

Type \widehat{C} Brauer loop schemes and loop model with boundaries

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Abstract. In this paper we study the Brauer loop model on a strip and the associated quantum Knizhnik–Zamolodchikov (qKZ) equation. We show that the minimal degree solution of the Brauer qKZ equation with one of four different possible boundary conditions, gives the multidegrees of the irreducible components of generalizations of the Brauer loop scheme of [16, Knutson–Zinn-Justin ’07] with one of four kinds of symplectic-type symmetry. This is accomplished by studying these irreducible components, which are indexed by link patterns, and describing the geometric action of Brauer generators on them. We also provide recurrence relations for the multidegrees and compute the sum rules (multidegrees of the whole schemes).

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1. Introduction

1.1. Background. Recently there has been renewed interest in the connection between quantum integrable systems and algebraic geometry, see e.g. [11, 12, 19, 22]. A particularly explicit realization of this correspondence can be found in the case of the Brauer loop model [18, 5, 9] and its geometric counterpart, the Brauer loop scheme [16, 17]. As part of the dictionary between these two subjects (cf. [19]), the type of symmetry of the geometric object determines the boundary conditions of the integrable model. In the setup of [16, 17], the Brauer loop scheme has symmetry governed by an algebra of type \hat{A} , so that the corresponding R -matrix satisfies the ordinary Yang–Baxter equation and the Brauer loop model has periodic boundary conditions. It is expected that other types will lead to other boundary conditions (see [8] for some experiments in that general direction). Among them, type \hat{C} is particularly natural: according to its Dynkin diagram (see Section 1.3), it should correspond to models defined on a strip, with a bulk defined in terms of ordinary R -matrices satisfying the Yang–Baxter equation, and two boundaries, each with a K -matrix satisfying the reflection (or boundary Yang–Baxter) equation.

The purpose of this paper is to validate this hypothesis by, on the one hand, introducing and studying the Brauer loop model on a strip with various boundary conditions, and on the other, describing their geometric counterparts, leading to four distinct type \hat{C} Brauer loop schemes.

1.2. Results. More specifically, we study the type \hat{C} quantum Knizhnik–Zamolodchikov (qKZ) equation – by this we mean the generalization due to Cherednik [2] of the original qKZ equation [10] to all types – associated to the Brauer R -matrix and to either a trivial or nontrivial K -matrix at both boundaries, with a further refinement consisting in identifying or not the two boundaries. Taking into account the obvious left/right symmetry, these boundary conditions lead to four possibilities, denoted by the superscripts i (identified), c (closed), o (open), m (mixed). We are interested in a polynomial solution for a specific value of the shift parameter of the qKZ equation.¹ Note that the Brauer qKZ equation is very nontrivial because, in contradistinction with the more usual case of the Hecke algebra, the R -matrix has three terms, so that a polynomial solution is not obviously related to the action of an algebra (say, the double affine Hecke algebra [3])

¹ We expect polynomiality to be only possible for discrete values of this parameter, and the one we choose to give the lowest possible degree among these polynomial solutions.

on polynomials, i.e., it does not immediately reduce to a standard representation theory problem.

We study in Section 2 *polynomial* solutions of the qKZ equation. We discuss various properties they possess, including recurrence relations. We determine in particular a lower bound on their degree, and that if there exists a solution which saturates this bound, then it is unique up to multiplication by a constant; however, contrary to type \widehat{A} , we cannot show at this stage the existence of such a solution in types \widehat{C} .

Before turning to the geometry, let us mention some motivation and physical applications. In Section 3, we recall that setting the loop weight to 1 results in the shift parameter of the qKZ equation being zero, and show that the polynomial solution mentioned above is an eigenvector of the inhomogeneous transfer matrix of the Brauer loop model with the same boundary conditions. In the physical range of parameters where the Boltzmann weights are positive, it is in fact the *ground state* of the transfer matrix, making it particularly interesting to calculate. Equivalently, since the transfer matrix is stochastic, the entries of the ground state can be interpreted as (unnormalized) probabilities of the connectivity of boundary points on a half-infinite strip. (The normalization is in fact computed in the present work). This paves the way to the calculation of more physically interesting quantities such as correlation functions. The model on a strip is particularly interesting because it should help with the computation of the boundary-to-boundary current, similarly to the work [6] on the noncrossing loop model.

Next we come to the geometric construction. In Section 4, we define four (conical, affine) schemes, which we call type \widehat{C} Brauer loop schemes, corresponding to the four cases {i, c, o, m} mentioned above. We provide different descriptions, either as infinite periodic matrices or as flat limits of certain orbits generalizing nilpotent orbits. We also define the group action that these schemes are naturally equipped with, and in particular the torus action. We then describe their irreducible components, following a similar study in [16] in type \widehat{A} , in terms of link patterns, giving a first hint of the connection to the loop model, since these link patterns form the natural basis of the space on which the Brauer algebra acts. The construction makes use of several type C analogues of the classification of B -orbits of upper triangular matrices which square to zero [20]. As a byproduct, we point out the connection to a symplectic analogue of the commuting variety, which was one of the motivations of [16].

Finally, Section 5 provides the exact connection between the Brauer loop model and the Brauer loop scheme in various types. This can be summarized by the following meta-theorem:

Theorem. *The multidegrees of the irreducible components of the Brauer loop scheme form a polynomial solution of the qKZ equation.*

(Multidegrees are a convenient reformulation of equivariant cohomology in our setting;² equivalently, they can be thought of as equivariant volumes up to overall normalization.) This statement will be made more precise later (see Theorem 5.1). The proof involves the detailed analysis of the action of certain SL_2 subgroups on the Brauer loop schemes.

An interesting feature is that in type \hat{C} , it is not possible to solve the qKZ equation explicitly in order to exhibit a polynomial solution (contrary to the case of type \hat{A} , see [9], where one can at least in principle compute it inductively). Therefore, the geometry provides an explicit solution of the qKZ problem, which by definition has all the desired properties (polynomiality in all variables, minimal degree).

The analysis of Section 5 is rather technical, and unfortunately, we have not been able to do without a case-by-case analysis depending on the boundary condition, a fully general approach (similar to the analysis of [19]) being outside the scope of this paper. We have therefore decided to give the full proofs only in types i, c. In addition, we have mentioned in parallel the case of type \hat{A} (denoted p for periodic), not only to summarize the main results of [16, 17] and for comparison purposes, but also because some of the proofs and results we give are new even in type \hat{A} .

The conclusion wraps up the proof of the main theorem and discusses *sum rules*. On the geometric side, these correspond to the multidegrees of the full Brauer loop schemes. (In fact, using a combination of flat deformation and equivariant localization, some formulae, albeit not particularly explicit, are already provided in Section 4.2.2). On the physical side, they are the normalization constants for the probabilities of the connectivity of boundary points on the half-infinite strip. Using recurrence relations, we provide alternative formulae for them as Pfaffians or determinants.

In the appendices we give some small size solutions of the qKZ system as well as some technical results that are needed in the proofs.

1.3. Dynkin diagrams. Since the models and geometry we consider are based on the affine Dynkin diagrams of type \hat{A} and \hat{C} , we briefly describe our conventions concerning these.

² Since our solution of the Yang–Baxter equation is *rational*, we obtain on the geometric side ordinary cohomology, as opposed to K -theory or elliptic cohomology.

First introduce the following notation: given an integer $L \geq 2$, define the group Γ acting on \mathbb{Z} by generators

- $i \mapsto i + L$ in type \widehat{A} ;
- $i \mapsto i + 2L, i \mapsto 2L - i + 1$ in type \widehat{C} .

Equivalence classes in \mathbb{Z}/Γ are canonically identified with edges in the Dynkin diagram of $\widehat{A}_{L-1}, \widehat{C}_L$, respectively. We denote by $\text{cl}(i)$ the class of $i \in \mathbb{Z}$ in \mathbb{Z}/Γ . The standard choice of representatives is $1, \dots, L$ (they also correspond to the choice of variables z_1, \dots, z_L in the weights, see below), and when there is no risk of confusion we identify such representatives i and $\text{cl}(i)$. For future purposes, also define the action of $\Gamma = \Gamma_e \cup \Gamma_o$ (where Γ_e are translations and Γ_o are reflections when acting on \mathbb{Z}) on $\mathbb{Z} \times \mathbb{Z}$ by $(i, j) \mapsto (\gamma(i), \gamma(j))$ if $\gamma \in \Gamma_e$, $(i, j) \mapsto (\gamma(j), \gamma(i))$ if $\gamma \in \Gamma_o$, and $\text{cl}(i, j)$ to be the class of (i, j) under this action.

We also choose the somewhat clumsy (but standard) convention to index a node by the edge to its left, except for the leftmost node of type \widehat{C} which is labelled 0, see Figure 1.

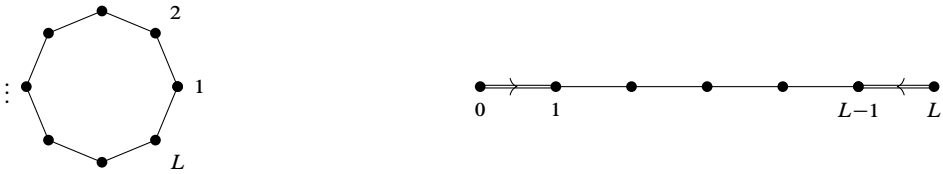


Figure 1. Dynkin diagrams for the affine \widehat{A}_{L-1} (left) and \widehat{C}_L (right) root systems.

Similarly, the root lattices are defined as follows. We start with a countable set of generators of the form s and $z_i, i \in \mathbb{Z}$. Then we take the quotient of the abelian group they generate with the relations (for all $i \in \mathbb{Z}$):

- $z_{i+L} = z_i + s$ in type \widehat{A}_{L-1} ;
- $z_{i+2L} = z_i + s, z_{2L-i+1} = -z_i$ in type \widehat{C} .

The result is isomorphic to \mathbb{Z}^{L+1} , a possible choice of generators being s, z_1, \dots, z_L .

In all types, the simple roots are then defined by $\alpha_i = z_i - z_{i+1}$, with i being the index of a node in the Dynkin diagram. More explicitly, and if we use the variables s, z_1, \dots, z_L , we have

- in type $\widehat{A}_{L-1}, \alpha_i = z_i - z_{i+1}$ for $1 \leq i \leq L - 1$, and $\alpha_L = z_L - z_1 - s$;
- in type $\widehat{C}_L, \alpha_i = z_i - z_{i+1}$ for $1 \leq i \leq L - 1, \alpha_0 = -2z_1 - s$ and $\alpha_L = 2z_L$.

We also need the commutative ring generated by s and z_i , $i \in \mathbb{Z}$, i.e., the quotient of $\mathbb{Z}[s, z_i, i \in \mathbb{Z}]$ with the same relations above. Reflections w.r.t. simple roots act on it in the following way (with the same choice of variables):

- in type \widehat{A}_{L-1} ,

$$\tau_i f(\dots, z_i, z_{i+1}, \dots) = f(\dots, z_{i+1}, z_i, \dots) \quad 1 \leq i \leq L-1,$$

$$\tau_L f(z_1, \dots, z_L) = f(z_L - s, \dots, z_1 + s);$$

- in type \widehat{C}_L

$$\tau_i f(\dots, z_i, z_{i+1}, \dots) = f(\dots, z_{i+1}, z_i, \dots) \quad 1 \leq i \leq L-1,$$

$$\tau_0 f(z_1, \dots) = f(-z_1 - s, \dots),$$

$$\tau_L f(\dots, z_L) = f(\dots, -z_L).$$

Finally, we define the divided difference operators, acting on functions of s, z_1, \dots, z_L :

$$\partial_i := \frac{1}{\alpha_i} (1 - \tau_i). \quad (1)$$

2. The quantum Knizhnik–Zamolodchikov equation

The qKZ equation³ that we will be studying is based on the Brauer algebra. We will look at five different boundary conditions, which we will refer to as periodic (p), closed (c), identified (i), open (o), and mixed (m). The first of these corresponds to the type \widehat{A} root system while the other four correspond to the type \widehat{C} root system. We state some previous results for periodic [5, 9] and closed [7], but for the other boundary conditions the results are original.

Our aim, as explained in the introduction, is to give a geometric meaning to the qKZ solutions of type \widehat{C} , as was done for the periodic case in [5, 9, 16, 17], however we will include the periodic case in all our statements in order to make references and comparisons to it.

³ also called “difference Knizhnik–Zamolochikov equation” because it is naturally expressed in terms of the *additive* spectral parameter. This is due to the fact that our solution of the Yang–Baxter equation is *rational*.

By the relations (2) the R -matrices satisfy unitarity

$$\check{R}_i(z)\check{R}_i(-z) = 1, \tag{4}$$

and the Yang–Baxter equation (YBE)

$$\check{R}_i(z)\check{R}_{i+1}(z+w)\check{R}_i(w) = \check{R}_{i+1}(w)\check{R}_i(z+w)\check{R}_{i+1}(z). \tag{5}$$

The R -matrix also has the important property

$$\check{R}_i(A) = \frac{1}{\beta}e_i. \tag{6}$$

An important remark is that equations (4) and (5) are nothing but the relations of the symmetric group for the operator $\tau_i \check{R}_i(z_i - z_{i+1})$. In the two paragraphs that follow, we shall extend the Brauer algebra (i.e., make a choice of boundary conditions for the loop model) in order to obtain relations of the *affine* Weyl groups \widehat{A} or \widehat{C} .

2.1.1. Type \widehat{A} . The periodic Brauer algebra has two additional generators, e_L and f_L , which act between sites L and 1 and satisfy all the relations (2) under the identification $L + 1 \equiv 1$. The graphical depiction is the natural analogue of the one for the ordinary Brauer algebra, where diagrams are drawn on a (periodic) strip.

One may wish to add the following relations involving idempotent elements I_1 and I_2 :

$$\begin{aligned} I_1 I_2 I_1 &= \beta^2 I_1 & I_2 I_1 I_2 &= \beta^2 I_2, \\ I_1 &:= e_1 e_3 \dots e_{L-1}, & I_2 &:= e_2 e_4 \dots e_L, & L \text{ even}, \\ I_1 &:= e_1 e_3 \dots e_L, & I_2 &:= e_2 e_4 \dots e_{L-1}, & L \text{ odd}. \end{aligned}$$

(in particular, they will be satisfied in the representation below).

Defining the R -matrix using the same formula (3) for $i = 1, \dots, L$, equations (4) and (5) are satisfied with indices mod L , i.e., we obtain the relations of the affine Weyl group of type \widehat{A}_{L-1} .

The periodic Brauer algebra has a representation on the vector space with canonical basis indexed by *link patterns*. In the periodic case, the latter are chord diagrams that connect the L points around a circle in pairs, see Figure 2. If L is odd one site is left unpaired, referred to either as a fixed point or as a connection to infinity. We refer to the set of periodic link patterns of size L as LP_L^{P} , and it has $(2\lceil L/2 \rceil - 1)!!$ elements.

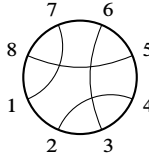
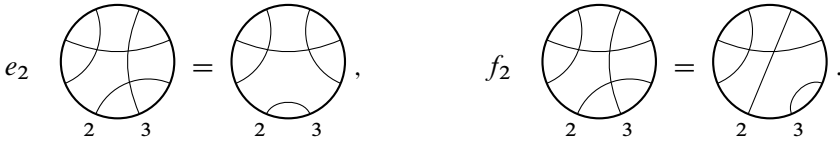


Figure 2. An example periodic link pattern for $L = 8$.

The Brauer generators act on the link patterns in the natural graphical way, by pasting the strip around the disk; for example,



2.1.2. Type \widehat{C} . Let us now add two more generators to the ordinary Brauer algebra, e_0 and e_L , with relations

$$e_0^2 = e_0, \quad e_L^2 = e_L, \tag{7a}$$

$$e_1 e_0 e_1 = e_1, \quad e_{L-1} e_L e_{L-1} = e_{L-1}, \tag{7b}$$

$$e_0 f_1 e_0 = e_0 e_1 e_0, \quad e_L f_{L-1} e_L = e_L e_{L-1} e_L. \tag{7c}$$

These do not appear in every version of the type \widehat{C} boundary conditions. In the closed case, we shall use neither, i.e., stick to the ordinary Brauer algebra; in the mixed case, we shall need the one-boundary Brauer algebra, i.e., add e_L ; and in the identified and open cases, we shall use the two-boundary Brauer algebra including both e_0 and e_L .

For identified and open boundaries we also have the idempotent relations

$$\begin{aligned} I_1 I_2 I_1 &= I_1 & I_2 I_1 I_2 &= I_2, \\ I_1 &:= e_0 e_2 \dots e_L, & I_2 &:= e_1 e_3 \dots e_{L-1}, & L \text{ even,} \\ I_1 &:= e_0 e_2 \dots e_{L-1}, & I_2 &:= e_1 e_3 \dots e_L, & L \text{ odd.} \end{aligned}$$

The type \widehat{C} link patterns are a string of sites numbered from 1 to L , connected to each other in pairs or (if allowed by the boundary conditions) to a boundary, or (in the odd size closed case) left unpaired. See Figure 3 for examples.



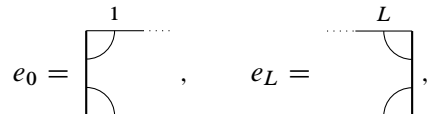
Figure 3. Example closed, identified, open and mixed link patterns for $L = 4$.

We can represent the link patterns by tuples of numbers, where the i th represents the site that site i is connected to, with l representing the left boundary, r representing the right, b representing the generic boundary, and \bullet representing the unpaired site (or connection to infinity) in the odd closed case. For example, the link pattern in Figure 2 is $(7, 4, 6, 2, 8, 3, 1, 5)$, and the link patterns in Figure 3 are $\{(3, 4, 1, 2), (3, b, 1, b), (3, r, 1, l), (3, r, 1, r)\}$. The link pattern sets and respective sizes are

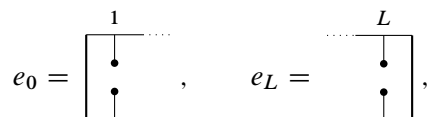
$$\begin{aligned}
 \text{LP}_L^i &: \sum_{j=0}^{\lfloor L/2 \rfloor} \binom{L}{2j} (2j - 1)!! \\
 \text{LP}_L^c &: (2\lceil L/2 \rceil - 1)!! \\
 \text{LP}_L^o &: \sum_{j=0}^{\lfloor L/2 \rfloor} 2^{L-2j} \binom{L}{2j} (2j - 1)!! \\
 \text{LP}_L^m &: \sum_{j=0}^{\lfloor L/2 \rfloor} \binom{L}{2j} (2j - 1)!! .
 \end{aligned}$$

We will sometimes refer to the number of chords in a link pattern: this refers to the number of links connecting sites to sites or sites to boundaries. It does not count the unpaired site in the odd closed or periodic cases. For example, the numbers of chords in the link patterns in Figure 3 are 2, 3, 3, and 3 respectively.

The standard graphical depiction for e_0 and e_L (used in the open and mixed cases) is



however for identified boundaries the graphical depiction is



where the dots signify a connection to a single generic boundary. In other words, identified and open Brauer algebras are the same, but we use different representations and therefore corresponding graphical depictions.

The boundary generators act on link pattern in a similar way to the bulk generators. The generator is placed on top of the link pattern, and a new link pattern is formed according to the resulting connections between the sites. For example, in the open case

$$e_0 \quad \begin{array}{|c|} \hline 1 \\ \hline \text{Diagram with arcs} \\ \hline \end{array} = \begin{array}{|c|} \hline 1 \\ \hline \text{Diagram with arcs} \\ \hline \end{array},$$

and in the identified case

$$e_L \quad \begin{array}{|c|} \hline L \\ \hline \text{Diagram with arcs} \\ \hline \end{array} = \begin{array}{|c|} \hline L \\ \hline \text{Diagram with arcs and dots} \\ \hline \end{array}.$$

We now define the K -matrices, graphically denoted by

$$\check{K}_0(w) =: \begin{array}{c} \text{Diagram with arcs and dot} \\ \text{Labels } w-s/2, -w-s/2 \end{array}, \quad \check{K}_L(w) =: \begin{array}{c} \text{Diagram with arcs and dot} \\ \text{Labels } -w, w \end{array}.$$

Let

$$k(w) := (A + 2w).$$

For a given boundary condition, if e_0 or e_L exists we define the K -matrix to be

$$\check{K}_{0,L}(w) := \frac{(A - 2w)}{k(w)} + \frac{4w}{k(w)} e_{0,L}. \tag{8}$$

If the boundary generator doesn't exist, we define the K -matrix to be the identity. With these two possible definitions, and by (7), the K -matrices satisfy unitarity

$$\check{K}_0(w)\check{K}_0(-w) = \check{K}_L(w)\check{K}_L(-w) = 1,$$

and the boundary Yang–Baxter equation (reflection equation)

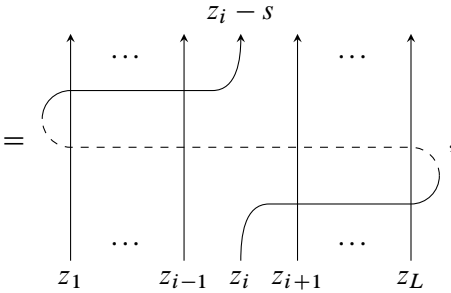
$$\begin{aligned} &\check{K}_0(z)\check{R}_1(z+w)\check{K}_0(w)\check{R}_1(w-z) \\ &= \check{R}_1(w-z)\check{K}_0(w)\check{R}_1(z+w)\check{K}_0(z), \\ &\check{K}_L(z)\check{R}_{L-1}(z+w)\check{K}_L(w)\check{R}_{L-1}(w-z) \\ &= \check{R}_{L-1}(w-z)\check{K}_L(w)\check{R}_{L-1}(z+w)\check{K}_L(z). \end{aligned}$$

Again, these relations are simply those satisfied by boundary generators of the affine Weyl group \widehat{C}_L .

2.2. The quantum Knizhnik–Zamolodchikov equation

2.2.1. Type $\hat{\mathbf{A}}$. For each i we define the scattering matrix

$$S_i := \check{R}_{i-1}(z_i - z_{i-1} - s) \dots \check{R}_1(z_i - z_1 - s) \sigma^{-1} \check{R}_{L-1}(z_i - z_L) \dots \check{R}_i(z_i - z_{i+1})$$



where σ is an operator that rotates a link pattern by one clockwise step,⁴ and s is a new (nonzero) parameter. The quantum Knizhnik–Zamolodchikov (qKZ) equation is then [10]

$$S_i |\Psi(\dots, z_i, \dots)\rangle = |\Psi(\dots, z_i - s, \dots)\rangle, \tag{9}$$

where $|\Psi\rangle$ is a vector belonging to the space spanned by LP_L^p ,

$$|\Psi(z_1, \dots, z_L)\rangle = \sum_{\pi \in LP_L^p} \psi_\pi(z_1, \dots, z_L) |\pi\rangle.$$

We will sometimes include a subscript to indicate the size L of the system if necessary. Here we are interested in a stronger version, called the qKZ system, which is the following system:

$$\check{R}_i(z_i - z_{i+1}) |\Psi(\dots, z_i, z_{i+1}, \dots)\rangle = |\Psi(\dots, z_{i+1}, z_i, \dots)\rangle, \quad 1 \leq i \leq L - 1, \tag{10}$$

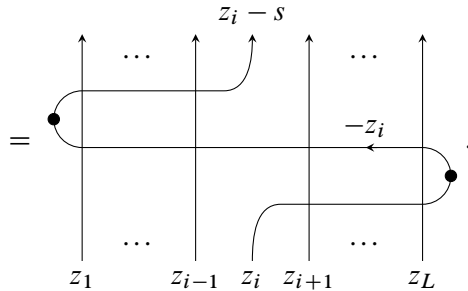
$$\sigma |\Psi(z_1, \dots, z_L)\rangle = |\Psi(z_2, \dots, z_L, z_1 + s)\rangle. \tag{11}$$

It is easy to show that (10)–(11) implies (9), though the converse is in general not true. We refer to (10) as exchange relations, and (11) as the rotation equation.

⁴ Adding σ , rather than simply the $\tau_i \check{R}_i$, corresponds to considering the *extended* affine Weyl group of type $\hat{\mathbf{A}}_{L-1}$.

2.2.2. Type \widehat{C} . Cherednik considered generalizations of the qKZ equation to other types [2]. For type \widehat{C} there is no longer a need for σ , but instead we use the boundary operators. The scattering matrix is defined for each i as

$$S_i := \check{R}_{i-1}(z_i - z_{i-1} - s) \dots \check{R}_1(z_i - z_1 - s) \\ \check{K}_0(z_i - s/2) \check{R}_1(z_1 + z_i) \dots \check{R}_{L-1}(z_L + z_i) \\ \check{K}_L(z_i) \check{R}_{L-1}(z_i - z_L) \dots \check{R}_i(z_i - z_{i+1})$$



The qKZ equation is then as before

$$S_i |\Psi(\dots, z_i, \dots)\rangle = |\Psi(\dots, z_i - s, \dots)\rangle, \tag{12}$$

where

$$|\Psi(z_1, \dots, z_L)\rangle = \sum_{\pi \in \text{LP}_L^a} \psi_\pi(z_1, \dots, z_L) |\pi\rangle, \quad a \in \{i, c, o, m\},$$

and the qKZ system is

$$\check{R}_i(z_i - z_{i+1}) |\Psi(\dots, z_i, z_{i+1}, \dots)\rangle = |\Psi(\dots, z_{i+1}, z_i, \dots)\rangle, \quad 1 \leq i \leq L - 1, \tag{13}$$

$$\check{K}_0(-z_1 - s/2) |\Psi(z_1, \dots, z_L)\rangle = |\Psi(-z_1 - s, \dots, z_L)\rangle, \tag{14}$$

$$\check{K}_L(z_L) |\Psi(z_1, \dots, z_L)\rangle = |\Psi(z_1, \dots, -z_L)\rangle, \tag{15}$$

which implies the qKZ equation. We will refer to (14) and (15) as boundary exchange relations.

2.3. Solution. In what follows we shall be interested in solutions of the systems (10)–(11) and (13)–(15) which are *polynomials* in their arguments z_1, \dots, z_L as well as A and ϵ . It is easily seen that these polynomials will be homogeneous. The shift parameter is taken to be $s^{p,i,c} = \epsilon$ or $s^{o,m} = \epsilon/2$, because as we shall show in Section 5, polynomial solutions exist at these values. We will also show that the minimal degree solution is unique up to a constant.

The qKZ system gives us a set of relationships between the components of $|\Psi\rangle$, as well as giving special linear factors and symmetries that must appear in some components based on the associated link pattern. We will explore these in detail in the next section. We will also show that any solution of the qKZ system must satisfy an infinite number of recurrence relations.

Remark. In the periodic and closed cases all the components can be written as divided difference operators acting on just one component, but in the other cases this is not possible. We will not use this approach, for more details see [7, 9].

2.3.1. Factors and symmetries. Here we will list all the symmetries and factors dictated by the qKZ system, both for the bulk and the boundaries. Note that for the periodic model only the bulk rules apply, and the rotation equation (11) is an extra restriction.

First we define the modified divided difference operator for $0 < i < L$, with $\alpha_i = z_i - z_{i+1}$ as in Section 1.3,

$$\partial'_i := (A + \alpha_i)\partial_i \frac{1}{A + \alpha_i}.$$

The rules obtained from the qKZ system (13)–(15) are listed below for a component ψ_π corresponding to a link pattern π . Again, the rules for the boundaries only apply if there is a non-trivial K -matrix, since if the K -matrix is trivial the boundary exchange relation merely implies a symmetry.

FOR $i = 0$.

- i. If $\pi(1) \neq l, b$, then there is no link pattern ρ for which $e_0 |\rho\rangle = |\pi\rangle$. So we have

$$\partial_0 \frac{\psi_\pi}{k(-z_1 - s/2)} = 0,$$

implying that $\psi_\pi = k(-z_1 - s/2)S_\pi^0$, where S_π^0 is a polynomial in z_1, \dots, z_L that is invariant under $(z_1 + s/2) \leftrightarrow (-z_1 - s/2)$.

- ii. Otherwise there is a small link from site 1 to the left boundary (or the generic boundary if identified), and the boundary exchange relation gives us the relationship

$$k(-z_1 - s/2)(-\partial_0)\psi_\pi = 2 \sum_{\rho \neq \pi: e_0|\rho\rangle=|\pi\rangle} \psi_\rho. \tag{16}$$

FOR $i = L$.

i. If $\pi(L) \neq r, b$, then there is no ρ for which $e_L |\rho\rangle = |\pi\rangle$. So we have

$$\partial_L \frac{\psi_\pi}{k(z_L)} = 0,$$

implying that $\psi_\pi = k(z_L)S_\pi^L$, where S_π^L is a polynomial in z_1, \dots, z_L that is even in z_L .

ii. Otherwise there is a small link from site L to the right boundary (or the generic boundary if identified), and the boundary exchange relation gives us the relationship

$$k(z_L)(-\partial_L)\psi_\pi = 2 \sum_{\rho \neq \pi: e_L |\rho\rangle = |\pi\rangle} \psi_\rho. \quad (17)$$

FOR GENERAL i , $0 < i < L$.

i. If $\pi(i) \neq i + 1$, then there is no ρ for which $e_i |\rho\rangle = |\pi\rangle$. So we have

$$(2A - z_i + z_{i+1} - \epsilon)(-\partial'_i)\psi_\pi = \psi_\pi + \psi_{f_i \pi}, \quad (18a)$$

$$(2A - z_i + z_{i+1} - \epsilon)(-\partial'_i)\psi_{f_i \pi} = \psi_\pi + \psi_{f_i \pi}. \quad (18b)$$

Specializing the coefficient of $|\pi\rangle$ in the i th exchange relation to

$$z_i = A + z_{i+1}$$

gives

$$\tau_i \psi_\pi \Big|_{z_i = A + z_{i+1}} = 0,$$

implying that ψ_π (and equivalently, $\psi_{f_i \pi}$) contains a factor of $(A + z_i - z_{i+1})$.

Additionally, if $(\pi(i), \pi(i + 1)) = (l, l), (r, r)$ or (b, b) , then $f_i |\pi\rangle = |\pi\rangle$, and (18) becomes

$$\partial_i \frac{\psi_\pi}{r(z_i - z_{i+1})} = 0,$$

implying that $\psi_\pi = r(z_i - z_{i+1})S_\pi^i$, where S_π^i is a polynomial in z_1, \dots, z_L that is symmetric in z_i and z_{i+1} .

- ii. Otherwise there is a small link from site i to site $i + 1$, and the i th exchange relation gives us the relationship

$$r(z_i - z_{i+1})(-\partial_i)\psi_\pi = (2A - \epsilon) \sum_{\rho \neq \pi: e_i|\rho|=|\pi} \psi_\rho. \quad (19)$$

2.3.2. Maximally factorized components. The rules above give many linear factors for certain components, some of which are a result of the symmetry conditions. In every case except closed, there are some components for which there are a quadratic (in L) number of these small factors. We refer to these as the maximally factorized components. In the even periodic case, there is one such component, from which all the others can be determined by means of the qKZ system (see however our remark at the start of Section 2.3). In the open, identified and odd periodic cases there is more than one maximally factorized component.

We use ψ_Ω to refer to the maximally factorized components, and they are labelled by the following link patterns:

$$\begin{aligned} \Omega^p &:= (L/2 + 1, \dots, L, 1, \dots, L/2), & L \text{ even,} \\ \Omega_k^p &:= \sigma^k ((L + 1)/2, \dots, L - 1, 1, \dots, (L - 1)/2, \bullet), & L \text{ odd,} \\ \Omega_1^i &:= (b, \dots, b), \quad \Omega_2^i := (L, b, \dots, b, 1), \\ \Omega_k^o &:= (\underbrace{r, \dots, r}_k, \underbrace{l, \dots, l}_{L-k}), & k = 0, \dots, L, \\ \Omega^m &:= (r, \dots, r). \end{aligned}$$

The explicit formulas for the components are (the periodic case comes from [9]; only $\psi_{\Omega_0^p}^L$ is given for odd size, the others can be obtained by application of (11)):

$$\psi_{\Omega^p}^L = \prod_{\substack{1 \leq i < j \leq L \\ j-i < L/2}} (A + z_i - z_j) \prod_{\substack{1 \leq i < j \leq L \\ j-i > L/2}} (A - z_i + z_j - s) S_{\Omega^p}^\emptyset(z_1, \dots, z_L), \quad (20a)$$

$$\begin{aligned} \psi_{\Omega_0^p}^L &= \prod_{\substack{1 \leq i < j \leq L \\ j-i < (L-1)/2 \text{ or } i > (L-1)/2}} (A + z_i - z_j) \\ &\quad \prod_{\substack{1 \leq i < j \leq L \\ j-i > (L-1)/2}} (A - z_i + z_j - s) S_{\Omega_0^p}^\emptyset(z_1, \dots, z_L), \end{aligned} \quad (20b)$$

$$\psi_{\Omega_1^i}^L = 2^L \prod_{1 \leq i < j \leq L} r(z_i - z_j) S_{\Omega_1^i}^{\{1, \dots, L-1\}}(z_1, \dots, z_L), \quad (20c)$$

$$\begin{aligned} \psi_{\Omega_2^L}^L &= 2^{L-1} k(-z_1 - s/2)k(z_L) \prod_{2 \leq i < j \leq L-1} r(z_i - z_j) \\ &\quad \prod_{i=2}^{L-1} ((A + z_1 - z_i)(A + z_i - z_L) \\ &\quad (A - z_1 - z_i - s)(A + z_i + z_L)) S_{\Omega_2^L}^{\{0,2,\dots,L-2,L\}}(z_1, \dots, z_L), \end{aligned} \quad (20d)$$

$$\begin{aligned} \psi_{\Omega_k^L}^L &= 2^L \prod_{i=1}^k k(-z_i - s/2) \prod_{i=k+1}^L k(z_i) \\ &\quad \prod_{1 \leq i < j \leq k} r(z_i - z_j)r(-z_i - z_j - s) \prod_{k+1 \leq i < j \leq L} r(z_i - z_j)r(z_i + z_j) \\ &\quad \prod_{i=1}^k \prod_{j=k+1}^L ((A + z_i - z_j)(A - z_i - z_j - s) \\ &\quad (A + z_i + z_j)(A - z_i + z_j - s)) \\ &\quad S_{\Omega_k^L}^{\{0,\dots,k-1,k+1,\dots,L\}}(z_1, \dots, z_L), \end{aligned} \quad (20e)$$

$$\psi_{\Omega^m}^L = 2^L \prod_{1 \leq i < j \leq L} r(z_i - z_j)r(-z_i - z_j - s) S_{\Omega^m}^{\{0,\dots,L-1\}}(z_1, \dots, z_L), \quad (20f)$$

where the S functions are polynomials whose superscripts denote their symmetries as defined in the previous section.

2.3.3. Recurrence relations. Here we will describe recurrence relations that are satisfied by solutions to the qKZ system (10)–(11) or (13)–(15). The first proposition describes a ‘bulk’ recurrence relation, which involves setting one of the variables to be dependent on another, and the second considers a ‘boundary’ recurrence relation, which is only valid in cases with a nontrivial boundary, and involves setting one variable to a constant. Recall that we have set $s^{p,i,c} = \epsilon$ and $s^{o,m} = \epsilon/2$.

The proofs of the two propositions depend on the following two lemmas.

Lemma 2.1. *If, for a polynomial vector $|\Phi^{(1)}\rangle$, some integer $0 < i < L$, and a polynomial f which is coprime to $\tau_i f$ and does not contain the factor $r(z_{i+1} - z_i)$, we have*

$$\check{R}_i(z_i - z_{i+1})|\Phi^{(1)}\rangle = \frac{f}{(\tau_i f)} \tau_i |\Phi^{(1)}\rangle,$$

then $|\Phi^{(1)}\rangle = f|\Phi^{(2)}\rangle$ where $|\Phi^{(2)}\rangle$ is a polynomial vector that satisfies the i th exchange relation.

If in addition $|\Phi^{(1)}\rangle$ satisfies the k th exchange relation for some $k \neq i$, then

$$\check{R}_k(z_k - z_{k+1})|\Phi^{(2)}\rangle = \frac{(\tau_k f)}{f} \tau_k |\Phi^{(2)}\rangle.$$

Thus if f does not contain the factor $r(z_k - z_{k+1})$, $|\Phi^{(1)}\rangle$ also contains any factor of $\tau_k f$ that is not in f .

Remark. An equivalent statement can be made for the K -matrices.

Proof. Straightforward, using polynomiality of the vectors and the fact that the only denominator in the equations which is not explicit is that of the R -matrix. \square

Lemma 2.2. We define the following operators on link patterns: Let φ_i acting on a link pattern insert a small loop from site i to $i + 1$ while increasing the size of the link pattern by 2, and let $\tilde{\varphi}_0$ (resp. $\tilde{\varphi}_L$) insert a small loop from the first (resp. last) site to the left (resp. right) boundary, which increases the size of the link pattern by 1.

We have the following identities:

$$\begin{aligned} & \check{R}_{j-1}(z_j - z_{j+2})\check{R}_j(A + z_j - z_{j+2}) \\ & \check{R}_{j+1}(z_{j-1} - z_{j+2})\check{R}_j(z_{j-1} - (A + z_j)) \\ & \check{R}_{j-1}(z_{j-1} - z_j)\varphi_j \\ & = \frac{r(A + z_j - z_{j-1})r(z_{j+2} - z_j)}{r(z_{j-1} - z_j)r(A + z_j - z_{j+2})} \varphi_j \circ \check{R}_{j-1}(z_{j-1} - z_{j+2}); \end{aligned} \quad (21)$$

$$\check{R}_j(z_{j-1} - (A + z_j))\check{R}_{j-1}(z_{j-1} - z_j)\varphi_j = \frac{r(A + z_j - z_{j-1})}{r(z_{j-1} - z_j)} \varphi_{j-1}; \quad (22)$$

$$\begin{aligned} & \check{K}_0(-A - z_1 - s/2)\check{R}_1(-A - 2z_1 - s)\check{K}_0(-z_1 - s/2) \varphi_1 \\ & = \frac{k(A + z_1 + s/2)k(-z_1 - s/2 - \epsilon/2)}{k(-z_1 - s/2)k(A + z_1 + s/2 - \epsilon/2)} \varphi_1; \end{aligned} \quad (23)$$

$$\begin{aligned} & \check{K}_L(z_{L-1})\check{R}_{L-1}(A + 2z_{L-1})\check{K}_L(A + z_{L-1}) \varphi_{L-1} \\ & = \frac{k(-z_{L-1})k(A + z_{L-1} - \epsilon/2)}{k(A + z_{L-1})k(-z_{L-1} - \epsilon/2)} \varphi_{L-1}; \end{aligned} \quad (24)$$

$$\begin{aligned} & \check{R}_1(-A/2 - z_2 - s/2) \check{K}_0(-z_2 - s/2) \check{R}_1(A/2 - z_2 - s/2) \tilde{\varphi}_0 \\ &= \frac{k(A + z_2 + s/2)k(A - z_2 - s/2 - \epsilon)}{k(A - z_2 - s/2)k(A + z_2 + s/2 - \epsilon)} \tilde{\varphi}_0 \circ \check{K}_0(-z_2 - s/2); \end{aligned} \quad (25)$$

$$\begin{aligned} & \check{R}_{L-1}(-A/2 + z_{L-1}) \check{K}_L(z_{L-1}) \check{R}_1(A/2 + z_{L-1}) \tilde{\varphi}_L \\ &= \frac{k(A - z_{L-1})k(A + z_{L-1} - \epsilon)}{k(A + z_{L-1})k(A - z_{L-1} - \epsilon)} \tilde{\varphi}_L \circ \check{K}_L(z_{L-1}); \end{aligned} \quad (26)$$

where (23)–(26) are only true in the boundary cases where the relevant K -matrix is nontrivial.

Proof. These are easily proved by the definitions, in the same way as the Yang–Baxter and boundary Yang–Baxter equations. \square

We will also repeatedly use the fact $r(A + z) = r(-z - \epsilon)$.

Proposition 2.3. *Given a polynomial solution $|\Psi_L\rangle$ of the q KZ system for size L , we can construct a polynomial solution $|\Psi_{L-2}\rangle$ of the q KZ system for size $L - 2$ by taking out any two neighbouring sites, by*

$$\begin{aligned} & |\Psi_L(z_1, \dots, z_j, A + z_j, \dots, z_L)\rangle \\ &= p_j(z_j | z_1, \dots, \hat{z}_j, \hat{z}_{j+1}, \dots, z_L) \varphi_j |\Psi_{L-2}(z_j | z_1, \dots, \hat{z}_j, \hat{z}_{j+1}, \dots, z_L)\rangle, \end{aligned} \quad (27)$$

where the notation \hat{z}_i means that z_i is missing from the list of arguments. The proportionality factors for different boundary conditions are

$$p_j^p(z_j | \dots, \hat{z}_j, \hat{z}_{j+1}, \dots) = 2 \prod_{i < j} r(z_i - z_j) \prod_{i > j+1} r(A + z_j - z_i), \quad (28a)$$

$$\begin{aligned} & p_j^i(z_j | \dots, \hat{z}_j, \hat{z}_{j+1}, \dots) \\ &= 2 k(-z_j - \epsilon/2)k(A + z_j) \prod_{i < j} r(z_i - z_j) \prod_{i > j+1} r(A + z_j - z_i) \\ & \quad \prod_{i \neq j, j+1} r(A + z_i + z_j), \end{aligned} \quad (28b)$$

$$\begin{aligned} & p_j^c(z_j | \dots, \hat{z}_j, \hat{z}_{j+1}, \dots) \\ &= 2 \prod_{i < j} r(z_i - z_j) \prod_{i > j+1} r(A + z_j - z_i) \prod_{i \neq j, j+1} r(A + z_i + z_j), \end{aligned} \quad (28c)$$

$$\begin{aligned}
& p_j^0(z_j | \dots, \hat{z}_j, \hat{z}_{j+1}, \dots) \\
&= 2(A - \epsilon/2)(2A - \epsilon/2)k(-z_j - \epsilon/2)k(A + z_j)k(-z_j - \epsilon/4) \\
&\quad k(A + z_j - \epsilon/4) \prod_{i < j} r(z_i - z_j) \prod_{i > j+1} r(A + z_j - z_i) \\
&\quad \prod_{i \neq j, j+1} r(A + z_i + z_j)r(z_i - z_j - \epsilon/2)r(-z_j - z_i - \epsilon/2),
\end{aligned} \tag{28d}$$

$$\begin{aligned}
& p_j^m(z_j | \dots, \hat{z}_j, \hat{z}_{j+1}, \dots) \\
&= 2(A - \epsilon/2)(2A - \epsilon/2)k(-z_j - \epsilon/2)k(A + z_j) \\
&\quad \prod_{i < j} r(z_i - z_j) \prod_{i > j+1} r(A + z_j - z_i) \\
&\quad \prod_{i \neq j, j+1} r(A + z_i + z_j)r(z_i - z_j - \epsilon/2)r(-z_j - z_i - \epsilon/2).
\end{aligned} \tag{28e}$$

(The constant factors are included for technical reasons that will be explained later.)

Proof of Proposition 2.3. First we note that the j th exchange relation implies

$$|\Psi_L(z_{j+1} = A + z_j)\rangle = \varphi_j |\Psi_{L-2,j}^{(1)}(z_j | z_1, \dots, z_{j-1} | z_{j+2}, \dots, z_L)\rangle, \tag{29}$$

for some vector in the space of link patterns of size $L-2$. We note that the exchange relations for $i \neq j-1, j, j+1$ are still valid for this new vector $|\Psi_{L-2,j}^{(1)}\rangle$.

We will drop the subscript $L-2$ here. Applying both sides of (21) to $|\Psi_j^{(1)}\rangle$, we have via (29)

$$\varphi_j \tilde{\tau}_{j-1} |\Psi_j^{(1)}\rangle = \frac{r(A + z_j - z_{j-1})r(z_{j+2} - z_j)}{r(z_{j-1} - z_j)r(A + z_j - z_{j+2})} \check{R}_{j-1}(z_{j-1} - z_{j+2}) \varphi_j |\Psi_j^{(1)}\rangle,$$

where $\tilde{\tau}_{j-1}$ swaps z_{j-1} and z_{j+2} . We define φ_j^\dagger as an upside-down loop between sites j and $j+1$, so that $\varphi_j \varphi_j^\dagger = e_j$ and $\varphi_j^\dagger \varphi_j = \beta$. We can then multiply by φ_j^\dagger and use Lemma 2.1 to get

$$|\Psi_j^{(1)}\rangle = \prod_{i < j} r(z_i - z_j) \prod_{i > j+1} r(A + z_j - z_i) |\Psi_j^{(2)}\rangle, \tag{30}$$

with $|\Psi_j^{(2)}\rangle$ satisfying the exchange relations

$$\begin{aligned}
& \check{R}_i(z_i - z_{i+1}) |\Psi_j^{(2)}\rangle = \tau_i |\Psi_j^{(2)}\rangle \quad i \neq j-1, j, j+1, \\
& \check{R}_{j-1}(z_{j-1} - z_{j+2}) |\Psi_j^{(2)}\rangle = \tilde{\tau}_{j-1} |\Psi_j^{(2)}\rangle.
\end{aligned}$$

By applying $\check{R}_j(z_{j-1} - (z_j + A))\check{R}_{j-1}(z_{j-1} - z_j)$ to $|\Psi_L(z_{j+1} = A + z_j)\rangle$, and using (22) and (29), we have

$$\begin{aligned} & |\Psi_j^{(2)}(z_j|z_1, \dots, z_{j-1}|z_{j+2}, \dots, z_L)\rangle \\ &= |\Psi_{j-1}^{(2)}(z_j|z_1, \dots, z_{j-2}|z_{j-1}, z_{j+2}, \dots, z_L)\rangle. \end{aligned}$$

In other words, $|\Psi_j^{(2)}\rangle$ and $|\Psi_k^{(2)}\rangle$ are related by an obvious rearrangement of arguments. Thus we can drop the subscript j and simplify the argument notation: $|\Psi^{(2)}(z_j|z_1, \dots, z_{j-1}, z_{j+2}, \dots, z_L)\rangle$.

Finally we consider the different boundary conditions separately.

- **PERIODIC.** The rotation equation for $|\Psi_L\rangle$ at $z_{j+1} = A + z_j$ leads to the rotation equation for $|\Psi^{(2)}\rangle$, so $|\Psi^{(2)}\rangle$ is a solution to the qKZ system of size $L - 2$. Thus $|\Psi_{L-2}\rangle = |\Psi^{(2)}\rangle$ and the proportionality factor (30) is the same as in (28).
- **CLOSED AND IDENTIFIED.** From (14) we have

$$\begin{aligned} & \check{K}_0(-z_1 - s/2)r(z_1 - z_j)|\Psi^{(2)}(z_j|\dots)\rangle \\ &= r(-z_1 - z_j - s)\tau_0|\Psi^{(2)}(z_j|\dots)\rangle, \quad j > 1, \end{aligned} \tag{31a}$$

$$\begin{aligned} & \check{K}_L(z_L)r(A + z_j - z_L)|\Psi^{(2)}(z_j|\dots)\rangle \\ &= r(A + z_j + z_L)\tau_L|\Psi^{(2)}(z_j|\dots)\rangle, \quad j < L - 1. \end{aligned} \tag{31b}$$

This implies via the K -matrix version of Lemma 2.1 that

$$|\Psi^{(2)}(z_j|\dots)\rangle = \prod_{i \neq j, j+1} r(A + z_i + z_j)|\Psi^{(3)}(z_j|\dots)\rangle, \quad \text{for all } j.$$

Applying (23) to $|\Psi^{(1)}\rangle$ leads to

$$|\Psi^{(3)}(-A - z_1 - \epsilon|z_3, \dots)\rangle = \frac{k(A + z_1 + \epsilon/2)k(-z_1 - \epsilon)}{k(-z_1 - \epsilon/2)k(A + z_1)}|\Psi^{(3)}(z_1|z_3, \dots)\rangle,$$

implying

$$|\Psi^{(3)}(z_j|\dots)\rangle = k(-z_j - \epsilon/2)k(A + z_j)|\Psi^{(4)}(z_j|\dots)\rangle.$$

A similar argument can be made for $j = L - 1$ using (24) but this results in the same factors.

Now $|\Psi_{L-2}\rangle = |\Psi^{(3)}\rangle$ for the closed case and $|\Psi_{L-2}\rangle = |\Psi^{(4)}\rangle$ for the identified case, and the proportionality factors are as given in (28).

- **MIXED AND OPEN.** We have again equations (31), but this time $s = \epsilon/2$ so they imply separate factors. Using Lemma 2.1 and its K -matrix version, we thus have

$$\begin{aligned} & |\Psi^{(2)}(z_j | \dots)\rangle \\ &= \prod_{i \neq j, j+1} r(A + z_i + z_j) r(-z_i - z_j - \epsilon/2) r(z_i - z_j - \epsilon/2) |\Psi^{(3)}(z_j | \dots)\rangle, \end{aligned}$$

for all j .

Again we can consider (23) and (24), and this time the two produce different factors, implying that for the mixed case

$$|\Psi^{(3)}(z_j | \dots)\rangle = k(-z_j - \epsilon/2) k(A + z_j) |\Psi^{(4)}(z_j | \dots)\rangle,$$

and for the open case,

$$\begin{aligned} & |\Psi^{(3)}(z_j | \dots)\rangle \\ &= k(-z_j - \epsilon/2) k(A + z_j) k(-z_j - \epsilon/4) k(A + z_j - \epsilon/4) |\Psi^{(4)}(z_j | \dots)\rangle. \end{aligned}$$

Now $|\Psi_{L-2}\rangle = |\Psi^{(4)}\rangle$ and the proportionality factors are as given in (28). \square

Note that the recurrence relation at j implies that if a link pattern does not have a small loop from j to $j + 1$, then the corresponding component in the solution of the $q\mathbf{KZ}$ system disappears when $z_{j+1} = A + z_j$, which is consistent with the factors found in Section 2.3.1. A similar statement will be true of the following proposition.

Proposition 2.4. *Given a polynomial solution $|\Psi_L\rangle$ of the type $\widehat{\mathbf{C}}$ $q\mathbf{KZ}$ system for size L , we can construct a polynomial solution $|\Psi_{L-1}\rangle$ of the $q\mathbf{KZ}$ system for size $L - 1$ by taking out the first or last site (iff the K -matrix at the chosen boundary is nontrivial), by*

$$|\Psi_L((A - s)/2, z_2, \dots, z_L)\rangle = p_0(z_2, \dots, z_L) \tilde{\varphi}_0 |\Psi_{L-1}(z_2, \dots, z_L)\rangle \quad (32)$$

and

$$|\Psi_L(z_1, \dots, z_{L-1}, -A/2)\rangle = p_L(z_1, \dots, z_{L-1}) \tilde{\varphi}_L |\Psi_{L-1}(z_1, \dots, z_{L-1})\rangle. \quad (33)$$

The proportionality factors for different boundary conditions are

$$p_0^i(z_2, \dots, z_L) = 2 \prod_{j=2}^L \frac{k(A - z_j - \epsilon/2)k(A + z_j - \epsilon/2)}{4}, \quad (34a)$$

$$p_0^o(z_2, \dots, z_L) = 2(2A - s) \prod_{j=2}^L \frac{\mathfrak{P}}{16}, \quad (34b)$$

where

$$\mathfrak{P} = k(A - z_j - \epsilon/4)k(A + z_j - 3\epsilon/4)k(A + z_j - \epsilon/4)k(A - z_j - 3\epsilon/4),$$

$$p_L^i(z_1, \dots, z_{L-1}) = 2 \prod_{j=1}^{L-1} \frac{k(A + z_j)k(A - z_j - \epsilon)}{4}, \quad (34c)$$

$$\begin{aligned} p_L^o(z_1, \dots, z_{L-1}) \\ = 2(2A - s) \prod_{j=1}^{L-1} \frac{k(A + z_j)k(A - z_j - \epsilon)k(A - z_j - \epsilon/2)k(A + z_j - \epsilon/2)}{16}, \end{aligned} \quad (34d)$$

$$\begin{aligned} p_L^m(z_1, \dots, z_{L-1}) \\ = 2 \prod_{j=1}^{L-1} \frac{k(A + z_j)k(A - z_j - \epsilon)k(A - z_j - \epsilon/2)k(A + z_j - \epsilon/2)}{16}. \end{aligned} \quad (34e)$$

(The constant factors are again included for technical reasons.)

Proof. The proof is very similar to the bulk case, so we will skip some details. Let us first consider the left boundary. The left boundary exchange relation implies

$$|\Psi_L(z_1 = (A - s)/2)\rangle = \tilde{\varphi}_0 |\Psi^{(1)}(\hat{z}_1)\rangle,$$

for some vector in the space of link patterns of size $L - 1$. Equation (25) leads by the K -matrix version of Lemma 2.1 to, for the identified case,

$$|\Psi^{(1)}\rangle = \prod_{j>1} k(A - z_j - \epsilon/2)k(A + z_j - \epsilon/2) |\Psi^{(2)}\rangle,$$

and for the open case,

$$|\Psi^{(1)}\rangle = \prod_{j>1} k(A - z_j - \epsilon/4)k(A + z_j - 3\epsilon/4)k(A + z_j - \epsilon/4)k(A - z_j - 3\epsilon/4) |\Psi^{(2)}\rangle.$$

Similarly, (26) leads to for the identified case

$$|\Psi^{(1)}\rangle = \prod_{j < L} k(A + z_j)k(A - z_j - \epsilon)|\Psi^{(2)}\rangle,$$

and for the open and mixed cases

$$|\Psi^{(1)}\rangle = \prod_{j < L} k(A + z_j)k(A - z_j - \epsilon)k(A - z_j - \epsilon/2)k(A + z_j - \epsilon/2)|\Psi^{(2)}\rangle.$$

In every case, $|\Psi_{L-1}\rangle = |\Psi^{(2)}\rangle$. □

2.3.4. Uniqueness of minimal degree solutions. We can show that any solution to the qKZ system satisfies an infinite number of recurrence relations. Choosing z_L as the specialization variable, via the qKZ system we can show that $|\Psi_L\rangle$ has two-site recurrences at the following points, where $1 \leq j \leq L - 1$ and t is a non-negative integer:

- for type \widehat{A} , $z_L = A + z_j - ts$ and $z_L = -A + z_j + (t + 1)s$;
- for type \widehat{C} , $z_L = \pm A \pm z_j \mp ts$ and $z_L = \pm A \mp z_j \mp (t + 1)s$;

and the type \widehat{C} cases have one site recurrences at the points:

- for i, o, $z_L = (\pm A \mp (2t + 1)s)/2$;
- for i, o, m, $z_L = -A/2 + ts$ and $z_L = A/2 - (t + 1)s$.

Two polynomial solutions of the qKZ system are proportional iff their proportionality factor is a constant, a fact which is a direct consequence of the exchange relations. However it is conceivable that this constant could include one of the extra variables from a larger solution, for example $|\Psi_1^0(z_1)\rangle$ could have an overall factor depending on z_2 . We will concentrate on solutions whose overall factor is a constant with respect to the variables of a system of any size. We call these minimal solutions.

The infinite number of recurrence relations and the requirement that the solution is polynomial indicates that the minimal solution is unique.

In each boundary case, we can calculate the minimal solution of the qKZ system of smallest meaningful size to provide a grounding for the recurrence relations. In p and c, because there is only a two-site recurrence, we take both $L = 1$ and $L = 2$. For i and m, we only need $L = 1$. In all of these cases the vector only has one element, which must therefore both be polynomial and satisfy $\psi(z_1) = \psi(z_1 + \epsilon)$, an impossible requirement unless ψ is in fact a constant. Thus for all these cases the smallest solution is a constant.

In o, the smallest meaningful system is $L = 1$, but $|\Psi_1^o(z_1)\rangle$ already has two components. However it is possible to show that the solution is linear, by assuming a polynomial form of arbitrary degree and trying to solve the system. The solution is (up to a constant)

$$\psi_\ell^o(z_1) = 2(A + 2z_1),$$

$$\psi_r^o(z_1) = 2(A - s - 2z_1).$$

The recurrence relations tell us the total degree (in the z_i as well as in A and ϵ , making the solutions homogeneous) of a solution of any qKZ system of size L given a solution of size $L - 2$ or $L - 1$. With the small size solutions given above, we thus know the degree of solutions to systems of all sizes:

Total degree	p	i	c	o	m
p_j	$2(L - 2)$	$2(2L - 3)$	$4(L - 2)$	$2(4L - 5)$	$4(2L - 3)$
$p_{0,L}$	-	$2(L - 1)$	-	$4L - 3$	$4(L - 1)$
$ \Psi_L\rangle$	$2\lfloor \frac{L}{2} \rfloor \lfloor \frac{L-1}{2} \rfloor$	$L(L - 1)$	$4\lfloor \frac{L}{2} \rfloor \lfloor \frac{L-1}{2} \rfloor$	$L(2L - 1)$	$2L(L - 1)$

Note that these degrees are exhausted by the factors listed in (20), thus all the unknown symmetric functions in those expressions must be constant.

3. The Brauer loop model

For this section we will set $\beta = 1$, which implies that $\epsilon = s = 0$. The Brauer loop model is a statistical mechanical model of crossing loops, based on the Brauer algebra. At this special point, the transfer matrix becomes stochastic and the ground state eigenvalue is 1. In this section we will show that the ground state eigenvector is a solution to the qKZ system at $\epsilon = s = 0$, and that the would-be unique minimal qKZ solution for general ϵ becomes the ground state eigenvector when $\epsilon \rightarrow 0$.

3.1. Definition and transfer matrix. The Brauer loop model is defined on a vertically semi-infinite square lattice, on a cylinder in type \widehat{A} or on a strip in type \widehat{C} . Loops are drawn on the faces in three possible ways, and the model is integrable via the Yang–Baxter equation when the probabilities of these configurations are

given by the (unchecked) R -matrix

$$\begin{aligned}
 R(w-z) &:= \frac{2A(A-w+z)}{r(w-z)} \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} + \frac{2A(w-z)}{r(w-z)} \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \\
 &+ \frac{(A-w+z)(w-z)}{r(w-z)} \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \\
 &=: \begin{array}{c} \uparrow \\ \square \\ \downarrow \\ z \end{array} \begin{array}{c} \leftarrow \\ \square \\ \rightarrow \\ w \end{array},
 \end{aligned}$$

with $r(z) = (A+z)(2A-z)$.

We describe the configuration probabilities of the smallest repeating element of the lattice by the transfer matrix $T(w|z_1, \dots, z_L)$, which acts on the vector space spanned by link patterns LP_N . We will define the transfer matrix explicitly in the following sections, separately for types \hat{A} and \hat{C} . The transfer matrix is stochastic, so the ground state eigenvalue is 1; one can check that in the physical range of parameters where Boltzmann weights are positive, the Perron–Frobenius theorem applies so that the associated eigenvector is unique up to normalization (so that this remains true for generic values of the parameters). We can also use the Yang–Baxter equation (and if necessary the boundary YBE) to show that two transfer matrices with different values of w commute, so the eigenvector does not depend on w . Additionally, the entries of the transfer matrix are homogeneous rational functions of w, z_i and A , thus the ground state can be normalized so that its entries are homogeneous polynomials in the z_i and A without a common factor. The ground state eigenvector equation is therefore

$$T(w|z_1, \dots, z_L) |\Psi(z_1, \dots, z_L)\rangle = |\Psi(z_1, \dots, z_L)\rangle,$$

and $|\Psi\rangle$ is written in the basis of link patterns as

$$|\Psi(z_1, \dots, z_L)\rangle = \sum_{\pi \in LP_N} \psi_\pi(z_1, \dots, z_L) |\pi\rangle,$$

where $\psi_\pi(z_1, \dots, z_L)$ are coprime polynomials.

From the YBE, the transfer matrix satisfies the interlacing relation

$$\check{R}_i(z_i - z_{i+1}) T(w|\dots, z_i, z_{i+1}, \dots) = T(w|\dots, z_{i+1}, z_i, \dots) \check{R}_i(z_i - z_{i+1}). \tag{35}$$

It also has the recurrence relation

$$T_L(w|z_1, \dots, z_i, A + z_i, \dots, z_L)\varphi_i = \varphi_i \circ T_{L-2}(w|z_1, \dots, \hat{z}_i, \hat{z}_{i+1}, \dots, z_L), \quad (36)$$

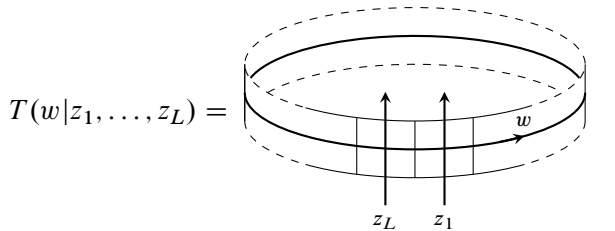
where φ_i is as defined in Proposition 2.3.

Each of the above statements is true for every boundary condition, but the proofs differ slightly. For further details see for example [7, 9].

3.1.1. Type \widehat{A} . The periodic Brauer loop model is drawn on a semi-infinite cylinder, of which the smallest repeating element is one row. The transfer matrix is therefore defined as

$$T(w|z_1, \dots, z_L) := \text{tr}_w (R(w - z_L) \dots R(w - z_1)),$$

which can be depicted graphically as



In addition to the properties already listed, this transfer matrix satisfies the rotation property

$$T(w|z_2, \dots, z_L, z_1) \sigma = \sigma T(w|z_1, \dots, z_L). \quad (37)$$

3.1.2. Type \widehat{C} . The Brauer loop model for type \widehat{C} is defined on a semi-infinite strip. At the boundaries the loop configurations are described by the (unchecked) K -matrices, which are depicted as

and defined as follows for the different boundary conditions, with $k(w) = A + 2w$:

Identified:

$$K_0^i(w) := \frac{-A + 2w}{k(A - w)} \left[\text{triangle with curved line} \right] + \frac{4(A - w)}{k(A - w)} \left[\text{triangle with two dots} \right],$$

$$K_L^i(w) := \frac{A - 2w}{k(w)} \left[\text{triangle with curved line} \right] + \frac{4w}{k(w)} \left[\text{triangle with two dots} \right];$$

Closed:

$$K_0^c(w) = K_L^c(w) := 1;$$

Open:

$$K_0^o(w) := \frac{-A + 2w}{k(A - w)} \left[\text{triangle with curved line} \right] + \frac{4(A - w)}{k(A - w)} \left[\text{triangle with horizontal line} \right],$$

$$K_L^o(w) := \frac{A - 2w}{k(w)} \left[\text{triangle with curved line} \right] + \frac{4w}{k(w)} \left[\text{triangle with horizontal line} \right];$$

Mixed:

$$K_0^m(w) := 1,$$

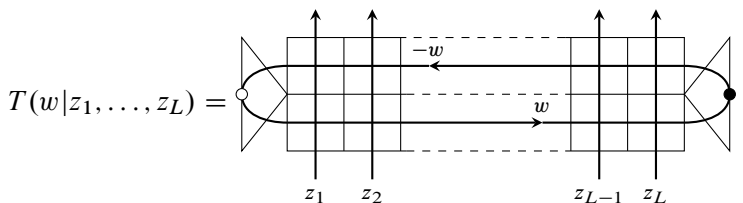
$$K_L^m(w) := \frac{A - 2w}{k(w)} \left[\text{triangle with curved line} \right] + \frac{4w}{k(w)} \left[\text{triangle with horizontal line} \right].$$

The transfer matrix for these models describes two rows of the lattice (the smallest repeating element),

$$T(w|z_1, \dots, z_L)$$

$$:= \text{tr}_w(K_0(w)R(z_1 + w) \cdots R(z_L + w)K_L(w)R(w - z_L) \cdots R(w - z_1)),$$

graphically depicted as



In addition to the interlacing condition (35), the boundary YBE implies the boundary interlacing conditions

$$\check{K}_0(-z_1)T(w|z_1, \dots) = T(w|-z_1, \dots)\check{K}_0(-z_1), \quad (39a)$$

$$\check{K}_L(z_L)T(w|\dots, z_L) = T(w|\dots, -z_L)\check{K}_L(z_L). \quad (39b)$$

In the cases where $K_{0,L} = 1$, these turn into symmetries. There are also boundary recurrence relations in addition to the bulk recurrence relation (36), which are only valid if the associated K -matrix is non-trivial:

$$T_L(w|A/2, z_2, \dots, z_L)\tilde{\varphi}_0 = \tilde{\varphi}_0 \circ T_{L-1}(w|z_2, \dots, z_L),$$

$$T_L(w|z_1, \dots, z_{L-1}, -A/2)\tilde{\varphi}_L = \tilde{\varphi}_L \circ T_{L-1}(w|z_1, \dots, z_{L-1}),$$

where $\tilde{\varphi}_0$ and $\tilde{\varphi}_L$ are defined as in Proposition 2.4.

3.2. Relationship between the loop model and the qKZ system. For the purposes of this section, we will use $|\Phi\rangle$ to denote a solution of the qKZ system with $s = 0$, and $|\Psi\rangle$ to denote the ground state of the Brauer loop model with loop weight $\beta = 1$.

Some of the statements made about $|\Phi\rangle$ in Section 2 are not valid at $s = 0$. In particular, the number of recurrence relations satisfied by the solution is no longer infinite – there are $2(L - 1)$ recurrence relations for the periodic case, $4(L - 1) + 2$ for identified, open and mixed, and $4(L - 1)$ for closed. Thus uniqueness of the solution is not guaranteed; indeed, any solution could be multiplied by a polynomial that has the symmetry of the appropriate Weyl group to make a new solution. Finally, we note that the prefactor in (21) is equal to 1, so the calculation of the proportionality factor is not valid.

However it is still true that any solution of the qKZ system at $s = 0$ has a recurrence to a smaller size solution. Thus there is a family of solutions to the qKZ system of different sizes, each of which can be obtained from a larger solution by recurrence. In this section we will show that any member of this family is a ground state of the Brauer loop model transfer matrix, and vice versa, and that there is a unique recursive family of solutions with coprime entries.

Further, any recursive family of polynomial solutions to the qKZ system for general s of the kind considered in Section 2, when taken at $s = 0$, will give the family of ground state eigenvectors of the Brauer transfer matrix, up to a symmetric factor.

Proposition 3.1. *The ground state of the Brauer transfer matrix with coprime entries is a solution to the qKZ system at $s = 0$.*

Proof for type \hat{A} . This proposition is also stated in [9]. We apply the interlacing condition (35) at any i to the eigenvector $|\Psi\rangle$,

$$\check{R}_i(z_i - z_{i+1}) |\Psi\rangle = T(w | \dots, z_{i+1}, z_i, \dots) \check{R}_i(z_i - z_{i+1}) |\Psi\rangle.$$

Since the eigenvector is unique, we can deduce that

$$\check{R}_i(z_i - z_{i+1}) |\Psi(\dots, z_i, z_{i+1}, \dots)\rangle = b_i(z_1, \dots, z_L) |\Psi(\dots, z_{i+1}, z_i, \dots)\rangle.$$

It is not hard to show that b is either 1 or $\frac{r(z_{i+1}-z_i)}{r(z_i-z_{i+1})}$, by the fact that the elements of $|\Psi\rangle$ are coprime polynomials. Similarly, using the rotation property (37) of the transfer matrix we find

$$\sigma |\Psi(z_1, \dots, z_L)\rangle \propto |\Psi(z_2, \dots, z_L, z_1)\rangle,$$

and the proportionality factor must be 1 by positivity of the ground state when the arguments are all set to 0. Thus $|\Psi\rangle$ satisfies (11).

Acting on $|\Psi\rangle$ with the scattering matrix S_1 , we thus find

$$S_1(z_1, \dots, z_L) |\Psi(z_1, \dots, z_L)\rangle = \prod_{j=1}^{L-1} b_j(z_1, \dots, z_L) |\Psi(z_1, \dots, z_L)\rangle.$$

But it is not hard to show that

$$S_i(z_1, \dots, z_L) = T(w = z_i | z_1, \dots, z_L), \tag{40}$$

which means that the product of b_j s should be equal to one, indicating that $b_j = 1$ for all j and showing that $|\Psi\rangle$ satisfies (10). \square

Proof for type \hat{C} . To show (40) for the identified and open cases we must note that while S_i uses the algebraic K -matrices defined in (8), T uses the graphical versions defined in (38). At the right boundary, we have $\check{K}_L(w) = K_L(w)$, but at the left boundary we need the relation

$$\check{K}_0(w) = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ \text{---} \end{array} \begin{array}{c} w \\ \text{---} \\ -w \end{array} = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \circ \\ \diagdown \quad \diagup \\ \text{---} \end{array} \begin{array}{c} w \\ \text{---} \\ -w \end{array}. \tag{41}$$

By applying the boundary interlacing conditions (39) to the eigenvector one can show that it satisfies the boundary exchange relations up to a proportionality factor, in the same way as for the bulk. Again we can act the scattering matrix S_1 on $|\Psi\rangle$ and show that the proportionality factors must all be 1. Thus $|\Psi\rangle$ satisfies (13)–(15). When the K -matrix is the identity the proof is trivial. \square

When $s = 0$ the qKZ equation becomes an invariance equation,

$$S_i^{(L)}(z_1, \dots, z_L) |\Phi_L(z_1, \dots, z_L)\rangle = |\Phi_L(z_1, \dots, z_L)\rangle \quad \text{for all } i. \quad (42)$$

Proposition 3.2. *Let $|\Phi_L\rangle$ and $|\Phi_{L-2}\rangle$ be solutions of (42) for sizes L and $L - 2$ respectively, such that*

$$|\Phi_L(\dots, z_i, A + z_i, \dots)\rangle = p_i \varphi_i |\Phi_{L-2}(\dots, \hat{z}_i, \hat{z}_{i+1}, \dots)\rangle, \quad \text{for all } i,$$

and such that $|\Phi_{L-2}\rangle$ has no overall symmetric factor. Then $|\Phi_{L-2}\rangle$ is a ground state eigenvector of the Brauer transfer matrix of size $L - 2$.

Proof. We consider the action of φ_{i-1}^\dagger on the scattering matrix when $z_{i-1} = -A + z_i$. For type \widehat{A} , by the property of the R -matrix (6), we have

$$\varphi_{i-1}^\dagger S_i(\dots, -A + z_i, z_i, \dots) \varphi_{i-1} = T(z_i | \dots, \hat{z}_{i-1}, \hat{z}_i, \dots).$$

For type \widehat{C} , this statement is also true, but to prove it we must again use (41). This is only necessary of course in the identified and open cases, where the left K -matrix is non-trivial.

Acting $\varphi_{i-1}^\dagger S_i(\dots, -A + z_i, z_i, \dots)$ on $|\Phi(\dots, -A + z_i, z_i, \dots)\rangle$ gives

$$T(z_i | \dots, \hat{z}_{i-1}, \hat{z}_i, \dots) |\Phi_{L-2}(\dots, \hat{z}_{i-1}, \hat{z}_i, \dots)\rangle = |\Phi_{L-2}(\dots, \hat{z}_{i-1}, \hat{z}_i, \dots)\rangle,$$

so that

$$|\Phi_{L-2}(\dots, \hat{z}_{i-1}, \hat{z}_i, \dots)\rangle \propto |\Psi_{L-2}(\dots, \hat{z}_{i-1}, \hat{z}_i, \dots)\rangle,$$

by uniqueness of the ground state. Since both $|\Phi\rangle$ and $|\Psi\rangle$ have no overall symmetric factors, and $|\Psi\rangle$ satisfies the qKZ system, they must be proportional by a constant. \square

3.3. Solution. The expressions for the maximally factorized components given in (20) apply to the Brauer loop model simply by setting $s = 0$. In addition, for $\epsilon = 0$ [7] gives a non-recursive expression for the component corresponding to the maximally crossing closed link pattern, which has $\pi(i) = i + L/2$. We will not need this expression.

3.3.1. Sum rules. Finally, we define the sum rule of the ground state as the sum of all its entries, noting again that the entries have been defined to be coprime (up to a constant factor that will be explained in Section 4.1.2),

$$Z_L^a := \sum_{\pi} \psi_{\pi}^a.$$

Because we have set $\epsilon = 0$, Z_L^a is a symmetric polynomial of the arguments z_1, \dots, z_L , and in type \hat{C} , an even polynomial in these variables. The proof is standard and we describe it briefly. First we write $Z_L^a = \langle v | \Psi^a \rangle$ where v is the covector with constant entries 1 in the basis of link patterns. Then we note that at $\epsilon = 0$, from (3), $\langle v | \check{R}_i(z) = \langle v |$, and similarly, from (8), $\langle v | \check{K}_{0,L}(z) = \langle v |$. We now apply (10) and conclude that τ_i leaves $\langle v | \Phi^a \rangle$ invariant for all $i = 0, \dots, L$, which is the desired symmetry.

This means that the recurrence relations satisfied by the components of the ground state extend to many more recurrence relations for Z_L :

- for all $i \neq j$,

$$Z_L^p(z_j = A + z_i) = p^p(z_i | \hat{z}_i, \hat{z}_j) Z_{L-2}^p(\hat{z}_i, \hat{z}_j);$$

- for all $i \neq j$, $a \in \{i, c, o, m\}$,

$$Z_L^a(z_j = A \pm z_i) = p^a(\pm z_i | \hat{z}_i, \hat{z}_j) Z_{L-2}^a(\hat{z}_i, \hat{z}_j);$$

- for all $i \neq j$, $a \in \{i, c, o, m\}$,

$$Z_L^a(z_j = -A \pm z_i) = p^a(\mp z_i | \hat{z}_i, \hat{z}_j) Z_{L-2}^a(\hat{z}_i, \hat{z}_j);$$

- for all i , $a \in \{i, o, m\}$,

$$Z_L^a(z_i = \pm A/2) = p^a(\hat{z}_i) Z_{L-1}^a(\hat{z}_i).$$

One of the results of this paper is explicit expressions for the i,o and m sum rules. These are given in Section 4.2.2 and Section 5.6, along with the p and c cases that have been done before [7, 9].

4. The Brauer loop schemes

Following [16], there are essentially two ways to define the Brauer loop schemes in various types: either in terms of infinite periodic matrices (i.e., loop algebras); or as flat limits of certain nilpotent orbit closures. We provide both interpretations below.

4.1. The infinite strip picture

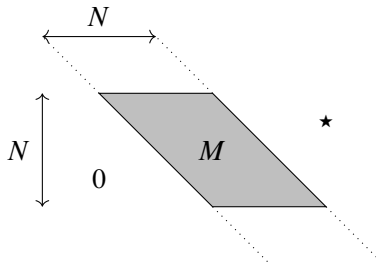
4.1.1. Definitions. Fix a positive integer N . Consider the algebra $R := \{M = (M_{ij})_{i,j \in \mathbb{Z}}\}$ of complex *upper triangular* matrices that are infinite in both directions, and the subalgebra $R_{\mathbb{Z} \bmod N}$ of the (N, N) -periodic ones:

$$R_{\mathbb{Z} \bmod N} := \{M \in R \mid MS^N = S^N M\},$$

where $S = (\delta_{i,j-1}) \in R_{\mathbb{Z} \bmod N}$ is the shift operator. Then we define the algebra \mathcal{M}_N to be the quotient of $R_{\mathbb{Z} \bmod N}$ by the ideal generated by S^N :

$$\mathcal{M}_N := R_{\mathbb{Z} \bmod N} / \langle S^N \rangle.$$

\mathcal{M}_N is of dimension N^2 . A fundamental domain for $M \in \mathcal{M}_N$ is



where the left diagonal is the main diagonal, and everything left of it is zero, while the right diagonal is the N^{th} diagonal, and everything on it or right of it is ignored. There is some freedom in choosing which N rows we put in the fundamental domain, i.e., in sliding the latter along the diagonal. In what follows, we identify an element of \mathcal{M}_N and any of its representatives when there is no risk of confusion.

The definitions above are directly relevant to type \widehat{A} (periodic boundary conditions), and so we shall also write $R_{\mathbb{Z} \bmod N} = R_{\mathbb{Z} \bmod N}^p$, $\mathcal{M}_N = \mathcal{M}_N^p$.

Assume now that $N = 2n$ is even. The new ingredient we introduce for type \widehat{C} is the antidiagonal symplectic form J :

$$J_{ij} := \delta_{i+j, N+1} \varepsilon_i \quad i, j \in \mathbb{Z}, \quad \varepsilon_i := \begin{cases} 1 & i = 1, \dots, n \pmod{N}, \\ -1 & i = n+1, \dots, N \pmod{N}. \end{cases}$$

Note that $S^n JS^n = -J$. Define the adjoint of M to be $M^\dagger := J^{-1} M^T J$ where $J^{-1} = -J$; explicitly,

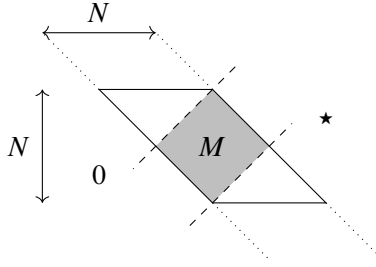
$$M_{ij}^\dagger = \varepsilon_i \varepsilon_j M_{N-j+1, N-i+1}.$$

Note that $R_{\mathbb{Z} \bmod N} = R_{\mathbb{Z} \bmod N}^\dagger$, and the same for $\langle S^N \rangle$. We thus define

$$\begin{aligned} R_{\mathbb{Z} \bmod N}^i &:= \{M \in R_{\mathbb{Z} \bmod N} \mid M = -M^\dagger\}, \\ \mathcal{M}_N^i &:= R_{\mathbb{Z} \bmod N}^i / (\langle S^N \rangle \cap R_{\mathbb{Z} \bmod N}^i), \\ R_{\mathbb{Z} \bmod N}^c &:= \{M \in R_{\mathbb{Z} \bmod N} \mid M = M^\dagger\}, \\ \mathcal{M}_N^c &:= R_{\mathbb{Z} \bmod N}^c / (\langle S^N \rangle \cap R_{\mathbb{Z} \bmod N}^c). \end{aligned}$$

(Note that the definitions above would be unaffected by the change $J \rightarrow S^n J S^{-n}$.) $R_{\mathbb{Z} \bmod N}^i$ is a Lie subalgebra of $R_{\mathbb{Z} \bmod N}$, and $R_{\mathbb{Z} \bmod N} = R_{\mathbb{Z} \bmod N}^i \oplus R_{\mathbb{Z} \bmod N}^c$ as a $R_{\mathbb{Z} \bmod N}^i$ -module.

A fundamental domain is (assuming that in the previous picture the chosen rows are from 1 to N)



where the dashed lines are symmetry axes (i.e., entries that are mirror images w.r.t. one of the axes are either equal or opposite). In $R_{\mathbb{Z} \bmod N}^c$, the entries on the symmetry axes are zero, whereas in $R_{\mathbb{Z} \bmod N}^i$, they are free.

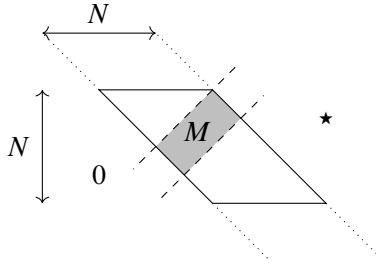
Assuming that N is a multiple of 4, i.e., that $n = 2m$, we can introduce a *second* symplectic form $J' = S^m J S^{-m}$ and a second notion of adjoint

$$M_{ij}^\ddagger = \varepsilon_{i+m} \varepsilon_{j+m} M_{n+1-j, n+1-i}.$$

This leads to more definitions:

$$\begin{aligned} R_{\mathbb{Z} \bmod N}^o &:= \{M \in R_{\mathbb{Z} \bmod N} \mid M = -M^\dagger = -M^\ddagger\}, \\ \mathcal{M}_N^o &:= R_{\mathbb{Z} \bmod N}^o / (\langle S^N \rangle \cap R_{\mathbb{Z} \bmod N}^o), \\ R_{\mathbb{Z} \bmod N}^m &:= \{M \in R_{\mathbb{Z} \bmod N} \mid M = M^\dagger = -M^\ddagger\}, \\ \mathcal{M}_N^m &:= R_{\mathbb{Z} \bmod N}^m / (\langle S^N \rangle \cap R_{\mathbb{Z} \bmod N}^m), \end{aligned}$$

i.e., a fundamental domain of the form



We finally define in each type the (unreduced) Brauer loop scheme to be

$$\widetilde{E}_N^a := \{M \in \mathcal{M}_N^a \mid M^2 = 0\}, \quad a \in \{p, i, c, o, m\}.$$

As in [16], noting that among the equations of the scheme are $M_{ii}^2 = 0$, we prefer to define the (generically reduced) Brauer loop scheme as

$$E_N^a := \{M \in \mathcal{M}_N^a \mid M^2 = 0 \text{ and } M_{ii} = 0 \text{ for all } i\}, \quad a \in \{p, i, c, o, m\}.$$

As sets, \widetilde{E}_N^a and E_N^a are identical, but as schemes, the latter is generically reduced (as we shall prove), and conjecturally reduced, whereas the former is neither. The distinction is rather inessential in type \widehat{A} (see however [16, Section 7]), but less so in type \widehat{C} , see the discussion before (44). Note that $E_N^a \subset (\mathcal{M}_N^a)_{\Delta=0}$ with the notation $(\mathcal{M}_N^a)_{\Delta=0} = \{M \in \mathcal{M}_N^a \mid M_{ii} = 0 \forall i\}$.

4.1.2. Group action and multidegrees. Invertible elements of R act by conjugation on R , and among them, the subgroup

$$B_{\mathbb{Z} \bmod N} := \{M \in R^\times \mid \text{there exists } \lambda \in \mathbb{C}^\times, S^N M = \lambda M S^N\},$$

leaves $R_{\mathbb{Z} \bmod N}$ and $\langle S^N \rangle$ invariant, thus acts on \mathcal{M}_N .

A maximal torus $T_{\mathbb{Z} \bmod N}$ of $B_{\mathbb{Z} \bmod N}$ consists of diagonal matrices with the same property:

$$T_{\mathbb{Z} \bmod N} := B_{\mathbb{Z} \bmod N} \cap \{\text{diagonal matrices}\}.$$

It is of dimension $N + 1$. Note however that scalar matrices act trivially by conjugation, so only $T_{\mathbb{Z} \bmod N}/\mathbb{C}^\times$, of dimension N , acts on \mathcal{M}_N^p .

We add to $T_{\mathbb{Z} \bmod N}$ an additional $\mathbb{C}^\times \ni q$ which acts on $M \in \mathcal{M}_N^p$ by scaling: $M \rightarrow qM$.

The corresponding Lie algebra $\mathfrak{t}_{\mathbb{Z} \bmod N}$ has elements of the form $\text{diag}(z_i)_{i \in \mathbb{Z}}$, where $z_{i+N} = z_i + \epsilon$ for all i , where $\lambda = \exp \epsilon$. Also introduce the generator A (where $q = \exp A$) of the extra \mathbb{C}^\times . Then the group of characters of $T_{\mathbb{Z} \bmod N} \times \mathbb{C}^\times$ (viewed as a lattice in $\mathfrak{t}_{\mathbb{Z} \bmod N} \oplus \mathbb{C}$) is the abelian group generated by the $z_i, i \in \mathbb{Z}$,

ϵ and A with relations $z_{i+N} = z_i + \epsilon$. Furthermore, the commutative ring they generate is the equivariant cohomology ring of a point, or of \mathcal{M}_N^p :

$$H_{T_{\mathbb{Z} \bmod N} \times \mathbb{C}^\times}^*(\mathcal{M}_N^p) \cong \mathbb{Z}[(z_i)_{i \in \mathbb{Z}}, A, \epsilon] / \langle z_i + \epsilon - z_{i+N}, i \in \mathbb{Z} \rangle.$$

Comparing with the notations of Section 1.3, we find that we must identify $L = N$, $s = \epsilon$, and then $H_{T_{\mathbb{Z} \bmod N} \times \mathbb{C}^\times}^*$ is the embedding ring for the root lattice of type \widehat{A} , with one additional variable A (corresponding to the extra circle).

A convenient algebraic framework for computations in equivariant cohomology of a vector space endowed with a linear group action is to use *multidegrees*; we refer to [21] for details. To any subvariety X of \mathcal{M}_N^p which is invariant by action of $T_{\mathbb{Z} \bmod N} \times \mathbb{C}^\times$, we can thus associate its multidegree $\text{mdeg } X$ in $H_{T_{\mathbb{Z} \bmod N} \times \mathbb{C}^\times}^*(\mathcal{M}_N^p)$. Because the real action is $(T_{\mathbb{Z} \bmod N} / \mathbb{C}^\times) \times \mathbb{C}^\times$, all our multidegrees depend only on $z_i - z_j$ (and more precisely, are sums of products of the weights $A + z_i - z_j, i \leq j < i + N$).

We now discuss type \widehat{C} . We define

$$\widetilde{B}_{\mathbb{Z} \bmod N} := \{M \in B_{\mathbb{Z} \bmod N} \mid \text{there exists } \zeta \in \mathbb{C}^\times, MM^\dagger = \zeta\},$$

$$\widetilde{T}_{\mathbb{Z} \bmod N} := \widetilde{B}_{\mathbb{Z} \bmod N} \cap T_{\mathbb{Z} \bmod N}.$$

We could set the scalar ζ to 1 because conjugation by a scalar is trivial, but we prefer to keep it for reasons which will become clear.

There are corresponding Lie algebras for which $M + M^\dagger = u, \zeta = \exp u$. In particular,

$$\widetilde{\mathfrak{t}}_{\mathbb{Z} \bmod N} := \{M = \text{diag}(z_i) \in \mathfrak{t}_{\mathbb{Z} \bmod N} \mid z_{N-i+1} = u - z_i\}.$$

Then we have

$$\begin{aligned} H_{\widetilde{T}_{\mathbb{Z} \bmod N} \times \mathbb{C}^\times}^*(\mathcal{M}_N^i) &\cong H_{\widetilde{T}_{\mathbb{Z} \bmod N} \times \mathbb{C}^\times}^*(\mathcal{M}_N^c) \\ &\cong \mathbb{Z}[(z_i)_{i \in \mathbb{Z}}, A, \epsilon, \alpha] / \langle z_i + \epsilon - z_{i+N}, z_i + z_{N-i+1} - u, i \in \mathbb{Z} \rangle. \end{aligned}$$

We can as before define multidegrees of subvarieties of $\mathcal{M}_N^{i,c}$; because of the trivial conjugation by scalar matrices, the parameter u is redundant, and will be set to 0 in what follows (it can be recovered by substituting $z_i \mapsto z_i - u/2$). We recover at this stage the embedding ring for the root lattice of type \widehat{C} of Section 1.3 with the following correspondence: $L = N/2 = n, s = \epsilon$.

Similarly, define

$$\widetilde{B}_{\mathbb{Z} \bmod N} := \{M \in B_{\mathbb{Z} \bmod N} \mid \text{there exists } \zeta, \xi \in \mathbb{C}^\times, MM^\dagger = \zeta, MM^\ddagger = \xi\},$$

$$\widetilde{T}_{\mathbb{Z} \bmod N} := \widetilde{B}_{\mathbb{Z} \bmod N} \cap T_{\mathbb{Z} \bmod N}.$$

Here the scalar factors become relevant: indeed, it is easy to see by combining the various equations (in particular, using $(J'J)^2 = S^N$) that $\zeta^2 = \lambda\xi^2$, so that, for $\lambda \neq 1$, one cannot set simultaneously ζ and ξ to 1. Writing $\xi = \exp v$, with the relation $2v = 2u - \epsilon$, the Lie algebra of the corresponding maximal torus is

$$\widetilde{t}_{\mathbb{Z} \bmod N} := \{M = \text{diag}(z_i) \in t_{\mathbb{Z} \bmod N} \mid z_{N-i+1} = u - z_i, z_{n-i+1} = v - z_i\}.$$

Finally, we have for $a \in \{o, m\}$

$$H_{\widetilde{T}_{\mathbb{Z} \bmod N} \times \mathbb{C}^\times}^* (\mathcal{M}_N^a) \cong \mathbb{Z}[(z_i)_{i \in \mathbb{Z}}, A, \epsilon/2, v] / \langle z_i + \epsilon - z_{i+N}, z_i + z_{N-i+1} - v - \epsilon/2, z_i + z_{n-i+1} - v, i \in \mathbb{Z} \rangle,$$

where we have replaced u with $v + \epsilon/2$. The parameter v is redundant and will be set to 0 (it can be recovered by $z_i \mapsto z_i - v/2$). We also recover the embedding ring for the root lattice of type \widehat{C} of Section 1.3, but with a different correspondence: $L = N/4 = m, s = \epsilon/2$.

At this stage, one would like to introduce the multidegrees of the components of the Brauer loop scheme. We note that \widetilde{E}_N^a and E_N^a are invariant by action of $T_{\mathbb{Z} \bmod N} \times \mathbb{C}^\times$ for $a = p$, $\widetilde{T}_{\mathbb{Z} \bmod N} \times \mathbb{C}^\times$ for $a \in \{i, c\}$, $\widetilde{T}_{\mathbb{Z} \bmod N} \times \mathbb{C}^\times$ for $a \in \{o, m\}$, and therefore so are their irreducible components.

The most natural quantities are the multidegrees of the primary top-dimensional components of the scheme \widetilde{E}_N^a (we conjecture equidimensionality, in which case “top-dimensional” can be omitted). However for practical reasons, it is much easier to deal with (reduced) varieties. Let us therefore define the E_π^a to be the (reduced) irreducible top-dimensional components of \widetilde{E}_N^a or E_N^a , where the indexing set for π will be determined to be the Brauer link patterns in Section 4.3. Then we write

$$\phi_\pi^a := m_\pi \text{ mdeg } E_\pi^a, \quad a \in \{p, i, c, o, m\}, \tag{44}$$

where the multidegree is relative to \mathcal{M}_N^a , and $m_\pi = 2^{\#\{\text{chords}(\pi)\} + \#\{\text{fixed points}(\pi)\}}$ (the number of fixed points is one if L is odd and $a \in \{p, c\}$, zero otherwise).⁵

⁵ In Section 4.3, we shall introduce another “periodic” diagram for π . It is important to note that in the definition of m_π we mean the number of chords and fixed points of the ordinary, nonperiodic (for $a \neq p$) diagram of π .

We shall show (see Appendix B) that m_π is the multiplicity of E_π^a in \tilde{E}_N^a , i.e., that ϕ_π^a is the multidegree of the primary component of \tilde{E}_N^a associated to E_π^a .

4.1.3. Relation to loop algebras. Since all matrices we consider commute with S^N , it is natural to consider $t = S^N$ as a scalar. We immediately conclude that $R_{\mathbb{Z} \bmod N} \cong R_N \oplus t \mathfrak{g}_N \oplus t^2 \mathfrak{g}_N \oplus \cdots$, where $\mathfrak{g}_N = \mathfrak{g}_N^p$ is the space of $N \times N$ matrices, i.e., the Lie algebra $\mathfrak{gl}_N(\mathbb{C})$, and R_N is the space of $N \times N$ upper triangular matrices; thus identifying $R_{\mathbb{Z} \bmod N}$ with the Borel subalgebra of the loop algebra $\mathfrak{gl}_N(\mathbb{C}[t, t^{-1}])$. Then $\mathcal{M}_N \cong R_{\mathbb{Z} \bmod N} / t R_{\mathbb{Z} \bmod N}$.

Similarly, denote by \mathfrak{g}_N^i the space of matrices $M \in \mathfrak{gl}_N$ that satisfy $JM + M^T J = 0$, where by abuse of notation we use the same letter J for the finite matrix

$$J := \begin{pmatrix} 0 & \cdots & & & 1 \\ \vdots & & & \ddots & \\ & & 1 & & \\ & & -1 & & \\ & \ddots & & & \vdots \\ -1 & & & \cdots & 0 \end{pmatrix}, \quad (45)$$

and its infinite periodic counterpart defined earlier.

\mathfrak{g}_N^i is the symplectic Lie algebra $\mathfrak{sp}_N(\mathbb{C})$. Now define \mathfrak{g}_N^c to be the space of matrices satisfying $JM - M^T J = 0$. One has $\mathfrak{g}_N = \mathfrak{g}_N^i \oplus \mathfrak{g}_N^c$ as a $\mathfrak{sp}_N(\mathbb{C})$ -module by conjugation.

Then $R_{\mathbb{Z} \bmod N}^a \cong R_N^a \oplus t \mathfrak{g}_N^a \oplus t^2 \mathfrak{g}_N^a \oplus \cdots$ for $a \in \{i, c\}$ (where $R_N^a = R_N \cap \mathfrak{g}_N^a$), which identifies $R_{\mathbb{Z} \bmod N}^i$ with the Borel subalgebra of the loop algebra $\mathfrak{sp}_N(\mathbb{C}[t, t^{-1}])$.

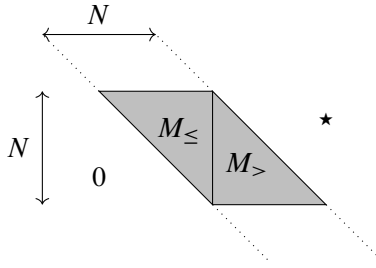
Finally, define $\mathfrak{sp}'_N(\mathbb{C})$ to be the Lie algebra of matrices satisfying $J'M + M^T J' = 0$, where

$$J' := \begin{pmatrix} 0 & & & & 1 \\ & \ddots & & & \\ -1 & & & & \\ & & 0 & & 1 \\ & & & \ddots & \\ & & -1 & & 0 \end{pmatrix}.$$

As a Lie algebra, $\mathfrak{sp}_N(\mathbb{C}) \cap \mathfrak{sp}'_N(\mathbb{C}) \cong \mathfrak{sp}_n(\mathbb{C}) \oplus \mathfrak{sp}_n(\mathbb{C})$, i.e., two copies of the symplectic Lie algebra. However, $R_N \cap \mathfrak{sp}_N(\mathbb{C}) \cap \mathfrak{sp}'_N(\mathbb{C})$ is *not* its Borel subalgebra. Define the spaces $\mathfrak{g}_N^o = \mathfrak{g}_N^i \cap \mathfrak{sp}'_N(\mathbb{C})$, $\mathfrak{g}_N^m = \mathfrak{g}_N^c \cap \mathfrak{sp}'_N(\mathbb{C})$, and $R_N^a = R_N \cap \mathfrak{g}_N^a$, so that $R_{\mathbb{Z} \bmod N}^a \cong R_N^a \oplus t \mathfrak{g}_N^a \oplus t^2 \mathfrak{g}_N^a \oplus \dots$ for $a \in \{o, m\}$.

4.2. The Brauer loop schemes as a flat limit

4.2.1. Orbit closures and their flat degeneration. Let us define the map from \mathfrak{g}_N to \mathcal{M}_N that takes M to $M_{\leq} + tM_{>}$, where M_{\leq} (resp. $M_{>}$) indicates the upper (resp. strict lower) triangle of M . (Equivalently, in terms of the strip picture, this amounts to the parameterization



of the fundamental domain).

The connection to loop algebras suggests that we should think of t as a numerical parameter which provides a one-parameter family of products on \mathfrak{g}_N . By varying the value of t , one interpolates between ordinary matrix product on \mathfrak{g}_N (at $t = 1$, a generic fiber) and a degenerate product at the special fiber $t = 0$ (denoted \bullet in [16]), which is the one on \mathcal{M}_N , which makes \mathcal{M}_N isomorphic to the semi-direct product $R_N \times (\mathfrak{g}_N / R_N)$, with multiplication $(\Upsilon, \Lambda)(\Upsilon', \Lambda') = (\Upsilon \Upsilon', \Upsilon \Lambda' + \Upsilon' \Lambda)$.

More explicitly, define for any $a \in \{p, i, c, o, m\}$

$$D_{N;t}^a := \{M \in \mathfrak{g}_N^a \mid (M_{\leq} + tM_{>})^2 = 0\}, \quad t \neq 0,$$

and $D_{N;0}^a = \lim_{t \rightarrow 0} D_{N;t}^a$ to be the flat limit as $t \rightarrow 0$.

If $a \in \{p, i, c\}$, this is equivalent to saying that $D_{N;0}^a$ is the *normal cone* of $D_N^a \cap R_N$ inside $D_N^a := D_{N;1}^a$. This is not so for $a \in \{o, m\}$ (note that it is M , not $M_{\leq} + tM_{>}$, that is in \mathfrak{g}_N^a).

Proposition 4.1. $D_{N;0}^a \subset \widetilde{E}_N^a$ as schemes.

Proof. First, one checks that in each case $a \in \{i, c, o, m\}$, the symmetry of $M \in \mathfrak{g}_N^a$ turns into the symmetry of $M_{\leq} + tM_{>} \in \mathcal{M}_N^a$. It then follows from the above (see also [16, Section 2.3]) that in the limit $t \rightarrow 0$, the equation $(M_{\leq} + tM_{>})^2 = 0$ in

\mathfrak{g}^a becomes, essentially tautologically, the equation $M^2 = 0$ in \mathcal{M}_N^a . This implies the inclusion of schemes. \square

In principle there may be more equations in the flat limit of the ideal generated by $(M_{\leq} + tM_{>})^2 = 0$ as $t \rightarrow 0$. In fact, we conjecture that there are not, so that the inclusion of Proposition 4.1 is an equality. It will be a consequence of the results below that $D_{N;0}^a = E_N^a$ as sets, so that the Brauer loop schemes can be defined alternatively as the normal cones $D_{N;0}^a$. We shall also prove (Appendix B) that the multiplicities of the E_{π}^a inside $D_{N;0}^a$ and \tilde{E}_N^a are equal. Note that some of these results are new even in type \hat{A} , proving some conjectures of [16].

The group $B_{\mathbb{Z} \bmod N}$ (or $\tilde{B}_{\mathbb{Z} \bmod N}, \tilde{\tilde{B}}_{\mathbb{Z} \bmod N}$) does not act on \mathfrak{g}_N^a ; instead, we have an action of GL_N (or Sp_N , or $Sp_N \cap Sp'_N$). At the level of the torus action, it is easy to see that the Cartan tori of the latter identify with subgroups of codimension one inside the Cartan tori of the former. In terms of Lie algebras, it means that we must restrict to the subalgebra given by $\epsilon = 0$. The degeneration respects that torus action, and therefore

$$\text{mdeg } D_{N;0}^a|_{\epsilon=0} = \text{mdeg } D_N^a. \tag{46}$$

By standard arguments, D_N^a is an orbit closure, and therefore irreducible. Furthermore, we check smoothness at a specific point (with the assumption that $n = N/2$ is even if $a = c$; see the discussion in the next section) by an elementary Zariski tangent space computation, and conclude that D_N^a is generically reduced. For $a \in \{p, i, c\}$, the smooth point is

$$M = \left(\begin{array}{c|c} & 1 \\ \hline 0 & \ddots \\ \hline & 1 \\ \hline 0 & 0 \end{array} \right), \quad a \in \{p, i\},$$

$$M = \left(\begin{array}{c|c} & & & 1 & 0 \\ & & & 0 & -1 \\ \hline 0 & & \ddots & & \\ & 1 & 0 & & \\ & 0 & -1 & & \\ \hline 0 & & & & 0 \end{array} \right), \quad a = c, N = 0 \pmod{4},$$

where blocks are $n \times n$.

The fixed points are the coordinate subspaces $V_I = \text{span}\{e_i, i \in I\}$ (where e_1, \dots, e_N is the standard basis of \mathbb{C}^N), indexed by n -subsets I of $\{1, \dots, N\}$, and localization gives

$$\text{mdeg } D_N^p = 2^r \prod_{i,j=1}^N (A + z_i - z_j) \sum_{\substack{I \subset \{1, \dots, N\} \\ |I|=n}} \prod_{i \in I, j \notin I} \frac{1}{(z_j - z_i)(A + z_i - z_j)}.$$

- a = i. Define the cotangent bundle of the Lagrangian (type C) Grassmannian:

$$Q_N^i = \{(V, M) \mid M \in \mathfrak{g}_N^i, V^\perp = V, \text{Im } M \subset V \subset \text{Ker } M\}.$$

Lagrangian coordinate subspaces are indexed by signs $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{+1, -1\}^n$: explicitly,

$$V_\varepsilon := \text{span}(\{e_i, 1 \leq i \leq n, \varepsilon_i = -1\} \cup \{e_{N-i+1}, 1 \leq i \leq n, \varepsilon_i = +1\}).$$

Then,

$$\begin{aligned} \text{mdeg } D_N^i &= A^{-n} \prod_{1 \leq i < j \leq n} (A \pm z_i \pm z_j) \sum_{\varepsilon \in \{+1, -1\}^n} \prod_{\substack{i,j=1 \\ i < j}}^n \frac{1}{(\varepsilon_i z_i + \varepsilon_j z_j)(A - \varepsilon_i z_i - \varepsilon_j z_j)}. \end{aligned}$$

- a = c. For $n = N/2$ even, the situation is similar to the identified case:

$$Q_N^c = \{(V, M) \mid M \in \mathfrak{g}_N^c, V^\perp = V, \text{Im } M \subset V \subset \text{Ker } M\},$$

$$\begin{aligned} \text{mdeg } D_N^c &= A^n \prod_{1 \leq i < j \leq n} (A \pm z_i \pm z_j) \sum_{\varepsilon \in \{+1, -1\}^n} \prod_{\substack{i,j=1 \\ i < j}}^n \frac{1}{\varepsilon_i z_i + \varepsilon_j z_j} \prod_{\substack{i,j=1 \\ i < j}}^n \frac{1}{A - \varepsilon_i z_i - \varepsilon_j z_j}. \end{aligned}$$

For n odd, the map $(V, M) \mapsto M$ is not generically one-to-one since the rank of a generic element of D_N^c is $n - 1$ (not n). The resolution of singularities is slightly more complicated and we shall skip the details, noting that it is simpler to use the recurrence relation (43) to deduce the sum rule in odd size from that in even size.

- $a = 0$. Recall that $D_N^0 = D_{N/2;1}^i \times D_{N/2;1}^i$, so the resolution of singularities is simply $Q_N^0 = Q_{N/2}^i \times Q_{N/2}^i$. We conclude immediately

$$\text{mdeg } D_N^0 = (\text{mdeg } D_{N/2}^i)^2.$$

- $a = m$. We use again the decomposition discussed at the end of last section, i.e., $M \sim \begin{pmatrix} 0 & X^\dagger \\ X & 0 \end{pmatrix}$ with $X \in \text{Hom}(W_+, W_-)$. The resolution of singularities is given by

$$Q_N^m = \{(V_+, V_-, X) \mid V_+^\perp = V_+, V_-^\perp = V_-, \\ \text{Im } X^\dagger \subset V_+ \subset \text{Ker } X, \\ \text{Im } X \subset V_- \subset \text{Ker } X^\dagger\},$$

and by localization one gets, writing $n = 2m$,

$$\text{mdeg } D_N^m = \prod_{1 \leq i, j \leq m} (A \pm z_i \pm z_j) \sum_{\substack{\varepsilon \in \{+1, -1\}^m \\ \varepsilon' \in \{+1, -1\}^m}} \prod_{\substack{i, j=1 \\ i \leq j}}^m \frac{1}{(\varepsilon_i z_i + \varepsilon_j z_j)(\varepsilon'_i z_i + \varepsilon'_j z_j)} \\ \prod_{i, j=1}^m \frac{1}{A - \varepsilon_i z_i - \varepsilon'_j z_j}.$$

Remark. It was shown in [16, section 7] that the type \widehat{A} localization formula can also be derived via an integral over the unitary group. Similar results can be obtained in type \widehat{C} , with integrals over the compact symplectic group.

4.3. Irreducible components. This section will outline the relationship between the irreducible components of the scheme E_N^a and the link patterns of the Brauer loop model with boundary condition a . We recall the notion of link patterns introduced in Section 2, see Figures 2 and 3. We now describe a map from link patterns of type \widehat{C} to link patterns of type \widehat{A} with certain symmetries.

Given a link pattern $\pi \in \text{LP}_L^a$, we define $\tilde{\pi}$ according to the following simple symmetry rules:

- (1) For $a = i, c$, we have $L = n = N/2$ and
 - if $\pi(j) = k \neq b$, then $\tilde{\pi}(j) = k$ and $\tilde{\pi}(N - j + 1) = N - k + 1$;
 - if $\pi(j) = b$, then $\tilde{\pi}(j) = N - j + 1$.

(2) For $a = o, m$, we have $L = m = N/4$ and

- if $\pi(j) = k \neq l, r$, then

$$\tilde{\pi}(j) = k,$$

$$\tilde{\pi}(n + j) = n + k,$$

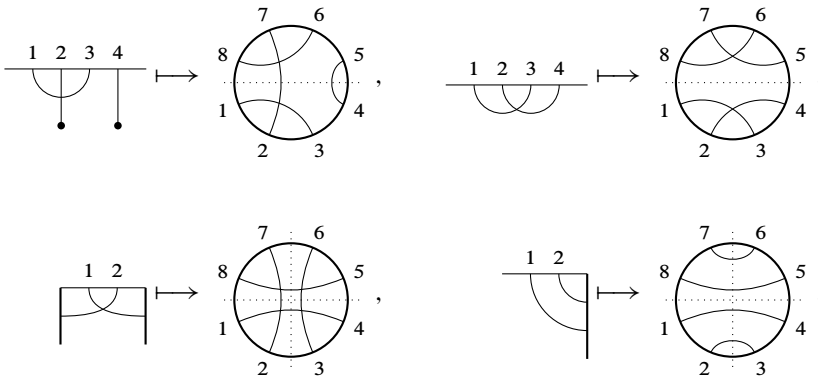
$$\tilde{\pi}(N - j + 1) = N - k + 1,$$

and

$$\tilde{\pi}(n - j + 1) = n - k + 1.$$

- If $\pi(j) = l$, then $\tilde{\pi}(j) = N - j + 1$ and $\tilde{\pi}(n - j + 1) = n + j$.
- If $\pi(j) = r$, then $\tilde{\pi}(j) = n - j + 1$ and $\tilde{\pi}(N - j + 1) = n + j$.

For example:



It is $\tilde{\pi}$, rather than π , that will appear naturally in the geometry, e.g., for defining the irreducible components, and as the distinction between π and $\tilde{\pi}$ is either irrelevant or obvious from context, we will drop the $\tilde{\cdot}$ notation. As to the pictorial description, we shall simply call this new representation the periodic diagram of the link pattern π .

Define $s_i(M) = \sum_{j=i}^{i+N} M_{ij} M_{j,i+N}$ for $M \in E_N^p$ (note that this is well-defined despite the quotient by $\langle S^N \rangle$).

Given a matrix M of size N , we define the so-called rank matrix $\text{rm}(M)$ as

$$\text{rm}(M)_{ij} := \text{rk } M_{i:N,1:j},$$

where $M_{i:N,1:j}$ denotes the submatrix south-west of entry (i, j) .

We recall from now on that detailed proofs are only provided for $a \in \{i, c\}$, and occasionally for $a = p$ when they are not already present in [16, 17].

4.3.1. Irreducible components of E_N^a for identified and closed boundaries.

In type \widehat{A} , it is known that E_N^p is equidimensional of dimension $2n^2$, and that E_N^p decomposes into its irreducible components E_π^p indexed by periodic link patterns [16, 23].

In this section we will prove a similar statement: that the top-dimensional irreducible components of E_N^i and E_N^c are indexed by link patterns in LP_n^i and LP_n^c respectively. (As mentioned previously, equidimensionality is only conjectured for these two cases.) We find that the dimensions of the two schemes are

$$\dim(E_N^i) = n(n + 1),$$

$$\dim(E_N^c) = 2 \left\lfloor \frac{n^2}{2} \right\rfloor.$$

We also show that the top-dimensional components are generically reduced.

We conjecture that the same statement is true for E_N^o and E_N^m , and that their respective dimensions are

$$\dim(E_N^o) = 2m(m + 1),$$

$$\dim(E_N^m) = m(2m + 1).$$

However the inductive proof of Theorem 4.2 (using an appropriately defined \widetilde{B}_N) presented technical challenges that obstructed the proofs of Theorem 4.3 and Theorem 4.5. Thus for the rest of this section $a \in \{p, i, c\}$ unless stated otherwise.

If $M \in E_N^a$, recall from Section 4.2.1 that we can write M as a pair (Υ, Λ) , where Υ belongs to $\mathcal{O}_N^a = \{\Upsilon \in R_N^a \mid \Upsilon^2 = 0\}$, and $\Lambda \in \mathfrak{g}_N^a / R_N^a$ such that $\Lambda\Upsilon + \Upsilon\Lambda$ (i.e., its strict lower triangular part) is 0. We will assume the diagonal of M to be zero (i.e., consider the reduced scheme E_N^a).

We define the Borel subgroups

$$B_N := \{M \in GL_N \mid M_{ij} = 0, j < i\},$$

$$\widetilde{B}_N := \{M \in B_N \mid M^{-1} = M^\dagger\},$$

where $M^\dagger = J^{-1}M^T J$ with the symplectic form as given in (45). Acting by conjugation with B_N leaves R_N^p and therefore \mathcal{O}_N^p invariant, and similarly for \widetilde{B}_N , $R_N^{i,c}$ and $\mathcal{O}_N^{i,c}$. We will use the notation \cdot for conjugation.

Definition 4.1.

$$\text{Inv}_N^i := \{\pi \text{ involution of } \{1, \dots, N\} \mid \pi(i) = N - \pi_{N-i+1} + 1\},$$

$$\text{Inv}_N^c := \{\pi \in \text{Inv}_N^i \mid \pi(i) \neq N - i + 1 \forall i\}.$$

Example 4.1.

$$\text{Inv}_4^i = \left\{ \begin{array}{c} \begin{array}{c} 4 \quad 3 \\ \circ \quad \circ \\ 1 \quad 2 \end{array}, \begin{array}{c} 4 \quad 3 \\ \circ \quad \circ \\ 1 \quad 2 \end{array}, \begin{array}{c} 4 \quad 3 \\ \circ \quad \circ \\ 1 \quad 2 \end{array}, \begin{array}{c} 4 \quad 3 \\ \circ \quad \circ \\ 1 \quad 2 \end{array}, \begin{array}{c} 4 \quad 3 \\ \circ \quad \circ \\ 1 \quad 2 \end{array}, \begin{array}{c} 4 \quad 3 \\ \circ \quad \circ \\ 1 \quad 2 \end{array} \end{array} \right\},$$

$$\text{Inv}_4^c = \left\{ \begin{array}{c} \begin{array}{c} 4 \quad 3 \\ \circ \quad \circ \\ 1 \quad 2 \end{array}, \begin{array}{c} 4 \quad 3 \\ \circ \quad \circ \\ 1 \quad 2 \end{array}, \begin{array}{c} 4 \quad 3 \\ \circ \quad \circ \\ 1 \quad 2 \end{array} \end{array} \right\}.$$

The following definitions give unique matrix representations of the involutions defined above.

Definition 4.2. For $\pi \in \text{Inv}_N^i$ we define the matrix $\pi_{<}^i \in \mathcal{O}_N^i$ as

$$(\pi_{<}^i)_{ij} := \begin{cases} 0 & j \leq i, \\ -\delta_{i,\pi(j)} & n < i < j, \\ \delta_{i,\pi(j)} & \text{else.} \end{cases}$$

For $\pi \in \text{Inv}_N^c$ we define $\pi_{<}^c \in \mathcal{O}_N^c$ as

$$(\pi_{<}^c)_{ij} := \begin{cases} 0 & j \leq i, \\ 0 & j + i = N + 1, \\ -\delta_{i,\pi(j)} & N - j + 1 < i \leq n. \\ \delta_{i,\pi(j)} & \text{else.} \end{cases}$$

Example 4.2.

$$(351624)_{<}^i = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (563421)_{<}^c = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 4.2. For $a \in \{i, c\}$, each \widetilde{B}_N -orbit of \mathcal{O}_N^a contains exactly one π_{\leq}^a where $\pi \in \text{Inv}_N^a$. Orbits are thus naturally labelled by these involutions.

Remark. This is a modification of the main theorem in [20], which applies to $a = p$, and the proof follows that given there.

Proof. We use induction from size $N - 2$ to size N . For $N = 2$ we could have that $\mathcal{O}_2^a \ni X = 0$ (which is trivial), or, for $a = i$, the upper-right entry of X is nonzero. The latter case is conjugate to $(2, 1)_{\leq}^i$ by a diagonal matrix.

We now consider general N . For any matrix $X \in \mathcal{O}_N^a$, we can form the matrix $\widehat{X} \in \mathcal{O}_{N-2}^a$ by truncating the first and last row and column of X . Assuming that the theorem is true for $N - 2$, there is a unique $\widehat{\pi} \in \text{Inv}_{N-2}^a$ such that $\widehat{w} := \widehat{\pi}_{\leq}^a = \widehat{U} \widehat{X} \widehat{U}^{-1}$ for some $\widehat{U} \in \widetilde{B}_{N-2}$. Define $U_0 \in \widetilde{B}_N$ as

$$U_0 := \begin{pmatrix} 1 & & \\ & \widehat{U} & \\ & & 1 \end{pmatrix}.$$

The matrix $Y = U_0 X U_0^{-1}$ has its middle $(N - 2) \times (N - 2)$ block equal to \widehat{w} . Its first row is free (not including entry $(1, 1)$, which equals 0), and its last column is decided from the first row by the symmetry of \mathcal{O}_N^a . Our aim is to find a transformation matrix $T \in \widetilde{B}_N$ so that $TYT^{-1} = \pi_{\leq}^a =: w$ for a unique $\pi \in \text{Inv}_N^a$. We note that rank is preserved by conjugation, so $\text{rk } Y = \text{rk } X = \text{rk } w$.

We now define $U_1 \in \widetilde{B}_N$ with first row

$$\left(1 \quad \left[- \sum_{s=2}^{N-1} \widehat{w}_{j-1, s-1} Y_{1, s} \right]_{2 < j < N-1} \quad 0 \right),$$

middle $(N - 2) \times (N - 2)$ block equal to the $(N - 2)$ identity matrix, and other entries decided by the symmetry of \widetilde{B}_N . Let $Z = U_1 Y U_1^{-1}$. If $\text{rk } Y = \text{rk } \widehat{X}$, then $\text{rk } w = \text{rk } \widehat{w}$, so the first and last row and column of w must be zero. Then $T = U_1$, and $Z = w$.

If $\text{rk } Y = \text{rk } \widehat{X} + 1$ (this is only possible for $a = i$), then $\text{rk } w = \text{rk } \widehat{w} + 1$, so $w_{1N} = 1$ and all other extra entries must be zero. We define $U_2 \in \widetilde{B}_N$ as

$$U_2 := \begin{pmatrix} \frac{1}{\sqrt{Z_{1N}}} & & \\ & I_{N-2} & \\ & & \sqrt{Z_{1N}} \end{pmatrix},$$

then $w = U_2 Z U_2^{-1}$ and $T = U_2 U_1$.

Finally, if $\text{rk } Y = \text{rk } \hat{X} + 2$, we define k to be the column index of the first nonzero entry in the first row of Z . We define $U_3 \in \tilde{B}_N$ as having diagonal 1 except for

$$(U_3)_{11} = \frac{1}{Z_{1k}},$$

$$(U_3)_{NN} = Z_{1k},$$

and k th row given by

$$(U_3)_{kj} = \frac{Z_{1j}}{Z_{1k}}, \quad k < j < N,$$

$$(U_3)_{kN} = \frac{Z_{1N}}{2Z_{1k}}.$$

By the symmetry of \tilde{B}_N the $(N - k + 1)$ th column is also nontrivial:

$$(U_3)_{ik} = \begin{cases} \frac{-Z_{1N}}{2Z_{1k}^2} & k > n, i = 1, \\ \frac{-Z_{1,N-i+1}}{Z_{1k}} & k > n, 1 < i < N - k + 1, \\ \frac{Z_{1N}}{2Z_{1k}^2} & k \leq n, i = 1, \\ \frac{Z_{1,N-i+1}}{Z_{1k}} & k \leq n, 1 < i \leq n, \\ \frac{-Z_{1,N-i+1}}{Z_{1k}} & k \leq n, n < i < N - k + 1, \end{cases}$$

and all other entries of U_3 are zero. Then $w = U_3 Z U_3^{-1}$ and $T = U_3 U_1$. \square

As a consequence, $\tilde{B}_N \cdot E_N^a$ for $a \in \{i, c\}$ breaks into disjoint components, labelled by involutions. We denote these by F_π^a :

Definition 4.3. We set

$$F_\pi^a := \{M = (\Upsilon, \Lambda) \in E_N^a \mid \text{there exists } U \in \tilde{B}_N: U \Upsilon U^{-1} = \pi_\prec^a\}.$$

Theorem 4.3. *The sets F_π^a with the highest dimension have π corresponding to a link pattern.*

Remark. This is the analog of [16, Theorem 3] for $a = p$.

Proof. We have

$$\dim(F_\pi^a) = \dim(\widetilde{B}_N \cdot \pi_{<}^a) + \dim\{\Lambda \in \mathfrak{g}_N^a / R_N^a \mid (\pi_{<}^a \Lambda + \Lambda \pi_{<}^a)_{>} = 0\}.$$

To obtain the dimension of the \widetilde{B}_N -orbit, we calculate the dimension of the tangent space at $\pi_{<}^a$. This is $\{(U\pi_{<}^a - \pi_{<}^a U)\}$, where U is in the Lie algebra of \widetilde{B}_N ; that is, U is weakly upper-triangular with $U = -U^\dagger$.

The second term is

$$\begin{aligned} \dim\{\Lambda \in \mathfrak{g}_N^a / R_N^a \mid (\pi_{<}^a \Lambda + \Lambda \pi_{<}^a)_{>} = 0\} \\ = \dim(\mathfrak{g}_N^a / R_N^a) - \dim\{(\pi_{<}^a \Lambda + \Lambda \pi_{<}^a)_{>} \mid \Lambda \in \mathfrak{g}_N^a / R_N^a\}. \end{aligned}$$

The dimension of \mathfrak{g}_N^a / R_N^a is n^2 for $a = i$ and $n(n - 1)$ for $a = c$.

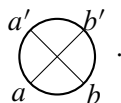
To calculate the dimensions of $\{(U\pi_{<}^a - \pi_{<}^a U)\}$ and $\{(\pi_{<}^a \Lambda + \Lambda \pi_{<}^a)_{>}\}$, we note that no more than two entries of $\pi_{<}^a$ will ever be involved in calculating a single entry of the matrices. Thus we can do the calculations for $N = 2, 4, 6, 8$ by brute force, and the results will extend easily to larger sizes. We find

$$\begin{aligned} \dim(F_\pi^i) &= n^2 + \# \begin{array}{c} a' \\ \bigcirc \\ a \end{array} + \# \begin{array}{c} a' \quad b' \\ \bigcirc \\ a \quad b \end{array} + 2\# \begin{array}{c} a' \quad b' \\ \bigcirc \\ a \quad b \end{array}, \\ \dim(F_\pi^c) &= n(n - 1) + \# \begin{array}{c} a' \quad b' \\ \bigcirc \\ a \quad b \end{array} + 2\# \begin{array}{c} a' \quad b' \\ \bigcirc \\ a \quad b \end{array}. \end{aligned}$$

The highest dimension is thus

$$\begin{aligned} \max(\dim(F_\pi^i)) &= n(n + 1), \\ \max(\dim(F_\pi^c)) &= 2 \left\lfloor \frac{n^2}{2} \right\rfloor, \end{aligned}$$

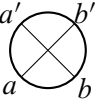
which occurs if π has as few fixed points as possible and no instances of

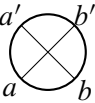


These π correspond to link patterns. □

Corollary 4.4. *The sets F_π^a with highest dimension minus one, i.e. $\dim(F_\pi^p) = 2n^2 - 1$, $\dim(F_\pi^i) = n(n + 1) - 1$, $\dim(F_\pi^c) = 2\left\lfloor \frac{n^2}{2} \right\rfloor - 1$, have π being an involution that looks like a link pattern except for:*

$a = p$: *one pair of fixed points,*

$a = i$: *one instance of , or one pair of fixed points at a and a',*

$a = c$: *one instance of .*

Definition 4.4. For π a link pattern $\in LP_L^a$, we define E_π^a as the closure of F_π^a .

The projection $(\Upsilon, \Lambda) \mapsto \Upsilon$ makes F_π a vector bundle over the orbit $\tilde{B}_N \cdot \pi_{<}^a$. As the closure of a vector bundle over the orbit of a connected group, E_π^a is irreducible. Theorem 4.3 says that the E_π^a are the top-dimensional components of E_N^a (we conjecture that there are no other components; this can probably be proved in a similar way as in the periodic case [23]).

Theorem 4.5. *Each E_π^a is generically reduced (as a component of E_N^a).*

Remark. This proof is similar to that of [16, Theorem 4].

Proof. We need to show that the Zariski tangent space at a generic point has the same dimension as E_N^a . The tangent space is given by the set of all matrices $P \in (\mathcal{M}_N^a)_{\Delta=0}$ that satisfy the derivative of the defining equation $M^2 = 0$:

$$P\underline{\pi}t + \underline{\pi}tP = 0,$$

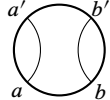
where $\underline{\pi}$ is the matrix representation of π (with diagonal zeroed out) that belongs to E_N^a , and $\underline{\pi}t$ is $\underline{\pi}$ multiplied by a generic diagonal matrix with restrictions necessary for the result to belong to \mathcal{M}_N^a . We note that both P and $\underline{\pi}t$ have zero diagonal. To compute the dimension of this set we consider the individual matrix components of the above equation:

$$\begin{aligned} 0 &= (P\underline{\pi}t + \underline{\pi}tP)_{ij} \\ &= P_{i\pi(j)}t_j [i < \pi(j) < j < i + N] + P_{\pi(i)j}t_{\pi(i)} [i < \pi(i) < j < i + N], \end{aligned} \tag{47}$$

where $[a]$ stands for 1 if a is true and 0 if a is false. In this equation, only two loops ever interact, so we only need to consider small size examples (up to $N = 8$). For $a = i$ there are 7 base cases that need to be considered. We will give 4 example calculations for $a = i$ here, the rest are similar.

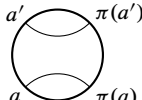
We first note that due to symmetry, the RHS of (47) is the same for (i, j) as for (j', i') , meaning that only one of these will contribute to the dimension. We also note that if $j = i'$, the RHS is automatically zero due to symmetry. Thus we will only give nontrivial equations where $j < i'$.

(1)

$\pi =$		i	j	
		a	b'	$0 = P_{ab}t_{b'}$
		b	a	$0 = P_{ba'}t_a + P_{b'a}t_{b'}$
		b'	a	$0 = P_{b'a'}t_a$

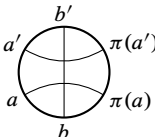
After applying the known symmetries of P_{ij} and t_j , there are only 2 independent equations in this list.

(2)

$\pi =$		i	j	
		a	$\pi(a')$	$0 = P_{\pi(a)\pi(a')}t_{\pi(a)}$
		$\pi(a')$	a	$0 = P_{a'a}t_{a'}$

These 2 equations are independent.

(3)

$\pi =$		i	j	
		a	b'	$0 = P_{ab}t_{b'} + P_{\pi(a)b'}t_{\pi(a)}$
		b	a	$0 = P_{b\pi(a)}t_a + P_{b'a}t_{b'}$
		$\pi(a)$	b	$0 = P_{\pi(a)b'}t_b + P_{ab}t_a$
		$\pi(a')$	b	$0 = P_{\pi(a')b'}t_b + P_{a'b}t_{a'}$

Here we have only included those equations that don't already appear in the previous example, thus for every pair of chords that are mirror images we must add 2 equations. The above four equations are independent, so in total there are 6 equations in this example.

(4)

$\pi =$

i	j	
a	$\pi(b)$	$0 = P_{ab}t_{\pi(b)} + P_{\pi(a)\pi(b)}t_{\pi(a)}$
a	$\pi(b')$	$0 = P_{\pi(a)\pi(b')}t_{\pi(a)}$
a	b'	$0 = P_{a\pi(b')}t_{b'} + P_{\pi(a)b'}t_{\pi(a)}$
b	a	$0 = P_{b\pi(a)}t_a + P_{\pi(b)a}t_{\pi(b)}$
$\pi(a)$	b'	$0 = P_{\pi(a)\pi(b')}t_{b'}$
$\pi(a)$	b	$0 = P_{\pi(a)\pi(b)}t_b + P_{ab}t_a$
$\pi(b)$	$\pi(a)$	$0 = P_{\pi(b)a}t_{\pi(a)} + P_{b\pi(a)}t_b$
$\pi(b')$	a	$0 = P_{b'a}t_{b'}$
$\pi(b')$	$\pi(a)$	$0 = P_{\pi(b')a}t_{\pi(a)} + P_{b'\pi(a)}t_{b'}$
b'	$\pi(a)$	$0 = P_{b'a}t_{\pi(a)}$

There are 8 independent equations in this list, and as before we must add 4 equations, giving a total of 12.

The other cases can be treated in the same way, and we find that there are 2 equations for every pair of chords in the *periodic* diagram of π , which comes to $n(n - 1)$ equations. The dimension of the larger space $(\mathcal{M}_N^i)_{\Delta=0}$ is $2n^2$, thus the dimension of the Zariski tangent space is $2n^2 - n(n - 1) = n(n + 1)$, the same as F_π^i .

For $a = c$ there are 6 base cases to consider. Recall that the symmetry implies that $P_{ii'} = 0$ and $t_i = 0$ if $\pi(i) = i'$. We give here two examples.

(1)

$\pi =$

i	j	
a	b'	$0 = P_{\pi(a)b'}t_{\pi(a)}$
b	a	$0 = P_{b\pi(a)}t_a$
$\pi(a)$	b	$0 = P_{ab}t_a$
$\pi(a')$	b	$0 = P_{a'b}t_{a'}$

These 4 equations are all independent.

(2)

	i	j			
$\pi =$			a	$\pi(b)$	$0 = P_{ab}t_{\pi(b)} + P_{\pi(a)\pi(b)}t_{\pi(a)}$
	a	$\pi(b')$		$0 = P_{\pi(a)\pi(b')}t_{\pi(a)}$	
	a	b'		$0 = P_{a\pi(b')}t_{b'} + P_{\pi(a)b'}t_{\pi(a)}$	
	b	a		$0 = P_{b\pi(a)}t_a + P_{\pi(b)a}t_{\pi(b)}$	
	$\pi(a)$	b'		$0 = P_{\pi(a)\pi(b')}t_{b'}$	
	$\pi(a)$	b		$0 = P_{\pi(a)\pi(b)}t_b + P_{ab}t_a$	
	$\pi(b)$	$\pi(a)$		$0 = P_{\pi(b)a}t_{\pi(a)} + P_{b\pi(a)}t_b$	
	$\pi(b')$	a		$0 = P_{b'a}t_{b'}$	
$\pi(b')$	$\pi(a)$		$0 = P_{\pi(b')a}t_{\pi(a)} + P_{b'\pi(a)}t_{b'}$		
b'	$\pi(a)$		$0 = P_{b'a}t_{\pi(a)}$		

There are 8 independent equations in this list.

We find that there are 2 equations for every pair of chords that are not mirror images, as well as 1 for every fixed point-chord pair, in total $n(n-1) - 2\lfloor \frac{n}{2} \rfloor$ equations. The dimension of the larger space $(\mathcal{M}_N^c)_{\Delta=0}$ is $2n(n-1)$, so the dimension of the Zariski tangent space is $2\lfloor \frac{n^2}{2} \rfloor$, the same as F_π^a . \square

4.3.2. Defining Equations. We first find another characterization of the E_π^a . From its definition, E_N^a , and therefore its irreducible components, are invariant by conjugation and scaling, i.e., under the action of the group $B_{\mathbb{Z} \bmod N} \times \mathbb{C}^\times$ (resp. $\widetilde{B}_{\mathbb{Z} \bmod N} \times \mathbb{C}^\times$, $\widetilde{B}_{\mathbb{Z} \bmod N} \times \mathbb{C}^\times$). The latter is a semi-direct product of $T_{\mathbb{Z} \bmod N} \times \mathbb{C}^\times$ and $U_{\mathbb{Z} \bmod N}$, where

$$U_{\mathbb{Z} \bmod N} := \{M \in B_{\mathbb{Z} \bmod N} \mid M_{ii} = 1 \ \forall i\},$$

and similarly for $\widetilde{B}_{\mathbb{Z} \bmod N}$ and $\widetilde{\widetilde{B}}_{\mathbb{Z} \bmod N}$, with $\widetilde{U}_{\mathbb{Z} \bmod N} = U_{\mathbb{Z} \bmod N} \cap \widetilde{B}_{\mathbb{Z} \bmod N}$ and $\widetilde{\widetilde{U}}_{\mathbb{Z} \bmod N} = U_{\mathbb{Z} \bmod N} \cap \widetilde{\widetilde{B}}_{\mathbb{Z} \bmod N}$. Since the use of the full group does not significantly simplify the orbit structure, we investigate below dense orbits under $\widetilde{U}_{\mathbb{Z} \bmod N}$ in the E_π^a , $a \in \{i, c\}$.

Theorem 4.6.

$$E_\pi^a = \overline{\widetilde{U}_{\mathbb{Z} \bmod N} \cdot \{\underline{\pi}t \mid t \text{ diagonal} \in \mathcal{M}_N^a\}}.$$

Proof. First we compute the dimension of $\widetilde{U}_{\mathbb{Z} \bmod N} \cdot \{\underline{\pi}t\}$. It is not hard to show that each $\widetilde{U}_{\mathbb{Z} \bmod N}$ -orbit contains only one $\underline{\pi}t$. Given this, we have

$$\dim(\widetilde{U}_{\mathbb{Z} \bmod N} \cdot \{\underline{\pi}t\}) = \dim(\{\underline{\pi}t\}) + \dim(\widetilde{U}_{\mathbb{Z} \bmod N} \cdot (\underline{\pi}t)),$$

where the second term, the dimension of a $\tilde{U}_{\mathbb{Z} \bmod N}$ -orbit for a generic choice of t , is equal to the number of equations defining the infinitesimal stabilizer

$$\{P \in \tilde{\mathfrak{u}}_{\mathbb{Z} \bmod N} \mid P\underline{\pi}t - \underline{\pi}tP = 0\},$$

where $\tilde{\mathfrak{u}}_{\mathbb{Z} \bmod N}$ is the Lie algebra of $\tilde{U}_{\mathbb{Z} \bmod N}$. We call this number d_P^a .

Unlike the periodic case, we cannot calculate the number of equations defining the infinitesimal stabilizer in the same way as for Theorem 4.5. Instead, we have

$$\begin{aligned} d_P^i &= n^2 - \#(a \sim a'), \\ d_P^c &= n(n-1), \end{aligned}$$

and

$$\begin{aligned} \dim(\{\underline{\pi}^i t\}) &= n + \#(a \sim a'), \\ \dim(\{\underline{\pi}^c t\}) &= 2 \left\lfloor \frac{n}{2} \right\rfloor, \end{aligned}$$

so

$$\begin{aligned} \dim(\tilde{U}_{\mathbb{Z} \bmod N} \cdot \{\underline{\pi}t\}) &= n(n+1), \\ \dim(\tilde{U}_{\mathbb{Z} \bmod N} \cdot \{\underline{\pi}t\}) &= 2 \left\lfloor \frac{n^2}{2} \right\rfloor. \end{aligned}$$

For π a link pattern, we note that $\tilde{U}_{\mathbb{Z} \bmod N} \cdot \{\underline{\pi}t \mid t \text{ invertible}\} \subset F_\pi^a$, because the upper triangular part of any matrix in the former is \tilde{B}_N -conjugate to $\pi_<$. Therefore, $\overline{\tilde{U}_{\mathbb{Z} \bmod N} \cdot \{\underline{\pi}t \mid t \text{ invertible}\}} = \overline{\tilde{U}_{\mathbb{Z} \bmod N} \cdot \{\underline{\pi}t\}} \subset \overline{F_\pi^a} = E_\pi^a$. Since $\overline{\tilde{U}_{\mathbb{Z} \bmod N} \cdot \{\underline{\pi}t\}}$ has the same dimension as E_π^a , and the latter is irreducible, they must be equal. \square

A similar statement holds for $a \in \{0, m\}$, i.e., $E_\pi^a = \overline{\tilde{U}_{\mathbb{Z} \bmod N} \cdot \{\underline{\pi}t\}}$.

Theorem 4.7. Any $M \in E_\pi^a$ satisfies the following equations:

- (1) $M \in \mathcal{M}_N^a$;
- (2) $M^2 = 0$;
- (3) $s_k(M) = s_l(M)$ when $l \in \text{cl}(k) \cup \text{cl}(\pi(k))$;
- (4) $\text{rm}(M) \leq \text{rm}(\underline{\pi})$.⁶

⁶Note that the symmetries in condition (1) implies $\text{rm}(M)_{ij} = \text{rm}(M)_{kl}$ when $(k, l) \in \text{cl}(i, j)$.

Proof. For $a = p$ this was proved in [16]. For $a \in \{i, c\}$ only equations (3) and (4) are new; one easily checks that they are satisfied by $\underline{\pi}t$, and that they are invariant by conjugation by $\widetilde{B}_{\mathbb{Z} \bmod N}$. \square

We conjecture that these are the defining equations of E_π^a . At least, we know that these equations define a set that is the union of E_π^a and of lower dimensional pieces, because the other top-dimensional components contain matrices that do not satisfy the equations. If this conjecture is true, it implies the following:

Conjecture 4.1. For any link pattern $\pi \in LP_L^a$ and its associated periodic link pattern $\tilde{\pi}$ as described at the start of Section 4.3, we have

$$E_{\tilde{\pi}}^p \cap \mathcal{M}_N^a = E_\pi^a.$$

Once again, one can prove the slightly weaker statement that $E_{\tilde{\pi}}^p \cap \mathcal{M}_N^a$ has E_π^a as its unique top-dimensional component, because intersecting with any other component of $E_{\tilde{\pi}}^a$ reduces its dimension, some equation of Theorem 4.7 for $E_{\tilde{\pi}}^p$ being violated.

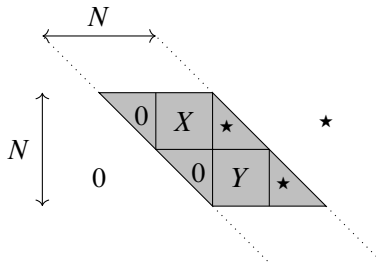
4.4. The permutation sector. For simplicity we assume in this section that $N = 2n = 4m$ (the case $n = 2m + 1$ can be treated analogously).

Define the permutation subspace $\mathcal{M}_N^{\text{perm}}$ to be the linear subspace of \mathcal{M}_N

$$\mathcal{M}_N^{\text{perm}} := \{M \in \mathcal{M}_N \mid M_{ij} = 0 \text{ for } m < i \leq j \leq 3m \text{ or } 3m < i < j \leq 5m\}.$$

(Compared to the definition in [16, Section 5], we have shifted by m along the diagonal in order for the subspace to be invariant under the symmetry of type \widehat{C} .)

In the strip picture, choosing the fundamental domain to be between rows $m + 1$ and $5m$, we have



where we label the two $n \times n$ submatrices X and Y for convenience.

Now define

$$E_N^{a,\text{perm}} := E_N^a \cap \mathcal{M}_N^{\text{perm}}, \quad a \in \{p, i, c\}.$$

In [16], it is explained how the equations satisfied by M in $E_N^{p,\text{perm}}$ only involve X and Y , and are:

$$E_N^{p,\text{perm}} = \{XY \text{ and } YX \text{ upper triangular}\},$$

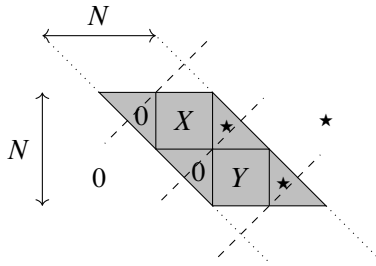
(so that it is isomorphic to the so-called “upper-upper scheme”

$$\{X, Y \in \mathfrak{gl}_n \mid XY \text{ and } YX \text{ upper triangular}\},$$

see [14], times some irrelevant vector space), and that $E_N^{p,\text{perm}}$ is a complete intersection, allowing us to compute its multidegree:

$$\text{mdeg } E_N^{p,\text{perm}} = A^N \prod_{\substack{m < i < j \leq 3m \\ \text{or} \\ 3m < i < j \leq 5m}} (A + z_i - z_j)(2A + z_j - z_i - \epsilon).$$

The same argument works for $a \in \{i, c\}$. The symmetry axes are



so that we find

$$E_N^{i,\text{perm}} = \{Y = X^\dagger, XY \text{ and } YX \text{ upper triangular}\},$$

$$E_N^{c,\text{perm}} = \{Y = -X^\dagger, XY \text{ and } YX \text{ upper triangular}\},$$

where $X^\dagger = J^{-1}X^T J$, and J denotes the $n \times n$ skew-symmetric matrix of the type of (45). That is, $E_N^{i,\text{perm}}$ and $E_N^{c,\text{perm}}$ are isomorphic to the “symplectic upper-upper scheme” $\{X \in \mathfrak{gl}_n \mid XX^\dagger \text{ and } X^\dagger X \text{ upper triangular}\}$ times some irrelevant vector spaces (the latter being due to the \star entries, being careful that the symmetry imposes linear relations between them, and in particular imposes zeroes on the symmetry axis in $E_N^{c,\text{perm}}$).

The counting of equations goes as follows: taking into account the symmetry of M , there are $2m(m+1)$ (resp. $2m^2$) linear equations defining $E_N^{i,\text{perm}}$ (resp. $E_N^{c,\text{perm}}$). In both cases, similarly taking into account the symmetry of XX^\dagger and $X^\dagger X$, there are $2m(m-1)$ quadratic equations. The total number of equations is therefore equal to $4m^2$ (resp. $2m(2m-1)$), which is the codimension of $E_N^a \supset E_N^{\text{perm}}$; therefore E_N^{perm} is a complete intersection, and its multidegree is the product of the weights of its equations:

$$\begin{aligned} \text{mdeg } E_N^{i,\text{perm}} = & \prod_{1 \leq i \leq j \leq m} (A + z_i - z_j)(A - z_i - z_j - \epsilon) \\ & \prod_{1 \leq i < j \leq m} (2A + z_j - z_i - \epsilon)(2A + z_i + z_j) \\ & \prod_{m+1 \leq i \leq j \leq 2m} (A + z_i - z_j)(A + z_i + z_j) \\ & \prod_{m+1 \leq i < j \leq 2m} (2A + z_j - z_i - \epsilon)(2A - z_i - z_j - \epsilon), \end{aligned} \quad (48a)$$

$$\begin{aligned} \text{mdeg } E_N^{c,\text{perm}} = & A^{2m} \prod_{1 \leq i < j \leq m} (A + z_i - z_j)(A - z_i - z_j - \epsilon) \\ & (2A + z_j - z_i - \epsilon)(2A + z_i + z_j) \\ & \prod_{m+1 \leq i < j \leq 2m} (A + z_i - z_j)(A + z_i + z_j) \\ & (2A + z_j - z_i - \epsilon)(2A - z_i - z_j - \epsilon). \end{aligned} \quad (48b)$$

As a complete intersection E_N^{perm} is equidimensional (of the same dimension as E_N^a), and therefore a union of top-dimensional components of E_N^a . In order to find which, we simply test whether $\underline{\pi}t$ belongs to $\mathcal{M}_N^{\text{perm}}$. We easily find

$$E_N^{\text{perm}} = \bigcup_{\pi: \pi(\{1, \dots, m\}) = \{m+1, \dots, 2m\}} E_\pi^a.$$

Such link patterns are in bijection with permutations of $\{1, \dots, m\}$.

Considering $m_\pi = 2^n$ for all such link patterns, we can also write

$$\sum_{\pi: \pi(\{1, \dots, m\}) = \{m+1, \dots, 2m\}} \phi_\pi^a = 2^n \text{mdeg } E_N^{\text{perm}}, \quad a \in \{i, c\},$$

where the RHS is given explicitly by (48).

Note that we have $E_\pi^i \cong E_\pi^c \times \mathbb{C}^n$ for all such π , or

$$\phi_\pi^i = \prod_{i=1}^m (A - 2z_i - \epsilon) \prod_{i=m+1}^{2m} (A + 2z_i) \phi_\pi^c, \quad \pi(\{1, \dots, m\}) = \{m + 1, \dots, 2m\}.$$

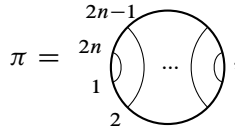
(The prefactor is due to the different embedding space.)

4.5. Commuting varieties. In [14], it is shown that one particular component of the upper-upper scheme is the singular fiber of a one-parameter flat family whose generic fiber is the *commuting variety*

$$C_n := \{X, Y \in \mathfrak{gl}_n \mid XY = YX\}.$$

In [16], this was used to provide a formula for the degree of the commuting variety; in our notations, one has

$$\deg C_n = \deg E_{(2n, 2n-1, \dots, 1)}^p,$$



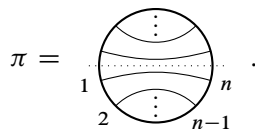
(and more generally, equality of multidegrees with the appropriate correspondence of torus actions).

Assume now n even. Using the exact same argument, one can show that a particular component of the “symplectic upper-upper scheme” (see previous section) is the singular fiber of a one-parameter flat family whose generic fiber is the “symplectic commuting variety”

$$\tilde{C}_n := \{X \in \mathfrak{gl}_n \mid XX^\dagger = X^\dagger X\}.$$

This implies that

$$\begin{aligned} \deg \tilde{C}_n &= \deg E_{(n, n-1, \dots, 1)}^i = \deg E_{(n, n-1, \dots, 1)}^c \\ &= 1, 11, 1583, 3186265, 92351668113 \dots, \quad n = 2, 4, \dots, \end{aligned}$$



In principle, an explicit formula for the (multi)degree of \widetilde{C}_n can be obtained by repeated application of divided difference operators and by using some formulae of [7] in type $a = c$. We shall not reproduce them here because they are rather cumbersome.

5. From the Brauer loop schemes to the loop model

We now provide the link between the geometric construction of Section 4 to the loop model of Sections 2 and 3. As explained in Section 2.3 and Section 4.1.2, the correspondence of parameters is as follows.

- The length of the loop model L is related to the size N of the matrices by: $L = N$ for $a = p$, $L = N/2$ for $a \in \{i, c\}$, $L = N/4$ for $a \in \{o, m\}$.
- The shift s of the qKZ equation is related to the equivariance parameter ϵ by: $\epsilon = s$ for $a \in \{p, i, c\}$, $\epsilon = s/2$ for $a \in \{o, m\}$.

The precise theorem, as advertised in the introduction is:

Theorem 5.1. *In all types $a \in \{p, i, c, o, m\}$, the vector $|\Phi\rangle = \sum_{\pi \in \text{LP}_L^a} \phi_\pi^a |\pi\rangle$ of multidegrees $\phi_\pi^a = m_\pi \text{mdeg } E_\pi^a$ of the irreducible components of the Brauer loop scheme E_N^a satisfies the qKZ system (10–11) or (13–15), as well as the recurrence relations (27), (32) and (33) (up to normalization), thus identifying it with the unique (up to normalization) minimal degree polynomial solution of the qKZ system.*

The rest of this section is dedicated to the proof of this theorem.

5.1. Lévy subgroups. The geometric interpretation of the quantum Knizhnik–Zamolodchikov equation follows the same general philosophy that was outlined in [9] and then developed in [16, 17]. It is based on a combination of “cutting” – intersecting with hypersurfaces – and “sweeping” – taking the image under Lévy subgroups, similarly to the pullbacks/pushforwards in convolution actions [4].

Let $B \subset \text{SL}_2$ be the group of 2×2 invertible upper triangular matrices inside the group of 2×2 matrices of determinant 1. We start with the following standard lemma: (see also [17, lemma 8] and [16, lemma 1]; the use of GL_2 instead of SL_2 makes no difference)

Lemma 5.2. *Let X be a variety in a vector space V equipped with an SL_2 -representation, such that X is B -invariant and conical. If the generic fiber of the map*

$$\mu : \mathrm{SL}_2 \times_B X \longrightarrow V$$

is finite over $\mathrm{Image} \mu$, call its cardinality k ; otherwise let $k = 0$. (The latter occurs iff X is SL_2 -invariant.) Then

$$k \text{ mdeg}(\mathrm{Image} \mu) = -\partial_i \text{ mdeg} X,$$

where ∂_i is the divided difference operator $\partial_i f = (f - \tau f)/\alpha$ as defined in Section 1.3, and α is the root of SL_2 .

The multidegree is w.r.t. the Cartan torus of SL_2 .

To each node i in the Dynkin diagram of \hat{A} or \hat{C} , we can associate groups $B^{(i)} \subset \mathrm{SL}_2^{(i)}$ (isomorphic to $B \subset \mathrm{SL}_2$) defined by

$$\mathrm{SL}_2^{(i)} = \left\{ M = (M_{jk})_{j,k \in \mathbb{Z}} \left| \begin{array}{l} MS^N = S^N M, \\ MM^\dagger = 1 \text{ if } a \in \{i, c, o, m\}, \\ MM^\ddagger = 1 \text{ if } a \in \{o, m\}, \\ M_{jk} = \delta_{jk} \text{ unless } (j, k) \in \mathrm{cl} \left(\left\{ \begin{array}{c} i \\ i+1 \end{array} \right\}, \left\{ \begin{array}{c} i \\ i+1 \end{array} \right\} \right), \\ \left| \begin{array}{cc} M_{ii} & M_{i,i+1} \\ M_{i+1,i} & M_{i+1,i+1} \end{array} \right| = 1 \end{array} \right\},$$

and

$$B^{(i)} = \mathrm{SL}_2^{(i)} \cap B_{\mathbb{Z} \bmod N}.$$

Note that in all types, the isomorphism from $\mathrm{SL}_2^{(i)}$ to SL_2 consists in extracting the 2×2 submatrix at rows and columns $i, i+1$. When there is no risk of confusion we shall identify $\mathrm{SL}_2^{(i)}$ and SL_2 via this isomorphism.

Next, given $P \in \mathrm{SL}_2^{(i)}$ and $M \in R_{\mathbb{Z} \bmod N}^a$, one can consider conjugating: PMP^{-1} . Two problems arise at this stage. Firstly, the result has entries below the diagonal. There are various ways to deal with this: the one we choose here is to restrict ourselves to matrices sitting in the subspace $R_{\mathbb{Z} \bmod N}^a \cap \{M_{ii} = M_{i+1,i+1} = M_{i,i+1} = 0\}$, which is stable under the $\mathrm{SL}_2^{(i)}$ action. Secondly, this *does not* descend to an action on (appropriate subspaces of) \mathcal{M}_N^a , because the action does not preserve the ideal generated by S^N . In [17], this difficulty is circumvented by working inside $R_{\mathbb{Z} \bmod N}^a$, i.e., taking preimages of subvarieties of

\mathcal{M}_N^a before taking their image (“sweeping” them) under $SL_2^{(i)}$ and then taking the image again in \mathcal{M}_N^a . Here, to slightly simplify the discussion, we shall by abuse of notation identify such a subvariety with its preimage.

Taking into account the fact that the multidegree depends on the embedding space, we are led to the following modification of the lemma.

- For “closed boundaries,” that is, for $a = c$ and $i = 0, L$ or for $a = m$ and $i = 0$, the lemma applies without any changes to the multidegrees w.r.t. \mathcal{M}_N^a , the divided difference operators being the ones defined in (1).
- In all other cases, to apply the lemma to multidegrees w.r.t. \mathcal{M}_N^a , the divided difference operator of (1) has to be conjugated, i.e., replaced with

$$\partial'_i = (A + \alpha_i) \partial_i \frac{1}{A + \alpha_i},$$

(the factor $A + \alpha_i$ being the weight of $M_{i,i+1}$).

5.2. Geometry of the exchange relations. We start with the exchange relations (10) or (13), $i = 1, \dots, L - 1$. Note that in type \widehat{A} , the rotation equation (11) is trivially satisfied due to the cyclic nature of the Brauer loop scheme (see [17]), so equation (10) is also valid at $i = L$.

We rewrite this equation here for convenience:

$$\check{R}_i(z_i - z_{i+1}) |\Phi(\dots, z_i, z_{i+1}, \dots)\rangle = |\Phi(\dots, z_{i+1}, z_i, \dots)\rangle, \tag{49}$$

where $|\Phi\rangle = \sum_{\pi \in LP_L^a} \phi_\pi^a |\pi\rangle$.

As explained in Section 2.3.1, when writing (49) in components, there are two cases to consider, depending on whether $\pi(i) \neq i + 1$ or $\pi(i) = i + 1$. We treat them separately below.

5.2.1. The f_i action. We assume that $\pi \in LP_L^a$ is such that $\pi(i) \neq i + 1$. Our goal is to prove (18) for the multidegrees ϕ_π^a . The geometric procedure is as follows:

- “Sweep” E_π^a with $SL_2^{(i)}$.
- “Cut” the result with $(M^2)_{i+1,i+N} = 0$, and show that it produces $E_\pi^a \cup E_{f_i \pi}^a$.

Equation (18) will then be a translation into multidegrees of this construction.

We shall need the following

Lemma 5.3. (1) *If M, M' are generic elements of components E_π^a and $E_{\pi'}^a$, with $\pi(i) \neq i + 1$, $\pi'(i) \neq i + 1$, such that $M' = PMP^{-1}$, $P \in \mathrm{SL}_2^{(i)}$, then $s_j(M) = s_j(M')$ if $j \in \mathrm{cl}(i), \mathrm{cl}(i + 1)$, and $\{s_i(M), s_{i+1}(M)\} = \{s_i(M'), s_{i+1}(M')\}$; and for a fixed M' (resp. M), the set of possible cosets of P in $\mathrm{SL}_2^{(i)}/B^{(i)}$ (resp. $B^{(i)} \setminus \mathrm{SL}_2^{(i)}$) consists of exactly two points, corresponding to whether these s_i, s_{i+1} are in the same order, or reversed.*

(2) *In the particular case where $M = \underline{\pi}t$, t generic diagonal, then the two classes of P have representatives $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; in the latter case,*

$$P\underline{\pi}tP^{-1} = \underline{f_i}\pi t',$$

with t' the diagonal matrix obtained from t by switching diagonal entries at $j \in \mathrm{cl}(i), \mathrm{cl}(i + 1)$. M and M' playing symmetric roles, an analogous result holds for $M' = \underline{\pi}t$.

Proof. P conjugates the matrices M^2, M'^2 , and in particular (restricting to rows $i, i + 1$ and columns $i + N, i + N + 1$), their 2×2 submatrices around the N^{th} diagonal, which are upper triangular with eigenvalues $s_i(M), s_{i+1}(M)$ and $s_i(M'), s_{i+1}(M')$. Generically these eigenvalues are distinct because $\pi(i) \neq i + 1$ or $\pi'(i) \neq i + 1$, so that $P = b'^{-1}P_0b$ where b, b' are upper triangular matrices which diagonalize these 2×2 submatrices, and P_0 is as in the second part of the lemma. The rest is a direct computation. \square

The degree of the sweeping map. We compute the cardinality of the generic fiber of the map $\mathrm{SL}_2^{(i)} \times_{B^{(i)}} E_\pi^a \rightarrow \mathrm{SL}_2^{(i)} \cdot E_\pi^a$. Since the $\mathrm{SL}_2^{(i)}$ action is a group action, we may assume that the fiber $\{(P, M) \mid PMP^{-1} = M'\}$ is that of an element M' of E_π^a , and furthermore that it is of the form $M' = \underline{\pi}t$. We are in the situation of Lemma 5.3 with $M, M' \in E_\pi^a$. We conclude that there are two possibilities:

- if $f_i\pi \neq \pi$, (i.e., if at least one of i or $i + 1$ is paired in π (not connected to the boundary), or in type a = 0 if they are connected to different boundaries), then only the first coset, namely, $P \in B^{(i)}$, leads to $M' \in E_\pi^a$, and therefore the fiber (in $\mathrm{SL}_2^{(i)} \times_{B^{(i)}} E_\pi^a$) is a point;
- if $f_i\pi = \pi$ (i.e., if both i and $i + 1$ are connected to the (same) boundary), then both cosets lead to $M' \in E_\pi^a$, and the fiber consists of two points.

In conclusion, we find that

$$\text{cardinality of a generic fiber of } \text{SL}_2^{(i)} \times_{B^{(i)}} E_\pi^a \rightarrow \text{SL}_2^{(i)} \cdot E_\pi^a = \begin{cases} 1 & \text{if } f_i \pi \neq \pi, \\ 2 & \text{if } f_i \pi = \pi. \end{cases} \quad (50)$$

Determination of the result of sweeping and cutting. The generic fiber being finite, the image has same dimension as the source, i.e., $\dim(\text{SL}_2^{(i)} \cdot E_\pi^a) = \dim E_\pi^a + 1$. An elementary calculation shows that the only equation of E_N^a that $\text{SL}_2^{(i)} \cdot E_\pi^a$ violates is $(M^2)_{i+1, i+N} = 0$. Noting that $\{(M^2)_{i+1, i+N} = 0\}$ is a Cartier divisor in the (irreducible) variety $\text{SL}_2^{(i)} \cdot E_\pi^a$, we conclude that $(\text{SL}_2^{(i)} \cdot E_\pi^a) \cap \{(M^2)_{i+1, i+N}\}$ is a subscheme of E_N^a of pure dimension $\dim E_N^a$, therefore a union of its top-dimensional components E_π^a .

In order to determine which, we apply again Lemma 5.3. We first compute the image of $M = \underline{\pi}t$:

$$(\text{SL}_2^{(i)} \cdot \{\underline{\pi}t\}) \cap \{(M^2)_{i+1, i+N} = 0\} = B^{(i)} \cdot \{\underline{\pi}t\} \cup B^{(i)} \cdot \{\underline{f_i \pi} t'\},$$

with t' as in the lemma. Finally, taking the union over t and the closure of the $B_{\mathbb{Z} \bmod N}$ (resp. $\widetilde{B}_{\mathbb{Z} \bmod N}$, $\widetilde{\widetilde{B}}_{\mathbb{Z} \bmod N}$) orbit, we obtain:

$$(\text{SL}_2^{(i)} \cdot E_\pi^a) \cap \{(M^2)_{i+1, i+N} = 0\} = E_\pi^a \cup E_{f_i \pi}^a. \quad (51)$$

The equation above is only an equality of sets; however since E_N^a is generically reduced in top dimension (Theorem 4.5), both sides of the equation are generically reduced (which is all that matters for multidegree purposes).

Multidegree equality. Using (50) and applying Lemma 5.2, we have

$$\text{mdeg } \text{SL}_2^{(i)} \cdot E_\pi^a = 2^{\delta_{\pi, f_i \pi}} (-\partial'_i) \text{mdeg } E_\pi^a.$$

Next we intersect the variety $\text{SL}_2^{(i)} \cdot E_\pi^a$ with the hypersurface

$$\{(M^2)_{i+1, i+N} = 0\};$$

by the properties of multidegrees, this multiplies its multidegree with

$$\text{mdeg}\{(M^2)_{i+1, i+N} = 0\} = 2A + z_{i+1} - z_{i+N} = 2A + z_{i+1} - z_i - \epsilon.$$

Finally, we apply (51), noting that the factor of 2 when $\pi = f_i \pi$ is compensated by the fact that $E_\pi = E_{f_i \pi}$, and find

$$(2A + z_{i+1} - z_i - \epsilon)(-\partial'_i) \text{mdeg } E_\pi^a = \text{mdeg } E_\pi^a + \text{mdeg } E_{f_i \pi}^a.$$

Equivalently, using (44) and noting that π and $f_i\pi$ have the same number of chords, we find

$$(2A + z_{i+1} - z_i - \epsilon)(-\partial'_i)\phi_\pi^a = \phi_\pi^a + \phi_{f_i\pi}^a. \tag{52}$$

5.2.2. The e_i action. We now assume that $\pi \in \text{LP}_L^a$ is such that $\pi(i) = i + 1$, and we wish to prove that (19) is satisfied by the multidegrees ϕ_π^a .

Fully interpreting geometrically the e_i equation (19) is rather complicated (see [26] and [17, arXiv v1] for the case of type \hat{A}). Given such a link pattern π , the geometric construction is:

- cut E_π^a with $M_{i,i+1} = 0$, producing F_1 ;
- throw away the $\text{SL}_2^{(i)}$ -invariant components (giving $\bigcup_{\rho \in \mathcal{E}(\pi,i)} X_{\rho,i}^a$, to be defined below), and then sweep with $\text{SL}_2^{(i)}$, producing F_2 ;
- finally, cut with $(M^2)_{i+1,i+N} = 0$, producing F_3 , and show that

$$F_3 = \bigcup_{\rho \neq \pi: e_i \rho = \pi} E_\rho^a \cap \{s_i(M) = s_{i+1}(M)\}.$$

The multidegree of F_3 is the desired expression.

Here we shall only provide a semi-geometric proof of (18), as in [17]: we shall stop at the first stage in the construction, i.e., only analyze F_1 above, and then use the (already proven) f_i equation (52) to conclude.

The auxiliary varieties $X_{\rho,i}^a$. Denote by $|\rho|$ the number of crossings of ρ .

Proposition 5.4. *Given ρ a link pattern such that $\rho(i) \neq i + 1$ and $|\rho| \geq |f_i \rho|$:*

- *If $f_i \rho \neq \rho$, then $E_\rho^a \cap \{s_i(M) = s_{i+1}(M)\}$ has a single geometric component. Call it $X_{\rho,i}^a$.*
- *If $f_i \rho = \rho$, $E_\rho^a \cap \{s_i(M) = s_{i+1}(M)\}$ has two geometric components, one of which is $\text{SL}_2^{(i)}$ -invariant. Call $X_{\rho,i}^a$ the other one.*

In both cases $E_\rho^a \cap \{s_i(M) = s_{i+1}(M)\}$ is generically reduced at $X_{\rho,i}^a$.

Proof. (We only give the full proof for $a \in \{p, i, c\}$.) This is similar to [17, Proposition 9]. Since $\rho(i) \neq i + 1$, $E_\rho^a \not\subset \{s_i(M) = s_{i+1}(M)\}$ (as can be checked on say $\underline{\rho}t$), so all the geometric components of $E_\rho^a \cap \{s_i(M) = s_{i+1}(M)\}$ have dimension $\dim E_N^a - 1$. We use the decomposition

$$E_\rho^a \cap \{s_i(M) = s_{i+1}(M)\} = \bigsqcup_{\sigma} (E_\rho^a \cap \{s_i(M) = s_{i+1}(M)\} \cap F_\sigma),$$

and consider only pieces of dimension $\dim E_N^a - 1$.

We start from F_σ itself. For it to have dimension $\geq \dim E_N^a - 1$, according to Corollary 4.4, σ can have one pair of the form

$$\begin{array}{c}
 \begin{array}{ccc}
 & a' & b' \\
 & \diagdown & / \\
 \circ & & \\
 & / & \diagdown \\
 a & & b
 \end{array}
 \end{array}
 \quad \text{for } a \in \{i, c\},$$

or a pair of fixed points for $a \in \{p, i\}$.

We then intersect with $\{s_i(M) = s_{i+1}(M)\}$. There are three possibilities.

(1) $\sigma(\{i, i + 1\}) \not\subset \{i, i + 1, N - i, N - i + 1\}$. The equation $s_i(M) = s_{i+1}(M)$ being \widetilde{B}_N -invariant and linear in Λ (in the $M = \Upsilon + \Lambda$ decomposition, cf. Section 4.3.1), $F_\sigma \cap \{s_i(M) = s_{i+1}(M)\}$ is a subvector bundle of F_σ , where the dimension of the fiber can be easily evaluated, say at $\Upsilon = \sigma^a$, where the extra equation $\Lambda_{i \leftrightarrow \sigma(i)} = \Lambda_{i+1 \leftrightarrow \sigma(i+1)}$ reduces it by one compared to that of F_σ . This implies that σ must be a link pattern (otherwise the dimension is too low).

(2) $\sigma(i) = N - i, \sigma(i + 1) = N - i + 1$, in which case $F_\sigma \subset \{s_i(M) = s_{i+1}(M)\}$. For the dimension to be right, σ cannot have any other crossing pairs of the same form or any fixed points.

(3) $\sigma(i) = i + 1, \sigma(i + 1) = i$.

Now we want to intersect with E_ρ^a . This immediately excludes case (3), because if $\sigma(i) = i + 1$, the rank condition (equations (4) in Theorem 4.7) of E_ρ^a at $(i, i + 1)$ is violated, so $F_\sigma \cap E_\rho^a$ is empty. We are left with cases (1) and (2); in both, $F_\sigma \cap \{s_i(M) = s_{i+1}(M)\}$ is a subvector bundle of F_σ , so is an open (irreducible) variety of the target dimension.

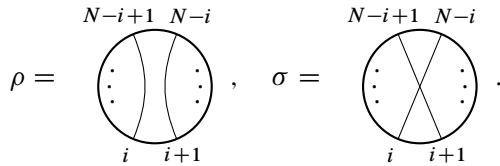
If σ differs from ρ outside $\{i, i + 1\}$, then it is easy to see that some of the equations of Theorem 4.7 of E_ρ^a are violated by say $\underline{\sigma}t$ with $t_i \neq 0, t_i t_{\sigma(i)} = t_{i+1} t_{\sigma(i+1)}$. Indeed, the $s_i(\underline{\sigma}t)$ only have the repeats of the pairings of σ , so ρ cannot have more pairings than σ (otherwise equations (3) of Theorem 4.7 would be violated). Inversely, assuming the pairings of ρ are a strict subset of those of σ , i.e., there exists $i < \sigma(i) = j$, but $\rho(i) \neq j$, then the rank condition (equation (4) of Theorem 4.7) of ρ at (i, j) would be violated (and in fact, in that case, the intersection would be empty). It follows that the dimension of $F_\sigma \cap \{s_i(M) = s_{i+1}(M)\} \cap E_\rho^a$ is less than the target. Therefore the two possibilities reduce to the following case.

(1) $\sigma = \rho$, in which case of course

$$E_\rho^a \cap \{s_i(M) = s_{i+1}(M)\} \cap F_\sigma = F_\rho \cap \{s_i(M) = s_{i+1}(M)\}.$$

- If $f_i \rho = \rho$, we show in Appendix C that $\overline{F_\rho \cap \{s_i(M) = s_{i+1}(M)\}}$ is $\text{SL}_2^{(i)}$ -invariant.
- If $f_i \rho \neq \rho$, we call $X_{\rho,i}^a$ the closure of $F_\rho \cap \{s_i(M) = s_{i+1}(M)\}$ (with its reduced structure).

(2) $\sigma(i) = N - i, \sigma(i + 1) = N - i + 1$ and is identical to ρ elsewhere, i.e.,



This situation can only occur when $f_i \rho = \rho$. In this case, we claim that

$$\overline{F_\sigma} = \overline{E_\rho^a \cap F_\sigma \cap \{s_i(M) = s_{i+1}(M)\}}.$$

That $F_\sigma = F_\rho \cap \{s_i(M) = s_{i+1}(M)\}$ is obvious.

Next, we note that

$$\overline{\tilde{B}_N \cdot \rho_{<}^i} \supset \tilde{B}_N \cdot \sigma_{<}^i.$$

Indeed, the matrix P with submatrix

$$\begin{pmatrix} i/t & 1/t & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 1/t & 1/t \\ 0 & 0 & 0 & -it \end{pmatrix}$$

at rows $i, i + 1, N - i, N - i + 1$ and identity elsewhere is symplectic and sends $\rho_{<}^i$ to $P\rho_{<}^i P^{-1} = \sigma_{<}^i + O(t^2)$.

This implies that $E_\rho \cap F_\sigma = \overline{F_\rho} \cap F_\sigma$ is a vector bundle over $\tilde{B}_N \cdot \sigma_{<}^i$; the dimension of its fiber is greater or equal to that of F_ρ by semi-continuity of rank, and less than or equal to that of F_σ by obvious inclusion. But according to the dimension count in the proof of Theorem 4.3, the latter two are equal, and therefore there is equality of dimensions, which implies $E_\rho \cap F_\sigma = F_\sigma$.

In this case, we call $\overline{F_\sigma}$ (with its reduced structure) $X_{\rho,i}^a$.

In all cases, note that, as the closure of a vector bundle over an open (irreducible) variety, $X_{\rho,i}^a$ is irreducible. Generic reducedness in $E_\rho^a \cap \{s_i(M) = s_{i+1}(M)\}$ is shown in Appendix B. □

Remark. One can also analyze the case $|\rho| < |f_i \rho|$, cf. [17, Appendix B] in type \widehat{A} , with similar conclusions as when $\rho = f_i \rho$, but we shall not need it here.

Determination of the result of cutting. We recall that a link pattern π such that $\pi(i) = i + 1$ is fixed. Define

$$\varepsilon(\pi, i) := \{\rho \neq \pi \mid e_i \rho = \pi, |\rho| \geq |f_i \rho|\},$$

e.g.,

$$\begin{aligned} a = i: \quad \pi &= \left(\text{Diagram 1} \right), \quad \varepsilon(\pi, 1) = \left\{ \left(\text{Diagram 2} \right), \left(\text{Diagram 3} \right), \left(\text{Diagram 4} \right) \right\}, \\ a = c: \quad \pi &= \left(\text{Diagram 5} \right), \quad \varepsilon(\pi, 2) = \left\{ \left(\text{Diagram 6} \right) \right\}. \end{aligned}$$

Then one has $e_i^{-1}(\pi) - \{\pi\} = \bigcup_{\rho \in \varepsilon(\pi, i)} \{\rho, f_i \rho\}$.

Proposition 5.5. *The geometric components of $E_\pi^a \cap \{M_{i, i+1} = 0\}$ are the $X_{\rho, i}^a$, $\rho \in \varepsilon(\pi, i)$, as well as one extra $SL_2^{(i)}$ -invariant component in the cases $a \in \{p, c\}$.*

The multiplicity of $X_{\rho, i}^a$ in $E_\pi \cap \{M_{i, i+1} = 0\}$ is 1 except for the single case (in type $a = o$) of ρ connecting $i, i + 1$ to distinct boundaries, i.e., $\rho(i) = r$, $\rho(i + 1) = \ell$, in which case the multiplicity is 2.

Proof. The proof is along the same lines as that of Proposition 5.4. Since $\pi(i) = i + 1$, $E_\pi^a \not\subset \{M_{i, i+1} = 0\}$ and $E_\pi^a \cap \{M_{i, i+1} = 0\}$ is of dimension $\dim E_N^a - 1$. Once again we use the decomposition

$$E_\pi^a \cap \{M_{i, i+1} = 0\} = \bigsqcup_{\sigma} (E_\pi^a \cap \{M_{i, i+1} = 0\} \cap F_\sigma).$$

Intersecting with $\{M_{i, i+1} = 0\}$ amounts to imposing $\sigma(i) \neq i + 1$ (assuming for $a = p$ that $i \neq N$, a case we can always avoid by cyclic symmetry), in which case this equation is automatically satisfied. So we have

$$E_\pi^a \cap \{M_{i, i+1} = 0\} = \bigsqcup_{\sigma: \sigma(i) \neq i+1} (E_\pi^a \cap F_\sigma),$$

and we must consider only pieces of dimension $\dim E_N^a - 1$.

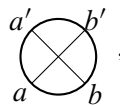
- (1) If σ is a link pattern, since among the equations (3) of Theorem 4.7 for E_π^a are $s_i(M) = s_{i+1}(M)$, we have $E_\pi^a \cap F_\sigma = E_\pi^a \cap F_\sigma \cap \{s_i(M) = s_{i+1}(M)\}$, where $F_\sigma \cap \{s_i(M) = s_{i+1}(M)\}$ is irreducible and of the target dimension. Now, using the exact same reasoning as in the proof of Proposition 5.4, if σ differs from π outside of $\{i, i + 1, \sigma(i), \sigma(i + 1)\}$ (and their images under the symplectic symmetry), then an equation among those of Theorem 4.7 for E_π^a is not satisfied, and the intersection has too low dimension.

This implies that $e_i\sigma = \pi$. Furthermore, if $|f_i\sigma| \geq |\sigma|$, i.e., if i and $i + 1$ are both connected to the boundary in σ , or the arches coming from them do not cross, then an additional equation of type (4) of Theorem 4.7, namely if say $i + 1$ is connected to j , the rank condition at $(i + 1, j)$, is violated, and similarly in other cases. So $|f_i\sigma| < |\sigma|$.

We conclude in the end that $\sigma \in \varepsilon(i, \pi)$, $f_i\sigma \neq \sigma$. In that case, from Theorem 4.7, $E_\pi^a \cap F_\sigma \subset F_\sigma \cap \{s_i(M) = s_{i+1}(M)\}$. In fact, one can easily show the equality – the proof is given in type \hat{A} in [17, Appendix B], but it works in all types, so we shall not repeat it here. In the proof of Proposition 5.4 we have seen that $F_\sigma \cap \{s_i(M) = s_{i+1}(M)\}$ is irreducible and that its closure is $X_{\sigma,i}^a$.

The other two cases are obtained by assuming that F_σ is of dimension $\dim E_N^a - 1$, applying Corollary 4.4 again, and eliminating cases by more of the same dimension considerations. In the end we find either:

- (2) σ has a crossing of the form



which forces it to be $\sigma(i) = N - i, \sigma(i + 1) = N - i + 1$, and $\sigma(j) = \pi(j)$ for $j \neq i, i + 1$. In this case we claim that $F_\sigma \cap E_\pi = F_\sigma$. This is identical to case (2) in the proof of Proposition 5.4. First we check that $\widetilde{B}_N \cdot \pi_{<}^i \supset \widetilde{B}_N \cdot \sigma_{<}^i$. Indeed, the matrix P with submatrix

$$\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 1/t & -1/t & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & 1/t \end{pmatrix}$$

at rows $i, i + 1, N - i, N - i + 1$ and identity elsewhere is symplectic and sends $\pi_{<}^i$ to $P\pi_{<}^iP^{-1} = \sigma_{<}^i + O(t^2)$. We then conclude that $E_\pi \cap F_\sigma = \overline{F_\pi} \cap F_\sigma$

is a vector bundle over $\widetilde{B}_N \cdot \sigma_{<}^i$, and that the dimension of its fiber matches that of F_σ , which implies $E_\pi \cap F_\sigma = F_\sigma$.

If $a = c$, we show in Appendix C that $\overline{F_\sigma}$ is $\mathrm{SL}_2^{(i)}$ -invariant.

If $a = i$, $\overline{F_\sigma} = X_{\rho,i}^a$ with $\rho(i) = N - i + 1$, $\rho(i + 1) = N - i$, i.e., this is the second case considered in the proof of Proposition 5.4. We find this way $X_{\rho,i}^a$ with $f_i \rho = \rho$; or,

- (3) σ has two fixed points which forces it to be $\sigma(i) = i$, $\sigma(i + 1) = i + 1$ and $\sigma(j) = \pi(j)$ for $j \neq i, i + 1$. This is only possible for $a = p$, and the corresponding component is $\mathrm{SL}_2^{(i)}$ -invariant, as discussed in [17, sect. 5.4].

What we have obtained is a set-theoretic decomposition of

$$E_\pi^a \cap \{M_{i,i+1} = 0\},$$

and we need to calculate multiplicities. This computation is performed in Appendix B. \square

Multidegree equality. As usual, intersections of (irreducible) varieties with hypersurfaces result in the multidegree identities

$$\mathrm{mdeg}(E_\pi^a \cap \{M_{i,i+1} = 0\}) = (A + z_i - z_{i+1}) \mathrm{mdeg} E_\pi^a,$$

$$\mathrm{mdeg}(E_\rho^a \cap \{s_i(M) = s_{i+1}(M)\}) = (2A - \epsilon) \mathrm{mdeg} E_\rho^a.$$

Finally, the decomposition of Proposition 5.5 combined with Proposition 5.4 translates into

$$(A + z_i - z_{i+1}) \mathrm{mdeg} E_\pi^a = \sum_{\rho \in \varepsilon(\pi,i)} 2^{\delta_{\rho_i} \cdot r \delta_{\rho_{i+1}} \cdot \ell} (2A - \epsilon) \mathrm{mdeg} E_\rho^a \pmod{\mathrm{Ker} \partial'_i},$$

where we have used Lemma 5.2 to take care of the $\mathrm{SL}_2^{(i)}$ -invariant terms.

We now apply $-\partial'_i$ and multiply by $2A + z_{i+1} - z_i - \epsilon$:

$$\begin{aligned} & (2A + z_{i+1} - z_i - \epsilon)(A + z_i - z_{i+1})(-\partial'_i) \mathrm{mdeg} E_\pi \\ &= (2A - \epsilon) \sum_{\rho \in \varepsilon(\pi,i)} 2^{\delta_{\rho_i} \cdot r \delta_{\rho_{i+1}} \cdot \ell} (2A + z_{i+1} - z_i - \epsilon)(-\partial'_i) \mathrm{mdeg} E_\rho \\ &= (2A - \epsilon) \sum_{\rho \in \varepsilon(\pi,i)} 2^{\delta_{\rho_i} \cdot r \delta_{\rho_{i+1}} \cdot \ell} (\mathrm{mdeg} E_\rho + \mathrm{mdeg} E_{f_i \rho}) \quad (\text{using eq. (52)}) \\ &= (2A - \epsilon) \sum_{\rho \in e_i^{-1}(\pi) - \{\pi\}} 2^{\delta_{\rho_i} \cdot b/\ell/r \delta_{\rho_{i+1}} \cdot b/\ell/r} \mathrm{mdeg} E_\rho. \end{aligned}$$

Now note that all the $\rho \in e_i^{-1}(\pi) - \{\pi\}$ such that i and $i + 1$ are not both connected to a boundary have the same number of chords as π , whereas the ones such that they are have one extra chord, which matches the power of 2 above; therefore, using (44), we find

$$(2A + z_{i+1} - z_i - \epsilon)(A + z_i - z_{i+1})(-\partial_i)\phi_\pi^a = (2A - \epsilon) \sum_{\rho \neq \pi: e_i \rho = \pi} \phi_\rho^a.$$

5.3. Geometry of the boundary exchange relations. In what follows we are necessarily in type \widehat{C} .

5.3.1. The invariant components

Proposition 5.6. *Given a link pattern π , E_π^a is $SL_2^{(L)}$ -invariant if and only if $\pi(L) \neq L + 1$.*

Proof. The $SL_2^{(L)}$ -invariance of E_π^a when $\pi(L) \neq L + 1$ is given in Appendix C.

Conversely, if $\pi(L) = L + 1$, $SL_2^{(L)} \cdot \pi t$ has entries below the diagonal, so the corresponding component cannot be invariant. \square

Multidegree equality. Assume now that $\pi(L) \neq L + 1$ (or, using the alternate notation, $\pi(L) \neq r$). Note that this is necessarily the case if $a \in \{c, m\}$.

We apply Lemma 5.2, minding, as explained right below it, the conjugation of the divided difference operator, and find:

$$\phi_\pi^a \text{ is an even polynomial in } z_L \times \begin{cases} 1 & a \in \{c, m\}, \\ A + 2z_L & a \in \{i, o\}, \quad \pi(L) \neq r. \end{cases}$$

With the exact same arguments, we find at the left boundary:

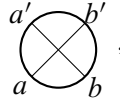
$$\phi_\pi^a \text{ is an even polynomial in } z_1 \times \begin{cases} 1 & a = c, \\ A - 2z_1 + \epsilon & a \in \{i, o, m\}, \quad \pi(1) \neq \ell. \end{cases}$$

5.3.2. The noninvariant components. We now assume that $\pi(L) = L + 1$ (which implies $a \in \{i, o\}$). The geometric construction corresponding to the boundary exchange relation (15) is the following:

- cut E_π^a with $M_{L,L+1} = 0$;
- throw away the $SL_2^{(L)}$ -invariant components, and sweep with $SL_2^{(L)}$, producing

$$\bigcup_{\rho \neq \pi: e_L \rho = \pi} E_\rho^a.$$

(2) σ has a crossing of the form



which forces it to be of the form ρ' described above, where ρ is a link pattern such that $e_L \rho = \pi$, $\rho \neq \pi$. In this case, according to Lemma 5.7, $F_\sigma \cap E_\pi^a = F_\sigma$ and its closure is Y_ρ^a .

(3) σ has two fixed points which forces it to be $\sigma(L) = L, \sigma(L + 1) = L + 1$ and $\sigma(j) = \pi(j)$ for $j \neq L, L + 1$. The corresponding component is $SL_2^{(L)}$ -invariant, as proved in Appendix C.

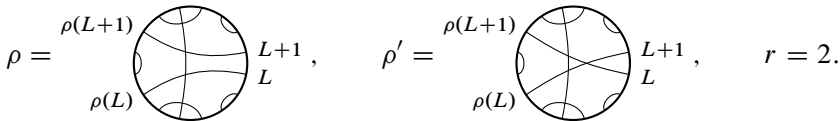
The multiplicity computation is performed in Appendix B. □

The degree of the sweeping map. The fiber of the map

$$SL_2^{(L)} \times_{B^{(L)}} Y_\rho^a \longrightarrow SL_2^{(L)} \cdot Y_\rho^a$$

is more difficult to study than in Section 5.2.1 because we have the identity $s_L(M) = s_{L+1}(M)$ (by symplectic symmetry), which means the block of M^2 on the N^{th} diagonal provides us no useful information. Instead we proceed as follows.

Since the $SL_2^{(L)}$ action is a group action, we can as usual look at the fiber $\{(P, M) \mid PMP^{-1} = M'\}$ of an element $M' \in Y_\rho^a$. Consider the ranks of successive submatrices of M' southwest of entries $(\rho(L), L-1), (\rho(L), L), (\rho(L), L+1)$, respectively. Since $M' \in Y_\rho^a$, these must be of the form $r, r, r + 1$, where r is the number of pairings inside $\{\rho(L + 1) + 1, \dots, L - 1\}$, e.g.,



Now conjugate M' with $P \in SL_2^{(L)}$; the effect is to mix columns L and $L + 1$, and the same ranks for $P^{-1}M'P$ (for generic P) are $r, r + 1, r + 1$. This violates the rank equations of M unless P does not send column $L + 1$ to L , i.e., $P \in B^{(L)}$. This is equivalent to saying that the fiber of the map $SL_2^{(L)} \times_{B^{(L)}} Y_\rho^a \rightarrow SL_2^{(L)} \cdot Y_\rho^a$ is of cardinality 1.

Determination of the result of sweeping. According to Lemma 5.7, $Y_\rho^a \subset E_\rho^a$, and E_ρ^a is $SL_2^{(L)}$ -invariant by Proposition 5.6. So $SL_2^{(L)} \cdot Y_\rho^a \subset E_\rho^a$, and since the fiber above is finite, and $\dim Y_\rho^a = \dim E_N^a - 1$, $SL_2^{(L)} \cdot Y_\rho^a$ and E_ρ^a have the same dimension. We conclude from irreducibility of E_ρ^a that

$$SL_2^{(L)} \cdot Y_\rho^a = E_\rho^a. \tag{53}$$

Multidegree identity. Proposition 5.8 implies that

$$(A + 2z_L) \text{mdeg } E_\pi^a = \sum_{\rho \neq \pi: e_L \rho = \pi} \text{mdeg } Y_\rho^a.$$

We then sweep with $SL_2^{(L)}$, apply Lemma 5.2 with the generic fiber of cardinality 1, and obtain using (53)

$$(A + 2z_L)(-\partial_L) \text{mdeg } E_\pi^a = \sum_{\rho \neq \pi: e_L \rho = \pi} \text{mdeg } E_\rho^a.$$

Finally, noting that π has one more chord than the $\rho \neq \pi$ such that $e_L \rho = \pi$, we have

$$(A + 2z_L)(-\partial_L) \phi_\pi^a = 2 \sum_{\rho \neq \pi: e_L \rho = \pi} \phi_\rho^a.$$

5.4. Geometry of the recurrence relations. This section follows closely [16, Section 6], which covers the periodic case. Here we consider $a \in \{i, c\}$.

5.4.1. The bulk recurrence

Proposition 5.9. Fix an i , $0 < i < n$. For a link pattern $\pi = \varphi_i \widehat{\pi}$,

$$\phi_\pi^a(z_{i+1} = A + z_i) = A^2 p_i^a(z_i | \dots, \widehat{z}_i, \widehat{z}_{i+1}, \dots) \phi_{\widehat{\pi}}^a(\dots, \widehat{z}_i, \widehat{z}_{i+1}, \dots),$$

where p_i^i and p_i^c are given in (28), and φ_i was defined in Lemma 2.2.

Remark. Note that if $\pi(i) \neq i + 1$, $M_{i,i+1} = 0$ so the multidegree disappears when $(A + z_i - z_{i+1}) = 0$.

Proof. We define the hyperplane

$$H^a = \mathcal{M}_N^a \cap \{M_{jk} = 0 \mid (j, k) \in \text{cl}(i, i + 1)\},$$

and the linear spaces

$$L^i = \mathbb{C}(e_{i,i+1} - e_{N-i,N-i+1}), \quad L^c = \mathbb{C}(e_{i,i+1} + e_{N-i,N-i+1}),$$

noting $\mathcal{M}_N^a = H^a \times L^a$. The equations defining E_π^a can be written as

$$q_j M_{i,i+1}^{d_j} + r_j = 0, \quad j = 0, 1, \dots, \tag{54}$$

where in the j th equation, d_j is the highest power of $M_{i,i+1}$, q_j is the coefficient of $M_{i,i+1}^{d_j}$, and r_j is the remainder. Now call Θ_π^a the scheme defined by the equations $q_j = 0, \forall j$. This can also be thought of as the result of taking $M_{i,i+1}$ to infinity in E_π^a . Then by [15, Corollary 2.6], we have

$$\text{mdeg}_{\mathcal{M}_N^a} E_\pi^a|_{A+z_i-z_{i+1}=0} = \text{mdeg}_{H^a} \Theta_\pi^a|_{A+z_i-z_{i+1}=0}.$$

We can now extract some factors of the RHS by examining the defining equations of Θ_π^a .

Amongst the defining equations of E_π^a are the defining equations of E_N^a :

$$(M^2)_{kl} = \sum_{j=k+1}^{l-1} M_{kj} M_{jl} = 0, \quad l - k < N, \quad l \neq N - k + 1.$$

Writing these equations in the form (54), we find that d_j can either equal 1 or 0. For $d_j = 1$ we must have $k = i$ or $l = i + 1$, meaning that the following equations form part of the definition of Θ_π^a :

$$\begin{aligned} M_{jk} &= 0, & (j, k) \in \text{cl}(i + 1, a), & \quad a \neq i, i + 1, N - i + 1, \\ M_{jk} &= 0, & (j, k) \in \text{cl}(b, i), & \quad b \neq i, i + 1, N - i, \end{aligned}$$

and with these substituted into the remaining equations we find no dependence on M_{jk} when $(j, k) \in \text{cl}(i, a)$ for all $a \neq i + 1$, $\text{cl}(b, i + 1)$ for all $b \neq i$, or $\text{cl}(i + 1, i)$ (see Figure 4).

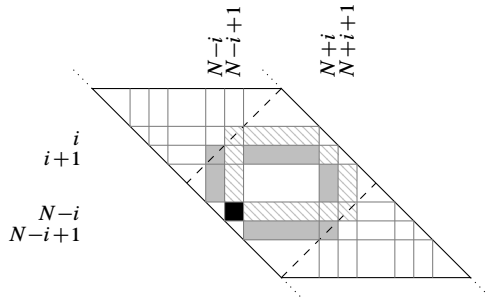


Figure 4. The fundamental region of a generic matrix with i or c -type symmetry. The black entry has been taken to infinity. In the defining equations of Θ_π^a , there is no dependence on those entries shaded with slanted lines, and the grey entries are equal to zero.

Taking the leading coefficient of $M_{i,i+1}$ in each of the other equations satisfied by E_π^a (as listed in Theorem 4.7), we find these are also independent of all the matrix elements with an index of $i, i+1, N-i$, or $N-i+1$. We further find that they are exactly the equations of Theorem 4.7 that are satisfied on $E_{\widehat{\pi}}^a$, where $\widehat{\pi}$ is the involution of size $N-4$ that is π with the links from i to $i+1$ and $N-i$ to $N-i+1$ removed. Since, as observed after Theorem 4.7, these equations define $E_{\widehat{\pi}}^a$ up to lower dimensional pieces, and the flat limit of E_π^a is equidimensional, we conclude that after removal of rows and columns $i, i+1, N-i$, and $N-i+1$, we obtain $E_{\widehat{\pi}}^a$ (up to embedded components, which are irrelevant for multidegree purposes).

We can therefore relate the multidegree of Θ_π^a to the multidegree of $E_{\widehat{\pi}}^a$, by intersecting Θ_π^a successively with a series of hyperplanes and using the inductive definition of the multidegree. The hyperplanes we use are the ones defined by $M_{jk} = 0$ for j or k in $\text{cl}(i)$ or $\text{cl}(i+1)$ (with the exception of $(j, k) \in \text{cl}(i, i+1)$ and, in the closed case, any choice of j and k for which $j = N - k + 1$, because the matrix entries on the symmetry axis are already zero by definition). The result of intersecting H^a with these hyperplanes is \mathcal{M}_{N-4}^a , so we have

$$\begin{aligned} & \text{mdeg}_{H^i} \Theta_\pi^i \Big|_{A+z_i-z_{i+1}=0} \\ &= A^2 (A - 2z_i - \epsilon)(3A + 2z_i) \prod_{a \neq i, i+1}^n (2A + z_a + z_i)(A - z_a - z_i - \epsilon) \\ & \quad \prod_{a=1}^{i-1} (A + z_a - z_i)(2A - z_a + z_i - \epsilon) \\ & \quad \prod_{a=i+2}^n (A + z_a - z_i - \epsilon)(2A - z_a + z_i) \text{mdeg}_{\mathcal{M}_{N-4}^i} E_{\widehat{\pi}}^i, \end{aligned}$$

$$\begin{aligned} & \text{mdeg}_{H^c} \Theta_\pi^c \Big|_{A+z_i-z_{i+1}=0} \\ &= A^2 \prod_{a \neq i, i+1}^n (2A + z_a + z_i)(A - z_a - z_i - \epsilon) \prod_{a=1}^{i-1} (A + z_a - z_i)(2A - z_a + z_i - \epsilon) \\ & \quad \prod_{a=i+2}^n (A + z_a - z_i - \epsilon)(2A - z_a + z_i) \text{mdeg}_{\mathcal{M}_{N-4}^c} E_{\widehat{\pi}}^c. \end{aligned}$$

Using (44), we have the result. \square

Similar arguments can be used to reproduce Proposition 5.9 in the cases $a \in \{0, m\}$.

5.4.2. The boundary recurrence

Proposition 5.10. *For a link pattern $\pi = \tilde{\varphi}_n \hat{\pi}$,*

$$\phi_\pi^i(z_n = -A/2) = A p_n^i(z_1, \dots, z_{n-1}) \phi_{\hat{\pi}}^i(z_1, \dots, z_{n-1}),$$

and for a link pattern $\pi = \tilde{\varphi}_0 \hat{\pi}$,

$$\phi_\pi^i(z_1 = (A - \epsilon)/2) = A p_0^i(z_2, \dots, z_n) \phi_{\hat{\pi}}^i(z_2, \dots, z_n),$$

where p_n^i and p_0^i are given in (34).

Remark. As before, if $\pi(n) \neq n + 1$ (resp. $\pi(1) \neq N$), then $M_{n,n+1} = 0$ ($M_{N1} = 0$) and the multidegree disappears when $z_n = -A/2$ ($z_1 = (A - \epsilon)/2$). Further, note that the proposition only refers to the identified case; in the closed case the below proof does not work because $M_{n,n+1}$ and M_{N1} are both zero by definition, and thus do not appear in the defining equations. Since we only consider the identified case we will drop the i superscript for the proof.

Proof. Right boundary: We define

$$H = \mathcal{M}_N \cap \{M_{n,n+1} = 0\}, \quad L = \text{Ce}_{n,n+1}.$$

As before we write the defining equations of E_π in the form $q_j M_{n,n+1}^{d_j} + r_j = 0$, and call Θ_π the scheme defined by $q_j = 0$ for all j . Again by [15, Corollary 2.6],

$$\text{mdeg } E_\pi|_{A+2z_n=0} = \text{mdeg}_H \Theta_\pi|_{A+2z_n=0}.$$

From the defining equations of E_N we find in the definition of Θ_π the following equations (see Figure 5):

$$M_{jk} = 0, \quad (j, k) \in \text{cl}(a, n), \quad a \neq n, n + 1,$$

and we also find no dependence on M_{jk} for $(j, k) \in \text{cl}(n, a) \forall a \neq n + 1$ or $(j, k) \in \text{cl}(n + 1, n)$. We again find that the rest of the equations defining Θ_π are exactly the equations defining $E_{\hat{\pi}}$, where $\hat{\pi}$ is the involution of size $N - 2$ that is π with the link from n to $n + 1$ removed.

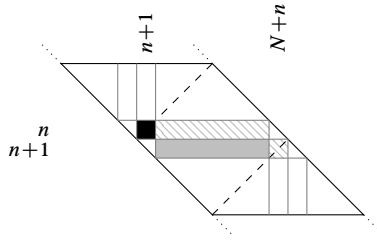


Figure 5. The fundamental region of a generic matrix with i-type symmetry. The black entry has been taken to infinity. In the defining equations of Θ_π , there is no dependence on those entries shaded with slanted lines, and the grey entries are equal to zero.

Intersecting Θ_π with hyperplanes defined by $M_{jk} = 0$ for j or k in $\text{cl}(n)$, excepting $(j, k) \in \text{cl}(n, n + 1)$, we find

$$\text{mdeg}_H \Theta_\pi|_{A+2z_n=0} = A \prod_{a=1}^{n-1} \frac{(3A + 2z_j)(3A - 2z_j - 2\epsilon)}{4} \text{mdeg}_{\mathcal{M}_{N-2}^i} E_{\hat{\pi}},$$

and again using (44), we have the result.

Left boundary: The previous argument can be slightly modified, to obtain

$$\text{mdeg } E_\pi|_{A-2z_1-\epsilon=0} = A \prod_{a=2}^n \frac{(3A - 2z_j - \epsilon)(3A + 2z_j - \epsilon)}{4} \text{mdeg}_{\mathcal{M}_{N-2}^i} E_{\hat{\pi}},$$

where $\hat{\pi}$ is the involution of size $N - 2$ that is π with the link from 1 to N removed. □

A similar argument can be made to reproduce Proposition 5.10 in the case $a = 0$, as well as in the case $a = m$ where only the right boundary statement applies.

5.5. The specialization $\epsilon = 2A$. We mention an interesting specialization of the ϕ_π , which is $\epsilon = 2A$. Note that this corresponds to the loop weight β being infinite.

Proposition 5.11. *One has*

$$\phi_{\pi}^{\text{p}}|_{\epsilon=2A} = A^L (-1)^{|\pi|} \prod_{\substack{1 \leq i < j \leq L \\ j \neq \pi(i)}} (A + z_i - z_j),$$

$$\phi_{\pi}^{\text{i}}|_{\epsilon=2A} = A^L (-1)^{|\pi|} \prod_{i=1}^L (A + 2z_i) \prod_{\substack{1 \leq i < j \leq L \\ j \neq \pi(i)}} (A + z_i - z_j)(A + z_i + z_j),$$

$$\phi_{\pi}^{\text{c}}|_{\epsilon=2A} = A^L (-1)^{|\pi|} \prod_{\substack{1 \leq i < j \leq L \\ j \neq \pi(i)}} (A + z_i - z_j)(A + z_i + z_j),$$

$$\phi_{\pi}^{\text{o}}|_{\epsilon=2A} = A^L \prod_{1 \leq i < j \leq L} (z_i - z_j)(z_i + z_j)(A + z_i - z_j)(A + z_i + z_j) \prod_{i=1}^L \begin{cases} A + 2z_i & \pi(i) = \ell, \\ -2z_i & \pi(i) = r, \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi_{\pi}^{\text{m}}|_{\epsilon=2A} = \begin{cases} A^L \prod_{1 \leq i < j \leq L} (z_i - z_j)(z_i + z_j)(A + z_i - z_j)(A + z_i + z_j) & \pi = (b, \dots, b), \\ 0 & \text{otherwise,} \end{cases}$$

where $|\pi|$ is defined as the number of crossings of π , plus (the location of the unpaired site minus one) in odd size for $a \in \{\text{p}, \text{c}\}$.

Proof. The recurrence relations (27), (32) and (33), combined with the qKZ system (10)–(11) or (13)–(15), provide an infinite number of values for the ϕ_{π} , even after the specialization $\epsilon = 2A$. So we only need to check that all these equations are satisfied by the expression in the proposition. We leave it as an exercise to the reader (see also [9, Lemma 2] for a similar proof). \square

It would be interesting to find a geometric interpretation of this specialization.

5.6. Conclusion. Noting that $A^L |\phi_{\pi}^{\text{a}}|$ for all π and all types a because this is nothing but the product of weights of the equations $M_{ii} = 0$, $i = 1, \dots, L$, it is natural to redefine the ϕ_{π}^{a} by dividing them by A^L ; since E_N^{a} actually sits in $(\mathcal{M}_N^{\text{a}})_{\Delta=0}$, we redefine

$$\psi_{\pi}^{\text{a}} = m_{\pi} \text{mdeg}_{(\mathcal{M}_N^{\text{a}})_{\Delta=0}} E_{\pi}^{\text{a}}.$$

The ψ_π^a satisfy the qKZ system (10)–(11) or (13)–(15), as well as recurrence relations of the form (27), (32) and (33). This shows existence of the would-be solution of the qKZ system that was studied in Section 2, where its uniqueness was proved.

In particular, by setting $\epsilon = 0$, we conclude from Proposition 3.2 that the ψ_π^a are the entries of the ground state of the Brauer loop model.

We also show:

Proposition 5.12. *The greatest common denominator of the $\psi_\pi^a|_{\epsilon=0}$, and therefore of the ψ_π^a , is 1.*

Here we ignore possible numerical factors, i.e., consider the gcd as polynomials with coefficients in \mathbb{Q} .

Proof. In types $a \in \{p, c\}$, we use Proposition 5.11: the greatest common denominator of the $\phi_\pi^a/A^L|_{\epsilon=A=0}$ is clearly 1.

In other types, we use Proposition 3.1: $|\Psi\rangle/\text{gcd}(\psi_\pi)$ is the ground state eigenvector and therefore a solution of the qKZ system. But so is $|\Psi\rangle$, so that the gcd must be a Weyl-group invariant polynomial. We then use the fully factorized components (20): from the degree of $|\Psi\rangle$, the factors S must be equal to 1. Furthermore, by inspection, no nontrivial Weyl-group invariant polynomial divides them. (Alternatively, we can use the form of the fully factorized components directly at $\epsilon = 0$, since as remarked at the beginning of Section 3.3 they are the same as those at generic ϵ in which we set $\epsilon = 0$, and conclude by degree). \square

Consider now the sum rule $Z_L^a = \sum_{\pi \in \text{LP}_L^a} \psi_\pi^a$ at $\epsilon = 0$. It satisfies the recurrence relations given in (43). These combined with the symmetry properties specify Z_L^a as a function of say z_1 at a certain number of points. In Appendix D, this number is carefully computed for each type a and compared to the degree in each variable of Z_L^a . The former is found to be strictly lower than that the latter, so that Z_L^a is specified uniquely by these recurrence relations, along with the initial condition 1 in size 0.

With the exception of the mixed case, the result can be written in determinant or Pfaffian form. The expression for the periodic case comes from [9], and the closed case comes from [7]. First define

$$b(z_i, z_j) = \frac{(A^2 - (z_i - z_j)^2)(A^2 - (z_i + z_j)^2)}{z_i^2 - z_j^2}.$$

The sum rules are

$$Z_L^p = \begin{cases} 2^{L/2} \prod_{1 \leq i < j \leq L} \mathfrak{A}_{i,j} \text{Pf } \mathfrak{B}_{i,j,L}, & L \text{ even,} \\ (-2)^{(L+1)/2} \prod_{1 \leq i < j \leq L} \mathfrak{A}_{i,j} \text{Pf} \begin{bmatrix} \mathfrak{B}_{i,j,L} & [-1]_{1 \leq j \leq L} \\ [1]_{1 \leq i \leq L} & 0 \end{bmatrix}, & L \text{ odd,} \end{cases} \quad (55a)$$

$$Z_L^i = \begin{cases} 2^{L/2} \prod_{1 \leq i < j \leq L} b(z_i, z_j) \text{Pf } \mathfrak{C}_{i,j,L}, & L \text{ even,} \\ 2^{(L+1)/2} \prod_{1 \leq i < j \leq L} b(z_i, z_j) \text{Pf} \begin{bmatrix} \mathfrak{C}_{i,j,L} & [1]_{1 \leq j \leq L} \\ [-1]_{1 \leq i \leq L} & 0 \end{bmatrix}, & L \text{ odd,} \end{cases} \quad (55b)$$

$$Z_L^c = \begin{cases} 2^{L/2} \prod_{1 \leq i < j \leq L} b(z_i, z_j) \text{Pf } \mathfrak{D}_{i,j,L}, & L \text{ even,} \\ 2^{(L-1)/2} \prod_{1 \leq i < j \leq L} b(z_i, z_j) \text{Pf} \begin{bmatrix} \mathfrak{D}_{i,j,L} & [1]_{1 \leq j \leq L} \\ [-1]_{1 \leq i \leq L} & 0 \end{bmatrix}, & L \text{ odd,} \end{cases} \quad (55c)$$

and

$$Z_L^o = \begin{cases} (2A)^L \prod_{1 \leq i < j \leq L} b(z_i, z_j)^2 \det \mathfrak{C}_{i,j,L}, & L \text{ even,} \\ 2(2A)^L \prod_{1 \leq i < j \leq L} b(z_i, z_j)^2 \det \begin{bmatrix} \mathfrak{C}_{i,j,L} & [1]_{1 \leq j \leq L} \\ [-1]_{1 \leq i \leq L} & 0 \end{bmatrix}, & L \text{ odd,} \end{cases} \quad (55d)$$

where

$$\begin{aligned} \mathfrak{A}_{i,j} &= \frac{A^2 - (z_i - z_j)^2}{z_i - z_j}, \\ \mathfrak{B}_{i,j,L} &= \left[\frac{z_i - z_j}{A^2 - (z_i - z_j)^2} \right]_{1 \leq i, j \leq L}, \\ \mathfrak{C}_{i,j,L} &= \left[\frac{(5A^2 - 2z_i^2 - 2z_j^2)}{b(z_i, z_j)} \right]_{1 \leq i, j \leq L}, \end{aligned}$$

and

$$\mathfrak{D}_{i,j,L} = \left[\frac{1}{b(z_i, z_j)} \right]_{1 \leq i, j \leq L}.$$

Now we can compare these results to the localization formulae of Section 4.2.2. It is an elementary check that based on them, the multidegree of D_N^a divided by A^L satisfies the same recurrence relations (43) and the same initial condition $\text{mdeg } D_0^a = 1$. We therefore conclude

Theorem 5.13. *We have the equality of multidegrees:*

$$\sum_{\pi \in \text{LP}_L^a} \phi_\pi^a|_{\epsilon=0} = \text{mdeg } D_N^a,$$

where $N = L, 2L, 4L$ depending on $a = p, i/c, o/m$.

Recall that this multidegree is also equal to that of the flat limit of D_N^a , namely $D_{N;0}^a$ (see Section 4.2.1 for its definition). It is shown in Appendix B.1 that this implies that $D_{N;0}^a = E_N^a$ as sets and that m_π is the multiplicity of E_π^a in either $D_{N;0}^a$ or \widetilde{E}_N^a (which are conjecturally equal as schemes).

Appendices

A. Small size examples

A.1. Identified case. $L = 2$:

$$\psi_{bb}^i = 4(2A - s + z_2 - z_1)(A + z_1 - z_2),$$

$$\psi_{21}^i = 2(A - s - 2z_1)(A + 2z_2).$$

$L = 3$:

$$\begin{aligned} \psi_{21b}^i &= 4(A - s - 2z_1)(A + z_2 - z_3)(A + z_2 + z_3) \\ &\quad (7A^3 - 9A^2s + 3As^2 - 3Asz_1 \\ &\quad + 2s^2z_1 - 3Az_1^2 + 2sz_1^2 - 4A^2z_2 + 3Asz_2 \\ &\quad + 2sz_1z_2 + 2z_1^2z_2 - Az_3^2 - 2z_2z_3^2), \end{aligned}$$

$$\begin{aligned} \psi_{bbb}^i &= 8(2A - s + z_3 - z_2)(2A - s + z_3 - z_1)(2A - s + z_2 - z_1) \\ &\quad (A + z_2 - z_3)(A + z_1 - z_3)(A + z_1 - z_2), \end{aligned}$$

$$\begin{aligned} \psi_{3b1}^i &= 4(A - s - z_1 - z_2)(A - s - 2z_1)(A + 2z_3) \\ &\quad (A + z_2 - z_3)(A + z_2 + z_3)(A + z_1 - z_2), \end{aligned}$$

$$\begin{aligned}\psi_{b32}^i &= 4(A - s - z_1 - z_2)(A + 2z_3)(A + z_1 - z_2) \\ &\quad (7A^3 - 7A^2s + 2As^2 - Asz_1 + s^2z_1 - Az_1^2 + sz_1^2 \\ &\quad + 4A^2z_2 - 3Asz_2 + s^2z_2 + 2sz_1z_2 + 2z_1^2z_2 \\ &\quad - 3Az_3^2 + sz_3^2 - 2z_2z_3^2).\end{aligned}$$

A.2. Closed case. $L = 3$:

$$\begin{aligned}\psi_{21\bullet}^c &= 4(A + z_2 - z_3)(A + z_2 + z_3) \\ &\quad (3A^2 - 3As + s^2 + sz_1 + z_1^2 - 2Az_2 + sz_2 - z_3^2),\end{aligned}$$

$$\begin{aligned}\psi_{\bullet32}^c &= 4(A - s - z_1 - z_2)(A + z_1 - z_2) \\ &\quad (3A^2 - 2As - sz_1 - z_1^2 + 2Az_2 - sz_2 + z_3^2),\end{aligned}$$

$$\psi_{3\bullet1}^c = 4(A - s - z_1 - z_2)(A + z_2 - z_3)(A + z_2 + z_3)(A + z_1 - z_2).$$

$L = 4$:

$$\begin{aligned}\psi_{3412}^c &= 4(A + z_1 - z_2)(A + z_2 - z_3)(A + z_3 - z_4) \\ &\quad (A + z_3 + z_4)(A - s - z_1 - z_2) \\ &\quad (5A^3 - 6A^2s + 3A^2z_2 - 3A^2z_3 + 2As^2 - Asz_1 - 3Asz_2 \\ &\quad + 2Asz_3 - Az_1^2 - 2Az_2z_3 - Az_4^2 + s^2z_1 + s^2z_2 + sz_1^2 \\ &\quad + sz_1z_2 + sz_1z_3 + sz_2z_3 + z_1^2z_2 + z_1^2z_3 - z_2z_4^2 - z_3z_4^2),\end{aligned}$$

$$\begin{aligned}\psi_{2143}^c &= 4(A + z_2 - z_3) \\ &\quad (23A^7 - 59sA^6 - 7z_2A^6 + 7z_3A^6 + 60s^2A^5 - 10z_1^2A^5 - 11z_2^2A^5 \\ &\quad - 11z_3^2A^5 - 10z_4^2A^5 - 10sz_1A^5 + 9sz_2A^5 - 19sz_3A^5 \\ &\quad + 2z_2z_3A^5 - 28s^3A^4 + 3z_2^3A^4 - 3z_3^3A^4 + 28sz_1^2A^4 + 23sz_2^2A^4 \\ &\quad + 19sz_3^2A^4 + z_2z_3^2A^4 + 14sz_4^2A^4 - 10z_2z_4^2A^4 - 18z_3z_4^2A^4 \\ &\quad + 28s^2z_1A^4 + 4s^2z_2A^4 + 18z_1^2z_2A^4 + 18sz_1z_2A^4 + 24s^2z_3A^4 \\ &\quad + 10z_1^2z_3A^4 - z_2^2z_3A^4 + 10sz_1z_3A^4 + 5s^4A^3 + 3z_1^4A^3 + 3z_4^4A^3 \\ &\quad + 6sz_1^3A^3 - 5sz_2^3A^3 + 3sz_3^3A^3 - 4z_2z_3^3A^3 - 20sz_1^2A^3 \\ &\quad - 18s^2z_2^2A^3 + 4z_1^2z_2^2A^3 + 4sz_1z_2^2A^3 - 11s^2z_3^2A^3 + 9z_1^2z_3^2A^3 \\ &\quad + 3z_2^2z_3^2A^3 + 9sz_1z_3^2A^3 - 2sz_2z_3^2A^3 - 9s^2z_4^2A^3 - 11z_1^2z_4^2A^3 \\ &\quad + 9z_2^2z_4^2A^3 + 4z_3^2z_4^2A^3 - 11sz_1z_4^2A^3 + 21sz_2z_4^2A^3 + 30sz_3z_4^2A^3 \\ &\quad + 16z_2z_3z_4^2A^3 - 23s^3z_1A^3 - 9s^3z_2A^3 - 18sz_1^2z_2A^3 - 18s^2z_1z_2A^3)\end{aligned}$$

$$\begin{aligned}
& -14s^3z_3A^3 - 4z_2^3z_3A^3 - 4sz_1^2z_3A^3 - s^2z_2z_3A^3 - 4s^2z_1z_3A^3 \\
& + 16z_1^2z_2z_3A^3 + 16sz_1z_2z_3A^3 - 5sz_1^4A^2 - 3sz_4^4A^2 + z_2z_4^4A^2 \\
& + 3z_3z_4^4A^2 - 10s^2z_1^3A^2 + 2s^2z_2^3A^2 - 4z_1^2z_2^3A^2 - 4sz_1z_2^3A^2 \\
& - s^2z_3^3A^2 - z_1^2z_3^3A^2 + z_2^2z_3^3A^2 - sz_1z_3^3A^2 + 5sz_2z_3^3A^2 \\
& + s^3z_1^2A^2 + 5s^3z_2^2A^2 - 11sz_1^2z_2^2A^2 - 11s^2z_1z_2^2A^2 + 2s^3z_3^2A^2 \\
& - z_2^2z_3^2A^2 - 11sz_1^2z_3^2A^2 - 3sz_2^2z_3^2A^2 - 11s^2z_1z_3^2A^2 + 2s^2z_2z_3^2A^2 \\
& + z_1^2z_2z_3^2A^2 + sz_1z_2z_3^2A^2 + 4s^3z_4^2A^2 + z_2^3z_4^2A^2 + 4z_3^3z_4^2A^2 \\
& + 14sz_1^2z_4^2A^2 - 10sz_2^2z_4^2A^2 - 3sz_3^2z_4^2A^2 + 4z_2z_3^2z_4^2A^2 \\
& + 14s^2z_1z_4^2A^2 - 15s^2z_2z_4^2A^2 + z_1^2z_2z_4^2A^2 + sz_1z_2z_4^2A^2 \\
& - 19s^2z_3z_4^2A^2 - z_1^2z_3z_4^2A^2 - z_2^2z_3z_4^2A^2 - sz_1z_3z_4^2A^2 \\
& - 19sz_2z_3z_4^2A^2 + 6s^4z_1A^2 + 3s^4z_2A^2 - 3z_1^4z_2A^2 - 6sz_1^3z_2A^2 \\
& - 3s^2z_1^2z_2A^2 + 3s^4z_3A^2 - z_1^4z_3A^2 - 2sz_1^3z_3A^2 + 4sz_2^3z_3A^2 \\
& - 7s^2z_1^2z_3A^2 + s^2z_2^2z_3A^2 - 4z_1^2z_2^2z_3A^2 - 4sz_1z_2^2z_3A^2 - 6s^3z_1z_3A^2 \\
& - 2s^3z_2z_3A^2 - 22sz_1^2z_2z_3A^2 - 22s^2z_1z_2z_3A^2 + 2s^2z_1^4A + s^2z_4^4A \\
& + z_1^2z_4^4A - 2z_2^2z_4^4A - z_3^2z_4^4A + sz_1z_4^4A - 2sz_2z_4^4A - 3sz_3z_4^4A \\
& - 2z_2z_3z_4^4A + 4s^3z_1^3A + 3sz_1^2z_2^3A + 3s^2z_1z_2^3A + 2z_2^3z_3^3A \\
& + sz_2^2z_3^3A - 2s^2z_2z_3^3A - 2z_1^2z_2z_3^3A - 2sz_1z_2z_3^3A + 2s^4z_1^2A \\
& - z_1^4z_2^2A - 2sz_1^3z_2^2A + 5s^2z_1^2z_2^2A + 6s^3z_1z_2^2A - 2z_1^4z_3^2A \\
& - 4sz_1^3z_3^2A + 2sz_2^3z_3^2A + s^2z_1^2z_3^2A + 2s^2z_2^2z_3^2A + z_1^2z_2^2z_3^2A \\
& + sz_1z_2^2z_3^2A + 3s^3z_1z_3^2A - s^3z_2z_3^2A - 2sz_1^2z_2z_3^2A - 2s^2z_1z_2z_3^2A \\
& - s^4z_4^2A + z_1^4z_4^2A + 2sz_1^3z_4^2A - sz_2^3z_4^2A - 3sz_3^3z_4^2A - 4s^2z_1^2z_4^2A \\
& + 4s^2z_2^2z_4^2A + z_1^2z_2^2z_4^2A + sz_1z_2^2z_4^2A + s^2z_3^2z_4^2A + z_1^2z_3^2z_4^2A \\
& + z_2^2z_3^2z_4^2A + sz_1z_3^2z_4^2A - 2sz_2z_3^2z_4^2A - 5s^3z_1z_4^2A + 5s^3z_2z_4^2A \\
& + sz_1^2z_2z_4^2A + s^2z_1z_2z_4^2A + 6s^3z_3z_4^2A - 2z_2^3z_3z_4^2A + 3sz_1^2z_3z_4^2A \\
& + 3s^2z_1z_3z_4^2A + 9s^2z_2z_3z_4^2A + 6z_1^2z_2z_3z_4^2A + 6sz_1z_2z_3z_4^2A \\
& + sz_1^4z_2A + 2s^2z_1^3z_2A + 4s^3z_1^2z_2A + 3s^4z_1z_2A - sz_1^4z_3A \\
& - 2s^2z_1^3z_3A - s^2z_2^3z_3A + 2s^3z_1^2z_3A + 3sz_1^2z_2^2z_3A + 3s^2z_1z_2^2z_3A \\
& + 3s^4z_1z_3A + s^4z_2z_3A - 2z_1^4z_2z_3A - 4sz_1^3z_2z_3A + 6s^2z_1^2z_2z_3A \\
& + 8s^3z_1z_2z_3A - z_3^3z_4^4 + sz_2^2z_4^4 - z_2z_3^2z_4^4 + s^2z_2z_4^4 + z_1^2z_2z_4^4 \\
& + sz_1z_2z_4^4 + s^2z_3z_4^4 + z_1^2z_3z_4^4 + sz_1z_3z_4^4 + sz_2z_3z_4^4 + z_1^4z_2^3 \\
& + 2sz_1^3z_2^3 + s^2z_1^2z_2^3 - sz_2^3z_2^3 - s^2z_2^2z_2^3 - z_1^2z_2^2z_2^3 - sz_1z_2^2z_2^3 \\
& + 2sz_1^4z_2^2 + 4s^2z_1^3z_2^2 + 2s^3z_1^2z_2^2 + sz_1^4z_2^2 + 2s^2z_1^3z_2^2 - s^2z_2^3z_2^2
\end{aligned}$$

$$\begin{aligned}
& -z_1^2 z_2^3 z_3^2 - s z_1 z_2^3 z_3^2 + s^3 z_1^2 z_3^2 - s^3 z_2^2 z_3^2 - 2s z_1^2 z_2^2 z_3^2 - 2s^2 z_1 z_2^2 z_3^2 \\
& -s z_1^4 z_4^2 - 2s^2 z_1^3 z_4^2 - z_1^2 z_2^3 z_4^2 - s z_1 z_2^3 z_4^2 + s^2 z_3^3 z_4^2 + z_1^2 z_3^3 z_4^2 \\
& + z_2^2 z_3^3 z_4^2 + s z_1 z_3^3 z_4^2 + s z_2 z_3^3 z_4^2 - s^3 z_1^2 z_4^2 - s^3 z_2^2 z_4^2 \\
& - 2s z_1^2 z_2^2 z_4^2 - 2s^2 z_1 z_2^2 z_4^2 + z_2^3 z_3^2 z_4^2 + s z_2^2 z_3^2 z_4^2 + s^2 z_2 z_3^2 z_4^2 \\
& + z_1^2 z_2 z_3^2 z_4^2 + s z_1 z_2 z_3^2 z_4^2 - s^4 z_2 z_4^2 - z_1^4 z_2 z_4^2 - 2s z_1^3 z_2 z_4^2 \\
& - 3s^2 z_1^2 z_2 z_4^2 - 2s^3 z_1 z_2 z_4^2 - s^4 z_3 z_4^2 - z_1^4 z_3 z_4^2 - 2s z_1^3 z_3 z_4^2 \\
& + s z_2^3 z_3 z_4^2 - 3s^2 z_1^2 z_3 z_4^2 - z_1^2 z_2^2 z_3 z_4^2 - s z_1 z_2^2 z_3 z_4^2 - 2s^3 z_1 z_3 z_4^2 \\
& - 2s^3 z_2 z_3 z_4^2 - 4s z_1^2 z_2 z_3 z_4^2 - 4s^2 z_1 z_2 z_3 z_4^2 + s^2 z_1^4 z_2 \\
& + 2s^3 z_1^2 z_2 + s^4 z_1^2 z_2 + s^2 z_1^4 z_3 + 2s^3 z_1^3 z_3 + s^4 z_1^2 z_3 + z_1^4 z_2^2 z_3 \\
& + 2s z_1^3 z_2^2 z_3 + s^2 z_1^2 z_2^2 z_3 + 2s z_1^4 z_2 z_3 + 4s^2 z_1^3 z_2 z_3 + 2s^3 z_1^2 z_2 z_3),
\end{aligned}$$

$$\begin{aligned}
\psi_{4321}^c &= 4(A + z_1 - z_2)(A + z_3 - z_4)(A + z_3 + z_4)(A - s - z_1 - z_2) \\
& (11A^4 - 18A^3 s + 8A^3 z_2 - 8A^3 z_3 + 10A^2 s^2 - 3A^2 s z_1 - 11A^2 s z_2 \\
& + 8A^2 s z_3 - 3A^2 z_1^2 + A^2 z_2^2 - 8A^2 z_2 z_3 + A^2 z_3^2 - 3A^2 z_4^2 - 2As^3 \\
& + 3As^2 z_1 + 5As^2 z_2 - 2As^2 z_3 + 3As z_1^2 + 2As z_1 z_3 - As z_2^2 \\
& + 6As z_2 z_3 + 2As z_4^2 + 2Az_1^2 z_3 - 2Az_2^2 z_3 + 2Az_2 z_3^2 - 2Az_2 z_4^2 \\
& - s^3 z_1 - s^3 z_2 - s^2 z_1^2 - s^2 z_1 z_2 - s^2 z_1 z_3 - s^2 z_2 z_3 - s z_1^2 z_2 \\
& - s z_1^2 z_3 - s z_1 z_2^2 + s z_1 z_4^2 + s z_2^2 z_3 + s z_2 z_4^2 - z_1^2 z_2^2 + z_1^2 z_4^2 \\
& + z_2^2 z_3^2 - z_3^2 z_4^2).
\end{aligned}$$

A.3. Open case. $L = 2$:

$$\begin{aligned}
\psi_{r\ell}^o &= 4(A - s - 2z_1)(A + 2z_2)(A + z_1 - z_2)(A + z_1 + z_2) \\
& (A - s - z_1 - z_2)(A - s - z_1 + z_2),
\end{aligned}$$

$$\begin{aligned}
\psi_{\ell\ell}^o &= 4(A + 2z_2)(A + z_1 - z_2)(A + z_1 + z_2)(A + 2z_1) \\
& (2A - 2s - z_1 - z_2)(2A - 2s - z_1 + z_2),
\end{aligned}$$

$$\begin{aligned}
\psi_{rr}^o &= 4(A + z_1 - z_2)(A - s - 2z_2)(A - s - z_1 - z_2)(A - s - 2z_1) \\
& (2A - 2s - z_1 + z_2)(2A - s + z_1 + z_2),
\end{aligned}$$

$$\begin{aligned}
\psi_{\ell r}^o &= 4(A - s - z_1 - z_2)(A + z_1 - z_2)(A + z_1 + z_2) \\
& (11A^3 - 26A^2 s + 19As^2 - 4s^3 - 3A^2 z_1 + As z_1 + 2s^2 z_1 - 2Az_1^2 \\
& + 2s z_1^2 + 3A^2 z_2 - 7As z_2 + 4s^2 z_2 - 8Az_1 z_2 + 10s z_1 z_2 \\
& + 4z_1^2 z_2 - 2Az_2^2 - 4z_1 z_2^2),
\end{aligned}$$

$$\begin{aligned}
\psi_{21}^o &= 2(A - s)(A - s - 2z_1)(A + 2z_2)(2A - s) \\
& (5A^2 - 7As + 2s^2 - 2s z_1 - 2z_1^2 - 2z_2^2).
\end{aligned}$$

A.4. Mixed case. $L = 2$:

$$\psi_{21}^m = 2(A - s)(A + 2z_2)(4A^2 - 5As + 2s^2 + 2sz_1 + 2z_1^2 - 2z_2^2),$$

$$\psi_{rr}^m = 4(2A - s + z_1 + z_2)(2A - 2s - z_1 + z_2)(A - s - z_1 - z_2)(A + z_1 - z_2).$$

B. Multiplicity computations

B.1. Multiplicity of E_π^a in \widetilde{E}_N^a . We consider here the *tangent cone* of \widetilde{E}_N^a at the point $\underline{\pi}t$ (t generic diagonal) of E_π^a , which is defined by taking the leading terms of the ideal of equations of \widetilde{E}_N^a expanded around that point. Let us write $M = \underline{\pi}t + P$. Among the equations for P , we obtain of course all the equations of the tangent space, which were computed in the proof of Theorem 4.5; furthermore, the diagonal entries now satisfy $M_{ii}^2 = 0$, or equivalently $P_{ii}^2 = 0$. Also, the equations $(P\underline{\pi}t + \underline{\pi}tP)_{ij} = 0$ are now nontrivial when $j = \pi(i)$, resulting in (after simplifying with $t_j \neq 0$)

$$P_{ii} + P_{jj} = 0, \quad i = \pi(j).$$

In principle there may be more equations in the ideal of leading terms; as we shall see, this is already enough to get a bound on the multiplicity, which we shall then show is saturated.

According to the above, the other nonlinear equations concern the diagonal entries of P ; they all satisfy $P_{ii}^2 = 0$, but those connected by either π or by the symplectic symmetry are equal, so the number of independent variables that square to zero is exactly the number of chords in the link pattern, plus the number of fixed points (the latter only occur for odd L , $a \in \{p, c\}$). So we find that the tangent cone sits inside a scheme of degree equal to $m_\pi = 2^{\#\{\text{chords}(\pi)\} + \#\{\text{fixed points}(\pi)\}}$, and is of the same dimension (the would-be extra equations cannot change the dimension of the tangent cone, since the point $\underline{\pi}t$ is smooth in E_π^a , as found in the proof of Theorem 4.5). Therefore the degree of the tangent cone is less than or equal to m_π . Since the equations of \widetilde{E}_N^a are invariant by conjugation and the union of orbits by conjugation of the $\underline{\pi}t$ is dense in E_π^a (Theorem 4.6), this is also true of the multiplicity of E_π^a in \widetilde{E}_N^a . Therefore, we have

$$\text{mdeg } \widetilde{E}_N^a \leq \sum_{\pi} m_\pi \text{mdeg } E_\pi^a = \sum_{\pi} \phi_\pi^a,$$

where inequality is here in the sense of multidegrees with positive multigrading, see e.g. [17, lemma 12] for details. We now specialize the multidegrees by setting

$\epsilon = 0$ (this corresponds to equivariance w.r.t. a codimension 1 subtorus, and does not spoil positivity of the multigrading).

According to Theorem 5.13 and (46),

$$\text{mdeg } D_{N;0}^a|_{\epsilon=0} = \sum_{\pi} \phi_{\pi}^a|_{\epsilon=0},$$

and according to Proposition 4.1, $D_{N;0}^a \subset \tilde{E}_N^A$ as schemes. Therefore the inequality, $\text{mdeg } \tilde{E}_N^a \leq \text{mdeg } D_{N;0}^a$ is an equality, and \tilde{E}_N^a and $D_{N;0}^a$ have the same top-dimensional irreducible components, with the same multiplicities. Finally, the equation above determines them to be:

$$\text{multiplicity of } E_{\pi}^a \text{ in } \tilde{E}_N^a = m_{\pi} = 2^{\#\{\text{chords}(\pi)\} + \#\{\text{fixed points}(\pi)\}}.$$

B.2. Multiplicity of $X_{\rho,i}^a$. In this section, given a link pattern ρ such that $|\rho| \geq |f_i \rho|$, we compute the multiplicity of $X_{\rho,i}^a$ in either $E_{\rho}^a \cap \{s_i(M) = s_{i+1}(M)\}$ or $E_{\pi}^a \cap \{M_{i,i+1} = 0\}$ (assuming $\rho \in \varepsilon(\pi, i)$).

We consider a point of the form $\underline{\rho}t$ where the t_j 's are nonzero and $t_i t_{\rho(i)} = t_{i+1} t_{\rho(i+1)}$. By direct inspection, $\underline{\rho}t \in X_{\rho,i}^a$. First we compute its Zariski tangent space in E_N^a . This is the exact same calculation that was performed in the proof of Theorem 4.5, so we do not repeat it here. In fact all the cases that we need here are given in the proof. For $a = i$:

- (2) If i and $i + 1$ are both connected to the boundary in ρ , then the counting is the same.
- (3) If one of the two is connected to the boundary in ρ , say $i + 1$, then naively there is one less independent equation because $t_i t_{\rho(i)} = t_{i+1} t_{\rho(i+1)}$. However, the two equations which become proportional involve the variable $P_{i,i+1}$, which is here equal to $M_{i,i+1} = 0$ either from Theorem 4.7 in E_{ρ}^a or by definition in $E_{\pi}^a \cap \{M_{i,i+1} = 0\}$, so that we are back to the original count.
- (4) Finally, if neither is connected to the boundary, then we lose two equations. Once again, reimposing $P_{i,i+1} = M_{i,i+1} = 0$ gives one more equation. Similarly, consider the rank condition on the interval $[i + 1, \rho(i)]$ for either E_{π}^a or E_{ρ}^a : $\text{rm}(M)_{i+1,\rho(i)} = \text{rm}(\underline{\pi})_{i+1,\rho(i)} = \text{rm}(\underline{\rho})_{i+1,\rho(i)} = r$, where $(j_1, \rho(j_1)), \dots, (j_r, \rho(j_r))$ are all the chords inside $[i + 1, \rho(i)]$. Now consider the $(r + 1) \times (r + 1)$ submatrix of M with row indices $i + 1, j_1, \dots, j_r$ and column indices $\rho(j_1), \dots, \rho(j_r), \rho(i)$. Its determinant must vanish; expanding at first order $M = \underline{\rho}t + P$, the only contribution in the expansion of the determinant is obtained by matching j_a with $\rho(j_a)$, $a = 1, \dots, r$, and therefore $i + 1$ with $\rho(i)$. This implies $P_{i+1,\rho(i)} = 0$, which recreates the second missing equation.

For $a = c$:

- (1) In odd size, if either i or $i + 1$ is a fixed point of ρ , then the counting is the same.
- (2) Otherwise, we lose two equations involving $P_{i,i+1}$ and $P_{i+1,\rho(i)}$, which just as above, are recovered by imposing $M_{i,i+1} = 0$ and the rank condition in the interval $[i + 1, \rho(i)]$.

So in all cases, we find that the number of equations is the same as in the proof of Theorem 4.5, so at this stage the dimension we would get out of this computation is $\dim E_N^a$, which is one more than the target dimension $\dim X_{\rho,i}^a$.

However, we have not yet used the additional equation: $s_i(M) = s_{i+1}(M)$, which is valid either by definition $E_\rho^a \cap \{s_i(M) = s_{i+1}(M)\}$ or from Theorem 4.7 in $E_\pi^a \cap \{M_{i,i+1} = 0\}$. If we expand at first order we obtain

$$t_{\pi(i)} P_{\pi(i),i} + P_{i,\pi(i)} t_i = t_{\pi(i+1)} P_{\pi(i+1),i+1} + P_{i+1,\pi(i+1)} t_{i+1}.$$

One can check explicitly in all cases that this equation is independent from the ones above, thus showing that the dimension is equal to that of $X_{\rho,i}^a$.

Note that the reasoning above *fails* in type $a = o$ in the case that $\rho(i) = r$, $\rho(i + 1) = \ell$, in the sense that the tangent space has dimension one more than that of the space. In this case one needs to consider the tangent cone itself, which turns out to be of degree 2.

B.3. Multiplicity of Y_ρ^a . Recall that given a link pattern ρ such that $\rho(L) \neq L + 1$, we can associate to it the involution ρ' where the images of L and $L + 1$ are swapped. By definition of Y_ρ , $\underline{\rho'}t$, t generic diagonal, is in Y_ρ^a . We wish to calculate the Zariski tangent space of $\underline{\rho'}t$ inside $E_\pi^a \cap \{M_{L,L+1} = 0\}$, where $\pi = e_L(\rho)$.

It is not hard to check that the counting of equations is the same as in the proof of Theorem 4.5 for $\underline{\rho}t$ in type $a = i$, except the equations involving only $\{L, L + 1, \rho(L), \rho(L + 1)\}$ which we redo here:

$$\rho' = \begin{matrix} \rho(L+1) \\ \circlearrowleft \\ \rho(L) \end{matrix} \begin{matrix} L+1 \\ L \end{matrix}, \quad t_L P_{L,L+1} + t_{L+1} P_{\rho(L+1),\rho(L)} = 0.$$

So we have one less equation than for $\underline{\rho}t$, whereas we need one more.

The first extra equation is obviously $M_{L,L+1} = 0$ which implies $P_{L,L+1} = 0$. The second equation comes from the rank condition (Theorem 4.7) for E_π^a in the interval $[\rho(L), L]$, which implies $P_{\rho(L),L} = 0$.

In total, we find that the Zariski tangent space has dimension one less than E_N^a , which is the dimension of Y_ρ^a .

C. $\mathrm{SL}_2^{(i)}$ -invariance of certain subvarieties of E_N^a

Consider a subvariety $V \subset E_N^a$, and its image under conjugation by a subgroup $\mathrm{SL}_2^{(i)}$, as defined in Section 5.1. As a first remark, to prove $\mathrm{SL}_2^{(i)}$ -invariance of V , we only need to show that $\dim(\mathrm{SL}_2^{(i)} \cdot V) = \dim V$, since $\mathrm{SL}_2^{(i)} \cdot V$ contains V and is irreducible of the same dimension as V .

Next, note that if V is to be $\mathrm{SL}_2^{(i)}$ -invariant, then in particular $\mathrm{SL}_2^{(i)} \cdot V \subset E_N^a$. It is easy to see that the only equation of E_N^a that is potentially violated by sweeping with $\mathrm{SL}_2^{(i)}$ is $(M^2)_{i+1, i+N} = 0$. Explicitly, if $M = PM'P^{-1}$ with $M' \in V$, then

$$(M^2)_{i+1, i+N} = P_{i+1, i+1} P_{i+1, i} ((M'^2)_{ii} - (M'^2)_{i+1, i+1}) - P_{i+1, i}^2 (M'^2)_{i, i+N+1}. \quad (56)$$

Note that $(M^2)_{i, i+N+1}$ is well-defined in the quotient space \mathcal{M}_N^a only if $M_{i, i+1} = 0$ in V , which will be the case below.

Also, if, as in all cases below, one has $V \subset \{s_i(M) = s_{i+1}(M)\}$, then the equation above simplifies to

$$(M^2)_{i+1, i+N} = -P_{i+1, i}^2 (M'^2)_{i, i+N+1}. \quad (57)$$

C.1. Bulk case. Here $i = 1, \dots, L-1$.

C.1.1. $F_\rho \cap \{s_i = s_{i+1}\}$ for $\rho(i) \neq i+1$, $f_i \rho = \rho$. This is case (1) of the proof of Proposition 5.4 with $f_i \rho = \rho$. Necessarily, $a = i$. Call

$$V = \overline{F_\rho \cap \{s_i(M) = s_{i+1}(M)\}}.$$

First we check that $(M^2)_{i+1, i+N} = 0$ in $\mathrm{SL}_2^{(i)} \cdot V$. Since all entries of M^2 to the left of and below $(i, i+N+1)$ are known to be zero, this equation is \tilde{B}_N -invariant and so we only need to use (57) with $M' = \rho_{<}^i + \Lambda$ where $\Lambda \in \mathfrak{g}_N^a / R_N^a$. Now

$$(M'^2)_{i, i+N+1} = (\rho_{<}^i)_{i, N-i+1} \Lambda_{N-i+1, N+i+1} = \Lambda_{-i+1, i+1}.$$

By a similar calculation,

$$0 = (M'^2)_{i+1, i+N} = \Lambda_{-i, i}.$$

These two entries are related by the symplectic symmetry $(a, b) \mapsto (1-b, 1-a)$ and are therefore equal. So $(M'^2)_{i, i+N+1} = 0$.

By density and \tilde{B}_N -invariance, this implies that $\mathrm{SL}_2^{(i)} \cdot V \subset (\mathrm{SL}_2^{(i)} \cdot E_\rho^a) \cap E_N^a = E_\rho^a$ from Section 5.2.1.

We also learnt in the proof of Lemma 5.3 that sweeping can only permute s_i and s_{i+1} , so we conclude that $\mathrm{SL}_2^{(i)} \cdot V \subset E_\rho^a \cap \{s_i = s_{i+1}\}$. The latter being of the same dimension as the former, we conclude that V is $\mathrm{SL}_2^{(i)}$ -invariant.

C.1.2. F_σ where σ is a link pattern except $\sigma(i) = N - i, \sigma(i + 1) = N - i + 1$. This is case (2) of the proof of Proposition 5.5. Here, $a = c$, and $V = \overline{F_\sigma}$.

We follow the same reasoning as above, and consider (56) with $M' = \sigma_{<}^i + \Lambda$. First we find that $s_i(M') = \Lambda_{-i,i}$ and $s_{i+1}(M') = \Lambda_{-i+1,i+1}$, and once again these entries are related by the symplectic symmetry so $s_i(M') = s_{i+1}(M')$.

Therefore, we are reduced to (57), and we have:

$$(M'^2)_{i,i+N+1} = (\sigma_{<}^i)_{i,N-i} \Lambda_{N-i,i+N+1}.$$

But $\Lambda_{N-i,i+N+1}$ is such that the sum of its row and column is equal to 1 mod N , which means it is on the axis of the symplectic symmetry, which means in type $a = c$ that it is zero.

So $\mathrm{SL}_2^{(i)} \cdot V \subset E_N^a$. Sweeping can at best permute s_i and s_{i+1} and leave the other s_j unchanged, and preserves the equation $M_{i,i+1} = 0$, so

$$\mathrm{SL}_2^{(i)} \cdot V \subset (E_\pi^a \cap \{M_{i,i+1} = 0\}) \cup \bigcup_{\rho \neq \pi} (E_\rho^a \cap \{s_j = s_{\pi(j)}, j \neq i, i + 1\}).$$

It is easy to check that the RHS is of dimension $E_N^a - 1$, just like V itself. By the same argument as above, we conclude that $V = \overline{F_\sigma}$ is $\mathrm{SL}_2^{(i)}$ -invariant.

C.2. Boundary case. We only do the right boundary, i.e., $i = L$. The left boundary can be treated similarly.

C.2.1. E_π^a where $\pi(L) \neq L + 1$. Here $V = E_\pi^a$. If $\pi(L) \neq L + 1$, then any $M \in E_\pi^a$ satisfies $M_{L,L+1} = 0$ by Theorem 4.7. This means that PMP^{-1} is still upper triangular for $P \in \mathrm{SL}_2^{(L)}$. Also, $s_L(M) = s_{L+1}(M)$ by symplectic symmetry, so once again we are reduced to (57). But $(M'^2)_{L,L+N+1} = 0$ because $(M'^2)^\dagger = M'^2$, and the sum of the row and column indices of that entry is equal to 1 mod N .

Therefore, $\mathrm{SL}_2^{(L)} \cdot E_\pi^a \subset E_N^a$. and by the usual dimension argument, it must be equal to E_π^a .

C.2.2. $E_\pi^a \cap F_\sigma$, where $\pi(L) = L + 1$ and σ is a link pattern, $\sigma(L) \neq L + 1$. Let V be an irreducible component of $E_\pi^a \cap F_\sigma$ of dimension $\dim E_N^a - 1$ (if there exists any, in which case it is necessary top-dimensional). By Proposition 5.6, whose proof is right above, E_σ^a is $\mathrm{SL}_2^{(L)}$ -invariant.

Now consider the rank equation southwest of the entry $(\sigma(L), L)$. Being in F_σ implies that the rank is *equal* to the number of pairings of σ inside $\{\sigma(L), \dots, L\}$, which we call r . For $E_\pi^a \cap F_\sigma$ to be nonempty, this implies (equations (4) of Theorem 4.7) that the number of pairings of π inside $\{\sigma(L), \dots, L\}$ must be *at least* r . But $\pi(L) = L + 1$ and $\sigma(L) \neq L + 1$, so the number of pairings of π in $\{\sigma(L), \dots, L - 1\}$ is the same, i.e., at least r , whereas that of σ is only $r - 1$. This means that π possesses at least one pairing that σ does not, say $i \leftrightarrow \pi(i)$, $\sigma(L) \leq i < L$. This means the equation (3) of Theorem 4.7 $s_i(M) = s_{\pi(i)}(M)$ is generically violated in E_σ^a , and since sweeping with $\text{SL}_2^{(L)}$ does not affect them, that $\dim(\text{SL}_2^{(L)} \cdot V) \leq \dim E_N^a - 1$.

C.2.3. F_σ , where σ is a link pattern except $\sigma(L) = L, \sigma(L + 1) = L + 1$. Set $V = \overline{F_\sigma}$, with σ as in the third case of the proof of Proposition 5.8. As explained in the proof for E_π^a above, $M_{L,L+1} = 0$ in V implies that $\text{SL}_2^{(L)} \cdot V \subset E_N^a$. Furthermore, as easily checked on a matrix of the form $M = \sigma_{<} + L$, one has $s_L(M) = s_{L+1}(M) = 0$ in V . This equation is preserved by sweeping (easily checked since the whole 2×2 submatrix of entries with row and column equal to $L, L + 1 \pmod N$ is actually zero). No top-dimensional component of E_N^a has this equation, so $E_N^a \cap \{s_L(M) = s_{L+1}(M)\}$ is of dimension $\dim E_N^a - 1$, which is the dimension of V .

D. Bound on the degree of the polynomials ϕ_π^a

We wish to bound the degree of $\phi_\pi^a = m_\pi \text{mdeg } E_\pi^a$ as a polynomial in one of the variables z_1, \dots, z_L .

Recall that from the definition of multidegrees [21], any multidegree in \mathcal{M}_N^a is a sum of products of (distinct) weights of \mathcal{M}_N^a . This gives a first “naive” bound on the degree of ϕ_π^a in a given variable z_i : it is less or equal to the number of coordinates in \mathcal{M}_N^a whose weight has a z_i -dependence. See Table 1.

However, this bound is not enough for our purposes. We can refine it as follows. We focus at first on the multidegree of the whole of E_N^a rather than ϕ_π^a . Suppose we apply the inductive definition of the multidegree by intersecting E_N^a with hyperplanes given by the vanishing of entries of the form M_{ij} and M_{ji} for fixed i . Each time the multidegree is multiplied by a factor of the weight of M_{ij} , the intersection is trivial and the dimension stays constant. Since all variables whose weight have a z_i -dependence belong to that row/column, we have a bound on the degree in z_i :

degree in z_i of $\text{mdeg } E_N^a$

$$\leq \text{number of entries of the form } M_{ij} \text{ or } M_{ji} - (\dim E_N^a - \dim E_{N-(1,2,4)}^a),$$

since the resulting variety (after all intersections) is simply the Brauer loop scheme one size below ($L \rightarrow L - 1$). More precisely, in all cases except $a \in \{m, o\}$, the entries of the form M_{ij} or M_{ji} are in fact exactly those whose weight have z_i -dependence, and the inequality above is an equality; if $a \in \{m, o\}$, $M_{i,i+n}$ does not have such a dependence, so the LHS is equal either to the RHS, or the RHS minus one. In other words,

degree in z_i of $\text{mdeg } E_N^a$

$$= \text{naive degree bound} - (\dim E_N^a - \dim E_{N-(1,2,4)}^a) + (0 \text{ or } 1)[a \in \{m, o\}].$$

a	naive degree bound	dim	dim shift
p	$2(L - 1)$	$L^2/2, (L^2 - 1)/2$	$L, L - 1$
i	$2(2L - 1)$	$L(L + 1)$	$2L$
c	$4(L - 1)$	$L^2, L^2 - 1$	$2L, 2(L - 1)$
o	$4(2L - 1)$	$2L(L + 1)$	$4L$
m	$2(4L - 3)$	$L(2L + 1)$	$4L - 1$

a	refined degree bound	# recurrences
p	$L - 2, L - 1$	$2(L - 1)$
i	$2(L - 1)$	$2(2L - 1)$
c	$2(L - 2), 2(L - 1)$	$4(L - 1)$
o	$4L - 3^*$	$2(2L - 1)$
m	$4(L - 1)^*$	$2(2L - 1)$

Table 1. Degree bounds for ϕ_π^a and number of available recurrences. The refined degree bound (fourth column) is equal to the first column minus the third, plus one when there is a *. If two numbers are shown they correspond to even/odd cases.

Finally, we can calculate the number of recurrence relations of the type of (43) and find that it is always greater than the degree. This allows us to derive the explicit expression (55), and shows that 1 is the correct choice for $a \in \{m, o\}$ in the equation above.

Now if we consider individual components E_π^a rather than the whole scheme, the same argument applies except after intersecting, we only have an upper bound on the resulting scheme (it is a subscheme of $E_{N-(1,2,4)}^a$), so that we obtain an upper bound for the degree:

degree in z_i of ϕ_π^a

$$\leq \text{naive degree bound} - (\dim E_N^a - \dim E_{N-(1,2,4)}^a) + 1[a \in \{m, o\}].$$

The result is shown in Table 1.

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