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Loop-weighted walk

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Abstract. Loop-weighted walk with parameter $\lambda \geq 0$ is a non-Markovian model of random walks that is related to the loop $O(N)$ model of statistical mechanics. A walk receives weight λ^k if it contains k loops; whether this is a reward or punishment for containing loops depends on the value of λ. A challenging feature of loop-weighted walk is that it is not purely repulsive, meaning the weight of the future of a walk may either increase or decrease if the past is forgotten. Repulsion is typically an essential property for lace expansion arguments. This article circumvents the lack of repulsion and proves, for any $\lambda > 0$, that loop-weighted walk is diffusive in high dimensions by lace expansion methods.

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Contents

1. Introduction and main results

Loop-weighted walk with parameter λ, abbreviated λ*-LWW*, is a model of self-interacting walks that can be informally defined as follows. Formal definitions will be given in Section 1.2. Let ω be a walk on a graph. A walk is called a *loop* if ω begins and ends at the same vertex. The *loop erasure* $\mathrm{LE}(\omega)$ is formed by chronologically removing loops from ω. If $n_L(\omega)$ denotes the number of loops removed, the λ-LWW weight of a walk ω is

$$w_\lambda(\omega) = \lambda^{n_L(\omega)}. \tag{1.1}$$

Throughout this article it will be assumed that $\lambda \geq 0$, so equation (1.1) defines a non-negative weight on walks. In particular, w_λ defines a probability measure on n-step walks that begin at a fixed vertex of a graph by defining the probability of ω to be proportional to $w_\lambda(\omega)$. If $0 \leq \lambda < 1$ the effect of the weight is to discourage walks from containing loops, and for this parameter range λ-LWW interpolates between the uniform measure on n-step self-avoiding walks (0-LWW) and the uniform measure on all n-step walks (1-LWW). If $\lambda > 1$ the weight encourages the existence of loops: walks are rewarded for returning to vertices that have been visited in the past. Note that λ-LWW for $\lambda \neq 1$ is not a Markovian model of walks.

In addition to being an interesting model of self-interacting random walks that encompasses the well-known models of self-avoiding and simple random walk, λ-LWW also has connections with spin models in statistical mechanics. The description of these connections will be deferred until after the results of the article are described, see Section 1.1.

This article consists of a lace expansion analysis of λ-LWW. The lace expansion, originally introduced by Brydges and Spencer [3], is a powerful tool for proving mean-field behaviour in high dimensions [14]. With few exceptions, see the discussion at the end of Section 1.2.2, walk models that have been successfully studied with the lace expansion have been *purely repulsive*. A walk model being purely repulsive means that the weight w on walks that defines the model satisfies the inequality

$$w(\omega \circ \eta) \leq w(\omega) w(\eta), \tag{1.2}$$

where $\omega \circ \eta$ is the concatenation of two walks ω and η. For example, self-avoiding walk is purely repulsive. In general λ-LWW is *not* purely repulsive if $\lambda \neq 0, 1$. See Figure 1.

The most significant step required to analyze λ-LWW with the lace expansion is therefore a technique to overcome the lack of repulsion. This is done by re-summing λ-LWW to obtain a model of self-interacting and self-avoiding walks.

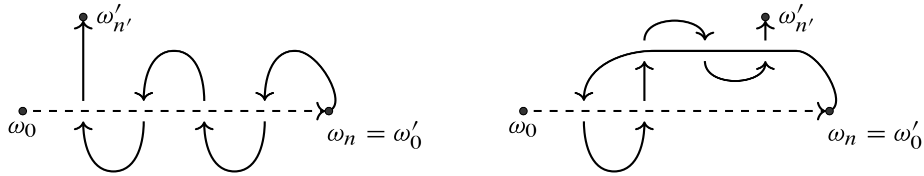

Figure 1. For each diagram consider (i) the concatenation of the dashed walk ω and the solid walk ω' and (ii) the two walks as being separate. On the left (i) results in four loops being erased, $n_L(\omega \circ \omega') = 4$, while (ii) results in no loops being erased, $n_L(\omega) = n_L(\omega') = 0$. On the right (i) results in one loop being erased, $n_L(\omega \circ \omega') = 1$, while (ii) results in three loops being erased, $n_L(\omega) = 0$, $n_L(\omega') = 3$. It follows that the λ-LWW weight is not purely repulsive for $\lambda \neq 0, 1$.

The particular form of the λ-LWW weight leads to a very explicit description of the self-interaction in terms of a generalization of the loop measure of [10], and this explicit description makes it clear that the self-interaction is repulsive. This enables a lace expansion to be performed. Further details about the proof follow after the statement of Theorem 1.1.

Some notation will be needed to state the results. Let $\langle \cdot \rangle_n^\lambda$ denote expectation with respect to the measure on n-step walks associated to w_λ. Let c_n^λ be the normalizing factor for the expectation, i.e., the sum over all n step walks weighted by $\lambda^{n_L(\omega)}$ as in (1.1). Let $\chi_\lambda(z) = \sum_n c_n^\lambda z^n$, and let $z_c(\lambda)$ be the radius of convergence of $\chi_\lambda(z)$. The main result of this article can be summarized as saying that, in high dimensions, λ-LWW has mean field behaviour at criticality.

Theorem 1.1. *Fix $\lambda \geq 0$ and consider λ-LWW on $\mathbb{Z}^d$. There exists $d_0 = d_0(\lambda)$ such that for $d \geq d_0$ there are constants A and D such that*

(1) *the susceptibility diverges linearly: $\chi_\lambda(z) \sim A z_c (z_c - z)^{-1}$ as $z \nearrow z_c$,*

(2) *$c_n^\lambda = A\,(z_c(\lambda))^{-n}\,(1 + O(n^{-\delta}))$ for any $\delta < 1$, and*

(3) *λ-LWW is diffusive: $\langle |\omega_n|^2 \rangle_n^\lambda = Dn(1 + O(n^{-\delta}))$ for any $\delta < 1$.*

For $\lambda = 0$, Theorem 1.1 has been proven with $d_0 = 5$ by Hara and Slade [8]. It is worth emphasizing that Theorem 1.1 holds for $\lambda > 1$ when λ-LWW is attractive in the sense that the formation of loops is encouraged.

Remark 1.2. No attempt has been made to track the value of d_0 that is required, and the proof presented in this article requires $d_0 \gg 9$. The true behaviour of $d_0(\lambda)$ is an interesting question for future study.

Let us say a few more words about the proof of Theorem 1.1. As described earlier, the key step is a resummation of λ-LWW into a self-interacting self-avoiding walk. The self-interaction of the self-avoiding walk is a many-body interaction, and this leads to a hypergraph-based lace expansion instead of the graph-based lace expansion that is used for self-avoiding walk. We stress that hypergraphs are merely an organizational tool, and no prior knowledge of hypergraphs is needed to understand the expansion. Once the lace expansion has been performed the various self-interacting self-avoiding walk quantities can be re-expressed in terms of λ-LWW. The diagrams that occur in analyzing the expansion generalize the diagrams for self-avoiding walk, and when $\lambda = 0$ they reduce to the diagrams for self-avoiding walk. With some effort it is possible to analyze the diagrams for $\lambda > 0$ with existing methods. Once the analysis of the diagrams is completed it is possible to apply established techniques to analyze λ-LWW, namely the trigonometric approach to the convergence of the lace expansion [14] and complex analytic methods for studying asymptotics.

In fact Theorem 1.1 holds in greater generality. Let $\lambda_\ell \geq 0$ be the weight of the loop ℓ. Replace the weight λ per loop in equation (1.1) with the product of λ_ℓ over the set of loops ℓ that are erased when performing loop erasure on ω. Assume the set of weights $\{\lambda_\ell\}$ satisfy a mild symmetry hypothesis, see Theorem 1.2.1, and are uniformly bounded above. Then the results of Theorem 1.1 continue to hold.

The remainder of the introduction is as follows. Section 1.1 describes an important connection between λ-LWW and the loop $O(N)$ model of statistical physics. Section 1.2 gives a formal definition of λ-LWW, relates λ-LWW to a self-interacting and self-avoiding walk, and outlines how this enables a lace expansion analysis. Lastly, Section 1.3 establishes a few conventions used in the remainder of the article.

1.1. Motivation from statistical mechanics. For $N \in \mathbb{N}$ the $O(N)$ *model* on a graph finite $G = (V, E)$ is a generalization of the Ising model. To each vertex $x \in V$ is associated a *spin* $\vec{s}_x$ taking values in the unit sphere in $\mathbb{R}^N$. The probability of a spin configuration is defined by

$$\mathbb{P}\left(\{\vec{s}_x\}_{x\in V}\right) \propto \exp\Big(\beta \sum_{x\sim y} \vec{s}_x \cdot \vec{s}_y\Big),$$

where β is a real parameter and the summation is over all edges $\{x, y\} \in E$. In [6] a simplification of the $O(N)$ model known as the *loop* $O(N)$ *model* was introduced. The loop $O(N)$ model is defined in terms of subgraph configurations on G. In the special case of a graph with vertex degree bounded by 3, the loop $O(N)$ model configurations are subgraphs that are disjoint unions of cycles of length at least 3, and the probability of a subgraph H is given by

$$\mathbb{P}(H) \propto z^{|E(H)|} N^{\#H}, \tag{1.3}$$

where $\#H$ denotes the number of connected components of H. Note that the probability in equation (1.3) may be negative if $N < 0$: equation (1.3) defines a signed measure in general.

The definition of the loop $O(N)$ model on an arbitrary graph G involves non-cyclic subgraphs, see for example [5]. The noncyclic subgraphs are predicted by non-rigorous renormalization group arguments to be irrelevant [12], at least for $|N| \leq 2$ on planar graphs. Call the model whose configurations are disjoint unions of cyclic subgraphs the $O(N)$ cycle gas. For $N \in \mathbb{N}$ this model has previously appeared in the physics literature as a model for melting transitions [13].

As described in Section A.4, λ-LWW is a walk representation of the $O(N)$ cycle gas. The two-point function of λ-LWW corresponds to a two-point correlation in the $O(N)$ cycle gas for $N = -2\lambda$. In other words, λ-LWW yields a *probabilistic* interpretation of the $O(N)$ cycle gas for $N < 0$. This is an example of a "negative activity isomorphism theorem": an equivalence between a statistical mechanics model at negative activity ($N < 0$) and a probability model. An important previous example of such a theorem is the Brydges–Imbrie isomorphism between branched polymers in $\mathbb{R}^{d+2}$ and the hard-core gas in $\mathbb{R}^d$ [4]. In the present work the isomorphism allows results about λ-LWW to be transferred to the $O(N)$ cycle gas for $N < 0$. For example, the isomorphism theorem combined with Theorem 1.1 immediately implies the following corollary.

Corollary 1.3. *For d sufficiently large the susceptibility of the $O(N)$ cycle gas on $\mathbb{Z}^d$ for $N < 0$ diverges linearly at the critical point.*

This section may be summarized as saying that λ-LWW can be viewed as a random walk representation of an approximation of the $O(N)$ model. Thus λ-LWW fits into a long history of random walk representations of spin models [1, 2, 7] inspired by the pioneering work of Symanzik [15].

1.2. Introduction to the loop-weighted walk model. The rest of the paper will be concerned with $\mathbb{Z}^d$, the simple cubic lattice in d dimensions. Edges $\{x, y\}$ will often be abbreviated xy. Two vertices x and y will be called *adjacent*, written $x \sim y$, if xy is an edge in $\mathbb{Z}^d$. Let

$$\Omega = \{y \in \mathbb{Z}^d \mid y \sim 0\},$$

so $|\Omega| = 2d$ is the number of vertices adjacent to the origin 0.

1.2.1. Model definition. The next paragraphs establish some conventions about walks. An *n-step walk* $\omega = (\omega_0, \omega_1, \dots, \omega_n)$ is a sequence of $n+1$ adjacent vertices in $\mathbb{Z}^d$. Given a walk ω, $|\omega|$ will denote the number of steps in ω. A walk is a *loop* if $\omega_{|\omega|} = \omega_0$, *self-avoiding* if $\omega_i = \omega_j$ implies $i = j$, and a *self-avoiding polygon* if $\omega_i = \omega_j$ and $i \neq j$ implies $\{i, j\} = \{0, |\omega|\}$.

A walk ω *begins* at ω_0 and *ends* at $\omega_{|\omega|}$. Let $\omega \colon x \to y$ denote the set of walks beginning at x and ending at y. Let Ω_{SAW} denote the set of self-avoiding walks beginning at x and ending at y; if $x = y$ this is taken to be the set of self-avoiding polygons beginning at x. Let

$$\Omega_{\mathrm{SAP}} = \bigcup_x \Omega_{\mathrm{SAW}}(x, x) \quad \text{and} \quad \Omega_{\mathrm{SAW}} = \bigcup_x \bigcup_y \Omega_{\mathrm{SAW}}(x, y).$$

If $\omega^{(i)} = (\omega_0^{(i)}, \dots, \omega_{k_i}^{(i)})$ for $i = 1, 2$ and $\omega_{k_1}^{(1)} = \omega_0^{(2)}$ the *concatenation* $\omega^{(1)} \circ \omega^{(2)}$ of $\omega^{(1)}$ with $\omega^{(2)}$ is defined by

$$\omega^{(1)} \circ \omega^{(2)} = (\omega_0^{(1)}, \dots, \omega_{k_1}^{(1)}, \omega_1^{(2)}, \dots, \omega_{k_2}^{(2)}).$$

To define λ-LWW precisely requires an explicit description of the loop erasure of a walk ω. Define

$$\begin{aligned} \tau_\omega &= \min\left\{i \mid \text{there exists } j < i \text{ such that } \omega_i = \omega_j\right\}, \\ \tau_\omega^\star &= \min\left\{j \mid \omega_j = \omega_{\tau_\omega}\right\}. \end{aligned}$$

If ω is a self-avoiding walk, define $\tau_\omega = \tau_\omega^\star = \infty$. The time τ_ω is the first time a walk visits a vertex twice.

Definition 1.4. Let ω be a walk of length n. The *single loop erasure* $\mathrm{LE}^1(\omega)$ of ω is given by

$$\mathrm{LE}^1(\omega) = (\omega_0, \dots, \omega_{\tau_\omega^\star \wedge n}, \omega_{\tau_\omega + 1}, \dots, \omega_n),$$

where $a \wedge b$ denotes the minimum of a and b. The walk $(\omega_{\tau_\omega^\star}, \omega_{\tau_\omega^\star + 1}, \dots, \omega_{\tau_\omega})$ is the *loop removed by loop erasure*. The *loop erasure* $\mathrm{LE}(\omega)$ of ω is the result of iteratively applying LE^1 until $\tau_\omega = \infty$.

By construction, each loop removed from a walk by loop erasure is a self-avoiding polygon.

Definition 1.5. The *loop vector* $n_L(\omega)$ of ω is the vector with coordinates

$$n_L^\eta(\omega) = \#\text{ of times } \eta \text{ is removed by loop erasure applied to } \omega,$$

with $\eta \in \Omega_{\mathrm{SAP}}$.

In what follows λ will denote a vector of activities $\lambda_\eta \geq 0$ for $\eta \in \Omega_{\mathrm{SAP}}$. Inequalities with respect to λ are to be interpreted pointwise in $\eta \in \Omega_{\mathrm{SAP}}$. Define

$$\lambda^{n_L(\omega)} = \prod_\eta \lambda_\eta^{n_L^\eta(\omega)}.$$

Definition 1.6. Let $\lambda \geq 0$, $z \geq 0$. The weight $w_{\lambda,z}$ of λ-LWW at activity z is given by

$$w_{\lambda,z}(\omega) = z^{|\omega|}\lambda^{n_L(\omega)}.$$

Definition 1.7. The *susceptibility* $\chi_\lambda(z)$ of λ-LWW is

$$\chi_\lambda(z) = \sum_{x\in\mathbb{Z}^d}\sum_{\omega:\, 0\to x} w_{\lambda,z}(\omega).$$

The *critical point* $z_c(\lambda)$ of λ-LWW is defined to be the radius of convergence of $\chi_\lambda(z)$.

If $0 \leq \lambda \leq 1$ then $\chi_\lambda(z) \leq \chi_1(z)$, and hence $\chi_\lambda(z)$ converges for $z < |\Omega|^{-1}$. The next proposition gives a mild condition under which the critical point is non-trivial.

Proposition 1.8. *Let* $\bar\lambda = \sup_\eta \lambda_\eta > 1$. *If* $z < (|\Omega|\sqrt{\bar\lambda})^{-1}$ *then* $\chi_\lambda(z)$ *is finite.*

Proof. An n-step walk contains at most $\lfloor n/2 \rfloor$ loops, and weighting each loop by $\bar\lambda$ yields an upper bound for $\chi_\lambda(z)$. Cancelling the factors of $\sqrt{\bar\lambda}$ gives the claim, as the resulting sum is $\chi_1(\bar z)$ for some $\bar z < |\Omega|^{-1}$. □

If $\mathcal{R}$ is an isometry of $\mathbb{Z}^d$, and $A \subset \mathbb{Z}^d$, let $\mathcal{R}A = \{\mathcal{R}a \mid a \in A\}$.

Assumption 1. Assume that $\lambda_\eta = \lambda_{\mathcal{R}\eta}$ for any isometry $\mathcal{R}$ and any $\eta \in \Omega_{\mathrm{SAP}}$. Further assume that $\lambda_\eta = \lambda_{\tilde\eta}$ if η and $\tilde\eta$ are self-avoiding polygons that differ only in terms of initial vertex and orientation.

Assumption 2. Assume $\sup_{\eta\in\Omega_{\text{SAP}}} \lambda_\eta < \infty$.

Theorem 1.9. *Fix $\lambda \geq 0$. If Assumption 1 and Assumption 2 hold, then there exists $d_0 = d_0(\lambda)$ such that for $d \geq d_0$ there are constants A and D such that the conclusions of Theorem 1.1 hold.*

Theorem 1.1 is the special case of Theorem 1.9 when the loop activities λ are constant. The constants A and D have explicit expressions, see Section 6.2. For the remainder of the article it will be assumed that Assumption 1 and Assumption 2 hold.

1.2.2. Aspects of proof. This section describes the basic facts about λ-LWW that allow for a lace expansion analysis, and gives an outline of the proof of Theorem 1.9.

Definition 1.10. The *loop-erased λ-LWW* weight $\bar{w}_{\lambda,z}$ on self-avoiding walks is

$$\bar{w}_{\lambda,z}(\eta) = \mathbb{1}_{\{\eta\in\Omega_{\text{SAW}}\}} \sum_{\omega\colon \text{LE}(\omega)=\eta} w_{\lambda,z}(\omega).$$

Note that the definition of $\bar{w}_{\lambda,z}$ assigns non-zero weight only to self-avoiding walks. The definition of $\bar{w}_{\lambda,z}$ implies that for any $x \in \mathbb{Z}^d$

$$\sum_{\omega\colon 0\to x} \bar{w}_{\lambda,z}(\omega) = \sum_{\omega\colon 0\to x} w_{\lambda,z}(\omega), \tag{1.4}$$

as the left-hand side is just a reorganization of the right-hand side. This identity will be important in what follows.

Definition 1.11. The *range*, range(ω), of a walk ω is the set of vertices visited by ω.

The *λ-LWW loop measure at activity z* of a closed walk ω is given by $w_{\lambda,z}(\omega)/\,|\omega|$. The next definition introduces a convenient shorthand for the loop measure of certain subsets of walks; note that $\mu_{\lambda,z}$ is not a measure.

Definition 1.12. Let $A, B \subset \mathbb{Z}^d$. The *$\lambda$-LWW loop measure* $\mu_{\lambda,z}(A; B)$ is

$$\mu_{\lambda,z}(A; B) = \sum_x \sum_{\substack{\omega\colon x\to x\\ |\omega|\geq 1}} \mathbb{1}_{\{\text{range}(\omega)\cap A\neq\emptyset\}} \mathbb{1}_{\{\text{range}(\omega)\cap B=\emptyset\}} \frac{w_{\lambda,z}(\omega)}{|\omega|}.$$

Define

$$\mu_{\lambda,z}(A) = \mu_{\lambda,z}(A; \emptyset).$$

For singleton sets $\{x\}, \{y\}$, let

$$\mu_{\lambda,z}(x; y) = \mu_{\lambda,z}(\{x\}; \{y\}).$$

For the special case of $\lambda = 1$ the next theorem is [10, Proposition 9.5.1].

Theorem 1.13. *The loop erased λ-LWW weight on self-avoiding walks can be written in terms of the λ-LWW loop measure*:

$$\bar{w}_{\lambda,z}(\eta) = \sum_{\omega :\, \mathrm{LE}(\omega)=\eta} w_{\lambda,z}(\omega) = z^{|\eta|} \exp(\mu_{\lambda,z}(\mathrm{range}(\eta))).$$

Proof. Deferred to Appendix A. □

A function f on subsets of $\mathbb{Z}^d$ is said to be *weakly increasing* if $A \subset B$ implies $f(A) \le f(B)$, and *weakly decreasing* if $f(A) \ge f(B)$.

Proposition 1.14. *Assume $z \ge 0$, $\lambda \ge 0$.*

(1) *Let $A, B \subset \mathbb{Z}^d$. Then for any isometry $\mathcal{R}$*

$$\mu_{\lambda,z}(\mathcal{R}A; \mathcal{R}B) = \mu_{\lambda,z}(A; B),$$

(2) *$\mu_{\lambda,z}(A; B)$ is weakly increasing in A and weakly decreasing in B.*

Proof. The first item follows from the isometry invariance of $w_{\lambda,z}$, which follows from Assumption 1. The second follows as increasing A (decreasing B) reduces (increases) the constraints on the set of walks that contribute to the defining sum, and $w_{\lambda,z}(\omega) \ge 0$. □

If $\eta = \eta_1 \circ \eta_2$ is self-avoiding then Theorem 1.13 and the definition of the loop measure imply

$$\bar{w}_{\lambda,z}(\eta) = z^{|\eta_1|} z^{|\eta_2|} \exp(\mu_{\lambda,z}(\mathrm{range}(\eta_1))) \exp(\mu_{\lambda,z}(\mathrm{range}(\eta_2); \mathrm{range}(\eta_1))).$$

By the second statement of Proposition 1.14 dropping the constraint in the second loop measure increases the weight, and hence *loop-erased λ-LWW* is purely repulsive. This enables a lace expansion analysis of λ-LWW as the two-point functions of λ-LWW and loop-erased λ-LWW coincide by equation (1.4). This is done as follows.

- Section 2 derives a lace expansion for λ-LWW. This is done by manipulating the identity

$$\bar{w}_{\lambda,z}(\eta) = z^{|\eta|}\mathbb{1}_{\{\eta\in\Omega_{\mathrm{SAW}}\}}\prod_{X\in\mathcal{X}}(1+\alpha_X)^{\mathbb{1}_{\{\ell(X)\cap\mathrm{range}(\eta)\neq\emptyset\}}}, \tag{1.5}$$

  where

$$\begin{aligned}\mathcal{X} &= \bigcup_{x\in\mathbb{Z}^d}\{\omega\colon x\to x, |\omega|\geq 1\}, \qquad (1.6)\\ \ell(\omega) &= \mathrm{range}(\omega),\\ \alpha_\omega &= \exp\left(\frac{w_{\lambda,z}(\omega)}{|\omega|}\right)-1.\end{aligned}$$

  In equation (1.6) the condition $|\omega|\geq 1$ can be relaxed to $|\omega|>1$ as all closed walks have length at least 2. Note that $\alpha_\omega\geq 0$ for any closed walk ω as $\lambda\geq 0$, and that the product in equation (1.5) converges for z sufficiently small by Proposition 1.8.

- Section 3 expresses the results of Section 2 in terms of $\mu_{\lambda,z}$, as opposed to the variables α_ω.

- Section 4 and Section 5 prove the convergence of the lace expansion at the critical point. The strategy is based on [14].

- Lastly, Section 6 proves the main theorem after establishing some further estimates on the lace expansion coefficients. The analysis is based on [11].

Before carrying out the arguments outlined above, let us briefly comment on other relevant non-repulsive random walks that have been studied. Ueltschi [16] has given a lace expansion analysis of a self-avoiding walk with nearest neighbour attractions; the attraction means his model is not repulsive. The analysis in [16] overcomes the lack of repulsion by exploiting the self-avoiding nature of the walk. Implementing this idea requires technical assumptions that (i) the attraction is sufficiently weak and (ii) the self-avoiding walk can take steps of unbounded range. A second non-repulsive model that has been studied is excited random walk: the analysis of this model in [17] is essentially a lace expansion analysis. These results have a somewhat different flavour as the walk being studied has non-zero speed. Roughly speaking, the lack of repulsion is overcome by using the transience of the walk in the excited direction.

1.3. Notation and conventions. Let $\mathbb{1}_{\{A\}}$ denote the indicator function of a set A. For notational ease we will occasionally also make use of the Kronecker delta $\delta_{x,y} = \mathbb{1}_{\{x=y\}}$. The *single step distribution* $D(x)$ is defined by $D(x) = |\Omega|^{-1}\,\mathbb{1}_{\{x\sim 0\}}$, where we recall that $|\Omega| = 2d$ and $x \sim 0$ indicates that x is a nearest neighbour of 0 in $\mathbb{Z}^d$.

The Fourier transform

$$\hat{f}\colon\ [-\pi,\pi]^d \longrightarrow \mathbb{C}$$

of a function f on $\mathbb{Z}^d$ is defined by

$$\hat{f}(k) = \sum_{x\in\mathbb{Z}^d} e^{ik\cdot x} f(x)$$

Subwalks of a walk ω can be identified by specifying the subinterval that defines them. That is, for $0 \leq a < b \leq |\omega|$ define $\omega\,[a,b] = (\omega_a,\ldots,\omega_b)$, $\omega\,[a,b) = \omega\,[a,b-1]$, $\omega\,(a,b] = \omega\,[a+1,b]$, and $\omega\,(a,b) = \omega\,[a+1,b-1]$. By convention $[a,a] = \{a\}$, so $\omega\,[a,a] = \omega_a$. To avoid some ungainly notation, let $\omega\,[a\!:\,] = \omega\,[a,|\omega|]$.

By convention $\inf\emptyset = \infty$ and $\sup\emptyset = -\infty$. The set $\{0,1,\ldots,n\}$ will be denoted $[n]$, and $[\omega]$ will denote $[|\omega|]$ when ω is a walk. Further, c will denote a positive constant independent of the dimension d and activity z; the precise value of c may change from line to line.

2. A lace expansion

Remark 2.1. The lace expansion presented here can be derived by other means, e.g., the technique developed for self-interacting walks in [17].

2.1. Graphical representations.

This section provides a representation of the weight $\overline{w}_{\lambda,z}$ in terms of graphs. The utility of such a representation is that it allows recursive identities to be derived.

2.1.1. Graph representation of self avoidance

Definition 2.2. Let A be a set. For $s,t \in A$, $s \neq t$, the pair $\{s,t\} \equiv st$ is called an *edge*. A *graph* Γ on A is a set of edges.

The condition $\omega \in \Omega_{\mathrm{SAW}}$ that a walk ω is self-avoiding can be expressed using graphs:

$$\begin{aligned} \mathbb{1}_{\{\omega\in\Omega_{\mathrm{SAW}}\}} &= \prod_{0\le s<t\le|\omega|} \mathbb{1}_{\{\omega_s\neq\omega_t\}} \\ &= \prod_{0\le s<t\le|\omega|} (1-\mathbb{1}_{\{\omega_s=\omega_t\}}) \\ &= \sum_{\Gamma}\prod_{st\in\Gamma}(-\mathbb{1}_{\{\omega_s=\omega_t\}}). \end{aligned} \tag{2.1}$$

The sum in the rightmost term is over all graphs Γ on $[\omega]$, where we recall the definition $[\omega] = [|\omega|] = \{0, 1, \dots, |\omega|\}$.

2.1.2. Hypergraph decomposition of LWW weight. A representation of the weight on self-avoiding walks due to the product over $\mathcal{X}$ in equation (1.5) is less straightforward than the graph representation of self-avoidance. This is because the condition of self-avoidance involves two distinct times, while the condition that $\mathrm{range}(\omega) \cap \ell(X) \neq \emptyset$ involves many distinct times. This issue can be handled by using inclusion-exclusion. A convenient way to represent the results of inclusion-exclusion is in terms of hypergraphs. We emphasize, however, that no prior knowledge of hypergraphs are needed to understand the expansion – they are only used as a bookkeeping instrument.

Lemma 2.3. *Let ω be a walk, and let $X \in \mathcal{X}$. Then*

$$(1+\alpha_X)^{\mathbb{1}_{\{\ell(X)\cap\mathrm{range}(\omega)\neq\emptyset\}}} = \prod_{J\subset[\omega]:\,|J|\ge1}(1+F_{J,X}(\omega)),$$

where

$$F_{J,X}(\omega) \equiv \begin{cases} \alpha_X \prod_{j\in J}\mathbb{1}_{\{\omega_j\in\ell(X)\}}, & |J|\in 2\mathbb{N}+1 \\ -\frac{\alpha_X}{1+\alpha_X}\prod_{j\in J}\mathbb{1}_{\{\omega_j\in\ell(X)\}} & |J|\in 2\mathbb{N}. \end{cases} \tag{2.2}$$

In equation (2.2) 0 is included in $2\mathbb{N}$.

Proof. Apply inclusion-exclusion to the condition $\ell(X) \cap \mathrm{range}(\omega) \neq \emptyset$:

$$\begin{aligned}
\mathbb{1}_{\{\ell(X)\cap\mathrm{range}(\omega)\neq\emptyset\}} &= 1 - \mathbb{1}_{\{\ell(X)\cap\mathrm{range}(\omega)=\emptyset\}} \\
&= 1 - \prod_{j=0}^{|\omega|}(1 - \mathbb{1}_{\{\omega_j\in\ell(X)\}}) \\
&= \sum_{J\subset[\omega]:\,|J|\geq 1} (-1)^{|J|+1} \prod_{j\in J} \mathbb{1}_{\{\omega_j\in\ell(X)\}}.
\end{aligned}$$

Then

$$\begin{aligned}
(1+\alpha_X)^{\mathbb{1}_{\{\ell(X)\cap\mathrm{range}(\omega)\neq\emptyset\}}} &= \prod_{J\subset[\omega]:\,|J|\geq 1} (1+\alpha_X)^{(-1)^{|J|+1}\prod_{j\in J}\mathbb{1}_{\{\omega_j\in\ell(X)\}}} \\
&= \prod_{J\subset[\omega]:\,|J|\geq 1} (1 + F_{J,X}(\omega)),
\end{aligned}$$

where the weights $F_{J,X}$ arise from the identities $(1+\alpha)^{-\mathbb{1}_A} = 1 - \frac{\alpha}{1+\alpha}\mathbb{1}_A$ and $(1+\alpha)^{\mathbb{1}_A} = 1 + \alpha\mathbb{1}_A$. □

Definition 2.4. A *hypergraph* G on a countable set A is a (possibly empty) finite subset of A. Each element of G is called a *hyperedge*.

To connect this definition with the more familiar notion of a graph, consider the case when A is $V^2 \setminus \{\{x,x\} \mid x \in V\}$ for V a finite set. A subset of A is then the edge set of a graph on V.

If $F(a)$ is an indeterminate associated to the hyperedge a then, as formal power series,

$$\prod_{a\in A}(1+F(a)) = \sum_G \prod_{a\in G} F(a), \tag{2.3}$$

where the sum on the right-hand side of (2.3) is over all hypergraphs on A. In what follows we perform calculations in the sense of formal power series. We will ultimately find that our final expressions have interpretations as convergent objects.

To represent the product over $\mathcal{X}$ in equation (1.5) in terms of hypergraphs take A in Definition 2.4 to be $(2^{[n]}\setminus\emptyset)\times\mathcal{X}$. If $a \in A$ then $a = (J, X)$ for J a non-empty subset of $[n]$ and $X \in \mathcal{X}$. Define $F(a) \equiv F_{J,X}$. This implies

$$\begin{aligned}
\prod_{X\in\mathcal{X}} (1+\alpha_X)^{\mathbb{1}_{\{\ell(X)\cap\mathrm{range}(\omega)\neq\emptyset\}}} &= \prod_{X\in\mathcal{X}} \prod_{J\subset[\omega]:\,|J|\geq 1} (1+F_{J,X}(\omega)) \\
&= \sum_G \prod_{(J,X)\in G} F_{J,X}(\omega),
\end{aligned} \tag{2.4}$$

where the sum in (2.4) is over all hypergraphs. The next corollary is a useful hypergraph representation of the weight carried by a subwalk.

Corollary 2.5. *Let ω be an n-step walk. For $k \leq n$, $X \in \mathcal{X}$,*

$$(1+\alpha_X)^{\mathbb{1}_{\{\mathrm{range}(\omega[0,k))\cap\ell(X)=\emptyset\}}\mathbb{1}_{\{\mathrm{range}(\omega[k,n])\cap\ell(X)\neq\emptyset\}}} = \prod_{\substack{J\subset[n]:\,|J|\geq 1,\\ J\cap[k,n]\neq\emptyset}} (1+F_{J,X}(\omega)).$$

Proof. Observe that $\mathbb{1}_{\{\mathrm{range}(\omega[k,n])\cap\ell(X)\neq\emptyset\}}\mathbb{1}_{\{\mathrm{range}(\omega[0,k))\cap\ell(X)=\emptyset\}}$ can be rewritten as $\mathbb{1}_{\{\mathrm{range}(\omega)\cap\ell(X)\neq\emptyset\}} - \mathbb{1}_{\{\mathrm{range}(\omega[0,k))\cap\ell(X)\neq\emptyset\}}$. The corollary follows by applying Lemma 2.3 to both ω and $\omega[0,k)$ and dividing. □

2.1.3. The full graphical representation

Definition 2.6. Let $J \subset [n]$ be non-empty and let X denote an element of $\mathcal{X}\cup\{\emptyset\}$. A pair (J,X) is *timelike* if $|J|=2$, $X=\emptyset$. A pair is *spacelike* if $X\neq\emptyset$.

The use of spacelike and timelike as labels has no relation to the use of these terms in physics. Extend the definition of $F_{J,X}$ by defining $F_{J,X}$ via (2.2) if (J,X) is spacelike, and defining $F_{st,\emptyset} = -\mathbb{1}_{\{\omega_s=\omega_t\}}$ for timelike hyperedges $(st,\emptyset)$. Let $\mathcal{G}[a,b]$ denote the set of hypergraphs whose hyperedges are pairs (J,X) such that (i) $X\in\mathcal{X}\cup\{\emptyset\}$, (ii) $J\subset\{a,a+1,\ldots,b\}$, $|J|\geq 1$, and (iii) $X=\emptyset$ implies $|J|=2$. Define $\mathcal{G}(n)\equiv\mathcal{G}[0,n]$. The decompositions of Section 2.1 imply that

$$\begin{aligned} c_n(0,x) &= \sum_{\substack{\omega:\,0\to x\\ |\omega|=n}} \mathbb{1}_{\{\omega\in\Omega_{\mathrm{SAW}}\}} \prod_{X\in\mathcal{X}} (1+\alpha_X)^{\mathbb{1}_{\{\ell(X)\cap\mathrm{range}(\omega)\neq\emptyset\}}} \\ &= \sum_{\substack{\omega:\,0\to x\\ |\omega|=n}} \sum_{G\in\mathcal{G}(n)} \prod_{(J,X)\in G} F_{J,X}(\omega). \end{aligned} \tag{2.5}$$

2.2. Lace graphs

Definition 2.7. A graph Γ on $[a,b]$ is (*lace*) *connected* if (i) $b>a+1$, (ii) for all $a<j<b$ there is an edge $st\in\Gamma$ such that $s<j<t$, and (iii) there are j_1, j_2 such that $aj_1, j_2b\in\Gamma$. Let $G[a,b]$ (resp. $G^c[a,b]$) denote the set of graphs (resp. lace connected graphs) on $[a,b]$.

We caution the reader that the definition of lace connectedness is not the same as the graph theoretical definition of connectedness. The adjective lace will be dropped in what follows, as the graph-theoretic notion of connectedness is not relevant in this section.

A function w on graphs on the discrete interval $[a, b]$ is called *multiplicative* if $w(G) = \prod_{st \in E(G)} w(st)$. Note that a multiplicative function on graphs assigns the empty graph weight 1. If w is a multiplicative function on graphs on $[a, b]$ define

$$K[a, b] = \sum_{G \in G[a,b]} w(G), \quad J[a, b] = \sum_{G \in G^c[a,b]} w(G),$$

and let $K[a, b] = J[a, b] = 0$ if $a > b$. For $a < b$ the observation that a graph on $[a, b]$ either contains a in a connected subgraph or does not and the definition of connectedness imply

$$K[a, b] = K[a, a+1]\, K[a+1, b] + \sum_{j \geq 2} J[a, a+j]\, K[a+j, b].$$

Definition 2.8. A graph is a *lace graph* if the removal of any edge results in a graph which is not connected.

A *labelled graph* is a graph where each edge is given a label of either spacelike or timelike; a labelled graph may contain both the edge $(st$, spacelike) and the edge $(st$, timelike). The definition of a lace graph applies to labelled graphs as the notion of connectedness does not depend on the labelling. The following procedure associates a unique lace L_Γ to each labelled connected graph Γ on $[a, b]$. The labelled lace L_Γ consists of the set of edges $s_i t_i$ along with their labellings, where $s_i t_i$ are determined by $s_1 = a$, $t_1 = \max\{v : s_1 v \in \Gamma\}$, $s_{i+1} = \min\{s : st_{i+1} \in \Gamma\}$, $t_{i+1} = \max\{v :$ there exists $s < t_i$ such that $sv \in \Gamma\}$. If this does not uniquely specify $s_i t_i$ then $s_i t_i$ is chosen to have the label spacelike. The procedure terminates when $t_{i+1} = b$. See Figure 2.

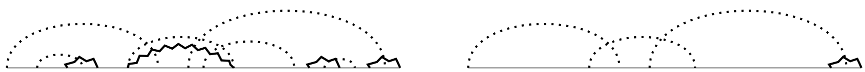

Figure 2. A labelled graph and the corresponding labelled lace graph. The left-hand side depicts a connected labelled graph, while the right-hand side depicts the corresponding labelled lace graph. The dotted black edges are labelled spacelike, while the solid black zigzag edges are labelled timelike.

A labelled edge st is said to be *compatible* with a lace L if $L_{L\cup\{st\}} = L$, i.e., if the addition of the labelled edge st does not alter the outcome of the above algorithm. Let $\mathcal{L}[a,b]$ denote the set of labelled lace graphs on $[a,b]$ and $\mathcal{C}(L)$ the set of compatible labelled edges for a lace $L \in \mathcal{L}$.

Lemma 2.9. *Let w be a weight on labelled edges st. Then*

$$\sum_{\Gamma\in G^c[a,b]} \prod_{st\in\Gamma} w(st) = \sum_{L\in\mathcal{L}[a,b]} \prod_{st\in L} w(st) \prod_{s't'\in\mathcal{C}(L)} (1+w(s't')),$$

where the sums are over labelled connected graphs and labelled laces, respectively.

Proof. The proof is the same as the proof for unlabelled graphs, see [3], [14], or [19]. □

Remark 2.10. Definition 2.7 is *not* the definition of lace connectedness typically used for self-avoiding walk, as the graph consisting of the single edge $\{a, a+1\}$ is not being considered connected. This change is entirely cosmetic for self-avoiding walk as graphs consisting of a single edge $\{a, a+1\}$ do not contribute.

2.3. Laces and hypergraphs. This section obtains an analogue of Lemma 2.9 for hypergraphs.

2.3.1. Recursion relation for hypergraphs

Definition 2.11. For a hyperedge (J, X) define $\mathrm{span}(J, X) = \{\min J, \max J\}$. If (J, X) is spacelike label $\mathrm{span}(J, X)$ spacelike, and if (J, X) is timelike label $\mathrm{span}(J, X)$ timelike. If G is a hypergraph the labelled graph Γ_G with labelled edges $\{\mathrm{span}(J, X) \mid (J, X) \in G\}$ will be called the *graph of spans* of G.

Definition 2.12. A hypergraph G on $[a,b]$ is *connected* if the graph of spans of G is connected on $[a,b]$. The set of connected hypergraphs on $[a,b]$ is denoted $\mathcal{G}^c[a,b]$.

The objects α and α_0 in the next definition have interpretations in terms of the loop measure, but for now should be thought of as convenient shorthand.

Definition 2.13. Let $\mathcal{X}_0 = \{X \in \mathcal{X} \mid 0 \in \ell(X)\}$ and let $y \in \Omega$ be a vertex adjacent to 0. Define

$$\alpha_0 = \alpha_0(\mathcal{X}) = \prod_{X \in \mathcal{X}_0} (1 + \alpha_X), \quad \alpha = \alpha(\mathcal{X}) = \prod_{X \in \mathcal{X}_0} (1 + \alpha_X)^{\mathbb{1}_{\{y \notin \ell(X)\}}}, \tag{2.6}$$

That α is independent of the vertex $y \in \Omega$ chosen follows from the isometry invariance of the loop-weighted walk weight.

By translation invariance α_0 is also given by the product over $X \in \mathcal{X}$ such that any single fixed vertex is contained in $\ell(X)$, and hence

$$\alpha_0 = \sum_{G \in \mathcal{G}[1,1]} w(G), \tag{2.7}$$

where $w(G) = \prod_{(J,X) \in G} F_{J,X}$. Using equation (2.7) and the definition of connectedness for hypergraphs implies that for $n \geq 1$

$$\begin{aligned} \sum_{G \in \mathcal{G}[0,n]} w(G) = \alpha_0^{-1} &\sum_{G_1 \in \mathcal{G}[0,1]} \sum_{G_2 \in \mathcal{G}[1,n]} w(G_1) w(G_2) \\ &+ \alpha_0^{-1} \sum_{j \geq 2} \sum_{G_1 \in \mathcal{G}^c[0,j]} \sum_{G_2 \in \mathcal{G}[j,n]} w(G_1) w(G_2). \end{aligned} \tag{2.8}$$

The factor of α_0^{-1} multiplying the first term arises since the hypergraphs $G \in \mathcal{G}[1,1]$ are double counted due to being present in both $\mathcal{G}[0,1]$ and $\mathcal{G}[1,n]$. The factor of α_0^{-1} multiplying the second factor arises similarly, due to double counting of the sum over $\mathcal{G}[j,j]$; translation invariance implies this is the same as the sum over $\mathcal{G}[1,1]$. The next lemma simplifies equation (2.8) by computing the sum over $\mathcal{G}[0,1]$.

Lemma 2.14. *Fix $n \geq 1$. Then*

$$\begin{aligned} &\sum_{G \in \mathcal{G}[0,n]} \prod_{(J,X) \in G} F_{J,X} \\ &\quad = \alpha \sum_{G \in \mathcal{G}[1,n]} \prod_{(J,X) \in G} F_{J,X} + \alpha_0^{-1} \sum_{j \geq 2} \sum_{G_1 \in \mathcal{G}^c[0,j]} \sum_{G_2 \in \mathcal{G}[j,n]} \prod_{(J,X) \in G_1} F_{J,X} \prod_{(J',X') \in G_2} F_{J',X'}. \end{aligned}$$

Proof. Let ω be a walk. Lemma 2.3 and (2.1) imply that

$$\sum_{G\in\mathcal{G}[0,1]}\prod_{(J,X)\in G} F_{J,X}(\omega) = \mathbb{1}_{\{\omega_0\neq\omega_1\}}\prod_{X\in\mathcal{X}}(1+\alpha_X)^{\mathbb{1}_{\{\{\omega_0,\omega_1\}\cap\ell(X)\neq\emptyset\}}}.$$

The constraint that $\omega_0\neq\omega_1$ is irrelevant as $\omega_{j+1}\neq\omega_j$ for any walk. Using the representation of α_0 in equation (2.7) gives

$$\frac{\sum_{G\in\mathcal{G}[0,1]}\prod_{(J,X)\in G} F_{J,X}(\omega)}{\sum_{G\in\mathcal{G}[1,1]}\prod_{(J,X)\in G} F_{J,X}(\omega)} = \prod_{X\in\mathcal{X}}(1+\alpha_X)^{\mathbb{1}_{\{\omega_1\in\ell(X)\}}\mathbb{1}_{\{\omega_0\notin\ell(X)\}}}, \tag{2.9}$$

and this last quantity is α by equation (2.6). Using (2.7) and (2.9) to rewrite equation (2.8) gives the claim. □

2.3.2. Laces for hypergraphs and weights on lace edges. The weight on hypergraphs, $w(G) = \prod F_{J,X}$, can be pushed forward to a weight $w_\star^\omega(st)$ on labelled graphs; recall that labelled graphs were introduced following Definition 2.8. Explicitly, the weight $w_\star^\omega(st)$ is defined by

$$w_\star^\omega(st,\text{timelike}) \equiv -\mathbb{1}_{\{\omega_s=\omega_t\}}, \tag{2.10a}$$

$$w_\star^\omega(st,\text{spacelike}) \equiv (1-\mathbb{1}_{\{\omega_s=\omega_t\}})\sum_{\{(J_i,X_i)\}:\ \mathrm{span}(J_i,X_i)=st}\ \prod_i F_{J_i,X_i}(\omega). \tag{2.10b}$$

The sum for a spacelike edge in (2.10) is over all non-empty collections of hyperedges, each of whose span is the labelled edge $(st,\text{spacelike})$. The factor $(1-\mathbb{1}_{\{\omega_s=\omega_t\}})$ accounts for the possibility that a timelike hyperedge exists when the edge st is given the label spacelike. Note that this weight neglects hyperedges (J,X) with $|J|=1$. For notational ease let $F_{j,X}=F_{\{j\},X}$.

Lemma 2.15. *The following identity holds for $a<b$:*

$$\begin{aligned}\sum_{G\in\mathcal{G}^c[a,b]}\prod_{(J,X)\in G}F_{J,X} \\ = \prod_{\substack{a\le j\le b\\ X\in\mathcal{X}}}(1+F_{j,X})\sum_{L\in\mathcal{L}[a,b]}\prod_{st\in L}w_\star(st)\prod_{\substack{(J',X'):\\ \mathrm{span}(J',X')\in\mathcal{C}(L)}}(1+F_{J',X'}).\end{aligned} \tag{2.11}$$

The left-hand sum is over all connected hypergraphs on $[a,b]$, while the right-hand sum is over labelled laces.

Proof. Apply Lemma 2.9 with the weight $w_\star$, and take the product of this equation with the first term on the right-hand side of (2.11):

$$\prod_{\substack{a\le j\le b\\ X\in\mathcal{X}}}(1+F_{j,X})\sum_{\Gamma\in G^c[a,b]}\prod_{st\in\Gamma}w_\star(st)$$
$$=\prod_{\substack{a\le j\le b\\ X\in\mathcal{X}}}(1+F_{j,X})\sum_{L\in\mathcal{L}[a,b]}\prod_{st\in L}w_{\star(st)}\prod_{s't'\in\mathcal{C}(L)}(1+w_\star(s't')).$$

Expanding the product over connected labelled graphs with weight $w_\star$ gives the left-hand side of (2.11) as hyperedges of the form $(\{j\},X)$ play no role in connectivity, and for each st the weight $w_\star$ is a sum of the possible collections of hyperedges whose span is st. Similarly, $1+w_\star(ij)$ for $ij\in\mathcal{C}(L)$ can be written in the product form used above, giving the right-hand side of (2.11). □

The next definition and lemma simplifies the sum over laces in (2.11) by resumming the contributions to the product over $st\in L$.

Definition 2.16. For $0\le s<t$,

- if $\omega_s=\omega_t$, define $I^\omega_{\mathcal{X}}(s,t)=1$, and
- if $\omega_s\neq\omega_t$, define

$$I^\omega_{\mathcal{X}}(s,t)=1-\prod_{X\in\mathcal{X}}\Big(1-\frac{\alpha_X}{1+\alpha_X}\mathbb{1}_{\{\omega_s\in\ell(X)\}}\mathbb{1}_{\{\omega_t\in\ell(X)\}}\mathbb{1}_{\{\ell(X)\cap\mathrm{range}(\omega(s,t))=\emptyset\}}\Big).\tag{2.12}$$

Lemma 2.17. *Let st be an edge. Then*

$$w^\omega_\star(st,\text{spacelike})+w^\omega_\star(st,\text{timelike})=-I^\omega_{\mathcal{X}}(s,t)$$

Proof. The case $\omega_s=\omega_t$ corresponds to the timelike edge. Consider the spacelike term. As any non-empty collection of spacelike hyperedges $\{(J_i,X_i)\}$ may be chosen in equation (2.10) the equation can be rewritten as

$$w^\omega_\star(st,\text{spacelike})=(1-\mathbb{1}_{\{\omega_s=\omega_t\}})\Big(\prod_{\substack{(J,X):\\ \mathrm{span}(J,X)=st}}(1+F_{J,X}(\omega))-1\Big).$$

A hyperedge with span st and second element X is equivalent to a possibly empty subset J of (s,t). Using $F_{J\cup\{ab\},X} = \mathbb{1}_{\{\omega_a\in\ell(X)\}}\mathbb{1}_{\{\omega_b\in\ell(X)\}}F_{J,X}$ gives

$$w_\star^\omega(st,\text{spacelike}) = \mathbb{1}_{\{\omega_s\neq\omega_t\}}\Big(\prod_{X\in\mathcal{X}}\prod_{J\subset(s,t)}(1+\mathbb{1}_{\{\omega_s\in\ell(X)\}}\mathbb{1}_{\{\omega_t\in\ell(X)\}}F_{J,X}(\omega))-1\Big),$$

where we recall that $F_{\emptyset,X}(\omega) = -\alpha_X(1+\alpha_X)^{-1}$. Putting the condition that ω_s and ω_t are in $\ell(X)$ into the product, separating the case $J=\emptyset$, and then applying Lemma 2.3 yields

$$\begin{aligned} w_\star^\omega(st,\text{spacelike}) &= \mathbb{1}_{\{\omega_s\neq\omega_t\}}\Bigg(\prod_{\substack{X\in\mathcal{X}:\\ \omega_s,\omega_t\in\ell(X)}}\Bigg(\Big(1-\frac{\alpha_X}{1+\alpha_X}\Big)\prod_{\substack{J\subset(s,t)\\|J|\geq 1}}(1+F_{J,X}(\omega))\Bigg)-1\Bigg)\\ &= \mathbb{1}_{\{\omega_s\neq\omega_t\}}\Bigg(\prod_{\substack{X\in\mathcal{X}:\\ \omega_s,\omega_t\in\ell(X)}}(1+\alpha_X)^{-\mathbb{1}_{\{\text{range}(\omega(s,t))\cap\ell(X)=0\}}}-1\Bigg), \end{aligned}$$

which is the second half of (2.12). □

2.4. The lace expansion equation. This section shows how the recursion for the interaction expressed in Lemmas 2.14 and 2.15 translates into a recursion for the c_n. By summing the resulting recursion over n the desired lace expansion is obtained.

2.4.1. Lace expansion equation. For $m\geq 2$ define

$$\begin{aligned} \pi_m^{(N)}(x) \equiv{}& z^m\alpha_0^{-1}\sum_{\substack{\omega:\,0\to x\\|\omega|=m}}\sum_{L\in\mathcal{L}^{(N)}[0,m]}\Big(\prod_{st\in L}I_{\mathcal{X}}^\omega(s,t)\Big)\\ &\prod_{\text{span}(J,X)\in\mathcal{C}(L)}(1+F_{J,X}(\omega))\prod_{\substack{a\leq j\leq b\\X'\in\mathcal{X}}}(1+F_{j,X'}(\omega)), \end{aligned}$$

where $\mathcal{L}^{(N)}[0,m]$ is the set of laces with N edges on the interval $[0,m]$. Let

$$\pi_m = \sum_{N\geq 1}(-1)^N\pi_m^{(N)}.$$

Define $c_m = 0$ for $m < 0$. Equation 2.5 combined with Lemmas 2.14 and 2.15 imply

- for $n \geq 1$,

$$z^n c_n(0,x) = z\alpha \sum_{y\sim 0} z^{n-1} c_{n-1}(y,x) + \sum_{j\geq 2}\sum_{y} \pi_j(y) z^{n-j} c_{n-j}(y,x), \tag{2.13a}$$

  and

- for $n = 0$,

$$z^n c_n(0,x) = \alpha_0 \delta_{0,x}. \tag{2.13b}$$

Let $G_z(x) = \sum_n z^n c_n(0,x)$. Summing (2.13) over n, using the translation invariance of $G_z(x)$, and taking the Fourier transform yields

$$\widehat{G}_z(k) = \alpha_0 + \alpha z\,|\Omega|\,\widehat{D}(k)\widehat{G}_z(k) + \widehat{\Pi}_z(k)\widehat{G}_z(k), \tag{2.14}$$

where $\Pi_z(x) = \sum_{m\geq 2} \pi_m(x)$.

The next two sections give expressions for $\pi_m^{(N)}(x)$ in terms of the quantities α_X.

2.4.2. Walk representation of $\pi_m^{(N)}(x)$ for $N = 1$. If $N = 1$ the lace consists of a single edge $0m$. If $x = 0$ then $\omega_0 = \omega_m$, $I_{\mathcal{X}}^{\omega}(0,m) = 1$, and

$$\pi_m^{(1)}(0) = z^m \alpha_0^{-1} \sum_{\substack{\omega:\,0\to 0\\ |\omega|=m}} \mathbb{1}_{\{\omega\in\Omega_{\mathrm{SAP}}\}} \prod_X (1+\alpha_X)^{\mathbb{1}_{\{\mathrm{range}(\omega)\cap\ell(X)\neq\emptyset\}}}. \tag{2.15}$$

If $x \neq 0$ the set of incompatible hyperedges are those that contain both 0 and m. Let $m_1 = m-1$. Corollary 2.5 implies that for $\omega \in \Omega_{\mathrm{SAW}}$

$$\prod_{(J,X)\in\mathcal{C}(0m)} (1 + F_{J,X}(\omega)) = \prod_{X\in\mathcal{X}} (1+\alpha_X)^{\mathbb{1}_{\{\mathrm{range}(\omega)\cap\ell(X)\neq\emptyset\}}+\mathbb{1}_A} \tag{2.16}$$

where

$$\mathbb{1}_A = \mathbb{1}_{\{\omega_0\in\ell(X)\}}\mathbb{1}_{\{\omega_m\in\ell(X)\}}\mathbb{1}_{\{\mathrm{range}(\omega[1,m_1])\cap\ell(X)=\emptyset\}},$$

while if ω is not self-avoiding the right-hand side of equation (2.16) is zero. To see these two claims, use Corollary 2.5 to compute the products over hyperedges (J,X) with (i) $J \subset [1,m_1]$, (ii) $J \subset [1,m]$ with $m \in J$, and (iii) $J \subset [0,m_1]$ with $0 \in J$. The product over compatible hyperedges is the product of these terms.

The definition of $I^{\omega}_{\mathcal{X}}(0,m)$ when $\omega_m = x \neq 0$ then gives a formula for $\pi^{(1)}_m(x)$:

$$\pi^{(1)}_m(x) = z^m \alpha_0^{-1} \sum_{\substack{\omega\colon 0\to x\\ |\omega|=m}} \mathbb{1}_{\{\omega\in\Omega_{\mathrm{SAW}}\}} \prod_X (1+\alpha_X)^{\mathbb{1}_{\{\mathrm{range}(\omega)\cap\ell(X)\neq\emptyset\}}} \Big(\prod_{X\in\mathcal{X}} (1+\alpha_X)^{\mathbb{1}_{\{\omega_0\in\ell(X)\}}\mathbb{1}_{\{\omega_m\in\ell(X)\}}\mathbb{1}_{\{\mathrm{range}(\omega[1,m_1])\cap\ell(X)=\emptyset\}}} - 1\Big). \tag{2.17}$$

2.4.3. Walk representation of $\pi^{(N)}_m(x)$ for $N \geq 2$. For $N \geq 2$ the central observation is that the edges of a lace on the discrete interval $[a,b]$ divides the interval $[a,b]$ into $2N-1$ subintervals, see Figure 2.

Definition 2.18. Let $m \in \mathbb{N}$. A vector $\vec{m}$ with integer components $m_1,\dots,m_{2N-1}$ is called *valid* if

(i) $m_1 \geq 1$, $m_{2N-1} \geq 1$, and $m_{2j} \geq 1$ for $1 \leq j \leq N-1$,

(ii) $m_{2j+1} \geq 0$ for $1 \leq j \leq N-1$, and

(iii) $\sum m_i = m$.

The lengths of the subintervals determined by a lace form valid vector $\vec{m}$. The restrictions on which m_i are strictly positive arise from the definition of connectedness, see [14, Section 3.3] for more details. The subintervals are given by

$$\begin{aligned} \overline{I}_1 &= [0, m_1]\,, \\ \overline{I}_2 &= [m_1, m_1+m_2]\,, \\ &\vdots \\ \overline{I}_{2N-1} &= [m_1+\dots m_{2N-2}, m_1+\dots m_{2N-1}]\,. \end{aligned}$$

To each interval $\overline{I}_k$ associate a walk $\omega^{(k)}$, e.g.

$$\omega^{(2)} = (\omega_{m_1}, \omega_{m_1+1}, \dots, \omega_{m_1+m_2}).$$

The walks $\omega^{(k)}$ interact with one another through the compatible edges.

To the k^{th} interval associate (i) all hyperedges whose span is contained in $\overline{I}_k$ and (ii) all compatible hyperedges (J,X) such that span(J,X) is not contained in $\overline{I}_k$ with $\max J \in \overline{I}_k$ *and* $\max J \neq \max \overline{I}_k$.

For the subinterval $2N-1$ omit the last condition. That is, if a hyperedge has $\max J = m$ associate this edge to $\overline{I}_{2N-1}$. Subintervals $\overline{I}_k$ for $k < 2N-1$ are missing hyperedges of the form $(\max \overline{I}_k, X)$. Including them, and dividing by α_0 to correct for this, shows the weight associated to the interval $\overline{I}_k$ is

$$\alpha_0^{-1} \prod_{\substack{(J,X)\\ J\subset \overline{I}_K}} (1+F_{J,X}) \prod_{\mathrm{span}(J',X')\in\mathcal{C}_k} (1+F_{J',X'}), \tag{2.18}$$

where the factor of α_0^{-1} for $k = 2N-1$ comes from the prefactor α_0^{-1} in the definition of $\pi_m^{(N)}$.

The last two factors can be evaluated together. A compatible hyperedge must have its minimum index be at least the second index of either $\omega^{(k-2)}$ or $\omega^{(k-3)}$. Suppose the first case; the second is similar. Corollary 2.5 implies the product in (2.18) forces $\omega^{(k)}$ to be self-avoiding, $\omega^{(k)}$ to avoid $\omega^{(k-1)}$ and $\omega^{(k-2)}[1:]$, and assigns $\omega^{(k)}$ the weight

$$\alpha_0^{-1} \prod_{X\in\mathcal{X}} (1+\alpha_X)^{\mathbb{1}_{\{\mathrm{range}(\omega^{(k)})\cap\ell(X)\neq\emptyset\}}\mathbb{1}_{\{\mathrm{range}(\omega^{(k-2)}\circ\omega^{(k-1)}[1:])\cap\ell(X)=\emptyset\}}}, \tag{2.19}$$

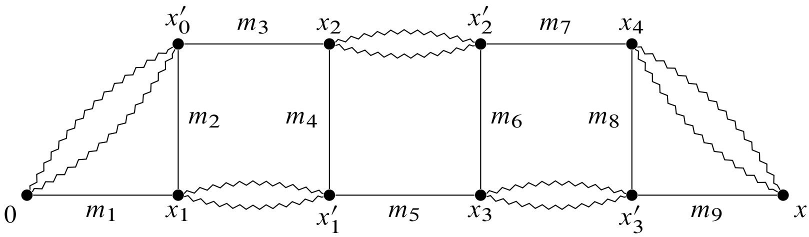

Figure 3. The diagrammatic representation of $\pi_m^{(5)}(x)$ with $m = \sum m_i$. The vertices $x_1,\dots,x_4$ and $x_0',\dots,x_3'$ are summed over.

As an explicit formula for $\pi_m^{(N)}$ detailing the constraints is unwieldy, let us explain the formula with a brief discussion of the diagrammatic representation of $\pi_m^{(N)}$ in Figure 3. The solid lines represent a subdivision of a walk ω into subwalks; these subwalks are subject to self-avoidance constraints detailed below. Pairs of zigzag lines represent $I_{\mathcal{X}}^{\omega}(t_i,t_{i+1})$, where t_i is the time x_i occurs in the walk $\omega = \omega^{(1)}\circ\cdots\circ\omega^{(2N-1)}$. Each walk $\omega^{(i)}$ has length m_i and is self-avoiding. Further, each walk $\omega^{(i)}$ avoids some of the previous walks $\omega^{(j)}$ for $j < i$, excluding the endpoint of $\omega^{(i-1)}$. To be precise, $\omega^{(2)}$ avoids $\omega^{(1)}$, $\omega^{(2k+1)}$ avoids $\omega^{(2k-1)}$ and $\omega^{(2k)}$, and $\omega^{(2k+2)}$ avoids $\omega^{(2k-1)}$, $\omega^{(2k)}$, and $\omega^{(2k+1)}$. The walk $\omega^{(j)}$ is weighted by those closed walks in $\mathcal{X}$ that do not intersect the $\omega^{(j)}$ which $\omega^{(i)}$ is forbidden to intersect; for example, in equation (2.19) the walk $\omega^{(k)}$ is being weighted by all closed walks that do not intersect $\omega^{(k-1)}$ or $\omega^{(k-2)}[1:]$.

Remark 2.19. Since $\alpha_X \geq 0$ for each X, ignoring the constraint that some closed walks do not weight a subwalk gives an upper bound for the weight on the subwalks $\omega^{(i)}$. Ignoring the constraint of avoiding $\omega^{(j)}$ for some $j < i$ gives a further upper bound on $\pi_m^{(N)}(x)$.

3. Concrete expressions for the lace expansion for λ-LWW

Quantities such as $\alpha_0(\mathcal{X})$ and $I_{\mathcal{X}}^{\omega}$ will be written as $\alpha_0(\lambda, z)$, $I_{\lambda,z}^{\omega}$ and similarly in what follows. The arguments λ and z may be omitted to lighten the notation. As emphasized earlier, $\lambda \geq 0$ and $z \geq 0$ implies $w_{\lambda,z}(\omega) \geq 0$, and hence $\alpha_\omega \geq 0$. In particular, by Remark 2.19 we can obtain upper bounds by ignoring constraints.

Definition 3.1. The *two point function* $G_{\lambda,z}(x, y)$ for λ-LWW is defined by

$$G_{\lambda,z}(x, y) = \sum_{\omega\colon x \to y} w_{\lambda,z}(\omega).$$

By Theorem 1.13 and equation (1.4) the two-point function $G_{\lambda,z}$ of λ-LWW is given by the two-point function of self-avoiding walks weighted as in equation (1.5). For future reference we state a reformulation of (2.14) as a proposition.

Proposition 3.2. *We have*

$$\widehat{G}_{\lambda,z}(k) = \frac{\alpha_0(\lambda, z)}{1 - \alpha(\lambda, z)\,|\Omega|\,\widehat{D}(k) - \widehat{\Pi}_{\lambda,z}(k)}. \tag{3.1}$$

To analyze the recursion (3.1) it will be convenient to rewrite the equation in terms of $w_{\lambda,z}$ and the loop measure $\mu_{\lambda,z}$. The quantities $\alpha_0(\lambda, z)$ and $\alpha(\lambda, z)$ can be expressed as, for $y \sim 0 \in \mathbb{Z}^d$,

$$\alpha_0(\lambda, z) = \exp(\mu_{\lambda,z}(0)), \quad \alpha(\lambda, z) = \exp(\mu_{\lambda,z}(0; y)).$$

Note that $\alpha_0 \geq \alpha \geq 1$. Let $I_{\lambda,z}^{\omega} = I_{\mathcal{X}}^{\omega}$. $I_{\lambda,z}^{\omega}(a, b)$ can be written in a loop measure like way:

$$I_{\lambda,z}^{\omega}(a, b) = \mathbb{1}_{\{\omega_a = \omega_b\}} + \mathbb{1}_{\{\omega_a \neq \omega_b\}}(1 - e^{-\mu_{\lambda,z}(\omega_a, \omega_b; \mathrm{range}(\omega(a,b)))}), \tag{3.2}$$

where

$$\mu_{\lambda,z}(A, B; C) = \sum_{x} \sum_{\substack{\omega\colon x \to x \\ |\omega| \geq 1}} \frac{w_{\lambda,z}(\omega)}{|\omega|} \mathbb{1}_{\{\mathrm{range}(\omega) \cap A \neq \emptyset\}} \mathbb{1}_{\{\mathrm{range}(\omega) \cap B \neq \emptyset\}} \mathbb{1}_{\{\mathrm{range}(\omega) \cap C = \emptyset\}}.$$

As with the loop measure, define $\mu_{\lambda,z}(A, B) = \mu_{\lambda,z}(A, B; \emptyset)$. The effect of this more complicated object is to require that both an element from A *and* B are in the range of the walk.

4. Convergence of the lace expansion I. Preliminaries

This section establishes the basic facts used to prove the convergence of the lace expansion. The strategy is that of [14], suitably adapted and modified for λ-LWW. An important role is played by the function $H_{\lambda,z}$ in the next definition.

Definition 4.1. The *reduced two point function* $H_{\lambda,z}(x, y)$ is defined by

$$H_{\lambda,z}(x, y) = (1 - \delta_{x,y})G_{\lambda,z}(x, y).$$

A useful fact that will be used repeatedly is that

$$G_{\lambda,z}(x, y) = \delta_{x,y}\alpha_0(\lambda, z) + H_{\lambda,z}(x, y). \tag{4.1}$$

The two-point functions $G_{\lambda,z}$ and $H_{\lambda,z}$ inherit the isometry invariance of the weight $w_{\lambda,z}$. By translation invariance $G_{\lambda,z}(x, y) = G_{\lambda,z}(0, y - x)$; it will be convenient to write $G_{\lambda,z}(x)$ for $G_{\lambda,z}(0, x)$.

4.1. Random walk quantities and bounds

Definition 4.2. The *random walk* 2*-point function* $C_z(x)$ and its Fourier transform $\widehat{C}_z(k)$ are given by

$$C_z(x) = \sum_{\omega\colon x\to x} z^{|\omega|}, \qquad \widehat{C}_z(k) = \frac{1}{1 - z\,|\Omega|\,\widehat{D}(k)}.$$

The following facts about the random walk two-point function will be useful. For notational clarity, let β be a quantity that is $O(|\Omega|^{-1})$. β is to be thought of as being a small parameter.

Lemma 4.3 (Lemma 5.5 of [11]). *Assume $d > 4$. Then for* $0 \le z \le |\Omega|^{-1}$

$$\sup_x D(x) \le \beta \tag{4.2}$$

$$\|C_z\|_2^2 \le 1 + c\beta \tag{4.3}$$

$$\|(1 - \cos(k \cdot x))C_z(x)\|_\infty \le 5(1 + c\beta)(1 - \widehat{D}(k))$$

Proposition 4.4. *Let $r \in \mathbb{N}$. There is a constant K independent of d such that for $d > 2r$.*

$$\int_{[-\pi,\pi]^d} \left(\frac{1}{1-\widehat{D}(k)}\right)^r \frac{d^d k}{(2\pi)^d} \le 1 + c\beta.$$

Proof. This follows by the argument used in the proof of [11, Lemma A.3]. ☐

4.2. Convergence strategy and basic bounds. The proof of convergence is based on comparing the behaviour of simple random walk and λ-LWW. Define $p(z)$ by

$$\frac{\widehat{G}_{\lambda,z}(0)}{\alpha_0(\lambda,z)} = \frac{1}{1-p(z)\,|\Omega|} = \widehat{C}_{p(z)}(0).$$

Roughly speaking, the intuition is that λ-LWW should behave like simple random walk. The definition of $p(z)$ serves to determine the activity of the simple random walk that matches λ-LWW with activity z. The following bootstrap lemma is what enables conclusions to be drawn for $z < z_c(\lambda)$.

Lemma 4.5 ([14, Lemma 5.9]). *Let $a < b$, let f be a continuous function on the interval $[z_1, z_2)$, and assume that $f(z_1) \le a$. Suppose for each $z \in (z_1, z_2)$ that $f(z) \le b$ implies $f(z) \le a$. Then $f(z) \le a$ for all $z \in [z_1, z_2)$.*

To describe the function f used in applying Lemma 4.5 some definitions are needed.

Definition 4.6. Define $\Delta_k \widehat{A}(\ell)$ by

$$-\frac{1}{2}\Delta_k \widehat{A}(\ell) = \widehat{A}(\ell) - \frac{1}{2}(\widehat{A}(\ell+k) + \widehat{A}(\ell-k)),$$

and define

$$\begin{aligned} U_{p(z)}(k,\ell) = 16\widehat{C}^{-1}_{p(z)}(k)(&\widehat{C}_{p(z)}(\ell-k)\widehat{C}_{p(z)}(\ell) + \widehat{C}_{p(z)}(\ell+k)\widehat{C}_{p(z)}(\ell) \\ &+ \widehat{C}_{p(z)}(\ell-k)\widehat{C}_{p(z)}(\ell+k)). \end{aligned}$$

The quantity $U_{p(z)}$ is a convenient upper bound for $\frac{1}{2}|\Delta_k \widehat{C}_{p(z)}(\ell)|$: this can be seen by [14, Lemma 5.7]. Define $f(z) = \max\{f_1(z), f_2(z), f_3(z)\}$, where

$$f_1(z) = z\alpha(\lambda,z)|\Omega|,$$

$$f_2(z) = \sup_{k\in[-\pi,\pi]^d} \frac{|\widehat{G}_{\lambda,z}(k)|}{\widehat{C}_{p(z)}(k)},$$

$$f_3(z) = \sup_{k,\ell\in[-\pi,\pi]^d} \frac{\Delta_k \widehat{G}_{\lambda,z}(\ell)}{U_{p(z)}(k,\ell)}.$$

The next lemma will be useful for estimating $G_{\lambda,z}$.

Lemma 4.7. *Assume $y \neq x$. The following inequality holds*:

$$G_{\lambda,z}(x,y) \le z\alpha(\lambda,z)\,|\Omega| \sum_u D(u) G_{\lambda,z}(u,y).$$

Proof. This can be proven using the loop measure representation. For η a walk beginning at $u \sim 0$, let $0\eta = (0,u) \circ \eta$. We get

$$\begin{aligned}
G_{\lambda,z}(0,y) &= \sum_{\eta\colon 0\to y} \mathbb{1}_{\{\eta\in\Omega_{\mathrm{SAW}}\}} z^{|\eta|} \exp(\mu_{\lambda,z}(\mathrm{range}(\eta))) \\
&= \sum_{u\sim 0} \sum_{\eta\colon u\to y} \mathbb{1}_{\{0\eta\in\Omega_{\mathrm{SAW}}\}} z \exp(\mu_{\lambda,z}(0;\mathrm{range}(\eta))) z^{|\eta|} \\
&\qquad\qquad \exp(\mu_{\lambda,z}(\mathrm{range}(\eta))) \\
&\le z\alpha(\lambda,z)\,|\Omega| \sum_u D(u) \sum_{\eta\colon u\to y} \mathbb{1}_{\{\eta\in\Omega_{\mathrm{SAW}}\}} z^{|\eta|} \exp(\mu_{\lambda,z}(\mathrm{range}(\eta))) \\
&= z\alpha(\lambda,z)\,|\Omega| \sum_u D(u) G_{\lambda,z}(u,y),
\end{aligned}$$

The inequality follows as (a) Proposition 1.14 implies $\mu_{\lambda,z}(0;\mathrm{range}(\eta))$ is bounded above by $\mu_{\lambda,z}(0;u) = \alpha_0$ and (b) $\mathbb{1}_{\{0\eta\in\Omega_{\mathrm{SAW}}\}}$ is bounded above by $\mathbb{1}_{\{\eta\in\Omega_{\mathrm{SAW}}\}}$. □

Proposition 4.8. *Assume $d > 4$. Fix $z \in (0, z_c)$ and assume $f(z) \le K$. Then there is a constant c_K independent of z and d such that*

$$\|(1-\cos(k\cdot x))H_{\lambda,z}\|_\infty \le c_K(1+\beta)\widehat{C}_{p(z)}(k)^{-1}, \tag{4.4}$$

$$\|H_{\lambda,z}\|_2^2 \le c_K\beta \tag{4.5}$$

$$\|H_{\lambda,z}\|_\infty \le c_K\beta. \tag{4.6}$$

Proof. The general fact that $\|g\|_\infty \le \|\hat{g}\|_1$ and the identity

$$\sum_x \cos(k\cdot x) f(x) e^{i\ell\cdot x} = \frac{1}{2}(\hat{f}(\ell+k) + \hat{f}(\ell-k))$$

imply

$$\begin{aligned}
\|(1-\cos(k\cdot x))H_{\lambda,z}(x)\|_\infty &= \|(1-\cos(k\cdot x))G_{\lambda,z}(x)\|_\infty \\
&\le \frac{1}{2}\|\Delta_k \widehat{G}_{\lambda,z}(\ell)\|_1.
\end{aligned}$$

The definition of U, the fact that $f_3 \leq K$, and the Cauchy–Schwarz inequality then imply

$$\|(1-\cos(k\cdot x))H_{\lambda,z}(x)\|_\infty \leq 16K\hat{C}_{p(z)}(k)^{-1}3\|\hat{C}_{p(z)}\|_2^2,$$

which yields (4.4) after using (4.3).

To estimate $\|H_{\lambda,z}\|_2^2$ note that Lemma 4.7 implies

$$H_{\lambda,z}(x) \leq z\alpha(\lambda,z)\,|\Omega|\,D * G_{\lambda,z}(x)$$

The factor $z\alpha\,|\Omega|$ is estimated using $f_1(z) \leq K$. To estimate $D * G_{\lambda,z}$ use Parseval's identity, $f_2(z) \leq K$, and Lemma 4.3:

$$\begin{aligned}\|H_{\lambda,z}\|_2^2 &\leq K^2\|D * G_{\lambda,z}\|_2^2\\ &\leq K^4\|\hat{D}\hat{C}_{|\Omega|^{-1}}\|_2^2\\ &= K^4(\|\hat{C}_{|\Omega|^{-1}}\|_2^2 - 1)\\ &\leq cK^4\beta.\end{aligned}$$

For the last inequality use the fact that $\sup_x H_{\lambda,z}(x) = \sup_{x\neq 0} G_{\lambda,z}(x)$, Lemma 4.7, equation (4.1) and then Lemma 4.7 again. Using $f_1 \leq K$ gives

$$H_{\lambda,z}(x) \leq K\alpha_0(\lambda,z)D(x) + K^2 D * D * G_{\lambda,z}(x).$$

A little manipulation shows that $\|D * D * G_{\lambda,z}\|_\infty \leq \|\hat{D}^2\hat{C}_{p(z)}^2\|_1$, so Lemma 4.3 implies

$$\|D * D * G_{\lambda,z}\|_\infty \leq cK\beta.$$

Equation 4.2 implies $D(x) \leq \beta$ so it suffices to show $\alpha_0(\lambda,z)$ is bounded above. This follows from $f_2 \leq K$:

$$\begin{aligned}\alpha_0 &= \int_{[-\pi,\pi]^d} \hat{G}_{\lambda,z}(k)\,\frac{d^dk}{(2\pi)^d}\\ &\leq K\int_{[-\pi,\pi]^d} \hat{C}_{p(z)}(k)\,\frac{d^dk}{(2\pi)^d}\\ &\leq K\|\hat{C}_{|\Omega|^{-1}}\|_1,\end{aligned}$$

and this last integral is finite for $d \geq 3$, and decreases as the dimension d increases.

□

5. Convergence of the lace expansion II. Diagrammatic bounds and convergence

To control the lace expansion it is necessary to show that $\widehat{\Pi}_{\lambda,z}$ is small. This is done by obtaining bounds on norms of $\Pi^{(N)}_{\lambda,z}$ in terms of $H_{\lambda,z}$, $G_{\lambda,z}$, and $I_{\lambda,z}$. These bounds are known as *diagrammatic bounds*. Coupled with Proposition 4.8 diagrammatic bounds are what make the hypothesis $f(z) \le K$ powerful.

Obtaining diagrammatic bounds requires bounding the weight of walks constrained to have $\omega_j = x$ in terms of unconstrained walks. This is best illustrated by an example. Consider obtaining a bound for $\frac{d}{dz} G_{\lambda,z}(0,x)$. For self-avoiding walk ($\lambda = 0$) this is straightforward: the Leibniz rule implies the derivative is a sum over all self-avoiding walks from 0 to x together with a marked edge. Splitting the walk at the marked edge and using the fact that self-avoiding walk is purely repulsive yields

$$\frac{d}{dz} G_{0,z}(0,x) \le z^{-1} G_{0,z} * H_{0,z}(0,x). \tag{5.1}$$

For $\lambda > 0$ a similar argument is possible, but the weight on the second half of the walk is not $w_{\lambda,z}$: memory of the first half of the walk is needed to know when loops are erased. Section 5.1 derives identities for walks that play the role of equation (5.1) for $\lambda > 0$. Section 5.2 uses these identities to derive the diagrammatic bounds necessary to apply Lemma 4.5.

5.1. Decompositions for λ-LWW.

The formulas presented in this section are the result of tracking what happens when loop erasure is performed. The reader may find it helpful to draw examples while reading the text.

5.1.1. Decompositions from loop erasure.

The loop erasure of a walk can be viewed as a last exit decomposition: if $\omega : x \to y$ then the second vertex in the loop erasure is the first vertex visited after the last visit to x. Iterating this implies the next proposition.

Proposition 5.1. *Let ω be a walk. Define $\ell_0 = 0$ and*

$$\ell_k = \sup\{j \mid \omega_j = \omega_{\ell_{k-1}}\} + 1$$

for $k \in \mathbb{N}$. Suppose there are $n+1$ finite values of ℓ_k such that $\ell_k \le |\omega|$. Then

$$\mathrm{LE}(\omega) = (\omega_{\ell_0}, \omega_{\ell_1}, \dots, \omega_{\ell_n}).$$

In Proposition 5.1 the restriction to finite values at most $|\omega|$ is due to the fact that there will be an $\ell_k = |\omega| + 1$, and then $\ell_{k+1} = -\infty$. The loop erasure of a walk ω induces a decomposition of ω. Let $\eta = \mathrm{LE}(\omega) = (\omega_{\ell_0}, \dots, \omega_{\ell_k})$. Define, for $0 \le r < s \le k$,

$$\eta^{-1}[r, s] = \omega[\ell_r, \ell_s - 1], \tag{5.2}$$

where, recalling Proposition 5.1, $\ell_{k+1} = |\omega| + 1$. See Figure 4.

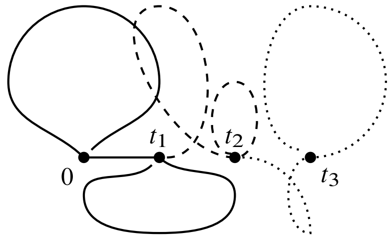

Figure 4. An illustration of the definition of $\eta^{-1}[r, s]$ and of the subdivision of a walk given by equation (5.3). The initial walk $\eta^{-1}[0, t_1]$, drawn with a solid line, is the preimage of the initial solid segment of the loop erasure of the walk. The dashed and dotted curves are $\eta^{-1}[t_1, t_2]$ and $\eta^{-1}[t_2, t_3]$ respectively. The small gaps in curves indicate the flow of time.

Remark 5.2. It would be more accurate to write $\mathrm{LE}(\omega)^{-1}[r, s]$ as the definition requires knowledge of the walk ω whose loop erasure is η. As the walk ω will be clear from context this will not cause any confusion.

The following extension of the notion of the concatenation of two walks will be notationally convenient. If $\omega^i : x_i \to y_i$ and $y_1 \sim x_2$ write $\omega^1 \diamond \omega^2$ for the walk that consists of ω^1 followed by a step from y_1 to x_2 followed by the walk ω^2.

Fix a walk ω whose loop erasure is k steps long. A sequence of times

$$0 = t_0 < t_1 < t_2 < \cdots < t_n = k$$

induces a decomposition of ω by using equation (5.2):

$$\omega = \eta^{-1}[t_0, t_1] \diamond \cdots \diamond \eta^{-1}[t_{n-1}, t_n]. \tag{5.3}$$

This decomposition has two notable features. First, the loop erasure of the segments of the decomposition yield $\eta[t_i, t_{i+1} - 1]$. Second, each segment, barring perhaps the first segment, never returns to its starting vertex. See Figure 4.

The next definitions serve to formalize the fact that given the loop erasure $\eta = \mathrm{LE}(\omega[0, j])$ of a walk ω up to time j, the remainder of ω has the effect of erasing some of η, and then extending the remainder of η to complete the formation of $\mathrm{LE}(\omega)$.

Definition 5.3. Let $A \subset \mathbb{Z}^d$. The *hitting time* $\tau_\omega(A)$ of A by ω is

$$\tau_\omega(A) = \inf\{j \geq 0 \mid \omega_j \in A\}.$$

Definition 5.4. Let $\eta\colon x \to y$ be a self-avoiding walk, and let ω be a walk beginning at y. Let $\eta^0 = \eta\,[0, |\eta|)$. For $k \geq 1$ inductively define

$$s^k_\eta(\omega) = \tau_\omega(\eta^{k-1}), \quad t^k_\omega(\eta) = \eta^{-1}(\omega_{s^k_\eta(\omega)}), \quad \eta^k = \eta[0, t^k_\omega(\eta))).$$

The times $s^k_\eta(\omega)$ are the *shrinking times of* η *by* ω.

See Figure 5 for an illustration of shrinking times. The walks η^k in the definition are decreasing in length, and it follows that the times $t^k_\omega(\eta)$ are decreasing in k.

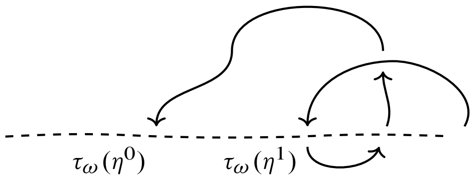

Figure 5. An illustration of the shrinking times of the self-avoiding walk η (dashed) by ω (solid). The gaps in ω are to indicate the progress of time. Note that the second hitting time of η is *not* a shrinking time as it occurs on a portion of η that is erased at the first hitting time.

5.1.2. Expected visits of $\boldsymbol{\lambda}$-LWW. The next proposition gives a formula for the expected number of visits of a closed λ-LWW to a given vertex y. We will first give an informal description of the formula. The number of visits by a walk ω to a vertex y can be expressed as

$$\left|\{j \geq 1 \mid \omega_j = y\}\right| = \sum_{j \geq 1} \mathbb{1}_{\{\omega_j = y\}}.$$

Consider a walk with $\omega_j = y$. This naturally splits into two pieces: the walk $\omega^{(a)}$ up to time j, and the walk $\omega^{(b)}$ after time j. The splitting times introduced in Section 5.1.1 then split each of $\omega^{(a)}$ and $\omega^{(b)}$ into k segments if there are k splitting times. In Proposition 5.5 the segments of $\omega^{(a)}$ are called $\omega^{(i)}$ for $i = 1, \ldots, k$, and the segments of $\omega^{(b)}$ are called $\omega^{(k+i)}$ for $i = 1, \ldots, k$. The conditions A_i and B_i are formalizations of the fact that these subwalks arise from splitting times.

Proposition 5.5. *Fix $x, y \in \mathbb{Z}^d$, $y \neq x$. Then*

$$\begin{aligned}
&\sum_{\omega\colon x\to x} \big|\{j \geq 1 \mid \omega_j = y\}\big| \, w_{\lambda,z}(\omega) \\
&= \alpha_0 \sum_{k\geq 1} \sum_{\substack{x_0,\dots,x_k \\ \text{distinct}}} \sum_{i=1}^{k} \mathbb{1}_{\{x_0=x\}}\mathbb{1}_{\{x_k=y\}}\lambda^k \\
&\quad \sum_{\substack{\omega^{(i)}\colon x_{i-1}\to x_i \\ \omega^{(k+i)}\colon x_{k-i+1}\to x_{k-i}}} \Big(\prod_{i=1}^{k} w_{\lambda,z}(\omega^{(i)})\mathbb{1}_{\{\omega^{(i)}\in A_i\}}\Big) \\
&\qquad \Big(\prod_{i=1}^{k} w_{\lambda,z}(\omega^{(k+i)})\mathbb{1}_{\{\omega^{(k+i)}\in B_i\}}\Big),
\end{aligned} \tag{5.4}$$

where A_i and B_i are defined as follows. A walk ω is in A_i if $\omega\,[1\colon]$ does not hit $\mathrm{LE}(\omega^{(j)})$ *for any $j < i$. A walk ω is in B_i if*

(1) *ω does not hit ω^{k-j} for $j > i + 1$,*

(2) *$\omega\,[1\colon]$ hits ω^{k-i} at ω_0^{k-i},*

(3) *ω hits ω^{k-i-1} at ω_0^{k-i}, and ω does not hit* $\mathrm{LE}(\omega^{k-i-1}) \setminus \{\omega_0^{k-i}\}$.

Proof. Rewrite $\big|\{j \geq 1 \mid \omega_j = y\}\big|$ as $\sum_{j\geq 1} \mathbb{1}_{\{\omega_j=y\}}$. To prove the claim it suffices to show that walks with $\omega_j = y$ are in bijection with the summands such that $\big|\omega^{(1)} \circ \cdots \circ \omega^{(k)}\big| = j$.

Suppose $\omega_j = y$, and let $\eta = \mathrm{LE}(\omega\,[0, j])$. Let t^ℓ, s^ℓ be $t^\ell_\omega(\eta)$ and $s^\ell_\eta(\omega)$, respectively. Assume there are k shrinking times for the walk ω. Observing that ω closed implies $t^k = 0$, $s^k = |\omega|$ implies

$$\omega\,[0, j] = \eta^{-1}[t^k, t^{k-1}] \diamond \cdots \diamond \eta^{-1}[t^2, t^1] \diamond \eta^{-1}[t^1, |\eta|] \tag{5.5}$$

$$\omega[j\colon] = \omega\,\big[j, s^1\big] \diamond \cdots \diamond \omega[s^{k-1}, s^k]. \tag{5.6}$$

Call the subwalks on the right-hand sides of (5.5) and (5.6) the *constituents* of $\omega\,[0, j]$ and $\omega\,[j\colon]$, respectively. Call a walk $\omega\colon x \to x$ an *excursion* if the only occurrences of x in ω are ω_0 and $\omega_{|\omega|}$.

Separating any initial excursions from x to x from the first subwalk comprising $\omega\,[0, j]$ gives the factor α_0. To complete the claim, notice that any excursions immediately after a shrinking time that occur prior to the next hitting time of η^ℓ can be transferred to the previous subwalk comprising $\omega\,[j:]$. In the case of the first constituent of $\omega\,[j:]$ the excursions can be transferred to the last constituent of $\omega\,[0, j]$. ☐

The next proposition handles the case of visits to the initial vertex of a walk.

Proposition 5.6. *We have*

$$\sum_{\substack{\omega:\, x\to x \\ |\omega|\geq 1}} \big|\{j \geq 1 \mid \omega_j = x\}\big|\, w_{\lambda,z}(\omega) = \alpha_0(\alpha_0 - 1). \tag{5.7}$$

Proof. Write $\big|\{j \geq 1 \mid \omega_j = x\}\big|$ as $\sum_{j\geq 1} \mathbb{1}_{\{\omega_j = x\}}$. Insert this into the left-hand side of (5.7) and split each walk ω at time j. Summing the remainder after time j gives a factor α_0. Summing over j gives $\alpha_0 - 1$ as $j \geq 1$ implies the empty walk is excluded. ☐

To avoid explicitly writing the cumbersome right-hand side of (5.4) repeatedly it will be convenient to introduce a short-hand definition:

Definition 5.7. The *bubble chain* $\mathrm{BC}_{\lambda,z}(x, y)$ *from* x *to* y is defined to be $\alpha_0(\alpha_0 - 1)$ if $x = y$ and the right-hand side of (5.4) if $x \neq y$.

The next decomposition formula is the analogue of Proposition 5.5 for walks ω that are not closed. Some notation will be needed: for η a self-avoiding walk ending at x define $\mathrm{BC}^{\eta}_{\lambda,z}(x, y)$ to be the bubble chain in $\mathbb{Z}^d \setminus \{\eta_0, \ldots, \eta_{|\eta|-1}\}$. See Figure 6.

Proposition 5.8. *Fix $x, y, b \in \mathbb{Z}^d$, $x \neq y$, $b \neq x$. Then*

$$\sum_{\omega:\, x\to y} \mathbb{1}_{\{x\notin\omega[1:]\}} \big|\{j \geq 1 \mid \omega_j = b\}\big|\, w_{\lambda,z}(\omega)$$
$$= \sum_{a\in\mathbb{Z}^d} \sum_{\substack{\omega^{(1)}:\, x\to a \\ x\notin\omega[1:]}} \sum_{\substack{\omega^{(2)}:\, a\to y \\ \omega^{(2)}[1:]\cap \mathrm{LE}(\omega^{(1)})=\emptyset}} (\delta_{a,b} + \mathrm{BC}^{\mathrm{LE}(\omega^{(1)})}_{\lambda,z}(a, b))\, w_{\lambda,z}(\omega^{(1)})\, w_{\lambda,z}(\omega^{(2)}). \tag{5.8}$$

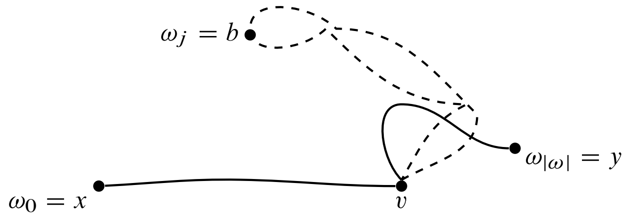

Figure 6. An illustration of a contribution to to the sum in equation (5.8). For clarity, only the loop erasure of each walk is shown. The dashed black path is the bubble chain portion of the walk. The vertex v indicates the division between the path prior to the bubble chain and after the bubble chain.

Proof. This follows by writing $\left|\{j \geq 1 \mid \omega_j = b\}\right|$ as $\sum_{j\geq 1} \mathbb{1}_{\{\omega_j = b\}}$ and noting that this splits, by applying equation (5.3) with $\eta = \mathrm{LE}(\omega\,[0, j])$, a walk ω into (i) an initial segment $\omega^{(1)}$ whose loop erasure is the subset of $\mathrm{LE}(\omega\,[0, j])$ that is contained in $\mathrm{LE}(\omega)$, (ii) a bubble chain from the endpoint of $\omega^{(1)}$ to b whose walks do not hit $\mathrm{LE}(\omega^{(1)})$; if the endpoint of $\omega^{(1)}$ is b then it is also possible this walk is null, and (iii) a walk $\omega^{(2)}$ from the endpoint of $\omega^{(1)}$ to y that does not, after the first vertex, hit $\mathrm{LE}(\omega^{(1)})$. □

The restriction in Proposition 5.8 to walks ω that do not return to their initial vertex is simply because this is the type of sum that will occur most frequently in what follows.

5.1.3. Two-point functions and their derivatives. The quantity $I^{\omega}_{\lambda,z}(a, b)$ defined in equation (3.2) is inconvenient due to its dependence on the details of ω; the next definition introduces a simple upper bound.

Definition 5.9. The *interaction two-point function* $I_{\lambda,z}(x, y)$ is the function

$$I_{\lambda,z}(x, y) = \mathbb{1}_{\{x=y\}} + \mathbb{1}_{\{x\neq y\}}(1 - e^{-\mu_{\lambda,z}(x,y;\emptyset)}).$$

Lemma 5.10. *Let ω be a walk of length n, and let $0 \leq a < b \leq n$.*

$$I^{\omega}_{\lambda,z}(a, b) \leq I_{\lambda,z}(\omega_a, \omega_b). \tag{5.9}$$

Proof. If $\omega_a = \omega_b$ then (5.9) is an equality. If $\omega_a \neq \omega_b$ the inequality follows because the loop measure is decreasing in its final argument. □

The important aspect of the next bound is that it is independent of η.

Proposition 5.11. *Let $\eta\colon x \to y$ be a self-avoiding walk. Then*

$$\frac{d}{dz} I^{\eta}_{\lambda,z}(x,y) \le \mathbb{1}_{\{x\neq y\}} z^{-1} \sum_{a\in\mathbb{Z}^d} \sum_{\substack{\omega\colon a\to a \\ |\omega|\ge 1}} \mathbb{1}_{\{x\in\omega\}} \mathbb{1}_{\{y\in\omega\}} w_{\lambda,z}(\omega). \tag{5.10}$$

Further, the right-hand side of (5.10) is an upper bound for $\frac{d}{dz} I_{\lambda,z}(x,y)$ as well.

Proof. Differentiate, and then use $e^{-x} \le 1$ for $x \ge 0$. □

Definition 5.12. The *scaled two-point functions* $\bar{G}_{\lambda,z}(x,y)$ and $\bar{H}_{\lambda,z}(x,y)$ are defined by

$$\bar{G}_{\lambda,z}(x,y) = \alpha_0(\lambda,z)^{-1} G_{\lambda,z}(x,y),$$
$$\bar{H}_{\lambda,z}(x,y) = \alpha_0(\lambda,z)^{-1} H_{\lambda,z}(x,y).$$

Let $\bar{B}_{\lambda,z}(x) = \bar{H}_{\lambda,z}(x)^2$. An upper bound on $\mathrm{BC}_{\lambda,z}$ is obtained by dropping the constraints A_i and B_i.

Definition 5.13. Define $\mathrm{B}^{\star}{}_{\lambda,z}(x,y)$ by

$$\mathrm{B}^{\star}{}_{\lambda,z}(x,y) = \alpha_0 \begin{cases} \sum_{k\ge 1} \lambda^k \underbrace{\bar{B}_{\lambda,z} * \cdots * \bar{B}_{\lambda,z}}_{k \text{ terms}}(x,y) & x \neq y \\ \alpha_0 - 1 & x = y \end{cases} \tag{5.11}$$

Proposition 5.14. *Let η be any self-avoiding walk ending at x. Then*

$$\mathrm{BC}^{\eta}_{\lambda,z}(x,y) \le \mathrm{BC}_{\lambda,z}(x,y) \le \mathrm{B}^{\star}{}_{\lambda,z}(x,y).$$

Proof. The first inequality follows as the set of summands is increasing from left to right and all summands are non-negative. For the second inequality note that relaxing the conditions A_i and B_i increases the set of summands. Using $H_{\lambda,z}(x,y) = H_{\lambda,z}(y,x)$, which follows from Theorem 1.13, to reverse the direction of the walks $\omega^{(k+i)}$ gives the upper bound $\mathrm{B}^{\star}{}_{\lambda,z}(x,y)$. □

The next lemma shows that if a sum over walks satisfying some constraints is upper bounded by relaxing the constraints, an upper bound on the derivative is obtained by differentiating the upper bound. This will be used frequently.

Lemma 5.15. *Suppose A, B are two sets of walks, and $A \subset B$. Then*

$$\frac{d}{dz}\sum_{\omega\in A} w_{\lambda,z}(\omega) \le \frac{d}{dz}\sum_{\omega\in B} w_{\lambda,z}(\omega).$$

Proof. Each summand is non-negative as the weight of a walk ω is proportional to $z^{|\omega|}$, and the set of summands on the right-hand side is larger. □

The formulas of Section 5.1.2 yield diagrammatic bounds on derivatives of two-point functions by applying the identity

$$|\omega| = \sum_{a\in\mathbb{Z}^d} |\{j \ge 1 \mid \omega_j = a\}|, \tag{5.12}$$

where $j = 0$ is not included because there $|\omega| + 1$ vertices in a walk.

Proposition 5.16. *For $x \in \mathbb{Z}^d$, $x \neq 0$,*

$$\frac{d}{dz}\bar{G}_{\lambda,z}(x) = \frac{d}{dz}\bar{H}_{\lambda,z}(x) \le z^{-1}(1 + \|\mathrm{B}^{\star}{}_{\lambda,z}\|_1)\bar{G}_{\lambda,z} * \bar{H}_{\lambda,z}(x).$$

Proof. The first equality is straightforward as $\bar{G}_{\lambda,z}(x) = \delta_{0,x} + \bar{H}_{\lambda,z}(x)$ by equation (4.1). For the inequality observe that

$$\frac{d}{dz}\bar{H}_{\lambda,z}(x) = z^{-1} \sum_{\substack{\omega\colon 0\to x\\ 0\notin\omega[1:]}} |\omega|\, w_{\lambda,z}(\omega)$$

Applying (5.12) and Proposition 5.8 yields

$$z^{-1}\sum_{b\in\mathbb{Z}^d}\sum_{a\in\mathbb{Z}^d}\sum_{\substack{\omega^{(1)}\colon 0\to a\\ 0\notin\omega[1:]}}\sum_{\substack{\omega^{(2)}\colon a\to x\\ \omega^{(2)}[1:]\cap \mathrm{LE}(\omega^{(1)})=\emptyset}} (\delta_{a,b} + \mathrm{BC}_{\lambda,z}^{\mathrm{LE}(\omega^{(1)})}(a,b))\; w_{\lambda,z}(\omega^{(1)})w_{\lambda,z}(\omega^{(2)}).$$

By Proposition 5.14 removing the restriction on the bubble chain gives an upper bound. The sum over b then gives the factor $1 + \|\mathrm{B}^{\star}{}_{\lambda,z}\|_1$. Dropping the constraint that $\omega^{(2)}$ does not intersect $\mathrm{LE}(\omega^{(1)})$ gives the claim. □

Proposition 5.17. *We have*

$$\frac{d}{dz}\alpha_0(\lambda,z) = z^{-1}\|\mathrm{B}^\star{}_{\lambda,z}\|_1.$$

Proof. As a zero step walk does not survive being differentiated,

$$\frac{d}{dz}\alpha_0(\lambda,z) = z^{-1}\sum_{\substack{\omega:\,0\to 0\\ |\omega|\geq 1}} |\omega|\, w_{\lambda,z}(\omega).$$

The proposition follows by (i) applying (5.12) to rewrite $|\omega|$, (ii) using Propositions 5.5 and 5.6 to recognize the resulting sum as the 1-norm of the bubble chain, and (iii) using Proposition 5.14 to upper bound the norm of the bubble chain. □

Proposition 5.18. *We have*

$$\frac{d}{dz}\|\mathrm{BC}_{\lambda,z}\|_1 \leq z^{-1}\|\mathrm{B}^\star{}_{\lambda,z}\|_1(3\alpha_0 - 1 - \alpha_0^2 + \|\mathrm{B}^\star{}_{\lambda,z}\|_1)$$
$$+ 2\alpha_0 z^{-1}\lambda\|\bar{H}_{\lambda,z}\cdot(\bar{G}_{\lambda,z} * \bar{H}_{\lambda,z})\|_1(1+\|\mathrm{B}^\star{}_{\lambda,z}\|_1)^3.$$

Proof. By Lemma 5.15 it suffices to obtain bounds on the derivative of $\|\mathrm{B}^\star{}_{\lambda,z}\|_1$. For the summand with $x = y$ an upper bound is

$$z^{-1}\|\mathrm{B}^\star{}_{\lambda,z}\|_1(\alpha_0 - 1) + z^{-1}\alpha_0\|\mathrm{B}^\star{}_{\lambda,z}\|_1$$

by Proposition 5.17.

For $x \neq y$ differentiating equation (5.11) and using Proposition 5.17 gives an upper bound $z^{-1}\|\mathrm{B}^\star{}_{\lambda,z}\|_1\alpha_0^{-1}(\|\mathrm{B}^\star{}_{\lambda,z}\|_1 - \alpha_0(\alpha_0-1))$ if the derivative is applied to α_0. The factor of α_0^{-1} can be dropped to give an upper bound as $\alpha_0 \geq 1$. When the derivative is not applied to α_0 we have, using Proposition 5.16, the upper bound

$$\frac{d}{dz}\|\mathrm{B}^\star{}_{\lambda,z}\|_1 = \alpha_0\frac{d}{dz}\sum_{k\geq 1}\sum_y \lambda^k \underbrace{\bar{H}^2_{\lambda,z} * \cdots * \bar{H}^2_{\lambda,z}}_{k \text{ terms}}(y)$$
$$\leq 2\alpha_0\sum_{k\geq 1}\sum_y k\lambda^k\left(\bar{H}_{\lambda,z}\frac{d}{dz}\bar{H}_{\lambda,z}\right) * \underbrace{\bar{H}^2_{\lambda,z} * \cdots * \bar{H}^2_{\lambda,z}}_{k-1 \text{ terms}}(y)$$
$$= 2\alpha_0 z^{-1}\lambda\|\bar{H}_{\lambda,z}\cdot(\bar{G}_{\lambda,z} * \bar{H}_{\lambda,z})\|_1(1+\|\mathrm{B}^\star{}_{\lambda,z}\|_1)^3.$$

Summing these upper bounds gives the result. □

5.2. Diagrammatic bounds. The bounds derived in this section will be obtained under the assumption that $f(z) \leq K$ for $z < z_c(\lambda)$. In particular the results of Proposition 4.8 hold. It will also be assumed that the dimension d is sufficiently large, i.e., β is sufficiently small.

5.2.1. Initial diagrammatic bounds

Proposition 5.19. *If $z < z_c$ and $f(z) \le K$ then $\alpha_0(\lambda, z) \le 1 + c\beta$.*

Proof. By definition and Theorem 1.13

$$\alpha_0(\lambda, z) = \exp(\mu_{\lambda,z}(0)) = 1 + \sum_{\substack{\omega\colon 0\to 0\\ |\omega|\ge 1}} w_{\lambda,z}(\omega).$$

The walks contributing to the sum have their last vertex a neighbour of 0, so

$$\sum_{\substack{\omega\colon 0\to 0\\ |\omega|\ge 1}} w_{\lambda,z}(\omega) = z\lambda\, |\Omega|\, D * H_{\lambda,z}(0), \tag{5.13}$$

which is bounded by $z\lambda\, |\Omega|\, \|H_{\lambda,z}\|_\infty$. The claim follows from $z\,|\Omega| \le f_1(z) \le K$ and (4.6). □

Proposition 5.20. *If $z < z_c$ and $f(z) \le K$ then $\|\mathrm{B}^\star{}_{\lambda,z}\|_1 \le c\beta$.*

Proof. Repeatedly using $\|f * g\|_1 \le \|f\|_1 \|g\|_1$ implies

$$\|\mathrm{B}^\star{}_{\lambda,z}\|_1 \le \alpha_0\Big((\alpha_0 - 1) + \sum_{k\ge 1} \lambda^k \|\bar{H}_{\lambda,z}\|_2^{2k}\Big),$$

The interchange of summations is valid as each term is non-negative. By Proposition 5.19, $\alpha_0 \le 1 + c\beta$ so $\alpha_0 - 1 \le c\beta$. Since $\alpha_0 \ge 1$, $\|\bar{H}_{\lambda,z}\|_2^2 \le \|H_{\lambda,z}\|_2^2$, so equation (4.5) implies that for β sufficiently small

$$\sum_{k\ge 1} \lambda^k \|\bar{H}_{\lambda,z}\|_2^{2k} \le c\beta. \qquad □$$

Proposition 5.21. *Let $I_{\lambda,z}(x) = I_{\lambda,z}(0, x)$. If $z < z_c$ and $f(z) \le K$ then $\|I_{\lambda,z}\|_1 \le 1 + c\beta$.*

Proof. The inequality $1 - e^{-x} \le x$ implies that $1 + \|\mathbb{1}_{\{x\ne 0\}}\mu_{\lambda,z}(0, x)\|_1$ is an upper bound for $\|I_{\lambda,z}\|_1$. The factor of 1 is from the term $\mathbb{1}_{\{x=0\}}$ in $I_{\lambda,z}$. Observe that $\|\mathbb{1}_{\{x\ne 0\}}\mu_{\lambda,z}(0, x)\|_1$ is bounded by

$$\sum_{x\ne 0}\sum_{y}\sum_{\substack{\omega\colon y\to y\\ |\omega|\ge 1}} \mathbb{1}_{\{0\in\omega\}}\mathbb{1}_{\{x\in\omega\}}\frac{w_{\lambda,z}(\omega)}{|\omega|} \le \sum_{y}\sum_{\substack{\omega\colon y\to y\\ |\omega|\ge 1}} \mathbb{1}_{\{0\in\omega\}} w_{\lambda,z}(\omega),$$

as $\sum_{x\neq 0} \mathbb{1}_{\{x\in\omega\}} \leq |\text{range}(\omega)| \leq |\omega|$. By translation invariance this is

$$\sum_{y} \sum_{\substack{\omega:\,0\to 0\\ |\omega|\geq 1}} \mathbb{1}_{\{-y\in\omega\}} w_{\lambda,z}(\omega) = \Big\| \sum_{\substack{\omega:\,0\to 0\\ |\omega|\geq 1}} \mathbb{1}_{\{y\in\omega\}} w_{\lambda,z}(\omega) \Big\|_1,$$

where the norm is with respect to y. To establish the proposition (i) bound $\mathbb{1}_{\{y\in\omega\}}$ by $\big|\{j \geq 1 \mid \omega_j = y\}\big|$, (ii) apply Proposition 5.6 for the summands with $y = 0$, (iii) apply Proposition 5.5 and Proposition 5.14 for the summands with $y \neq 0$, and (iv) observe that the sum of these two bounds is $\|\mathrm{B}^\star{}_{\lambda,z}\|_1$ and apply Proposition 5.20. □

5.2.2. Bounds for $\pi^{(1)}$

Proposition 5.22. *We have*

$$\pi^{(1)}(x) = \sum_m \pi_m^{(1)} \begin{cases} = z\lambda\,|\Omega|\,D * \bar{H}_{\lambda,z}(0) & x = 0 \\ \leq \bar{H}_{\lambda,z}(x) I_{\lambda,z}(0,x) e^{\mu_{\lambda,z}(0,x)} & x \neq 0, \end{cases}$$

Proof. For $x = 0$ the claim follows from the identities in equations (1.4), (1.5), (2.15), and (5.13). For $x \neq 0$ use equation (2.17). Recall the loop measure representation of the second product, i.e., the expression for $I_{\mathcal{X}}^{\omega}$ given by equation (3.2). The desired bound follows by forgetting the constraint in the loop measure and the rearrangement $e^{\mu_{\lambda,z}(0,x)} - 1 = e^{\mu_{\lambda,z}(0,x)} I_{\lambda,z}(0,x)$. □

Proposition 5.23. *Suppose $f(z) \leq K$. The following bounds hold for $u = 0, 1$ and $k \in [-\pi,\pi]^d$:*

$$\|\,|x|^{2u}\,\pi^{(1)}\|_1 \leq c\beta(\mathbb{1}_{\{u=0\}} + \|\,|x|^{2u}\,\bar{H}_{\lambda,z}\|_\infty),$$

and

$$\|(1-\cos k\cdot x)\pi^{(1)}(x)\|_1 \leq c\beta\|(1-\cos k\cdot x)\bar{H}_{\lambda,z}\|_\infty.$$

Proof. The triangle inequality, $\|f * g\|_1 \leq \|f\|_\infty \|g\|_1$ with $g = I_{\lambda,z}$, and $1 - \cos 0 = 0$ imply

$$\begin{aligned}\|\,|x|^{2u}\,\pi^{(1)}\|_1 &\leq \mathbb{1}_{\{u=0\}} z\,|\Omega|\,\|\bar{H}_{\lambda,z}\|_\infty \\ &\quad + \|I_{\lambda,z}(0,x)e^{\mu_{\lambda,z}(0,x)}\mathbb{1}_{\{x\neq 0\}}\|_1 \|\,|x|^{2u}\,\bar{H}_{\lambda,z}\|_\infty\end{aligned}$$

and

$$\begin{aligned}&\|(1-\cos k\cdot x)\pi^{(1)}(x)\|_1 \\ &\quad \leq \|I_{\lambda,z}(0,x)e^{\mu_{\lambda,z}(0,x)}\mathbb{1}_{\{x\neq 0\}}\|_1 \|(1-\cos k\cdot x)\bar{H}_{\lambda,z}\|_\infty.\end{aligned}$$

Using $z\,|\Omega| \le f_1(z) \le K$ by $f_3 \le K$ and $\sup_x e^{\mu_{\lambda,z}(0,x)} \le \alpha_0(\lambda, z)$ implies

$$\begin{aligned}\|I_{\lambda,z}(0,x)e^{\mu_{\lambda,z}(0,x)}\mathbb{1}_{\{x\neq 0\}}\|_1 &\le \alpha_0 \|I_{\lambda,z}(0,x)\mathbb{1}_{\{x\neq 0\}}\|_1 \\ &\le \alpha_0(\|I_{\lambda,z}(0,x)\|_1 - 1).\end{aligned}$$

The conclusion now follows from Propositions 5.19 and 5.21. □

5.2.3. Bounds for $\pi^{(N)}$, $N \ge 2$

Proposition 5.24. *Let $m \ge 2$, $x \in \mathbb{Z}^d$, and $N \ge 2$. Let $x_0 = 0, x'_{N-1} = x$. Then*

$$\begin{aligned}|\pi_m^{(N)}(x)| \le \sum_{\vec{m}} \sum_{\substack{x_1,\dots,x_{N-1} \\ x'_0,\dots,x'_{N-2}}} \sum_{\substack{\omega^{(1)}:\,0\to x_1 \\ |\omega^{(1)}|=m_1}} \sum_{\substack{\omega^{(2)}:\,x_1\to x'_0 \\ |\omega^{(2)}|=m_2}} \cdots \\ \sum_{\substack{\omega^{(2N-2)}:\,x_{N-1}\to x'_{N-2} \\ |\omega^{(2N-2)}|=m_{2N-2}}} \sum_{\substack{\omega^{(2N-1)}:\,x'_{N-2}\to x \\ |\omega^{(2N-1)}|=m_{2N-1}}} \\ \prod_{j=0}^{N-1} |I_{\lambda,z}(x_j, x'_j)| \prod_{k=1}^{2N-1} \alpha_0^{-1} \exp(\mu_{\lambda,z}(\mathrm{range}(\omega^{(k)}))),\end{aligned} \tag{5.14}$$

where the summation is over valid vectors $\vec{m}$ (recall Definition 2.18) of subinterval lengths such that $\sum m_i = m$.

Proof. This follows from Section 2.4.3. By Lemma 5.10 the factors of $I^{\omega}_{\lambda,z}$ can be replaced by $I_{\lambda,z}$. As $\alpha_\omega \ge 0$ for any walk ω, by Remark 2.19 the constraints on subwalks can be ignored to give an upper bound. This proves the claim. □

Upper bounds on $\|\pi^{(N)}(x)\|_1$ can be efficiently found by formulating Proposition 5.24 in terms of multiplication and convolution operators. Let $\mathcal{M}_g$ and $\mathcal{C}_g$ denote multiplication and convolution by g, respectively: $\mathcal{M}_g f = gf$ and $\mathcal{C}_g f = g * f$.

Lemma 5.25. *Fix $N \ge 2$ and let $\bar{H} = \bar{H}_{\lambda,z}$, $\bar{G} = \bar{G}_{\lambda,z}$, and $I = I_{\lambda,z}$. Then*

$$\sum_x |\pi^{(N)}(x)| \le \|(\mathcal{C}_{\bar{H}*I}\mathcal{M}_{\bar{H}})(\mathcal{C}_{\bar{G}*I}\mathcal{M}_{\bar{H}})^{N-2}\bar{H} * I\|_\infty. \tag{5.15}$$

Proof. The definition of a valid vector of lengths implies that summing (5.14) over all valid vectors of lengths results in the sums over walks with indices 1, $2j$, and $2N-1$ being replaced by $\bar{H}_{\lambda,z}$, and the remaining sums of walks are replaced by $\bar{G}_{\lambda,z}$. Consulting Figure 3, this means that all horizontal solid lines except the leftmost and rightmost are weighted by $\bar{G}_{\lambda,z}$, while the rest are weighted by $\bar{H}_{\lambda,z}$. Formally,

$$\left|\pi^{(N)}(x)\right| \leq \sum_{\substack{x_1,\dots,x_{N-1}\\ x'_0,\dots,x'_{N-2}}} \Big(\prod_{j=0}^{N-1} I_{\lambda,z}(x_j,x'_j)\Big)\bar{H}_{\lambda,z}(x_0,x_1) \Big(\prod_{j=0}^{N-2} \bar{H}_{\lambda,z}(x'_j,x_{j+1})\Big)\Big(\prod_{j=0}^{N-3} \bar{G}_{\lambda,z}(x'_j,x_{j+2})\Big)\bar{H}_{\lambda,z}(x'_{N-2},x'_{N-1}). \tag{5.16}$$

Replace the factor $I_{\lambda,z}(x_0,x'_0)$ by $I_{\lambda,z}(y,x'_0)$ in (5.16) and call the resulting function $F(x,y)$. As $\sum_x |F(x,0)| = \sum_x |\pi^{(N)}(x)|$ the quantity $\sup_y \sum_x |F(x,y)|$ is an upper bound for the left-hand side of (5.15). The associativity of convolution implies

$$\sum_x |F(x,y)| = ((\mathcal{C}_{\bar{H}}\mathcal{C}_I)\mathcal{M}_{\bar{H}}(\mathcal{C}_{\bar{G}}\mathcal{C}_I\mathcal{M}_{\bar{H}})^{N-2}\bar{H} * I)(y). \tag{5.17}$$

Equation (5.15) follows as $\mathcal{C}_{\bar{G}}\mathcal{C}_I = \mathcal{C}_{\bar{G}*I}$ and $\mathcal{C}_{\bar{H}}\mathcal{C}_I = \mathcal{C}_{\bar{H}*I}$. □

The right-hand side of Lemma 5.25 can be easily estimated with the help of the next lemma.

Lemma 5.26 (Lemma 4.6 of [14]). *Given a sequence of non-negative even functions $f_0, f_1, \dots, f_{2M}$ on $\mathbb{Z}^d$, define $\mathcal{C}_j$ and $\mathcal{M}_j$ to be the operations of convolution with f_{2j} and multiplication by f_{2j-1} for $j = 1,\dots,M$. Then for any $k \in \{0,\dots,2M\}$,*

$$\|\mathcal{C}_M\mathcal{M}_M \dots \mathcal{C}_1\mathcal{M}_1 f_0\|_\infty \leq \|f_k\|_\infty \prod \|f_j * f_{j'}\|_\infty, \tag{5.18}$$

where the product is over disjoint consecutive pairs j, j' taken from the set $\{0,\dots,2M\} \setminus \{k\}$.

The strange formatting of the bounds in the next proposition are strictly for typographic convenience; in applications we multiply through by the denominators of the left-hand sides.

Proposition 5.27. *Let $N \geq 2$. Then for $z < z_c$ and $u \in \{0, 1\}$*

$$\frac{\| \, |x|^{2u} \, \pi^{(N)}(x)\|_1}{(2N-1)^u} \leq \| \, |x|^{2u} \, \bar{H}_{\lambda,z}\|_\infty (c\beta)^{N-2+\mathbb{1}_{\{N=2\}}}(1+c\beta)^{N+\mathbb{1}_{\{N\geq 3\}}}$$

and

$$\frac{\|(1-\cos(k\cdot x))\pi^{(N)}(x)\|_1}{(4N-1)(2N-1)} \leq \|(1-\cos(k\cdot x))\bar{H}_{\lambda,z}(x)\|_\infty (c\beta)^{N-2+\mathbb{1}_{\{N=2\}}}(1+c\beta)^{N+\mathbb{1}_{\{N\geq 3\}}}$$

Proof. Suppose that both

$$\frac{\| \, |x|^{2u} \, \pi^{(N)}(x)\|_1}{(2N-1)^u} \leq \| \, |x|^{2u} \, \bar{H}_{\lambda,z}\|_\infty \|\bar{G}_{\lambda,z} * \bar{G}_{\lambda,z}\|_\infty \|\bar{G}_{\lambda,z} * \bar{H}_{\lambda,z}\|_\infty^{N-2} \|I_{\lambda,z}\|_1^N \tag{5.19}$$

and

$$\frac{\|(1-\cos(k\cdot x))\pi^{(N)}(x)\|_1}{(4N-1)(2N-1)} \leq \|(1-\cos(k\cdot x))\bar{H}_{\lambda,z}(x)\|_\infty \|\bar{G}_{\lambda,z} * \bar{G}_{\lambda,z}\|_\infty \|\bar{G}_{\lambda,z} * \bar{H}_{\lambda,z}\|_\infty^{N-2} \|I_{\lambda,z}\|_1^N. \tag{5.20}$$

Suppose further that if $N = 2$ the same bounds hold with each term $\bar{G}_{\lambda,z}$ replaced by $\bar{H}_{\lambda,z}$. The claim then follows, as equation (4.1), the triangle inequality, Cauchy–Schwarz, and $\bar{H}_{\lambda,z} \leq H_{\lambda,z}$ imply

$$\|\bar{H}_{\lambda,z} * \bar{G}_{\lambda,z}\|_\infty = \|\bar{H}_{\lambda,z} + \bar{H}_{\lambda,z} * \bar{H}_{\lambda,z}\|_\infty \leq \|\bar{H}_{\lambda,z}\|_\infty + \|\bar{H}_{\lambda,z}\|_2^2 \leq c\beta, \tag{5.21}$$

and Proposition 5.21 implies $\|I_{\lambda,z}\|_1 \leq 1 + c\beta$. The rest of the proof establishes equations (5.19) and (5.20).

First observe that the difference between $N = 2$ and $N \geq 3$ is only that all two-point functions in Lemma 5.25 are $\bar{H}_{\lambda,z}$ for $N = 2$, while for $N \geq 3$ factors of $\bar{G}_{\lambda,z}$ arise.

If $u = 0$ equation (5.19) follows by applying Lemma 5.26 to the right-hand side of Lemma 5.25, putting the sup norm on the final $I_{\lambda,z} * \bar{H}_{\lambda,z}$, and using the inequality

$$\|\bar{G}_{\lambda,z} * \bar{H}_{\lambda,z} * I_{\lambda,z}\|_\infty \leq \|\bar{G}_{\lambda,z} * \bar{H}_{\lambda,z}\|_\infty \|I_{\lambda,z}\|_1.$$

For $u = 1$, note that $x = x_1 + \dots x_{2N-1}$, where x_j is the displacement along the j^{th} subwalk in a summand contributing to $\pi^{(N)}$. As $|x|^2 \leq \sum |x_i|^2$ it follows that an upper bound is given by

$$\sum_{j=1}^{2N-1} \|(\mathcal{C}_{\bar{H}*I}\mathcal{M}_{\bar{H}})(\mathcal{C}_{\bar{G}*I}\mathcal{M}_{\bar{H}})^{N-2}\bar{H} * I\|_\infty,$$

where the j^{th} two-point function $\bar{G}$ or $\bar{H}$ is replaced with $|x|^2\,\bar{H}$. The claim follows by (i) applying Lemma 5.26 and putting the sup norm on the term involving the factor of $|x|^2$ (ii) noting that the resulting norms are of the form $\|\bar{H} * \bar{H} * I\|_\infty$, $\|I * \bar{G} * \bar{H}\|_\infty$, $\|I * I * \bar{G} * \bar{G}\|_\infty$, or $\|\bar{H} * I\|_\infty$ and (iii) iterating $\|f * g\|_\infty \leq \|f\|_\infty\|g\|_1$. The uniform upper bound follows by using $\|\bar{H} * \bar{H}\|_\infty \leq \|\bar{H} * \bar{G}\|_\infty$.

To prove equation (5.20) let $t = \sum_{j=1}^n t_j$. Then (see [14, Section 4.2.3])

$$(1 - \cos t) \leq (2n + 1)\sum_{j=1}^{n}(1 - \cos t_j). \tag{5.22}$$

Letting $t_j = k \cdot x_j$ where x_j is the displacement along the j^{th} subwalk the argument used to prove (5.19) with $u = 1$ can be applied to give (5.20). The prefactor $(4N - 1)(2N - 1)$ arises as for an N edge lace there are $2N - 1$ subwalks, so $n = 2N - 1$ in equation (5.22). □

5.3. Completion of the bootstrap. This section begins by using the diagrammatic bounds of Sections 5.2.2 and 5.2.3 to establish that Π is small under the hypothesis $f(z) \leq K$.

Lemma 5.28. *Fix $z \in (0, z_c)$ and assume d is sufficiently large. If $f(z) \leq K$, then there is a constant $\bar{c}_K$ independent of z and d such that*

$$\sum_{x\in\mathbb{Z}^d} |\Pi_z(x)| \leq \bar{c}_K\beta \tag{5.23}$$

and

$$\sum_{x\in\mathbb{Z}^d} (1 - \cos(k \cdot x))\,|\Pi_z(x)| \leq \bar{c}_K\beta\widehat{C}_{p(z)}(k)^{-1}. \tag{5.24}$$

Proof. This follows by combining the bounds of Propositions 5.23 and 5.27 for $u = 0$ with the bound $\|(1 - \cos k \cdot x)H_{\lambda,z}(x)\|_\infty \leq c_K(1 + \beta)\widehat{C}^{-1}_{p(z)}(k)$ of equation (4.4). □

The remainder of this section is devoted to verifying the hypothesis of Lemma 4.5 for $z_1 = 0$, $z_2 = z_c(\lambda)$, $a = 4$ and $b = 1 + O(\beta)$.

Lemma 5.29. *The function* f *obeys* $f(0) = 1$.

Proof. Clearly $f_1(0) = 0$. The definition of $p(z)$ implies $p(0) = 0$ as we have $\alpha_0(\lambda, 0) = 1$, so $f_2(0) = 1$. Lastly, $f_3(0) = 0$: $U_0 = 48$ while $\Delta_k \widehat{G}_{\lambda,0} = 0$. □

Lemma 5.30. *The function* f *is continuous on* $[0, z_c)$.

Proof. It suffices to show f_1, f_2, f_3 are continuous on $[0, r]$ for any $r < z_c$. For f_1 this follows as $\alpha(\lambda, z) \le \alpha_0(\lambda, z) \le \chi_\lambda(z)$, i.e., $\alpha(\lambda, z)$ has a convergent power series representation.

Recall (see [14, Lemma 5.13]) that the supremum of an equicontinuous family of functions over a compact interval is a continuous function, provided this supremum is finite. It follows that it is enough to prove a bound uniform in k on the derivative of f_2 (resp. f_3) with respect to z. Since equicontinuity of a family $\{|g_\alpha|\}$ is equivalent to equicontinuity of $\{g_\alpha\}$, the absolute value on $\widehat{G}_{\lambda,z}$ (resp. $\Delta_k \widehat{G}_{\lambda,z}$) can be ignored. For f_2 the derivative is

$$\frac{d}{dz}\frac{\widehat{G}_{\lambda,z}(k)}{\widehat{C}_{p(z)}(k)} = \frac{1}{\widehat{C}_{p(z)}(k)^2}\left(\widehat{C}_{p(z)}(k)\frac{d\widehat{G}_{\lambda,z}(k)}{dz} - \widehat{G}_{\lambda,z}(k)\frac{d\widehat{C}_{p(z)}(k)}{dp}\Big|_{p=p(z)}\frac{dp(z)}{dz}\right).$$

Now note $|\widehat{G}_{\lambda,z}(k)| \le \chi_\lambda(r)$, $\left|\frac{d}{dz}\widehat{G}_{\lambda,z}(k)\right| \le \left|\frac{d}{dz}\chi_\lambda(r)\right|$, $|\partial_p \widehat{C}_p(k)| \le |\Omega|\, \chi_\lambda(r)^2$. Further,

$$\left|\frac{dp(z)}{dz}\right| = \left|\frac{d}{dz}|\Omega|^{-1}\left(1 - \frac{\alpha_0(\lambda, z)}{\chi_\lambda(z)}\right)\right|$$
$$\le |\Omega|^{-1}\,\alpha_0(\lambda, r)\frac{d}{dz}\chi_{\lambda,}(r)\chi_\lambda^{-2}(0) + \chi_\lambda^{-1}(1)\frac{d}{dz}\alpha_0(\lambda, r),$$

and $\frac{d}{dz}\alpha_0(\lambda, r)$ is bounded above by $\frac{d}{dz}\chi_\lambda(r)$ by Lemma 5.15. A uniform bound on the derivative then follows from

$$\frac{1}{2} \le \widehat{C}_{p(z)}(k) \le \widehat{C}_{p(z)}(0) = \frac{\chi_\lambda(z)}{\alpha_0(\lambda, z)} \le \chi_\lambda(r),$$

where the second last equality follows from the definition of $p(z)$, and the last inequality from $\alpha_0(\lambda, z) \ge 1$.

For f_3 the calculation is essentially the same. Calculating the derivative shows that what is needed is upper bounds on $|\widehat{G}_{\lambda,z}(k)|$, $\left|\frac{d}{dz}\widehat{G}_{\lambda,z}(k)\right|$, $|\partial_p \widehat{C}_p(k)|$, and $\left|\frac{d}{dz}p(z)\right|$, along with upper and lower bounds on $\widehat{C}_{p(z)}$. These bounds have already been obtained. □

The next lemma completes the bootstrap argument.

Lemma 5.31. *Suppose d is sufficiently large. Fix $z \in (0, z_c)$, and suppose that $f(z) \leq 4$. Then there is a constant c independent of z and d such that $f(z) \leq 1 + c\beta$.*

Proof. We prove $f_j(z) \leq 1 + c\beta$ for $j = 1, 2, 3$ in sequence.

Since $\alpha_0(\lambda, z)$ and $\chi_\lambda(z)$ are both positive and finite it follows that

$$\frac{\alpha_0(\lambda, z)}{\chi_\lambda(z)} = 1 - z\alpha(\lambda, z)\,|\Omega| - \widehat{\Pi}_{\lambda,z}(0) > 0. \tag{5.25}$$

Equation (5.25) and Lemma 5.28 together imply

$$f_1(z) = z\alpha(\lambda, z)\,|\Omega| \leq 1 + \widehat{\Pi}_{\lambda,z}(0) \leq 1 + \bar{c}_4\beta.$$

Proposition 5.19 implies $\alpha_0 \leq 1 + \bar{c}\beta$, so $f_2 \leq 1 + O(\beta)$ follows if

$$\frac{\widehat{G}_{\lambda,z}(k)}{\alpha_0(\lambda, z)\widehat{C}_{p(z)}(k)} = 1 + \frac{1 - p(z)\,|\Omega|\,\widehat{D}(k) - \widehat{F}_{\lambda,z}(k)}{\widehat{F}_{\lambda,z}(k)} \tag{5.26}$$

is $1 + O(\beta)$, where

$$\widehat{F}_{\lambda,z}(k) \equiv \widehat{G}_{\lambda,z}(k)^{-1} = 1 - z\alpha(\lambda, z)\,|\Omega|\,\widehat{D}(k) - \widehat{\Pi}_{\lambda,z}(k).$$

By definition, $p(z)\,|\Omega| = z\alpha(\lambda, z)\,|\Omega| + \widehat{\Pi}_{\lambda,z}(0)$. Hence the numerator of the right-hand side of (5.26) is

$$1 - p(z)\,|\Omega|\,\widehat{D}(k) - \widehat{F}_{\lambda,z}(k) = \widehat{\Pi}_{\lambda,z}(0)(1 - \widehat{D}(k)) - (\widehat{\Pi}_{\lambda,z}(0) - \widehat{\Pi}_{\lambda,z}(k)), \tag{5.27}$$

which is bounded above by $4\bar{c}_4\beta$. An alternative upper bound of the right hand side of (5.27) follows from equations (5.23) and (5.24):

$$\begin{aligned}\widehat{\Pi}_{\lambda,z}(0)(1 - \widehat{D}(k)) &- (\widehat{\Pi}_{\lambda,z}(0) - \widehat{\Pi}_{\lambda,z}(k)) \\ &\leq \bar{c}_4\beta(1 - \widehat{D}(k)) + \bar{c}_4\beta(1 - p(z)\,|\Omega|\,\widehat{D}(k)).\end{aligned}$$

Since

$$(1 - \widehat{D}(k))\widehat{C}_{p(z)}(k) = 1 + \underbrace{\widehat{D}(k)}_{\leq 1} \underbrace{\frac{p(z)\,|\Omega| - 1}{1 - p(z)\,|\Omega|\,\widehat{D}(k)}}_{\leq 1} \leq 2, \tag{5.28}$$

the numerator of (5.26) is bounded by

$$3\bar{c}_4\beta(1-p(z)\,|\Omega|\,\widehat{D}(k)) \le 3\bar{c}_4\beta(\widehat{F}_{\lambda,z}(0)+(1-\widehat{D}(k))).$$

The denominator of (5.26) is

$$\begin{aligned}\widehat{F}_{\lambda,z}(k) &= \widehat{F}_{\lambda,z}(0) + (\widehat{F}_{\lambda,z}(k) - \widehat{F}_{\lambda,z}(0)) \\ &= \widehat{F}_{\lambda,z}(0) + z\alpha(\lambda,z)\,|\Omega|\,(1-\widehat{D}(k)) + (\widehat{\Pi}_{\lambda,z}(0) - \widehat{\Pi}_{\lambda,z}(k)).\end{aligned}$$

Let $\bar{\lambda} = \sup_{\eta\in\Omega_{\mathrm{SAP}}}\lambda_\eta$, and $\lambda^\star = \max(1,\bar{\lambda})$. For $z \le (2\,|\Omega|\,\sqrt{\lambda^\star})^{-1}$ Proposition 1.8 (if $\lambda^\star > 1$) or neglecting loops (if $\lambda^\star \le 1$) implies

$$\widehat{F}_{\lambda,z}(0) \ge \widehat{C}_{z\sqrt{\lambda^\star}}(0)^{-1} \ge \frac{1}{2}.$$

Then $1-\widehat{D}(k) \ge 0$ and (5.24) imply

$$\widehat{F}_{\lambda,z}(k) \ge \widehat{F}_{\lambda,z}(0) - 2\bar{c}_4\beta \ge \frac{1}{2} - 2\bar{c}_4\beta.$$

For $(2\,|\Omega|\,\lambda^\star)^{-1} \le z < z_c(\lambda)$ equation (5.24), $\widehat{F}_z(0) > 0$, and $\alpha(\lambda,z) \ge 1$ imply

$$1-p(z)\,|\Omega|\,\widehat{D}(k) = 1-(1-\widehat{F}_{\lambda,z}(0))\widehat{D}(k) \le 1-\widehat{D}(k)+\widehat{F}_{\lambda,z}(0)$$

and hence

$$\begin{aligned}\widehat{F}_{\lambda,z}(k) &\ge \widehat{F}_{\lambda,z}(0) + \frac{1}{2\sqrt{\lambda^\star}}(1-\widehat{D}(k)) - \bar{c}_4\beta(1-p(z)\,|\Omega|\,\widehat{D}(k)) \\ &\ge \Big(\frac{1}{2\sqrt{\lambda^\star}} - \bar{c}_4\beta\Big)(\widehat{F}_{\lambda,z}(0)+(1-\widehat{D}(k))).\end{aligned}$$

For $z \le (2\,|\Omega|\,\lambda^\star)^{-1}$ or $(2\,|\Omega|\,\lambda^\star)^{-1} \le z < z_c$ these lower and upper bounds combine to imply the right-hand side of (5.26) is $1+O(\beta)$, and hence

$$f_2(z) = 1+O(\beta).$$

Lastly consider $f_3(z)$. As for f_2, it suffices to prove the claim for f_3/α_0. Let

$$\hat{g}_{\lambda,z}(k) = z\alpha(\lambda,z)\,|\Omega|\,\widehat{D}(k) + \widehat{\Pi}_{\lambda,z}(k),$$

so

$$\frac{\widehat{G}_{\lambda,z}(k)}{\alpha_0(\lambda,z)} = \frac{1}{1-\hat{g}_{\lambda,z}(k)}.$$

The symmetry of $D(x)$ and $\Pi_{\lambda,z}(x)$ implies that $g_{\lambda,z}(x) = g_{\lambda,z}(-x)$, so applying Lemma 5.7 of [14] (a general fact about even functions) gives

$$\begin{aligned}\frac{1}{2}|\Delta_k \widehat{G}_{\lambda,z}(\ell)| \le \frac{1}{2}(\widehat{G}_{\lambda,z}(\ell-k) + \widehat{G}_{\lambda,z}(\ell+k))\widehat{G}_{\lambda,z}(\ell)(|\hat{g}_{\lambda,z}(0)| - |\hat{g}_{\lambda,z}(k)|) \\ + 4\widehat{G}_{\lambda,z}(\ell-k)\widehat{G}_{\lambda,z}(\ell)\widehat{G}_{\lambda,z}(\ell+k)(\left|\hat{g}_{\lambda,z}(0)\right| - \left|\hat{g}_{\lambda,z}(k)\right|) \\ (\left|\hat{g}_{\lambda,z}(0)\right| - |\hat{g}_{\lambda,z}(\ell)|).\end{aligned}$$

Using $f_2(z) \le 1 + O(\beta)$ bounds each factor of $\widehat{G}_{\lambda,z}$ by $(1 + O(\beta))\, \widehat{C}_{p(z)}$. Further,

$$\begin{aligned}\left|\hat{g}_{\lambda,z}(0)\right| - \left|\hat{g}_{\lambda,z}(k)\right| &\le \sum_x (1 - \cos(k \cdot x))\, (z\alpha(\lambda, z)\, |\Omega| + |\Pi_z(x)|) \\ &\le z\alpha(\lambda, z)\, |\Omega|\, (1 - \widehat{D}(k)) + \bar{c}_4 \beta \widehat{C}_{p(z)}(k)^{-1} \\ &\le (2 + O(\beta))\, \widehat{C}_{p(z)}(k)^{-1},\end{aligned}$$

where the second inequality is by (5.24) and the third is by $f_1(z) \le 1 + O(\beta)$ and (5.28). Combining the bounds and using the definition of $U_{p(z)}$ gives $f_3(z) \le 1 + O(\beta)$. ☐

Corollary 5.32. *For d sufficiently large, λ-LWW satisfies a **k-space infrared bound**: there is a constant $K = 1 + O(\beta)$ such that for $0 \le z \le z_c(\lambda)$*

$$\widehat{G}_{\lambda,z}(k) \le K \widehat{C}_{p(z)}(k).$$

Proof. The proof of Lemma 5.31 showed that $f_2(z) \le 1 + O(\beta)$ without absolute values on $\widehat{G}_{\lambda,z}$, uniformly for $z < z_c$. Taking a limit gives the claim. ☐

The fact that the quantities $T_{\lambda,z}$ and $S_{\lambda,z}$ defined below are small will be important in what follows.

Definition 5.33. The *triangle diagram* $T_{\lambda,z}$ and *square diagram* $S_{\lambda,z}$ are the quantities

$$T_{\lambda,z} = \|\widehat{H}^3_{\lambda,z}\|_1, \quad S_{\lambda,z} = \|\widehat{H}^4_{\lambda,z}\|_1.$$

Corollary 5.34. *For d sufficiently large and $z \le z_c$ the triangle and square diagrams are bounded above by $c\beta$.*

Proof. For notational convenience write $\bar{H}_{\lambda,z} = \alpha_0^{-1} H_{\lambda,z}$, and similarly for $\bar{G}_{\lambda,z}$. By equation (4.1), $\alpha_0^{-1}\widehat{H}_{\lambda,z} = \alpha_0^{-1}\widehat{G}_{\lambda,z} - 1$. Corollary 5.32 implies

$$\alpha_0^{-1}\widehat{G}_{\lambda,z} \le (1 + O(\beta))\widehat{C}_{p(z)}$$

since $\alpha_0 \le 1 + O(\beta)$. The claim follows from Proposition 4.4. ☐

6. Proofs of the main results

To go beyond the k-space infrared bound of Corollary 5.32 requires control of the derivatives of $G_{\lambda,z}$ and $\Pi_{\lambda,z}$ with respect to z. This control is established in Section 6.1. The remainder of the section establishes Theorem 1.9 using arguments based on [11, Chapter 6]. Throughout let $z_c = z_c(\lambda)$.

6.1. Further diagrammatic bounds. Having verified that the bounds of Section 5.2 holds for $z < z_c$, the monotone convergence theorem implies they continue to hold at z_c.

Proposition 6.1. *For d sufficiently large and* $0 < z \leq z_c$

$$\frac{d}{dz}\|\mathrm{B}^{\star}{}_{\lambda,z}\|_1 \leq z_c^{-1} c\beta.$$

Proof. The left-hand side is a polynomial with positive coefficients, so it suffices to obtain an upper bound at $z = z_c$. By Proposition 5.18, $\alpha_0(\lambda, z_c) \leq 1 + c\beta$, $\|\mathrm{B}^{\star}{}_{\lambda,z_c}\|_1 \leq c\beta$, and Corollary 5.34, the claim follows. □

Proposition 6.2. *Let d be sufficiently large,* $0 < z \leq z_c$, *and* $v = 1, 2$. *Then*

$$\|\partial_z^v \bar{G}_{\lambda,z}\|_\infty = \|\partial_z^v \bar{H}_{\lambda,z}\|_\infty \leq c\beta z_c^{-v}$$

Proof. As for Proposition 6.1 it suffices to consider $z = z_c$. The equality of the first two terms follows from equation (4.1). Proposition 5.16 implies

$$\frac{d}{dz}\bar{H}_{\lambda,z} \leq z^{-1}(1 + \|\mathrm{B}^{\star}{}_{\lambda,z}\|_1)\bar{H}_{\lambda,z} * \bar{G}_{\lambda,z}.$$

The claim follows for $v = 1$ as $\|\bar{H}_{\lambda,z} * \bar{G}_{\lambda,z}\|_\infty \leq c\beta$ by equation (5.21) and $\|\mathrm{B}^{\star}{}_{\lambda,z}\|_1 \leq c\beta$ by Proposition 5.20.

For $v = 2$ apply Lemma 5.15. After computing the derivative and using the triangle inequality (i) argue as for $v = 1$ for the term from differentiating z^{-1}, (ii) use Proposition 6.1 when differentiating $\|\mathrm{B}^{\star}{}_{\lambda,z}\|_1$, and (iii) when differentiating either of the two-point functions use Proposition 5.16 and

$$\|\bar{H}_{\lambda,z} * \bar{H}_{\lambda,z} * \bar{G}_{\lambda,z}\|_\infty \leq \|\bar{H}_{\lambda,z} * \bar{H}_{\lambda,z}\|_\infty + \|\bar{H}_{\lambda,z} * \bar{H}_{\lambda,z} * \bar{H}_{\lambda,z}\|_\infty,$$

and Corollary 5.34 to see that this is bounded by $c\beta$. Each term is therefore bounded by $c\beta z_c^{-2}$. □

Proposition 6.3. *Let d be sufficiently large,* $0 < z \leq z_c$, *and* $v = 1, 2$. *Then*

$$\|\partial_z^v I_{\lambda,z}\|_1 \leq c\beta z_c^{-v}$$

Proof. For $v = 1$ note

$$\frac{d}{dz} I_{\lambda,z} = \frac{d}{dz}(1 - e^{-\mu_{\lambda,z}(0,x)}) \leq \frac{d}{dz}\mu_{\lambda,z}(0,x).$$

This bound is increasing in z, so considering z_c is enough. Translation invariance, as in the proof of Proposition 5.21, implies this is equal to the derivative in z of $\|\mathrm{B}^{\star}{}_{\lambda,z}\|_1$. The claim follows for $v = 1$ from Proposition 6.1.

For $v = 2$ it is enough to bound the derivative of the bound of Proposition 5.18. This is similar to the arguments already given; the only new terms that arise occur when differentiating $\|\bar{H}_{\lambda,z} \cdot \bar{G}_{\lambda,z} * \bar{H}_{\lambda,z}\|_1$, which is $\bar{H}_{\lambda,z} * \bar{G}_{\lambda,z} * \bar{H}_{\lambda,z}(0)$. By Proposition 5.16 after taking a derivative the result is, up to a factor of $(1 + O(\beta))$, a square diagram $\bar{H}_{\lambda,z} * \bar{G}_{\lambda,z} * \bar{H}_{\lambda,z} * \bar{G}_{\lambda,z}(0)$. Repeatedly using equation (4.1) and Corollary 5.34 shows this is at most $c\beta$. □

Proposition 6.4. *For d sufficiently large,* $0 < z < z_c$*, and* $v = 1, 2$

$$\|\partial_z^v \Pi_{\lambda,z}\|_1 \leq c\beta z_c^{-v} \tag{6.1}$$

Proof. The Leibniz rule and Lemma 5.15 imply that the result of differentiating Π is a sum of terms of the form of the bounds of Proposition 5.11, but where each term has one of the factors of $\bar{G}_{\lambda,z}$, $\bar{H}_{\lambda,z}$ or $I_{\lambda,z}$ differentiated. Given this, the argument is as in the proofs of Proposition 5.23 and Proposition 5.27. Let us describe the proof for $N = 2$. For $N = 1$ the proof is similar as $e^{\mu_{\lambda,z}(0,x)} \leq \alpha_0$.

Consider $v = 1$. There are $3N - 1$ terms arising when differentiating $\pi^{(N)}_{\lambda,z}$. If $\bar{G}_{\lambda,z}$ or $\bar{H}_{\lambda,z}$ is differentiated apply Proposition 5.16 and place the sup norm on this term when applying Lemma 5.26, and then use Proposition 6.2 to bound this norm. If $I_{\lambda,z}$ is differentiated use Lemma 5.26 placing the sup norm on a term $H_{\lambda,z}$ and use Proposition 6.3 to bound the one norm of the derivative of $I_{\lambda,z}$. This yields the claim as the factor of $3N - 1$ is irrelevant for the convergence of the series.

If $v = 2$ there are $(3N - 1)^2$ terms. If both derivatives fall on a single factor proceed as in the previous paragraph and use Proposition 6.2 or Proposition 6.3. If the derivatives fall on distinct factors, one factor being $I_{\lambda,z}$, proceed as before. For the remaining case, where two distinct factors of $\bar{H}_{\lambda,z}$ (or $\bar{G}_{\lambda,z}$) are differentiated, place a sup norm on one term. The new term to bound when applying Lemma 5.26 is of the form $\|\bar{H}_{\lambda,z} * \bar{G}_{\lambda,z} * \bar{G}_{\lambda,z} * I_{\lambda,z}\|_\infty$. It suffices to bound $\|\bar{H} * \bar{G} * \bar{G}\|_\infty$, and this is bounded above by

$$\|\bar{H} * \bar{G}\|_\infty + \|\bar{H} * \bar{H}\|_\infty + \|\bar{H} * \bar{H} * \bar{H}\|_\infty,$$

all of which are bounded by $c\beta$ by Corollary 5.34. □

Corollary 6.5. *Let d be sufficiently large and $0 < z \leq z_c$. Then*

$$-\frac{d}{dz}\widehat{F}_z(0) \geq c > 0.$$

Proof. The derivative is

$$-\frac{d}{dz}\widehat{F}_{\lambda,z}(0) = |\Omega|\,\alpha(\lambda, z) + z\,|\Omega|\,\frac{d}{dz}\alpha(\lambda, z) + \frac{d}{dz}\widehat{\Pi}_{\lambda,z}(0). \tag{6.2}$$

By Proposition 6.4, $\left|\frac{d}{dz}\widehat{\Pi}_{\lambda,z}(k)\right|$ is bounded above by a constant since z_c is bounded below by a term of order β by Proposition 1.8. An argument as for Proposition 5.17 shows the magnitude of the second term is bounded by a constant. As $\alpha(\lambda, z) \geq 1$ the first term dominates for d sufficiently large. □

6.1.1. Derivatives of moments.

The next proposition (for $\lambda = 0$) is [14, Exercise 5.17].

Lemma 6.6. *For d sufficiently large and $0 \leq z < z_c$*

$$\|\,|x|^2\,\Pi_{\lambda,z}(x)\|_1 \leq c\beta$$

Proof. This follows from $\widehat{C}^{-1}_{p(z)} \leq 1 - \widehat{D}(k)$ and (5.24). □

Proposition 6.7. *For $0 \leq z \leq z_c$ the following bounds hold:*

$$\|\,|x|^2\,H_{\lambda,z}(x)\|_\infty \leq c\beta \tag{6.3}$$

and

$$\|\,|x|^2\,H_{\lambda,z}(x)\|_2 \leq c. \tag{6.4}$$

Proof. The proof relies on the identity

$$|x_\mu|^2\,H_{\lambda,z}(x) = -\int_{[-\pi,\pi]^d} \partial^2_{k_\mu}\widehat{H}_{\lambda,z}(k)e^{-ik\cdot x}\,\frac{d^dk}{(2\pi)^d}, \tag{6.5}$$

where μ is a unit basis vector of $\mathbb{Z}^d$. Omitting the subscripts λ and z and letting a subscript μ denote partial differentiation with respect to k_μ the derivative can be calculated:

$$\widehat{G}_{\mu,\mu}(k) = z\alpha\,|\Omega|\,\frac{\widehat{D}_{\mu,\mu}(k)}{\widehat{F}^2(k)} + 2(z\alpha\,|\Omega|)^2\frac{\widehat{D}^2_\mu(k)}{\widehat{F}^3(k)} + \frac{\widehat{\Pi}_{\mu,\mu}(k)}{\widehat{F}^2(k)} + 4z\alpha\,|\Omega|\,\frac{\widehat{D}_\mu(k)\widehat{\Pi}_\mu(k)}{\widehat{F}^3(k)} + 2\frac{\widehat{\Pi}^2_\mu(k)}{\widehat{F}^3(k)}.$$

To obtain an estimate of $\| |x|^2 H_{\lambda,z}\|_\infty$ take the absolute value of (6.5) inside of the integral and estimate the resulting one norms. Using $z\alpha|\Omega| \le 1 + O(\beta)$ an upper bound for the first two terms is

$$(1+O(\beta))\Big(\Big\|\frac{\widehat{D}_{\mu,\mu}(k)}{(1-\widehat{D}(k))^2}\Big\|_1 + 2\Big\|\frac{\widehat{D}^2_\mu(k)}{(1-\widehat{D}(k))^3}\Big\|_1\Big) \le c\beta,$$

where the second inequality follows by estimating the integrals, see [11, Appendix A].

For the remaining terms, $\| |x|^2 \Pi_{\lambda,z}\|_1 \le c\beta$ implies $\|\widehat{\Pi}_{\mu,\mu}\|_\infty \le c\beta$. Since $\widehat{\Pi}_\mu(k) = 0$ when $k_\mu = 0$ Taylor's theorem and the above bound on $\|\widehat{\Pi}_{\mu,\mu}\|_\infty$ imply $\|\widehat{\Pi}_\mu\|_\infty \le c\beta\,|k_\mu|$. Lastly, $|\widehat{D}_\mu(k)|_\infty \le c\,|k_\mu|$. These bounds combined with the k-space infrared bound Corollary 5.32 imply each of the remaining terms are bounded by $c\beta$. This proves equation (6.3).

For $\| |x|^2 H_{\lambda,z}\|_2$ use Parseval's identity: $\|\widehat{|x|^2 H_{\lambda,z}}\|_2 = \|\partial_k^2 \widehat{H}_{\lambda,z}\|_2$. The previously described bounds for the numerators along with Corollary 5.32 and Proposition 4.4 imply that $\widehat{G}_{\mu,\mu}(k)$ is square integrable in sufficiently high dimensions. This implies equation (6.4). □

Proposition 6.8. *For d sufficiently large and* $0 < z \le z_c$

$$\|\partial_z^v |x|^2 \Pi_{\lambda,z}\|_1 \le c\beta z_c^{-v}$$

Proof. Distribute the factor $|x|^2$ along the factors of $\bar{H}_{\lambda,z}$ and $\bar{G}_{\lambda,z}$ as in the proof of Proposition 5.27. The proof is now essentially the same as for Proposition 6.4. For each term place the sup norm on the factor with the term $|x|^2$.

If a factor $|x|^2 G_{\lambda,z}$ has been differentiated once or twice, then the resulting term whose norm must be estimated has the form of either $\bar{H}_{\lambda,z} * \bar{G}_{\lambda,z}$ or $\bar{H}_{\lambda,z} * \bar{G}_{\lambda,z} * \bar{G}_{\lambda,z}$. In either case the factor $|x|^2$ can again be split along the factors in the convolution. In the first case use equation (4.1), the triangle inequality, and Young's inequality to obtain

$$\|(|x|^2 \bar{H}_{\lambda,z}) * \bar{G}_{\lambda,z}\|_\infty \le \| |x|^2 \bar{H}_{\lambda,z}\|_\infty + \| |x|^2 \bar{H}_{\lambda,z}\|_2 \|\bar{H}_{\lambda,z}\|_2,$$

and then use Proposition 6.7 to see that this is bounded by $c\beta$. For the second case arguing similarly gives

$$\begin{aligned}\|(|x|^2 \bar{H}_{\lambda,z}) * \bar{G}_{\lambda,z} * \bar{G}_{\lambda,z}\|_\infty \le{}& \|(|x|^2 \bar{H}_{\lambda,z}) * \bar{G}_{\lambda,z}\|_\infty \\ &+ \|(|x|^2 \bar{H}_{\lambda,z}) * \bar{H}_{\lambda,z}\|_\infty \\ &+ \| |x|^2 \bar{H}_{\lambda,z}\|_2 \|\bar{H}_{\lambda,z} * \bar{H}_{\lambda,z}\|_2.\end{aligned}$$

The first case analysis implies the first two terms are bounded above by $c\beta$. Parseval's identity combined with Corollary 5.34 implies the last term is bounded by $c\beta$. The rest of the analysis of these terms is in the proof of Proposition 6.4.

The cases in which all derivatives fall on factors without the term $|x|^2$ can be handled in the same manner as in the proof of Proposition 6.4 by using Young's inequality, the triangle inequality, and Corollary 5.34. □

6.2. Linear divergence of $\chi_\lambda(z)$ as $z \nearrow z_c$. Before proving the linear divergence of the susceptibility it will be helpful to verify that it is only infinite at the critical point $z = z_c$ itself.

Lemma 6.9. *For d sufficiently large and $|z| \leq z_c$ the inverse susceptibility $\widehat{F}_{\lambda,z}(0)$ satisfies*

$$|\widehat{F}_{\lambda,z}(0)| \geq \frac{|\Omega|}{2}\,|z_c - z|\,.$$

Proof. As $\widehat{F}_{\lambda,z_c}(0) = 0$ the fundamental theorem of calculus implies

$$|F_{\lambda,z}(0)| = \left|\int_{z_c}^{z} -\frac{d}{dz}\widehat{F}_z(0)\,dz\right|.$$

Using $\widehat{F}_{\lambda,z_c}(0) = 0$, equation (6.2), and integrating from z_c to z along the straight line $z_t = (1-t)z_c + tz$ implies

$$\left|\widehat{F}_{\lambda,z}(0)\right| = |\Omega|\,|z - z_c|\left|\int_0^1 \alpha(\lambda, z_t) + z_t\frac{d}{dz}\alpha(\lambda, z_t) + |\Omega|^{-1}\,\frac{d}{dz}\widehat{\Pi}_{\lambda,z_t}(0)\,dt\right|. \tag{6.6}$$

The last two terms are bounded by $c\beta$, see the proof of Corollary 6.5. The claim follows by taking the dimension sufficiently large as $\int \alpha = 1 + O(\beta)$. □

Define constants $A = A(\lambda)$ and $D = D(\lambda)$ by

$$A(\lambda) = z_c^{-1}\Big(\alpha(\lambda, z_c)\,|\Omega| + z_c\,|\Omega|\,\frac{d}{dz}\alpha(\lambda, z_c) + \frac{d}{dz}\widehat{\Pi}_{\lambda,z_c}(0)\Big)^{-1} \tag{6.7}$$

and

$$D(\lambda) = A(\lambda)(-z_c\,|\Omega|\,\alpha(\lambda, z_c)\nabla_k^2\widehat{D}(0) - \nabla_k^2\widehat{\Pi}_{\lambda,z_c}(0)). \tag{6.8}$$

Theorem 6.10. *For d large enough, the susceptibility of λ-LWW diverges linearly as $z \nearrow z_c$:*

$$\chi_\lambda(z) \sim \frac{Az_c}{z_c - z}. \tag{6.9}$$

The constant A in (6.9) is as in equation (6.7).

Proof. Recall $\widehat{F}_{\lambda,z}(0) = \widehat{G}_{\lambda,z}(0)^{-1}$ is zero at z_c since $\chi_\lambda(z) \nearrow \infty$ as $z \nearrow z_c$. We get

$$\begin{aligned}\chi_\lambda(z) &= \frac{1}{\widehat{F}_{\lambda,z}(0) - \widehat{F}_{\lambda,z_c}(0)} \\ &= \frac{1}{z_c - z}\Big(\alpha(\lambda, z_c)\,|\Omega| + z\,|\Omega|\,\frac{\alpha(\lambda, z_c) - \alpha(\lambda, z)}{z_c - z} \\ &\qquad + \frac{\widehat{\Pi}_{\lambda,z_c}(0) - \widehat{\Pi}_{\lambda,z}(0)}{z_c - z}\Big)^{-1}.\end{aligned}$$

The claim follows from Proposition 6.8 and Proposition 5.17 combined with $\alpha_0 \le 1 + c\beta$ for $z \le z_c$, which implies differentiability of α_0 at z_c. □

6.3. Growth rate and diffusive scaling. To establish the growth rate of λ-LWW, as well as the diffusive scaling, a Tauberian type theorem is needed. The statement and proof of the next lemma in [11] involve fractional derivatives of order $1 + \epsilon$ for $0 < \epsilon < 1$, but the arguments apply without modification for two ordinary derivatives.

Lemma 6.11 (Lemma 6.3.4 of [11]). *Let*

$$f(z) = \frac{1}{\phi(z)} = \sum_{n=0}^{\infty} b_n z^n,$$

where $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$. *Suppose that*

$$\sum_{n=0}^{\infty} n^2\,|a_n|\,R^n < \infty,$$

so in particular, $\phi(z)$, $\phi'(z)$, *and* $\phi''(z)$ *are finite when* $|z| = R$. *Assume in addition that* $\phi'(R) \ne 0$. *Suppose that* $\phi(R) = 0$ *and* $\phi(z) \ne 0$ *for* $|z| \le R$, $z \ne R$. *Then*

$$f(z) = \frac{1}{-\phi'(R)}\frac{1}{R - z} + O(1)$$

uniformly in $|z| \le R$, *and*

$$b_n = R^{-n-1}\Big(\frac{1}{-\phi'(R)} + O(n^{-\alpha})\Big) \quad \text{as } n \to \infty,$$

for every $\alpha < 1$.

Recall that c_n^λ is the total mass of n-step λ-LWW, i.e.,

$$c_n^\lambda = \sum_x \sum_{\substack{\omega : 0 \to x \\ |\omega| = n}} \lambda^{n_L(\omega)}.$$

Theorem 6.12. *For d sufficiently large and any $\delta < 1$*

$$c_n^\lambda = A(\lambda) z_c(\lambda)^{-n}(1 + O(n^{-\delta})).$$

Proof. Apply Lemma 6.11 to $\widehat{F}_{\lambda,z}(0)$. The verification of the hypotheses of the theorem are the conclusions of Proposition 6.4, Corollary 6.5, and Lemma 6.9. □

The proof of the next theorem is essentially the proof for self-avoiding walk in [11] verbatim; it is reproduced here for the sake of completeness. The next lemma, which will be used several times, is stated here for the convenience of the reader.

Lemma 6.13 (Lemma 6.3.2 of [11]). *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Let $R > 0$, and suppose $f'(R) = \sum_{n=0}^{\infty} n \, |a_n| \, R^{n-1} < \infty$, so in particular $f(z)$ converges for $|z| \leq R$. Then for $|z| \leq R$*

$$|f(z) - f(R)| \leq f'(R) \, |R - z| \, .$$

If $f''(z)(R) < \infty$, then for $|z| \leq R$

$$\left| f(z) - f(R) - f'(R)(z - R) \right| \leq \frac{1}{2} f''(R) \, |R - z|^2 \, .$$

Theorem 6.14. *For d sufficiently large λ-LWW is diffusive:*

$$\langle |\omega(n)|^2 \rangle_n^\lambda = Dn(1 + O(n^{-\delta}))$$

as $n \to \infty$ for any $\delta < 1$. The constant D is that of (6.8).

Proof. Let ∇_k^2 denote the k-space Laplacian. Then

$$\langle |\omega(n)|^2 \rangle_{\lambda,n} = -\frac{\nabla_k^2 \hat{c}_n^\lambda(0)}{c_n^\lambda}.$$

Since $\hat{c}_n^\lambda(k)$ is the coefficient of z^n in $\widehat{G}_{\lambda,z}(k)$ Cauchy's formula implies

$$-\nabla_k^2 \hat{c}_n^\lambda(0) = \frac{1}{2\pi i} \oint \frac{\nabla_k^2 \widehat{F}_{\lambda,z}(0)}{\widehat{F}_{\lambda,z}(0)^2} \frac{dz}{z^{n+1}}, \tag{6.10}$$

where the integral is around a small origin centred circle. Define $E(z)$ by

$$\frac{\nabla_k^2 \widehat{F}_{\lambda,z}(0)}{\widehat{F}_{\lambda,z}(0)^2} = \frac{\nabla_k^2 \widehat{F}_{z_c}(0)}{\left(\frac{d}{dz}\widehat{F}_{z_c}(0)\right)^2 (z_c - z)^2} + E(z).$$

Making this substitution into equation (6.10) and calculating the first integral implies

$$-\nabla_k^2 \hat{c}_n^\lambda(0) = \frac{\nabla_k^2 \widehat{F}_{z_c}(0)}{\left(\frac{d}{dz}\widehat{F}_{z_c}(0)\right)^2}(n+1)z_c^{-n-2} + \frac{1}{2\pi i}\oint E(z)\frac{dz}{z^{n+1}}.$$

Assuming the integral of $E(z)$ is $O(n^\delta z_c^{-n})$ for every $\delta > 0$ implies the theorem by inserting the behaviour of c_n^λ given by Theorem 6.12.

To verify the assumption it suffices by Lemma 6.13 to prove

$$|E(z)| \le \text{const.}\, |z_c - z|^{-1} \quad \text{for all } |z| \le z_c.$$

Split $E(z)$ as $E(z) = T_1(z) + T_2(z)$ with

$$T_1(z) = \left(\frac{d}{dz}\widehat{F}_{\lambda,z_c}(0)\right)^{-2} \frac{\nabla_k^2 \widehat{F}_{\lambda,z}(0) - \nabla_k^2 \widehat{F}_{\lambda,z_c}(0)}{(z_c - z)^2}$$

and

$$T_2(z) = \frac{-\nabla_k^2 \widehat{F}_{\lambda,z}(0)\left(\widehat{F}_{\lambda,z}(0)^2 - \left(\frac{d}{dz}\widehat{F}_{\lambda,z_c}(0)\right)^2 (z_c - z)^2\right)}{\left(\frac{d}{dz}\widehat{F}_{\lambda,z_c}(0)\right)^2 \widehat{F}_{\lambda,z}(0)^2 (z_c - z)^2}.$$

The numerator of $T_1(z)$ is differentiable in z by Proposition 6.8, so (i) of Lemma 6.13 implies the numerator is bounded above by a constant times $|z_c - z|$. It follows that $|T_1| \le O(|z_c - z|^{-1})$.

For T_2 note that $\widehat{F}_{\lambda,z}(0)^2 \ge \text{const.}\, |z_c - z|^2$ by Lemma 6.9 so

$$|T_2(z)| \le \text{const.}\, |z_c - z|^{-4}\left(\widehat{F}_{\lambda,z}(0) + \frac{d}{dz}\widehat{F}_{\lambda,z_c}(0)(z_c - z)\right)$$
$$\left(\widehat{F}_{\lambda,z}(0) - \frac{d}{dz}\widehat{F}_{\lambda,z_c}(0)(z_c - z)\right),$$

as $\nabla_k^2 \widehat{F}_{\lambda,z}(0)$ is bounded by a constant by Proposition 6.7. By (ii) of Lemma 6.13, Proposition 6.8, and $\widehat{F}_{\lambda,z_c}(0) = 0$, the middle term is $O(|z_c - z|^2)$. Using (i) of Lemma 6.13 for $\widehat{F}_{\lambda,z}(0)$ in the last term shows the last term is $O(|z_c - z|)$. Thus $|T_2(z)| \le O(|z_c - z|^{-1})$, which proves the claim. □

Appendix A. Loop measure representation of λ-LWW

The purpose of this appendix is to provide a proof of Theorem 1.13. A fundamental property of λ-LWW is that it admits a loop measure representation. The representation follows from a theorem of Viennot [18] and is proved via the theory of heaps of pieces in Section A.3.

Remark A.1. The methods of [10, Chapter 9] are sufficient to derive formulas that would suffice for the lace expansion analysis of λ-LWW. These methods have the benefit of brevity, but they do not reveal the connection with the loop $O(N)$ model. For this reason we have chosen to present a more scenic route here.

The rest of this section will take place in the context of an arbitrary graph G, as specializing to $\mathbb{Z}^d$ does not provide any simplification. The theory of heaps of pieces will be freely used; see [18] or [9] for an introduction.

A.1. Viennot's theorem

Definition A.2. A *trivial cycle* is a single edge of G. An *oriented cycle* is either (i) an oriented cyclic subgraph of G or (ii) a trivial cycle in G.

An oriented cycle corresponds to an equivalence class of self-avoiding polygons, where a self-avoiding polygon $\omega = (\omega_0, \dots, \omega_k = \omega_0)$ is equivalent to any cyclic permutation $\tilde{\omega} = (\omega_r, \omega_{r+1}, \dots, \omega_k, \omega_1, \dots, \omega_r)$. For example, a trivial cycle $\{x, y\}$ corresponds to the self-avoiding polygons (x, y, x) and (y, x, y), while an oriented 3-cycle corresponds to walks of the form (x, y, z, x) and cyclic permutations thereof for x, y, z distinct.

Definition A.3. A *heap of (oriented) cycles* is a heap of pieces whose labels are oriented cycles. Two oriented cycles C_1, C_2 are concurrent if $V(C_1) \cap V(C_2) \neq \emptyset$, i.e., if the cycles share a vertex.

Definition A.4. A pair (η, H) where η is a self-avoiding walk from a to b and H is a heap of cycles whose maximal elements' labels each contain a vertex in η is called a *legal* (a, b) *pair*. Let $\mathcal{V}(a, b)$ denote the set of legal (a, b) pairs, and $\mathcal{V}$ denote the set of all legal pairs.

Theorem 1.13, the loop measure representation of λ-LWW, is a byproduct of the proof of the following theorem of Viennot.

Theorem A.5 ([18, Proposition 6.3]). *There is a bijection ϕ_{ab} from the set $\mathcal{V}(a,b)$ of legal (a,b) pairs to the set of walks $\Omega(a,b)$ from a to b. Further,*

(1) *the multi-set of edges in a legal (a,b) pair (η, H) is the same as the multi-set of edges in the walk $\phi_{ab}((\eta, H))$;*

(2) *the multi-set of oriented cycles $\{\ell(x) \mid x \in H\}$ for a heap $(H, \ell, \preceq)$ is the same as the multi-set of oriented cycles that are erased by applying loop erasure to $\phi_{ab}((\eta, H))$.*

Theorem A.5 is not proven in [18]. For the sake of completeness and the convenience of the reader a proof is given in Section A.2. The remainder of this section consists of a heuristic description of the proof; see also Figure 7 which depicts the proof strategy.

Let ω be a walk from a to b. Trace ω until the first time a vertex is visited twice. This identifies a first closed subwalk $C_1 = (\omega_{\tau_\omega^\star}, \dots, \omega_{\tau_\omega})$. Remove C_1 by performing a single loop erasure, and form a heap of pieces consisting of a single piece labelled C_1. The first time a vertex is visited twice by the walk $\mathrm{LE}^1(\omega)$ identifies a second closed subwalk, call this C_2. Remove C_2 and form a new heap of pieces by adding a second piece labelled C_2 to the heap consisting of C_1. Continuing in this manner removes all of the closed subwalks from ω, resulting in a self-avoiding walk η from a to b. Each maximal piece in the heap is labelled by a cycle that shares a vertex with η. In other words, this procedure converts each walk from a to b into a legal pair (η, H).

Conversely, consider a legal pair (η, H). To invert the procedure what is required is a way to reduce the heap to the empty heap one piece at a time, while inserting the labels of the removed pieces into the (initially) self-avoiding walk η. This is relatively straightforward: the maximal pieces of the heap H have labels that share a vertex with η, and hence the maximal pieces can be ordered by using the linear order on vertices in η. Take the maximal piece in this order, remove it from the heap to get a heap H', and glue the corresponding label into η to get a walk η'. The maximal elements of H' have labels that share a vertex with η', and hence this procedure can be iterated.

These operations are in fact inverses of one another. The next section makes the preceding discussion precise.

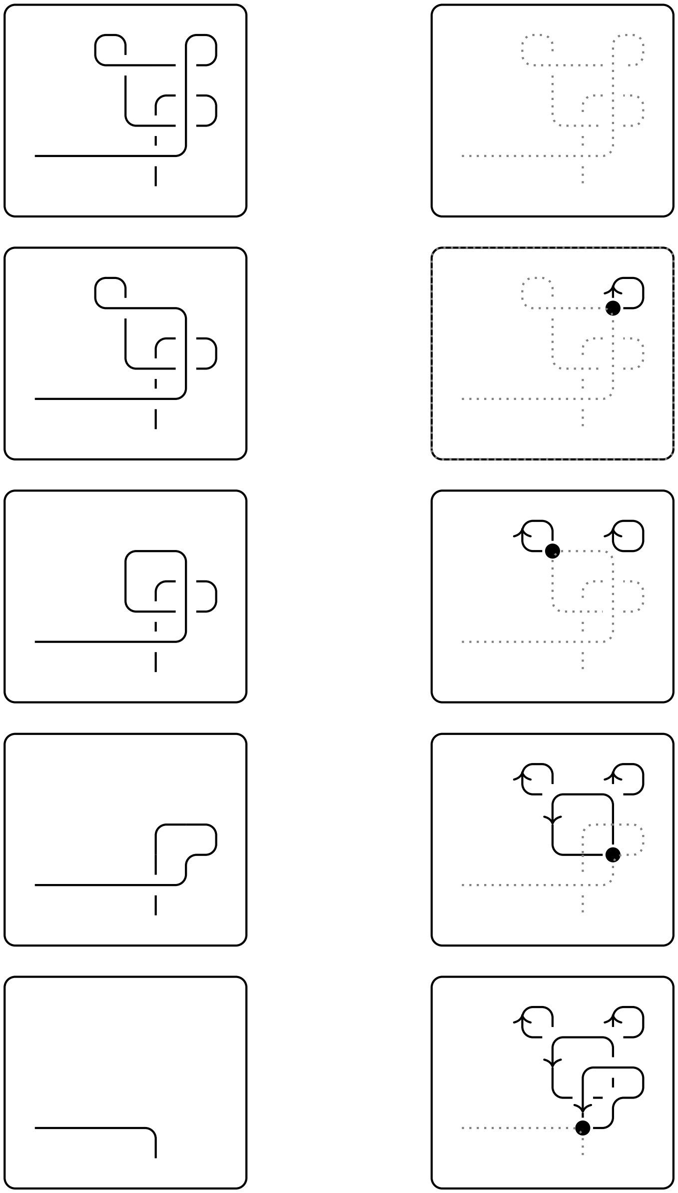

Figure 7. The figure illustrates the bijection between valid pairs (η, H) and walks ω whose loop erasure is η. The left-hand side shows the results of successive applications of LE^1, culminating in a self-avoiding walk. The right hand side shows the heaps of oriented cycles generated, with the walk displayed in dotted gray. Each heap has been given a distinguished vertex. The vertex indicates the oriented cycle that is maximal in the walk order as well as the location at which this oriented cycle is glued in to the corresponding walk when performing loop addition.

A.2. Proof of Viennot's theorem. The theorem requires two algorithms, one which inserts oriented cycles into a given walk, and one which removes oriented cycles from a walk. Removing oriented cycles is achieved by loop erasure. The other algorithm is introduced now.

Definition A.6. Let ω be a walk of length n, and let C be an oriented cycle of length k. Assume that C and ω have a vertex in common, and let i be the minimal index such that ω_i is a vertex in C. Let $(c_0, \dots, c_k)$ be the unique representative of C such that $c_0 = \omega_i$. The *loop insertion* $\omega \oplus C$ *of* C *into* ω is the walk

$$(\omega_0, \dots, \omega_{i-1}, c_0, \dots, c_k, \omega_{i+1}, \dots, \omega_n).$$

In words, to insert a loop C into a walk ω we find the first vertex ω_i in ω that is contained in C. C is then rooted at ω_i, ω is traversed until just before reaching ω_i, C is traversed, and then the remainder of ω is traversed.

Lemma A.7. *Let ω be a walk, and let C be the oriented cycle removed to create* $\mathrm{LE}^1(\omega)$. *Then* $\mathrm{LE}(\omega) \oplus C = \omega$.

Proof. The definition of $\tau^\star_\omega$ and the definition of loop erasure implies that the first vertex in common between $\mathrm{LE}^1(\omega)$ and C is $\omega_{\tau^\star_\omega}$, and hence the closed self-avoiding walk representing C that is inserted by loop insertion is $(\omega_{\tau^\star_\omega}, \dots, \omega_{\tau_\omega})$. □

Given a collection of oriented cycles that intersect a walk it is necessary to determine the order in which the cycles should be inserted. The next definition gives the correct order for inverting loop erasure.

Definition A.8. Let ω be a walk, and $C_1, \dots, C_k$ a collection of oriented cycles that each share a vertex with ω. Let $t_j = \min\{i \mid \omega_i \in C_j\}$. The *walk order* on the oriented cycles is given by setting $C_m \geq C_n$ if $t_m \geq t_n$.

The following algorithm, called the *loop addition algorithm*, constructs a walk beginning at the vertex a and ending at the vertex b from a legal (a, b) pair (η, H).

(1) Set $\omega^0 = \eta$.
(2) Suppose $H^{i-1} \neq \emptyset$. Set $\omega^i = \omega^{i-1} \oplus C$, where C is maximal in the walk order among the labels of the maximal pieces of H^{i-1}. Let y be the piece whose label is C, and set $H^i = H^{i-1} \setminus \{y\}$.
(3) If $H^{i-1} = \emptyset$, output $\omega = \omega^{i-1}$. Otherwise go to 2.

The algorithm is well-defined as the labels of the maximal pieces in a heap must be vertex disjoint, so the walk order is a strict total order on the maximal pieces of the heap. Note that at each step of the algorithm the walk ω^i begins at the vertex a and ends at b, so ω is a walk from a to b as claimed.

Lemma A.9. *Suppose $(\eta, H) \in \mathcal{V}$. Suppose the output of the loop addition algorithm is ω. If C is the last oriented cycle inserted, then the oriented cycle removed by loop erasure applied to ω is C.*

Proof. The proof is by induction on the size of H. Suppose C was the $(k+1)^{\mathrm{st}}$ oriented cycle added.

(1) If C was the label of a maximal element in H^{k-1} then C is vertex disjoint from the k^{th} added oriented cycle C'. The definition of the walk order implies that the first vertex C shares with ω^{k-1} occurs prior to the first vertex in C' because C is disjoint from C'. It follows that C is the oriented cycle erased by loop erasure, as C closes prior to C', which was previously (by induction) the first oriented cycle to close.

(2) If C was not the label of a maximal piece in H^{k-1} then C is the label of a piece that was below the k^{th} inserted piece. Suppose the k^{th} piece had label C'. As C intersects C', C is inserted into the subwalk C' of ω^{k-1}. By induction C' was the first oriented cycle to close in ω^{k-1}, so C is the first oriented cycle to close in ω^k. □

To construct a legal pair (η, H) from a walk is fairly straightforward. By applying loop erasure oriented cycles are removed, and they naturally form a heap by using the heap composition operation. More precisely, we have the (*total*) *loop erasure algorithm*.

(1) Set $\omega^0 = \eta$ and $H^0 = \emptyset$, where $\emptyset$ is the empty heap of oriented cycles.

(2) If ω^{i-1} is not a self-avoiding walk, set $\omega^i = \mathrm{LE}^1(\omega^{i-1})$, and if C is the closed self-avoiding walk removed from ω^{i-1}, let $H^i = H \circ \{\overline{C}\}$ where $\overline{C}$ is the oriented cycle corresponding to C.

(3) If ω^{i-1} is a self-avoiding walk, output (ω^{i-1}, H^{i-1}). Otherwise go to 2.

Single loop erasure removes a subwalk of length at least 2 from any non-simple walk at each step, so iteratively applying LE^1 stabilizes on a self-avoiding walk in a finite number of iterations. It follows that the total loop erasure is well defined.

Lemma A.10. *The output of the loop erasure algorithm applied to a walk $\omega = (\omega_0, \dots, \omega_n)$ is a pair $(\eta, H) \in \mathcal{V}(\omega_0, \omega_n)$.*

Proof. At each step of the algorithm the maximal pieces of the heap H^i share a vertex with the remaining walk ω^i, and the algorithm only terminates once the remaining walk is self-avoiding. Removing a cycle cannot change the initial vertex of a walk, so $\eta_0 = \omega_0$. If the final vertex of ω is removed it must be that visiting the final vertex completes a cycle, and hence η ends at ω_n. □

Proof of Theorem A.5. We claim that loop erasure and loop addition are inverses of one another, and prove the claim by induction. Suppose the claim holds between walks whose loop erasure removes k oriented cycles and pairs $(\eta, H) \in \mathcal{V}(a, b)$ whose heap H has k pieces.

On the one hand, inserting the final oriented cycle C in the loop addition algorithm yields a walk, and C is the first oriented cycle removed by loop erasure by Lemma A.9. By induction it follows that loop erasure applied to the loop addition of a pair $(\eta, H) \in \mathcal{V}(a, b)$ returns (η, H).

On the other hand when a single oriented cycle C is removed from ω the cycle C is minimal in the walk order and the removed oriented cycle is the label of a maximal piece. So the reconstruction of the heap formed by loop erasure proceeds as if the piece with label C was not present, and hence (by induction) recreates $\mathrm{LE}^1(\omega)$ correctly. Lemma A.7 then implies that ω is the output of applying loop erasure and then loop addition. □

A.3. Proof of Theorem 1.13. The proof of Theorem 1.13 follows from two calculations. The first is a straightforward consequence of the fact that the bijection between walks and legal pairs is given by loop erasure. Let $\mathcal{T}$ denote the set of trivial heaps of oriented cycles, and $\mathcal{H}$ the set of heaps of oriented cycles. Let $\vec{\mathcal{C}}(\eta)$ denote the set of oriented cycles that *do not* share a vertex with the set η, and let $\mathcal{H}_\eta$ denote the set of heaps H such that (η, H) is a legal pair. The definition of λ-LWW, Theorem A.5, and the heap theorem [18, Proposition 5.3] imply

$$\bar{w}_{\lambda,z}(\eta) = \sum_{\omega :\, \mathrm{LE}(\omega)=\eta} w_{\lambda,z}(\omega) \tag{A.1}$$

$$= z^{|\eta|} \sum_{H \in \mathcal{H}_\eta} w_{\lambda,z}(H)$$

$$= z^{|\eta|} \frac{\displaystyle\sum_{T \in \mathcal{T}(\vec{\mathcal{C}}(\eta))} (-1)^{|T|} w_{\lambda,z}(T)}{\displaystyle\sum_{T \in \mathcal{T}} (-1)^{|T|} w_{\lambda,z}(T)}, \tag{A.2}$$

where

$$w_{\lambda,z}(H) = \prod_{x\in H} w_{\lambda,z}(\ell(x))$$

for a heap $(H, \ell, \preceq)$. In particular note that this definition assigns a weight $z^2\lambda$ to a trivial cycle.

The second calculation is an expression for sums over trivial heaps of oriented cycles. Theorem 1.13 follows by applying Proposition A.11 to the numerator and denominator of (A.2) and cancelling common factors. This calculation is a calculation involving formal power series; to see that it holds as a relation between power series, note that for z sufficiently small the final expressions are bounded by random walk quantities, which converge.

Proposition A.11. *We have*

$$\sum_{T\in\mathcal{T}(\vec{\mathcal{C}}(A))} (-1)^{|T|} w_{\lambda,z}(T) = \exp\Big(- \sum_{x\in\mathbb{Z}^d} \sum_{\substack{\omega\colon x\to x\\ |\omega|\geq 1}} \mathbb{1}_{\{\mathrm{range}(\omega)\cap A=\emptyset\}} \frac{w_{\lambda,z}(\omega)}{|\omega|}\Big).$$

Proof of Theorem 1.13. Let $\bar{z} = sz$. Then $w_{\lambda,z}(\omega) = w_{\lambda,\bar{z}}(\omega)$ when $s = 1$. Using this observe that

$$\sum_{T\in\mathcal{T}(\vec{\mathcal{C}}(A))} (-1)^{|T|} w_{\lambda,z}(T) = \exp \int_0^1 \frac{d}{ds} \log \sum_{T\in\mathcal{T}(\vec{\mathcal{C}}(A))} (-1)^{|T|} w_{\lambda,\bar{z}}(T).$$

In calculating the derivative the Leibniz rule for differentiating s^k can be interpreted as selecting one of the k vertices contained in the cycles of a trivial heap. The selected vertex distinguishes a self-avoiding polygon. Theorem A.5 can be applied to transform this into a walk weighted by $w_{\lambda,z}$. The factor of -1 in the exponent arises from the application of Theorem A.5, as the distinguished cycle carried a factor of -1. Lastly, the term $|\omega|^{-1}$ arises from the integration of $s^{|\omega|-1}$ from 0 to 1; the missing factor of s is due to the differentiation which distinguished a vertex. □

A.4. Relation to correlations of the $O(N)$ cycle gas. Note that equation (A.1) and equation (A.2) imply that $G_{\lambda,z}(0, x)$ is given by a ratio of partition functions. The denominator is a sum over oriented mutually disjoint cyclic subgraphs, where the weight of a subgraph H is $z^{|E(H)|}(-\lambda)^{\#H}$, where $\#H$ denotes the number of cyclic subgraphs contained in H. The numerator is a sum over self-avoiding walks from 0 to x along with disjoint cyclic subgraphs; the weight is the same as for the denominator except for the fact that the walk does not receive a factor of λ.

For each cycle of length at least 3 summing over the possible orientations of the cycles results in a model of unoriented cycle, where each unoriented cycle has weight -2λ, except for trivial cycles, which have weight $-\lambda$. *Defining* the two-point correlation in the $O(N)$ cycle gas to be the ratio described in the previous paragraph gives the relation between the $O(N)$ cycle gas and λ-LWW. Note that if cycles of length two are assigned loop activity 0 this yields a precise correspondence between λ-LWW and the $O(-2\lambda)$ cycle gas.

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