

# Perfect and separating hash families: new bounds via the algorithmic cluster expansion local lemma

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**Abstract.** We present new lower bounds for the size of perfect and separating hash families ensuring their existence. Such new bounds are based on the algorithmic cluster expansion improved version of the Lovász local lemma, which also implies that the Moser–Tardos algorithm finds such hash families in polynomial time.

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## 1. Introduction and results

In this initial section we will review rapidly the state of the art of the Lovász local lemma, a powerful tool in the framework of the probabilistic method in combinatorics, focusing specifically on the recent cluster expansion improvement of the Moser–Tardos algorithmic version of the Lemma. We then will recall the main results in the literature concerning Perfect Hash Families and Separating Hash Families. Finally we will present the results of the paper.

**1.1. Lovász local lemma: state of the art.** The Lovász local lemma (LLL) was originally formulated by Erdős and Lovász in [8] and since then it has turned out to be one of the most powerful tools in the framework of the probabilistic

method in combinatorics to prove the existence of combinatorial objects with certain desirable properties. The philosophy of the Lemma is basically to consider a collection of “bad” events in some suitably defined probability space whose occurrence, even of just one of them, prevents the existence of a certain “good” event (i.e. the combinatorial object under analysis). Then the Lemma provides a sufficient condition which, once satisfied, guarantees that there is a strictly positive probability that none of the bad events occurs (so that the good event exists). Such sufficient condition can be inferred from the so-called *dependency graph* of the collection of events. We remind that a dependency graph for a or a collection of random events  $\mathcal{B}$  is a (simple and undirected) graph  $G$  with vertex set  $\mathcal{B}$  such that each event  $B \in \mathcal{B}$  is independent from the  $\sigma$ -algebra generated by the collection of events  $\mathcal{B} \setminus \Gamma_G^*(B)$  where  $\Gamma_G^*(B) = \Gamma_G(B) \cup \{B\}$ , with  $\Gamma_G(B)$  denoting the neighborhood of  $B$  in  $G$ , i.e. the set of vertices of  $G$  which are connected to the vertex  $B$  by an edge of  $G$ .

The connection between the LLL and the cluster expansion of the abstract polymer gas, implicitly implied by an old paper by Shearer [15], has been sharply pointed out in [14] by Scott and Sokal who also showed that the LLL (with dependency graph  $G$ ) can be viewed as a reformulation of the Dobrushin criterion [7] for the convergence of the cluster expansion of the hard-core lattice gas (on the same graph  $G$ ).

In a later paper [9] Fernández and Procacci improved the Dobrushin criterion and this has then been used straightforwardly by Bissacot et al. in [3] to obtain a correspondent improved cluster expansion version of the LLL (shortly CLLL). Such new version of the LLL has been already implemented to get new bounds on several graph coloring problems (see [12] and [5]).

As the original Lovász local lemma by Erdős-Lovász, the improved cluster expansion version by Bissacot et al. given in [3] is “non-constructive”, in the sense that it claims the existence of a certain event without explicitly exhibiting it. Nevertheless, an algorithmic version of the CLLL, based on a breakthrough paper by Moser and Tardos [11], has been recently provided in [13] and [1].

**1.1.1. Moser–Tardos setting (general case).** In the Moser–Tardos setting all events in the collection  $\mathcal{B}$  depend on a finite family  $\mathcal{V}$  of mutually independent random variable with  $\Omega$  being the sample space determined by these variables so that a outcome  $\omega \in \Omega$  is just a random evaluation of all variables of the family  $\mathcal{V}$ . Each event  $B \in \mathcal{B}$  is supposed to depend only on some subset of the variables  $\mathcal{V}$ , denoted by  $\text{vbl}(B)$ . Since variables in  $\mathcal{V}$  are assumed to be mutually independent, any two events  $B, B' \in \mathcal{B}$  such that  $\text{vbl}(B) \cap \text{vbl}(B') = \emptyset$  are necessarily

independent. Therefore the family  $\mathcal{B}$  has a natural dependency graph, i.e. the graph  $G$  with vertex-set  $\mathcal{B}$  and edge-set constituted by the pairs  $\{B, B'\} \subset \mathcal{B}$  such that  $\text{vbl}(B) \cap \text{vbl}(B') \neq \emptyset$ .

In this setting Moser and Tardos define the following random algorithm, whose output, when (and if) it stops, is an evaluation of the variables of the family  $\mathcal{V}$  (i.e. an outcome  $\omega \in \Omega$ ) which avoids all the events in the collection  $\mathcal{B}$ .

**MT-algorithm** (general case)

– STEP 0. Sample all random variables in the family  $\mathcal{V}$ .

Let  $\omega_0 \in \Omega$  be the output.

– STEP  $k$  ( $k \geq 1$ ).

a) Take  $\omega_{k-1} \in \Omega$  and check all bad events in the family  $\mathcal{B}$ .

b) i) If some bad event occurs, choose one, say  $B$ , and resample its variables  $\text{vbl}(B)$  leaving unchanged the remaining variables.

ii) If no bad event occurs, stop the algorithm.

Let  $\omega_k \in \Omega$  be the output.

We are now in a position to state the algorithmic version of the CLLL, which will be the basic tool to get our results on perfect and separating hash families. We remind that an independent set in a graph  $G$  is a set of vertices of  $G$  no two of which are connected by an edge of  $G$ .

**Theorem 1.1** (algorithmic CLLL). *Given a finite set  $\mathcal{V}$  of mutually independent random variables, let  $\mathcal{B}$  be a finite set of events determined by these variables with natural dependency graph  $G$ . Let  $\mu = \{\mu_B\}_{B \in \mathcal{B}}$  be a sequence of real numbers in  $[0, +\infty)$ . If, for each  $B \in \mathcal{B}$ ,*

$$\text{Prob}(B) \leq \frac{\mu_B}{\sum_{\substack{Y \subseteq \Gamma_G^*(B) \\ Y \text{ independent in } G}} \prod_{B' \in Y} \mu_{B'}}$$

*then the MT-algorithm reaches an assignment of values of the variables  $\mathcal{V}$  such that none of the events in  $\mathcal{B}$  occurs. Moreover the expected total number of resampling steps made by the MT-algorithm to reach this assignment is at most  $\sum_{B \in \mathcal{B}} \mu_B$ .*

The proof of Theorem 1.1 can be found in [13] and [1].

**1.2. Perfect hash families and separating hash families.** Given a finite set  $U$  we denote by  $|U|$  its cardinality. Given an integer  $k$ , we denote shortly  $[k] = \{1, 2, \dots, k\}$ . A collection of sets  $\{W_1, \dots, W_k\}$  such that  $W_i \cap W_j = \emptyset$  for all  $\{i, j\} \subset [k]$  will be called hereafter a “disjoint family”.

Let  $n, w$  be integers such that  $2 \leq w \leq n$ . We denote by  $P_w([n])$  the set of all subsets of  $[n]$  with cardinality  $w$ .

Given  $s, w_1, w_2, \dots, w_s$  integers such that  $\sum_{i=1}^s w_i = w$ , we denote by  $P_w^*([n])$  the set whose elements are the disjoint families  $S = \{W_1, \dots, W_s\}$  such that  $W_i \subset [n]$  and  $|W_i| = w_i$  for  $i = 1, \dots, s$ .

Let  $A$  be a  $N \times n$  matrix. Given  $W \in P_w([n])$  we denote by  $A|_W$  the  $N \times w$  matrix formed by the  $w$  columns of the matrix  $A$  with indices in  $W$ . Analogously, given a disjoint family  $S = \{W_1, \dots, W_s\} \in P_w^*([n])$ , we denote by  $A|_S$  the  $N \times w$  matrix formed by the  $w$  columns of the matrix  $A$  with indices in  $W_1 \cup \dots \cup W_s$ .

**Perfect hash family.** Let  $X$  and  $Y$  be finite sets with cardinality  $|X| = n$  and  $|Y| = m$ . Let  $w \in \mathbb{N}$  such that  $2 \leq w \leq n$ . Then a perfect hash family of size  $N$  is a sequence  $f_1, \dots, f_N$  of functions from  $X$  to  $Y$  such that for any subset  $W \subset X$  with cardinality  $|W| = w$  there exists  $i \in \{1, \dots, N\}$  such that  $f_i$  is injective when restricted to  $W$ . Such perfect hash family will be denoted by  $\text{PHF}(N; n, m, w)$ .

A perfect hash family  $\text{PHF}(N; n, m, w)$  is usually viewed as a matrix  $A$  with  $N$  rows and  $n$  columns, with entries in the set of integers  $[m] \equiv \{1, 2, \dots, m\}$  such that for any set  $W \in P_w([n])$ , the  $N \times w$  matrix  $A|_W$  formed by the  $w$  columns of the matrix  $A$  with indices in  $W$  has at least one line with distinct entries.

**Separating hash family.** Given  $X$  and  $Y$  finite sets with cardinality  $|X| = n$  and  $|Y| = m$  and the integers  $w_1, \dots, w_s$  such that  $w = w_1 + \dots + w_s \leq n$ , a separating hash family of size  $N$  is a sequence  $f_1, \dots, f_N$  of functions from  $X$  to  $Y$  such that for all disjoint families of subsets  $\{W_1, \dots, W_s\}$  of  $X$  such that  $|W_j| = w_j$  ( $j = 1, \dots, s$ ), there exists  $i \in \{1, \dots, N\}$  such that  $\{f_i(W_1), \dots, f_i(W_s)\}$  is a disjoint family of subsets of  $Y$ .

A separating hash family  $\text{SHF}(N; n, m, \{w_1, \dots, w_s\})$  can be viewed as a matrix  $A$  with  $N$  rows and  $n$  columns, with entries in the set of integers  $[m]$  such that for any disjoint family  $S = \{W_1, \dots, W_s\} \in P_w^*([n])$ , the  $N \times w$  matrix  $A|_S$  formed by the  $w$  columns of the matrix  $A$  with indices in  $W_1 \cup \dots \cup W_s$  has at least one line which “separate  $A|_{W_1}, \dots, A|_{W_s}$ ”, i.e., for any unordered pair  $\{r, r'\} \subset [s]$ , the entries of this line belonging to  $A|_{W_r}$  are different from the entries of the same line belonging to  $A|_{W_{r'}}$ .

The probabilistic method has been already used several times in the past to face the problem of the existence of perfect and separated hash families. In particular lower bounds for  $N$ , ensuring the existence of a perfect hash family with fixed values  $n, m, w$  have been first obtained by Mehlhorn in [10] using standard techniques of the probabilistic method. The Local Lovász Lemma has been subsequently used by Blackburn [4] to improve the Mehlhorn bound. In the same year, another technique in the framework of the probabilistic method, the so-called expurgation method, has been used to get alternative bounds for perfect hash families [16]. Later the Lovász local lemma has also been used in [6] to get similar bounds also for separating hash families. In the same paper [6] the authors also outlined a comparison between the LLL and the expurgation method for perfect hash families suggesting that the expurgation method yields better bounds than the LLL. In a related paper [17] an alternative technique still based on the expurgation method has been used to obtain new lower bounds for  $N$ , for fixed values  $n, m, \{w_1, w_2\}$ , guaranteeing the existence of separating hash families. We finally mention that there have been also several results regarding upper bounds for  $N$  ensuring the non-existence of Separating and Hash families (see, e.g., [2] and references therein)

**1.3. Results.** We conclude this introductory section by presenting our main results which consist in new bounds for perfect hash families and separating hash families.

Our first result concerns a lower bound for perfect hash families.

**Theorem 1.2.** *Let  $N, n, m, w$  be integers such that  $2 \leq w \leq n$ . Then there exists a perfect hash family  $\text{PHF}(N; n, m, w)$  as soon as*

$$N \geq \frac{\ln[\varphi'_{w,n}(\tau)] + (w - 1) \ln(n - w) - \ln(w - 1)!}{\ln(m^w) - \ln(m^w - w! \binom{m}{w})} \tag{1.1}$$

where  $\tau$  is the first positive solution of the equation  $\varphi_{w,n}(x) - x\varphi'_{w,n}(x) = 0$  and

$$\varphi_{w,n}(x) = 1 + \sum_{k=1}^{\lfloor n/w \rfloor \wedge w} \binom{w}{k} \tilde{\Gamma}_k(w, n) x^k \tag{1.2}$$

with

$$\begin{aligned} \tilde{\Gamma}_k(w, n) &= \sum_{j=0}^{w-k} \binom{w-k}{j} \prod_{\ell=1}^{k(w-1)-j-1} \left(1 - \frac{\ell}{n-w}\right) \left[\frac{w}{n-w}\right]^j \\ &\quad \sum_{\substack{i_1+\dots+i_k=j \\ i_s \geq 0}} \frac{j!}{i_1! \dots i_k!} \prod_{\ell=1}^k \left[ \frac{1}{i_\ell+1} \prod_{s=1}^{i_\ell} \left(1 - \frac{s}{w}\right) \right]. \end{aligned} \tag{1.3}$$

Moreover the MT-algorithm (described in Section 1.3.1 below) finds such perfect hash family  $\text{PHF}(N; n, m, w)$  in an expected time which is polynomial in the input parameters  $N, n$  and  $m$  for any fixed  $w$ .

The second result concerns a similar lower bound for separating hash families. To state this result we need to introduce the following definition. Given a multi-set  $w_1, \dots, w_s$  of integers such that  $w_1 + \dots + w_s = w$ , we denote by  $m_p$  the multiplicity of the integer  $p \in \{1, 2, \dots, w\}$  in the multi-set  $w_1, \dots, w_s$ , i.e.  $m_p = \sum_{i=1}^s \mathbf{1}_{\{w_i=p\}}$ .

**Theorem 1.3.** *Let  $N, n, w$  be integers and let  $w$  be an integer such that  $2 \leq w \leq n$ . Let  $s \geq 2$  and let  $\{w_1, \dots, w_s\}$  be a family of integers such that  $w_1 + \dots + w_s = w$ . Then there exists a separating hash family  $\text{SHF}(N; n, m, \{w_1, \dots, w_s\})$  as soon as*

$$N \geq \frac{\ln[\varphi'_{w,n}(\tau)] + (w-1) \ln(n-w) - \ln(w-1)! + \ln(m_w)}{\ln\left(\frac{1}{q}\right)} \tag{1.4}$$

where  $\varphi'_{w,n}(\tau)$  is the same number introduced in Theorem 1.2,

$$m_w = \frac{1}{\prod_{p=1}^w m_p!} \frac{w!}{w_1! \dots w_k!} \tag{1.5}$$

and

$$q = 1 - \frac{\pi_{\mathcal{G}_s}(m)}{m^w}$$

with  $\pi_{\mathcal{G}_s}(m)$  being the chromatic polynomial of the complete  $s$ -partite graph  $\mathcal{G}_s$  with  $w_1, \dots, w_s$  vertices. Moreover the MT-algorithm (described in Section 1.3.1 below) finds such separating hash family  $\text{SHF}(N; n, m, \{w_1, \dots, w_s\})$  in an expected time which is polynomial in the input parameters  $N, n$  and  $m$  for any fixed  $w$ .

As claimed in the abstract, we will use the algorithmic version of the CLLL, i.e., Theorem 1.1, to prove Theorems 1.2 and 1.3. Let us thus conclude this section by describing how to adapt the Moser–Tardos setting and the MT-algorithm to the case of Perfect and separated hash families.

**1.3.1. Moser–Tardos setting for (perfect [separating] hash families).** In the present case of Hash families. the finite family  $\mathcal{V}$  of the Moser–Tardos setting is constituted by a set of  $Nn$  mutually independent random variable taking values in the set  $[m]$  according to the uniform distribution and representing the possible entries of a  $N \times n$  matrix. The sample space generated by the family  $\mathcal{V}$  is thus  $\Omega = [m]^{N \times n}$  and an outcome in  $\Omega$  is a  $N \times n$  matrix  $A$ .

**The bad events.** For each  $W \in P_w([n])$  [for each  $S = \{W_1, \dots, W_s\} \in P_w^*([n])$ ], let  $E_W$  be the event such that in any line of  $A|_W$  at least two entries are equal [let  $E_S$  be the event such that for any line of  $A|_S = A|_{\cup_{r=1}^s W_r}$  there is a pair  $\{r, r'\} \subset [s]$  such that two entries of this line, one in  $A|_{W_r}$  and the other in  $A|_{W_{r'}}$ , are equal]. We have thus a family  $\mathcal{W} \equiv \{E_W\}_{W \in P_w([n])}$  [a family  $\mathcal{S} \equiv \{E_S\}_{S \in P_w^*([n])}$ ] of bad events containing  $\binom{n}{w}$  members [containing  $\binom{n}{w} m_w$  members, with  $m_w$  defined in (1.5)]. If  $A$  is a sampled matrix such that no bad event of the family  $\mathcal{W}$  [ $\mathcal{S}$ ] occurs, then for every  $W \in P_w([n])$  [for every  $S \in P_w^*([n])$ ] at least one line of  $A|_W$  [of  $A|_S$ ] has distinct entries [separates  $S = \{W_1, \dots, W_s\}$ ], that is to say  $A$  is a PHF( $N; n, m, w$ ) [SHF( $N; n, m, \{w_1, \dots, w_s\}$ )].

We are now in the position to outline the MT-algorithm for PHF [SHF]

**MT-algorithm** (for PHF [for SHF])

– STEP 0. Pick an evaluation all  $Nn$  variables of the family  $\mathcal{V}$  (the matrix entries). Let  $A_0$  be the output matrix.

For  $k \geq 1$

– STEP  $k$  ( $k \geq 1$ ).

a) Take the matrix  $A_{k-1}$  and check all bad events of the family  $\mathcal{W}$  [of the family  $\mathcal{S}$ ].

b) i) If some bad event occurs, choose one, say  $E_W$  [ $E_S$ ], and take a new random evaluation of the entries of  $A_{k-1}|_W$  [of the entries of  $A_{k-1}|_S$ ] leaving unchanged the remaining entries of  $A_{k-1}$ .

ii) If no bad event occurs, stop the algorithm.

Let  $A_k$  be the output matrix.

Note that when the algorithm stops the output matrix is a PHF( $N; n, m, w$ ) [the output matrix is a SHF( $N; n, m, w$ )]. We will see in the next section this algorithm stops after an expected number of steps equal to  $\binom{n}{w}$ .

The rest of the paper is organized as follows. In Section 2 we give the proofs of Theorems 1.2 and 1.3. Finally, in Section 3 we discuss some comparisons with previous bounds given in the literature.

## 2. Proofs of Theorems 1.2 and 1.3

**2.1. Proof of Theorem 1.2.** Let us apply Theorem 1.1 for the family of events  $\mathcal{W} = \{E_W\}_{W \in P_w([n])}$  introduced in the previous section (Section 1.3.1). Clearly two events  $E_W, E_{W'} \in \mathcal{W}$  are independent if  $W \cap W' = \emptyset$ . Therefore the dependency graph  $G$  of the family of events  $\{E_W\}_{W \in P_w([n])}$  can be identified with the graph whose vertices are the elements  $W$  of  $P_w([n])$ , i.e. the subsets  $W$  of  $[n]$  with cardinality  $w$ , and two vertices  $W$  and  $W'$  of  $G$  are connected by an edge of the dependency graph  $G$  if and only if  $W \cap W' \neq \emptyset$ . Thus the neighborhood of  $W$  in  $G$  is the set

$$\Gamma_G(W) = \{W' : W' \in P_w([n]) \text{ and } W' \cap W \neq \emptyset\}$$

The probability of an event  $E_W$  is

$$P(E_W) = \frac{[m^w - m(m-1) \cdots (m-w+1)]^N}{m^{wN}} = \frac{\left[ m^w - w! \binom{m}{w} \right]^N}{m^{wN}}$$

and, according to Theorem 1.1, the MT-algorithm (as it was described in Section 1.3.1) finds a perfect hash family PHF( $N; n, m, w$ ) if, for some  $\mu > 0$

$$P(E_W) = \frac{\left[ m^w - w! \binom{m}{w} \right]^N}{m^{wN}} \leq \frac{\mu_W}{\sum_{Y \subseteq \Gamma_G^*(W)} \prod_{W' \in Y} \mu_{W'}} \quad (2.1)$$

$Y$  independent in  $G$

We set  $\mu_W = \mu$  for all  $W \in P_w([n])$  so that

$$\sum_{\substack{Y \subseteq \Gamma_G^*(W) \\ Y \text{ independent in } G}} \prod_{W' \in Y} \mu_{W'} = \sum_{Y \subseteq \Gamma_G^*(W)} \mu^{|Y|} = 1 + \sum_{k=1}^{\lfloor n/w \rfloor \wedge w} \Gamma_k(w, n) \mu^k$$

where

$$\Gamma_k(w, n) = \sum_{\substack{Y \subseteq \Gamma_G^*(W) : |Y|=k, \\ Y \text{ independent in } G}} 1 \quad (2.2)$$

Hence (2.1) rewrites

$$P(E_W) = \frac{\left[ m^w - w! \binom{m}{w} \right]^N}{m^{wN}} \leq \frac{\mu}{1 + \sum_{k=1}^{\lfloor n/w \rfloor \wedge w} \Gamma_k(w, n) \mu^k} \quad (2.3)$$

Let us now calculate explicitly the number  $\Gamma_k(w, n)$  defined in (2.2). We have

$$\begin{aligned}
 \Gamma_k(w, n) &= \frac{1}{k!} \sum_{\substack{i_0+i_1+\dots+i_k=w \\ i_s \geq 1, s \geq 1}} \frac{w!}{i_0!i_1!\dots i_k!} \binom{n-w}{w-i_1} \binom{n-w-(w-i_1)}{w-i_2} \dots \\
 &\quad \binom{n-w-(w-i_1)-\dots-(w-i_{k-1})}{w-i_k} \\
 &= \frac{1}{k!} \left(\frac{1}{w}\right)^k \sum_{i_0=0}^{w-k} \frac{w!}{i_0!} \sum_{\substack{i_1+\dots+i_k=w-i_0 \\ i_s \geq 1}} \prod_{l=1}^k \binom{w}{i_l} \frac{(n-w)!}{(n-kw-i_0)!} \\
 &= \binom{w}{k} \left[\frac{(n-w)^{w-1}}{w!}\right]^k \sum_{j=0}^{w-k} \binom{w-k}{j} \frac{\prod_{\ell=1}^{k(w-1)-j-1} \left(1 - \frac{\ell}{n-w}\right)}{(n-w)^j} \\
 &\quad \frac{j! \sum_{\substack{i_1+\dots+i_k=j \\ i_s \geq 0}} \prod_{l=1}^k \binom{w}{i_l+1}}{(n-w)^j} \\
 &= \binom{w}{k} \left[\frac{(n-w)^{w-1}}{w!}\right]^k \sum_{j=0}^{w-k} \binom{w-k}{j} \frac{\prod_{\ell=1}^{k(w-1)-j-1} \left(1 - \frac{\ell}{n-w}\right)}{(n-w)^j} \\
 &\quad \sum_{\substack{i_1+\dots+i_k=j \\ i_s \geq 0}} \frac{j!}{i_1!\dots i_k!} \prod_{l=1}^k \left[\frac{w!}{(i_l+1)(w-i_l-1)!}\right] \\
 &= \binom{w}{k} \left[\frac{(n-w)^{w-1}}{(w-1)!}\right]^k \sum_{j=0}^{w-k} \binom{w-k}{j} \left[\frac{w}{n-w}\right]^j \prod_{\ell=1}^{k(w-1)-j-1} \left(1 - \frac{\ell}{n-w}\right) \\
 &\quad \sum_{\substack{i_1+\dots+i_k=j \\ i_s \geq 0}} \frac{j!}{i_1!\dots i_k!} \prod_{l=1}^k \left[\frac{\prod_{s=1}^{i_l} \left(1 - \frac{s}{w}\right)}{(i_l+1)}\right].
 \end{aligned}$$

I.e. we get

$$\Gamma_k(w, n) = \binom{w}{k} \left[\frac{(n-w)^{w-1}}{(w-1)!}\right]^k \tilde{\Gamma}_k(w, n), \tag{2.4}$$

with  $\tilde{\Gamma}_k(w, n)$  given by (1.3). Therefore, setting  $\alpha = \frac{(n-w)^{w-1}}{(w-1)!} \mu$  the condition (2.3) becomes

$$\frac{(n-w)^{w-1}}{(w-1)!} \left[ \frac{m^w - w! \binom{m}{w}}{m^w} \right]^N \leq \max_{\alpha > 0} \frac{\alpha}{\varphi_{w,n}(\alpha)} = \frac{1}{\varphi'_{w,n}(\tau)} \quad (2.5)$$

where

$$\varphi_{w,n}(\alpha) = 1 + \sum_{k=1}^{\lfloor n/w \rfloor \wedge w} \binom{w}{k} \tilde{\Gamma}_k(w, n) \alpha^k$$

and  $\tau$  is the first positive solution of the equation  $\varphi(x) - x\varphi'(x) = 0$ . Taking the logarithm on both sides of (2.5), condition (2.3) is thus implied by the following inequality

$$N \geq \frac{A_n(w)}{D_m(w)}, \quad (2.6)$$

where

$$A_n(w) = \ln[\varphi'_{w,n}(\tau)] + (w-1) \ln(n-w) - \ln(w-1)! \quad (2.7)$$

and

$$D_m(w) = \ln(m^w) - \ln\left(m^w - w! \binom{m}{w}\right). \quad (2.8)$$

In conclusion once (2.6) is satisfied then also (2.1) is satisfied and therefore, according to Theorem 1.1, a PHF( $N; n, m, w$ ) exists and the MT-algorithm (as described in Section 1.3.1) finds it in an expected number of steps

$$\sum_{W \in P_w([n])} \mu = |P_w([n])| \mu = \binom{n}{w} \mu \leq \binom{n}{w},$$

where the last inequality follows from the fact that the optimum  $\mu$  which maximize the right hand side of (2.3) is surely less than one.

Now, at each step  $k \geq 1$  of the MT-algorithm described in Section 1.3.1, in order to check in item a) whether or not a bad event of the family  $\{E_W\}_{P_w([n])}$  occurs, we need to consider all the  $N$  lines of (at worst) all the  $\binom{n}{w}$  matrices  $A|_W$  with  $W \in P_w([n])$ , and for each line of a given matrix  $A|_W$  we need to compare all pair of entries of the line to check whether they are equal or not. This is done in (at most)  $N \binom{w}{2} \binom{n}{w}$  operations.

This concludes the proof of Theorem 1.2.

**2.2. Proof of Theorem 1.3.** We first recall that, for a fixed sequence of integers  $w_1, w_2, \dots, w_s$  such that  $w = w_1 + \dots + w_s \leq n$ ,  $P_w^*([n])$  is the set whose elements are the disjoint families  $S = \{W_1, \dots, W_s\}$  of subsets of  $[n]$  with cardinality  $w_1, \dots, w_s$  resp. and  $A|_S$  is the  $N \times w$  matrix formed by the  $w$  columns of the

matrix  $A$  with indices in  $\bigcup_{i=1}^s W_i$ . Given two disjoint families  $S = \{W_1, \dots, W_k\}$  and  $S' = \{W'_1, \dots, W'_k\}$ , we also denote shortly  $S \cap S' \doteq (\bigcup_{i=1}^s W_i) \cap (\bigcup_{i=1}^s W'_i)$ .

Let us apply Theorem 1.1 for the family of events  $\mathcal{S} = \{E_S\}_{S \in P_w^*([n])}$  introduced in Section 1.3.1.

For  $S = \{W_1, \dots, W_s\} \in P_w^*([n])$ , the probability of the event  $E_S$  is given by

$$P(E_S) = \prod_{i=1}^N P_i(S),$$

where  $P_i(S)$  is the probability that the line  $i$  of the matrix  $A|_S$  do not separate  $S$ . To calculate  $P_i(S)$  just observe that

$$P_i(S) = \frac{\# \text{ favorable cases}}{\# \text{ number of all cases}} = 1 - \frac{\# \text{ unfavorable cases}}{\# \text{ number of all cases}}.$$

Setting  $w = w_1 + w_2 + \dots + w_s$ ,

$$\# \text{ number of all cases} = m^w.$$

To count the number of unfavorable cases, consider the complete  $s$ -partite graph  $\mathcal{G}_s$  with vertex set  $W$  and independent sets of vertices set  $W_1, \dots, W_s$ . Then,

$$\# \text{ unfavorable cases} = \# \text{ proper colorings of } \mathcal{G}_s \text{ with } m \text{ colors} = \pi_{\mathcal{G}_s}(m),$$

where  $\pi_{\mathcal{G}_s}(m)$  is the chromatic polynomial of the graph  $\mathcal{G}_s$ . Thus

$$P_i(S) = 1 - \frac{\pi_{\mathcal{G}_s}(m)}{m^w} \doteq q$$

and therefore

$$P(E_S) = q^N.$$

As before two events  $E_S, E_{S'} \in \mathcal{S}$  are independent if  $S \cap S' = \emptyset$ . Therefore The dependency graph for the family of events  $\mathcal{S} = \{E_S\}_{S \in P_w^*([n])}$  can be identified with the graph  $G$  with vertex set  $P_w^*([n])$  such that two vertices  $S = \{W_1, \dots, W_s\}$  and  $S' = \{W'_1, \dots, W'_s\}$  are connected by an edge of  $G$  if and only if  $S \cap S' \neq \emptyset$ , where recall that  $S \cap S' \doteq (\bigcup_{i=1}^s W_i) \cap (\bigcup_{i=1}^s W'_i)$ . This implies that the neighbor  $\Gamma_G^*(S)$  of a vertex  $S$  of  $G$  is given by

$$\Gamma_G^*(S) = \{S': S' \in P_w^*([n]) \text{ and } S' \cap S \neq \emptyset\}.$$

By Theorem 1.1, the Moser–Tardos algorithm (as described in Section 1.3.1) finds a separating hash family SHF( $N; n, m, \{w_1, \dots, w_s\}$ ) if the following condition is satisfied: there exists  $\nu > 0$  such that

$$q^N \leq \frac{\nu}{\sum_{\substack{Y \subseteq \Gamma_G^*(S) \\ Y \text{ independent in } G}} \nu^{|Y|}}. \tag{2.9}$$

Note that, as we did in the previous section, we have set  $\mu_S = \nu$  for all  $S \in P_w^*([n])$ .

The denominator of the right hand side of (2.9) can be evaluated similarly as we did for the case of perfect hash families. Indeed, given a disjoint family  $S = \{W_1, \dots, W_s\}$ , the neighbor of  $S$  in  $G$  is formed by all vertices  $S' = \{W'_1, \dots, W'_s\}$  such that  $(\bigcup_{r=1}^s W_r) \cap (\bigcup_{r=1}^s W'_r) \neq \emptyset$ . The only thing that changes respect to the calculations done for case of the perfect hash families is that now, fixed a set of columns  $W$  with cardinality  $w = w_1 + \dots + w_s$ , the number of different disjoint families  $S = \{W_1, \dots, W_s\}$  such that  $W = \bigcup_{r=1}^s W_r$  and  $|W_r| = w_r$  for  $r = 1, \dots, s$  is given by the quantity  $m_w$  defined in (1.5). Therefore we have

$$\sum_{\substack{Y \subseteq \Gamma_G^*(S) \\ Y \text{ independent in } G}} \nu^{|Y|} = 1 + \sum_{k=1}^{\lfloor n/w \rfloor \wedge w} \Gamma_k(w, n) (m_w \nu)^k,$$

where  $\Gamma_k(w, n)$  is exactly the same number defined in (2.2). Hence posing  $m_w \nu = \mu$  we have that a separating hash family SHF( $N; n, m, \{w, \dots, w_s\}$ ) exists and can be found in polynomial time by the Moser–Tardos algorithm if

$$m_w q^N \leq \frac{\mu}{1 + \sum_{k=1}^{\lfloor n/w \rfloor \wedge w} \Gamma_k(w, n) \mu^k}. \tag{2.10}$$

Note that the right hand side of inequality (2.10) and the right hand side of inequality (2.3) are the same. Hence we get that the condition (2.9) becomes

$$\frac{(n-w)^{w-1}}{(w-1)!} m_w q^N \leq \frac{1}{\varphi'_{w,n}(\tau)},$$

where  $\varphi'_{w,n}(\tau)$  is the same number as in (2.5). In other word the condition (2.9) rewrites as

$$N \geq \frac{S_n(w)}{\ln\left(\frac{1}{q}\right)} \tag{2.11}$$

with

$$S_n(w) = \ln[\varphi'_{w,n}(\tau)] + (w - 1) \ln(n - w) - \ln(w - 1)! + \ln(m_w). \quad (2.12)$$

According to Theorem 1.1 the MT-algorithm (as described in Section 1.3.1) finds a SHF( $N; n, m, \{w_1, \dots, w_s\}$ ) satisfying (2.11) and hence (2.9) in an expected number of steps

$$\sum_{S \in P_w^*([n])} \nu = |P_w^*([n])| \nu = \binom{n}{w} m_w \nu = \binom{n}{w} \mu \leq \binom{n}{w},$$

where the last inequality follows from the fact that the optimum  $\mu$  which maximize the right hand side of (2.10) is surely less than one. The number  $\binom{n}{w}$  of these steps, similarly to what done for PHF, has to be multiplied by  $N \binom{w}{2} m_w \binom{n}{w}$  which is the number of operations needed to check item a) of step  $i \geq 1$  of the MT-algorithm for SHF described in Section 1.3.1.

This concludes the proof of Theorem 1.3.

### 3. Comparison with previous bounds

Let us first observe that the polynomial  $\varphi_{w,n}(x)$  introduced in (1.2) is such that

$$\lim_{n \rightarrow \infty} \varphi_{w,n}(x) \longrightarrow (1 + x)^w,$$

and therefore the number  $\varphi'_{w,n}(\tau)$  appearing in bounds (1.1) and (1.4) for perfect hash families and separating hash families resp. is such that

$$\lim_{n \rightarrow \infty} \varphi'_{w,n}(\tau) = w \left(1 + \frac{1}{w-1}\right)^{w-1}. \quad (3.1)$$

**3.1. Perfect hash families.** We first recall the previous lower bounds obtained in the literature via the Probabilistic method.

First, via the usual Lovász local Lemma (see, e.g., [6]) one obtains

$$N \geq \frac{L_n(w)}{D_m(w)}, \quad (3.2)$$

where

$$L_n(w) = \ln \left[ e \left( \binom{n}{w} - \binom{n-w}{w} \right) \right]. \quad (3.3)$$

On the other hand, via the expurgation method (see [16] and [6]) one gets

$$N \geq \frac{E_n(w)}{D_m(w)}, \quad (3.4)$$

where

$$E_n(w) = \ln \binom{2n}{w} - \ln n. \quad (3.5)$$

Let's first compare our bound (2.6) with (3.4) obtained via expurgation method. Note that the numerator  $E_n(w)$  appearing in the right hand side of (3.4) can be written as

$$\begin{aligned} E_n(w) &= \ln \binom{2n}{w} - \ln n \\ &= w \ln 2 + (w-1) \ln n + \sum_{j=1}^{w-1} \ln \left(1 - \frac{j}{2n}\right) - \ln(w!) \\ &= w \ln 2 + (w-1) \ln(n-w) - \ln(w!) + \sum_{j=1}^{w-1} \ln \left(\frac{1 - \frac{j}{2n}}{1 - \frac{j}{n}}\right). \end{aligned} \quad (3.6)$$

So that asymptotically

$$E_n(w) \sim \mathcal{E}_w + (w-1) \ln n - \ln[(w-1)!] \quad (3.7)$$

with

$$\mathcal{E}_w = w \ln 2 - \ln w.$$

On the other hand, in view of (3.1), the numerator of the right hand side of (2.6) is asymptotic to

$$A_n(w) \sim \mathcal{A}_w + (w-1) \ln(n) - \ln[(w-1)!] \quad (3.8)$$

with

$$\mathcal{A}_w = \ln w + \ln \left[ \left(1 + \frac{1}{w-1}\right)^{w-1} \right].$$

Observe that while  $\mathcal{E}_w$  grows linearly in  $w$ , the factor  $\mathcal{A}_w$  in (3.8) grows only logarithmically  $w$ . Thus, to compare (2.6) with (3.4) as  $n \rightarrow \infty$  we have that

$$A_n(w) < B_n(w) \iff 2 \ln w + \ln \left[ \left(1 + \frac{1}{w-1}\right)^{w-1} \right] < w \ln 2$$

which happens as soon as

$$w > 6.91043.$$

This means that, asymptotically in  $n$ , for all  $w \geq 7$  our bound beats the expurgation bound. Moreover, numerical evidence suggests that the function

$$\Delta_n(w) = E_n(w) - A_n(w) = \ln \binom{2w}{w} + \sum_{j=1}^{w-1} \ln \left(\frac{1 - \frac{j}{2n}}{1 - \frac{j}{n}}\right) - \ln[\phi'_{w,n}(\tau)] \quad (3.9)$$

is decreasing in  $n$  for fixed  $w$ . If this were the case (we do not have a proof of that), our bound would be always better as long as  $w \geq 7$  (see Table 1). For values of  $w \leq 6$ , one can perform numerical calculations with the bounds (2.6) and (3.4) and see that our bound beats the bound obtained via expurgation only for low values of  $n$  and the lower is  $w$  the lower is the  $n$  for which we win. In particular, we get that for  $w = 6$ ,  $w = 5$ ,  $w = 4$ , and  $w = 3$  our bound beats expurgation bound for all  $n \leq 97$ ,  $n \leq 34$ ,  $n \leq 15$ , and  $n \leq 7$  resp. (see also Table 2).

Table 1. Bounds on the cardinality of perfect hash families with  $w \geq 7$ .

| $n$  | $m$ | $w$ | Theorem 1.2 | Expurgation |
|------|-----|-----|-------------|-------------|
| 15   | 7   | 7   | 1437        | 1926        |
| 50   | 7   | 7   | 3034        | 3191        |
| 200  | 7   | 7   | 4529        | 4572        |
| 1000 | 7   | 7   | 6139        | 6152        |
| 50   | 8   | 8   | 8463        | 9159        |
| 200  | 8   | 8   | 12965       | 13282       |
| 1000 | 8   | 8   | 17774       | 17988       |
| 1000 | 12  | 8   | 900         | 911         |
| 1000 | 50  | 8   | 53          | 54          |
| 1000 | 50  | 15  | 730         | 781         |
| 1000 | 50  | 18  | 2812        | 3037        |

Table 2. Bounds on the cardinality of perfect hash families with  $w < 7$ .

| $n$ | $m$ | $w$ | Theorem 1.2 | Expurgation |
|-----|-----|-----|-------------|-------------|
| 10  | 4   | 4   | 57          | 62          |
| 15  | 4   | 4   | 76          | 77          |
| 50  | 4   | 4   | 121         | 114         |
| 10  | 5   | 5   | 144         | 187         |
| 15  | 5   | 5   | 211         | 234         |
| 50  | 5   | 5   | 369         | 364         |
| 15  | 6   | 6   | 558         | 681         |
| 50  | 6   | 6   | 1072        | 1092        |
| 90  | 6   | 6   | 1284        | 1287        |
| 200 | 6   | 6   | 1557        | 1546        |

By theoretical reasons (see [3]) the cluster expansion Lovász Lemma is always better than the usual Lovász local lemma, so that bound (2.6) always beats the bound (3.2) for any pair  $(w, n)$ . In any case, it is interesting to compare our bound with the Lovász Lemma bound asymptotically as  $n \rightarrow \infty$ . We have that they are

not equivalent, i.e. our bound beats LLL even asymptotically since

$$\lim_{n \rightarrow \infty} \frac{\ln[\varphi'_{n,w}(\tau)] + (w - 1) \ln(n - w) - \ln(w - 1)!}{\ln \left[ e \left( \binom{n}{w} - \binom{n-w}{w} \right) \right]} = \frac{w - 1}{w}.$$

**3.2. Separating hash families.** We can compare the bound (1.4) obtained in Theorem 1.3 with that obtained via expurgation method and see that the improvement is completely analogous to that obtained for the perfect hash families. First observe that, recalling (1.5), the quantity  $S_n(w)$  defined in (2.12) can be rewritten as follows:

$$S_n(w) = \ln[w\varphi'_{w,n}(\tau)] + (w - 1) \ln(n - w) - \sum_{i=1}^s \ln(w_i!) - \sum_{p=1}^w \ln(m_p!).$$

Let us then consider for simplicity the case  $k = 2$  and  $w_1 \neq w_2$ . For this case  $S_n(w)$  becomes

$$S_n(w) = \ln[w\varphi'_{w,n}(\tau)] + (w - 1) \ln(n - w) - \ln(w_1!w_2!).$$

Via the expurgation method (see [6]) one gets

$$N \geq \frac{F_n(w)}{\ln\left(\frac{1}{q}\right)}, \tag{3.10}$$

where

$$F_n(w) = \ln\binom{2n}{w_1} - \ln\binom{2n - w_1}{w_2} - \ln n. \tag{3.11}$$

As before we can write

$$F_n(w) = w \ln 2 + (w - 1) \ln(n - w) + \sum_{j=1}^{w-1} \ln\left(\frac{1 - \frac{j}{2n}}{1 - \frac{j}{n}}\right) - \ln(w_1!w_2!).$$

So we have

$$F_n(w) - S_n(w) = \Delta_n(w),$$

where  $\Delta_n(w)$  is the same quantity defined in (3.9). This means that all that we discussed for perfect hash families holds also for separating hash families. In particular, our bound beats the bound obtained via expurgation method reported in [6] asymptotically in  $n$  as soon as  $w > 6$ .

We finally compare our bound with still another bound given by Stinson and Zaverucha [17] in 2008. These authors claim that a SHF( $N; n, m, \{w_1, w_2\}$ ) exists if

$$n \leq \left(1 - \frac{1}{C_w}\right) \left[\frac{1}{q}\right]^{\frac{N}{w-1}}, \quad (3.12)$$

where  $w = w_1 + w_2$  and

$$C_w = \begin{cases} w_1!w_2! & \text{if } w_1 \neq w_2, \\ 2w_1!w_2! & \text{if } w_1 = w_2. \end{cases}$$

Our bound (1.4) on the other hand implies that a SHF( $N; n, m, \{w_1, w_2\}$ ) exists if

$$(n - w)^{w-1} \leq \frac{C_w}{w\varphi'(\tau)} \left[\frac{1}{q}\right]^N.$$

Once again, for sake of simplicity we perform this comparison asymptotically as  $N$  (and hence  $n$ ) large. In this case we have seen that  $\varphi'(\tau) \sim w(1 + \frac{1}{w-1})^{w-1}$ . Therefore asymptotically we have the bound

$$n \leq \left(\frac{C_w}{w^2}\right)^{\frac{1}{w-1}} \frac{w-1}{w} \left[\frac{1}{q}\right]^{\frac{N}{w-1}}. \quad (3.13)$$

The comparison of this with bound (3.12) is now straightforward. We see that our bound (3.13) beats (3.12) as soon as  $w$  is larger than 6.

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## References

- [1] R. G. Alves and A. Procacci, Witness trees in the Moser–Tardos algorithmic Lovász local lemma and Penrose trees in the hard-core lattice gas. *J. Stat. Phys.* **156** (2014), no. 5, 877–895. [MR 3231046](#) [Zbl 1302.60021](#)

- [2] M. Bafraşhan and T. van Trung, Improved bounds for separating hash families. *Des. Codes Cryptogr.* **69** (2013), no. 3, 369–382. [MR 3092767](#) [Zbl 1273.94339](#)
- [3] R. Bissacot, R. Fernández, A. Procacci, and B. Scoppola, An improvement of the Lovász local lemma via cluster expansion. *Combin. Probab. Comput.* **20** (2011), no. 5, 709–719. [MR 2825585](#) [Zbl 1233.05196](#)
- [4] S. R. Blackburn, Perfect hash families: probabilistic methods and explicit constructions. *J. Combin. Theory Ser. A* **92** (2000), no. 1, 54–60. [MR 1783938](#) [Zbl 0962.68042](#)
- [5] J. Böttcher, Y. Kohayakawa, and A. Procacci, Properly coloured copies and rainbow copies of large graphs with small maximum degree. *Random Structures Algorithms* **40** (2012), no. 4, 425–436. [MR 2925305](#) [Zbl 1244.05087](#)
- [6] D. Deng, D. R. Stinson, and R. Wei, The Lovász local lemma and its applications to some combinatorial arrays. *Des. Codes Cryptogr.* **32** (2004), no. 1-3, 121–134. [MR 2072321](#) [Zbl 1046.05080](#)
- [7] R. L. Dobrushin, Perturbation methods of the theory of Gibbsian fields. In P. Bernard (ed.), *Lectures on probability theory and statistics*. Lectures from the 24<sup>th</sup> Saint-Flour Summer School held July 7–23, 1994. Lecture Notes in Mathematics, 1648. Springer-Verlag, Berlin, 1996, 1–66. [MR 1600880](#) [Zbl 1600892](#)
- [8] P. Erdős and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions. In A. Hajnal, R. Rado and, V. T. Sós (eds.), *Infinite and finite sets*. Vol. II. Proceedings of a Colloquium held at Keszthely, June 25–July 1, 1973. Dedicated to Paul Erdős on his 60<sup>th</sup> birthday. North-Holland Publishing Co., Amsterdam and London, and János Bolyai Mathematical Society, Budapest, 1975, 609–627. [MR 0382050](#) [Zbl 0315.05117](#)
- [9] R. Fernández and A. Procacci, Cluster expansion for abstract polymer models. New bounds from an old approach. *Comm. Math. Phys.* **274** (2007), no. 1, 123–140. [MR 2318850](#) [Zbl 1206.82148](#)
- [10] K. Mehlhorn, *Data structures and algorithms*. 1. Sorting and searching. EATCS Monographs on Theoretical Computer Science. Springer-Verlag, Berlin, 1984. [MR 0756413](#) [Zbl 0556.68001](#)
- [11] R. Moser and G. Tardos, A constructive proof of the general Lovász local lemma. *J. ACM* **57** (2010), no. 2, Art. 11, 15 pp. [MR 2606086](#) [Zbl 1300.60024](#)
- [12] S. Ndreca, A. Procacci and B. Scoppola, Improved bounds on coloring of graphs. *European J. Combin.* **33** (2012), no. 4, 592–609. [MR 2864444](#) [Zbl 1244.05094](#)
- [13] W. Pegden, An extension of the Moser–Tardos algorithmic local lemma. *SIAM J. Discrete Math.* **28** (2014), no. 2, 911–917. [MR 3215059](#) [Zbl 1301.60013](#)
- [14] A. Scott and A. D. Sokal, The repulsive lattice gas, the independent-set polynomial, and the Lovász local lemma. *J. Stat. Phys.* **118** (2005), no. 5-6, 1151–1261. [MR 2130890](#) [Zbl 1107.82013](#)
- [15] J. B. Shearer, On a problem of Spencer. *Combinatorica* **5** (1985), no. 3, 241–245. [MR 0837067](#) [Zbl 0587.60012](#)

- [16] D. R. Stinson, T. van Trung, and R. Wei, Secure frameproof codes, key distribution patterns, group testing algorithms and related structures. *J. Statist. Plann. Inference* **86** (2000), no. 2, 595–617. [MR 1768292](#) [Zbl 1054.94013](#)
- [17] D. R. Stinson and G. M. Zaverucha, Some improved bounds for secure frameproof codes and related separating hash families. *IEEE Trans. Inform. Theory* **54** (2008), no. 6, 2508–2514. [MR 2449258](#) [Zbl 1304.94110](#)

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