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Bipartite dimer representation of squares of 2d-Ising correlations

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Abstract. The combinatorial Bosonisation identities of [5] show that the square of 2d-Ising order and disorder correlations are equal to $\pm$ the ratio of bipartite dimer partition functions. In this self-contained paper, we give an alternative proof of these identities using the approach of [2]. Our proof is more direct and allows to see the effect of order and disorder on XOR-Ising polygon configurations.

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Keywords. Bipartite dimer model, two-dimensional Ising model, correlation functions, XOR-Ising model.

1. Introduction

Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a finite, planar embedded graph. Consider the Ising model on the graph G with coupling constants $\mathsf{J} = (\mathsf{J}_e)_{e\in\mathsf{E}}$, and denote by $Z_{\mathrm{Ising}}(\mathsf{G}, \mathsf{J})$ the Ising partition function.

Following Kadanoff and Ceva [9], we introduce *order* and *disorder* in the model: *order* amounts to adding $i\frac{\pi}{2}$ to coupling constants along n paths of the graph G joining $2n$, pairwise distinct, vertices $u_1, \dots, u_{2n}$, see Figure 1 (left, blue paths); *disorder* amounts to negating coupling constants of dual edges of m paths of the dual graph G^* joining $2m$, pairwise distinct, faces $f_1, \dots, f_{2m}$ of G, see Figure 1 (left, green paths); all paths are assumed to be pairwise disjoint, see also Remark 2. Denote by $\overline{\mathsf{J}} = (\overline{\mathsf{J}}_e)_{e\in\mathsf{E}}$ the modified coupling constants, and by $\langle \sigma_{u_1} \dots \sigma_{u_{2n}}\ \mu_{f_1} \dots \mu_{f_{2m}} \rangle_{(\mathsf{G},\mathsf{J})}$ the Ising correlation defined as the ratio $Z_{\mathrm{Ising}}(\mathsf{G}, \overline{\mathsf{J}})/Z_{\mathrm{Ising}}(\mathsf{G}, \mathsf{J})$.

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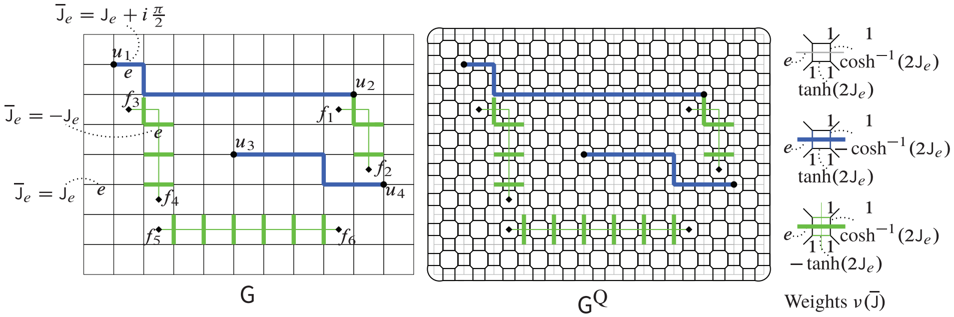

Figure 1. (Colors in on-line version). Left: order and disorder in the Ising model on G. Center: image of order and disorder in the bipartite graph G^Q. Right: dimer weight function $\nu(\overline{J})$ associated to the modified coupling constants $\overline{J}$, expressed as a function of the original coupling constants J. Note that $\nu(\overline{J})$ is defined locally, implying that dimer weights of quadrangle-edges that are not crossed by an order or disorder line, are the usual dimer weights.

When there is no disorder, the correlation $\langle \sigma_{u_1} \dots \sigma_{u_{2n}} \rangle_{(G,J)}$ is, up to an explicit constant, the usual $2n$-spin correlation, see [9] or equation (14); σ is the field consisting of the values of the spins at vertices of the graph. Disorder has the effect of favoring spins to be different across the disorder paths. It amounts to modifying the state space and cannot, strictly speaking, be thought of as a field. Nevertheless, since the effect on the partition function is independent of the choice of paths [9], it is commonly written as a field μ. This way of writing is also favored by the duality relation which exists between the fields σ and μ, see Point 5. of Remark 1. Mixed correlations involving σ and μ are very classical objects of study in the Ising model, see for example [12, 1, 14].

Consider the dimer model on the finite, planar, bipartite graph $G^Q = (V^Q, E^Q)$ constructed from G, see Figure 1 (center). Suppose that edges of G^Q are assigned the weight function $\nu(J) = (\nu(J)_e)_{e \in E^Q}$, defined in Figure 1 (top right), and denote by $Z_{\text{dimer}}(G^Q, \nu(J))$ the corresponding dimer partition function. Consider also the modified weight function $\nu(\overline{J})$ obtained from the modified coupling constants $\overline{J}$: $\nu(\overline{J})$ is defined as in Figure 1 (top right, with J replaced by $\overline{J}$); expressing $\nu(\overline{J})$ as a function of the coupling constants J yields Figure 1 (right: top, middle and bottom). Written in the notation of this paper, the combinatorial Bosonisation identities of Dubédat, see [5] (Point 1 of Lemma 3, p. 16), are stated as follows.

Theorem 1.1 ([5]). *The squared Ising correlation*

$$\langle \sigma_{u_1} \dots \sigma_{u_{2n}} \, \mu_{f_1} \dots \mu_{f_{2m}} \rangle^2_{(\mathsf{G},\mathsf{J})}$$

is equal to $\pm$ *the following ratio of bipartite dimer partition functions*:

$$\langle \sigma_{u_1} \dots \sigma_{u_{2n}} \, \mu_{f_1} \dots \mu_{f_{2m}} \rangle^2_{(\mathsf{G},\mathsf{J})} = (-1)^{|\Gamma|} \frac{Z_{\mathrm{dimer}}(\mathsf{G}^{\mathsf{Q}}, \nu(\overline{\mathsf{J}}))}{Z_{\mathrm{dimer}}(\mathsf{G}^{\mathsf{Q}}, \nu(\mathsf{J}))}, \tag{1}$$

where $|\Gamma|$ *is the number of edges in the union of the* n *paths defining order* (*the blue paths of Figure* 1).

Remark 1. (1) Bosonisation identities in conformal field theory consist in expressing squares of free-fermionic correlations as correlations of the free field [20]. The result of Dubédat is important because it proves that Bosonisation also holds true in the setting of discrete models. Indeed, the Ising model is the archetype of a lattice free-fermionic model, and the dimer model is a classical example of lattice bosonic model. More references on the subject, as well as a description of the right hand-side of equation (1) using a discrete version of the free field, can be found in the original paper by Dubédat [5].

The other important aspect of Dubédat's result is that, since explicit computations can be done in the dimer model using Kasteleyn techniques [7, 8, 16], it gives a way of computing squares of critical Ising spin correlations in the plane [5, 4]; see [3] for the proof of conformal invariance of critical Ising spin correlations.

(2) The proof of Theorem 1.1 uses the argument of [2]. The common thread between the approaches of [5] and [2] is that they use a sequence of expansions starting from two independent Ising models. The advantage of the approach of [2] is that it uses one layer less of Kramers and Wannier duality [10, 11], and it allows to geometrically keep track of XOR-Ising polygon configurations. The latter arise from the low-temperature expansion [10, 11] of the XOR-Ising model, also known as the *polarization* of the model, obtained by taking the product of the spins of two independent Ising models, see [6, 15, 18]. The paper [2] thus provides a coupling between the XOR-Ising model on G and the bipartite dimer model on G^{Q}.

The point of this paper is to highlight that the expansions used in [2] also hold for the modified weights $\overline{\mathsf{J}}$. This is apparent as we go through the different steps of the proof in Section 3. As a consequence, we have that the coupling between the XOR-Ising model and the dimer model also holds for the modified weights, and we can identify the effect of order and disorder (the modified weights $\overline{\mathsf{J}}$) on the XOR-Ising model.

(3) Theorem 1.1 is stated for the Ising model with free boundary conditions. In Section 2.2 we explain how, by transforming the graph and keeping it planar, and possibly adding disorder, all boundary conditions enter the framework of free boundary ones.

(4) Consequences of Theorem 1.1 are expressions as ratio of dimer partition functions for: the square of Ising spinor variables correlations, spin correlations, and magnetization. This is explained in Section 4.

(5) By [9], order and disorder correlations satisfy Kramers and Wannier duality:

$$(-1)^{|\Gamma|}\langle\sigma_{u_1}\dots\sigma_{u_{2n}}\ \mu_{f_1}\dots\mu_{f_{2m}}\rangle_{(\mathsf{G},\mathsf{J})} = (-1)^{|\Gamma^*|}\langle\sigma_{f_1}\dots\sigma_{f_{2m}}\ \mu_{u_1}\dots\mu_{u_{2n}}\rangle_{(\mathsf{G}^*,\mathsf{J}^*)},$$

where edges of the dual graph G^* are assigned *dual coupling constants* $\mathsf{J}^* = \mathsf{J}^*(\mathsf{J})$, defined by: $\mathsf{J}^* = \big(\mathsf{J}^*_{e^*} = -\frac{1}{2}\ln(\tanh \mathsf{J}_e)\big)_{e^*\in\mathsf{E}^*}$. In particular, when the graph G has no disorder, order correlations of G are mapped to disorder correlations of the dual graph G^*, with dual coupling constants. Theorem 1.1 allows to recover a new proof of Kramers and Wannier duality. Indeed, it holds for the numerator and the denominator of the right-hand-side of (1) as a consequence of the following two facts: modified coupling constants also satisfy the duality relation, *i.e.*, $\overline{\mathsf{J}^*}_{e^*} = -\frac{1}{2}\ln(\tanh\overline{\mathsf{J}}_e)$; and $\cosh^{-1}(2\mathsf{J}^*_{e^*}) = \tanh(2\mathsf{J}_e)$, for all choices of coupling constants J.

Outline

- **Section 2**. Definition of the Ising model, of order and disorder. Treatment of other boundary conditions. Definition of the dimer model on the bipartite graph G^{Q}.
- **Section 3**. Proof of Theorem 1.1 using the approach of [2].
- **Section 4**. Consequences of Theorem 1.1 for squares of Ising spinor variables correlations, spin correlations, and magnetization.

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2. Definitions and boundary conditions

2.1. Two-dimensional Ising model, order and disorder. Consider a finite, planar graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$, together with a collection of positive *coupling constants* $\mathsf{J} = (\mathsf{J}_e)_{e\in\mathsf{E}}$ indexed by edges of G. The *Ising model on* G*, with coupling constants* J, is defined as follows. A *spin configuration* σ is a function of the vertices of G taking values in $\{-1,1\}$. The probability on the set of spin configurations $\{-1,1\}^{\mathsf{V}}$, is given by the *Ising Boltzmann measure* $\mathbb{P}_{\mathrm{Ising}}$, defined by

$$\mathbb{P}_{\mathrm{Ising}}(\sigma) = \frac{1}{Z_{\mathrm{Ising}}(\mathsf{G},\mathsf{J})}\exp\Big(\sum_{e=uv\in\mathsf{E}} \mathsf{J}_e\sigma_u\sigma_v\Big), \quad \text{for all } \sigma\in\{-1,1\}^{\mathsf{V}},$$

where

$$Z_{\mathrm{Ising}}(\mathsf{G},\mathsf{J}) = \sum_{\sigma\in\{-1,1\}^{\mathsf{V}}} \exp\Big(\sum_{e=uv\in\mathsf{E}} \mathsf{J}_e\sigma_u\sigma_v\Big)$$

is the normalizing constant, known as the *Ising partition function*.

It is convenient to consider the graph G as embedded in the sphere. Suppose that the embedding of the dual graph G^* is such that dual vertices are in the interior of the faces of G, and such that primal and dual edges cross exactly once. Following Kadanoff and Ceva [9], we introduce *order* and *disorder* in the system. Given positive integers n and m, let $u_1,\dots,u_{2n}$ be $2n$, pairwise distinct, vertices of G and $f_1,\dots,f_{2m}$ be $2m$, pairwise distinct, vertices of the dual graph G^*. Consider n loop-free paths $\gamma_1,\dots,\gamma_n$ of G, such that γ_j has endpoints u_{2j-1}, u_{2j}, and m loop-free paths $\gamma_1^*,\dots,\gamma_m^*$ of G^*, such that γ_j^* has endpoints f_{2j-1}, f_{2j}, see Figure 1 (left). Denote by Γ the set of edges of the paths $\gamma_1,\dots,\gamma_n$, and by Γ^* the set of edges dual to edges of the paths $\gamma_1^*,\dots,\gamma_m^*$. Note that Γ^* is indeed a subset of edges of the primal graph G and not of the dual graph. All paths are assumed to be pairwise disjoint, see also Remark 2.

Define the following modified coupling constants $\bar{\mathsf{J}} = (\bar{\mathsf{J}}_e)_{e\in\mathsf{E}}$:

$$\bar{\mathsf{J}}_e = \begin{cases} \mathsf{J}_e + i\dfrac{\pi}{2} & \text{if } e\in\Gamma,\\ -\mathsf{J}_e & \text{if } e\in\Gamma^*,\\ \mathsf{J}_e & \text{otherwise,}\end{cases} \qquad \text{for all } e\in\mathsf{E}. \tag{2}$$

Then, $Z_{\mathrm{Ising}}(\mathsf{G},\bar{\mathsf{J}}) = \sum_{\sigma\in\{-1,1\}^{\mathsf{V}}}\exp\big(\sum_{e=uv\in\mathsf{E}}\bar{\mathsf{J}}_e\sigma_u\sigma_v\big)$ is the corresponding *modified Ising partition function*.

Remark 2. If only order or only disorder is introduced in the system, the modified Ising partition function is independent of the choice of paths of Γ or Γ^*. If both order and disorder are considered, then changing the paths might induce a sign change [9]. In writing the proofs, it is nevertheless convenient to have pairwise disjoint paths. We thus assume that the embedded planar graph G and its dual G^* are such that paths of Γ and Γ^* can be chosen to be pairwise disjoint. This is the only assumption we make on the graphs G and G^*; examples use a piece of $\mathbb{Z}^2$ simply because it is easier to draw.

2.2. Boundary conditions. The Ising model introduced in Section 2.1 is also known as the Ising model with *free-boundary conditions*. We now discuss how to handle other boundary conditions. Since the graph G is embedded in the sphere, fixing boundary conditions amounts to fixing spins on boundary vertices of a face F of G. We suppose that boundary edges of the face F are not covered by edges of Γ or Γ^*.

Consider first *plus-boundary conditions*, meaning that all spins on boundary vertices of F are +1. Denote by $\mathsf{E}_{\partial\mathsf{F}}$ the set of boundary edges of the face F. Then, up to the constant[1] $\prod_{e\in\mathsf{E}_{\partial\mathsf{F}}} e^{\mathsf{J}_e}$, the modified Ising partition function is equal to the one of the graph G' obtained from G by contracting the face F into a single vertex, and where this vertex is fixed to having spin $+1$, see Figure 2 (left). Since the modified partition function is invariant under the transformation $\sigma \leftrightarrow -\sigma$, it is up to a factor $\frac{1}{2}$, the modified partition function of the graph G' with free boundary conditions. The graph G' is also planar and embedded in the sphere, so that it enters the framework of this paper. A mixture of plus and free-boundary conditions can be handled in a similar way, by contracting all edges with fixed +1 spins, see Figure 2 (right).

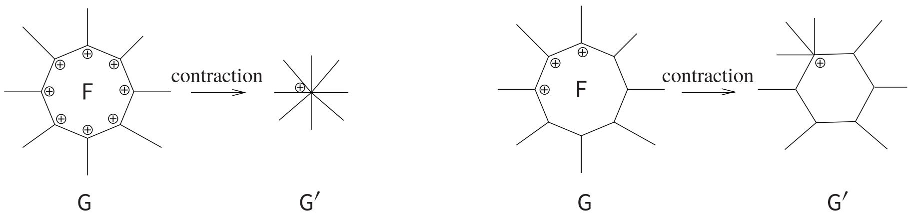

Figure 2. Contraction of vertices and edges to handle plus-boundary conditions (left) and plus-free-boundary conditions (right).

[1] the constant does not depend on the modified coupling constants $\overline{\mathsf{J}}$ because, by assumption, boundary edges of F are not covered by Γ and Γ^*

Consider now *Dobrushin boundary conditions*, meaning that the boundary of the face F is split into two connected components, one having +1 spins and the other -1 spins. Up to a constant, the modified Ising partition function is equal to the one of the graph G' obtained by contracting all vertices and edges of the face F having respectively +1 spins and -1 spins, see Figure 3 (center). Let u be the vertex with fixed -1 spin, then the modified partition function of G' is equal to the one where the spin at u is +1 and coupling constants on edges incident to u are negated. We now have an Ising model with two vertices on the boundary of a face of degree 2 with fixed +1 spins. Up to a constant, the modified Ising partition function is equal to the one of the graph G'' obtained by contracting the two vertices into a single vertex with +1 spin, and adding a disorder line, see Figure 3 (right). Up to a constant $\frac{1}{2}$ it is equal to the modified partition function of the graph G'' with free boundary conditions, and enters again the framework of this paper.

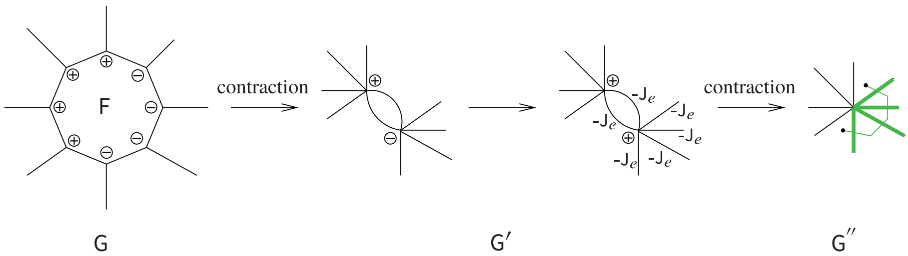

Figure 3. Contraction of vertices and edges, and introduction of a disorder line to handle Dobrushin boundary conditions.

2.3. Dimer model on the bipartite graph G^{Q}. The bipartite graph $\mathsf{G}^{\mathrm{Q}} = (\mathsf{V}^{\mathrm{Q}}, \mathsf{E}^{\mathrm{Q}})$ is constructed from the graph G and its dual G^* as follows. Let us first define the *quad-graph*, denoted $\mathsf{G}^\diamond$, whose vertices are those of G and of the dual graph G^*. A dual vertex is then joined to all primal vertices on the boundary of the corresponding face. The embedding of $\mathsf{G}^\diamond$ is chosen such that its edges do not intersect those of G and G^*, see Figure 4 (left, grey lines). Consider the graph obtained by superimposing the primal graph G, the dual graph G^*, the quad-graph $\mathsf{G}^\diamond$, and by adding a vertex at the crossing of each primal and dual edge. Then, the dual of this graph, denoted by G^{Q}, is the graph on which the dimer model lives, see Figure 4 (right). It is bipartite and consists of *quadrangles* and *legs* connecting the quadrangles: legs are edges crossing those of the quad-graph $\mathsf{G}^\diamond$; in each quadrangle, two edges are "parallel" to an edge e of G and two edges are "parallel" to the dual edge e^* of G^*, see Figure 5.

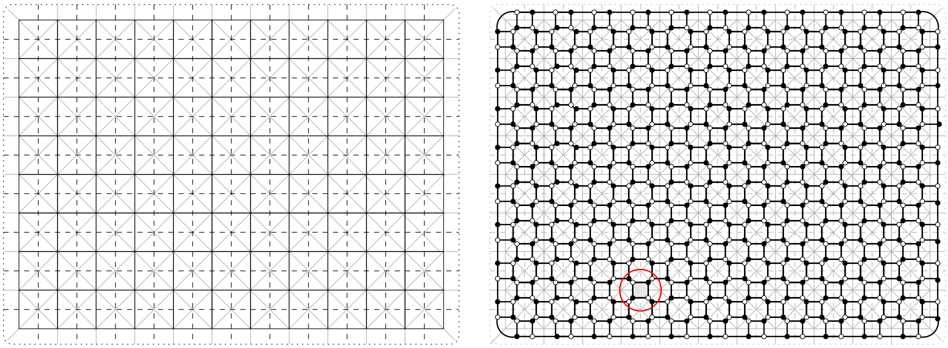

Figure 4. Left: planar embedding of the graph G (plain black lines), dual graph G^* (wide-dotted black lines), the "tiny-dotted" line is a spread out way of representing the vertex of G^* corresponding to the outer-face (in the planar embedding) of G, and the quad-graph G^{Q} (grey lines). Right: the bipartite graph G^{Q} (plain black lines).

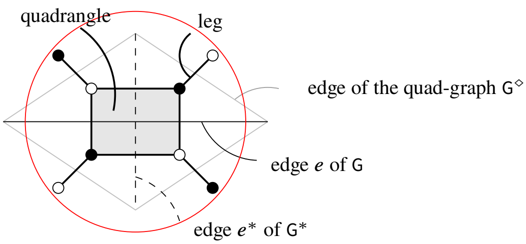

Figure 5. Zoomed-in picture of the circled region of Figure 4.

Suppose that edges of G^{Q} are assigned a positive weight function $\nu = (\nu_{\mathsf{e}})_{\mathsf{e}\in\mathsf{G}^{\mathrm{Q}}}$. The *dimer model on* G^{Q} *with weight function* ν, is defined as follows. A *dimer configuration* of G^{Q}, also known as a *perfect matching*, is a subset of edges M of G^{Q} such that every vertex is incident to exactly one edge of M, see Figure 8. Let us denote by $\mathcal{M}(\mathsf{G}^{\mathrm{Q}})$ the set of dimer configurations of the graph G^{Q}. The probability measure we consider on the set of dimer configurations $\mathcal{M}(\mathsf{G}^{\mathrm{Q}})$ is the *dimer Boltzmann measure* $\mathbb{P}_{\mathrm{dimer}}$, defined by

$$\mathbb{P}_{\mathrm{dimer}}(\mathsf{M}) = \frac{\prod_{\mathsf{e}\in\mathsf{M}} \nu_{\mathsf{e}}}{Z_{\mathrm{dimer}}(\mathsf{G}^{\mathrm{Q}}, \nu)} \quad \text{for all } \mathsf{M} \in \mathcal{M}(\mathsf{G}^{\mathrm{Q}}),$$

where

$$Z_{\mathrm{dimer}}(\mathsf{G}^{\mathrm{Q}}, \nu) = \sum_{\mathsf{M}\in\mathcal{M}(\mathsf{G}^{\mathrm{Q}})} \prod_{\mathsf{e}\in\mathsf{M}} \nu_{\mathsf{e}}$$

is the normalizing constant known as the *dimer partition function*.

3. Proof of Theorem 1.1

We now turn to the proof of Theorem 1.1. The key result is Theorem 5.5 of [2], which in the case of a finite, planar graph G embedded in the sphere (genus 0 case), reads

$$[Z_{\text{Ising}}(\mathsf{G},\mathsf{J})]^2 = 2^{|\mathsf{V}|}\Big(\prod_{e\in\mathsf{E}}\cosh(2\mathsf{J}_e)\Big)\cdot Z_{\text{dimer}}(\mathsf{G}^{\mathsf{Q}},\nu(\mathsf{J})), \tag{3}$$

where the dimer weight function $\nu(\mathsf{J})$ is defined from the coupling constants J as follows:

$$\nu(\mathsf{J})_{\mathsf{e}} = \begin{cases} 1 & \text{if } \mathsf{e} \text{ is a leg,}\\ \tanh(2\mathsf{J}_e) & \text{if } \mathsf{e} \text{ is “parallel” to a primal edge } e \text{ of } \mathsf{G},\\ \cosh^{-1}(2\mathsf{J}_e) & \text{if } \mathsf{e} \text{ is “parallel” to the dual edge } e^* \text{ of an edge } e \text{ of } \mathsf{G}. \end{cases} \tag{4}$$

This result is also proved in the genus 0 and 1 case in [5]. Both proofs use a sequence of expansions starting from two independent Ising models. Dubédat starts from one Ising model living on the graph G and the other on the dual graph G^*. He then uses Kramers and Wannier duality to have the square of the Ising model partition function. One of the expansions he uses is of the “high-temperature type”, in the sense that there is no geometrical mapping between configurations before and after the expansion.

The approach of [2] is more transparent: it starts from two independent Ising models living on the *same* graph G, and while doing expansions, allows to keep track of XOR-Ising polygon configurations. The latter are polygon configurations separating clusters of ± 1 spins of the XOR-Ising model [6, 15, 18], obtained by taking the product of the spins of the two independent Ising models. It provides a coupling between the bipartite dimer model on G^{Q} and the XOR-Ising model on G.

By looking at the proof of equation (3) in [2], we see that this equation actually holds for all choices of coupling constants, in particular negative or complex. Thus, for the modified coupling constants $\bar{\mathsf{J}}$ of equation (2), we have

$$[Z_{\text{Ising}}(\mathsf{G},\bar{\mathsf{J}})]^2 = 2^{|\mathsf{V}|}\Big(\prod_{e\in\mathsf{E}}\cosh(2\bar{\mathsf{J}}_e)\Big)\cdot Z_{\text{dimer}}(\mathsf{G}^{\mathsf{Q}},\nu(\bar{\mathsf{J}})), \tag{5}$$

where $\nu(\bar{\mathsf{J}})$ is given by equation (4) with J replaced by $\bar{\mathsf{J}}$.

Let us express the modified weights $\nu(\bar{\mathsf{J}})$ as a function of the original coupling constants J. By equation (4), we need to compute $\tanh(2\bar{\mathsf{J}}_e)$ and $\cosh(2\bar{\mathsf{J}}_e)$, for an edge e of G. We have

$$\tanh(2\bar{\mathsf{J}}_e) = \begin{cases} \tanh(2\mathsf{J}_e) & \text{if } e \notin \Gamma \cup \Gamma^*, \\ \tanh(2\mathsf{J}_e + i\pi) = \tanh(2\mathsf{J}_e) & \text{if } e \in \Gamma, \\ \tanh(-2\mathsf{J}_e) = -\tanh(2\mathsf{J}_e) & \text{if } e \in \Gamma^*. \end{cases} \tag{6}$$

and

$$\cosh(2\bar{\mathsf{J}}_e) = \begin{cases} \cosh(2\mathsf{J}_e) & \text{if } e \notin \Gamma \cup \Gamma^*, \\ \cosh(2\mathsf{J}_e + i\pi) = -\cosh(2\mathsf{J}_e) & \text{if } e \in \Gamma, \\ \cosh(-2\mathsf{J}_e) = \cosh(2\mathsf{J}_e) & \text{if } e \in \Gamma^*. \end{cases} \tag{7}$$

As a consequence, we have that the weight function $\nu(\bar{\mathsf{J}})$ written as a function of the original coupling constants is given by, see also Figure 1 (right),

$$\nu(\bar{\mathsf{J}})_{\mathsf{e}} = \begin{cases} 1 & \text{if } \mathsf{e} \text{ is an external edge,} \\ \tanh(2\mathsf{J}_e) & \text{if } \mathsf{e} \text{ is ``parallel'' to an edge } e \text{ of } \mathsf{G},\ e \notin \Gamma^*, \\ -\tanh(2\mathsf{J}_e) & \text{if } \mathsf{e} \text{ is ``parallel'' to an edge } e \text{ of } \mathsf{G},\ e \in \Gamma^*, \\ \cosh^{-1}(2\mathsf{J}_e) & \text{if } \mathsf{e} \text{ is ``parallel'' to the dual edge } e^* \\ & \text{of an edge } e \text{ of } \mathsf{G},\ e \notin \Gamma, \\ -\cosh^{-1}(2\mathsf{J}_e) & \text{if } \mathsf{e} \text{ is ``parallel'' to the dual edge } e^* \\ & \text{of an edge } e \text{ of } \mathsf{G},\ e \in \Gamma. \end{cases} \tag{8}$$

From equation (7), we also have that the term $\prod_{e\in\mathsf{E}} \cosh(2\bar{\mathsf{J}}_e)$ in equation (5) is equal to $(-1)^{|\Gamma|}\prod_{e\in\mathsf{E}}\cosh(2\mathsf{J}_e)$. Taking the ratio of equations (5) and (3) yields Theorem 1.1. □

In the paper [2], equation (3) is proved for graphs embedded in surfaces of genus g. We now give an outline of the proof in the genus 0 case. We do so for several reasons: first, in the genus 0 case the proof greatly simplifies, it is thus rather short and makes this paper self-contained; second, it is by looking at the proof that one sees that equation (3) does not require positivity of the coupling constants; finally, it allows to see the effect of order and disorder (of the modified weights $\bar{\mathsf{J}}$) on XOR-Ising polygon configurations.

The proof of equation (3) has two steps. The first, based on an idea of [13], is to show that the square of the Ising partition function is equal, up to an explicit constant, to a weighted sum over pairs of non-intersecting polygon configurations of the graph G and of its dual graph G^*, where polygon configurations of the graph G^* arise from the low-temperature expansion of the XOR-Ising model. The second step, based on ideas of [13, 19, 5], is to map the weighted sum over pairs of non-intersecting polygon configurations of the graphs G and G^* onto the dimer model on the bipartite graph G^{Q}.

3.1. Step 1: polygonal representation of the square of the Ising partition function. A *polygon configuration* of the graph G is a subset of edges P such that every vertex of G is incident to an even number of edges of P. The set of polygon configurations of G is denoted by $\mathcal{P}(\mathsf{G})$. The set of polygon configurations $\mathcal{P}(\mathsf{G}^*)$ of the dual graph G^* of G is defined similarly.

In Proposition 1.1 of [2], we prove that the square of the Ising partition function is equal, up to a constant, to a weighted sum over pairs of non-intersecting polygon configurations of the graph G and of its dual graph G^*, see Figure 6. The result holds for any choice of coupling constants. It is proved for graphs embedded in surfaces of genus g, using an idea of Nienhuis [13]. In particular, for graphs embedded in the sphere, it reads

$$[Z_{\text{Ising}}(\mathsf{G},\mathsf{J})]^2 = \mathcal{C} \sum_{\{(\mathsf{P},\mathsf{P}^*)\in\mathcal{P}(\mathsf{G})\times\mathcal{P}(\mathsf{G}^*):\ \mathsf{P}\cap\mathsf{P}^*=\emptyset\}} \Big(\prod_{e^*\in\mathsf{P}^*}\cosh^{-1}(2\mathsf{J}_e)\Big)\Big(\prod_{e\in\mathsf{P}}\tanh(2\mathsf{J}_e)\Big), \tag{9}$$

where

$$\mathcal{C} = 2^{|\mathsf{V}|+1}\Big(\prod_{e\in\mathsf{E}}\cosh(2\mathsf{J}_e)\Big).$$

Remark 3. From [13], see also [2], we know that polygon configurations of the dual graph G^* arise from the low-temperature expansion [10, 11] of the XOR-Ising model.

The proof of equation (9) in the genus g case is rather complicated because of homology considerations which come into play. In the genus 0 case, it greatly simplifies, and can be written as follows.

Proof of equation (9) *in the genus* 0 *case.* The square of the modified partition function is equal to

$$[Z_{\text{Ising}}(\mathsf{G},\mathsf{J})]^2 = \sum_{\sigma,\sigma'\in\{-1,1\}^{\mathsf{V}}}\ \prod_{e=uv\in\mathsf{E}} e^{\mathsf{J}_e\sigma_u\sigma_v} e^{\mathsf{J}_e\sigma'_u\sigma'_v}.$$

For every pair of spin configurations σ,σ', denote by τ the XOR-Ising configuration, obtained by taking the product $\sigma\sigma'$,

$$\tau_u = \sigma_u\sigma'_u \quad \text{for all } u\in\mathsf{V},$$

whence $\tau\in\{-1,1\}^{\mathsf{V}}$.

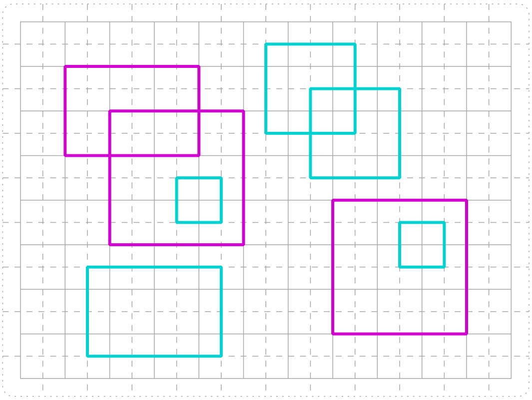

Figure 6. The square of the modified Ising partition function can be written as a weighted sum over pairs of non-intersecting primal (pink) and dual (turquoise) polygon configurations of G and G^*.

Since σ and σ' take values in $\{-1,1\}$, we have $\sigma' = \tau\sigma$, and the square of the modified partition function can be written as

$$[Z_{\text{Ising}}(\mathsf{G},\mathsf{J})]^2 = \sum_{\tau,\sigma\in\{-1,1\}^{\mathsf{V}}} \Big(\prod_{e=uv\in\mathsf{E}} e^{\mathsf{J}_e\sigma_u\sigma_v(1+\tau_u\tau_v)} \Big).$$

Let us now fix a XOR-spin configuration τ. Denote by $\mathsf{V}_\tau^1,\cdots,\mathsf{V}_\tau^{k_\tau}$ the partition of vertices of V corresponding to clusters of ± 1 spins of τ. For every $\ell\in\{1,\cdots,k_\tau\}$, let E_τ^ℓ be the subset of edges joining vertices of V_τ^ℓ. Set $\mathsf{E}_\tau = \bigcup_{\ell=1}^{k_\tau}\mathsf{E}_\tau^\ell$ and $(\mathsf{E}_\tau)^c = \mathsf{E}\setminus\mathsf{E}_\tau$. Then, for every $e=uv\in\mathsf{E}_\tau$, $\tau_u\tau_v=1$, and for every $e\in(\mathsf{E}_\tau)^c$, $\tau_u\tau_v=-1$, implying that

$$[Z_{\text{Ising}}(\mathsf{G},\mathsf{J})]^2 = \sum_{\tau\in\{-1,1\}^{\mathsf{V}}}\sum_{\sigma\in\{-1,1\}^{\mathsf{V}}}\prod_{e=uv\in\mathsf{E}_\tau} e^{2\mathsf{J}_e\sigma_u\sigma_v}.$$

Exchanging the sum over spins σ's and the product over edges of E_τ yields

$$[Z_{\text{Ising}}(\mathsf{G},\mathsf{J})]^2 = \sum_{\tau\in\{-1,1\}^{\mathsf{V}}}\prod_{\ell=1}^{k_\tau}\Big[\sum_{\sigma^\ell\in\{-1,1\}^{\mathsf{V}_\tau^\ell}}\Big(\prod_{e=uv\in\mathsf{E}_\tau^\ell} e^{2\mathsf{J}_e\sigma_u\sigma_v}\Big)\Big]. \tag{10}$$

That is, for every $\ell\in\{1,\cdots,k_\tau\}$, we have the partition function of an Ising model on $\mathsf{G}_\tau^\ell = (\mathsf{V}_\tau^\ell,\mathsf{E}_\tau^\ell)$, with modified, doubled coupling constants $2\mathsf{J}$. Using Kramers and Wannier high-temperature expansion [10, 11, 17] for each of these modified

Ising models, we obtain

$$\prod_{\ell=1}^{k_\tau}\Big[\sum_{\sigma^\ell\in\{-1,1\}^{\mathsf{V}_\tau^\ell}}\Big(\prod_{e=uv\in\mathsf{E}_\tau^\ell} e^{2\mathsf{J}_e\sigma_u\sigma_v}\Big)\Big]$$
$$= \prod_{\ell=1}^{k_\tau}\Big[2^{|\mathsf{V}_\tau^\ell|}\Big(\prod_{e\in\mathsf{E}_\tau^\ell}\cosh(2\mathsf{J}_e)\Big)\sum_{\mathsf{P}\in\mathcal{P}(\mathsf{G}_\tau^\ell)}\Big(\prod_{e\in\mathsf{P}}\tanh(2\mathsf{J}_e)\Big)\Big] \tag{11}$$
$$= 2^{|\mathsf{V}|}\Big(\prod_{e\in\mathsf{E}_\tau}\cosh(2\mathsf{J}_e)\Big)\prod_{\ell=1}^{k_\tau}\Big[\sum_{\mathsf{P}\in\mathcal{P}(\mathsf{G}_\tau^\ell)}\Big(\prod_{e\in\mathsf{P}}\tanh(2\mathsf{J}_e)\Big)\Big].$$

Plugging (11) into the square of the modified partition function (10) yields

$$[Z_{\text{Ising}}(\mathsf{G},\mathsf{J})]^2 = \mathcal{C}'\sum_{\tau\in\{-1,1\}^{\mathsf{V}}}\Big(\prod_{e\in(\mathsf{E}_\tau)^c}\cosh^{-1}(2\mathsf{J}_e)\Big)\prod_{\ell=1}^{k_\tau}\Big(\sum_{\mathsf{P}\in\mathcal{P}(\mathsf{G}_\tau^\ell)}\prod_{e\in\mathsf{P}}\tanh(2\mathsf{J}_e)\Big),$$

where $\mathcal{C}' = 2^{|\mathsf{V}|}\big(\prod_{e\in\mathsf{E}}\cosh(2\mathsf{J}_e)\big)$.

The proof is concluded by assigning, as in the low-temperature expansion, dual polygon configurations separating clusters of spins of XOR-Ising configurations. Note that the constant $\mathcal{C}'$ and the constant $\mathcal{C}$ of the statement differ by a factor 2 because two spin configurations are assigned to a given dual polygon configuration. □

In particular, when the coupling constants are the modified weights $\bar{\mathsf{J}}$, the result is

$$[Z_{\text{Ising}}(\mathsf{G},\bar{\mathsf{J}})]^2 = \mathcal{C}\sum_{\{(\mathsf{P},\mathsf{P}^*)\in\mathcal{P}(\mathsf{G})\times\mathcal{P}(\mathsf{G}^*):\,\mathsf{P}\cap\mathsf{P}^*=\emptyset\}}\Big(\prod_{e^*\in\mathsf{P}^*}\cosh^{-1}(2\bar{\mathsf{J}}_e)\Big)\Big(\prod_{e\in\mathsf{P}}\tanh(2\bar{\mathsf{J}}_e)\Big), \tag{12}$$

where

$$\mathcal{C} = 2^{|\mathsf{V}|+1}\Big(\prod_{e\in\mathsf{E}}\cosh(2\bar{\mathsf{J}}_e)\Big).$$

Returning to the computations of equations (6) and (7), the modified weights $\tanh(2\bar{\mathsf{J}}_e)$ and $\cosh^{-1}(2\bar{\mathsf{J}}_e)$ can be expressed as a function of the original weights $\tanh(2\mathsf{J}_e)$ and $\cosh^{-1}(2\mathsf{J}_e)$, see Figure 7.

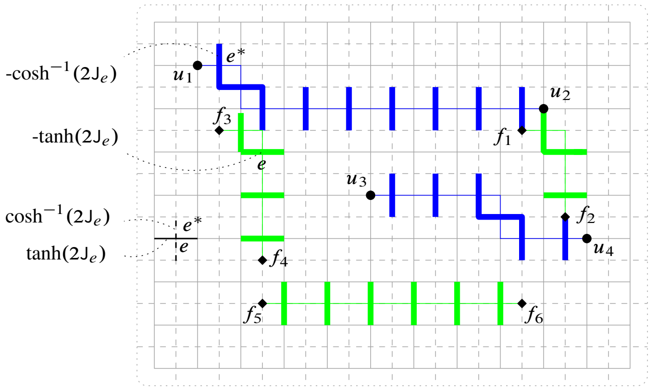

Figure 7. Modified edge-weights for the weighted sum over pairs of non-intersecting polygon configurations of the graph G and its dual graph G^*, induced by the modified coupling constants $\overline{\mathsf{J}}$.

We see that adding order and disorder into the system only affects weights of edges along, or crossing, Γ and Γ^*. Since polygon configurations of the dual graph are the XOR-Ising polygon configurations, we understand the effect of order and disorder on the XOR-Ising model.

3.2. Step 2: bipartite dimer representation of the polygon representation. We proceed as in [2]. The weighted sum over pairs of non-intersecting primal and dual polygon configurations naturally maps to a 6-vertex model [13]. This 6-vertex model is free-fermionic when polygon edge-weights arise from two independent Ising models (because $[\cosh^{-1}(2\mathsf{J}_e)]^2 + [\tanh(2\mathsf{J}_e)]^2 = 1$, which holds for *any* choice of coupling constants). The free-fermionic 6-vertex model then maps to the dimer model on the graph G^{Q} defined in Section 2.3 [19, 5]. The mapping from pairs of non-intersecting primal and dual polygon configurations of G and G^*, to dimer configurations of G^{Q} can be explained without going through the 6-vertex model. It is summarized in Figure 8.

As a consequence, we obtain

$$\sum_{\{(\mathsf{P},\mathsf{P}^*)\in\mathcal{P}(\mathsf{G})\times\mathcal{P}(\mathsf{G}^*):\,\mathsf{P}\cap\mathsf{P}^*=\emptyset\}}\Big(\prod_{e^*\in\mathsf{P}^*}\cosh^{-1}(2\mathsf{J}_e)\Big)\Big(\prod_{e\in\mathsf{P}}\tanh(2\mathsf{J}_e)\Big) = \frac{1}{2}Z_{\mathrm{dimer}}(\mathsf{G}^{\mathsf{Q}},\nu(\mathsf{J})), \tag{13}$$

where the dimer weight function $\nu(\mathsf{J})$ is given by equation (4).

The proof of equation (3) is concluded by combining equations (9) and (13).

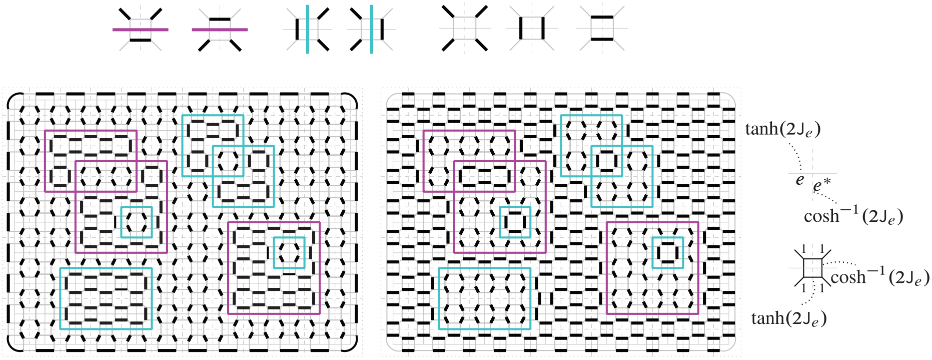

Figure 8. Mapping between pairs of non-intersecting primal and dual polygon configurations of G and G^* and dimer configurations of G^{Q} [13, 19, 5]. Top: mapping on the local level. Bottom: mapping on the global level. Given a pair of non-intersecting primal and dual polygon configurations, there are two possible leg configurations for the corresponding dimer configuration; then the configuration of quadrangles with 2 or 4 matched legs is fixed, and quadrangles having 0 matched leg each have two possible dimer configurations. Right: mapping of the weights.

4. Consequences

As a consequence of Theorem 1.1, we obtain expressions as ratio of bipartite dimer partition functions for squares of quantities of interest in the study of the Ising model. Throughout this section, we use the notation of Sections 1, 2, and 3.

First, following [9], we consider $2n$-*spinor variables*. This amounts to taking $m = n$ and choosing u_j, f_j in such a way that u_j is on the boundary of the face of G defined by the dual vertex f_j. Then, specifying Theorem 1.1 to this choice of vertices yields an expression for squares of spinor variables correlations as the ratio of bipartite dimer partition functions.

Next, let us consider $2n$-*spin correlations* $\mathbb{E}[\sigma_{u_1} \dots \sigma_{u_{2n}}]$. This enters the framework of this paper by conditioning Γ^* to be the empty set. More precisely, by Kadanoff and Ceva [9], $2n$-spin correlations are equal to

$$\mathbb{E}[\sigma_{u_1} \dots \sigma_{u_{2n}}] = (-i)^{|\Gamma|} \frac{Z_{\mathrm{Ising}}(\mathsf{G}, \bar{\mathsf{J}})}{Z_{\mathrm{Ising}}(\mathsf{G}, \mathsf{J})} = (-i)^{|\Gamma|} \langle \sigma_{u_1} \dots \sigma_{u_{2n}} \rangle_{(\mathsf{G},\mathsf{J})}, \tag{14}$$

where

$$\bar{\mathsf{J}}_e = \begin{cases} \mathsf{J}_e + i\dfrac{\pi}{2} & \text{if } e \in \Gamma, \\ \mathsf{J}_e & \text{otherwise.} \end{cases}$$

Computing the corresponding dimer weight function $\nu(\bar{\mathsf{J}})$ using equation (8) yields

$$\nu(\bar{\mathsf{J}})_{\mathsf{e}} = \begin{cases} 1 & \text{if } \mathsf{e} \text{ is an external edge,} \\ \tanh(2\mathsf{J}_e) & \text{if } \mathsf{e} \text{ is "parallel" to an edge } e \text{ of } \mathsf{G}, \\ \cosh^{-1}(2\mathsf{J}_e) & \text{if } \mathsf{e} \text{ is "parallel" to the dual edge } e^* \\ & \text{of an edge } e \text{ of } \mathsf{G},\ e \notin \Gamma, \\ -\cosh^{-1}(2\mathsf{J}_e) & \text{if } \mathsf{e} \text{ is "parallel" to the dual edge } e^* \\ & \text{of an edge } e \text{ of } \mathsf{G},\ e \in \Gamma. \end{cases} \tag{15}$$

As a consequence of Theorem 1.1, we obtain the following.

Corollary 1. *The square of $2n$-spin correlations $\mathbb{E}[\sigma_{u_1} \dots \sigma_{u_{2n}}]^2$ is the following ratio of bipartite dimer partition functions*:

$$\mathbb{E}[\sigma_{u_1} \dots \sigma_{u_{2n}}]^2 = \frac{Z_{\text{dimer}}(\mathsf{G}^{\mathsf{Q}}, \nu(\bar{\mathsf{J}}))}{Z_{\text{dimer}}(\mathsf{G}^{\mathsf{Q}}, \nu(\mathsf{J}))},$$

where the dimer weight function $\nu(\bar{\mathsf{J}})$ is given by equation (15).

Finally, let us express the *magnetization* $\mathbb{E}^+[\sigma_u]$, which is the expected value of a single spin u under plus-boundary conditions (if free boundary conditions were considered, the magnetization would be equal to zero by symmetry). Recall that fixing plus-boundary amounts to taking all spins on boundary vertices of a face F of G to be $+1$. Magnetization enters the framework of this paper by conditioning Γ^* to be the empty set, Γ to be a single path γ, and by using the procedure of Section 2.2 for treating plus-boundary conditions. More precisely, by [9], the magnetization is equal to

$$\mathbb{E}^+[\sigma_u] = (-i)^{|\gamma|} \frac{Z^+_{\text{Ising}}(\mathsf{G}, \bar{\mathsf{J}})}{Z^+_{\text{Ising}}(\mathsf{G}, \mathsf{J})},$$

where $Z^+_{\text{Ising}}(\mathsf{G}, \bar{\mathsf{J}})$ is the modified, plus-boundary condition Ising partition function, modified along a single path γ of the graph G, where γ joins a vertex on the boundary of F to the vertex u; this quantity is independent of the choice of boundary vertex. Let us suppose that γ does not use edges on the boundary of F, and let G' be the graph obtained from G by contracting the face F into a single vertex v, see Figure 2 (left). Note that in G', the path γ joins the vertices v and u. Using the argument of Section 2.2 for handling plus-boundary conditions, we obtain

$$\frac{Z^+_{\text{Ising}}(\mathsf{G}, \bar{\mathsf{J}})}{Z^+_{\text{Ising}}(\mathsf{G}, \mathsf{J})} = \frac{\frac{1}{2}\Big(\prod_{e \in \mathsf{E}_{\partial \mathsf{F}}} e^{\mathsf{J}_e}\Big) Z_{\text{Ising}}(\mathsf{G}', \bar{\mathsf{J}})}{\frac{1}{2}\Big(\prod_{e \in \mathsf{E}_{\partial \mathsf{F}}} e^{\mathsf{J}_e}\Big) Z_{\text{Ising}}(\mathsf{G}', \mathsf{J})} = \frac{Z_{\text{Ising}}(\mathsf{G}', \bar{\mathsf{J}})}{Z_{\text{Ising}}(\mathsf{G}', \mathsf{J})}.$$

Since γ does not use boundary edges of the face F, it has the same number of edges in the graphs G and G'. We have thus proved the following.

Lemma 4.1. *The magnetization $\mathbb{E}^+[\sigma_u]$ in the graph G, is equal to the pair-spin correlation $\mathbb{E}[\sigma_u\sigma_v]$ in the graph G'.*

As a consequence, the expression as ratio of bipartite dimer partition functions for the square of the magnetization is a specific case of Corollary 1.

References

[1] R. J. Baxter, *Exactly solved models in statistical mechanics.* Reprint of the 1982 original. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1989. MR 0998375 Zbl 0538.60093 (1982 ed.)

[2] C. Boutillier and B. de Tilière, Height representation of XOR-Ising loops via bipartite dimers. *Electron. J. Probab.* **19** (2014), article id. 80, 33 pp. MR 3256880 Zbl 1302.82014

[3] D. Chelkak, C. Hongler, and K. Izyurov, Conformal invariance of spin correlations in the planar Ising model. *Ann. of Math.* (2) **181** (2015), no. 3, 1087–1138. MR 3296821 Zbl 1318.82006

[4] J. Dubédat, Dimers and analytic torsion I. Preprint 2011. arXiv:1110.2808 [math.PR]

[5] J. Dubédat, Exact bosonization of the Ising model. Preprint 2011. arXiv:1112.4399 [math.PR]

[6] Y. Ikhlef and M. A. Rajabpour, Discrete holomorphic parafermions in the Ashkin–Teller model and SLE. *J. Phys. A* **44** (2011), no. 4, article id. 042001, 11 pp. MR 2754707 Zbl 1206.81113

[7] P. W. Kasteleyn, The statistics of dimers on a lattice. I: The number of dimer arrangements on a quadratic lattice. *Physica* **27** (1961), 1209–1225. Zbl 1244.82014

[8] P. W. Kasteleyn, Graph theory and crystal physics. In *Graph Theory and Theoretical Physics.* Academic Press, London, 1967, 43–110. MR 0253689 Zbl 0205.28402

[9] L. P. Kadanoff and H. Ceva, Determination of an operator algebra for the two-dimensional Ising model. *Phys. Rev. B* (3) **3** (1971), 3918–3939. MR 0389111

[10] H. A. Kramers and G. H. Wannier, Statistics of the two-dimensional ferromagnet. I. *Phys. Rev.* (2) **60** (1941), 252–262. MR 0004803 Zbl 0027.28505

[11] H. A. Kramers and G. H. Wannier, Statistics of the two-dimensional ferromagnet. Part II. *Phys. Rev.* (2) **60** (1941), 263–276. MR 0004804 Zbl 0027.28506

[12] B. McCoy and F. Wu, *The two-dimensional Ising model.* Havard University Press, Cambridge, MA, 1973. Zbl 1094.82500

[13] B. Nienhuis, Critical behavior of two-dimensional spin models and charge asymmetry in the Coulomb gas. *J. Statist. Phys.* **34** (1984), no. 5-6, 731–761. MR 0751711 Zbl 0595.76071

[14] J. Palmer, *Planar Ising correlations.* Progress in Mathematical Physics, 49. Birkhäuser Boston, Boston, MA, 2007. MR 2332010 Zbl 1136.82001

[15] M. Picco and R. Santachiara, Critical interfaces and duality in the Ashkin–Teller model. *Phys. Rev. E* **83**, article id. 061124.

[16] H. N. V. Temperley and M. E. Fisher, Dimer problem in statistical mechanics – an exact result. *Philos. Mag.* (8) **6** (1961), 1061–1063. MR 0136398 Zbl 0126.25102

[17] B. L. van der Waerden, Die lange Reichweite der regelmässigen Atomanordnung in Mischkristallen. *Z. Physik* **118** (1941), 473–488. Zbl 0026.28301 JFM 67.0961.03

[18] D. B. Wilson, XOR-Ising loops and the Gaussian free field. Preprint 2011. arXiv:1102.3782 [cond-mat.stat-mech]

[19] F. Y. Wu and K. Y. Lin, Staggered ice-rule vertex model – The Pfaffian solution. *Phys. Rev. B* **12** (1975), 419–428.

[20] J. B. Zuber and C. Itzykson, Quantum field theory and the two-dimensional Ising model. *Phys. Rev. D* **15** (1997), 2875–2884.

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