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# Edge correlation function of the 8-vertex model when $a + c = b + d$

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**Abstract.** This paper is devoted to the 8-vertex model and its edge correlation function. In some particular (integrable) cases, we find a closed form of the edge correlation function and we deduce also its asymptotics. In addition, we quantify influence of boundary conditions on this function.

To do this, we introduce a system of particles in interaction related to the 8-vertex model. This system, studied using various tools from analytic combinatorics, random walks and conics, permits to compute the correlation function. To study the influence of boundary conditions, we involve probabilistic cellular automata of order 2.

**Mathematics Subject Classification (2010).** Primary: 82B20, 82B31; Secondary 60J10, 05A15.

**Keywords.** 8-vertex model, correlation function, system of particles, probabilistic cellular automata.

## Contents

## 1. Introduction

**Vertex-models.** We start with some formal definitions. Let $K_N$ be the graph whose set of vertices is

$$V_N = \{(i, j)\colon -1 \le i, j \le N + 1\}$$

and set of edges is

$$\begin{aligned} E_N =&\{((i, j), (i, j + 1))\colon 0 \le i \le N, -1 \le j \le N\} \\ &\cup \{((i, j), (i + 1, j))\colon -1 \le i \le N, 0 \le j \le N\} \end{aligned}$$

(see Figure 1). The vertices

$$V_N^i = \{(i, j)\colon 0 \le i, j \le N\}$$

are called internal vertices of $K_N$ (it is also the set of vertices adjacent to 4 edges). The edges

$$\begin{aligned} E_N^e =&\{((-1, j), (0, j))\colon 0 \le j \le N\} \cup \{((N, j), (N + 1, j))\colon 0 \le j \le N\} \\ &\cup \{((i, -1), (i, 0))\colon 0 \le i \le N\} \cup \{((i, N), (i, N + 1))\colon 0 \le i \le N\} \end{aligned}$$

are called external (or boundary) edges (it is also the set of edges whose one end vertex is in $V_N^i$ and the other one in $V_N \setminus V_N^i$). The graph $K_N$ have $2N^2+2N$ edges that could be classified in two groups: the $4N$ external edges $E_N^e$ and the $2N^2-2N$ internal edges $E_N \setminus E_N^e$. Each of the edges could be oriented: either "from bottom to top" or "from top to bottom" if the edge is vertical, either "from left to right" or "from right to left" it the edge is horizontal. We call an *orientation* of $K_N$, the graph $K_N$ with an orientation for every of its edges. There exists $2^{2N^2+2N}$ orientations of $K_N$ and we denote $\Omega_N$ the set of these orientations.

In the following, we call vertices of $K_N$ only its internal vertices.

In the 8-vertex model, we consider the subset $\Omega_N^8 \subset \Omega_N$ of $K_N$'s orientations such that, around each vertex of $K_N$, there is an even number (0, 2 or 4) of incoming edges. Hence, for any vertex $(i, j)$ in a $K_N$'s orientation $O \in \Omega_N^8$, the $(i, j)$'s four adjacent oriented edges are oriented like one of the eight local configurations of Figure 2. For any $k \in \{1, \dots, 8\}$, a vertex is said to be *of type k* if its four adjacent edges are in the local configuration $k$. At each local configuration $k$, we associate a local weight $w_k$. Based on local weights $(w_k\colon k \in \{1, \dots, 8\})$, we define a global weight (of Boltzmann type) $W$ on $\Omega_N^8$: let $O \in \Omega_N^8$, the weight of the orientation $O$ is

$$W(O) = \prod_{k=1}^{8} w_k^{n_k(O)} \tag{1}$$

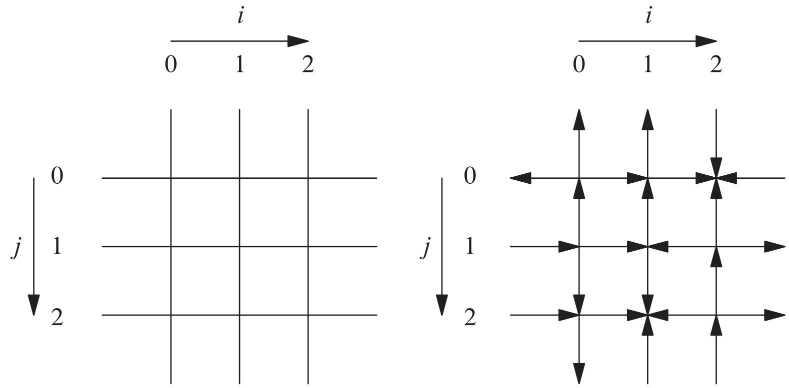



Figure 1. Left: $K_3$. Right: an orientation of $K_3$.

where $n_k(O)$ is the number of vertices of type $k$ in $O$. From $W$ on $\Omega_N^8$, we define a probability measure $P_N^8$ on $\Omega_N^8$: for any $O \in \Omega_N^8$,

$$P_N^8(O) = \frac{W(O)}{Z_N^8} \quad \text{where } Z_N^8 = \sum_{O' \in \Omega_N^8} W(O').$$

The quantity $Z_N^8$ is called *partition function* of the 8-vertex model on $K_N$. Under the probability $P_N^8$, the probability to get an orientation $O$ is then proportional to its weight $W(O)$.

When $w_7 = w_8 = 0$, the 8-vertex model becomes the 6-vertex model. Orientations of the 6-vertex model with a non-zero global weight are $K_N$'s orientations whose nodes have two incoming and two outgoing edges. The 6-vertex model is historically the first vertex-model introduced by Pauling in 1935 [17] to study a model of ice on plane. Indeed, in the 6-vertex model, nodes represent molecules of water, and oriented edges, polarities of hydrogen bonds between these molecules. It is a model of statistical physics widely studied and we recommend [2, Chapter 8], [19], [11] and references therein to the interesting reader.

The 8-vertex model is a generalization of the 6-vertex model introduced by Sutherland [22] and Fan and Wu [13] in 1970. Its partition function was computed by Baxter in 1972 using Bethe's Ansatz methods [1]. One important property of the 8-vertex model in comparison with the 6-vertex model is that it is less dependent on boundary conditions [6]. It is also related to Ising models as expressed by Baxter in [2, Section 10.3]. Let us present some results of Baxter on the 8-vertex model.

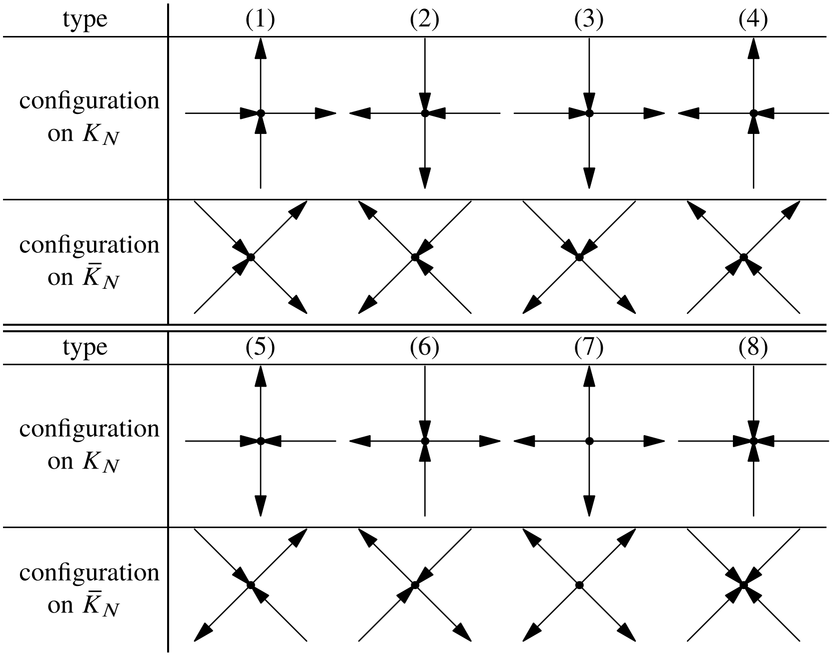



Figure 2. The 8 available local configurations around any vertex of an orientation in $\Omega_N^8$ and their corresponding configuration (rotated by an angle $-\frac{\pi}{4}$) in $\bar{\Omega}_N^8$.

Due to symmetries of $K_N$, the numbers of vertices of type 5 and of type 6 in an admissible configuration differ by less than $N$ ($|n_5 - n_6| \le N$), this permits to chose $w_5 = w_6 = c$ without loss of generality in asymptotics (in the asymptotic case, an interesting type $k$ of nodes is one for which we have that $n_k = \Theta(N^2)$). For similar reasons, $|n_7 - n_8| \le 2N$ and, hence, we consider in the following $w_7 = w_8 = d$. We will also suppose that we are in a “zero-field” case (a classical hypothesis in a first study of a vertex-model), i.e. we suppose $w_1 = w_2 = a$ and $w_3 = w_4 = b$. In that case, the partition function $Z_N^8$ of the 8-vertex model was studied by Baxter in 1972 [1] and he describes a phase transition behavior related to the value of $(a, b, c, d)$ when $N \to \infty$. He proved that the 8-vertex model has 5 different asymptotic behaviors [2, Section 8.10]:

- if $a > b+c+d$ (state I), then it is a ferromagnetic state, in which $N^2 - o(N^2)$ vertices are either of type 1, or of type 2 a.s. when $N \to \infty$;
- if $b > a+c+d$ (state II), then it is a ferromagnetic state, in which $N^2 - o(N^2)$ vertices are either of type 3, or of type 4 a.s. when $N \to \infty$;
- if $c > a+b+d$ (state IV), then it is an anti-ferromagnetic state, $\frac{N^2}{2} - o(N^2)$ vertices are of type 5 and $\frac{N^2}{2} - o(N^2)$ vertices are of type 6 a.s. when $N \to \infty$;

- if $d > a + b + c$ (state V), then it is an anti-ferromagnetic state, $\frac{N^2}{2} - o(N^2)$ vertices are of type 7 and $\frac{N^2}{2} - o(N^2)$ vertices are of type 8 a.s. when $N \to \infty$;
- otherwise (i.e. if $a, b, c, d < \frac{a+b+c+d}{2}$) (state III), then it is a disordered state, there are $\Theta(N^2)$ vertices of each type a.s. when $N \to \infty$.

Until now, we consider only the 8-vertex model on $K_N$ with free boundary conditions because external edges $E^e_N$ of $K_N$ are not constrained. In some other cases, edges of $E^e_N$ are constrained and so we do not consider all the orientations of $\Omega^8_N$ but a subset of them. The constraint imposed to the external edges is called a boundary condition. Some classical examples of boundary conditions are (see [6]):

- free boundary condition (FBC): no constraint on $E^e_N$;
- periodic boundary condition: external edges of a same line or a same column are oriented in a same direction;
- "wall" boundary condition (see [25]): horizontal external edges are oriented to the inside of the graph and vertical ones to the outside (see Figure 3);
- special boundary condition (SBC): orientation of external edges is arbitrary imposed ("'wall" is an example).

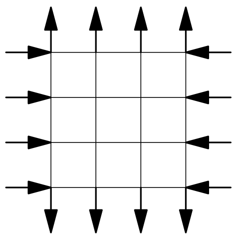


Figure 3. $K_4$ with a "wall" boundary condition.

In this article, main results are about the 8-vertex model on $K_N$ with FBC and the following condition on local weights

$$a + c = b + d. \tag{2}$$

Our aim is to compute the law of orientations of two distant edges, that is the edge correlation function. This computation has been realized for the 6-vertex model in some particular cases: the free fermion limit case [21] and the $a + c = b$ case [14] (see also [4] for references on the 6-vertex model with $a + c = b$). More generally, correlation functions are important subject in statistical physics, see [20, Chapter 10] for an overview.

To obtain these results, we consider the 8-vertex model on the graph $\bar{K}_N$. Let us define it. The set of $\bar{K}_N$'s vertices is

$$\begin{aligned}\bar{V}_N = &\Big\{\Big(-\frac{1}{2}+j,-\frac{1}{2}\Big): 0 \le j \le N\Big\} \\ &\cup \{(j,0): 0 \le j \le N-1\} \\ &\cup \Big\{\Big(\frac{i}{2}-\frac{1}{2}+j,\frac{i}{2}+\frac{1}{2}\Big): i \le 0 \le N-1, 0 \le j \le N-i\Big\}\end{aligned}$$

and the set of its edges is

$$\begin{aligned}\bar{E}_N = &\Big\{\Big(\Big(j-\frac{1}{2},-\frac{1}{2}\Big),(j,0)\Big),\Big(\Big(j+\frac{1}{2},-\frac{1}{2}\Big),(j,0)\Big): 0 \le j \le N-1\Big\} \\ &\cup \Big\{\Big(\Big(\frac{i}{2}-\frac{1}{2}+j,\frac{i}{2}+\frac{1}{2}\Big),\Big(\frac{i}{2}+j,\frac{i}{2}\Big)\Big), \\ &\quad \Big(\Big(\frac{i}{2}+\frac{1}{2}+j,\frac{i}{2}+\frac{1}{2}\Big),\Big(\frac{i}{2}+j,\frac{i}{2}\Big)\Big): \\ &\quad 0 \le i \le N-1,\ 0 \le j \le N-1-i\Big\}\end{aligned}$$

(see Figure 4). As before, we distinguish two types of vertices: internal vertices of degree 4 (called vertices in the following) and the other vertices of degree 1 or 2, and two types of edges: internal edges whose two end vertices are internal and external edges whose only one end vertex is internal. In addition, edges are labeled: the edge whose end vertices are $\{(i,t),(i',t')\}$ is labeled by $(2\max(i,i'),2\max(t,t'))$. And, as before, we can define the 6- and 8-vertex models on $\bar{K}_N$. Definitions are the same; only local configurations change, they are rotated of an angle $-\frac{\pi}{4}$ (see Figure 2) from the ones in the $K_N$ case. Notations are also the same as in the $K_N$ case except that they are overlined.

In the following, orientations of edges are denoted by their vertical orientations: if the edge $(i,t)$ is oriented like $\nwarrow$ (if $i+t$ is even) or like $\nearrow$ (if $i+t$ is odd), then edge $(i,t)$ is said to be *up-oriented* and, if the edge is oriented like $\searrow$ or $\swarrow$, it is said to be *down-oriented*. This information is encoded in a state $e(i,t)$:

$$e(i,t) = \begin{cases} 1 & \text{if the edge } (i,t) \text{ is up-oriented,} \\ 0 & \text{if the edge } (i,t) \text{ is down-oriented.}\end{cases} \tag{3}$$

We introduce the following probabilistic boundary condition on the 6- and 8-vertex models on $\bar{K}_N$: orientations of edges on the top side

$$(e(i,0): 0 \le i \le 2N-1)$$

are distributed according to a product measure with parameter $q$, i.e. they are i.i.d. of common law the Bernoulli law with parameter $q$ (denoted $\mathcal{B}(q)$): for any $i$,

$$P(e(i,0)=1) = q \quad \text{and} \quad P(e(i,0)=0) = 1-q;$$

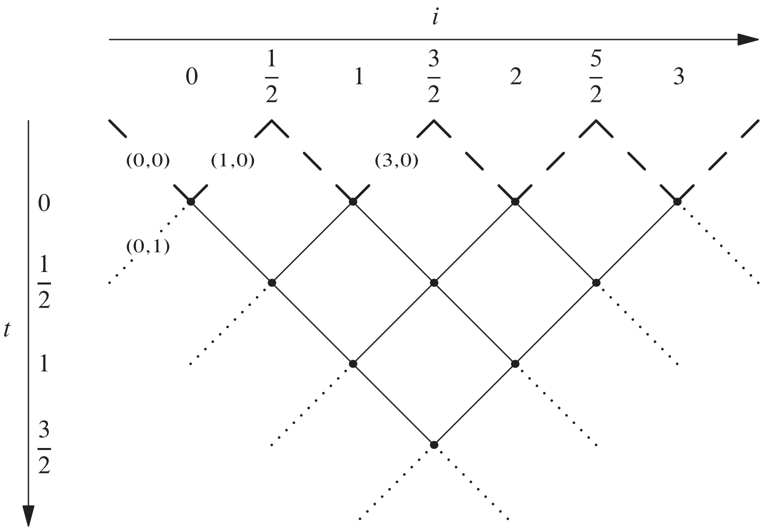



Figure 4. The graph $\bar{K}_4$. On axes: vertices coordinates. On edges: edges labels. Boundary conditions: dashed edges are i.i.d. (for the orientation up/bottom) and dotted edges are free.

the other external edges

$$\{(i,t): t - i = 1,\ 1 \le t \le N\} \cup \{(i,t): t + i = 2N,\ 1 \le t \le N\}$$

are free oriented. We call this boundary condition half product measure with parameter $q$ (denoted HPMBC($q$) in the following). Formally, the 8-vertex model on $\bar{K}_N$ with HPMBC($q$) defines a probability measure $\bar{P}_N^{8,q}$ on $\bar{\Omega}_N^8$ in the following way: for any $O \in \bar{\Omega}_N^8$,

$$\bar{P}_N^{8,q}(O) = \sum_{B \in \{0,1\}^{2N}} \prod_{i=0}^{2N-1} q^{e(i,0)}(1-q)^{1-e(i,0)} \frac{a^{n_1+n_2} b^{n_3+n_4} c^{n_5+n_6} d^{n_7+n_8}}{\bar{Z}_N^{8,B}} \mathbf{1}_{(e(i,0):0 \le i \le 2N-1)=B} \tag{4}$$

where $\bar{Z}_N^{8,B}$ is the partition function of the 8-vertex model on $\bar{K}_N$ with SBC $B$ on edges $((i,0): 0 \le i \le 2N)$ and FBC on other edges of $\bar{E}_N^e$. Denoting the subset of $\bar{\Omega}_N^8$ such that edges $((i,0): 0 \le i \le 2N)$ are oriented as $B$ by $\bar{\Omega}_N^{8,B}$,

$$\bar{Z}_N^{8,B} = \sum_{O' \in \bar{\Omega}_N^{8,B}} a^{n_1(O')+n_2(O')} b^{n_3(O')+n_4(O')} c^{n_5(O')+n_6(O')} d^{n_7(O')+n_8(O')}. \tag{5}$$

Our aim is to describe properties of the 6- and 8-vertex models on $\bar{K}_N$ in the thermodynamic limit (when $N \to \infty$) under the constraint $a + c = b + d$. A first remarkable property is that, when $a + c = b + d$, the 8-vertex models on $\bar{K}_N$ with HPMBC $\left(\frac{1}{2}\right)$ or with FBC are the same.

**Proposition 1.1.** *For any $N$, if $a + c = b + d$, then $\bar{P}_N^{8,\frac{1}{2}} = \bar{P}_N^8$.*

We define now the graph $\bar{K}_\infty$ on the half-plane $\mathbb{Z} \times \mathbb{N}$ and a probabilistic boundary condition on this graph. Later (in Proposition 1.3) this graph and its boundary condition will appear as the limit of the sequence of graphs $(\bar{K}_N : N \geq 1)$ with HPMBC$(q)$ when $N \to \infty$ (in a sense that we will precise). The set of $\bar{K}_\infty$'s vertices is

$$\bar{V}_\infty = \left\{\left(i - \frac{1}{2}, t - \frac{1}{2}\right), (i,t) : i \in \mathbb{Z}, t \in \mathbb{N}\right\}$$

and its set of edges is

$$\begin{aligned}\bar{E}_\infty = \Big\{ &\left(\left(i - \frac{1}{2}, t - \frac{1}{2}\right), (i,t)\right), \left(\left(i - \frac{1}{2}, t + \frac{1}{2}\right), (i,t)\right), \\ &\left(\left(i + \frac{1}{2}, t - \frac{1}{2}\right), (i,t)\right), \left(\left(i + \frac{1}{2}, t + \frac{1}{2}\right), (i,t)\right) : i \in \mathbb{Z}, t \in \mathbb{N}\Big\}\end{aligned}$$

(see Figure 5). As before, there are two types of vertices: internal vertices (called vertices in the following) of degree 4 and the other vertices $\left\{\left(i - \frac{1}{2}, -\frac{1}{2}\right) : i \in \mathbb{Z}\right\}$ of degree 2. As for $\bar{K}_N$, edges are labeled: edge whose end vertices is $\{(i,t), (i',t')\}$ is labeled by $(2\max(i,i'), 2\max(t,t'))$. Edges $\{(i,0) : i \in \mathbb{Z}\}$ are external and others are internal. On this graph, we call product measure boundary condition with parameter $q$ (PMBC$(q)$) the probabilistic boundary condition such that $(e(i,0) : i \in \mathbb{Z})$ are i.i.d. of common law $\mathcal{B}(q)$.

Let $\bar{\Omega}_\infty^8$ be the set of $\bar{K}_\infty$'s orientations such that any vertex of an orientation $O \in \bar{\Omega}_\infty^8$ has 0, 2 or 4 incoming edges. We can define a probability measure $\bar{P}_\infty^8$ on $\bar{\Omega}_\infty^8$ associated to the 8-vertex model on $\bar{K}_\infty$ with PMBC $\left(\frac{1}{2}\right)$. In the general case (for any $a$, $b$, $c$ and $d$), this measure must be seen as limit law of probability measures $\bar{P}_N^{8,\frac{1}{2}}$ when $N \to \infty$ in a certain sense. In the case $a + c = b + d$, there is a simpler way to prove its existence by considering the law $\mathcal{L}\left(\text{PM}\left(\frac{1}{2}\right); \frac{a}{a+c}, \frac{b}{b+d}\right)$, defined in the next paragraph.

**The law $\mathcal{L}(\mu; p, r)$ on $\bar{\Omega}_\infty$.** In [14], Kandel, Domany and Nienhuis defined a law on $\bar{\Omega}_\infty^6$ (the set of orientations of $\bar{K}_\infty$ with exactly 2 incoming edges around each vertex) as the law of a Markov chain whose state space is $\{0,1\}^{\mathbb{Z}}$. This law is in fact the limit law of the 6-vertex model on $\bar{K}_N$ with HPMBC$(q)$ when $N \to \infty$. Here, we generalize their idea to define laws $\mathcal{L}(\mu; p, r)$ on $\bar{\Omega}_\infty^8$, whose one specification is $\bar{P}_\infty^8$.

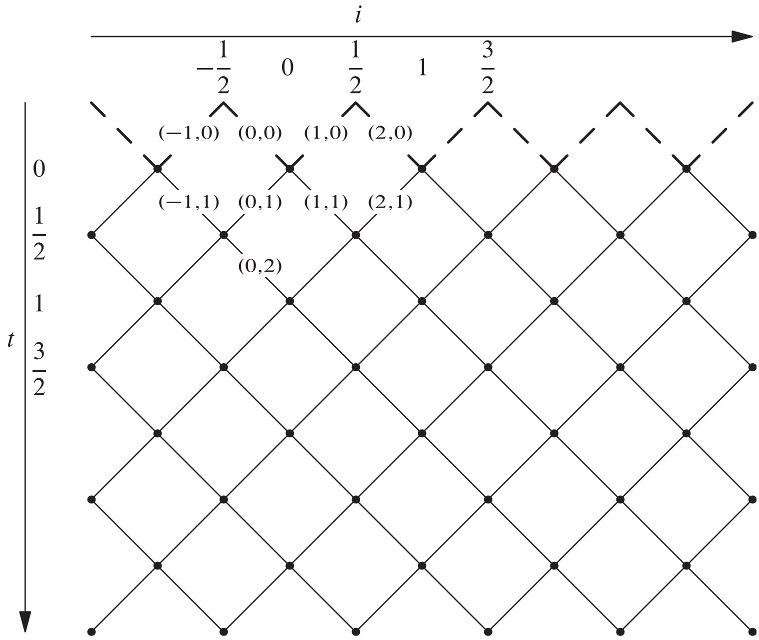



Figure 5. The graph $\bar{K}_\infty$. On axes: vertices coordinates. On edges: edges labels. Boundary condition PMBC($q$): vertical orientations of dashed edges $(e(i,0)\colon i \in \mathbb{N})$ are i.i.d.

**Definition 1.2** (law $\mathcal{L}(\mu; p, r)$). Let $\mu$ be any probability measure on $\{0,1\}^{\mathbb{Z}}$. Let $p, r \in [0,1]$, we define the law $\mathcal{L}(\mu; p, r)$ on $\bar{\Omega}^8_\infty$ as follows.

- Law of orientations $(e(i,0)\colon i \in \mathbb{Z})$ of edges on the first line is $\mu$.
- For any even $t$, starting with orientations $(e(i,t)\colon i \in \mathbb{Z})$ on line $t$, we obtain orientations $(e(i,t+1)\colon i \in \mathbb{Z})$ on line $t+1$ by the following way: for any $i \in \mathbb{Z}$, orientations of pair $(e(2i,t+1), e(2i+1,t+1))$ depends only on pair $(e(2i,t), e(2i+1,t))$ and local transition probabilities are, for any $i \in \mathbb{Z}$, any $k \in \{0,1\}$,

$$\begin{aligned}
&P((e(2i,t+1), e(2i+1,t+1)) = (k,k) \mid \\
&\quad (e(2i,t), e(2i+1,t)) = (k,k)) = r, \\
&P((e(2i,t+1), e(2i+1,t+1)) = (1-k,1-k) \mid \\
&\quad (e(2i,t), e(2i+1,t)) = (k,k)) = 1-r, \\
&P((e(2i,t+1), e(2i+1,t+1)) = (1-k,k) \mid \\
&\quad (e(2i,t), e(2i+1,t)) = (k,1-k)) = p, \\
&P((e(2i,t+1), e(2i+1,t+1)) = (k,1-k) \mid \\
&\quad (e(2i,t), e(2i+1,t)) = (k,1-k)) = 1-p
\end{aligned}$$

and local transitions from $(e(2i,t), e(2i+1,t))$ to $(e(2i,t+1), e(2i+1,t+1))$ are independent of one another, i.e. for any $i_1, i_2 \in \mathbb{Z}$ such that $i_1 < i_2$, for any $(k_{2i_1}, k_{2i_1+1}, \dots, k_{2i_2}, k_{2i_2+1}) \in \{0,1\}^{2(i_2-i_1+1)}$,

$$P((e(j,t+1) = k_j : 2i_1 \le j \le 2i_2+1) \mid (e(i,t) : i \in \mathbb{Z})) \\ = \prod_{i=i_1}^{i_2} P((e(2i,t+1), e(2i+1,t+1)) = (k_{2i}, k_{2i+1}) \mid \\ (e(2i,t), e(2i+1,t))).$$

We denote by $T_0$ this operator on $\mathcal{M}(\{0,1\}^{\mathbb{Z}})$, the set of $\{0,1\}^{\mathbb{Z}}$'s probability measures:

$$T_0(\mu_t) = \mu_{t+1}. \tag{6}$$

where, for any $t$, $\mu_t$ is the law of $(e(i,t) : i \in \mathbb{Z})$.

- For any odd $t$, transition is the same as in the case where $t$ is even with the difference that we consider pairs of edges of abscissas $(2i-1, 2i)$ instead of pairs of edges of abscissas $(2i, 2i+1)$. We denote by $T_1$ this operator.

Local transitions of these two operators are illustrated on Figure 6.

Finally, the law $\mathcal{L}(\mu; p, r)$ on $\overline{\Omega}^8_\infty$ is the law of $(e(i,t) : i \in \mathbb{Z}, t \in \mathbb{N})$.

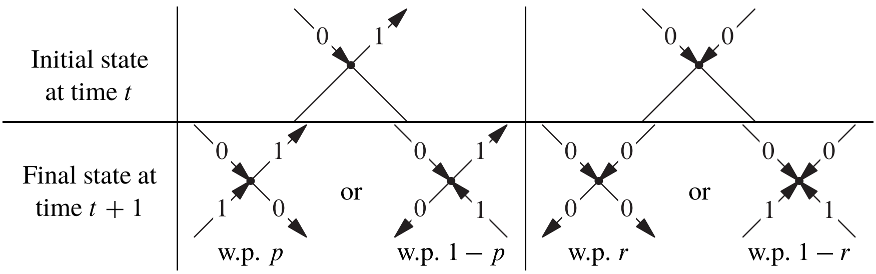



Figure 6. Operators $T_0$ and $T_1$. In all case, the left-up arrow is labeled by $(i,t)$ with $i+t$ even. Orientation on edge correspond to a choice of $k = 0$.

In the following, the considered measure $\mu$ will be the product measure with parameter $\frac{1}{2}$ (denoted PM $\left(\frac{1}{2}\right)$) and, sometimes, with parameter $q \in [0,1]$ (denoted PM$(q)$). To fix notations, $(e_i : i \in \mathbb{Z})$ is distributed according to PM$(q)$, if $(e_i : i \in \mathbb{Z})$ are i.i.d. and $e_0 \sim \mathcal{B}(q)$.

If $r = 1$ and $\mu = \mathrm{PM}(q)$, we recover the result of [14] on the stochastic 6-vertex model. For any value of $r$, we get the following generalization when $q = \frac{1}{2}$:

**Proposition 1.3.** *Let $O \sim \mathcal{L}\left(\mathrm{PM}\left(\frac{1}{2}\right); \frac{a}{a+c}, \frac{b}{b+d}\right)$ with $a+c = b+d$. For any $N$, the law of oriented edges of $O$ in the subset $\overline{K}_N$ $(O|_{\overline{K}_N})$ is distributed as $\overline{P}^8_N$.*

For this reason and the invariance of the 8-vertex model by any horizontal translation, $\mathcal{L}\big(\mathrm{PM}\left(\frac{1}{2}\right); \frac{a}{a+c}, \frac{b}{b+d}\big)$ could be seen as the limit of $\bar{P}^8_N$, thus it is denoted by $\bar{P}^8_\infty$ in the following. Moreover, we obtain that

**Proposition 1.4.** *Let $S_N = (V_N, E_N)$ be any subset of $\bar{K}_\infty$ isomorphic to $K_N$ rotated by an angle $-\frac{\pi}{4}$ (see Figure 7). Let $O \sim \bar{P}^8_\infty$ with $a + c = b + d$, then the law of $O_{|S_N}$ ($O$ restricted to edges in $S_N$ and rotated by an angle $\frac{\pi}{4}$) is $P^8_N$.*

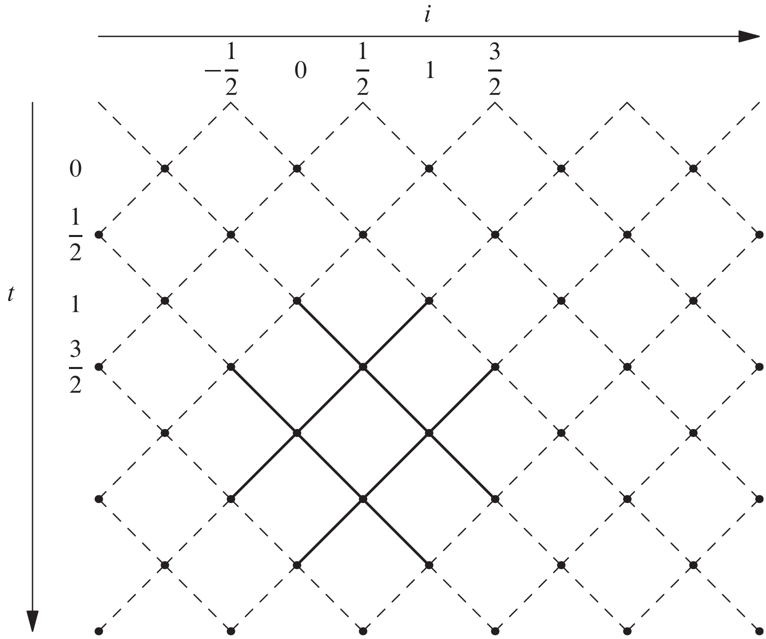



Figure 7. In full line, a subset $S_2$ of $\bar{K}_\infty$ isomorphic to $K_2$.

The reason why we consider only $\mu = \mathrm{PM}\left(\frac{1}{2}\right)$ is due to the fact that $\mathrm{PM}\left(\frac{1}{2}\right)$ has remarkable properties according to the Markov chain on $E^{\mathbb{Z}}$ with operators $T_0$ and $T_1$ used in Definition 1.2. Indeed, it is an invariant law of this Markov chain.

**Proposition 1.5.** *Let*

$$O \sim \mathcal{L}\Big(\mathrm{PM}\left(\frac{1}{2}\right); p, r\Big).$$

*Then, for any $t$,*

$$(e(i,t): i \in \mathbb{Z}) \sim \mathrm{PM}\left(\frac{1}{2}\right).$$

And, moreover, if $(p, r) \in (0, 1)^2$ and $p + r \neq 1$, it is the unique invariant law and the Markov chain is ergodic.

**Proposition 1.6.** *Let $\mu$ be any measure on $\{0,1\}^{\mathbb{Z}}$ and let $(p,r) \in (0,1)^2$. Let $O \sim \mathcal{L}(\mu; p, r)$. Then, by denoting $\mu_t$ the law of $(e(i,t) : i \in \mathbb{Z})$,*

$$\mu_t \xrightarrow{(1)} \mathrm{PM}\left(\frac{1}{2}\right) \quad \text{as } t \to \infty.$$

Proposition 1.5 generalizes the following one (Proposition 1.7) of [14] about the 6-vertex model when $q = \frac{1}{2}$, that is the only interesting case in the 8-vertex model.

**Proposition 1.7** ([14]). *Suppose that $a + c = b$. Let*

$$O \sim \overline{P}_{\infty}^{6,q} = \mathcal{L}\left(\mathrm{PM}(q); \frac{a}{a+c}, 1\right)$$

(*limit law of $\overline{P}_N^{6,q}$ when $N \to \infty$*). *Then, for any $t$,*

$$(e(i,t) : i \in \mathbb{Z}) \sim \mathrm{PM}(q).$$

Propositions 1.5 and 1.7 are easy to prove by coming back to Definition 1.2. Proposition 1.6 is more complicated and is proved in Section 5 using new results about probabilistic cellular automata.

**Edge correlation function of the 8-vertex model when $a + c = b + d$ and its special case the 6-vertex model when $a + c = b$.** Let $\mu$ be any probability distribution on $\{0,1\}^{\mathbb{Z}}$. Let $O \sim \mathcal{L}\left(\mu; \frac{a}{a+c}, \frac{b}{b+d}\right)$. The edge correlation function is the function $C((i,t);(i',t'))$ defined by, for any $(i,t), (i',t') \in \overline{E}_{\infty}$,

$$\begin{aligned} C((i,t);(i',t')) &= \frac{\mathrm{Cov}(e(i,t), e(i',t'))}{\sqrt{\mathrm{Var}(e(i,t))}\sqrt{\mathrm{Var}(e(i',t'))}} \\ &= \frac{E[e(i,t)e(i',t')] - E[e(i,t)]E[e(i',t')]}{\sqrt{\mathrm{Var}(e(i,t))}\sqrt{\mathrm{Var}(e(i',t'))}}. \end{aligned} \tag{7}$$

We can remark that, as $e(i,t) \in \{0,1\}$ for any $(i,t)$, knowing $C((i,t),(i',t'))$ is equivalent to knowing the joint law of $e(i,t)$ and $e(i',t')$.

Our main objective is to determine $\overline{C}_{\infty}^8$ that is the edge correlation function $C$ of the 8-vertex model when $a + c = b + d$ and with FBC, i.e. when $O \sim \overline{P}_{\infty}^8$.

First of all, $\overline{C}_{\infty}^8$ has some invariance properties.

**Proposition 1.8.** *For any $(i,t), (i',t') \in \overline{E}_{\infty}$,*

$$\overline{C}_{\infty}^8((i,t);(i',t')) = \overline{C}_{\infty}^8((i',t');(i,t)), \tag{8}$$

$$\overline{C}_{\infty}^8((0,t);(i',t')) = \overline{C}_{\infty}^8((1,t);(1-i',t')), \tag{9}$$

$$\overline{C}_{\infty}^8((i,t);(i',t')) = \overline{C}_{\infty}^8((i+2,t);(i'+2,t')), \tag{10}$$

$$\overline{C}_{\infty}^8((i,t);(i',t')) = \overline{C}_{\infty}^8((i+1,t+1);(i'+1,t'+1)). \tag{11}$$

This is a consequence of Proposition 1.5. Proposition 1.8 permits to determine $\bar{C}^8_\infty$ for any values $(i, t)$ and $(i', t')$ if the set $\{\bar{C}^8_\infty((0, 0); (i, t)) \colon (i, t) \in \bar{E}_\infty\}$ is known. Hence, in the following, we determine and denote $\bar{C}^8_\infty(i, t) = \bar{C}^8_\infty((0, 0); (i, t))$.

For the 6-vertex model under similar conditions, these invariant properties were already proved in [14]. And, moreover, the edge correlation function of the 6-vertex model under some conditions was evaluated in the same article:

**Theorem 1.9** ([14]). *For any $q \in (0, 1)$, let $O \in \bar{\Omega}^6_\infty$ distributed according to $\bar{P}^{6,q}_\infty$ (the limit law of $\bar{P}^{6,q}_N$, laws of the* 6*-vertex model on $\bar{K}_N$ with* PMBC$(q)$*). If $a + c = b$, the edge correlation function $C(i, t)$ is*

$$C(i, 2t) = \begin{cases} \frac{1}{2^{2t}} \dbinom{2t-1}{\frac{2t - i - \Delta(i)}{2}} & \text{if } 2t \geq i + \Delta(i), \\ 0 & \text{otherwise}, \end{cases} \tag{12}$$

*with*

$$\Delta(i) = \begin{cases} 1 & \text{if } i \text{ is odd}, \\ 2 & \text{if } i \text{ is even}. \end{cases}$$

*Moreover, for any $i$, when $t \to \infty$,*

$$C(i, 2t) = \Theta(t^{-\frac{1}{2}}). \tag{13}$$

In their article, they consider two lines by two lines, that's why the edge correlation function is the one of $C(i, 2t)$ instead of $C(i, t)$.

Just before to state the main theorem of the paper, we introduce some notations that are used all along the paper:

$$p = \frac{a}{a + c}, \quad r = \frac{b}{b + d} \tag{14}$$

and

$$\Delta = 1 - (p + r) = \frac{c - b}{a + c}, \tag{15a}$$

$$D = r - p = \frac{b - a}{a + c}, \tag{15b}$$

$$P = (2p - 1)(2r - 1) = \frac{(a - c)(b - d)}{(a + c)^2}. \tag{15c}$$

**Theorem 1.10.** *The edge correlation function* $\overline{C}^8_\infty$ *of the* 8*-vertex model on* $\overline{K}_\infty$ *with FBC and* $a + c = b + d$ *(i.e. when* $O \sim \overline{P}^8_\infty$*) is*

- *if* $i + t$ *is odd,*

$$\overline{C}^8_\infty(i,t) = (-1)^{t+1} D \sum_{k=0}^{\frac{t-1-|i|}{2}} (-1)^k \binom{t-1-k}{k, \frac{t-1+i}{2} - k, \frac{t-1-i}{2} - k} \Delta^{t-1-2k} P^k; \tag{16}$$

- *if* $t$ *is even and* $i = 0$,

$$\overline{C}^8_\infty(0,t) = \sum_{k=0}^{\frac{t}{2}} (-1)^k \binom{t-1-k}{\frac{t}{2} - k} \binom{\frac{t}{2}}{k} \Delta^{t-2k} P^k; \tag{17}$$

- *if* $i + t$ *is even and* $i < 0$,

$$\overline{C}^8_\infty(i,t) = \sum_{k=0}^{\frac{t+i}{2}-1} (-1)^{t+k} \binom{t-1-k}{\frac{t-i}{2} - k} \binom{\frac{t+i}{2}}{k} \Delta^{t-2k} P^k + (-1)^{\frac{t-i}{2}} \binom{\frac{t-i}{2} - 1}{\frac{t+i}{2} - 1} \Delta^{-i} P^{\frac{t+i}{2}}; \tag{18}$$

- *if* $i + t$ *is even and* $i > 0$,

$$\overline{C}^8_\infty(i,t) = \sum_{k=0}^{\frac{t-i}{2}-1} (-1)^{t+k} \binom{t-1-k}{\frac{t-i}{2} - k} \binom{\frac{t+i}{2}}{k} \Delta^{t-2k} P^k + (-1)^{\frac{t+i}{2}} \binom{\frac{t+i}{2}}{\frac{t-i}{2}} \Delta^{i} P^{\frac{t-i}{2}}. \tag{19}$$

**Remark 1.11.** There are some particular cases for which $\overline{C}^8_\infty$ is simpler.

- If ($i \geq 0$ and $t \leq i - 1$) or ($i \leq -1$ and $t \leq -i$) (i.e. the edge $(i,t)$ is not in a kind of cone starting from $(0,0)$, see Figure 8), then $\overline{C}^8_\infty(i,t) = 0$.
- If $a = d$ and $b = c$ (i.e. $p = 1 - r$, and so $\Delta = 0$), then

$$\overline{C}^8_\infty(i,t) = \begin{cases} 0 & \text{if } i \neq 0, \\ (1-2p)^t & \text{if } i = 0. \end{cases}$$

- If $a = b$ and $c = d$ (i.e. $p = r$, and so $D = 0$), then

$$\overline{C}^8_\infty(i,t) = \begin{cases} 0 & \text{if } i \neq t, \\ (2p-1)^t & \text{if } i = t. \end{cases}$$

- If $a = c$ (i.e. $p = \frac{1}{2}$, and so $P = 0$), then

$$\overline{C}^8_\infty(i,t) = \begin{cases} \left(-\frac{1}{2}\right)^t \binom{t-1}{\frac{t-i}{2}} (1-2r)^t & \text{if } i \text{ is even}, \\ \left(-\frac{1}{2}\right)^t \binom{t-1}{\frac{t-1-i}{2}} (1-2r)^t & \text{if } i \text{ is odd}. \end{cases}$$

- If $b = d$ (i.e. $r = \frac{1}{2}$, and so $P = 0$), then

$$\overline{C}^8_\infty(i,t) = \begin{cases} \left(-\frac{1}{2}\right)^t \binom{t-1}{\frac{t-i}{2}} (1-2p)^t & \text{if } i \text{ is even}, \\ -\left(-\frac{1}{2}\right)^t \binom{t-1}{\frac{t-1-i}{2}} (1-2p)^t & \text{if } i \text{ is odd}. \end{cases}$$

Moreover, we obtain the asymptotics of $\overline{C}^8_\infty(i,t)$ for any $i$ when $t \to \infty$.

**Theorem 1.12.** *For any $i$,*

- *if $a = d$ and $b = c$ (i.e. $p + r = 1$), for any $t$,*

$$\overline{C}^8_\infty(i,t) = \begin{cases} 0 & \textit{if } i \neq 0, \\ (1-2p)^t & \textit{if } i = 0. \end{cases}$$

- *otherwise, when $t \to \infty$,*

$$\overline{C}^8_\infty(i,t) = O\Big(\frac{\lambda(p,r)^t}{\sqrt{t}}\Big) \tag{20}$$

*with*

$$\lambda(p,r) = \max(|1-2p|, |1-2r|) = \frac{\max(|a-c|, |b-d|)}{|a+c|}. \tag{21}$$

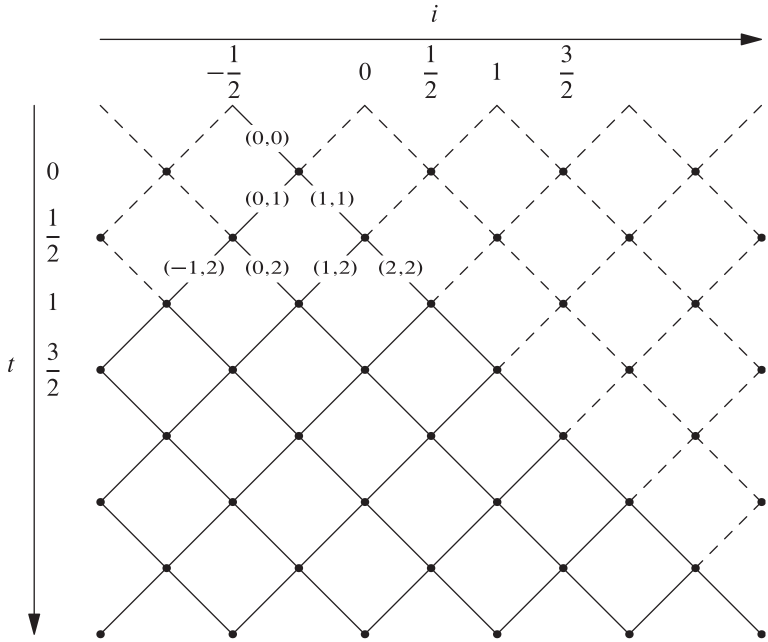



Figure 8. The "influence cone" of edge $(0,0)$. In full line: the set of edges for which $\overline{C}^8_\infty(i,t) \neq 0$ in general. In dashed line: the set of edges for which $\overline{C}^8_\infty(i,t) = 0$ when $a+c = b+d$.

For $d = 0$, we find (13), the asymptotic result of [14], that is a square-root decreasing of the edge correlation function in the 6-vertex model. In the generic case of the 8-vertex model when $a, b, c, d$ are all different of 0, decreasing becomes exponential.

In Theorem 1.10 and 1.12, the regime is supposed to be stationary. When the regime is not stationary, i.e. when we start with any initial law $\mu$ at time 0, we can obtain bounds on $C((0,0);(i,t))$.

**Proposition 1.13.** *Let $\mu$ be any probability measure on $\{0,1\}^{\mathbb{Z}}$ and let $O \sim \mathcal{L}(\mu; p, r)$. If $a+c = b+d$, then the edge correlation function $C$ satisfies, for any $i$, any $t$,*

$$\left| \sqrt{\frac{\mathrm{Var}(e(i,t))}{\mathrm{Var}(e(0,0))}}\, C((0,0);(i,t)) - \overline{C}^8_\infty(i,t) \right| \leq 2\left(\lambda(p,r)^{t-1-\left\lfloor \frac{t}{2} \right\rfloor} + \lambda(p,r)^{\left\lfloor \frac{t}{2} \right\rfloor} - \lambda(p,r)^{t-1}\right) \tag{22}$$

*with* $\lambda(p,r) = \max(|1-2p|, |1-2r|) = \frac{\max(|a-c|,|b-d|)}{|a+c|}$.

**Content.** Section 2 is about links between the 6- and 8-vertex models when $a + c = b + d$ and laws $\mathcal{L}(\mu; p, r)$. In particular, we express, in those cases, partition functions to prove Propositions 1.1, 1.3 and 1.4.

In Section 3, Theorem 1.10, the main theorem of this paper, is proved. First, in Section 3.1, we establish and prove Proposition 3.1 that relate $\overline{C}^8_\infty(i,t)$ to a rational fraction. The proof of this proposition is based on a fundamental lemma (Lemma 3.2) that permits to understand precisely behaviors of correlations in the 8-vertex model when $a + c = b + d$. Then, in Section 3.2, we extract coefficients of this rational fraction to end the proof of Theorem 1.10. And, in Section 3.3, we evaluate the influence of the boundary conditions on the correlation function by proving Proposition 1.13.

Section 4 contains the asymptotics of $\overline{C}^8_\infty(i,t)$ and Theorem 1.12 is proved. In Section 4.1, the case $r = 0$ is done using properties on random walks. Then, in Section 4.2, the general case is proved using both results of Section 4.1 and Theorem 1.10.

Section 5 is dedicated to some discussions about links between vertex models, colorings of the plane and probabilistic cellular automata. In particular, the aim is to prove that, when $a + c = b + d$, far away from the boundary the behavior of the 8-vertex model is closed to the one of the 8-vertex model with FBC (Proposition 5.2). For this, in Section 5.1, we introduce the general theory of probabilistic cellular automata and define triangular probabilistic cellular automata (TPCA, a new type of PCA at the best knowledge of the author). In Section 5.2, we present a family of TPCA that emulates the 8-vertex model with $a+c = b+d$. We prove that PCA of this family are egodic that gives a proof of Proposition 1.6. In Section 5.3, we goes on the discussion by presenting a family of TPCA that emulates specifically the 6-vertex model with $a + c = b$. In Section 5.4, theorems and properties stated in Section 5.1, 5.2 and 5.3 are proved.

Finally, in Section 6, we conclude this article giving some additional comments on vertex models.

## 2. From $\overline{K}_N$ to $\overline{K}_\infty$: laws $\mathcal{L}(\mu; p, r)$

The major aim of this section is to prove Propositions 1.1, 1.3 and 1.4. These propositions are in fact consequences of properties of partition functions of the 8-vertex model with $a + c = b + d$ and FBC on $\overline{K}_N$ and on $K_N$. To establish these properties, we need first to observe that the partition function $Z^8_G$ of the 8-vertex model with $a + c = b + d$ and FBC of a graph $G$ could be express according to the one $Z^8_{G\setminus v}$ of the graph $G$ without one of its node $v$.

**Lemma 2.1.** *For any graph $G$, finite subgraph of $\bar{K}_\infty$, and any vertex $v = (i, j)$.*

(1) *If the edges $(2i, 2j)$ and $(2i + 1, 2j)$ are internal edges and the edges $(2i, 2j + 1)$ and $(2i + 1, 2j + 1)$ are external edges of $G$, then the partition function $Z_G^8$ of the 8-vertex model on $G$ with FBC and $a + c = b + d$ satisfies*

$$Z_G^8 = (a + c) Z_{G \setminus v}^8. \tag{23}$$

(2) *If the edge $(2i, 2j)$ (resp. $(2i + 1, 2j)$) is internal and the edges $(2i + 1, 2j)$ (resp. $(2i, 2j)$), $(2i, 2j + 1)$ and $(2i + 1, 2j + 1)$ are external edges of $G$, then the partition function $Z_G^8$ satisfies*

$$Z_G^8 = 2(a + c) Z_{G \setminus v}^8. \tag{24}$$

*Proof.* (1) Let us decompose $\Omega_G^8$ (resp. $\Omega_{G \setminus v}^8$) the set of 8-vertex model configurations of $G$ (resp. $G \setminus v$) into four subsets $\Omega_G^{8,(e_1,e_2)}$ (resp. $\Omega_{G \setminus v}^{8,(e_1,e_2)}$) with $e_1, e_2 \in \{0, 1\}$ where $\Omega_G^{8,(e_1,e_2)}$ (resp. $\Omega_{G \setminus v}^{8,(e_1,e_2)}$) is the set of 8-vertex model configurations of $G$ (resp. $G \setminus v$) such that $e(2i, 2j) = e_1$ and $e(2i + 1, 2j) = e_2$. Then

$$\begin{aligned}
Z_G^8 &= \sum_{O \in \Omega_G^8} \prod_{v' \in V_G} w_{\text{type}_O(v')} \\
&= \sum_{O \in \Omega_G^{8,(0,0)}} \prod_{v' \in V_G} w_{\text{type}_O(v')} + \sum_{O \in \Omega_G^{8,(0,1)}} \prod_{v' \in V_G} w_{\text{type}_O(v')} \\
&\quad + \sum_{O \in \Omega_G^{8,(1,0)}} \prod_{v' \in V_G} w_{\text{type}_O(v')} + \sum_{O \in \Omega_G^{8,(1,1)}} \prod_{v' \in V_G} w_{\text{type}_O(v')} \\
&= \sum_{O \in \Omega_{G \setminus v}^{8,(0,0)}} \prod_{v' \in V_{G \setminus v}} w_{\text{type}_O(v')} (b + d) \\
&\quad + \sum_{O \in \Omega_{G \setminus v}^{8,(0,1)}} \prod_{v' \in V_{G \setminus v}} w_{\text{type}_O(v')} \, (a + c) \\
&\quad + \sum_{O \in \Omega_{G \setminus v}^{8,(1,0)}} \prod_{v' \in V_{G \setminus v}} w_{\text{type}_O(v')} \, (a + c) \\
&\quad + \sum_{O \in \Omega_{G \setminus v}^{8,(1,1)}} \prod_{v' \in V_{G \setminus v}} w_{\text{type}_O(v')} \, (b + d)
\end{aligned}$$

(by decomposition according to the possible orientations of edges $(2i, 2j+1)$ and $(2i+1, 2j+1)$)

$$= (a+c) \sum_{O \in \Omega^8_{G \setminus v}} \prod_{v \in V_{G \setminus v}} w_{\mathrm{type}_O(v')}$$
$$= (a+c) Z^8_{G \setminus v}.$$

(2) The proof is similar to the previous one. We treat the case where $(2i, 2j)$ is the internal edge of $v$. Let us decompose the set $\Omega^8_{G\setminus v}$ into two subsets $\Omega^{8,0}_{G\setminus v}$ and $\Omega^{8,1}_{G\setminus v}$ such that, for any $k \in \{0,1\}$, $\Omega^{8,k}_{G\setminus v}$ is the set of 8-vertex model configurations of $G \setminus v$ such that $e(2i, 2j) = k$. Then, decomposing according to possible orientations of $v$,

$$\begin{aligned}
Z^8_G &= a \sum_{O \in \Omega^{8,0}_{G\setminus v}} W(O) + b \sum_{O \in \Omega^{8,0}_{G\setminus v}} W(O) + c \sum_{O \in \Omega^{8,0}_{G\setminus v}} W(O) + d \sum_{O \in \Omega^{8,0}_{G\setminus v}} W(O) \\
&\quad + a \sum_{O \in \Omega^{8,1}_{G\setminus v}} W(O) + b \sum_{O \in \Omega^{8,1}_{G\setminus v}} \prod_{v' \in V_G} W(O) \\
&\quad + c \sum_{O \in \Omega^{8,1}_{G\setminus v}} W(O) + d \sum_{O \in \Omega^{8,1}_{G\setminus v}} \prod_{v' \in V_G} W(O) \\
&= (a+b+c+d) \Big( \sum_{O \in \Omega^{8,0}_G} W(O) + \sum_{O \in \Omega^{8,1}_G} W(O) \Big) \\
&= 2(a+c) Z^8_{G\setminus v}. \qquad \square
\end{aligned}$$

Now, we can compute the partition function of the 8-vertex model with $a + c = b + d$.

## 2.1. Partition function on $\bar{K}_N$

**Lemma 2.2.** *If $a + c = b + d$, then for any $N$, the partition function $\bar{Z}^8_N$ of the 8-vertex model on $\bar{K}_N$ with FBC is*

$$\bar{Z}^8_N = 2^{2N} (a+c)^{\frac{N(N+1)}{2}}. \tag{25}$$

*Proof.* First of all, for $N = 1$, we have that

$$\bar{Z}^8_1 = a + a + b + b + c + c + d + d = 2^2 (a+c). \tag{26}$$

Now, let define $L_N$ the subgraph of $\bar{K}_N$ that contains only the internal nodes $\{(i,0): 0 \leq i \leq N-1\} \subset \bar{V}_N$ and their adjacent edges, labeled $\{(i,t): 0 \leq i \leq 2N-1, t \in \{0,1\}\}$. Because the internal nodes of $L_N$ do not share edges, their local configurations are independent and, thus, the partition function of the 8-vertex model on $L_N$ with FBC is

$$Z^8_{L_N} = (\bar{Z}^8_1)^N = 2^{2N}(a+c)^N. \tag{27}$$

Now using point 1 of Lemma 2.1 and induction, we get that

$$Z_{\bar{K}_N} = (a+c)^{\frac{N(N-1)}{2}} Z_{L_N}. \qquad \square$$

Now, we can prove Proposition 1.1.

*Proof of Proposition* 1.1. Let $O \in \bar{\Omega}^8_N$. Set

$$\mathfrak{S} = a^{n_1(O)+n_2(O)} b^{n_3(O)+n_4(O)} c^{n_5(O)+n_6(O)} d^{n_7(O)+n_8(O)}.$$

Then

$$\bar{P}^8_N(O) = \frac{\mathfrak{S}}{2^{2N}(a+c)^{\frac{N(N+1)}{2}}} \tag{28}$$

and, by (4),

$$\bar{P}^{8,\frac{1}{2}}_N(O) = \sum_{B \in \{0,1\}^{2N}} \frac{1}{2^{2N}} \frac{\mathfrak{S}}{\bar{Z}^{8,B}_N} \mathbf{1}_{(e(i,0):0 \leq i \leq 2N-1)=B}. \tag{29}$$

But, for any $B \in \{0,1\}^{2N}$, $\bar{Z}^{8,B}_N = (a+c)^{\frac{N(N+1)}{2}}$ by point 1 of Lemma 2.1, induction and the following remark: for $N=1$,

$$\bar{Z}^{8,B}_1 = \begin{cases} (b+d), & \text{if } B \in \{(0,0),(1,1)\}, \\ (a+c), & \text{if } B \in \{(0,1),(1,0)\}, \end{cases}$$
$$= a+c.$$

Thus,

$$\bar{P}^{8,\frac{1}{2}}_N(O) = \frac{1}{2^{2N}} \frac{\mathfrak{S}}{(a+c)^{\frac{N(N+1)}{2}}} = \bar{P}^8_N(O). \qquad \square$$

Proposition 1.3 is a direct consequence of Lemma 2.2.

*Proof of Proposition* 1.3. Let $O \sim \mathcal{L}\big(\mathrm{PM}\left(\frac{1}{2}\right); \frac{a}{a+c}, \frac{b}{b+d}\big)$ with $a + c = b + d$. Then, orientation $O_{|\bar{K}_N}$ is distributed according to the following probability, by Definition 1.2,

$$
\begin{aligned}
P(O_{|\bar{K}_N}) &= \frac{1}{2^{2N}}\left(\frac{a}{a+c}\right)^{n_1+n_2}\left(\frac{b}{b+d}\right)^{n_3+n_4}\left(\frac{c}{a+c}\right)^{n_5+n_6}\left(\frac{d}{b+d}\right)^{n_7+n_8} \\
&= \frac{a^{n_1+n_2} b^{n_3+n_4} c^{n_5+n_6} d^{n_7+n_8}}{2^{2N}(a+c)^{\frac{N(N+1)}{2}}} \\
&= \bar{P}_N^8(O_{|\bar{K}_N}).
\end{aligned}
$$
□

## 2.2. Partition function on $K_N$

**Lemma 2.3.** *If $a + c = b + d$, then for any $N$, the partition function $Z_N^8$ of the 8-vertex model on $K_N$ with FBC is*

$$
Z_N^8 = 2^{2N}(a+c)^{N^2} \tag{30}
$$

*Proof.* The proof is similar to the one of Lemma 2.2: induction and both points of Lemma 2.1 permit to complete it. □

Now, we present a useful property of $\bar{P}_\infty^8 = \mathcal{L}\big(\mathrm{PM}\left(\frac{1}{2}\right); p, r\big)$.

**Lemma 2.4.** *Let $O \sim \bar{P}_\infty^8$. Let $(t_i : i \in \mathbb{Z}) \in \mathbb{N}^{\mathbb{Z}}$ such that $t_{i+1} - t_i \in \{0, (-1)^{i+1+t_i}\}$. Then $(e(i, t_i) : i \in \mathbb{N}) \sim \mathrm{PM}\left(\frac{1}{2}\right)$.*

Its proof is a corollary of Proposition 3.10, which is presented and proved in Section 3.3.

Now, with these two lemmas, we can prove Proposition 1.4.

*Proof of Proposition* 1.4. Let $O \sim \bar{P}_\infty^8$ and let $S_N = (V_N, E_N)$. Then, there exists $(i_0, t_0) \in \mathbb{Z} \times \mathbb{N}$, $i_0 + t_0$ even, ($(i_0, t_0)$ is the label of the edge isomorphic to the edge $((-1, 0), (0, 0))$ in $K_N$: in Figure 7, $(i_0, t_0) = (1, 3)$) such that

$$
E_N = \{(i', t') : i_0 + t_0 \le i' + t' \le i_0 + t_0 + 2N, i_0 - t_0 - 2N + 1 \le i' - t' \le i_0 + 1 - t_0\}.
$$

In particular, $(t_j = t_0 + j : -N + 1 \leq j \leq 0) \cup (t_j = t_0 - j + 1 : 1 \leq j \leq N)$ satisfies condition of Lemma 2.4 and, so, $(e(i_0 + j, t_j) : -N + 1 \leq j \leq N)$ are i.i.d. and of law $\mathcal{B}\left(\frac{1}{2}\right)$. Hence,

$$P(O_{|S_N}) = \frac{1}{2^{2N}}\left(\frac{a}{a+c}\right)^{n_1+n_2}\left(\frac{b}{b+d}\right)^{n_3+n_4}\left(\frac{c}{a+c}\right)^{n_5+n_6}\left(\frac{d}{b+d}\right)^{n_7+n_8}$$
$$= \frac{a^{n_1+n_2}b^{n_3+n_4}c^{n_5+n_6}d^{n_7+n_8}}{Z_N^8}. \qquad \square$$

## 3. Exact computation of the edge correlation function

To prove Theorem 1.10, we need to prove first the following proposition.

**Proposition 3.1.** *The edge correlation function $\bar{C}^8_\infty(i,t)$ (of the 8-vertex model with FBC and $a + c = b + d$) is the coefficient of $l^t x^{i+t}$ in the formal series of the following rational fraction*:

$$\frac{1 + l(1 - (p + r) + x(r - p))}{x^2 l^2 (2p - 1)(2r - 1) + l(1 - (p + r))(1 + x^2) + 1} \tag{31}$$

*with* $p = \frac{a}{a+c}$ *and* $r = \frac{b}{b+d}$.

After proving this proposition, we will extract coefficients of (31) to prove Theorem 1.10.

### 3.1. Proof of Proposition 3.1.

Let $O \sim \bar{P}^8_\infty$. We recall that, in Definition 1.2, there are two operators $T_0$ and $T_1$ that give orientations line by line according to parity of time.

In the following, we suppose that $r \leq 1 - p$. The case $1 - p \leq r$ can be treated in a similar way with some differences that are commented in Remark 3.5. First, let us compute $\bar{C}^8_\infty(i,t)$:

$$\begin{aligned}
\bar{C}^8_\infty(i,t) &= \frac{E[e(0,0)e(i,t)] - E[e(0,0)]E[e(i,t)]}{\sqrt{\mathrm{Var}(e(i,t))}\sqrt{\mathrm{Var}(e(0,0))}} \\
&= 4\left(P(e(i,t) = 1 \text{ and } e(0,0) = 1) - \frac{1}{4}\right) \\
&= 2P(e(i,t) = 1 \mid e(0,0) = 1) - 1 \\
&= P(e(i,t) = 1 \mid e(0,0) = 1) - P(e(i,t) = 0 \mid e(0,0) = 1).
\end{aligned} \tag{32}$$

Hence, we need to compute

$$P(e(i,t) = 1 \mid e(0,0) = 1) \text{ or } P(e(i,t) = 0 \mid e(0,0) = 1).$$

This is done using Definition 1.2 and the following crucial lemma. In few words, this lemma tells us that after a transition $T_0$ or $T_1$, orientation $e(i,t)$ of edge $(i,t)$ influences orientation $e(j,t+1)$ of a unique random edge on the line $((j,t+1)\colon j \in \mathbb{Z})$.

**Lemma 3.2.** *Let $i \in \mathbb{Z}$. Let $(s(j)\colon j \in \mathbb{Z})$ be a sequence of random variables whose values are in $\{0,1\}$ such that $s(i)$ is independent of $(s(j)\colon j \neq i)$. For any $u \in \{0,1\}$, we denote $s_u = T_u(s)$, then*

- *with probability $r$, $s_u(i) = s(i)$ and $(s_u(j)\colon j \neq i)$ are independent of $s(i)$;*
- *with probability $1-p-r$, $s_u(i+(-1)^{i+u}) = 1 - s(i)$ and $(s_u(j)\colon j \neq i+(-1)^{i+u})$ are independent of $s(i)$;*
- *with probability $p$, $s_u(i) = 1 - s(i)$ and $(s_u(j)\colon j \neq i)$ are independent of $s(i)$.*

*Proof.* We establish the proof for $u = 0$ and $i$ even, the other cases are proved in similar ways.

By definition of $T_0$,

- if $s(i+1) = 1 - s(i)$, then
  - with probability $1-p$, $s_0(i) = s(i) = 1 - s(i+1)$ and $s_0(i+1) = s(i+1) = 1 - s(i)$, and
  - with probability $p$, $s_0(i) = 1 - s(i)$ and $s_0(i+1) = 1 - s(i+1)$; but
- if $s(i+1) = s(i)$, then
  - with probability $r$, $s_0(i) = s(i)$ and $s_0(i+1) = s(i+1)$, and
  - with probability $1-r$, $s_0(i) = 1 - s(i) = 1 - s(i+1)$ and $s_0(i+1) = 1 - s(i+1) = 1 - s(i)$.

Now, observe, from the fact that $r \leq 1-p$ and, by a coupling argument, that

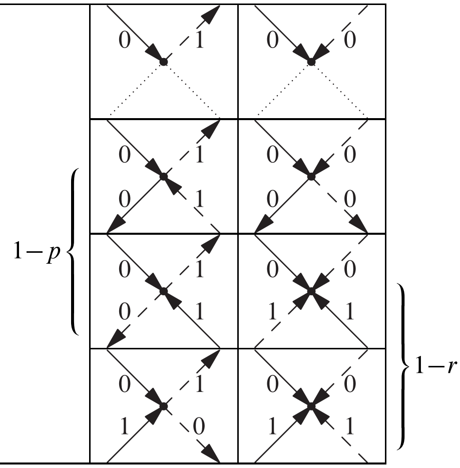
- with probability $r$, $s_0(i) = s(i)$ and $s_0(i)$ depends only on $s(i)$ and not on $s(i+1)$;
- with probability $1-p-r$, $s_0(i+1) = 1 - s(i)$ and $s_0(i+1)$ depends only on $s(i)$ and not on $s(i+1)$;
- with probability $p$, $s_0(i) = 1 - s(i)$ and $s_0(i)$ depends only on $s(i)$ and not on $s(i+1)$.

Table 1 shows this coupling argument. This coupling will be reused in Section 3.3.

Finally, by definition of $T_0$, $(s_0(j)\colon j \neq i, i+1)$ depend only on $(s(j)\colon j \neq i, i+1)$ and not on $s(i)$. That is ending the proof. $\square$

Table 1. Coupling in the proof of Lemma 3.2. Figures on the fifth and sixth columns represents the coupling when $s(i) = 1 - s(i+1) = 0$ in the fifth and when $s(i) = s(i+1) = 0$ in the sixth. Plain and dashed line represent particles discussed in Section 3.3.

| probability | $s_0(i)$ | $s_0(i+1)$ |
|---|---|---|
| $r$ | $s(i)$ | $s(i+1)$ |
| $1-p-r$ | $1-s(i+1)$ | $1-s(i)$ |
| $p$ | $1-s(i)$ | $1-s(i+1)$ |




Lemma 3.2 is crucial and central. Indeed, it permits itself to understand exact influences of orientation $e(0,0)$ on an orientation $O \sim \overline{P}^8_\infty$. This influence could be rewritten in term of a non-homogeneous random walk.

**Definition 3.3.** Let $p, r \in [0,1]$, $i_0 \in \mathbb{Z}$ and $k_0 \in \{0,1\}$. We denote by $(X_t : t \geq 0)$ the following stochastic process with value on $\mathbb{Z} \times \{0,1\}$:

- $X_0 = (i_0, k_0)$ a.s.;
- if $X_t = (i,k)$, then

$$X_{t+1} = \begin{cases} (i,k) & \text{w.p. } r, \\ (i + (-1)^{i+t}, 1-k) & \text{w.p. } 1-p-r, \\ (i, 1-k) & \text{w.p. } p. \end{cases} \tag{33}$$

This stochastic process is a Markov chain, but not a homogeneous Markov chain because its transitions depend on time's parity.

**Lemma 3.4.** *We have*

$$\overline{C}^8_\infty(i,t) = P(X_t = (i,1) | X_0 = (0,1)) - P(X_t = (i,0) | X_0 = (0,1)). \tag{34}$$

*Proof.* By (32), it is sufficient to prove that if $O \sim \overline{P}^8_\infty$, then for, any $i \in \mathbb{Z}$, any $t \in \mathbb{N}$,

$$\begin{aligned} &P(e(i,t)=1 \mid e(0,0)=1) - P(e(i,t)=0 \mid e(0,0)=1) \\ &\quad = P(X_t=(i,1) \mid X_0=(0,1)) - P(X_t=(i,0) \mid X_0=(0,1)). \end{aligned} \tag{35}$$

This is done by induction on $t$. When $t=0$, we have that, for any $i$, for any $k$,

$$\begin{aligned} &P(e(i,0)=k \mid e(0,0)=1) - P(e(i,0)=0 \mid e(0,0)=1) \\ &\quad = \begin{cases} 1-0, & \text{if } i=0, \\ \frac{1}{2}-\frac{1}{2}, & \text{otherwise,} \end{cases} \\ &\quad = P(X_0=(i,1)|X_0=(0,1)) - P(X_0=(i,0)|X_0=(0,1)). \end{aligned}$$

Now, let $t \in \mathbb{N}$ and suppose that (35) holds for any $i$, then

$$\begin{aligned} &P(e(i,t+1)=1 \mid e(0,0)=1) - P(e(i,t+1)=0 \mid e(0,0)=1) \\ &\quad = r\,P(e(i,t)=1 \mid e(0,0)=1) - r\,P(e(i,t)=0 \mid e(0,0)=1) \\ &\qquad + (1-p-r)P(e(i+(-1)^{i+t},t)=0 \mid e(0,0)=1) \\ &\qquad - (1-p-r)P(e(i+(-1)^{i+t},t)=1 \mid e(0,0)=1) \\ &\qquad + p\,P(e(i,t)=0 \mid e(0,0)=1) \\ &\qquad - p\,P(e(i,t)=1 \mid e(0,0)=1) \end{aligned}$$

(by Lemma 3.2)

$$\begin{aligned} &\quad = r(P(X_t=(i,1)|X_0=(0,1)) - P(X_t=(i,0)|X_0=(0,1))) \\ &\qquad + (1-p-r)(P(X_t=(i+(-1)^{i+t},0)|X_0=(0,1)) \\ &\qquad - P(X_t=(i+(-1)^{i+t},1)|X_0=(0,1))) \\ &\qquad + p(P(X_t=(i,0)|X_0=(0,1)) \\ &\qquad - P(X_t=(i,1)|X_0=(0,1))) \end{aligned}$$

(by Definition 3.3)

$$= P(X_{t+1}=(i,1)|X_0=(0,1)) - P(X_{t+1}=(i,0)|X_0=(0,1)). \qquad \square$$

Hence, computations of probabilities that $X_t$ is in a certain state is equivalent to compute $\overline{C}^8_\infty$. To do that, we use generating functions and methods of analytic combinatorics. For references to these methods, we recommend the book of Flajolet-Sedgewick [12].

Let $\widetilde{E}$ be the set of paths that start from $(0,0)$ and go to any point $(i,t) \in \mathbb{Z}\times\mathbb{N}$ using only three steps $(-1,1)$, $(0,1)$ or $(1,1)$ and such that, if the path is in a node $(i',t')$, then the next allowed steps are $((-1)^{i'+t'},1)$ and $(0,1)$. In other words, $\widetilde{E}$ is the set of paths of the graph represented on Figure 9 with starting point $(0,0)$. As the second coordinate of a path $w$ in $\widetilde{E}$ increases, we can mix up the trajectory of $w$ that is the sequence $((0,0),(i_1,1),\dots,(i_t,t))$ with its trace that is the set $\{(0,0),(i_1,1),\dots,(i_t,t)\}$ that are both a representation of $w$.

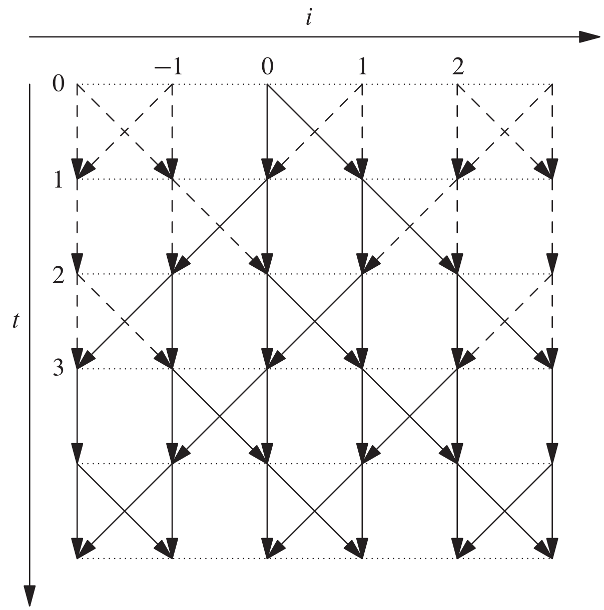



Figure 9. The directed graph on which $\widetilde{E}$ is supported. In full line, the first possible steps of paths in $\widetilde{E}$.

A colored path of $\widetilde{E}$ is a pair $(w,s)$ where $w \in \widetilde{E}$ and $s$ is a function from $w$ to $\{0,1\}$ (for any $u \in w$, $s(u)$ is called the color of $u$). We denote by $E$ the set of colored paths $(w = ((0,0),(i_1,1),\dots,(i_{t-1},t-1),(i,t)),s)$ of $\widetilde{E}$ that satisfy the following constraints:

- $s((0,0)) = 1$ and
- for any $1 \le j \le t$, if $|i_j - i_{j-1}| = 1$, then $s((i_j,j)) = 1 - s((i_{j-1},j-1))$; in other words: if the step from $(i_{j-1},j-1)$ to $(i_j,j)$ is diagonal ($(-1,1)$ or $(1,1)$), then the color changes.

To be able to count elements of $E$ according to statistics defined later, we decompose $E$ into four subsets that form a partition of $E$: for any $k_1,k_2 \in \{0,1\}$, $E_{k_1,k_2}$ is the subset of $E$ of colored paths that finish in a node $(i,t)$ such that $i+t = k_1 \bmod 2$ and $s(i,t) = k_2$.

In Flajolet-Sedgewick's symbolism, relations between these sets are

$$E_{0,1} = 1 + \underset{1}{\overset{E_{0,0}}{\searrow}} + \underset{1}{\overset{E_{1,1}}{\downarrow}} + \underset{1}{\overset{E_{1,0}}{\downarrow}}, \quad E_{0,0} = \underset{0}{\overset{E_{0,1}}{\searrow}} + \underset{0}{\overset{E_{1,0}}{\downarrow}} + \underset{0}{\overset{E_{1,1}}{\downarrow}}, \tag{36}$$

$$E_{1,1} = \underset{1}{\overset{E_{1,0}}{\swarrow}} + \underset{1}{\overset{E_{0,1}}{\downarrow}} + \underset{1}{\overset{E_{0,0}}{\downarrow}}, \quad E_{1,0} = \underset{0}{\overset{E_{1,1}}{\swarrow}} + \underset{0}{\overset{E_{0,0}}{\downarrow}} + \underset{0}{\overset{E_{0,1}}{\downarrow}}. \tag{37}$$

Now, we enumerate these four subsets according to six statistics:

- $n_v(w)$, number of vertical edges ((0, 1)) in colored path $(w, s)$,
- $n_d(w)$, number of diagonal edges ((−1, 1) or (1, 1)) in colored path $(w, s)$,
- $t(w)$, number of edges in colored path $(w, s)$,
- $i(w) = i + t$ where $(i, t)$ is the final node of colored path $(w, s)$,
- $n_c(w, s)$, number of color changes that occur on a vertical edge in colored path $(w, s)$ and
- $n_k(w, s)$, number of no color changes that occur on a vertical edge in colored path $(w, s)$.

Some of these statistics are redundant, e.g. $n_c + n_k = n_v$ or $n_v + n_d = t$.

To enumerate subsets $E_{k_1,k_2}$ according to these six statistics, we define generating functions by, for any $k_1, k_2 \in \{0, 1\}$,

$$F_{k_1,k_2}(z_v, z_d, l, x, z_c, z_k) = \sum_{(w,s)\in E_{k_1,k_2}} z_v^{n_v(w)} z_d^{n_d(w)} l^{t(w)} x^{i(w)} z_c^{n_c(w,s)} z_k^{n_k(w,s)}. \tag{38}$$

We can remark that if we take $z_d = (1 - (p + r))$, $z_v = p + r$, $z_k = \frac{r}{p+r}$ and $z_c = \frac{p}{p+r}$, then, for any $k_1, k_2 \in \{0, 1\}$,

$$F_{k_1,k_2} = \sum_{t\in\mathbb{N}, i\in\mathbb{Z} | i+t=k_1 \bmod 2} P(X_t = (i, k_2)|X_0 = (0, 1)) l^t x^{i+t}. \tag{39}$$

Hence, $\bar{C}^8_\infty(i, t)$ is the coefficient of $l^t x^{i+t}$ in series development of $F_{0,1} - F_{0,0} + F_{1,1} - F_{1,0}$ evaluated in $z_d = (1 - (p + r))$, $z_v = p + r$, $z_k = \frac{r}{p+r}$ and $z_c = \frac{p}{p+r}$.

To compute these generating functions, we use equations (36) and (37) that are translated at level of generating functions into

$$\begin{aligned}
F_{0,1} &= 1 + R\ F_{0,0} + K\ F_{1,1} + C\ F_{1,0},\\
F_{0,0} &= R\ F_{0,1} + K\ F_{1,0} + C\ F_{1,1},\\
F_{1,1} &= L\ F_{1,0} + K\ F_{0,1} + C\ F_{0,0},\\
F_{1,0} &= L\ F_{1,1} + K\ F_{0,0} + C\ F_{0,1}.
\end{aligned}$$

with $R = z_d l x^2$, $L = z_d l$, $K = z_k z_v l x$ and $C = z_c z_v l x$.

This system is solved (by hand or with the help of a formal computation software as Sage) and its resolution gives

$$F_{0,1} = \frac{1 - 2CRK - C^2 - R^2 - K^2}{H}, \tag{40}$$

$$F_{0,0} = \frac{C^2R - LR^2 + RK^2 + 2CK + L}{H}, \tag{41}$$

$$F_{1,1} = \frac{-K^3 + C(R + L) + K(C^2 + RL + 1)}{H}, \tag{42}$$

$$F_{1,0} = \frac{-C^3 + K(R + L) + C(K^2 + RL + 1)}{H}, \tag{43}$$

with $H = ((C + K)^2 - (1 - R)(1 - L))((C - K)^2 - (1 + R)(1 + L))$. Thus,

$$F_{0,1} - F_{0,0} + F_{1,1} - F_{1,0} = \frac{C - R - K - 1}{(C - K)^2 - (1 + R)(1 + L)} \tag{44}$$

that, evaluated in $C = plx$, $K = rlx$, $R = (1 - p - r)l$ and $L = (1 - p - r)lx^2$, gives the rational fraction (31). That ends the proof of Proposition 3.1 in the case $r \leq 1 - p$.

**Remark 3.5.** In the case $1 - p \leq r$, Lemma 3.2 is changed by the following lemma (changes between the two lemmas are boxed).

**Lemma 3.6.** *Let $i \in \mathbb{Z}$. Let $(s(j): j \in \mathbb{Z})$ a sequence of random variables whose values are in $\{0, 1\}$ such that $s(i)$ is independent of $(s(j): j \neq i)$. For any $u \in \{0, 1\}$, we denote $s_u = T_u(s)$, then*

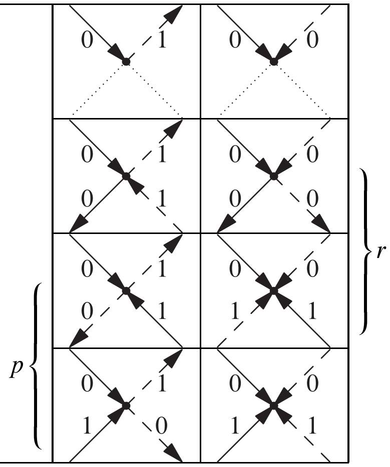
- *with probability $\boxed{1 - p}$, $s_u(i) = s(i)$ and $(s_u(j): j \neq i)$ are independent of $s(i)$;*
- *with probability $\boxed{r + p - 1}$, $s_u(i + (-1)^{i+u}) = \boxed{s(i)}$ and $(s_u(j): j \neq i + (-1)^i)$ are independent of $s(i)$;*
- *with probability $\boxed{1 - r}$, $s_u(i) = 1 - s(i)$ and $(s_u(j): j \neq i)$ are independent of $s(i)$.*

The proof of this lemma is similar to the one of Lemma 3.2. Table 1 becomes Table 2.

Then, these changes impact proof as follows. First, we need to change Definition 3.3 of the random walk $X$ on $\mathbb{Z} \times \{0, 1\}$ accordingly. Then, we enumerate set $E'$ of colored paths $(w, s)$ that have the following constraint: if $|i_j - i_{j-1}| = 1$, then $s(0, 0) = 1$ and $s(i_j, j) = s(i_{j-1}, j)$. We separate them as before in four subsets of colored paths whose generating functions $F'$ can be computed. We evaluate $F'_{0,1} - F'_{0,0} + F'_{1,1} - F'_{1,0}$ in $z_d = (p + r - 1)$, $z_v = 2 - (p + r)$, $z_k = \frac{1-p}{2-(p+r)}$ and $z_c = \frac{1-r}{2-(p+r)}$ to finally obtain the same rational fraction (31).

Table 2. Coupling in the Lemma 3.6. Figures on the fifth and sixth columns represents the coupling when $s(i) = 1 - s(i + 1) = 0$ in the fifth and when $s(i) = s(i + 1) = 0$ in the sixth.

| probability | $s_0(i)$ | $s_0(i + 1)$ |
|---|---|---|
| $1 - p$ | $s(i)$ | $s(i + 1)$ |
| $r + p - 1$ | $s(i + 1)$ | $s(i)$ |
| $1 - r$ | $1 - s(i)$ | $1 - s(i + 1)$ |




### 3.2. Proof of Theorem 1.10.

To prove Theorem 1.10, we develop in formal series the rational fraction (31) according to $l$ and $x$ and we extract coefficients of $l^t x^{i+t}$ to get $\overline{C}^8_\infty(i, t)$. With notations of (15), the rational fraction (31) is

$$\frac{1 + (\Delta + xD)l}{1 + \Delta(1 + x^2)l + Px^2l^2} \tag{45}$$

Let us begin the development of this rational fraction according to $l$.

**Lemma 3.7.** *Take $t \geq 0$. The coefficient of $l^t$ in formal series of the rational fraction (45) is*

$$f(t) + (\Delta + xD)f(t - 1) \tag{46}$$

*with, for any $t \geq 0$,*

$$f(t) = \sum_{k=0}^{\lfloor \frac{t}{2} \rfloor} \binom{t-k}{k} (-1)^{t-k} \Delta^{t-2k} P^k x^{2k} (1 + x^2)^{t-2k} \text{ and } f(-1) = 0. \tag{47}$$

*Thus, for any $t \geq 1$,*

$$\begin{aligned} &f(t) + (\Delta + xD) f(t-1) \\ &\quad = (-1)^{\frac{t}{2}} P^{\frac{t}{2}} x^t \mathbf{1}_{t=0 \mod 2} \\ &\qquad + \sum_{k=0}^{\lfloor \frac{t-1}{2} \rfloor} (-1)^{t+k} \Bigg( \binom{t-1-k}{k-1} \Delta \\ &\qquad\qquad - \binom{t-1-k}{k} Dx \\ &\qquad\qquad + \binom{t-k}{k} \Delta x^2 \Bigg) \Delta^{t-1-2k} P^k x^{2k} (1+x^2)^{t-1-2k} \end{aligned} \tag{48}$$

*with convention that* $\binom{n}{-1} = 0$ *for any* $n \in \mathbb{N}$.

*Proof.* We develop (45) in formal series according to $l$.

$$\begin{aligned} &\frac{1 + l(\Delta + xD)}{x^2 l^2 P + l\Delta(1+x^2) + 1} \\ &\quad = (1 + l(\Delta + xD)) \Big( \sum_{i=0}^{\infty} (-x^2 l^2 P - l\Delta(1+x^2))^i \Big) \\ &\quad = (1 + l(\Delta + xD)) \Big( \sum_{i=0}^{\infty} (-1)^i l^i (Px^2 l + \Delta(1+x^2))^i \Big) \\ &\quad = \Big(1 + l(\Delta + xD)\Big) \Big( \sum_{i=0}^{\infty} (-1)^i l^i \sum_{k=0}^{i} \binom{i}{k} (Px^2 l)^k (\Delta(1+x^2))^{i-k} \Big) \\ &\quad = (1 + l(\Delta + xD)) \Big( \sum_{i=0}^{\infty} \sum_{k=0}^{i} \binom{i}{k} (-1)^i \Delta^{i-k} l^{i+k} P^k x^{2k} (1+x^2)^{i-k} \Big) \\ &\quad = (1 + l(\Delta + xD)) \Big( \sum_{t=0}^{\infty} l^t \underbrace{\sum_{k=0}^{\lfloor \frac{t}{2} \rfloor} \binom{t-k}{k} (-1)^{t-k} \Delta^{t-2k} P^k x^{2k} (1+x^2)^{t-2k}}_{f(t)} \Big) \\ &\quad = \sum_{t=0}^{\infty} l^t (f(t) + (\Delta + xD) f(t-1)). \qquad \square \end{aligned}$$

For any $t \geq 0$, $f(t) + (\Delta + xD)f(t-1)$ is a polynomial in the variable $x$. Now, we extract coefficients of $x^j$ for any $j \geq 0$.

**Lemma 3.8.** *The coefficient of $x^j$ in (48) is*

- *if $j$ is odd (we denote $j' = \lfloor \frac{j}{2} \rfloor$),*

$$(-1)^{t+1} D \sum_{k=0}^{\min(j',t-1-j')} (-1)^k \binom{t-1-k}{k,\, j'-k,\, t-1-j'-k} \Delta^{t-1-2k} P^k; \tag{49}$$

- *if $j = t$ is even,*

$$\sum_{k=0}^{\frac{t}{2}} (-1)^k \binom{t-1-k}{\frac{t}{2}-k} \binom{\frac{t}{2}}{k} \Delta^{t-2k} P^k; \tag{50}$$

- *if $j$ is even (we denote $j' = \frac{j}{2}$) and $j \neq t$,*

$$\begin{aligned} &\sum_{k=0}^{\min(j'-1,t-1-j')} (-1)^{t+k} \binom{t-1-k}{t-j'-k} \binom{j'}{k} \Delta^{t-2k} P^k \\ &\quad + \mathbf{1}_{2j'<t} (-1)^{t-j'} \binom{t-1-j'}{j'-1} \Delta^{t-2j'} P^{j'} \\ &\quad + \mathbf{1}_{2j'>t} (-1)^{j'} \binom{j'}{t-j'} \Delta^{2j'-t} P^{t-j'}. \end{aligned} \tag{51}$$

*Proof.* We expand the sum in (48) as a sum of monomials in $x$.

$$\begin{aligned} &\sum_{k=0}^{\lfloor \frac{t-1}{2} \rfloor} \left( \binom{t-1-k}{k-1} \Delta - \binom{t-1-k}{k} D x \right. \\ &\qquad \left. + \binom{t-k}{k} \Delta x^2 \right) (-1)^{t+k} \Delta^{t-1-2k} P^k x^{2k} (1+x^2)^{t-1-2k} \\ &= \sum_{k=0}^{\lfloor \frac{t-1}{2} \rfloor} \left( \binom{t-1-k}{k-1} \Delta - \binom{t-1-k}{k} D x \right. \\ &\qquad \left. + \binom{t-k}{k} \Delta x^2 \right) (-1)^{t+k} \Delta^{t-1-2k} P^k x^{2k} \left( \sum_{j=0}^{t-1-2k} \binom{t-1-2k}{j} x^{2j} \right) \\ &= \sum_{k=0}^{\lfloor \frac{t-1}{2} \rfloor} \sum_{j=0}^{t-1-2k} \binom{t-1-2k}{j} \left( \binom{t-1-k}{k-1} \Delta - \binom{t-1-k}{k} D x \right. \\ &\qquad \left. + \binom{t-k}{k} \Delta x^2 \right) (-1)^{t+k} \Delta^{t-1-2k} P^k x^{2(k+j)}. \end{aligned}$$

First, we make the change of variable $j' = k + j$,

$$\sum_{k=0}^{\lfloor \frac{t-1}{2} \rfloor} \sum_{j'=k}^{t-1-k} \binom{t-1-2k}{j'-k} \left( \binom{t-1-k}{k-1} \Delta - \binom{t-1-k}{k} Dx \right.$$
$$\left. + \binom{t-k}{k} \Delta x^2 \right) (-1)^{t+k} \Delta^{t-1-2k} P^k x^{2j'}$$

then, permuting the sums,

$$\sum_{j'=0}^{t-1} x^{2j'} \sum_{k=0}^{\min(j', t-1-j')} \binom{t-1-2k}{j'-k} \left( \binom{t-1-k}{k-1} \Delta - \binom{t-1-k}{k} Dx \right.$$
$$\left. + \binom{t-k}{k} \Delta x^2 \right) (-1)^{t+k} \Delta^{t-1-2k} P^k.$$

The coefficient of $x^{2j'+1}$ is then

$$-D \sum_{k=0}^{\min(j', t-1-j')} \binom{t-1-2k}{j'-k} \binom{t-1-k}{k} (-1)^{t+k} \Delta^{t-1-2k} P^k$$

that is (49).

The coefficient of $x^{2j'}$ is

$$\sum_{k=0}^{\min(j', t-1-j')} \binom{t-1-2k}{j'-k} \binom{t-1-k}{k-1} (-1)^{t+k} \Delta^{t-2k} P^k$$
$$+ \sum_{k=0}^{\min(j'-1, t-j')} \binom{t-1-2k}{j'-1-k} \binom{t-k}{k} (-1)^{t+k} \Delta^{t-2k} P^k$$

that is, if $2j' = t$,

$$\sum_{k=0}^{\frac{t}{2}-1} (-1)^{t+k} \left( \binom{t-1-2k}{\frac{t}{2}-k} \binom{t-1-k}{k-1} + \binom{t-1-2k}{\frac{t}{2}-1-k} \binom{t-k}{k} \right) \Delta^{t-2k} P^k$$

and adding $(-1)^{\frac{t}{2}} P^{\frac{t}{2}} x^t \mathbf{1}_{t=0 \mod 2}$, we get (50).

And, in the case $2j' \neq t$, we obtain,

$$\sum_{k=0}^{\min(j'-1,t-1-j')}(-1)^{t+k}\left(\binom{t-1-2k}{j'-k}\binom{t-1-k}{k-1} + \binom{t-1-2k}{j'-1-k}\binom{t-k}{k}\right)\Delta^{t-2k}P^k$$
$$+ \mathbf{1}_{\min(j',t-j')=j'}(-1)^{t-j'}\binom{t-1-j'}{j'-1}\Delta^{t-2j'}P^{j'}$$
$$+ \mathbf{1}_{\min(j',t-j')=t-j'}(-1)^{j'}\binom{j'}{t-j'}\Delta^{2j'-t}P^{t-j'},$$

equivalent to

$$\sum_{k=0}^{\min(j'-1,t-1-j')}(-1)^{t+k}\binom{t-1-k}{t-j'-k}\binom{j'}{k}\Delta^{t-2k}P^k$$
$$+ \mathbf{1}_{\min(j',t-j')=j'}(-1)^{t-j'}\binom{t-1-j'}{j'-1}\Delta^{t-2j'}P^{j'}$$
$$+ \mathbf{1}_{\min(j',t-j')=t-j'}(-1)^{j'}\binom{j'}{t-j'}\Delta^{2j'-t}P^{t-j'}$$

that is (51).

N B. The fact that

$$\left(\binom{t-1-2k}{j'-k}\binom{t-1-k}{k-1} + \binom{t-1-2k}{j'-1-k}\binom{t-k}{k}\right) = \binom{t-1-k}{t-j'-k}\binom{j'}{k}$$

for any $t, k, j'$ could be proved using factorial notation of binomials and it is trivial if $k = 0$. □

*Proof of Theorem* 1.10. We recall that $\overline{C}^8_\infty(i,t)$ is the coefficient of $x^{i+t}l^t$ in the rational fraction (31) (see Proposition 3.1). These coefficients are given by Lemma 3.8. To conclude, we change variables from $(j = i + t, t)$ to $(i = j - t, t)$. □

**3.3. Particle system and proof of Proposition 1.13.** In this section, we suppose that $p + r \leq 1$. The case $p + r \geq 1$ could be treated in a similar way as explained in Remark 3.5.

First, we define a particle system that is related to the 8-vertex model.

**Definition 3.9** (particle system $\mathcal{P}(\mu; p, r)$). Let $\mu$ be a probability measure on $\{0,1\}^{\mathbb{Z}}$ and $p, r \in [0,1]$. The law $\mathcal{P}(\mu; p, r)$ is the following law on the set $(\mathbb{Z} \times \{0,1\})^{\mathbb{Z}\times\mathbb{N}}$. At time $t = 0$, for any $i \in \mathbb{Z}$, there is exactly one particle (named) $\alpha_i$ in position $p(\alpha_i, 0) = i$ and in a random state $s(\alpha_i, 0) \in \{0,1\}$ and $(s(\alpha_i, 0) : i \in \mathbb{Z}) \sim \mu$. Then, from time $t$ to time $t+1$, for any $i \in \mathbb{Z}$ such that $i + t$ is even, particles $\alpha$ and $\beta$ such that $p(\alpha, t) = i$ and $p(\beta, t) = i + 1$ interact in the following way:

- with probability $p$, particles do not move and theirs states do not change:

$$\begin{aligned} p(\alpha, t+1) &= i, & s(\alpha, t+1) &= s(\alpha, t), \\ p(\beta, t+1) &= i+1, & s(\beta, t+1) &= s(\beta, t); \end{aligned}$$

- with probability $1 - p - r$, particles exchange theirs positions and change their states:

$$\begin{aligned} p(\alpha, t+1) &= i+1, & s(\alpha, t+1) &= 1 - s(\alpha, t), \\ p(\beta, t+1) &= i, & s(\beta, t+1) &= 1 - s(\beta, t). \end{aligned}$$

- with probability $r$, particles do not move and change their states:

$$\begin{aligned} p(\alpha, t+1) &= i, & s(\alpha, t+1) &= 1 - s(\alpha, t), \\ p(\beta, t+1) &= i+1, & s(\beta, t+1) &= 1 - s(\beta, t). \end{aligned}$$

To see a representation of these transitions, see fifth and sixth columns on Table 1: plain line represents the particle $\alpha$ and dashed line the particle $\beta$. Moreover, all these transitions are independent. The law $\mathcal{P}(\mu; p, r)$ is then the law of the random variable $((p(\alpha_i, t), s(\alpha_i, t)) : i \in \mathbb{Z}, t \in \mathbb{N})$.

By definition, there is exactly one particle $\alpha$ at each time $t$ in position $i$; this particle will be denoted $\alpha(i, t)$ (and, simply, $\alpha_i$ if $t = 0$).

This particles system is related to the 8-vertex model with $a + c = b + d$ via the following proposition.

**Proposition 3.10.** *If* $((p(\alpha_i, t), s(\alpha_i, t)) : i \in \mathbb{Z}, t \in \mathbb{N}) \sim \mathcal{P}(\mu; p, r)$, *then* $(s(\alpha(i, t), t) : i \in \mathbb{Z}, t \in \mathbb{N}) \sim \mathcal{L}(\mu; p, r)$.

*Proof.* This is a consequence of the coupling defined in Lemma 3.2. □

A first consequence of this proposition is the Lemma 2.4.

*Proof of Lemma* 2.4. First, we begin by the following remark: if at time $t = 0$, $\{s(\alpha_i, 0): i \in \mathbb{Z}\}$ are independent, then for any $i \in \mathbb{Z}$, $\{s(\alpha_i, t): t \in \mathbb{N}\}$ is independent of $\{s(\alpha_j, t): j \in \mathbb{Z} \setminus \{i\}, t \in \mathbb{N}\}$ (by Lemma 3.2).

Now, take any sequence $(t_i : i \in \mathbb{Z})$ such that $t_{i+1} - t_i \in \{0, (-1)^{i+1+t_i}\}$. We know that the set of allowed positions for the particle $\alpha(i, t_i)$ is, if $i + t_i$ is even,

$$\begin{aligned}&\{(j,t): P(p(\alpha(i,t_i),t) = j) > 0\}\\ &\quad = \{(j,t): i - t_i \leq j - t, j + t \leq i + t_i - 1, t < t_i\} \cup \{(i,t_i)\}\\ &\qquad \cup \{(j,t): j - t \leq i - t_i, i + t_i + 1 \leq j + t, t_i < t\};\end{aligned}$$

if $i + t_i$ is odd,

$$\begin{aligned}&\{(j,t): P(p(\alpha(i,t_i),t) = j) > 0\}\\ &\quad = \{(j,t): i - t_i + 1 \leq j - t, j + t \leq i + t_i, t < t_i\} \cup \{(i,t_i)\}\\ &\qquad \cup \{(j,t): j - t \leq i - t_i - 1, i + t_i \leq j + t, t_i < t\}.\end{aligned}$$

that intersects the set $\{(j, t_j): j \in \mathbb{Z}\}$ in only one point that is $(i, t_i)$, see Figure 10. Hence, particle $\alpha(i, t_i)$ cannot be in any position $(j, t_j)$ for any $j \neq i$ and, so, for any $i, j \in \mathbb{Z}, i \neq j, \alpha(i, t_i) \neq \alpha(j, t_j)$.

To conclude, as we have $(s(\alpha_i, 0): \alpha_i \in \mathbb{Z}) \sim \mathrm{PM}\left(\frac{1}{2}\right)$ (because $O \sim \overline{P}^8_\infty$), we get that $(e(i, t_i) = s(\alpha(i, t_i), t_i): i \in \mathbb{Z})$ are independent, see above. And, by Proposition 1.5, for any $i \in \mathbb{Z}$, $P(e(i, t_i) = 0) = P(e(i, t_i) = 1) = \frac{1}{2}$. □

A second consequence of Proposition 3.10 is Proposition 1.13. Before its proof, we introduce the same particles system as before but with a slight difference on the transition kernel.

**Definition 3.11** (particle system $\mathcal{P}'(\mu; p, r)$). Let $p, r \in (0, 1)$. Let $B = \{B_{i,t}: i \in \mathbb{Z}, t \in \mathbb{N}\}$ be a set of i.i.d. variables of law $\mathcal{B}\left(\frac{1}{2}\right)$, and $Z = \{Z_{i,t}: i \in \mathbb{Z}, t \in \mathbb{N}\})$ be a set of i.i.d. variables of law $\mathcal{B}(2m)$ with $m = \min(p, r)$.

At time $t = 0$, for any $i \in \mathbb{Z}$, there is exactly one particle (named) $\alpha_i$ in position $p(\alpha_i, 0) = i$ and in a random state $s(\alpha_i, 0) \in \{0, 1\}$ and $(s(\alpha_i, 0): \alpha_i \in \mathbb{Z}) \sim \mu$, and such that $B$, $Z$ and $(s(\alpha_i, 0): i \in \mathbb{Z})$ are mutually independent. Then, from time $t$ to time $t + 1$, for any $i \in \mathbb{Z}$ such that $i + t$ is even, particles $\alpha$ and $\beta$ such that $p(\alpha, t) = i$ and $p(\beta, t) = i + 1$ interact in the following way:

(1) with probability $2m$ (that is when $Z_{i,t} = 1$),

$$\begin{aligned}&p(\alpha, t+1) = i, && s(\alpha, t+1) = B_{i,t},\\ &p(\beta, t+1) = i+1, && s(\beta, t+1) = \begin{cases} B_{i,t} & \text{if } s(\alpha,t) = s(\beta,t),\\ 1 - B_{i,t} & \text{if } s(\alpha,t) = 1 - s(\beta,t);\end{cases}\end{aligned}$$

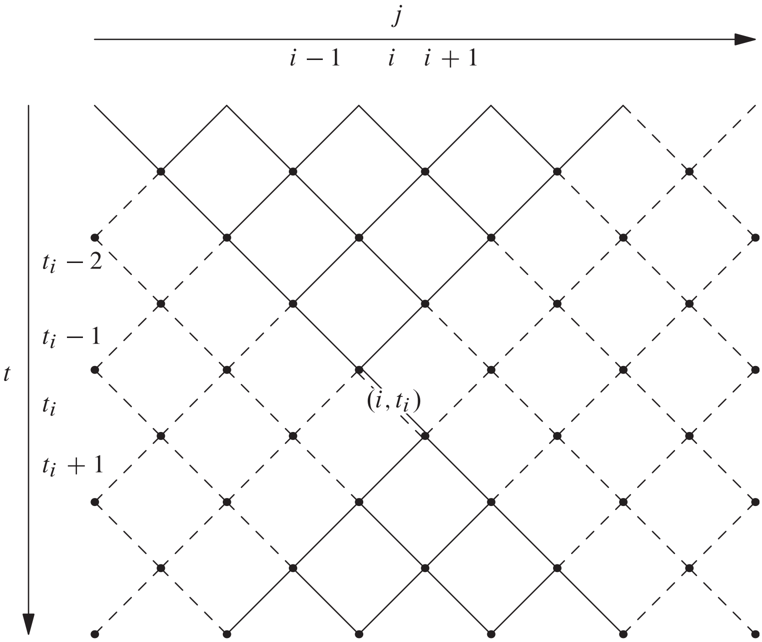



Figure 10. In double plain-dashed line: the position $(i, t_i)$. In plain line: the set of allowed positions for the particle $\alpha(i, t_i)$. In dashed line: available edges for a set $((j, t_j): j \in \mathbb{Z})$ that respects the condition $t_{i+1} - t_i \in \{0, (-1)^{i+1+t_i}\}$. There is no common position for plain and dashed line except in $(i, t_i)$.

(2) with probability $p - m$ (a $Z_{i,t} = 0$ case),

$$\begin{aligned} p(\alpha, t+1) &= i, & s(\alpha, t+1) &= s(\alpha, t), \\ p(\beta, t+1) &= i+1, & s(\beta, t+1) &= s(\beta, t); \end{aligned}$$

(3) with probability $1 - p - r$ (a $Z_{i,t} = 0$ case),

$$\begin{aligned} p(\alpha, t+1) &= i+1, & s(\alpha, t+1) &= 1 - s(\alpha, t), \\ p(\beta, t+1) &= i, & s(\beta, t+1) &= 1 - s(\beta, t); \end{aligned}$$

(4) with probability $r - m$ (a $Z_{i,t} = 0$ case),

$$\begin{aligned} p(\alpha, t+1) &= i, & s(\alpha, t+1) &= 1 - s(\alpha, t), \\ p(\beta, t+1) &= i+1, & s(\beta, t+1) &= 1 - s(\beta, t). \end{aligned}$$

Moreover, all these transitions are independent. We denote by $\mathcal{P}'(\mu; p, r)$ the law of $((p(\alpha_i, t), s(\alpha_i, t)): i \in \mathbb{Z}, t \in \mathbb{N})$.

**Lemma 3.12.** *For any* $\mu, p, r$, $\mathcal{P}'(\mu; p, r) = \mathcal{P}(\mu; p, r)$.

*Proof.* To prove the lemma, we have to show that when we are in Case 1 in Definition 3.11, we can obtain with probability $\frac{1}{2}$ and $\frac{1}{2}$ conclusion of Cases 2 and 4. This is the case because $B_{i,t} = \begin{cases} s(\alpha, t) & \text{w.p. } \frac{1}{2}, \\ 1 - s(\alpha, t) & \text{w.p. } \frac{1}{2}. \end{cases}$ □

In the following, for any particle $\alpha$, any $t, t_1, t_2 \in \mathbb{N}$ with $t_1 < t_2$, we denote the event $I(\alpha, t) = \{\omega : Z_{p(\alpha_i, t), t} = 1\}$: "particle $\alpha$ does the transition 1 at time $t$ (from time $t$ to $t+1$)" and we denote $I_2(\alpha, t_1, t_2)$ the event: "$\alpha$ does the transition 1 at time $t_1$ and $t_2$ with two different neighbor particles (i.e. $I(\alpha, t_1)$ and $I(\alpha, t_2)$ and $\alpha(p(\alpha, t_1) + (-1)^{p(\alpha, t_1)+t_1}, t_1) \neq \alpha(p(\alpha, t_2) + (-1)^{p(\alpha, t_2)+t_2}, t_2)$)". Finally, we denote

$$I_2(\alpha, t) = \bigcup_{(t_1, t_2): 0 \leq t_1 < t_2 < t} I_2(\alpha, t_1, t_2),$$

the event that $\alpha$ does at least once the transition 1 with two different neighbor particles before time $t$. Now, the correlation function of the 8-vertex model with initial law $\mu$ can be rewritten in this system of particles. First, we need the following lemma.

**Lemma 3.13.** *Let* $((p(\alpha_i, t), s(\alpha_i, t)): i \in \mathbb{Z}, t \in \mathbb{N}) \sim \mathcal{P}(\mu; p, r)$. *Then, for any* $i \in \mathbb{Z}$, *for any* $t$, $I_2(\alpha_i, t)$ *is independent of* $s(\alpha_0, 0)$.

*Proof.* This is induced by the fact that for any $i \in \mathbb{Z}$, $I(\alpha_i, t)$ is independent of $s(\alpha_0, 0)$, that is true because $Z$ is independent of $(s(\alpha_i, 0): i \in \mathbb{Z})$. □

Now, we can express the correlation function of the 8-vertex model with any boundary condition and $a + c = b + d$.

**Lemma 3.14.** *Let* $\mu$ *be any probability measure on* $\{0,1\}^{\mathbb{Z}}$ *and let* $O \sim \mathcal{L}(\mu; p, r)$. *Then the correlation function* $C$ *satisfies*

$$\left| \sqrt{\frac{\operatorname{Var}(e(i,t))}{\operatorname{Var}(e(0,0))}}\, C((0,0);(i,t)) - \overline{C}^{8}_{\infty}(i,t) \right| \leq 2P(I_2(\alpha(i,t),t)^c) \tag{52}$$

*Proof.* Let $O = (e(i,t): i \in \mathbb{Z}, t \in \mathbb{N}) \sim \mathcal{L}(\mu; p, r)$. Then,

$$\begin{aligned}
&\operatorname{Cov}(e(0,0), e(i,t)) \\
&\quad = E[e(0,0)e(i,t)] - E[e(0,0)]E[e(i,t)] \\
&\quad = P(e(0,0) = 1 = e(i,t)) - P(e(0,0) = 1)P(e(i,t) = 1) \\
&\quad = P(e(0,0) = 1)[P(e(i,t) = 1|e(0,0) = 1) \\
&\qquad\qquad - P(e(i,t) = 1|e(0,0) = 1)P(e(0,0) = 1) \\
&\qquad\qquad - P(e(i,t) = 1|e(0,0) = 0)P(e(0,0) = 0)] \\
&\quad = \operatorname{Var}(e(0,0))[P(e(i,t) = 1|e(0,0) = 1) - P(e(i,t) = 1|e(0,0) = 0)] \\
&\quad = \operatorname{Var}(e(0,0))[P(s(\alpha(i,t),t) = 1|s(\alpha_0,0) = 1) \\
&\qquad\qquad - P(s(\alpha(i,t),t) = 1|s(\alpha_0,0) = 0)].
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sqrt{\frac{\operatorname{Var}(e(i,t))}{\operatorname{Var}(e(0,0))}}\, C((0,0);(i,t)) \\
&\quad = \underbrace{P(s(\alpha_0,t) = 1 \text{ and } \alpha(i,t) = \alpha_0|s(\alpha_0,0) = 1) - P(s(\alpha_0,t) = 1 \text{ and } \alpha(i,t) = \alpha_0|s(\alpha_0,0) = 0)}_{=\bar{C}^{8}_{\infty}(i,t)} \\
&\qquad + P(s(\alpha(i,t),t) = 1 \text{ and } \alpha(i,t) \neq \alpha_0 \text{ and } I_2(\alpha(i,t),t)|s(\alpha_0,0) = 1) \\
&\qquad\quad - P(s(\alpha(i,t),t) = 1 \text{ and } \alpha(i,t) \neq \alpha_0 \text{ and } I_2(\alpha(i,t),t)|s(\alpha_0,0) = 0) \\
&\qquad + P(s(\alpha(i,t),t) = 1 \text{ and } \alpha(i,t) \neq \alpha_0 \text{ and } I_2(\alpha(i,t),t)^c|s(\alpha_0,0) = 1) \\
&\qquad - P(s(\alpha(i,t),t) = 1 \text{ and } \alpha(i,t) \neq \alpha_0 \text{ and } I_2(\alpha(i,t),t)^c|s(\alpha_0,0) = 0).
\end{aligned}$$

But,

$$P(s(\alpha(i,t),t) = 1 \text{ and } \alpha(i,t) \neq \alpha_0 \text{ and } I_2(\alpha(i,t),t)|s(\alpha_0,0) = 1)$$

(because $I_2(\alpha(i,t),t) \implies s(\alpha(i,t),t) \sim \mathcal{B}\left(\frac{1}{2}\right)$)

$$= \frac{1}{2} P(\alpha(i,t) \neq \alpha_0 \text{ and } I_2(\alpha(i,t),t)|s(\alpha_0,0) = 1)$$

(by Lemma 3.13 and the fact that trajectories of particles are independent of their initial states)

$$= \frac{1}{2} P(\alpha(i,t) \neq \alpha_0 \text{ and } I_2(\alpha(i,t),t)).$$

Similarly,

$$\begin{aligned}&P(s(\alpha(i,t),t) = 1 \text{ and } \alpha(i,t) \neq \alpha_0 \text{ and } I_2(\alpha(i,t),t)|s(\alpha_0,0) = 0)\\&\quad = \frac{1}{2}P(\alpha(i,t) \neq \alpha_0 \text{ and } I_2(\alpha(i,t),t)).\end{aligned}$$

Hence,

$$\begin{aligned}&\left|\sqrt{\frac{\mathrm{Var}(e(i,t))}{\mathrm{Var}(e(0,0))}}\,C((0,0);(i,t)) - \bar{C}^8_\infty(i,t)\right|\\&\quad \leq P(I_2(\alpha(i,t),t)^c|s(\alpha_0,0) = 1) + P(I_2(\alpha(i,t),t)^c|s(\alpha_0,0) = 0)\end{aligned}$$

(by Lemma 3.13)

$$\begin{aligned}&\quad = 2P((I_2(\alpha(i,t),t))^c)\\&\quad = 2P((I_2(\alpha_0,t)^c)\end{aligned}$$

because event $I_2(\alpha_i,t)$ occurs with the same probability for any particle $\alpha_i$. □

Now, to prove Proposition 1.13, we have to find a bound on $P(I_2(\alpha_0,t)^c)$.

**Lemma 3.15.** *Let* $m = \min(p,r)$.

$$P(I_2(\alpha_0,t)^c) \leq (1-m)^{t-1-\lfloor\frac{t}{2}\rfloor} + (1-m)^{\lfloor\frac{t}{2}\rfloor} - (1-m)^{t-1} \tag{53}$$

*Proof.* We denote $T_1(\omega) = \inf\{t\colon \omega \in I(\alpha_0,t)\}$. By definition of $I(\alpha_0,t)$, the law of $T_1$ is the geometric law on $\{1,2,\ldots\}$ whose success probability is $2m$, i.e. $P(T_1 = k) = 2m(1-2m)^{k-1}$. Now, denote $T_2(\omega) = \inf\{t\colon \omega \in I_2(\alpha_0, T_1(\omega), t)\}$. Due to the fact that two particles cannot interact twice during two successive steps of time, $\lceil\frac{T_2-T_1}{2}\rceil \leq T$ in law where $T$ is distributed as $T_1$. Hence,

$$\begin{aligned}P(I_2(\alpha_0,t)) &\geq P(T_2 \leq t)\\&\geq P(2T + T_1 \leq t)\\&= \sum_{t'=1}^{\lfloor\frac{t}{2}\rfloor} P(T = t')P(T_1 \leq t - 2t')\\&= \sum_{t'=1}^{\lfloor\frac{t}{2}\rfloor} 2m(1-2m)^{t'-1}(1-(1-2m)^{t-2t'})\end{aligned}$$

$$= 2m \sum_{t'=1}^{\lfloor \frac{t}{2} \rfloor} ((1-2m)^{t'-1} - (1-2m)^{t-t'-1})$$
$$= 2m \Big( \frac{1-(1-2m)^{\lfloor \frac{t}{2} \rfloor}}{2m} - (1-2m)^{t-1} \frac{(1-2m)^{-\lfloor \frac{t}{2} \rfloor} - 1}{2m} \Big)$$
$$= 1 - (1-2m)^{\lfloor \frac{t}{2} \rfloor} + (1-2m)^{t-1} - (1-2m)^{t-\lfloor \frac{t}{2} \rfloor}. \qquad \square$$

Now, we can do the proof of Proposition 1.13.

*Proof of Proposition* 1.13. In the case $p + r \leq 1$, by lemmas 3.14 and 3.15,

$$\left| \sqrt{\frac{\mathrm{Var}(e(i,t))}{\mathrm{Var}(e(0,0))}} C((0,0),(i,t)) - \bar{C}^8_\infty(i,t) \right| \leq 2((1-2m)^{\lfloor \frac{t}{2} \rfloor} + (1-2m)^{t-\lfloor \frac{t}{2} \rfloor - 1} - (1-2m)^{t-1}), \tag{54}$$

where $m = \min(p, r)$.

In the case $p + r \geq 1$, a similar proof permits to find (54) with $m = \min(1-p, 1-r)$.

To conclude, we remark that when $p+r \leq 1$, then $p \leq 1-r$ and $r \leq 1-p$, so $\min(1-p, 1-r, p, r) = \min(p, r)$; and when $p+r \geq 1$, $\min(p, r, 1-p, 1-r) = \min(1-p, 1-r)$. And, finally, we remark that $1-2\min(p, r, 1-p, 1-r) = \lambda(p, r)$. That is ending the proof. $\square$

## 4. Asymptotics of $\bar{C}^8_\infty(i,t)$: proof of Theorem 1.12

In this section, we suppose that $p + r \neq 1$. The case $p + r = 1$ has been treated in Remark 1.11. The proof of Theorem 1.12 is done in two steps. In a first step, the asymptotics is proved when $b = 0$ (i.e. when $r = 0$ and $0 \leq p < 1$). And, in a second step, it is generalized for any $(a, b, c, d)$ such that $a + c = b + d$.

**4.1. Case $r = 0$ and $0 \leq p < 1$.** In this case, $\lambda(p, 0) = 1$, see (21), and, so, Theorem 1.12 is

**Proposition 4.1.** *If $r = 0$ and $0 \leq p < 1$, then there exists $c > 0$ such that, for any $t \in \mathbb{N}$, for any $i \in \mathbb{Z}$*

$$\bar{C}^8_\infty(i,t) \leq \frac{c}{\sqrt{t}}.$$

To prove this proposition, we prove first two lemmas on the asymptotic behavior of random walks.

**Lemma 4.2.** *Let $S = (S_t : t \in \mathbb{N})$ be a simple random walk on $\mathbb{Z}$, i.e.*

- $S_0 = 0$ *a.s.,*
- $(S_{t+1} - S_t : t \in \mathbb{N})$ *are i.i.d.,*
- *for any $t \geq 0$,*

$$P(S_{t+1} - S_t = 1) = P(S_{t+1} - S_t = -1) = \frac{1}{2}.$$

*Then, there exists a constant $c > 0$ such that for any $t \in \mathbb{N}$, for any $i \in \mathbb{Z}$,*

$$P(S_t = i) \leq \frac{c}{\sqrt{t}}.$$

*Proof.* It is a classical result in probability theory. Proofs of generalization of this lemma exists for sum of i.i.d. random variables, see [18, Chapter 3]. A proof in that simple case can be obtained by enumeration of binary paths and application of Stirling's formula. □

The second lemma generalizes Lemma 4.2 to some processes constructed with a simple random walk.

**Lemma 4.3.** *Let $X = (X_t : t \in \mathbb{N})$ a process with values in $\mathbb{Z}$. If, for any $t \in \mathbb{N}$,*

$$X_t \overset{\mathrm{d}}{=} S_{N_t} + R_{t,N_t}$$

*such that $N = (N_t : t \in \mathbb{N})$ where $N_t$ follows a binomial law with parameters $(t, q)$ $(q \neq 0)$, $S = (S_u : u \in \mathbb{N})$ is a simple random walk on $\mathbb{Z}$ and $R = (R_{t,u} : t \in \mathbb{N}, u \in \mathbb{N})$ is any collection of any random variables on $\mathbb{Z}$ and $(N, S, R)$ are independent, then, there exists $c > 0$ such that for any $i$, for any $t$,*

$$P(X_t = i) \leq \frac{c}{\sqrt{t}}.$$

*Proof.* Let $t \in \mathbb{N} \setminus \{0\}$, $i \in \mathbb{Z}$ and $\epsilon > 0$,

$$\begin{aligned} P(X_t = i) &= P(S_{N_t} + R_{t,N_t} = i) \\ &\leq P\left(|N_t - qt| > \frac{qt}{2}\right) + \max_{n \in \left[\frac{qt}{2}, \frac{3qt}{2}\right]} P(S_n + R_{t,n} = i) \end{aligned}$$

(by Chebyshev's inequality)

$$\leq \frac{q(1-q)t}{\left(\frac{qt}{2}\right)^2} + \max_{n \in \left[\frac{qt}{2}, \frac{3qt}{2}\right]} P(S_n = i - R_{t,n})$$

(by Lemma 4.2)

$$\leq \frac{4(1-q)}{qt} + \max_{n\in[\frac{qt}{2},\frac{3qt}{2}]} \frac{c}{\sqrt{n}}$$

$$\leq \frac{c_1}{\sqrt{t}} + \frac{c\sqrt{2}}{\sqrt{q}}\frac{1}{\sqrt{t}}$$

$$= \frac{c_2}{\sqrt{t}}.$$

□

Now, we define a homogeneous Markov chain $Y$ with values on $\mathbb{Z}$.

**Definition 4.4.** The process $Y = (Y_t : t \in \mathbb{N})$ have the following properties:

- $Y_0 = 0$ a.s.;
- for any $t \in \mathbb{N}$,

$$Y_{t+1} = Y_t + \begin{cases} -1 & \text{w.p. } p(1-p), \\ 0 & \text{w.p. } p^2, \\ 1 & \text{w.p. } p(1-p), \\ 2(-1)^{Y_t} & \text{w.p. } (1-p)^2. \end{cases}$$

This Markov chain is related to the Markov chain $X$ defined in Definition 3.3.

**Proposition 4.5.** *Let $p \in [0,1)$ and $r = 0$. In that case, the non-homogeneous Markov chain $X$ (defined in Definition 3.3) satisfies, for any $t \in \mathbb{N}$,*

$$X_{2t} \overset{\mathrm{d}}{=} (Y_t, 1) \text{ and } X_{2t+1} \overset{\mathrm{d}}{=} \begin{cases} (1+Y_t, 0) & \text{w.p. } 1-p, \\ (-Y_t, 0) & \text{w.p. } p. \end{cases} \tag{55}$$

*In particular, (34) gives, for any $t$,*

$$\overline{C}^8_\infty(i, 2t) = P(Y_t = 0) \tag{56}$$

*and*

$$\overline{C}^8_\infty(i, 2t+1) = -(pP(Y_t = 0) + (1-p)P(Y_t = -1)). \tag{57}$$

*Proof.* We suppose that $r = 0$, we obtain, for any $t \in \mathbb{Z}$, applying twice (33), that if $X_t = (i, k)$, then

$$X_{t+2} = \begin{cases} (i-1, k) & \text{w.p. } p(1-p), \\ (i, k) & \text{w.p. } p^2, \\ (i+1, k) & \text{w.p. } p(1-p), \\ (i + 2(-1)^{i+t}, k) & \text{w.p. } (1-p)^2. \end{cases}$$

As $X_0 = (0, 1)$, we obtain that, for any $t \in \mathbb{Z}$, first coordinate of $X_{2t}$ is equal in distribution to $Y_t$ and its second coordinate is 1 a.s. And, as

$$X_1 = \begin{cases} (0, 0) & \text{w.p. } p, \\ (1, 0) & \text{w.p. } 1 - p, \end{cases}$$

first coordinate of $X_{2t+1}$ is equal in distribution to $-Y_t$ w.p. $p$ and to $1 + Y_t$ w.p. $1 - p$ and its second coordinate is 0 a.s. □

Now, we decompose $Y$ so that $Y$ satisfies conditions of Lemma 4.3. We define first a law on $\mathbb{Z}$.

**Definition 4.6.** Let $p \in [0, 1]$, $q \in [0, 1]$, $t \in \mathbb{N}$, $n \in \mathbb{N}$. Let $(L_j : 0 \leq j \leq n)$ be $n+1$ i.i.d. random variables distributed according to geometric law on $\{0, 1, 2, \dots\}$ of success parameter $q$, i.e., for any $j \in \mathbb{N}$, for any $k \in \mathbb{N}$,

$$P(L_j = k) = (1 - q)^k q.$$

We denote by $\mathcal{L}_{t,n}(q)$ the law on $\mathbb{N}^{n+1}$ of $(L_j : 0 \leq j \leq n)$ conditioned by $\sum_{j=0}^{n} L_j = t - n$. Let $(L_j : 0 \leq j \leq n) \sim \mathcal{L}_{t,n}(q)$. For any $j$, we set $G_j$ to be a random variable distributed according to binomial law with parameters $(L_j, p)$, and we suppose that $(G_j : 0 \leq j \leq n)$ knowing $(L_j : 0 \leq j \leq n)$ are independent. We define then by $\mathcal{R}_{t,n}(p, q)$ the law of

$$R_{t,n} = \sum_{j=0}^{n} (-1)^j G_j.$$

Now, we can check that $Y$ satisfies conditions of Lemma 4.3.

**Lemma 4.7.** *The process $Y$, as defined above, satisfies, for any $t \in \mathbb{N}$, $Y_t \overset{\mathrm{d}}{=} S_{N_t} + 2R_{t,N_t}$ where*

- *$N_t$ follows a binomial law with parameters $(t, 2p(1 - p))$,*
- *$S = (S_u : u \in \mathbb{N})$ is a simple random walk on $\mathbb{Z}$,*
- *$R = (R_{t,n})$ is a collection of (independent) random variables of mono-dimensional law: for any $t$, for any $n$, $R_{t,n}$ is distributed according to $\mathcal{R}_{t,n}\left(\frac{(1-p)^2}{p^2+(1-p)^2}, 2p(1 - p)\right)$,*
- *$N$, $S$ and $R$ are mutually independent.*

*Proof.* Let $t \in \mathbb{N}$ and $Y_t$. For any $1 \leq i \leq t$, we denote $\Delta(Y_u) = Y_u - Y_{u-1}$. We remark that $(\Delta(Y_u): 1 \leq u \leq t)$ is a sequence of independent random variables of law

$$\Delta(Y_i) = \begin{cases} -1 & \text{w.p. } p(1-p), \\ 0 & \text{w.p. } p^2, \\ 1 & \text{w.p. } p(1-p), \\ -2 & \text{w.p. } (1-p)^2 \text{ if } Y_i \text{ if odd}, \\ 2 & \text{w.p. } (1-p)^2 \text{ if } Y_i \text{ is even}. \end{cases}$$

As $Y_0 = 0$, we have $Y_t = \sum_{i=1}^{t} \Delta(Y_i)$.

First, we study the set

$$E_1 = \{u: 1 \leq u \leq t \text{ and } |\Delta(Y_u)| = 1\}.$$

We denote by $(u_1, \dots, u_N)$ the elements of $E_1$ sorted in increasing order. $E_1$ is the set of instants for which $Y_{t_i-1}$ and $Y_{t_i}$ have different parities. Cardinal $N$ of $E_1$ follows a binomial law with parameters $(t, 2p(1-p))$ (indeed, at each step of times, $|\Delta(Y_i)| = 1$ with probability $2p(1-p)$). The random variable $S_N = \sum_{i=1}^{N} \Delta(Y_{t_i})$ is distributed as a simple random walk finishing at a random time $N$.

We insist on the fact that, for any $0 \leq j \leq N-1$, every element of $(Y_i: u_j \leq i \leq u_{j+1} - 1)$ (setting $u_0 = 0$) are of the same parity as $j$, in particular $|Y_{u_{j+1}-1} - Y_{u_j}|$ is even. Let, for any $j \geq 0$, $L_j = u_{j+1} - 1 - u_j$. By construction of $Y$, random variables $(L_j: 0 \leq j \leq N)$ follow the law $\mathcal{L}_{t,N}(2p(1-p))$ (see Definition 4.6). We denote $G_j = \frac{|Y_{u_{j+1}-1} - Y_{u_j}|}{2}$. By construction of $Y$,

$$G_j = \sum_{i=1}^{L_j} X_i^{(j)}$$

where $(X_i^{(j)}: 1 \leq i \leq L_j, 0 \leq j \leq N)$ are i.i.d. of law: for any $i, j$,

$$P(X_i^{(j)} = 1) = 1 - P(X_i^{(j)} = 0) = \frac{(1-p)^2}{p^2 + (1-p)^2}.$$

In other words, $G_j$ follows a binomial law with parameters $\left(L_j, \frac{(1-p)^2}{p^2+(1-p)^2}\right)$.

Hence, we effectively obtain that

$$Y_t \stackrel{\mathrm{d}}{=} S_N + 2 \underbrace{\sum_{j=0}^{N} (-1)^j G_j}_{R_{t,N}}$$

where $N$ follows a binomial law with parameters $(t, 2p(1-p))$. $\square$

*Proof of Proposition* 4.1. Lemmas 4.3 and 4.7 and Proposition 4.5 have for immediate consequence Proposition 4.1. □

**4.2. General case: proof of Theorem 1.12.** In this section, we prove only the case $i = 0$. The cases $i \neq 0$ could be proved in a similar way but with some sections more technical. We denote by $H$ the function defined by, for any $(p, r) \in [0, 1]^2$ such that $p + r \neq 1$,

$$H(p, r) = \frac{(1 - 2p)(1 - 2r)}{(1 - (p + r))^2} \tag{58}$$

and we let, for any $n \geq 0$,

$$M_n^{(0)}(X) = \sum_{k=0}^{n} (-1)^k \binom{2n - 1 - k}{n - k} \binom{n}{k} X^k \tag{59}$$

and

$$M_n^{(1)}(X) = \sum_{k=0}^{n} (-1)^k \binom{2n - k}{k, n - k, n - k} X^k. \tag{60}$$

With these notation, when $p + r \neq 1$, $\overline{C}^8_\infty(0, t)$ (in Theorem 31) becomes, for any $t$,

- if $t$ is even,

$$\overline{C}^8_\infty(0, t) = (1 - (p + r))^t M^{(0)}_{\frac{t}{2}}(H(p, r)), \tag{61}$$

- if $t$ is odd,

$$\overline{C}^8_\infty(0, t) = (r - p)(1 - (p + r))^{t-1} M^{(1)}_{\frac{t-1}{2}}(H(p, r)). \tag{62}$$

Hence, the asymptotics of $\overline{C}^8_\infty(0, t)$ as $t \to \infty$ is related to those, as $n \to \infty$, of sequences of polynomials $(M_n^{(0)}(X): n \geq 0)$ and $(M_n^{(1)}(X): n \geq 0)$ when $X = H(p, r)$. To evaluate those asymptotic behaviors, we let the function $m$ that is, for any $K \leq 1$,

$$m(K) = \frac{1}{1 + \sqrt{1 - K}}. \tag{63}$$

**Lemma 4.8.** *For any* $K \leq 1$, *when* $n \to \infty$,

$$M_n^{(0)}(K) = O\Big(\frac{m(K)^{-2n}}{\sqrt{2n}}\Big) \tag{64}$$

*and*

$$M_n^{(1)}(K) = O\Big(\frac{m(K)^{-2n}}{\sqrt{2n}}\Big). \tag{65}$$

*Proof.* By Proposition 4.1 applied to $p = 1 - m(K)$ and $r = 0$ and (61), for any $t = 2n$ even,

$$m(K)^{2n} M_n^{(0)}(H(1-m(K),0)) = \bar{C}^8_\infty(0,2n) = O\Big(\frac{1}{\sqrt{2n}}\Big).$$

As $H(1-m(K),0) = K$, multiplying by $m(K)^{-2n}$,

$$M_n^{(0)}(K) = O\Big(\frac{m(K)^{-2n}}{\sqrt{2n}}\Big).$$

We prove asymptotics of $M_n^{(1)}(K)$ as $n \to \infty$ with same arguments. □

*Proof of Theorem* 1.12. Equations (61) and (62) and Lemma 4.8 imply that, for any $t$,

$$\bar{C}^8_\infty(0,t) = O\Big(\frac{(1-(p+r))^t m(H(p,r))^{-t}}{\sqrt{t}}\Big) = O\Big(\frac{\lambda(p,r)^t}{\sqrt{t}}\Big), \tag{66}$$

where $\lambda(p,r) = \big|\frac{1-(p+r)}{m(H(p,r))}\big|$. Now, let us compute $\lambda(p,r)$.

First, we compute $m(H(p,r))$.

$$\begin{aligned} m(H(p,r)) &= \frac{1}{1+\sqrt{1-\frac{(1-2p)(1-2r)}{(1-(p+r))^2}}} \\ &= \frac{|1-(p+r)|}{|1-(p+r)|+|p-r|} \end{aligned}$$

This last quantity is $\frac{1-(p+r)}{1-2r}$ if $(1-(p+r) \geq 0$ and $p-r \geq 0)$ or $(1-(p+r) \leq 0$ and $p-r \leq 0)$ (i.e. if $(1-(p+r))(p-r) = p-p^2-(r-r^2) \geq 0$), and it is $\frac{1-(p+r)}{1-2p}$ otherwise. Hence,

$$m(H(p,r)) = \begin{cases} \dfrac{1-(p+r)}{1-2p} & \text{if } p(1-p) \leq r(1-r), \\ \dfrac{1-(p+r)}{1-2r} & \text{if } r(1-r) \leq p(1-p). \end{cases}$$

And so,

$$\lambda(p,r) = \begin{cases} |1-2p| & \text{if } p(1-p) \leq r(1-r), \\ |1-2r| & \text{if } r(1-r) \leq p(1-p). \end{cases}$$

As $p + (1-p) = 1 = r + (1-r)$, we can use classical results about areas of rectangles with same perimeter, to conclude that $\lambda(p,r) = \max(|1-2p|, |1-2r|)$. □

**Remark 4.9.** The value of $m(K)$ has been originally obtained by studying the conic equation $\mathcal{H}_K = \{(p,r) \colon H(p,r) = K\}$ that is the union of two lines.

## 5. Vertex models and triangular probabilistics cellular automata

The aim of this section is to prove Proposition 1.6 that said, informally, that far away from the boundary, the 8-vertex model on $\bar{K}_\infty$ with $a + c = b + d$ looks like one with FBC whatever are the boundary conditions. To prove it, we use a coding, introduced by Baxter [2], between colorings of the faces of $\bar{K}_\infty$ and the 8-vertex model configurations $\bar{K}_\infty$, see Section 5.2. We prove that in the case $a + c = b + d$, the corresponding random coloring is the space-time diagram of a (new kind of) probabilistic cellular automata (PCA), called triangular PCA (TPCA). Thanks to that and the fact that this PCA is ergodic, we obtain thus a proof of Proposition 1.6.

Hence, we give first (in Section 5.1) an overview of the theory of PCA and some new results on TPCA. Then, in Section 5.2, we focus our attention on the family of TPCA that are interesting to study the 8-vertex model with $a+c = b+d$ and we prove Proposition 1.6. Section 5.3 is a short section about links between TPCA and the 6-vertex model with $a + c = b$. Finally, Section 5.4 contains some proofs of propositions and theorems stated in Section 5.1, 5.2 and 5.3.

**5.1. Triangular probabilistic cellular automata.** We define first probabilistic cellular automata (PCA) of order 2 whose triangular PCA (TPCA) are special cases. A PCA **A** of order 2 is a quintuple $(E, \mathbb{L}, N_1, N_2, T)$ where

- $E$ is a finite set;
- $\mathbb{L}$ is a lattice;
- $N_1$ is a neighborhood function of $\mathbb{L}$, i.e. there exists a finite subset $I_1$ of $\mathbb{L}$ such that, for any $i \in \mathbb{L}$, $N_1(i) = (i + j : j \in I_1)$, we denote $|N_1|$ the cardinal of $I_1$ that is the one of $N_1(i)$ for any $i \in \mathbb{L}$;
- $N_2$ is another neighborhood function of $\mathbb{L}$ and
- $T$ is a transition matrix from $E^{|N_2|} \times E^{|N_1|}$ to $E$, i.e. for any $(x, y) \in E^{|N_2|} \times E^{|N_1|}$, for any $z \in E$, $T(x, y; z) \geq 0$, and $\sum_{z' \in E} T(x, y; z') = 1$.

From this quintuple, we define a Markov chain $(S_t : t \geq 0)$ of order 2 on $E^{\mathbb{L}}$ in the following way: for any subset $C \subset \mathbb{L}$, for any $(z_i : i \in C) \in E^C$,

$$P((S_{t+2}(i) = z_i : i \in C) \mid S_t = (x_i : i \in \mathbb{L}), S_{t+1} = (y_i : i \in \mathbb{L})) \\ = \prod_{i \in C} T((x_j : j \in N_2(i)), (y_{j'} : j' \in N_1(i)); z_i). \tag{67}$$

The process $S_{t+2}$ is well defined because its law is defined on a compatible way on all cylinders of $E^{\mathbb{L}}$. $(S_t(i) : i \in \mathbb{L}, t \geq 0)$ is called *space-time diagram* of **A**.

An other way to see PCA of order 2 is to consider them as a deterministic map from $\mathcal{M}(E^{\mathbb{L}} \times E^{\mathbb{L}})$ (the set of probability measure on $E^{\mathbb{L}} \times E^{\mathbb{L}}$) to $\mathcal{M}(E^{\mathbb{L}} \times E^{\mathbb{L}})$.

Let $\mathbf{A}$ be a PCA $(E, \mathbb{L}, N_1, N_2, T)$. Let $\mu \in \mathcal{M}(E^{\mathbb{L}} \times E^{\mathbb{L}})$ and $(S_{t_0}, S_{t_0+1}) \sim \mu$. We denote by $\nu$ the law of $(S_{t_0+1}, S_{t_0+2})$ where $S_{t_0+2}$ is the image of $(S_{t_0}, S_{t_0+1})$ by $\mathbf{A}$. Then, for any subset $C \in \mathbb{L}$ and any $(y, z) \in E^{\mathbb{L}} \times E^{\mathbb{L}}$,

$$\begin{aligned}
&\nu((y_i : i \in N_1(C)), (z_i : i \in C)) \\
&\quad = \sum_{(x_i : i \in N_2(C)) \in E^{N_2(C)}} \mu((x_i : i \in N_2(C)), (y_i \in N_1(C))) \\
&\qquad \prod_{i \in C} T((x_j : j \in N_2(i)), (y_j : j \in N_1(i)); z_i)
\end{aligned}$$

where

$$N_k(C) = \bigcup_{i \in C} N_k(i) \quad \text{for any } k \in \{1, 2\}.$$

We denote by $\Phi_{\mathbf{A}}$ the function that maps $\mu$ to $\nu = \Phi_{\mathbf{A}}(\mu)$. We say that $\mu$ is an *invariant probability measure* of $\mathbf{A}$ if $\mu = \Phi_{\mathbf{A}}(\mu)$.

In the following, we consider only the cases where $\mathbb{L} = \mathbb{Z}$, $N_1(i) = (i, i+1)$ and $N_2(i) = (i+1)$. Such PCA of order 2 are called, in this article, triangular probabilistic cellular automata (TPCA). The name comes from the fact that their space-time diagrams are triangular lattices (see Figure 11). To simplify reading, transitions $T((x_{i+1}), (y_i, y_{i+1}); z_i)$ of TPCA are denoted now $T(y_i, x_i, y_{i+1}; z_i)$.

Before seeing new results on TPCA, we recall some theorems about "classical" PCA. The "classical" PCA, considered here, are PCA of order 2 for which $N_1(i) = (i, i+1)$ and $N_2(i) = \emptyset$. We will call them *square PCA* (SPCA) in the following because their space-time diagram are homeomorphic to $\mathbb{Z}^2$ (see Figure 11). To simplify reading, transitions $T(\emptyset, (x_i, x_{i+1}); y_i)$ of SPCA are denoted in the following $T(x_i, x_{i+1}; y_i)$.

Cellular automata and "classical" PCA have been studied since 1940s. For more information on PCA, we refer the interested reader to the recent survey of Mairesse and Marcovici [15]. In the present work, we focus our attention on results on SPCA whose one of its invariant probability measures is a Markovian distribution [3, 24, 23, 10, 5, 16, 8, 7]. In particular, we need to recall Theorem 2.6 of [8] that characterizes SPCA whose one of its invariant probability measures is a $(D, U)$-HZMC (Horizontal Zigzag Markov Chain).

A law $\mu$ on $E^{\mathbb{Z}} \times E^{\mathbb{Z}}$ is a $(D, U)$-HZMC distribution if there exists a pair $(D, U)$ of stochastic matrices from $E$ to $E$ and a family $(\rho_i : i \in \mathbb{Z})$ of probability measures on $E$ such that, for any $k_1, k_2 \in \mathbb{Z}$, $k_1 < k_2$, for any $(x_i : k_1 \leq i \leq k_2)$, $(y_i : k_1 \leq i \leq k_2 - 1)$,

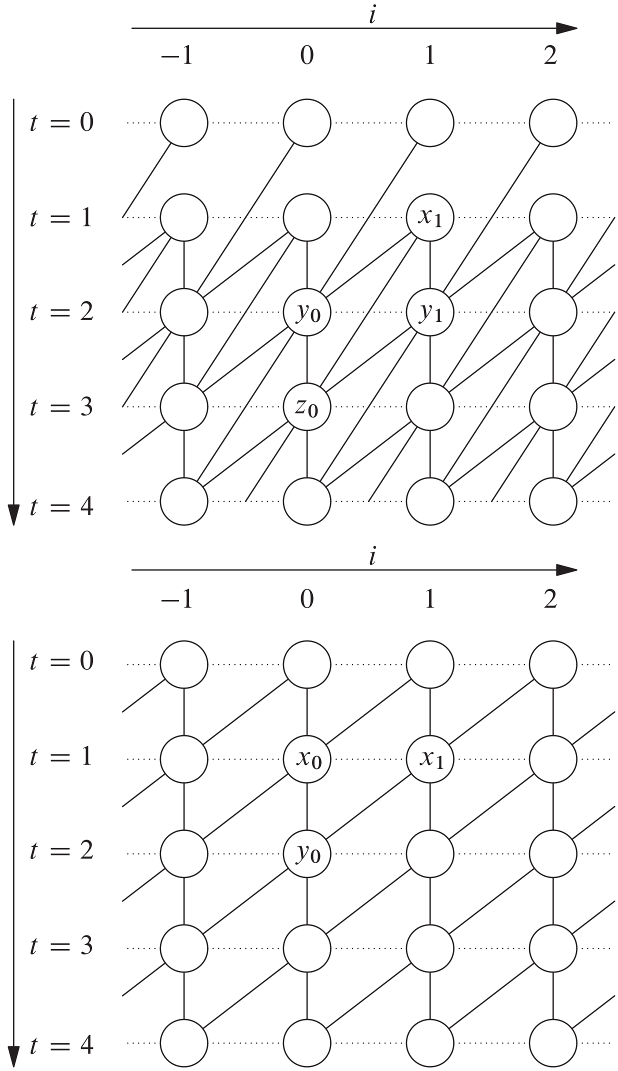



Figure 11. Top: (empty) space-time diagram of a TPCA. Bottom: (empty) space-time diagram of a SPCA.

$$\mu((x_i : k_1 \leq i \leq k_2), (y_i : k_1 \leq i \leq k_2 - 1)) = \rho_{k_1}(x_{k_1}) \prod_{i=k_1}^{k_2-1} D(x_j; y_j) U(y_j; x_{j+1}) \tag{68}$$

and, for any $i \in \mathbb{Z}$ and $x_{i+1} \in E$,

$$\rho_{i+1}(x_{i+1}) = \sum_{x_i \in E} \rho_i(x_i) \sum_{y_i \in E} D(x_i; y_i) U(y_i; x_{i+1}). \tag{69}$$

In other words, a HZMC distribution is a Markovian distribution on states of two consecutive lines crossed from bottom to top and left to right (see Figure 12). In the following, we denote $(x, y) \sim \xi_{(D,U)}$ if $(x, y)$ is distributed according to a $(D, U)$-HZMC distribution.

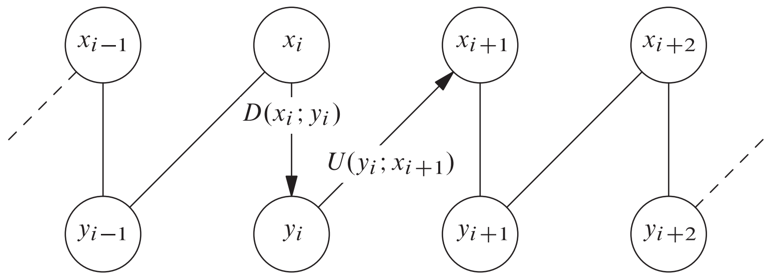



Figure 12. Representation of a horizontal zigzag Markov chain (HZMC).

Now, we define some quantities needed to state Theorem 2.6 of [8]. Let $T$ be any stochastic Markov kernel from $E^2$ to $E$ with positive coefficients. Let $\nu = (\nu(x) : x \in E)$ be the stochastic (i.e. normalized such that $\sum_{x\in E} \nu(x) = 1$) left eigenvector associated to the eigenvalue 1 of the following stochastic matrix

$$(T(x, x; y) : x \in E, y \in E)$$

(this eigenvector is unique due to the Perron–Frobenius theorem) and $\gamma$ be the stochastic left eigenvector of the matrix

$$\left(\nu(y)\frac{T(y, y; 0)}{T(y, x; 0)} : x \in E, y \in E\right)$$

associated with $\lambda$, its maximal eigenvalue. In this case, $\gamma$ is solution of

$$\sum_{x\in E} \frac{\gamma(x)}{T(y, x; 0)} = \lambda \frac{\gamma(y)}{T(y, y; 0)\nu(y)}. \tag{70}$$

Define further for any $\eta = (\eta(x) : x \in E) \in \mathcal{M}(E)$ with full support, the transition matrices $D^\eta$ and $U^\eta$ from $E$ to $E$:

$$D^\eta(x; y) = \frac{\sum_{x'\in E} \eta(x')\frac{T(x,x';y)}{T(x,x';0)}}{\sum_{x''\in E} \frac{\eta(x'')}{T(x,x'';0)}} \text{ and } U^\eta(y; x') = \frac{\eta(x')\frac{T(0,x';y)}{T(0,x';0)}}{\sum_{x''\in E} \eta(x'')\frac{T(0,x'';y)}{T(0,x'';0)}}. \tag{71}$$

**Theorem 5.1** (Theorem 2.6 of [8]). *Let* $\mathbf{A}$ *be a SPCA with finite alphabet* $E = \{0, \ldots, \kappa\}$ *and transition matrix* $T$ *such that, for any* $x_0, x_1, y_0 \in E$, $T(x_0, x_1; y_0) > 0$. *One of the invariant probability measures of* $\mathbf{A}$ *is a HZMC distribution if and only if* $T$ *satisfies the two following conditions*:

**Condition 1.** *for any* $x, x', y \in E$,

$$T(x, x'; y)T(x, 0; 0)T(0, x'; 0)T(0, 0; y) \\ = T(0, 0; 0)T(x, x'; 0)T(0, x'; y)T(x, 0; y);$$

**Condition 2.** *the equality* $D^\gamma U^\gamma = U^\gamma D^\gamma$ *holds (for* $\gamma$ *as defined in* (70) *and* $(D^\gamma, U^\gamma)$ *in* (71)*).*

*In this case,* $(D^\gamma, U^\gamma)$*-HZMC distribution is invariant by* $\mathbf{A}$.

We can present, now, two new theorems on PCA that characterize TPCA whose one of its invariant probability measures is a $(D, U)$-HZMC. We establish these characterizations in two particular cases. The first case is when $D = U$.

**Theorem 5.2.** *Let* $\mathbf{A}$ *be a TPCA on* $E$ *a finite alphabet of transition matrix* $T = (T(y, x, y'; z) \colon y, x, y', z \in E)$ *with positive rate (i.e.* $T(y, x, y'; z) > 0$ *for any* $y, x, y', z \in E$*). For any* $y, y' \in E$*, we denote* $(\widetilde{T}(y, y'; x) \colon x \in E)$ *the unique left stochastic eigenvector (associated to eigenvalue 1) of the stochastic matrix* $(T(y, x, y'; z) \colon x, z \in E)$*. One of the invariant probability measures of* $\mathbf{A}$ *is a* $(D, D)$*-HZMC distribution if and only if the SPCA* $\widetilde{\mathbf{A}}$ *on* $E$ *with transition matrix* $\widetilde{T} = (\widetilde{T}(y, y'; z) \colon y, y', z \in E)$ *satisfies Condition 1 and Condition 2 of Theorem 5.1 with* $D^\gamma = U^\gamma$*. In this case,* $(D^\gamma, D^\gamma)$*-HZMC distribution is invariant by* $\mathbf{A}$.

The second case is when $E$ is of size 2.

**Theorem 5.3.** *Let* $\mathbf{A}$ *be a TPCA on* $E = \{0, 1\}$ *of transition matrix* $T = (T(y, x, y'; z) \colon y, x, y', z \in E)$ *with positive rate. For any* $y, y' \in E$*, we denote* $(\widetilde{T}(y, y'; x) \colon x \in E)$ *the left eigenvector (associated to eigenvalue 1) of* $(\sum_u T(y', x, y; u)T(y, u, y'; z) \colon x, z \in E)$*. One of the invariant probability measures of* $\mathbf{A}$ *is a* $(D, U)$*-HZMC distribution if and only if* $\widetilde{T}$ *satisfies Condition 1 and*

**Condition 3.** *for the pair* $(D^\gamma, U^\gamma)$ *founded by application of Theorem 5.1 to SPCA* $\widetilde{\mathbf{A}}$ *on E with transition matrix* $\widetilde{T}$*, we have, for any* $y, y', z \in \{0, 1\}$,

$$D^\gamma(y; z)U^\gamma(z; y') = \sum_{x \in \{0,1\}} U^\gamma(y; x)D^\gamma(x; y')T(y, x, y'; z).$$

*In this case,* $(D^\gamma, U^\gamma)$*-HZMC distribution is an invariant probability measure of* $\mathbf{A}$.

Proofs of these two theorems are done in Section 5.4.2 and 5.4.3. These two theorems applied to two particular TPCA, $\mathbf{A_8}$ and $\mathbf{A_6}$ defined in Sections 5.2 and 5.3, give another way to prove Propositions 1.5 and 1.7 (see Section 5.4.4).

**5.2. TPCA $\mathbf{A_8}$ and 8-vertex models.** Now, we consider a family of TPCA related to the 8-vertex model when $a + c = b + d$. $\mathbf{A_8}$ is a TPCA with alphabet $E = \{0, 1\}$ and transition matrix $T$ such that, for any $k \in \{0, 1\}$,

- $T(k, k, k; k) = T(k, 1 - k, k; 1 - k) = r$,
- $T(k, k, k; 1 - k) = T(k, 1 - k, k, k) = 1 - r$,
- $T(k, 1 - k, 1 - k; k) = T(k, k, 1 - k; 1 - k) = p$,
- $T(k, 1 - k, 1 - k; 1 - k) = T(k, k, 1 - k; k) = 1 - p$.

To show their links with vertex models, we define first $\overline{K}_\infty$'s faces and coloring of $\overline{K}_\infty$. We call internal faces of $\overline{K}_\infty$ any square whose vertices are $\{(i, t), (i - \frac{1}{2}, t - \frac{1}{2}), (i, t - 1), (i + \frac{1}{2}, t - \frac{1}{2})\}$ for any $(i, t) \in \overline{V}_\infty$ such that $t \neq 0$; such a face is labeled by $(i - t, 2t)$. And we call external faces any triangle whose vertices are $\{(i - \frac{1}{2}, -\frac{1}{2}), (i, 0), (i + \frac{1}{2}, -\frac{1}{2})\}$ for any $i \in \mathbb{Z}$; such a face is labeled by $(i, 0)$. Set of (internal and external) faces of $\overline{K}_\infty$ is denoted $\overline{F}_\infty$. A 2-coloring of $\overline{K}_\infty$ is any function $C$ from $\overline{F}_\infty$ to $\{0, 1\}$. The set of 2-colorings is denoted $\mathcal{C}_2$.

Now we can remark that any realization of the space-time diagram of $\mathbf{A_8}$ is a 2-coloring of $\overline{K}_\infty$ (see Figure 13). Baxter[2, Section 10.2] presents a function, denoted here $\Theta_8$, from $\mathcal{C}_2$ to $\overline{\Omega}^8_\infty$. This function is the following: starting with any $C \in \mathcal{C}_2$, we obtain an orientation $O = \Theta_8(C) \in \overline{\Omega}^8_\infty$ by the following rule: take any edge $(i, t)$ (this edge is adjacent to 2 faces $f$ and $f'$), the orientation $e(i, t)$ is

$$e(i, t) = \mathbf{1}_{C(f) = C(f')}. \tag{72}$$

Conversely, starting with an orientation $O \in \overline{K}_\infty$, we can obtain two distinct 2-colorings $C$ and $C' \in \mathcal{C}_2$ ($\{C, C'\} = \Theta_8^{-1}(\{O\})$) by this way: first, color any face $f$ by any color 0 or 1, then color adjacent faces to the previous one respecting (72), and make it recursively to color any face. Two distinct 2-colorings $C$ and $C'$, obtained from the same orientation $O$, satisfy the following property: for any face $f$, $C(f) \neq C'(f)$. See Figure 13 as an example of $\Theta_8$.

First, note that, for any $t$, knowing $C_{|(F_t, F_{t+1})}$, the coloring of the set of faces $\{(i, t') : i \in \mathbb{Z}, t' \in \{t, t + 1\}\}$, is enough to know the orientations of $(e(i, t) : i \in \mathbb{Z}) = \Theta_8(C_{|(F_t, F_{t+1})})$. Hence, we can define $\mu_t = \Theta_8(\nu_t)$, a law on $(e(i, t) : i \in \mathbb{Z})$ according to $\nu_t$ a law on 2 coloring faces of $\{(i, t') : i \in \mathbb{Z}, t' \in \{t, t + 1\}\}$ by, for any $n \in \mathbb{N}$, for any $e_n = (e_{i,t} : i \in [\![-n, n]\!] = [-n, n] \cap \mathbb{Z})$,

$$\mu_t(e_n) = P((e(i, t) = e_{i,t} : i \in [\![-n, n]\!])) = \sum_{C \in \{C_1, C_2\} = \Theta_8^{-1}(e_n)} \nu_t(C) \tag{73}$$

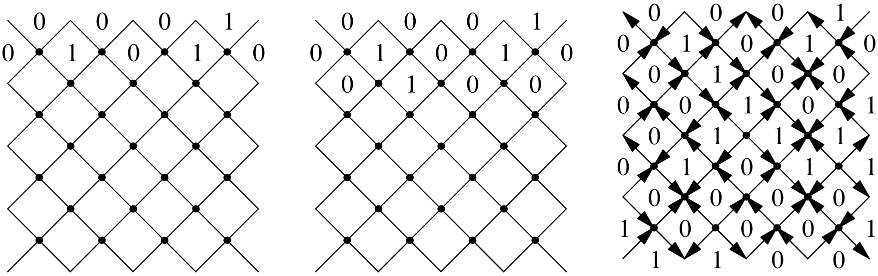



Figure 13. One realization of the space time diagram of the TPCA $\mathbf{A_8}$ and its associated 8-vertex model configuration.

Now, we can show the reason of our choice for $\mathbf{A_8}$.

**Lemma 5.4.** *Let $\nu_0$ be any probability measure on $\{0,1\}^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}}$ and let $C$ be the space-time diagram of $\mathbf{A_8}$ such that $C_{|(F_0,F_1)} \sim \nu_0$, then $\Theta_8(C) \sim \mathcal{L}(\Theta_8(\nu_0); p, r)$.*

*Proof.* The proof of this lemma is based on the fact that images by $\Theta_8$ of initial laws and transitions of $\mathbf{A_8}$ are those that define $\mathcal{L}(\mu; p, r)$ in Definition 1.2. □

Hence, $\mathbf{A_8}$ is related to laws $\mathcal{L}(\mu; p, r)$ and thus to the 8-vertex model. Now, the study of invariant Markovian laws of $\mathbf{A_8}$ give us a unique element.

**Proposition 5.5.** *For any $r \in (0,1)$ and $p \in (0,1)$. The set of invariant HZMC of $\mathbf{A_8}$ has a unique element that is the $(D,U)$-HZMC whose kernels $D$ and $U$ are such that $D = U$ and, for any $i, j \in \{0,1\}$, $D(i; j) = \frac{1}{2}$.*

The proof of this proposition is done in Section 5.4.4. Proposition 1.5 is, then, an immediate consequence of Lemma 5.4 and of this proposition. In addition,

**Lemma 5.6.** *For any $r \in (0,1)$ and $p \in (0,1)$, $\mathbf{A_8}$ is ergodic: for any initial law $\nu_0$, let $C$ be the space-time diagram of $\mathbf{A_8}$ such that $C_{|(F_0,F_1)} \sim \nu_0$ and denote $\nu_t$ the law of $C_{|(F_t,F_{t+1})}$, then $\nu_t \to \mathrm{PM}\left(\frac{1}{2}\right)$ as $t \to \infty$.*

The proof of this lemma is done in a more general context in an incoming paper on triangular probabilistic cellular automata [9, Theorem 8]. The idea of the proof is close to the one in [24]. This lemma permits to prove Proposition 1.6.

*Proof of Proposition* 1.6. Let $\mu$ be any law on $\{0,1\}^{\mathbb{Z}}$. Now, we have the choice for our initial law $\nu_0$ on coloring. We choose here the one that is symmetric: for any $n$, for any $C = (c_i : i \in [\![-2n-1, 2n]\!]) \in \{0,1\}^{[\![-n,n]\!]\times[\![-n-1,n]\!]}$,

$$\nu_0(C) = \nu_0((1-c_i : i \in [\![-2n-1, 2n]\!])) = \frac{1}{2}\mu(\Theta_8(C)). \tag{74}$$

Now, by Lemma 5.6, $\nu_t \to \mathrm{PM}\left(\frac{1}{2}\right)$. Thus, $\mu_t = \Theta_8(\nu_t) \to \mathrm{PM}\left(\frac{1}{2}\right)$, that is the law of $(e(i,t) : i \in \mathbb{Z})$. □

**5.3. TPCA $\mathbf{A_6}$ and 6-vertex models.** In this section, we show relations between the 6-vertex model when $a + c = b$ and TPCA $\mathbf{A_6}$. $\mathbf{A_6}$ is a TPCA with alphabet $E = \{0,1,2\}$ and transition matrix $T$ such that, for any $i \in \{0,1,2\}$,

- $T(i, i+1, i+2; i+1) = 1$,
- $T(i, i+1, i; i+2) = p$,
- $T(i, i+1, i; i+1) = 1 - p$

where additions on $E$ are done modulo 3.

Links between $\mathbf{A_6}$ and the 6-vertex model are similar to the ones between $\mathbf{A_8}$ and the 8-vertex model, instead of that 2-coloring is replaced by proper 3-coloring. A proper 3-coloring of $\overline{K}_\infty$ is any function $C$ from $\overline{F}_\infty$ to $\{0,1,2\}$ such that if two different faces $f, f'$ have a common edge then $C(f) \neq C(f')$. The set of proper 3-colorings is denoted $\mathcal{C}_3$.

We can remark that if we start iterations of $\mathbf{A_6}$ with an initial state $(S_0, S_1)$ such that, for any $i \in \mathbb{Z}$, $S_0(i) \neq S_1(i)$ and $S_1(i) \neq S_0(i+1)$ a.s., then the same condition is satisfied for any $t \geq 0$, i.e., for any $t \in \mathbb{N}$, for any $i \in \mathbb{Z}$, $S_t(i) \neq S_{t+1}(i)$ and $S_{t+1}(i) \neq S_t(i+1)$ a.s. Hence, a space-time diagram realization of $\mathbf{A_6}$ is a proper 3-coloring of $\overline{K}_\infty$. Moreover, there exists a function, denoted here $\Theta_6$, between $\mathcal{C}_3$ and $\overline{\Omega}^6_\infty$ [2, Section 8.13]. This function is obtained as follows. let $C \in \mathcal{C}_3$, take any edge $(i,t)$ of $\overline{K}_\infty$, edge $(i,t)$ is oriented such that if we look the oriented edge in front of us oriented to the top, then the value of the right face of the edge is equal (modulo 3) to the value of the left face $+1$ (see Figure 14). Conversely, starting with an orientation $O \in \overline{\Omega}^6_\infty$, we get three distinct proper 3-colorings $\{C, C', C''\} = \Theta_6^{-1}(\{O\})$. These three distinct 3-colorings $C$, $C'$ and $C''$ satisfy: for any face $f \in \overline{F}_\infty$, $\{C(f), C'(f), C''(f)\} = \{0,1,2\}$.

In a similar way that has been done in (73), we can define $\Theta_6$ as a function on measures of $\{0,1,2\}^{\mathbb{Z}}$.

**Lemma 5.7.** *Let $\mu$ be any measure on $\{0,1,2\}^{\mathbb{Z}} \times \{0,1,2\}^{\mathbb{Z}}$ such that if $(S_0, S_1) \sim \mu$ then for any $i \in \mathbb{Z}$, $S_0(i) \neq S_1(i)$ and $S_1(i) \neq S_0(i+1)$ a.s. Let $C$ be the space-time diagram of $\mathbf{A_6}$ such that $(S_0, S_1) \sim \mu$, then $\Theta_6(C) \sim \mathcal{L}(\Theta_6(\mu); p, 1)$.*

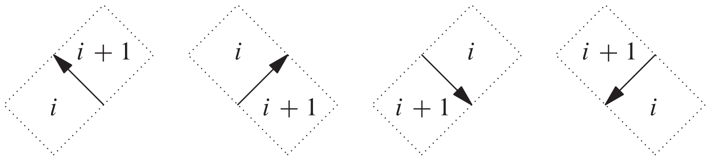



Figure 14. The relations observed by $\Theta_6$ permitting to go from a proper 3-coloring of $\bar{K}_\infty$ to a configuration of the 6-vertex model. Here, $i$ is any element of $\{0, 1, 2\}$ and the sum $i + 1$ is taken modulo 3.

*Proof.* Images by $\Theta_6$ of initial laws and transition of $\mathbf{A_6}$ are those that define $\mathcal{L}(\Theta_6(\mu); p, 1)$. □

An interesting property of $\mathbf{A_6}$ is described in the following theorem.

**Proposition 5.8.** *For any $p \in [0, 1]$. The set of invariant HZMC of $\mathbf{A_6}$ contains the set of $(D, U)$-HZMC whose kernels $D$ and $U$ are such that $D = U$ and, for any $i \in \{0, 1, 2\}$, $D(i; i + 1 \mod 3) = q$ and $D(i; i - 1 \mod 3) = 1 - q$ for any $q \in [0, 1]$.*

This property associated to Lemma 5.7 permits to get an alternative proof of Proposition 1.7. The proof of Proposition 5.8 is done in Section 5.4.4.

## 5.4. Proofs of previous results on TPCA

### 5.4.1. Preliminary results on TPCA and invariant HZMC distributions.

First of all, we recall necessary and sufficient conditions for a $(D, U)$-HZMC to be an invariant probability measure of a SPCA.

**Proposition 5.9** (Proposition 1.2 of [8]). *Let $E$ be a finite set. Let $\mathbf{A}$ be a PCA with positive rate and transition matrix $T$ and $(D, U)$ be two transition matrices from $E$ to $E$. The $(D, U)$-HZMC distribution is an invariant probability measure of $\mathbf{A}$ if and only if the two following conditions hold:*

**Condition 4.** *for any $x, x', y \in E$,*

$$T(x, x'; y) = \frac{D(x; y)U(y; x')}{(DU)(x; x')} \tag{75}$$

**Condition 5.** *we have*

$$DU = UD. \tag{76}$$

This proposition is weaker than Theorem 5.1 in the sense that Condition 4 and Condition 5 hold both on $T$ and $(D, U)$ and not just only on $T$. The first step to prove Theorems 5.2 and 5.3 is to generalize this proposition to TPCA.

**Lemma 5.10.** *Let $E$ be a finite set. Let $\mathbf{A}$ be a TPCA of transition matrix $T$ with positive rate and let $D$ and $U$ be two transition matrices from $E$ to $E$. The $(D, U)$-HZMC distribution is an invariant probability measure of $\mathbf{A}$ if and only if*

**Condition 6.** *for any $y, y', z \in E$,*

$$D(y;z)U(z;y') = \sum_{x \in E} U(y;x)D(x;y')T(y,x,y';z). \tag{77}$$

*Proof.* Let $\mathbf{A}$ be a TPCA of transition matrix $T$ with positive rate and $(D, U)$ two transition matrices from $E$ to $E$.

• Suppose that the $(D, U)$-HZMC distribution is an invariant probability measure of $\mathbf{A}$. Suppose that a pair of lines $(x_0, x_1) \sim \xi_{(D,U)}$, then $(x_1, x_2) \sim \xi_{(D,U)}$ where $x_2$ is the image of $(x_0, x_1)$ by $\mathbf{A}$. Now, for any $a, c, d \in E$, we compute $P(x_1(0) = a, x_2(0) = d, x_1(1) = c | x_1(0) = a)$. On one hand, the lines $(x_1, x_2) \sim \xi_{(D,U)}$, so

$$P(x_1(0) = a, x_2(0) = d, x_1(1) = c | x_1(0) = a) = D(a;d)U(d;c) \tag{78}$$

and, on the other hand, the pair $(x_0, x_1) \sim \xi_{(D,U)}$ and $x_2$ is their image by $\mathbf{A}$, so

$$\begin{aligned}
&P(x_1(0) = a, x_2(0) = d, x_1(1) = c | x_1(0) = a) \\
&= \sum_{b \in E} P(x_1(0) = a, x_0(1) = b, x_2(0) = d, x_1(1) = c | x_1(0) = a) \\
&= \sum_{b \in E} P(x_1(0) = a, x_0(1) = b, x_2(0) = d | x_1(0) = a) \\
&\qquad P(x_1(1) = c | x_1(0) = a, x_0(1) = b, x_1(1) = c) \\
&= \sum_{b \in E} U(a;b)D(b;c)T(a,b,c;d).
\end{aligned} \tag{79}$$

By (78) and (79), we finally obtain that Condition 6 is necessary.

• Conversely, suppose that Condition 6 holds, and take a pair of lines $(x_0, x_1) \sim \xi_{(D,U)}$ and $x_2$ their image by $\mathbf{A}$. Then, for any $k_1, k_2 \in \mathbb{Z}$, $k_1 < k_2$ for any $(b_i : k_1 \leq i \leq k_2) \in E^{k_2-k_1+1}$ and $(c_i : k_1 \leq i \leq k_2-1) \in E^{k_2-k_1}$,

$$
\begin{aligned}
P(&(x_1(i) = b_i : k_1 \leq i \leq k_2), (x_2(i) = c_i : k_1 \leq i \leq k_2 - 1)) \\
&= \sum_{a_i \in E : k_1 \leq i \leq k_2+1} \rho_{k_1}(a_{k_1}) \left( \prod_{i=k_1}^{k_2} D(a_i; b_i) U(b_i; a_{i+1}) \right) \\
&\qquad \left( \prod_{i=k_1}^{k_2-1} T(b_i, a_{i+1}, b_{i+1}; c_i) \right) \\
&= \left( \sum_{a_{k_1} \in E} \rho_{k_1}(a_{k_1}) D(a_{k_1}; b_{k_1}) \right) \\
&\qquad \prod_{i=k_1}^{k_2-1} \sum_{a_{i+1} \in E} U(b_i; a_{i+1}) D(a_{i+1}; b_{i+1}) T(b_i, a_{i+1}, b_{i+1}; c_i) \\
&= \rho_{k_1}(b_{k_1}) \prod_{i=k_1}^{k_2-1} D(b_i; c_i) U(c_i; b_{i+1})
\end{aligned}
\tag{80}
$$

Then, the pair $(x_1, x_2) \sim \xi_{(D,U)}$. □

**Remark 5.11.** When the transition matrix has not positive rates, Condition 6 implies always that the $(D, U)$-HZMC is an invariant probability measure of $\mathbf{A}$, but converse is not true because Condition 6 can hold on a subset of $E$, but not $E$ entirely.

We continue proving Theorems 5.2 and 5.3 by seeing that, in Proposition 5.9, for any $(D, U)$-HZMC, there exists a unique SPCA $\mathbf{A}^{\mathbf{S}}$ that lets the $(D, U)$-HZMC invariant, its transition matrix $T^S$ is, for any $y, y', z \in E$,

$$T^S(y, y'; z) = \frac{D(y; z) U(z; y')}{(DU)(y; y')}. \tag{81}$$

For the same reason, there exists a unique SPCA $\mathbf{A}^R$ that lets the $(U, D)$-HZMC invariant, its transition matrix $T^R$ is, for any $y, y', x \in E$,

$$T^R(y, y'; x) = \frac{U(y; x) D(x; y')}{(DU)(y; y')}. \tag{82}$$

Then, Condition 6 is equivalent, dividing by $(DU)(y; y')$ (not equal to zero in the positive rates cases), to

**Condition 7.** *for any* $y, y', z$,

$$T^S(y, y'; z) = \sum_{x \in E} T^R(y, y'; x) T(y, x, y'; z). \tag{83}$$

**Corollary 5.12.** *Let $E$ be a finite set. Let $\mathbf{A}$ be a TPCA of transition matrix $T$ with positive rates and let $D$ and $U$ be two transition matrices from $E$ to $E$. The $(D, U)$-HZMC is an invariant probability measure of $\mathbf{A}$ if and only if Condition 7 is satisfied with $T^S$ the transition matrix of the unique SPCA $\mathbf{A^S}$ that lets the $(D, U)$-HZMC invariant and $T^R$ the transition matrix of the unique SPCA $\mathbf{A^R}$ that lets the $(U, D)$-HZMC invariant.*

The main idea to prove Theorems 5.2 and 5.3 is to find, for a fixed transition matrix $T$ from $E^3$ to $E$, all the pair of transition matrices $(T^S, T^R)$ from $E^2$ to $E$ such that Condition 7 is satisfied, and then verify if $(T^S, T^R)$ satisfies (or not) the other wanted properties: conservation of a $(D, U)$-HZMC and of a $(U, D)$-HZMC thanks to Theorem 5.1. In the particular cases where $D = U$ or $E = \{0, 1\}$, we are able to find a unique possible pair of $(T^S, T^R)$ related to $T$ that can satisfy Condition 7. All other cases are open problems.

**5.4.2. Proof of Theorem 5.2.** Let $T$ be a transition matrix from $E^3$ to $E$ of a TPCA with positive rate. We denote, for any $y, y'$, $(\widetilde{T}(y, y'; x) : x \in E)$, the unique left eigenvector related to the eigenvalue 1 of $(T(y, x, y'; z) : x \in E, z \in E)$ normalized such that $\sum_{x \in E} \widetilde{T}(y, y'; x) = 1$, i.e., for any $y, y'$,

$$\widetilde{T}(y, y'; z) = \sum_{x \in E} \widetilde{T}(y, y'; x) T(y, x, y'; z) \tag{84}$$

and $\widetilde{T}$ is a transition matrix from $E^2$ to $E$, this eigenvector exists due to Perron–Frobenius theorem. Moreover, we suppose that $\widetilde{T}$ satisfies Condition 1 and Condition 2 of Theorem 5.1 with $D^\eta = U^\eta$, i.e. there exists $D^\eta$ such that the $(D^\eta, D^\eta)$-HZMC distribution is an invariant probability measure of $\widetilde{\mathbf{A}}$, the SPCA with transition matrix $\widetilde{T}$. In this case, we remark that SPCAs that let invariant $(D, U)$-HZMC and $(U, D)$-HZMC are the same, i.e. $T^R = T^S = \widetilde{T}$ in Corollary 5.12 and so (84) imply Condition 7. We finish the proof using Corollary 5.12.

Conversely, if the $(D, D)$-HZMC distribution is an invariant probability measure of $T$, by Lemma 5.10, for any $y, y', z \in E$,

$$D(y; z) D(z; y') = \sum_{x \in E} D(y; x) D(x; y') T(y, x, y'; z),$$

i.e. for any $y, y'$, $(D(y; x) D(x; y') : x \in E)$ is a left eigenvector of $(T(y, x, y'; z) : x \in E, z \in E)$ associated to the eigenvalue 1. By Perron–Frobenius theorem, the eigenspace associated to eigenvalue 1 is of dimension 1. Thus, for any $y, y', x$, $D(y;x)D(x;y') = \lambda_{y,y'}\widetilde{T}(y, y'; x)$. Moreover, as we want that $(\widetilde{T}(y, y'; x) : x \in E)$ is a probability vector, we obtain

$$\widetilde{T}(y, y'; x) = \frac{D(y;x)D(x;y')}{(DD)(y;y')}.$$

Hence, by Proposition 5.10, the $(D, D)$-HZMC distribution is an invariant probability measure of SPCA $\widetilde{\mathbf{A}}$ of transition matrix $\widetilde{T}$. And so, by Theorem 5.1, $\widetilde{\mathbf{A}}$ needs to satisfy Condition 1 and Condition 2 with $D = U$.

**5.4.3. Proof of Theorem 5.3.** In the case $E = \{0, 1\}$, we have the following algebraic property on $T^S$ and $T^R$.

**Lemma 5.13.** *Let $E = \{0, 1\}$ and $T^S$ and $T^R$ be two transition matrices from $E^2$ to $E$ such that the $(D, U)$-HZMC is an invariant probability measure of $\mathbf{A}^{\mathbf{S}}$ with transition matrix $T^S$ and the $(U, D)$-HZMC is an invariant probability measure of $\mathbf{A}^{\mathbf{R}}$ with transition matrix $T^R$. Then, for any $y, y', x \in E$,*

$$T^S(y, y'; x) = T^R(y', y; x). \tag{85}$$

*Proof.* As $\mathbf{A}^{\mathbf{S}}$ lets invariant the $(D, U)$-HZMC distribution, by Proposition 5.9, for any $y, x, y'$, (81) holds and, due to similar reasons, for any $y, x, y'$, (82) holds too.

When $y = y' = x$, by (81) and (82), $T^S(y, y; y) = T^R(y, y; y)$, and, moreover, as $T^S(y, y; .)$ and $T^R(y, y; .)$ are probability measures on $\{0, 1\}$, eq. (85) holds if $y = y'$.

Now, we look the more complicated case $y = 0$ and $y' = 1$ (the case $y = 1$ and $y' = 0$ is similar replacing $S$ by $R$) and $x = 0$ ($x = 1$ will be then immediate because $T^S(0, 1; 0) + T^S(0, 1; 1) = 1 = T^R(1, 0; 0) + T^R(1, 0; 1)$). By Proposition 5.9, $DU = UD$, so $(UD)(0; 0) = (DU)(0; 0)$ that simplifies in $U(0; 1)D(1; 0) = D(0; 1)U(1; 0)$, that implies $U(0; 1)(UD)(1; 0) = U(1; 0)(DU)(0; 1)$ and, finally,

$$T^S(0, 1; 0) = \frac{D(0;0)U(0;1)}{(DU)(0;1)} = \frac{U(1;0)D(0;0)}{(UD)(1;0)} = T^R(1, 0; 0). \qquad \square$$

Now, we can prove Theorem 5.3.

*Proof of Theorem 5.3.* • Suppose that one of the invariant probability measures of TPCA $\mathbf{A}$ of transition matrix $T$ with positive rates is a $(D, U)$-HZMC distribution and denote $(\widetilde{T}(y, y'; x) : x \in E)$ the left eigenvector related to the eigenvalue 1 of $\left(\sum_{k \in E} T(y', x, y; k)T(y, k, y'; z) : x \in E, z \in E\right)$ and such that $\sum_{x \in E} T(y, y'; x) = 1$.

By Lemma 5.10, Condition 6 holds. As $E = \{0, 1\}$, by Lemma 5.13, this condition rewrites: for any $y, y', z$,

$$T^S(y, y'; z) = \sum_{x \in E} T^S(y', y; x) T(y, x, y'; z). \tag{86}$$

Applying this equation twice establishes that, for any $y, y', z \in \{0, 1\}$,

$$\begin{aligned} &\sum_{x \in E} T^S(y, y'; x)\Big(\sum_{u \in E} T(y', x, y; u) T(y, u, y'; z)\Big) \\ &\quad = \sum_{x \in E} T^S(y', y; x) T(y, x, y'; z) = T^S(y, y'; z). \end{aligned} \tag{87}$$

In other words, for any $y, y'$, $(T^S(y, y'; x) : x \in E)$ is a left eigenvector related to eigenvalue 1 of $\big(\sum_{u \in E} T(y', x, y; u) T(y, u, y'; z) : x \in E, z \in E\big)$. So, by Perron–Frobenius theorem, for any $y, x, y'$, $T^S(y, y'; x) = \lambda_{y,y'} \widetilde{T}(y, y'; x)$ with $\lambda_{y,y'} = 1$ because the sum in $x$ is equal to 1 in both sides. Then, the TPCA $\widetilde{\mathbf{A}}$ with transition matrix $\widetilde{T} = T^S$ lets invariant the $(D, U)$-HZMC distribution. Hence, by Theorem 5.1 or results of Belyaev [3], $\widetilde{T}$ satisfies

$$\widetilde{T}(0,0;0)\widetilde{T}(0,0;1)\widetilde{T}(1,0;0)\widetilde{T}(0,1;0) = \widetilde{T}(1,1;1)\widetilde{T}(1,1;0)\widetilde{T}(0,1;1)\widetilde{T}(1,0;1) \tag{88}$$

And so Condition 3 holds by Lemma 5.10.

• Conversely, if $\widetilde{T}$ satisfies Condition 1 (i.e. (88) if $E = \{0, 1\}$), then we apply Theorem 5.1 to find a pair $(D^\eta, U^\eta)$ such that the $(D^\eta, U^\eta)$-HZMC distribution is an invariant probability measure of $\widetilde{\mathbf{A}}$ of transition matrix $\widetilde{T}$. If, moreover, this pair $(D^\eta, U^\eta)$ satisfies Condition 3 then, by Lemma 5.10, the $(D^\eta, U^\eta)$-HZMC is an invariant probability measure of $\mathbf{A}$. □

### 5.4.4. Proofs of Proposition 5.5 and 5.8

*Proof of Proposition* 5.5. To prove Proposition 5.5, we apply Theorem 5.3 to $\mathbf{A_8}$. First, let us compute matrices

$$\Big(\sum_{k \in E} T(y', x, y; k) T(y, k, y'; z) : x \in E, z \in E\Big)$$

for any $y, y'$. We obtain the following four matrices:

| $y \backslash y'$ | 0 | 1 |
|---|---|---|
| 0 | $\begin{pmatrix} p^2 + (1-p)^2 & 2p(1-p) \\ 2p(1-p) & p^2 + (1-p)^2 \end{pmatrix}$ | $\begin{pmatrix} r^2 + (1-r)^2 & 2r(1-r) \\ 2r(1-r) & r^2 + (1-r)^2 \end{pmatrix}$ |
| 1 | $\begin{pmatrix} r^2 + (1-r)^2 & 2r(1-r) \\ 2r(1-r) & r^2 + (1-r)^2 \end{pmatrix}$ | $\begin{pmatrix} p^2 + (1-p)^2 & 2p(1-p) \\ 2p(1-p) & p^2 + (1-p)^2 \end{pmatrix}$ |

Left eigenvectors related to eigenvalue 1 are all equal for these four matrices and their common value is $(\frac{1}{2}\ \frac{1}{2})$. So, we know have to study the SPCA whose transition matrix is, for any $y, x, y' \in \{0, 1\}$, $T(y, y'; x) = \frac{1}{2}$. We observe easily that this SPCA as for unique invariant HZMC distribution, the $(D, U)$-HZMC such that, for any $x, y \in \{0, 1\}$, $D(x; y) = U(x; y) = \frac{1}{2}$. Then, Condition 3 holds for this pair $(D, U)$. We deduce, by Theorem 5.3, that $\mathbf{A_8}$ lets invariant this $(D, U)$-HZMC distribution and, moreover, it is the unique HZMC distribution that is invariant by $\mathbf{A_8}$. □

*Proof of Proposition* 5.8. To prove Proposition 5.8, we check that Condition 6 holds with $T$, the transition matrix of $\mathbf{A_6}$, and for any $(D, U)$ such that $D = U$ and, for any $i \in \mathbb{Z}/3\mathbb{Z}$, $D(i, i + 1) = 1 - D(i, i - 1) = q$. And, then, Remark 5.11 concludes the proof. □

## 6. Conclusion

We have computed the edge correlation function of the 8-vertex model on $\bar{K}_\infty$ with free boundary conditions and $a + c = b + d$ and we have bounded the influence of being not in a free boundary conditions case.

Moreover, as stated in Proposition 1.4, edge correlation function of Theorem 1.10 is the one of the 8-vertex model on $K_N$ with $a + c = b + d$ and free boundary conditions. If, instead of a rotation of an angle $-\frac{\pi}{4}$ to pass from the 8-vertex model on $K_N$ to the 8-vertex model on $\bar{K}_N$, we have done a rotation by an angle $\frac{\pi}{4}$, then we would have obtained the correlation function of the 8-vertex model on $K_N$ with $a + d = b + c$ and free boundary conditions.

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