in \Sigma_{\sigma}$ and $\omega \in \Omega$, we have

$$|B_{\omega}(s)| \leq B e^{\frac{s}{R}}.$$

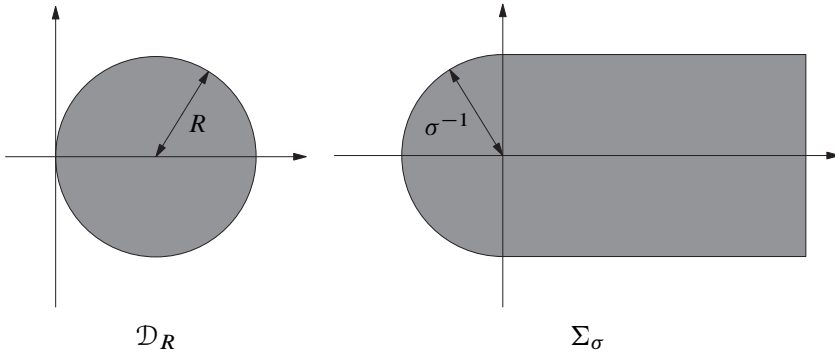


Figure 4. Domain of analyticity of F and of its Borel transform B .

Finally, for any $\lambda \in \mathcal{D}_R$, $F_\omega(\lambda)$ is given by the absolutely convergent integral

$$F_\omega(\lambda) = \int_0^\infty ds B_\omega(s) e^{-\frac{s}{\lambda}}. \tag{11}$$

If the assumption of Theorem 6 are fulfilled, F_ω is said to be Borel summable at $\lambda = 0$, uniformly in ω . In this case, F_ω can be uniquely recovered from the coefficients $a_n(\omega)$ using its Borel transform $s \mapsto B_\omega(s)$ and eq. (11).

For any $\lambda \in \mathcal{D}_R \subset \mathbb{C}$, $(\cos \frac{\arg \lambda}{2}) \geq 2^{-1/2}$ and there exists R such that the perturbative reminder in Theorem 5 is bounded as in eq. (10).

Corollary 1 (Borel summability). *The rescaled cumulants $N^{-2+|\pi|} K_\pi(\lambda, N)$ (with $|\pi|$ the number of parts in the partition π) are Borel summable in λ at the origin, uniformly in N , so that*

$$K_\pi(\lambda, N) = \int_0^\infty ds e^{-\frac{s}{\lambda}} \left(\sum_{n \geq k} \frac{a_{\pi,n}(N)}{n!} s^n \right),$$

in a disc included in \mathbb{C} tangent to the imaginary axis at the origin and independent of N .

Topological expansion for cumulants. The Taylor expansion at the origin of the cumulants leads to ribbon graphs drawn on surfaces with boundary. The Euler characteristic of a surface determines the power of N . This is known as the topological expansion. While it is well known that the contributions of Feynman graphs of fixed genus are analytic functions in a disk of fixed radius $\frac{1}{12}$, less is

known about the remainder. We state an analyticity results and a bound for the remainder.

Theorem 7 (topological expansion). *The cumulants $K_\pi(\lambda, N)$ are expanded in inverse powers of N as*

$$K_\pi(\lambda, N) = \sum_{h=0}^g N^{2-2h-|\pi|} K_{\pi,h}(\lambda) + \tilde{R}_{\pi,g}(\lambda, N),$$

where $K_{\pi,h}(\lambda)$ is a sum over ciliated ribbon graphs of genus h whose broken faces correspond to the partition π , convergent for $|\lambda| < \frac{1}{12}$:

$$K_{\pi,h}(\lambda) = \sum_{\substack{G \text{ ribbon graph with} \\ \text{genus } h \text{ and broken faces corresponding to } \pi}} \frac{(-\lambda)^{|E(G)|}}{|\text{Aut } G|}.$$

The topological remainder $\tilde{R}_{\pi,g}(\lambda, N)$ is a sum over LVE graphs with broken faces corresponding to π , genus $g + 1$ and such that, if we remove the loop edge of highest label, we get a genus g graph

$$\tilde{R}_{\pi,g}(\lambda, N) = \sum_{\substack{(G,T) \text{ LVE graphs with broken faces corresponding to } \pi \\ g(G)=g+1 \text{ and } g(G-e_{|L(G,T)|})=g}} A_{(G,T)}^\pi(\lambda, N).$$

This series converges for $\lambda \in \tilde{\mathcal{C}}$ and in this domain the topological remainder is bounded by

$$|\tilde{R}_{\pi,g}(\lambda, N)| \leq N^{2-2(g+1)-|\pi|} \frac{2^{3k} k!}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^k} C_{g+1}'' \left(\frac{12|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2}\right)^{2g+2} \frac{(4g+k+1)!}{\left(1 - \frac{12|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2}\right)^{4g+k}},$$

with C_g'' a constant depending only on the genus.

3. Intermediate field representation

To begin with, we introduce the intermediate field A (a $N \times N$ Hermitian matrix) and write the quartic interaction as a Gaußian integral:

$$\exp\left\{-\frac{\lambda}{2N} \text{Tr}(MM^\dagger MM^\dagger)\right\} = \int dA \exp\left\{-\frac{1}{2} \text{Tr}(A^2) + i\sqrt{\frac{\lambda}{N}} \text{Tr}(M^\dagger AM)\right\},$$

where the integral is over Hermitian $N \times N$ matrices and is assumed to be normalized. The new field A propagates with the trivial Gaussian measure and the four valent interaction is traded for a three valent interaction involving an A field and a M and a M^\dagger field. This is illustrated in Figure 5.

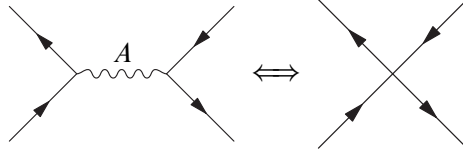


Figure 5. Intermediate field representation.

The generating function is thus

$$\begin{aligned} \mathcal{Z}[J, J^\dagger; \lambda, N] = \int dM dA \exp \left\{ -\frac{1}{2} \text{Tr}(A^2) - \text{Tr} \left[M^\dagger \left(1 - i\sqrt{\frac{\lambda}{N}} A \right) M \right] \right. \\ \left. + \sqrt{N} \text{Tr}(JM^\dagger) + \sqrt{N} \text{Tr}(MJ^\dagger) \right\}, \end{aligned} \quad (12)$$

The integral over the original matrices M and M^\dagger is a (non normalized) Gaussian integral with covariance $(1 - i\sqrt{\frac{\lambda}{N}} A) \otimes 1$. Taking into account that

$$\begin{aligned} \det \left[\left(1 - i\sqrt{\frac{\lambda}{N}} A \right) \otimes 1 \right] &= \left[\det \left(1 - i\sqrt{\frac{\lambda}{N}} A \right) \right]^N \\ &= \exp \left\{ N \text{Tr} \log \left(1 - i\sqrt{\frac{\lambda}{N}} A \right) \right\}, \end{aligned}$$

we obtain

$$\begin{aligned} \mathcal{Z}[J, J^\dagger; \lambda, N] = \int dA \exp \left\{ -\frac{1}{2} \text{Tr}(A^2) - N \text{Tr} \log \left(1 - i\sqrt{\frac{\lambda}{N}} A \right) \right. \\ \left. - N \text{Tr} \left[J \left(1 - i\sqrt{\frac{\lambda}{N}} A \right)^{-1} J^\dagger \right] \right\}. \end{aligned} \quad (13)$$

We thus have three different expressions (1), (12) and (13) for the generating function of the cumulants $\log \mathcal{Z}[J, J^\dagger; \lambda, N]$. Their Feynman graph expansions are constructed as follows.

The expression (12) involves two types of fields A and M so that the Feynman graphs have two types of edges. The M edges (solid edges) are oriented from M^\dagger to M since M is a complex matrix while the A edges (wavy edges) are not

because A is Hermitian. There are 3-valent vertices corresponding to $\text{Tr}(M^\dagger AM)$ and univalent vertices, also viewed as extra half-edges (external legs in the physics literature) corresponding to $\text{Tr}(JM^\dagger)$ and $\text{Tr}(MJ^\dagger)$. Note that all the variables we integrate over are matrices so that we have a cyclic ordering at each vertex and the Feynman graphs are ribbon graphs. We embed the trivalent vertices turning in the clockwise direction so that the A edges are on the right when we follow the orientation of the M edges.

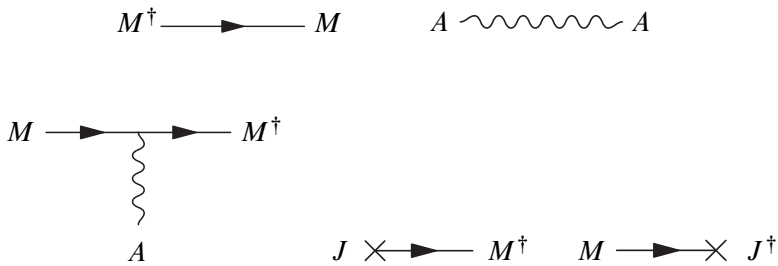


Figure 6. Propagators and interaction.

Integrating over A in (12) before proceeding to the perturbative expansion, we recover the integral (1). Its Feynman rules involve only the M edges (which are oriented) as well as an even number of univalent vertices (external legs) and tetravalent vertices. The latter involve two incoming edges and two outgoing ones, alternating in cyclic order around the vertex.

Integrating over M in (12) before proceeding to the perturbative expansion, we recover the integral (13). Its Feynman rules involve the A edges and two types of vertices of arbitrary valence (see Figure 7).

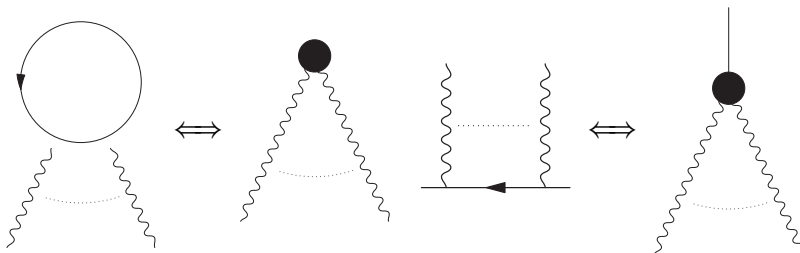


Figure 7. Intermediate field vertices.

The first one is an ordinary ribbon vertex, arising from the term $\text{Tr} \log (1 - i\sqrt{\frac{\lambda}{N}} A)$. The second one comes from the coupling to the source $\text{Tr} J (1 - i\sqrt{\frac{\lambda}{N}} A)^{-1} J^\dagger$. It is a ribbon vertex with a cilium on a corner (the insertion of the source). We illustrate the three representations for a graph contributing to the order 2 cumulant in Figure 8.

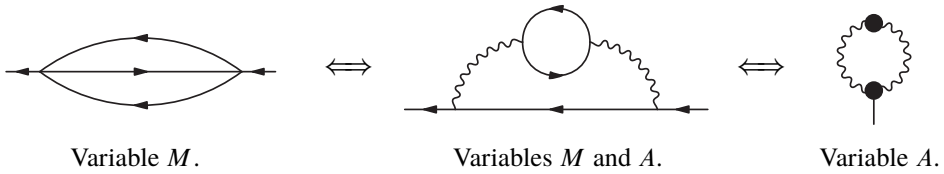


Figure 8. Three equivalent graphs contributing to the order 2 cumulant.

The perturbative expansion can be performed either starting from (1) or starting from (13), using

$$\left(1 - i\sqrt{\frac{\lambda}{N}} A\right)^{-1} = \sum_{n=0}^{\infty} \left(i\sqrt{\frac{\lambda}{N}}\right)^n A^n$$

and performing the Gaussian integral over A .

Comparing the two perturbative expansion for the order 2 cumulant yields the following bijection.

Proposition 3. *The intermediate field representation yields the following bijection*

$$\left\{ \begin{array}{l} \text{connected alternating} \\ \text{2-in 2-out ribbon} \\ \text{graphs with } n \text{ vertices} \\ \text{and } 2m \text{ external edges} \end{array} \right\} \iff \left\{ \begin{array}{l} \text{connected ribbon} \\ \text{graphs with } n \text{ edges} \\ \text{and } m \text{ ciliated vertices} \end{array} \right\}.$$

This bijection can be described explicitly as follows. Starting with a connected alternating 2-in 2-out ribbon graph with n vertices and $2m$ external edges, we observe that its faces come in three types. If the face does not contain an external edge, either it is on the left or it is on the right of all the edges which bound it. We color the first kind of faces in black and the second kind white. If a face is broken, then it contains an even number of external edges that are alternatively incoming and outgoing. The pieces of the face comprised between two consecutive external edges are either on the left or on the right of all the edges which bound them. We color these pieces of faces in black and respectively in white. We join pairs of incoming and outgoing external edges separated by black pieces

of broken faces into cilia. The black faces (ciliated or not) define the vertices of the intermediate field graphs. Two such vertices are joined by an intermediate field edge if and only if the associated faces meet at vertex.

Conversely, given a intermediate field graph, we expand its vertices into (black) faces, and we cut the cilia into two. We then form tetravalent vertices by contracting the intermediate field edges.

This construction is a generalization of the medial graph construction to graphs with external edges (or equivalently, cilia). Indeed, if there is no cilium on the A graph, then the associated M graph is its medial graph. The basic features of this bijection are summarized in Table 1.

Table 1. Matrix model graphs – intermediate field graphs correspondence.

matrix model	intermediate field
vertex	edge
black face	vertex
white face	face
edge	corner
pair of external legs	cilium

Let us end this section by giving two consequences of the intermediate field representation of the matrix model.

First, the number of planar graphs with n vertices contributing to the order 2 cumulant (2-point function in physics parlance) can be evaluated explicitly using the Schwinger–Dyson equation for the intermediate field. The details of this computation are relegated to the Appendix A and the result is

$$\frac{2 \cdot 3^n}{n+2} C_n \quad \text{with} \quad C_n = \frac{(2n)!}{n!2(n+1)} \quad (\text{Catalan numbers}).$$

This is nothing but the number of planar bipartite quadrangulations with n quadrangles, rooted at an edge. Bipartiteness means that the vertices of the quadrangulation are colored in black and white and the edges only connect vertices of different colors. The M graph is the dual of the quadrangulation. The black/white coloring of the faces of the M graph induces an orientation of the M edges in such a way that all the M vertices are alternating 2-in 2-out.

The intermediate field graphs are in bijection with bipartite quadrangulations with m marked edges. The intermediate field graph is obtained by connecting the pair of black vertices on each quadrangle by a wavy A edge (and adding a cilium for every incidence of a marked edge at a black vertex). We thus obtain:

Proposition 4. *There is a bijection*

$$\left\{ \begin{array}{l} \text{bipartite} \\ \text{quadrangulations with } n \\ \text{faces of genus } g \text{ having} \\ \text{ } m \text{ marked edges} \end{array} \right\} \iff \left\{ \begin{array}{l} \text{connected ribbon graphs} \\ \text{of genus } g \text{ with } n \text{ edges} \\ \text{and } m \text{ cilia} \end{array} \right\}.$$

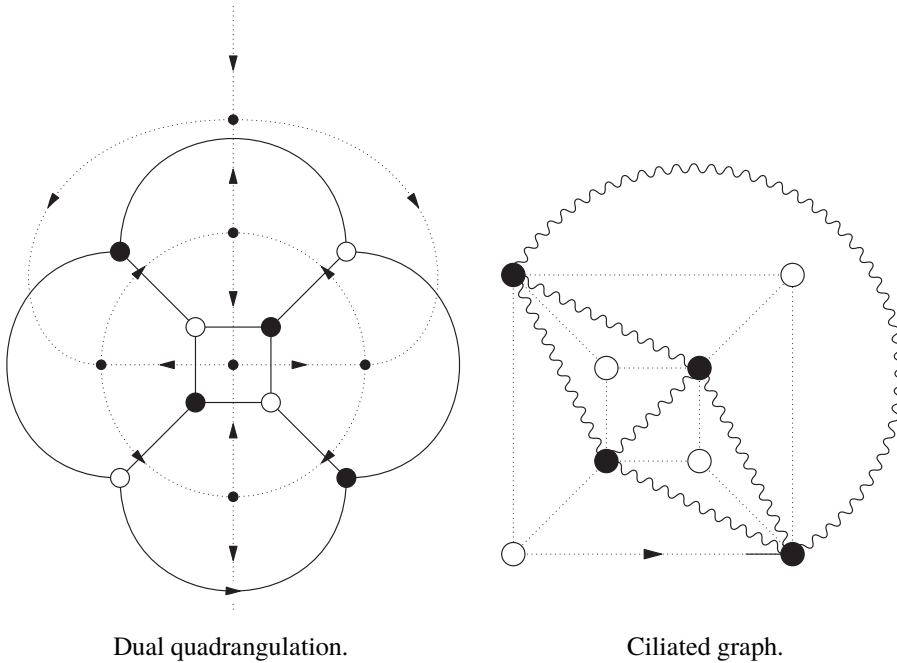


Figure 9. Bijection between bipartite quadrangulations with marked edges and ribbon graphs with cilia.

Second, the intermediate field can be used to study the analyticity properties of \mathcal{Z} . We first have the following bound.

Lemma 1. *Writing $\lambda = \rho e^{i\theta}$ with $\rho > 0$, we have*

$$\left\| \left(1 - i \sqrt{\frac{\lambda}{N} A} \right)^{-1} \right\| \leq \frac{1}{\cos\left(\frac{\theta}{2}\right)}, \tag{14}$$

where $\| \cdot \|$ denotes the operator norm.

Proof. To prove the lemma, it is convenient to factor $\sqrt{\lambda}$ and write

$$\left(1 - i\sqrt{\frac{\lambda}{N}}A\right)^{-1} = \frac{1}{\sqrt{\lambda}} \int_0^\infty d\alpha \exp\left\{-\alpha\frac{1}{\sqrt{\lambda}} + \alpha\frac{i}{\sqrt{N}}A\right\},$$

therefore, the operator norm is bounded by

$$\begin{aligned} \left\|\left(1 - i\sqrt{\frac{\lambda}{N}}A\right)^{-1}\right\| &\leq \frac{1}{|\sqrt{\lambda}|} \int_0^\infty \exp\left\{-\alpha\operatorname{Re}\left(\frac{1}{\sqrt{\lambda}}\right)\right\} \exp\left\{\alpha\frac{iA}{\sqrt{N}}\right\} \\ &= \frac{1}{\cos\left(\frac{\theta}{2}\right)}. \end{aligned} \quad \square$$

We can then rewrite (13) as

$$\mathcal{Z}[J, J^\dagger; N, \lambda] = \int dA \frac{\exp\left\{-\frac{1}{2}\operatorname{Tr}(A^2) + N\operatorname{Tr}J\left(1 - i\sqrt{\frac{\lambda}{N}}A\right)^{-1}J^\dagger\right\}}{\left[\det\left(1 - i\sqrt{\frac{\lambda}{N}}A\right)\right]^N},$$

and use Lemma 1 to show that this integral is convergent for $\theta \in (-\pi, \pi)$. As the integrand is analytic for $\theta \in (-\pi, \pi)$ we have the following result.

Proposition 5. $\mathcal{Z}[J, J^\dagger; N, \lambda]$ is analytic in λ on the cut plane $\mathbb{C} - \mathbb{R}^-$.

However, analyticity of $\mathcal{Z}[J, J^\dagger; N, \lambda]$ in the cut plane does not imply analyticity of its logarithm, as $\mathcal{Z}[J, J^\dagger; N, \lambda]$ may have zeros. In the next section we will see that in order to establish an analyticity result for the logarithm one needs to work some more.

4. Proofs of the theorems regarding the generating function

In this section, we establish the constructive Theorems 1, 2, and 3 regarding the generating function of the cumulants.

4.1. The Loop Vertex Expansion (proof of Theorem 1). The basic ingredient in establishing the constructive theorems stated in Section 2 is the loop vertex expansion, introduced by Rivasseau in [11]. Starting with (13), we expand the exponential as a power series, convergent if $\lambda \in \mathbb{C} - \mathbb{R}^-$,

$$\begin{aligned} \mathcal{Z}[J, J^\dagger] &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d\mu(A) \left[N\operatorname{Tr} \log\left(1 - i\sqrt{\frac{\lambda}{N}}A\right) \right. \\ &\quad \left. + N\operatorname{Tr}J\left(1 - i\sqrt{\frac{\lambda}{N}}A\right)^{-1}J^\dagger \right]^n, \end{aligned}$$

where $d\mu(A) = dA \exp -\frac{1}{2} \text{Tr}(A^2)$ is the normalized Gaussian measure on Hermitian matrices (and we dropped the arguments λ and N of \mathcal{Z} in order to simplify the notation).

We then use the replica trick and replace (for the term of order n) the integral over a single matrix A by integral over a n -tuple of $N \times N$ Hermitian matrices $A = (A_i)_{1 \leq i \leq n}$. The replicated Gaussian integral is performed with a normalized Gaussian measure $d\mu_C(A)$ with a degenerated covariance $C_{ij} = 1$. Recall that for any real positive symmetric matrix C_{ij} the Gaussian integral is

$$\int d\mu_C(A) A_{i|ab} A_{j|cd} = C_{ij} \delta_{ad} \delta_{bc},$$

where $A_{i|ab}$ denotes the matrix element in the row a and column b of the matrix A_i . The Gaussian integral with a degenerated covariance, is equivalent to inserting $n-1$ Dirac distributions $\delta(A_1 - A_2) \cdots \delta(A_{n-1} - A_n)$, since all the $n-1$ matrices $A_1 - A_2, \dots, A_{n-1} - A_n$ span the kernel of C_{ij} . This can easily be seen by regularizing the covariance as $C_{ij} + \epsilon \delta_{ij}$ and letting $\epsilon \rightarrow 0$. Equivalently, at the perturbative level, the uniform covariance generates all the edges (with the appropriated weights) in the Feynman graph expansion that connect the various replicas together.

The generating function then reads

$$\begin{aligned} \mathcal{Z}[J, J^\dagger] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d\mu_C(A) \prod_{i=1}^n \left[N \text{Tr} \log \left(1 - i \sqrt{\frac{\lambda}{N}} A_i \right) \right. \\ \left. + N \text{Tr} J \left(1 - i \sqrt{\frac{\lambda}{N}} A_i \right)^{-1} J^\dagger \right]. \end{aligned}$$

Remark that the Gaussian measure can alternatively be written as the differential operator:

$$\int d\mu_C(A) F(A) = \left[e^{\frac{1}{2} \sum_{i,j} \text{Tr} \left[\frac{\partial}{\partial A_i} \frac{\partial}{\partial A_j} \right]} F(A) \right]_{A_i=0}.$$

We now apply the Bridges–Kennedy–Abdessalam–Rivasseau forest formula (see Appendix B). We start by replacing the covariance $C_{ij} = 1$ by $C_{ij}(x) = x_{ij}$ (and $x_{ij} = x_{ji}$) evaluated at $x_{ij} = 1$ for $i \neq j$ and $C_{ii}(x) = 1$. Then $\mathcal{Z}[J, J^\dagger]$ is given as a sum over forests:

$$\mathcal{Z}[J, J^\dagger] = \sum_{F \text{ labeled forest}} \frac{(-1)^n}{n!} \int_0^1 \prod_{(i,j) \in F} dt_{ij} \left(\prod_{(i,j) \in F} \frac{\partial}{\partial x_{ij}} \right) \mathfrak{A},$$

where

$$\mathfrak{A} = \left\{ \int d\mu_{C(x)}(A) \prod_{i=1}^n \left[N \operatorname{Tr} \log \left(1 - i \sqrt{\frac{\lambda}{N}} A_i \right) + N \operatorname{Tr} J \left(1 - i \sqrt{\frac{\lambda}{N}} A_i \right)^{-1} J^\dagger \right] \right\} \Big|_{x_{ij} = v_{ij}^F},$$

and where n is the number of vertices of F , i and j label the vertices of the forest, there is one weakening parameter t_{ij} per edge (i, j) of the forest and

$$v_{ij}^F = \begin{cases} \inf_{(k,l) \in P_{i \leftrightarrow j}^F} t_{kl} & \text{if } P_{i \leftrightarrow j}^F \text{ exists,} \\ 0 & \text{if } P_{i \leftrightarrow j}^F \text{ does not exist.} \end{cases}$$

where $P_{i \leftrightarrow j}^F$ is the unique path in F joining i and j (and the infimum is set to 1 if $i = j$). The Gaussian measure can alternatively be written as the differential operator:

$$\int d\mu_{C(x)}(A) F(A) = [e^{\frac{1}{2} \sum_{i,j} x_{ij} \operatorname{Tr} \left[\frac{\partial}{\partial A_i} \frac{\partial}{\partial A_j} \right]} F(A)]_{A_i=0}.$$

In order to extract the logarithm we use the following lemma.

Lemma 2. *Let $\mathcal{W}(T)$ be a weight associated to a tree T , independent of the labels of its vertices and define the weight of a forest $\mathcal{W}(F)$ to be the product of the weights of its trees (connected components). Then, as formal series,*

$$\log \sum_{F \text{ labeled forests}} \frac{\mathcal{W}(F)}{|V(F)|!} = \sum_{T \text{ labeled trees}} \frac{\mathcal{W}(T)}{|V(T)|!},$$

with $|V(F)|$ and $|V(T)|$ the number of vertices in F and T .

Proof. This identity is equivalent to

$$\sum_{F \text{ labeled forests}} \frac{\mathcal{W}(F)}{|V(F)|!} = \exp \sum_{T \text{ labeled trees}} \frac{\mathcal{W}(T)}{|V(T)|!},$$

which follows by expanding the right hand side using the multinomial formula, and taking due care of the relabeling of the vertices. \square

As both the differential operator and the Gaussian measure factor over the trees in the forest F we obtain

$$\log \mathfrak{Z}[J, J^\dagger] = \sum_{T \text{ labeled trees}} \frac{(-1)^n}{n!} \int_0^1 \prod_{(i,j) \in T} dt_{ij} \left(\prod_{(i,j) \in T} \frac{\partial}{\partial x_{ij}} \right) \mathfrak{B}$$

and

$$v_{ij}^T = \inf_{(k,l) \in P_{i \leftrightarrow j}^T} t_{kl},$$

where

$$\mathfrak{B} = \left\{ \int d\mu_{C(x)}(A) \prod_{i=1}^n \left[N \operatorname{Tr} \log \left(1 - i\sqrt{\frac{\lambda}{N}} A_i \right) + N \operatorname{Tr} J \left(1 - i\sqrt{\frac{\lambda}{N}} A_i \right)^{-1} J^\dagger \right] \right\} \Big|_{v_{ij}^T}$$

where $P_{i \leftrightarrow j}^T$ is the unique path in the tree T joining i and j . Starting from the expression of the Gaussian integral as a differential operator it is immediate to see that

$$\frac{\partial}{\partial x_{ij}} \left(\int d\mu_{C(x)}(A) F(A) \right) = \int d\mu_{C(x)}(A) \operatorname{Tr} \left[\frac{\partial}{\partial A_i} \frac{\partial}{\partial A_j} \right] F(A).$$

This differential operator acts on two vertices (i and j) and generates an edge connecting them. Taking into account that

$$\frac{\partial}{\partial A_i|_{ab}} \left(1 - i\sqrt{\frac{\lambda}{N}} A_i \right)^{-1}_{cd} = \left(\frac{-\lambda}{N} \right) \left(1 - i\sqrt{\frac{\lambda}{N}} A_i \right)^{-1}_{ca} \left(1 - i\sqrt{\frac{\lambda}{N}} A_i \right)^{-1}_{bd},$$

we observe that a resolvent operator

$$\left(1 - i\sqrt{\frac{\lambda}{N}} A_i \right)^{-1}$$

is associated to each corner of a vertex. Multiple derivatives acting on the same vertex (corresponding to multiple edges hooked to it) can act on either of the corners of the vertex and split it into two. The logarithm of \mathcal{Z} becomes thus a sum over plane trees and the source terms JJ^\dagger correspond to cilia decorating some of the vertices of the trees. We thus obtain

$$\log \mathcal{Z}[J, J^\dagger] = \sum_{\substack{T \\ \text{LVE tree}}} \mathcal{A}_T[J, J^\dagger, \lambda, N],$$

and

$$\begin{aligned} \mathcal{A}_T[J, J^\dagger, \lambda, N] &= \frac{(-\lambda)^{|E(T)|} N^{|V(T)| - |E(T)|}}{|V(T)|!} \int_0^1 \prod_{e \in E(T)} dt_e \\ &\int d\mu_{C_T}(A) \operatorname{Tr} \left[\prod_{c \in \partial T \text{ corner}} \left(1 - i\sqrt{\frac{\lambda}{N}} A_{i_c} \right)^{-1} (JJ^\dagger)^{\eta_c} \right], \end{aligned} \tag{15}$$

where i_c is the label of the vertex the corner c is attached to, $\eta_c \in \{0, 1\}$ depending on whether the corner c is followed ($\eta_c = 1$) or not ($\eta_c = 0$) by a source insertion and the covariance C_T is

$$(C_T)_{ij} = \inf_{(k,l) \in P_{i \leftrightarrow j}^T} t_{kl}$$

and the infimum is set to 1 if $i = j$.

We have thus established the expansion (4) in Theorem 1. In order to establish Theorem 1 it remains to study the domain on which the expansion (15) is convergent.

We first bound the amplitude of each tree using Lemma 1:

$$\begin{aligned} & \left| \text{Tr} \left[\prod_{c \in \partial T \text{ corner}}^{\rightarrow} \left(1 - i \sqrt{\frac{\lambda}{N}} A_{i_c} \right)^{-1} (JJ^\dagger)^{\eta_c} \right] \right| \\ & \leq N \prod_{c \in \partial T \text{ corner}}^{\rightarrow} \left\| \left(1 - i \sqrt{\frac{\lambda}{N}} A_{i_c} \right)^{-1} \right\| \|JJ^\dagger\|^{\eta_c} \\ & \leq \frac{N \|JJ^\dagger\|^k}{\left(\cos \left(\frac{\arg \lambda}{2} \right) \right)^{2(|E(T)|+k)}}, \end{aligned} \quad (16)$$

where k denotes the number of cilia of the tree.

Then we bound the number of LVE trees with a given number of edges and cilia.

Lemma 3 (counting LVE trees). *The number of LVE trees with n edges and k cilia*

$$\begin{aligned} \mathcal{N}(n, k) &= \frac{(2n + k - 1)! (n + 1)!}{(n + k)! (n + 1 - k)! k!} \\ &\leq 2^{2n+k-1} (n - 1)! \frac{(n + 1)!}{(n + 1 - k)! k!}, \end{aligned}$$

Proof. The number of LVE trees with $n + 1$ vertices and k cilia on a fixed set of vertices labeled i_1, \dots, i_k is $\frac{(2n+k-1)!}{(n+k)!}$ (see [9]). To obtain $\mathcal{N}(n, k)$ one simply multiplies the latter by the possible choices of k vertices among $n + 1$. The bound follows by the binomial formula $\frac{(2n+k-1)!}{(n+k)! (n-1)!} \leq 2^{2n+k-1}$. \square

Consequently, the sum over LVE trees is bounded by

$$\begin{aligned}
 & \left| \sum_{\text{LVE tree}} \mathcal{A}_T[J, J^\dagger, \lambda, N] \right| \\
 & \leq \sum_{n=0}^{\infty} \sum_{k=1}^{n+1} \frac{N^2 |\lambda|^n \|JJ^\dagger\|^k}{(n+1)! \left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^{2n+k}} 2^{2n+k-1} (n-1)! \frac{(n+1)!}{(n+1-k)! k!} \quad (17) \\
 & \leq N^2 \sum_{n=0}^{\infty} \frac{2^{2n-1} |\lambda|^n}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^{2n}} \left(1 + \frac{2\|JJ^\dagger\|}{\cos\left(\frac{\arg \lambda}{2}\right)}\right)^{n+1}.
 \end{aligned}$$

Each $\mathcal{A}_T[J, J^\dagger, \lambda, N]$ is analytic in the cut plane $\mathbb{C} - \mathbb{R}^-$. Furthermore, For every λ inside the cardioid:

$$\mathcal{C} = \left\{ \lambda \in \mathbb{C}, 4|\lambda| < \cos^2\left(\frac{\arg \lambda}{2}\right) \right\},$$

it is possible to find a $\epsilon_\lambda > 0$ such that,

$$\frac{4|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2} \left(1 + \frac{2\epsilon_\lambda}{\cos\left(\frac{\arg \lambda}{2}\right)}\right) < 1,$$

hence Theorem 1 follows.

Remark furthermore that for $\lambda \in \mathcal{C}$ and $\|JJ^\dagger\| < \epsilon_\lambda$, $\log \mathcal{Z}[J, J^\dagger]$ is analytic in λ .

4.2. Perturbative expansion with remainder (proof of Theorem 2). Our starting point is eq. (15):

$$\log \mathcal{Z}[J, J^\dagger] = \sum_{\substack{T \\ \text{LVE tree}}} \mathcal{A}_T[J, J^\dagger, \lambda, N],$$

and

$$\begin{aligned}
 \mathcal{A}_T[J, J^\dagger, \lambda, N] &= \frac{(-\lambda)^{|E(T)|} N^{|V(T)|-|E(T)|}}{|V(T)|!} \int_0^1 \prod_{e \in E(T)} dt_e \\
 & \int d\mu_{C_T}(A) \text{Tr} \left[\prod_{c \in \partial T \text{ corner}} \vec{\Pi} \left(1 - i\sqrt{\frac{\lambda}{N}} A_{ic}\right)^{-1} (JJ^\dagger)^{\eta_c} \right],
 \end{aligned}$$

and the Gaussian measure $d\mu_{C_T}(A)$ can also be written as

$$\int d\mu_{C_T}(A) F(A) = [e^{\frac{1}{2} \sum_{ij} (\inf_{(k,l) \in P_{i \leftrightarrow j}^T} t_{kl})} \text{Tr} \left[\frac{\partial}{\partial A_i} \frac{\partial}{\partial A_j} \right] F(A)]_{A_i=0}.$$

A Taylor expansion at first order with an uniform parameter of the Gaussian measure leads to

$$\begin{aligned}
& e^{\frac{1}{2} \sum_{ij} [\inf_{(k,l) \in P_{i \leftrightarrow j}^T} t_{kl}] \text{Tr} \left[\frac{\partial}{\partial A_i} \frac{\partial}{\partial A_j} \right]} \\
&= e^{\frac{s}{2} \sum_{ij} (\inf_{(k,l) \in P_{i \leftrightarrow j}^T} t_{kl}) \text{Tr} \left[\frac{\partial}{\partial A_i} \frac{\partial}{\partial A_j} \right]} \Big|_{s=1} \\
&= 1 + \int_0^1 ds_1 \left[\frac{d}{ds} e^{\frac{s}{2} \sum_{ij} (\inf_{(k,l) \in P_{i \leftrightarrow j}^T} t_{kl}) \text{Tr} \left[\frac{\partial}{\partial A_i} \frac{\partial}{\partial A_j} \right]} \right]_{s=s_1} \\
&= 1 + \int_0^1 ds_1 \mathfrak{C} e^{\frac{s_1}{2} \sum_{ij} (\inf_{(k,l) \in P_{i \leftrightarrow j}^T} t_{kl}) \text{Tr} \left[\frac{\partial}{\partial A_i} \frac{\partial}{\partial A_j} \right]},
\end{aligned}$$

where

$$\mathfrak{C} = \frac{1}{2} \sum_{ij} (\inf_{(k,l) \in P_{i \leftrightarrow j}^T} t_{kl}) \text{Tr} \left[\frac{\partial}{\partial A_i} \frac{\partial}{\partial A_j} \right].$$

The term with the Gaussian measure set to 1 corresponds to setting all the replicated fields $A_i = 0$. Consequently all the resolvents in the trace are replaced by the identity and the trace becomes just a trace over a product of the external sources.

The rest term is more involved. The new derivatives with respect to the replicated fields A_i and A_j act on the resolvents in the trace. As before, a $\frac{1}{2} \text{Tr} \left[\frac{\partial}{\partial A_i} \frac{\partial}{\partial A_j} \right]$ operator creates a ribbon edge and brings an overall factor $\frac{-\lambda}{N}$. As the edge connects two vertices already present in the tree, the new edge is necessarily a loop edge. The sums over i and j yields a sum over all the possible ways to add such a loop edge to the tree T , hence we obtain a sum over all the LVE graphs (G, T) one can build over T having $|L(G, T)| = 1$ loop edges.

Iterating L times we obtain

$$\begin{aligned}
\mathcal{A}_T[J, J^\dagger, \lambda, N] &= \sum_G \frac{(-\lambda)^{|E(G)|} N^{\chi(G)}}{|V(T)|!} \prod_{f \in B(G)} \text{Tr}[(JJ^\dagger)^{c(f)}] \\
&\quad (G, T) \text{ LVE graph, } |L(G, T)| \leq L-1 \\
&\int_0^1 \prod_{e \in E(T)} dt_e \left(\prod_{e=(i,j) \in L(G,T)} (\inf_{(k,l) \in P_{i \leftrightarrow j}^T} t_{kl}) \right) \int_{1 \geq s_1 \geq \dots \geq s_{|L(G,T)|} \geq 0} \prod_{e \in L(G,T)} ds_e \\
&+ \sum_G \mathcal{A}_{(G,T)}[J, J^\dagger; \lambda, N], \\
&\quad (G, T) \text{ LVE graph, } |L(G, T)| = L
\end{aligned}$$

where

$$\begin{aligned} \mathcal{A}_{(G,T)}[J, J^\dagger; \lambda, N] &= \frac{(-\lambda)^{|E(G)|} N^{|V(G)|-|E(G)|}}{|V(G)|!} \int_{1 \geq s_1 \geq \dots \geq s_{|L(G,T)|} \geq 0} \prod_{e \in L(G,T)} ds_e \\ &\int_{[0,1]} \prod_{e \in E(T)} dt_e \left(\prod_{e=(i,j) \in L(G,T)} \inf_{e' \in P_{i \leftrightarrow j}^T} t_{e'} \right) \\ &\int d\mu_{s|L(G,T)|} \mathcal{C}_T(A) \prod_{f \in F(G)} \text{Tr} \left\{ \overrightarrow{\prod}_{c \in \partial f} \left(1 - i \sqrt{\frac{\lambda}{N}} A_{ic} \right)^{-1} (JJ^\dagger)^{\eta_c} \right\}. \end{aligned}$$

yielding the expression eq. (3) for the amplitude of an LVE graph.

We prove Theorem 2 by induction on n . We start with the tree expansion in (4) and select the unique LVE tree without any edge. This tree is a single vertex with one cilium (the vertex without a cilium is absent because of the normalization $\mathcal{Z}[0, 0, \lambda, N]$). We perform an expansion up to $L = 1$ loop edges for this term. All the other LVE trees with edges are included in the rest term. We obtain

$$\begin{aligned} \log \mathcal{Z}[J, J^\dagger] &= N \text{Tr}[JJ^\dagger] + \sum_{\substack{(G,T) \text{ LVE graph,} \\ |V(G)|=1, |L(G,T)|=1}} \mathcal{A}_{(G,T)}[J, J^\dagger; \lambda, N] \\ &+ \sum_{\substack{T \text{ LVE tree} \\ E(T) \geq 1}} \mathcal{A}_T[J, J^\dagger, \lambda, N]. \end{aligned}$$

As the trees with exactly one edge are LVE graphs themselves we can move them to the first rest term and write

$$\begin{aligned} \log \mathcal{Z}[J, J^\dagger] &= N \text{Tr}[JJ^\dagger] + \sum_{\substack{(G,T) \text{ LVE graph,} \\ E(G)=1}} \mathcal{A}_{(G,T)}[J, J^\dagger; \lambda, N] \\ &+ \sum_{\substack{T \text{ LVE tree} \\ E(T) \geq 2}} \mathcal{A}_T[J, J^\dagger, \lambda, N], \end{aligned}$$

reproducing eq. (5) for $n = 0$. The first term in this expression is the contribution of an ordinary Feynman graph (without any resolvents). If we write the theory in terms only of M , this graph has two univalent vertices J and J^\dagger connected by an edge. The second term is the amplitude for a LVE graph with one vertex, one cilium and one edge (which can either be a loop edge or a tree edge). The last term is the contribution of all the LVE trees with at least two edges.

Let us assume that the theorem has been established up to order n hence the perturbative remainder at order n , $R_n[J, J^\dagger, \lambda, N]$ is the sum of two terms:

$$R'_n[J, J^\dagger, \lambda, N] = \sum_{\substack{(G,T) \text{ LVE graph} \\ |E(G)|=n+1}} \mathcal{A}_{(G,T)}[J, J^\dagger, \lambda, N],$$

$$R''_n[J, J^\dagger, \lambda, N] = \sum_{\substack{T \text{ LVE trees} \\ |E(T)| \geq n+2}} \mathcal{A}_T[J, J^\dagger, \lambda, N].$$

The trees contributing to R''_n having at least $n + 3$ edges give exactly R''_{n+1} (where we omit the arguments in order to simplify the notation). The trees having exactly $n + 2$ edges are transferred to R'_{n+1} . Now consider the LVE graphs in R'_n . The amplitude of each of these graphs is written as a Gaußian integral with covariance $s_{|L(G,T)|} C_T$. For each of these graphs we expand one more loop edge using

$$e^{\frac{s_{|L(G,T)|}}{2} \sum_{ij} (\inf_{(k,l) \in P_{i \leftrightarrow j}^T} t_{kl}) \text{Tr} \left[\frac{\partial}{\partial A_i} \frac{\partial}{\partial A_j} \right]} \Big|_{s=1}$$

$$= 1 + \int_{s_{|L(G,T)|} \geq s_{|L(G,T)|+1} \geq 0} ds_{|L(G,T)|+1} \left(\frac{1}{2} \sum_{ij} (\inf_{(k,l) \in P_{i \leftrightarrow j}^T} t_{kl}) \text{Tr} \left[\frac{\partial}{\partial A_i} \frac{\partial}{\partial A_j} \right] \right)$$

$$e^{\frac{s_{|L(G,T)|+1}}{2} \sum_{ij} (\inf_{(k,l) \in P_{i \leftrightarrow j}^T} t_{kl}) \text{Tr} \left[\frac{\partial}{\partial A_i} \frac{\partial}{\partial A_j} \right]}.$$

The rest terms are all collected to yield the remaining terms in R'_{n+1} , as they are all LVE graphs of order $n + 2$, with the correct amplitude.

It remains to check that the evaluation of the new explicit terms reproduces exactly the perturbative evaluation of the amplitude of the graphs with exactly $E(G) = n + 1$ edges. As all the A_i 's are set to zero in the explicit terms, the product of traces yields just

$$N^{|F(G)|-|B(G)|} \prod_{f \in B(G)} \text{Tr}[(JJ^\dagger)^{c(f)}].$$

The integral over the loop parameters s_e can be trivially performed:

$$\int_{1 \geq s_1 \geq \dots \geq s_{|L(G,T)|} \geq 0} \prod_{e \in L(G,T)} ds_e = \frac{1}{|L(G, T)|!}.$$

Since there are precisely $|L(G, T)|!$ ways to label the loop edges, the sum becomes a sum over graphs with unlabeled loop edges. We are left with ciliated ribbon graphs with labels on their vertices and a distinguished spanning tree.

The integral over the weakening parameters t_e is more subtle (see [18]). A Hepp sector α in a graph G is a total order of the edges of G . For any Hepp sector α , the dominant tree $T(\alpha)$ in the sector is obtained by iteratively choosing the highest edges in the sector which do not form cycles. It turns out ([18]) that for fixed G and T the integral over the parameters t_e is the percentage of Hepp sectors of G in which the tree T is dominant. The following lemma is then trivial.

Lemma 4. *For any vertex labeled graph G ,*

$$\sum_{\substack{T \subset G \\ T \text{ spanning tree}}} \int \prod_{e \in E(T)} dt_e \left(\prod_{e=(i,j) \in L(G,T)} \inf_{e' \in P_{i \leftrightarrow j}^T} t_{e'} \right) = 1.$$

Collecting these two results together, the explicit terms are a sum over vertex labeled ciliated ribbon graphs with exactly $n + 1$ edges. Taking into account the amplitude does not depend on the labeling of the vertices, we collect together all the terms corresponding to different labellings of the same ribbon graph and use

$$\begin{aligned} & \frac{1}{|V(G)|!} \sum_{\text{labellings of } V(G)} (-\lambda)^{|E(G)|} N^{\chi(G)} \prod_{f \in B(G)} \text{Tr}[(JJ^\dagger)^{c(f)}] \\ &= \frac{(-\lambda)^{|E(G)|} N^{\chi(G)}}{|\text{Aut}(G)|} \prod_{f \in B(G)} \text{Tr}[(JJ^\dagger)^{c(f)}], \end{aligned}$$

where $|\text{Aut } G|$ corresponds to the order of the group of permutations of the vertices that leave the graph invariant to obtain the explicit terms in eq. (5).

Finally, the analyticity of the remainder is obvious since $R'_n[J, J^\dagger, \lambda, N]$ is a finite sum of analytic functions in $\mathbb{C} - \mathbb{R}^-$ and $R''_n[J, J^\dagger, \lambda, N]$ is bounded by the bound in eq. 17, hence converges and is analytic under the same hypothesis.

4.3. Topological expansion (proof of Theorem 3). In the expansion Theorem 2, we have recursively added loop edges to the trees irrespective of the genus of the graph (G, T) we obtained. In order to prove the topological expansion Theorem 3, we use the same algorithm, except that we stop adding loop edges to a graph if its genus reaches $g + 1$. We thus obtain

$$\begin{aligned} & \log \mathcal{Z}[J, J^\dagger; \lambda, N] \\ &= \sum_{\substack{G \text{ ciliated ribbon graph} \\ |E(G)| \leq n \text{ and } g(G) \leq g}} \frac{(-\lambda)^{|E(G)|} N^{2-2g(G)-B(G)}}{|\text{Aut}(G)|} \prod_{f \in B(G)} \text{Tr}[(JJ^\dagger)^{c(f)}] \\ & \quad + \tilde{\mathcal{R}}_{g,n}[J, J^\dagger; \lambda, N] + \tilde{\mathcal{R}}'_{g,n}[J, J^\dagger; \lambda, N] + \tilde{\mathcal{R}}''_n[J, J^\dagger; \lambda, N], \end{aligned}$$

where now there are three classes of remainder terms.

The first remainder term is made of LVE graphs with less than $n + 1$ edges such that the addition of the last loop edge (with label $|L(G, T)|$) increases the genus from g to $g + 1$:

$$\tilde{\mathcal{R}}_{g,n}[J, J^\dagger; \lambda, N] = \sum_{\substack{(G,T) \text{ LVE graph with } |E(G)| \leq n+1 \\ g(G)=g+1 \text{ and } g(G-e_{|L(G,T)|})=g}} \mathcal{A}_{(G,T)}[J, J^\dagger; \lambda, N].$$

The second remainder term is a summation over LVE graphs with $n + 1$ edges and genus less than g :

$$\tilde{\mathcal{R}}'_{g,n}[J, J^\dagger; \lambda, N] = \sum_{\substack{(G,T) \text{ LVE graph with} \\ |E(G)|=n+1 \text{ and } g(G) \leq g}} \mathcal{A}_{(G,T)}[J, J^\dagger; \lambda, N].$$

Finally, the last remainder term is a sum over LVE trees with at least $n + 2$ edges on which the loop generating algorithm has not yet been applied:

$$\tilde{\mathcal{R}}''_n[J, J^\dagger; \lambda, N] = \sum_{\substack{T \text{ LVE tree} \\ |E(T)| \geq n+2}} \mathcal{A}_T[J, J^\dagger; \lambda, N].$$

In order to take the limit $n \rightarrow \infty$, we bound the LVE amplitudes as well as the number of graphs contributing to the remainders.

To bound the LVE amplitude in eq. (3) we observe that the latter is a product over faces of traces of products of resolvents. Bounding each trace as $|\text{Tr}(O)| \leq N \|O\|$ and using Lemma 1 for the norm of the resolvents, we get

$$\begin{aligned} & |\mathcal{A}_{(G,T)}[J, J^\dagger, \lambda, N]| \\ & \leq \int \prod_{e \in E(T)} dt_e \left(\prod_{e=(i,j) \in L(G,T)} \inf_{e' \in P_{i \leftrightarrow j}^T} t_{e'} \right) \\ & \quad \frac{N^{|F(G)|+|V(G)|-|E(G)|} |\lambda|^{|E(G)|}}{|V(G)|! |L(G, T)|!} \left(\frac{1}{\cos\left(\frac{\arg \lambda}{2}\right)} \right)^{2E(G)+k} \|JJ^\dagger\|^k. \end{aligned}$$

This bound is very similar to the one of the tree amplitude (16), except that we get one factor of N for each internal face G and the integral over loop parameters s_e yields a factor $\frac{1}{|L(G,T)|!}$. Note that we have left the integral over the weakening parameters t_e since it allows to cancel the choice of the spanning tree. Let us denote by $\tilde{N}(g, n, k)$ the number of ribbon graphs with unlabeled vertices having genus g , n edges and k cilia. Using Proposition 4 and noticing that $\frac{1}{|L(G,T)|!}$ cancels the

labeling of the loop edges, we get

$$\begin{aligned}
 & \sum_{\substack{(G,T) \text{ LVE graph with } k \text{ cilia} \\ |E(G)|=n \text{ and } g(G)=g}} \int \prod_{e \in E(T)} dt_e \left(\prod_{e=(i,j) \in L(G,T)} \inf_{e' \in P_{i \leftrightarrow j}^T} t_{e'} \right) \\
 & \qquad \qquad \qquad \frac{1}{|V(G)|! |L(G, T)|!} \\
 & = \sum_{\substack{G \text{ ribbon graph with labeled vertices} \\ |E(G)| = n \text{ and } g(G) = g}} \frac{1}{|V(G)|!} \\
 & \leq \tilde{\mathcal{N}}(g, n, k).
 \end{aligned}$$

The inequality (instead of an equality) in the last line comes from the possibility to have different labellings of the vertices leading to the same unlabeled graph (this is also the origin of the factor $1/|\text{Aut}(G)|$ in the explicit terms).

Now we need a bound on $\tilde{\mathcal{N}}(g, n, k)$. We obtain all graph with k cilia by adding $k - 1$ cilia to a graph with 1 cilia. Relaxing the condition that there is at most one cilium per vertex to the condition that two cilia are not adjacent, in a graph with $n > 0$ edges and 1 cilium, there are $2n + 1 - 2 = 2n - 1$ corners on which we can add the second cilium (the graph has $2n + 1$ corners, but the 2 corners adjacent to the first cilium are forbidden). When adding the new cilium we create a new corner, but the two corners adjacent to the new cilium (which are distinct since $n > 0$) are forbidden. Therefore, there are $2n - 2$ corners on which we can add a third cilium and so on up to the last cilium for which we have $2n - (k - 1) = 2n + 1 - k$ available corners Therefore,

$$\begin{aligned}
 \tilde{\mathcal{N}}(g, n, k) & \leq \frac{\overbrace{(2n - 1) \dots (2n + 1 - k)}^{k-1 \text{ terms}}}{k!} \tilde{\mathcal{N}}(g, n, 1) \\
 & \leq \frac{(2n)!}{k!(2n - k)!} \tilde{\mathcal{N}}(g, n, 1),
 \end{aligned}$$

where we have divided by $k!$ since all the cilia are indistinguishable (i.e. adding cilia on the same corners but in a different order leads to the same graph). Note that we only obtain an inequality since adding cilia to different graphs can lead to the same ciliated graph.

The number of genus g graphs with n edges and a single cilium is equal to the number of rooted bipartite quadrangulations (due to the bijection in Proposition 4 applied to maps with one marked edge) which is known [14].

Lemma 5. *The number $\tilde{N}(g, n)$ of rooted maps with n edges and genus g has the asymptotic behavior*

$$\tilde{N}(g, n) \underset{n \rightarrow \infty}{\sim} C_g 12^n n^{\frac{5}{2}(g-1)},$$

with C_g a constant that only depends on the genus.

Consequently, there is a constant C'_g such that, for n large enough $\tilde{N}(g, n) = \tilde{N}(g, n, 1) \leq C'_g 12^n n^{\frac{5}{2}(g-1)}$, so that

$$\tilde{N}(g, n, k) \leq C'_g 12^n n^{\frac{5}{2}(g-1)} \frac{(2n)!}{k!(2n-k)!}. \quad (18)$$

We thus obtain the bound

$$\begin{aligned} & \sum_{\substack{(G,T) \text{ LVE graph with} \\ |E(G)|=n \text{ and } g(G)=g}} |\mathcal{A}_{(G,T)}[J, J^\dagger, \lambda, N]| \\ & \leq \sum_{k=1}^{n+1} N^{2-2g} |\lambda|^n \left(\frac{1}{\cos\left(\frac{\arg \lambda}{2}\right)} \right)^{2n+k} \|JJ^\dagger\|^k \frac{(2n)!}{k!(2n-k)!} \tilde{N}(g, n) \\ & \leq C'_g 12^n n^{\frac{5}{2}(g-1)} N^{2-2g} \left(\frac{|\lambda|}{\cos^2\left(\frac{\arg \lambda}{2}\right)} \right)^n \left(1 + \frac{\|JJ^\dagger\|}{\cos\left(\frac{\arg \lambda}{2}\right)} \right)^{2n}, \end{aligned}$$

where we have extended the sum over k from 0 to $2n$ (instead of $n+1$) and used the binomial formula.

For every $\lambda \in \tilde{\mathcal{C}}$, there exists $\epsilon_\lambda > 0$ such that

$$\left(\frac{|\lambda|}{\cos^2\left(\frac{\arg \lambda}{2}\right)} \right) \left(1 + \frac{\epsilon_\lambda}{\cos\left(\frac{\arg \lambda}{2}\right)} \right)^2 = \xi < \frac{1}{12}.$$

We choose $\|JJ^\dagger\| \leq \epsilon_\lambda$. We then have the following bounds:

- we bound the term

$$\tilde{\mathcal{R}}_{g,n}[J, J^\dagger; \lambda, N] = \sum_{\substack{(G,T) \text{ LVE graph with } |E(G)| \leq n+1 \\ g(G)=g+1 \text{ and } g(G-e_{L(G,T)})=g}} \mathcal{A}_{(G,T)}[J, J^\dagger; \lambda, N].$$

by a sum over graphs of genus $g+1$ having at most $n+1$ edges:

$$|\tilde{\mathcal{R}}_{g,n}[J, J^\dagger; \lambda, N]| \leq C'_{g+1} N^{2-2(g+1)} \sum_{m=0}^{n+1} m^{\frac{5}{2}g} (12\xi)^m,$$

which is convergent as $n \rightarrow \infty$ while keeping N fixed;

- we bound the term

$$\tilde{\mathcal{R}}'_{g,n}[J, J^\dagger; \lambda, N] = \sum_{\substack{(G,T) \text{ LVE graph with} \\ |E(G)|=n+1 \text{ and } g(G)\leq g}} \mathcal{A}_{(G,T)}[J, J^\dagger; \lambda, N].$$

by

$$|\tilde{\mathcal{R}}'_{g,n}[J, J^\dagger; \lambda, N]| \leq N^2 \left(\sum_{h=0}^g \frac{C'_h}{N^{2h}} (n+1)^{\frac{5}{2}(h-1)} \right) (12\xi)^{n+1},$$

hence this term goes to zero when sending $n \rightarrow \infty$ while keeping N fixed'

- as the sum over LVE trees is convergent, the last reminder term

$$\tilde{\mathcal{R}}''_n[J, J^\dagger; \lambda, N] = \sum_{\substack{T \text{ LVE tree} \\ |E(T)|\geq n+2}} \mathcal{A}_T[J, J^\dagger; \lambda, N],$$

also goes to zero when sending $n \rightarrow \infty$ while keeping N fixed;

This achieves the proof of Theorem 3.

5. Proofs of the theorems regarding the cumulants

5.1. Cumulants and their structure (proofs of Propositions 1 and 2). Before establishing the Proposition 1 we detail some properties of the Weingarten functions.

Lemma 6 (convolution inverse). *For $N > k$ and any permutations $\sigma, \tau \in \mathfrak{S}_k$, one has*

$$\sum_{\tau \in \mathfrak{S}_k} N^{|\mathcal{C}(\tau\rho^{-1})|} \text{Wg}(\tau\sigma^{-1}, N) = \begin{cases} 1 & \text{if } \rho = \sigma, \\ 0 & \text{otherwise,} \end{cases}$$

where $|\mathcal{C}(\sigma)|$ is the number of cycles in the decomposition of σ .

Proof. For any permutation ρ ,

$$\begin{aligned} & \sum_{a,c} \left(\prod_{i=1}^k \delta_{a_{\rho(i)}c_i} \right) \int dU U_{a_1 b_1} \dots U_{a_k b_k} U_{c_1 d_1}^* \dots U_{c_k d_k}^* \\ &= \sum_{a,c} \left(\prod_{i=1}^k \delta_{a_{\rho(i)}c_i} \right) \sum_{\sigma, \tau \in \mathfrak{S}_k} \left(\prod_{i=1}^k \delta_{a_{\tau(i)}c_i} \delta_{b_{\sigma(i)}d_i} \right) \text{Wg}(\tau\sigma^{-1}, N) \end{aligned}$$

implies

$$\left(\prod_{i=1}^k \delta_{b_{\rho(i)} d_i}\right) = \sum_{\sigma, \tau \in \mathfrak{S}_k} N^{|C(\tau\rho^{-1})|} \left(\prod_{i=1}^k \delta_{b_{\sigma(i)} d_i}\right) \text{Wg}(\tau\sigma^{-1}, N).$$

Applying this equality for $b_{\sigma(i)} = d_i = i$ the left hand side is non zero only for $\rho = \sigma$ and in this case it equals 1. \square

In our context, the Weingarten functions are used in order to write any unitary invariant homogeneous polynomial of degree k of a $N \times N$ Hermitian matrix H as a linear combination of products of traces of powers of H . Let P be such a polynomial:

$$P(H) = \sum_{1 \leq p_1, q_1, \dots, p_k, q_k \leq N} A_{p_1, q_1, \dots, p_k, q_k} H_{p_1 q_1} \cdots H_{p_k q_k},$$

with $P(UHU^\dagger) = P(H)$ for any $U \in U(N)$.

Lemma 7 (expansion over trace invariants). *Any unitary invariant degree k homogeneous polynomial can be written as*

$$P(H) = \sum_{\pi \in \Pi_k} P_\pi \text{Tr}_\pi(H),$$

with

$$P_\pi = \sum_{\substack{\sigma, \tau \in \mathfrak{S}_k \\ C(\sigma) = \pi}} \sum_{1 \leq p_1, \dots, p_k \leq N} A_{p_1 p_{\tau(1)}, \dots, p_k p_{\tau(k)}} \text{Wg}(\tau\sigma^{-1}, N)$$

and the trace invariant (that only depend on the cycle structure $C(\sigma)$) are

$$\text{Tr}_\pi(H) = \sum_{1 \leq a_1, \dots, a_k \leq N} H_{a_1 a_{\sigma(1)}} \cdots H_{a_k a_{\sigma(k)}}.$$

In particular, if P is already a trace invariant associated to the partition π_0 then $P_\pi = 1$ if $\pi = \pi_0$ and 0 otherwise.

Proof. Due to unitary invariance of P we have

$$\begin{aligned}
 P(H) &= \int [dU] P(UHU^\dagger) \\
 &= \sum_{1 \leq p_1, q_1, \dots, p_k, q_k \leq N} A_{p_1, q_1, \dots, p_k, q_k} \left(\prod_{i=1}^k U_{p_i a_i} U_{q_i b_i}^* H_{a_i b_i} \right) \\
 &= \sum_{\tau, \sigma \in \mathfrak{S}_k} \left(\sum_{1 \leq p_1, q_1, \dots, p_k, q_k \leq N} A_{p_1, q_1, \dots, p_k, q_k} \prod_{i=1}^k \delta_{p_\tau(i) q_i} \right) \\
 &\quad \left(\prod_{i=1}^k \delta_{a_{\sigma(i) b_i} H_{a_i b_i}} \right) \text{Wg}(\tau \sigma^{-1}, N).
 \end{aligned}$$

If P is the trace invariant associated to $\pi_0 = (k_1, \dots, k_{|\pi_0|})$ then

$$A_{p_1, q_1, \dots, p_k, q_k} = \frac{1}{\sum_{\substack{\rho \in \mathfrak{S}_k \\ C(\rho) = \pi_0}} 1} \sum_{\substack{\rho \in \mathfrak{S}_k \\ C(\rho) = \pi_0}} \prod_{i=1}^k \delta_{q_i p_{\rho(i)}}$$

and

$$\begin{aligned}
 P_\pi &= \sum_{\substack{\sigma, \tau \in \mathfrak{S}_k \\ C(\sigma) = \pi}} \left(\sum_{1 \leq p_1, \dots, p_k \leq N} \frac{1}{\sum_{\substack{\rho \in \mathfrak{S}_k \\ C(\rho) = \pi_0}} 1} \sum_{\substack{\rho \in \mathfrak{S}_k \\ C(\rho) = \pi_0}} \prod_{i=1}^k \delta_{p_\tau(i) p_{\rho(i)}} \right) \text{Wg}(\tau \sigma^{-1}, N) \\
 &= \frac{1}{\sum_{\substack{\rho \in \mathfrak{S}_k \\ C(\rho) = \pi_0}} 1} \sum_{\substack{\rho \in \mathfrak{S}_k \\ C(\rho) = \pi_0}} \sum_{\substack{\sigma, \tau \in \mathfrak{S}_k \\ C(\sigma) = \pi}} N^{|C(\tau \rho^{-1})|} \text{Wg}(\tau \sigma^{-1}, N).
 \end{aligned}$$

By the convolution inverse identity, the sum over τ enforces $\rho = \sigma$ and the lemma follows. \square

We choose a permutation $\zeta \in \mathfrak{S}_k$ whose cycle decomposition reproduces the contribution of the broken faces to the amplitude of a LVE graph,

$$\zeta = (i_1^1 \dots i_{k_1}^1) \cdots (i_1^b \dots i_{k_b}^b),$$

if there are $b = |B(G)|$ broken faces with k_1, \dots, k_b cilia. Denoting X^l the product of the resolvents in between the cilia l and $\zeta(l)$ and Y^m the product of the

resolvents around the unbroken face labeled m the amplitude can be written as

$$\begin{aligned}
 & A_{(G,T)}[J, J^\dagger, \lambda, N] \\
 &= \frac{(-\lambda)^{|E(G)|} N^{|V(G)|-|E(G)|}}{|V(G)|!} \int_{1 \geq s_1 \geq \dots \geq s_{|L(G,T)|} \geq 0} \prod_{e \in L(G,T)} ds_e \\
 & \int \prod_{e \in E(T)} dt_e \left(\prod_{e=(i,j) \in L(G,T)} \inf_{e' \in P_{i \leftrightarrow j}^T} t_{e'} \right) \\
 & \int d\mu_{s_{|L(G,T)|}} c_T(A) \prod_{1 \leq m \leq B(G)} \text{Tr} \left[J J^\dagger \prod_{1 \leq r \leq k_m}^{\overrightarrow{}} X^{i_r^m} \right] \prod_{1 \leq m \leq F(G)-B(G)} \text{Tr}[Y^m].
 \end{aligned}$$

This is a degree k homogeneous polynomial which is invariant under the unitary transformation $J \rightarrow UJ$ and $J^\dagger \rightarrow J^\dagger U^\dagger$. Indeed, because of the invariance of $d\mu_{c_T}(A)$ the transformation of J can be compensated by a transformation on the matrices A_1, \dots, A_n . Therefore, we may apply Lemma 7 to expand it over trace invariants,

$$A_{(G,T)}[J, J^\dagger, \lambda, N] = \sum_{\pi \in \Pi_k} \mathcal{A}_{(G,T)}^\pi(\lambda, N) \text{Tr}_\pi(JJ^\dagger),$$

with

$$\begin{aligned}
 & \mathcal{A}_{(G,T)}^\pi(\lambda, N) \\
 &= \frac{(-\lambda)^{|E(G)|} N^{|V(G)|-|E(G)|}}{|V(G)|!} \int_{1 \geq s_1 \geq \dots \geq s_{|L(G,T)|} \geq 0} \prod_{e \in L(G,T)} ds_e \\
 & \int \prod_{e \in E(T)} dt_e \left(\prod_{e=(i,j) \in L(G,T)} \inf_{e' \in P_{i \leftrightarrow j}^T} t_{e'} \right) \int d\mu_{s_{|L(G,T)|}} c_T(A) \\
 & \sum_{\substack{\tau, \sigma \in \mathfrak{S}_k \\ C(\sigma) = \pi}} \sum_{1 \leq p_1, \dots, p_k \leq N} \text{Wg}(\tau\sigma^{-1}, N) \prod_{1 \leq m \leq F(G)-B(G)} \text{Tr}[Y^m] \prod_{1 \leq l \leq k} X_{p_{\tau(l)} p_{\xi(l)}}^l.
 \end{aligned}$$

This proves Proposition 1.

In order to prove Proposition 2, we use the expression of $\log \mathcal{Z}[J, J^\dagger, \lambda, N]$ from Theorem 1. As the sum over LVE trees is convergent we can derive term by term with respect to the sources. Before deriving we express the amplitude of a LVE tree as in Proposition 1. Proposition 2 follows from the remark that for any two permutations $\tau, \xi \in \mathfrak{S}_k$ such that $\tau\xi^{-1}$ has cycle decomposition correspond-

ing to the partition π , we have

$$\begin{aligned} \text{Tr}_\pi(JJ^\dagger) &= \sum_{1 \leq p_1, q_1 \dots \leq N} \prod_{1 \leq l \leq k} J_{p_l q_l} J_{p_{\xi^{-1}(l)} q_l}^* \\ &= \sum_{1 \leq p_1, q_1 \dots \leq N} \prod_{1 \leq l \leq k} J_{p_l q_l} J_{p_{\tau(l)} q_{\xi(l)}}^*, \end{aligned}$$

hence the derivative with respect to the sources is

$$\begin{aligned} &\frac{\partial^k}{\partial J_{a_1 b_1} \dots \partial J_{a_k b_k}} \frac{\partial^k}{\partial J_{c_1 d_1}^* \dots \partial J_{c_k d_k}^*} \text{Tr}_\pi(JJ^\dagger) \\ &= \sum_{\rho, \sigma \in \mathfrak{S}_k} \sum_{1 \leq p_1, q_1 \dots \leq N} \prod_{1 \leq l \leq k} \delta_{a_{\rho(l)}, p_l} \delta_{b_{\rho(l)}, q_l} \delta_{c_{\sigma(l)}, p_{\tau(l)}} \delta_{d_{\sigma(l)}, q_{\xi(l)}}, \end{aligned}$$

and summing over p_l and q_l we obtain

$$\begin{aligned} &\frac{\partial^k}{\partial J_{a_1 b_1} \dots \partial J_{a_k b_k}} \frac{\partial^k}{\partial J_{c_1 d_1}^* \dots \partial J_{c_k d_k}^*} \text{Tr}_\pi(JJ^\dagger) \\ &= \sum_{\rho, \sigma \in \mathfrak{S}_k} \prod_{1 \leq l \leq k} \delta_{c_l, a_{\rho\tau\sigma^{-1}(l)}} \delta_{d_l, b_{\rho\xi\sigma^{-1}(l)}}. \end{aligned}$$

5.2. Constructive theorems for cumulants (proofs of Theorems 4 and 5).

In order to prove the constructive theorem for the cumulants we need to bound the amplitude in eq. (8). The contribution of the unbroken faces and of the broken faces are made of products of resolvents. The summation over the indices reproduces a product of $|C(\zeta\tau^{-1})|$ traces for the broken faces and $F(G) - B(G)$ traces for the unbroken faces.

We bound the Weingarten functions using the following lemma.

Lemma 8. *For N large enough,*

$$|Wg(\sigma, N)| < \frac{2^{2k}}{N^{2k - |C(\sigma)|}}.$$

This lemma can be deduced from the asymptotic behavior of the Weingarten functions [15, 16].

We bound the norm of the resolvents using Lemma 1 (recall that we have a resolvent per corner and there are $2|E(G)| + k$ corners on a LVE graph with $|E(G)|$ edges and k cilia). Taking into account that each trace produces a factor of N we

obtain

$$\left| \text{Wg}(\tau\sigma^{-1}, N) \sum_{1 \leq p_1, \dots, p_k \leq N} \prod_{1 \leq m \leq f-b} \text{Tr}[Y^m] \prod_{1 \leq l \leq k} X_{p_l p_{\xi\tau^{-1}(l)}}^l \right| \leq \frac{2^{2k} N^{|C(\xi\tau^{-1})| + |C(\tau\sigma^{-1})| - 2k + F(G) - B(G)}}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^{2E(G) + k}}.$$

In order to bound the scaling in N of the amplitude of a LVE graph we use the following Lemma [9].

Lemma 9. *Let σ and τ two permutations of k elements. Then, the numbers of cycles in the decompositions of σ , τ and $\sigma\tau$ obey*

$$|C(\sigma)| + |C(\tau)| \leq k + |C(\sigma\tau)|.$$

Proof. We will prove the following more general inequality: for any three permutations σ, τ and ξ of k elements we have

$$|C(\sigma\xi^{-1})| + |C(\xi\tau)| \leq k + |C(\sigma\tau)|.$$

Let us represent k white vertices labeled 1 to k and k black vertices labeled 1 to k . We connect the white vertex p with the black vertex $\tau(p)$ and the black vertex q with the white vertex $\xi(q)$. The cycles of $\xi\tau$ are the cycles made of alternating ξ and τ edges (the connected components of this graph).

If there exists a p such that $\xi\tau(p) \neq p$ we compare $|C(\xi\sigma^{-1})| + |C(\xi\tau)|$ with $|C(\sigma\tilde{\xi}^{-1})| + |C(\tilde{\xi}\tau)|$ where

$$\tilde{\xi}(q) = \begin{cases} \xi(q) & \text{for } q \neq \xi^{-1}(p), \tau(p), \\ \xi(\tau(p)) = \xi\tau\xi(q) & \text{for } q = \xi^{-1}(p), \\ p = \tau^{-1}(q) & \text{for } q = \tau(p). \end{cases}.$$

The number of cycles of $\xi\tau$ goes up by one:

$$|C(\xi\tau)| = 1 + |C(\tilde{\xi}\tau)|,$$

while the number of cycles of $\sigma\xi^{-1}$ can not decrease by more than one (looking at the graph corresponding to ξ and σ^{-1} we see that the change from ξ to $\tilde{\xi}$ can at most collapse two connected components into one). As $|C(\xi\sigma^{-1})| = |C(\sigma\tilde{\xi}^{-1})|$ we conclude that

$$|C(\sigma\tilde{\xi}^{-1})| + |C(\tilde{\xi}\tau)| \leq |C(\sigma\xi^{-1})| + |C(\xi\tau)|,$$

and now $\tilde{\xi}\tau(p) = p$. Iterating we obtain

$$|C(\sigma\xi^{-1})| + |C(\xi\tau)|(|C(\sigma\xi^{-1})| + |C(\xi\tau)|)_{\xi=\tau^{-1}} = k + |C(\sigma\tau)|. \quad \square$$

A double application of Lemma 9 leads to

$$|C(\xi\tau^{-1})| + |C(\tau\sigma^{-1})| \leq k + |C(\xi\sigma^{-1})| \leq 2k + |C(\xi)| - |C(\sigma)|,$$

and taking into account that $|C(\sigma)| = |\pi|$ (the number of integers in the partition π) and $|C(\xi)| = |B(G)|$ is the number of broken faces, we arrive at

$$|\mathcal{A}_{(G,T)}^\pi(\lambda, N)| \leq \frac{2^{2k} (k!)^2 |\lambda|^{|E(G)|} N^{|V(G)| - |E(G)| + |F(G)| - |\pi|}}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^{2|E(G)| + k} |V(G)|! (|E(G)| - |V(G)| + 1)!} \quad (19)$$

In particular, for a tree we have

$$\left| \mathcal{A}_T^\pi(\lambda, N) \right| \leq \frac{2^{2k} (k!)^2 |\lambda|^{|E(T)|} N^{2 - |\pi|}}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^{2|E(T)| + k} |V(T)|!}.$$

This establishes Theorem 4.

In order to prove Theorem 5 we apply the same algorithm as before and obtain the perturbative series with remainder

$$\mathcal{R}_{\pi,n}(\lambda, N) = \sum_{\substack{(G,T) \text{ LVE graph} \\ |E(G)| = n+1 \text{ and } |K(G)| = k}} \mathcal{A}_{(G,T)}^\pi(\lambda, N) + \sum_{\substack{T \text{ LVE tree} \\ |E(T)| \geq n+2 \text{ and } |K(T)| = k}} \mathcal{A}_T^\pi(\lambda, N),$$

where $K(G)$ denotes the number of cilia of G . In order to bound this remainder we use eq. (19) and bound separately the contributions of the trees and of the LVE graphs with loop edges.

The contribution of trees with $E(T) \geq n + 2$ edges is bounded

$$\left| \sum_{\substack{T \text{ LVE tree} \\ |E(T)| \geq n+2 \text{ and } |K(T)| = k}} \mathcal{A}_T^\pi(\lambda, N) \right| \leq \sum_{n' \geq n+2} \mathcal{N}(n', k) \frac{2^{2k} (k!)^2 |\lambda|^{n'} N^{2 - |\pi|}}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^{2n' + k} (n' + 1)!}$$

The number of LVE trees with n' edges and k cilia $\mathcal{N}(n', k)$ has been evaluated in Lemma 3 and we get

$$\begin{aligned} & \left| \sum_{\substack{T \text{ LVE tree} \\ |E(T)| \geq n+2 \text{ and } |K(T)| = k}} \mathcal{A}_T^\pi(\lambda, N) \right| \\ & \leq \frac{N^{2 - |\pi|} 2^{3k-1} k!}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^k} \sum_{n' \geq n+2} \frac{(n' - 1)!}{(n' + 1 - k)!} \frac{2^{2n'} |\lambda|^{n'}}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^{2n'}}. \end{aligned}$$

At fixed k the ratio $\frac{(n'-1)!}{(n'+1-k)!}$ is a polynomial in n' so that the series on the right hand side is convergent for $4|\lambda| < \cos^2 \frac{\arg \lambda}{2}$. It can be rewritten as

$$\begin{aligned} & \sum_{n' \geq n+2} \frac{(n'-1)!}{(n'+1-k)!} \frac{2^{2n'} |\lambda|^{n'}}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^{2n'}} \\ &= \sum_{m \geq 0} \frac{(m+n+1)!}{(m+n+3-k)!} \left(\frac{4|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2}\right)^{m+n+2}, \end{aligned}$$

and, as $n-1 \geq k$ (as the LVE trees have at most a cilium per vertex) we have $m! \geq (m+n+3-k)!$, hence

$$\begin{aligned} & \left| \sum_{\substack{T \text{ LVE tree} \\ |E(T)| \geq n+2 \text{ and } |K(T)|=k}} \mathcal{A}_T^\pi(\lambda, N) \right| \\ & \leq \frac{N^{2-|\pi|} 2^{3k-1} k!}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^k} (n+1)! \sum_{m \geq 0} \binom{m+n+1}{m} \left(\frac{4|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2}\right)^{m+n+2} \\ &= \frac{N^{2-|\pi|} 2^{3k-1} k!}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^k} (n+1)! \frac{\left(\frac{4|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2}\right)^{n+2}}{\left(1 - \frac{4|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2}\right)^{n+2}}. \end{aligned}$$

Denoting by n' the number of edges in a spanning tree (so that the graph has $n'+1$ vertices) and n'' the number of loop edges of a LVE graphs with loop edges, the contribution to the rest term of these graphs is bounded by

$$\begin{aligned} & \left| \sum_{\substack{(G,T) \text{ LVE graph} \\ |E(G)|=n+1 \text{ and } |K(G)|=k}} \mathcal{A}_{(G,T)}^\pi(\lambda, N) \right| \\ & \leq \sum_{n'+n''=n+1} \mathcal{N}(n', n'', k) \frac{2^{2k} (k!)^2 |\lambda|^{n'+n''} N^{2-|\pi|}}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^{2n'+2n''+k} (n'+1)! (n'')!}, \end{aligned}$$

where $\mathcal{N}(n', n'', k)$ is the number of LVE graphs with n'' loop edges, $n'+1$ vertices and k cilia.

The following lemma is an immediate consequence of the counting of LVE graph with given number of vertices, cilia and loop edges performed in [9].

Lemma 10 (counting LVE graphs). *The number of LVE graphs with $n'+1$ vertices, n'' loop edges and k cilia reads*

$$\mathcal{N}(n', n'', k) = \frac{(2n' + 2n'' + k - 1)! (n' + 1)!}{(n' + k)! 2^{n''} k! (n' + 1 - k)!}.$$

Proof. First notice that the number of LVE graphs with $n' + 1$ vertices, k cilia on a specific set of vertices and n'' loop edges reads [9]

$$\frac{(2(n' + n'') + k - 1)!}{2^{n''}(n' + k)!}.$$

Then, we obtain $\mathcal{N}(n', n'', k)$ by counting the configurations of vertices that can carry cilia, so that

$$\begin{aligned} \mathcal{N}(n', n'', k) &= \frac{(2(n' + n'') + k - 1)!}{2^{n''}(n' + k)!} \times \frac{(n' + 1)!}{(n' + 1 - k)!k!} \\ &= \frac{(2n' + 2n'' + k - 1)!(n' + 1)!}{(n' + k)!2^{n''}k!(n' + 1 - k)!}. \end{aligned} \quad \square$$

Using the binomial bound

$$\frac{1}{(n' + k)!(n'')!} \leq \frac{2^{n'+n''+k}}{(n' + n'' + k)!}$$

and the trivial inequality

$$\frac{1}{(n' + k - 1)!} \leq 1,$$

we arrive at

$$\begin{aligned} &\left| \sum_{\substack{(G,T) \text{ LVE graph} \\ |E(G)|=n+1}} \mathcal{A}_{(G,T)}^\pi(\lambda, N) \right| \\ &\leq \frac{2^{3k+n+1}k!|\lambda|^{n+1}N^{2-|\pi|}(2n+k+1)!}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^{2n+2+k}(n+k+1)!} \sum_{n'+n''=n+1} 1. \end{aligned}$$

Another use of the binomial formula shows that $\frac{(2n+k+1)!}{(n+k+1)!} \leq 2^{2n+k+1}n!$ so that

$$\left| \sum_{\substack{(G,T) \text{ LVE graph} \\ |E(G)|=n+1 \text{ and } |K(T)|=k}} \mathcal{A}_{(G,T)}^\pi(\lambda, N) \right| \leq \frac{2^{4k+3n+2}k!|\lambda|^{n+1}N^{2-|\pi|}n!(n+2)}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^{2n+2+k}}.$$

Since $n + 2 < 2(n + 1)$, this also implies

$$\left| \sum_{\substack{(G,T) \text{ LVE graph} \\ |E(G)|=n+1 \text{ and } |K(T)|=k}} \mathcal{A}_{(G,T)}^\pi(\lambda, N) \right| \leq \frac{2^{4k+3n+3}k!|\lambda|^{n+1}N^{2-|\pi|}(n+1)!}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^{2n+2+k}}.$$

Summing up the two bounds we obtain

$$\begin{aligned}
& |\mathcal{R}_{\pi,n}(\lambda, N)| \\
& \leq \frac{N^{2-|\pi|} 2^{3k-1} k!}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^k} (n+1)! \frac{\left(\frac{4|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2}\right)^{n+2}}{\left(1 - \frac{4|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2}\right)^{n+2}} \\
& \quad + \frac{2^{4k+3n+3} k! |\lambda|^{n+1} N^{2-|\pi|} (n+1)!}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^{2n+2+k}} \\
& = N^{2-|\pi|} \left(\frac{2^{3k-1} k!}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^k}\right) (n+1)! \left(\frac{4|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2}\right)^{n+1} \\
& \quad \left(\frac{\frac{4|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2}}{\left(1 - \frac{4|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2}\right)^{n+2}} + 2^{k+n+2}\right).
\end{aligned}$$

This establishes Theorem 5.

5.3. Topological expansion for the cumulants (proof of Theorem 7). The proof of Theorem 7 proceeds along the same lines as that of Theorem 5, except that the perturbative expansion involves contributions of all graphs up to genus g and the remainder contains graphs of genus $g + 1$. As before, the main idea is to express the contribution of Feynman graphs and LVE in terms of trace invariants.

Starting with eq. (7), we collect terms homogeneous of degree k in JJ^\dagger . The first term is obviously written as a sum of trace invariants

$$\begin{aligned}
& \sum_{\substack{G \text{ ribbon graph with } k \text{ cilia} \\ \text{broken faces corresponding to } \pi \text{ and } g(G) \leq g}} \frac{(-\lambda)^{|E(G)|} N^{\chi(G)}}{|\text{Aut}(G)|} \prod_{f \text{ broken face}} (JJ^\dagger)^{c(f)} \\
& = \sum_{\substack{G \text{ ribbon graph with } k \text{ cilia} \\ \text{broken faces corresponding to } \pi \text{ and } g(G) \leq g}} \frac{(-\lambda)^{|E(G)|} N^{\chi(G)}}{|\text{Aut}(G)|} \text{Tr}_\pi(JJ^\dagger).
\end{aligned}$$

After derivation with respect to the sources, it yields a contribution to $K_{\pi,g}(\lambda, n)$ of the form

$$\sum_{\substack{G \text{ ribbon graph with } k \text{ cilia} \\ \text{broken faces corresponding to } \pi \text{ and } g(G) \leq g}} \frac{(-\lambda)^{|E(G)|} N^{\chi(G)}}{|\text{Aut}(G)|},$$

which is just the sum over Feynman graph of genus less than g and broken faces corresponding to π . Recall that the number of graphs of genus g with n edges is bounded as in (18):

$$\tilde{N}(g, n, k) \leq C_g'' 12^n n^{\frac{5}{2}(g-1)} \frac{(2n)!}{k!(2n-k)!}.$$

Accordingly, the series $\sum_n \tilde{N}(g, n, k) z^k$ converges for $|z| < \frac{1}{12}$.

Consider now the remainder $\tilde{R}_{\pi,g}(\lambda, N)$, containing only genus $g + 1$ LVE graphs and express it in terms of trace invariants:

$$\tilde{\mathcal{R}}_{\pi,g}(\lambda, N) = \sum_{\substack{(G,T) \text{ LVE graphs with broken faces corresponding to } \pi \\ g(G)=g+1 \text{ and } g(G-e_{|L(G,T)|})=g}} \mathcal{A}_{(G,T)}^\pi(\lambda, N).$$

We bound each LVE graph as in (19). Since this bound only depends on the graph G and not on the choice of the spanning tree T , we use Lemma 4 to get

$$|\tilde{\mathcal{R}}_{\pi,g}(\lambda, N)| \leq \sum_{n=2(g+1)}^\infty \tilde{N}(g+1, n, k) \frac{2^{2k} (k!)^2 |\lambda|^{|n|} N^{2-2(g+1)-|\pi|}}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^{2n+k}}$$

hence

$$\begin{aligned} & |\tilde{\mathcal{R}}_{\pi,g}(\lambda, N)| \\ & \leq N^{2-2(g+1)-|\pi|} \frac{2^{3k} k!}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^k} C_{g+1}'' \sum_{n=2(g+1)}^\infty n^{\frac{5}{2}(g-1)+k} \left(\frac{12|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2}\right)^n \\ & = N^{2-2(g+1)-|\pi|} \frac{2^{3k} k!}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^k} C_{g+1}'' \left(\frac{12|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2}\right)^{2g+2} \\ & \quad \times \sum_{m \geq 0} (m+g+1)^{\frac{5}{2}(g-1)+k} \left(\frac{12|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2}\right)^m. \end{aligned}$$

Bounding

$$(m+g+1)^{\frac{5}{2}(g-1)+k} \leq (m+g+1)^{3g+k} \leq \frac{(m+4g+k+1)!}{m!},$$

we obtain the rough bound

$$\begin{aligned}
 &|\tilde{R}_{\pi,g}(\lambda, N)| \\
 &\leq N^{2-2(g+1)-|\pi|} \frac{2^{3k} k!}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^k} C''_{g+1} \left(\frac{12|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2}\right)^{2g+2} \\
 &\quad \frac{(4g+k+1)!}{\left(1 - \frac{12|\lambda|}{\left(\cos\left(\frac{\arg \lambda}{2}\right)\right)^2}\right)^{4g+k}}.
 \end{aligned}$$

This achieves the proof of Theorem 7.

Appendixes

A. Schwinger–Dyson equations for the intermediate field

In this Appendix, we derive the explicit formula for the order 2 cumulant in the large N limit using the Schwinger–Dyson equation for the intermediate field. This allows us to express it as an explicit power series collecting all the planar graphs, with two external legs in the matrix model formulation or, equivalently, with one cilium in the intermediate field representation. This result is classical and goes back to Koplik, Neveu and Nussinov [19] which treated the ϕ^4 interaction without recourse to the intermediate field.

Let us collect all planar graphs contributing to the order 2 cumulant in the power series

$$G(\lambda, q) = \sum_{m,n} G_{m,n} q^m \lambda^n,$$

with $G_{m,n}$ the number of connect planar graphs with n vertices (intermediate field edges or equivalently, matrix vertices $\text{Tr}(MM^\dagger MM^\dagger)$) and m boundary matrix lines. The indeterminate q only appears in the formulation of the Schwinger–Dyson equation and the order 2 cumulant is recovered in the limit $q \rightarrow 1$.

It obeys the planar Schwinger–Dyson equation

$$G(\lambda, q) = q + \lambda q G^2(\lambda, q) + \lambda q^2 \frac{G(\lambda, q) - G(\lambda, 1)}{q - 1}. \tag{20}$$

Graphically, this equation can be derived as follows. Starting from the incoming M line, there are three possibilities, see Figure 10.

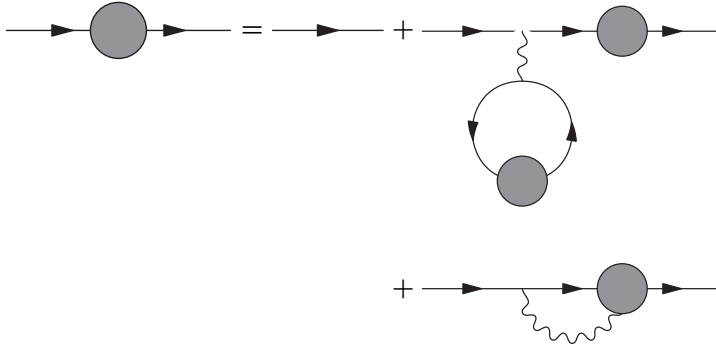


Figure 10. Schwinger–Dyson equations in the intermediate field representation.

- We do not meet an interaction with the intermediate field so that the graph reduces to a single matrix line (first term in Figure 10 and (20)).
- The removal of the first AMM^\dagger vertex we encounter disconnects the graph (second term in Figure 10 and (20)).
- The graph remains connected after the removal of the first vertex encountered (third term in Figure 10 and (20)). In this case, we have to keep track of the various possibilities of attaching the intermediate field line on the external boundary of the graph. This is the origin of the q -derivative term

$$\frac{G(\lambda, q) - G(\lambda, 1)}{q - 1} = \sum_{m \geq 1, n \geq 0} G_{m,n} \left(\sum_{k=0}^{m-1} q^k \right) \lambda^n,$$

since the insertion of the A line encloses k M -lines for $0 \leq k \leq m - 1$.

At lowest order in n , the explicit expressions of $G_n(q) = \sum_m G_{m,n} q^n$ read

$$\begin{aligned} G_0(q) &= q, \\ G_1(q) &= q^2 [2]_q, \\ G_2(q) &= 2q^4 [2]_q + q^2 ([2]_q + [3]_q), \\ G_3(q) &= q^5 ([2]_q)^2 + 4q^5 [2]_q + 2q^3 ([2]_q + [3]_q) \\ &\quad + 2([4]_q + [5]_q) + (2[2]_q + 2[3]_q + [4]_q), \end{aligned}$$

with $[n]_q = \frac{q^n - 1}{q - 1}$.

The planar Schwinger–Dyson equation (20) $G(1, \lambda)$ is quadratic in $G(q)$ and can be used to determine $G(q)$ in terms of the variables q and λ and of $G(1, \lambda)$ treated as an independent variable

$$G(q, \lambda) = \frac{q - 1 - \lambda q^2 - \sqrt{(q - 1 - \lambda q^2)^2 - 4\lambda q(q - 1)[q(q - 1) - \lambda q^2 G(1, \lambda)]}}{\lambda q(q - 1)},$$

where we have retained the solution with a well defined limit at $\lambda = 0$.

In order for $G(q)$ to be analytic in q and λ , $G(1, \lambda)$ which is itself a series in λ , must be such that the polynomial under the square root has a double root in q . This polynomial reads

$$(\lambda^2 - 4\lambda + 4\lambda^2 G(1, \lambda))q^4 + (6\lambda - 4\lambda^2 G(1, \lambda))q^3 + (1 - 2\lambda)q^2 - 2q + 1,$$

and its discriminant factorizes as

$$\lambda^4(1 - \lambda G(1, \lambda))^2(1 - G(1, \lambda) - 16\lambda + 18\lambda G(1, \lambda) - 27\lambda^2 G^2(1, \lambda)).$$

Discarding the solution $G(1, \lambda) = \frac{1}{\lambda}$, the solution of the quadratic part yields the planar contribution to the order 2 cumulant

$$G(1, \lambda) = \frac{-1 + 18\lambda - (1 - 12\lambda)^{3/2}}{54\lambda^2}.$$

Its expansion as a power series in λ reads

$$G(1, \lambda) = \sum_n \frac{2 \cdot 3^n}{n + 2} C_n \lambda^n \quad \text{with } C_n = \frac{(2n)!}{n!^2(n + 1)}.$$

This reproduces the counting of planar ribbon graphs with one cilium, or equivalently, rooted, bipartite quadrangulations.

B. The BKAR forest formula

In this appendix, we briefly review the Brydges-Kennedy-Abdesselam-Rivasseau (BKAR) forest formula [12] which allows us to expand $\log \mathcal{Z}[J, J^\dagger, \lambda, N]$ as a sum over trees.

Let ϕ be a function of $\mathbb{R}^{\frac{n(n-1)}{2}}$ whose arguments u_{ij} are associated to the edges of the complete graph on n vertices labeled $\{1, 2, \dots, n\}$. The following theorem yields an expansion of $\phi(1, \dots, 1)$ as a sum over forests with n labeled vertices. Recall that a forest is a subset of edges of the complete graph that does not contain any cycle.

For every forest F , let us denote (if it exists) $P_{i \leftrightarrow j}^F$ the unique path in the forest F joining the vertices i and j .

Theorem 8 (Brydges–Kennedy–Abdesselam–Rivasseau). *Let $\phi : \mathbb{R}^{\frac{n(n-1)}{2}} \rightarrow \mathbb{C}$ be a smooth, sufficiently derivable function. Then*

$$\phi(1, \dots, 1) = \sum_{F \text{ forest}} \int_0^1 \prod_{(i,j) \in F} du_{ij} \left(\frac{\partial^{|E(F)|} \phi}{\prod_{(i,j) \in F} \partial x_{ij}} \right) (v_{ij}^F),$$

where v_{ij}^F is given by

$$v_{ij}^F = \begin{cases} \inf_{(k,l) \in P_{i \leftrightarrow j}^F} u_{kl} & \text{if } P_{i \leftrightarrow j}^F \text{ exists,} \\ 0 & \text{otherwise,} \end{cases}$$

and $|E(F)|$ is the number of edges in the forest F .

This theorem is a broad generalization of the fundamental theorem of calculus, to which it reduces when $n = 2$. Indeed, in this case there are two forests on the complete graph with 2 vertices (thus with a single edge, see Figure 11) and we get

$$\phi(1) = \phi(0) + \int_0^1 dt_{12} \left(\frac{\partial \phi}{\partial x_{12}} \right) (t_{12})$$

The first term corresponds to the empty forest ($|E(F)| = 0$) and the second one to the full forest ($|E(F)| = 1$).



Figure 11. The two forests built on two vertices.

For $n = 3$, we have seven forests (see Figure 12) and we get

$$\begin{aligned} \phi(1, 1, 1) &= \phi(0, 0, 0) + \int_{[0,1]} dt_{12} \left(\frac{\partial \phi}{\partial x_{12}} \right) (t_{12}, 0, 0) \\ &\quad + \int_{[0,1]} dt_{23} \left(\frac{\partial \phi}{\partial x_{23}} \right) (0, t_{23}, 0) \\ &\quad + \int_{[0,1]} dt_{13} \left(\frac{\partial \phi}{\partial x_{13}} \right) (0, 0, t_{13}) \\ &\quad + \int_{[0,1]^2} dt_{12} dt_{23} \left(\frac{\partial^2 \phi}{\partial x_{12} \partial x_{23}} \right) (t_{12}, t_{23}, \inf(t_{12}, t_{23})) \end{aligned}$$

$$\begin{aligned}
 &+ \int_{[0,1]^2} dt_{12} dt_{13} \left(\frac{\partial^2 \phi}{\partial x_{12} \partial x_{13}} \right) (t_{12}, \inf(t_{12}, t_{13}), t_{13}) \\
 &+ \int_{[0,1]^2} dt_{23} dt_{13} \left(\frac{\partial^2 \phi}{\partial x_{23} \partial x_{13}} \right) (\inf(t_{23}, t_{13}), t_{23}, t_{13}).
 \end{aligned}$$

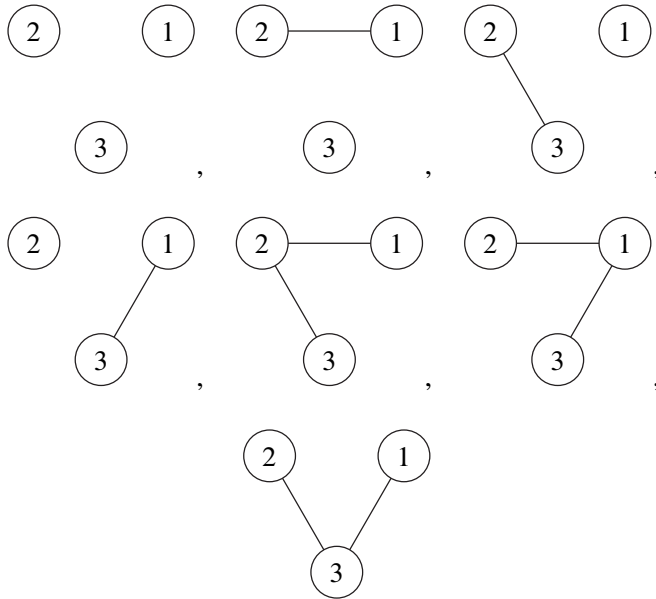


Figure 12. The seven forests built on three vertices.

The first term corresponds to the empty forest, the next three to the forests with one edge and the last three to the forests with two edges.

In quantum field theory, the main interest of this formula lies in the fact that it provides an expansion for the partition function with sources $\mathcal{Z}[J]$ as a sum over forests. Its logarithm is then readily computed as a *convergent* sum over trees.

C. Some examples of LVE graphs and their amplitudes

Here we illustrate how the LVE graph amplitude (3) is computed on a few examples. We use a double line representation for the intermediate field instead of a wavy line, $\text{~~~~~} \rightarrow \text{=====}$.

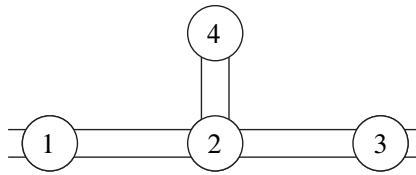


Figure 13. A LVE tree.

For the tree in Figure 13, the amplitude reads

$$\begin{aligned}
 \mathcal{A} &= \int_0^1 dt_{12} dt_{23} dt_{24} \int d\mu_C(A) \\
 &= \frac{N(-\lambda)^4}{3!} \int_0^1 dt_{12} dt_{23} dt_{24} \int d\mu_C(A) \\
 &= \text{Tr} \left[\left(1 - i\sqrt{\frac{\lambda}{N}} A_3 \right)^{-1} \left(1 - i\sqrt{\frac{\lambda}{N}} A_2 \right)^{-1} \left(1 - i\sqrt{\frac{\lambda}{N}} A_4 \right)^{-1} \right. \\
 &\quad \left. \left(1 - i\sqrt{\frac{\lambda}{N}} A_2 \right)^{-1} \left(1 - i\sqrt{\frac{\lambda}{N}} A_1 \right)^{-1} J J^\dagger \left(1 - i\sqrt{\frac{\lambda}{N}} A_1 \right)^{-1} \right. \\
 &\quad \left. \left(1 - i\sqrt{\frac{\lambda}{N}} A_2 \right)^{-1} \left(1 - i\sqrt{\frac{\lambda}{N}} A_3 \right)^{-1} J J^\dagger \right],
 \end{aligned}$$

with covariance matrix

$$C = \begin{pmatrix} 1 & t_{12} & \inf(t_{12}, t_{23}) & \inf(t_{12}, t_{24}) \\ t_{12} & 1 & t_{23} & t_{24} \\ \inf(t_{12}, t_{23}) & t_{23} & 1 & \inf(t_{23}, t_{24}) \\ \inf(t_{12}, t_{24}) & t_{24} & \inf(t_{23}, t_{24}) & 1 \end{pmatrix}.$$

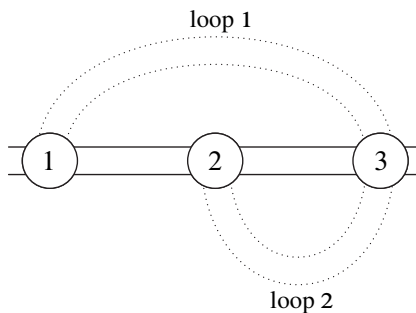
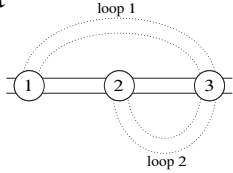


Figure 14. A planar LVE graph.

For the planar LVE graph in Figure 14, the amplitude reads

\mathcal{A}



$$\begin{aligned}
 &= \frac{N^{-1}(-\lambda)^4}{3!} \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0 ds_3 dt_{12} dt_{23} dt_{24} \inf(t_{12}, t_{23}) t_{23} \int d\mu_{s_2 C}(A) \\
 &\text{Tr} \left[\left(1 - i\sqrt{\frac{\lambda}{N}} A_3\right)^{-1} \left(1 - i\sqrt{\frac{\lambda}{N}} A_1\right)^{-1} J J^\dagger \right. \\
 &\quad \left. \left(1 - i\sqrt{\frac{\lambda}{N}} A_1\right)^{-1} \left(1 - i\sqrt{\frac{\lambda}{N}} A_2\right)^{-1} \left(1 - i\sqrt{\frac{\lambda}{N}} A_3\right)^{-1} J J^\dagger \right] \\
 &\text{Tr} \left[\left(1 - i\sqrt{\frac{\lambda}{N}} A_1\right)^{-1} \left(1 - i\sqrt{\frac{\lambda}{N}} A_2\right)^{-1} \left(1 - i\sqrt{\frac{\lambda}{N}} A_3\right)^{-1} \right] \\
 &\quad \text{Tr} \left[\left(1 - i\sqrt{\frac{\lambda}{N}} A_2\right)^{-1} \left(1 - i\sqrt{\frac{\lambda}{N}} A_3\right)^{-1} \right],
 \end{aligned}$$

with covariance matrix

$$s_2 C = s_2 \begin{pmatrix} 1 & t_{12} & \inf(t_{12}, t_{23}) \\ t_{12} & 1 & t_{23} \\ \inf(t_{12}, t_{23}) & t_{23} & 1 \end{pmatrix}.$$

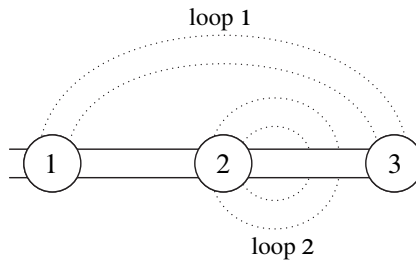
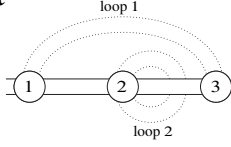


Figure 15. A non planar LVE graph.

For the non planar LVE graph in Figure 15, the amplitude reads

\mathcal{A}



$$\begin{aligned}
 &= \frac{N^{-1}(-\lambda)^4}{3!} \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} dt_{12} dt_{23} dt_{24} \inf(t_{12}, t_{23}) \int d\mu_{s_2} C(A) \\
 &\quad \text{Tr} \left[\left(1 - i\sqrt{\frac{\lambda}{N}} A_1\right)^{-1} \left(1 - i\sqrt{\frac{\lambda}{N}} A_2\right)^{-1} \left(1 - i\sqrt{\frac{\lambda}{N}} A_2\right)^{-1} \left(1 - i\sqrt{\frac{\lambda}{N}} A_1\right)^{-1} \right. \\
 &\quad \quad \left(1 - i\sqrt{\frac{\lambda}{N}} A_3\right)^{-1} \left(1 - i\sqrt{\frac{\lambda}{N}} A_2\right)^{-1} \left(1 - i\sqrt{\frac{\lambda}{N}} A_2\right)^{-1} \\
 &\quad \quad \left. \left(1 - i\sqrt{\frac{\lambda}{N}} A_3\right)^{-1} \left(1 - i\sqrt{\frac{\lambda}{N}} A_1\right)^{-1} J J^\dagger \right],
 \end{aligned}$$

with covariance matrix

$$s_2 C = s_2 \begin{pmatrix} 1 & t_{12} & \inf(t_{12}, t_{23}) \\ t_{12} & 1 & t_{23} \\ \inf(t_{12}, t_{23}) & t_{23} & 1 \end{pmatrix}. \quad (21)$$

D. Analyticity domain for the vector model

In the case of the vector model, the cardioid can be extended to reach the real negative axis. This result is not new, see [20]. An argument similar to ours, based on the LVE and contour rotation can also be found in [13].

The cumulants of the vector model are defined by the generating $\mathcal{W}[J, J^\dagger] = \log \mathcal{Z}[J, J^\dagger]$ with

$$\mathcal{Z}[J, J^\dagger] = \int d\Phi \exp - \left\{ \Phi^\dagger \Phi + \frac{\lambda}{2N} (\Phi^\dagger \Phi)^2 + \sqrt{N} \Phi^\dagger J + \sqrt{N} J^\dagger \Phi \right\}$$

where $\Phi \in \mathbb{C}^N$ is a N component vector and $\Phi^\dagger \Phi = |\Phi_1|^2 + \dots + |\Phi_N|^2$. As for the matrix model, the integral is normalized such that $\mathcal{Z}[J, J^\dagger] = 1$ at $\lambda = 0$. The sources J and J^\dagger are also N component complex vectors.

The vector model admits an intermediate field representation based on

$$\exp - \frac{\lambda}{2N} (\Phi^\dagger \Phi)^2 = \int dA \exp - \left\{ \frac{1}{2} A^2 - i\sqrt{\frac{\lambda}{N}} A \Phi^\dagger \Phi \right\}$$

where the integral is over a real scalar A . Thus, the Gaussian integral over Φ can be performed

$$\begin{aligned} \mathcal{Z}[J, J^\dagger] &= \int d\Phi dA \exp - \left\{ \frac{1}{2} A^2 + \left[\Phi^\dagger \left(1 - i \sqrt{\frac{\lambda}{N}} A \right) \Phi \right] + \sqrt{N} \Phi^\dagger J + \sqrt{N} J^\dagger \Phi \right\} \\ &= \int dA \exp - \left\{ \frac{1}{2} A^2 + N \log \left(1 - i \sqrt{\frac{\lambda}{N}} A \right) + N J^\dagger J \left(1 - i \sqrt{\frac{\lambda}{N}} A \right)^{-1} \right\} \end{aligned}$$

Let us notice two differences with respect to the matrix integral eq. (13): There is no power of N in front of the resolvent and A and $J^\dagger J$ are scalars, so that they can be commuted. However, the perturbative expansion of the vector model in the intermediate field also involves ribbon graphs, even if all quantities are scalars. This is so because the interaction vertices are based on resolvents, which have a cyclic ordering of their half edges.

Then, we perform the loop vertex expansion and expand $\log \mathcal{Z}[J, J^\dagger]$ over ciliated ribbon trees

$$\begin{aligned} \log \mathcal{Z}[J, J^\dagger] &= \sum_{T \text{ tree}} \frac{N(-\lambda)^{|E(T)|} (J^\dagger J)^k}{|V(T)|!} \int dt \int d\mu_{C_T}(A) \prod_{i=1}^n \left(1 - i \sqrt{\frac{\lambda}{N}} A_i \right)^{-l_i}, \end{aligned}$$

where n is the number of vertices of T , k the number of cilia and l_i the number of corners attached to vertex i . This expression has the same domain of convergence as the matrix integral, stated in Theorem 1.

However, the domain of analyticity can be enlarged. To proceed, let us write the powers of the resolvent as

$$\left(1 - i \sqrt{\frac{\lambda}{N}} A \right)^{-l} = \frac{1}{\Gamma(l)(\sqrt{\lambda})^l} \int_0^\infty d\alpha \alpha^{l-1} \exp - \alpha \left\{ \frac{1}{\sqrt{\lambda}} - \frac{i}{\sqrt{N}} A \right\}$$

Inserting this representation in the integral over replicas yields

$$\begin{aligned} &\int d\mu_{C_T}(A) \prod_{i=1}^n \left(1 - i \sqrt{\frac{\lambda}{N}} A_i \right)^{-l_i} \\ &= \int_0^\infty \prod_i d\alpha_i \frac{\prod_i \alpha_i^{l_i-1}}{(\sqrt{\lambda})^{2n-2+k} \prod_i \Gamma(l_i)} \int d\mu_{C_T}(A) \exp - \sum_i \left\{ \frac{\alpha_i}{\sqrt{\lambda}} - \frac{i\alpha_i A_i}{\sqrt{N}} \right\}. \end{aligned}$$

The main simplification in the vector model case is that the integral over the replicas is Gaussian and can be performed explicitly

$$\int d\mu_{C_T}(A) \exp - \sum_i \left\{ \frac{\alpha_i}{\sqrt{\lambda}} - \frac{i\alpha_i A_i}{\sqrt{N}} \right\} = \exp - \left\{ \sum_i \frac{\alpha_i}{\sqrt{\lambda}} + \sum_{i,j} \frac{C_{ij}\alpha_i\alpha_j}{2N} \right\}.$$

Therefore, the loop vertex expansion of the vector model reads

$$\begin{aligned} \log \mathcal{Z}[J, J^\dagger] &= \sum_T \frac{N(-1)^{(n-1)}(J^\dagger J)^k}{n!(\sqrt{\lambda})^k \prod_i \Gamma(l_i)} \int dt \int d\alpha \prod_i \alpha_i^{l_i-1} \exp \\ &\quad - \left\{ \sum_i \frac{\alpha_i}{\sqrt{\lambda}} + \sum_{i,j} \frac{C_{ij}\alpha_i\alpha_j}{2N} \right\} \end{aligned}$$

where we recall that the sum runs over ribbon trees with n labeled vertices and k cilia and that l_i is the number of corners attached to vertex i .

Next, we write $\lambda = \rho e^{i\theta}$ and rotate the α_i integrations by an angle $\frac{\phi}{2}$ in their complex planes, $\alpha_i \rightarrow e^{i\frac{\phi}{2}}\alpha_i$. This is possible since the integrand is holomorphic in all the α_i but requires that the integrand goes to 0 on the arcs $\alpha_i = R e^{i\chi}$ with $\chi \in [0, \frac{\phi}{2}]$ and $R \rightarrow +\infty$. This last condition imposes that the real part of the argument of the exponential be always positive. A sufficient condition is to impose that the linear and quadratic terms are separately positive, $\cos(\frac{\theta-\phi}{2}) > 0$ and $\cos\phi > 0$. After rotation of the contour, the generating function of the cumulants is expanded as

$$\begin{aligned} \log \mathcal{Z}[J, J^\dagger] &= e^{i\phi\frac{(2n-2+k)}{2}} \sum_T \frac{N(-1)^{(n-1)}(J^\dagger J)^k}{n!(\sqrt{\lambda})^k \prod_i \Gamma(l_i)} \\ &\quad \int dt \int d\alpha \prod_i \alpha_i^{l_i-1} \exp - \left\{ e^{\frac{i}{2}(\phi-\theta)} \sum_i \frac{\alpha_i}{\sqrt{\rho}} + e^{i\phi} \sum_{i,j} \frac{C_{ij}\alpha_i\alpha_j}{2N} \right\}. \end{aligned}$$

Let us emphasize that none of the terms in the sum over trees depend on ϕ . We may therefore conveniently choose ϕ to enlarge the domain of analyticity. To this aim, let us bound the exponential as

$$\begin{aligned} &\left| \exp - \left\{ e^{\frac{i}{2}(\phi-\theta)} \sum_i \frac{\alpha_i}{\sqrt{\rho}} + e^{i\phi} \sum_{i,j} \frac{C_{ij}\alpha_i\alpha_j}{2N} \right\} \right| \\ &= \exp - \left\{ \cos \frac{\phi-\theta}{2} \sum_i \frac{\alpha_i}{\sqrt{\rho}} + \cos \phi \sum_{i,j} \frac{C_{ij}\alpha_i\alpha_j}{2N} \right\} \\ &\leq \exp - \left\{ \cos \frac{\phi-\theta}{2} \sum_i \frac{\alpha_i}{\sqrt{\rho}} \right\} \end{aligned}$$

since the covariance is a positive matrix. Now we perform the integral over the Schwinger parameters α_i

$$|\log \mathcal{Z}[J, J^\dagger]| \leq \sum_T \frac{N |\lambda|^{(n-1)} (J^\dagger J)^k}{n! \cos^{2n-2+k} \left(\frac{\theta-\phi}{2}\right)}.$$

Since we have to stay away from the critical half line $\theta = \pm\pi$, the best bound (given the conditions imposed by the positivity of the argument of the exponential in the contour rotation) is obtained for $\phi = \theta$ for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ $\phi = \frac{\pi}{2}$ for $\theta \in [0, \frac{\pi}{2}]$ and $\phi = -\frac{\pi}{2}$ for $\theta \in [-\frac{\pi}{2}, 0]$. Therefore, the generating function is bounded as

$$|\log \mathcal{Z}[J, J^\dagger]| \leq \sum_T \frac{N |\lambda|^{(n-1)} (J^\dagger J)^k}{n!} \quad \text{for } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$|\log \mathcal{Z}[J, J^\dagger]| \leq \sum_T \frac{N |\lambda|^{(n-1)} (J^\dagger J)^k}{n! \left[\cos\left(\frac{|\theta|}{2} - \frac{\pi}{4}\right)\right]^{2n-2+k}} \quad \text{for } \theta \in \left[-\pi, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right]$$

Finally, an argument similar to the one presented in Section 4.1 and leading to Theorem 1 establishes the following analyticity theorem for the vector model.

Theorem 9 (Constructive theorem for the vector model). *Cumulants are analytic functions of λ inside the curve*

$$\mathcal{C}' = \left\{ \rho e^{i\theta} \text{ with } 4\rho < 1 \text{ for } \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \right. \\ \left. \text{and } 4\rho < \cos^2\left(\frac{|\theta|}{2} - \frac{\pi}{4}\right) \text{ for } \theta \in \left[-\pi, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right] \right\}$$

with cut on the negative real axis.

\mathcal{C}' is limited by a portion of a circle and two portions of cardioids, rotated by angles $\pm\frac{\pi}{2}$ as illustrated on Figure 16.

Note that the analyticity domain intersects the negative real axis, on which the function has a cut because $\theta = \pm\pi$ has to be excluded.

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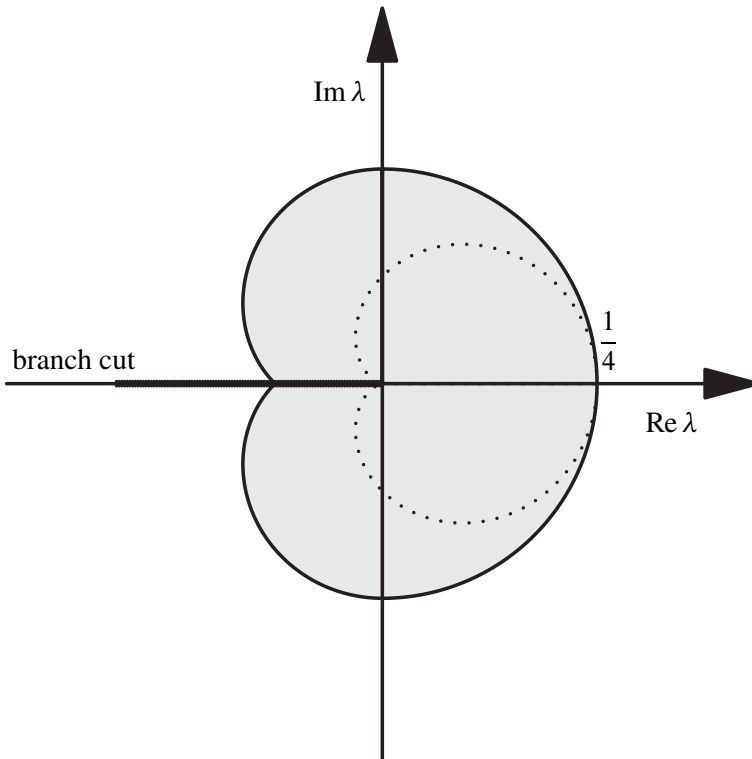


Figure 16. Analyticity domain for the vector model.

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