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Topology of the Configuration Space of Polygons as a Codimension One Submanifold of a Torus

By

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Abstract

We study the topology of polygons with fixed side length in Euclidean plane by Morse theory.

§ 1. Introduction

We study the space of oriented congruence classes of polygons with fixed side length in Euclidean plane E^2. Set

$$X_{n+2}=\left\{(z_1, \cdots, z_{n+2}) \in \mathbf{C}^{n+2}: \sum_{i=1}^{n+2} z_i=0\right\}$$

and

$$N_{n+2}=X_{n+2}-\{(0, \cdots, 0)\}/\mathbf{C}^*=\left\{[z_1, \cdots, z_{n+2}] \in \mathbf{C}P^{n+1}: \sum_{i=1}^{n+2} z_i=0\right\},$$

where $\mathbf{C}^*(=\mathbf{C}-\{0\})$ acts on X diagonally. Then N_{n+2} may be regarded as the space of oriented similarlity classes of $(n+2)$-gons in E^2.

For a positive number r, we set

$$M_{n+2,r}=\left\{[z_1, \cdots, z_{n+2}] \in N_{n+2}: |z_1|=|z_2|=\cdots=|z_n|=|z_{n+2}| \neq 0, |z_{n+1}|=r|z_{n+2}|\right\}.$$

Then by setting $\frac{z_i}{z_{n+2}}=w_i$, we have

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$$M_{n+2,r} \simeq \left\{ (w_1, w_2, \cdots, w_{n+1}, 1) \in (\mathbb{C}^*)^{n+1} : \begin{array}{l} |w_1| = \cdots = |w_n| = 1, \\ w_{n+1} = -(w_1 + \cdots + w_n + 1), \\ |w_{n+1}| = r \end{array} \right\}$$

$$\simeq \{ (w_1, w_2, \cdots, w_n) \in T^n : |w_1 + \cdots + w_n + 1| = r \}$$

where $\simeq$ means diffeomorphic. Hence the space of similarlity classes of $(n+2)$-gons whose $(n+1)$-sides have equal length may be regarded as a torus T^n, if we regard $(n+1)$-gons as a degenerate $(n+2)$-gons.
Set

$$W_{n+2,r} = \{ (w_1, \cdots, w_n) \in T^n : |w_1 + \cdots + w_n + 1| \geq r \},$$

where r is a nonnegative real number. $W_{n+2,r}$ is the configuration space of $(n+2)$-gons whose $(n+1)$-side length are 1 and the other one is greater than or equal to r. We define a smooth function f on T^n as follows :

$$f(e^{ix_1}, \cdots, e^{ix_n}) = -(\cos x_1 + \cdots + \cos x_n + 1)^2 - (\sin x_1 + \cdots + \sin x_n)^2.$$

Then $f^{-1}(-1) = M_{n+2,1}$ may be regarded as the space of oriented congruence classes of $(n+2)$-gons whose length of $(n+2)$-sides are 1. The critical points of f: $T^n \to \mathbb{R}$ are non degenerate except maximum. The value -1 is regular value if n is odd and is critical value if n is even. By using standard Morse theory for f, we have a handlebody decomposition of $W_{n+2,r}$. If $-r^2$ is a regular value of f, $M_{n+2,r} = \partial W_{n+2,r}$ is obtained from a sphere by succesive surgery. We give a cell structure of T^{n+1} by the product complex of $S^1 = e^0 \cup e^1$.
By observing attaching maps of handlebodies, we get the following results.

Theorem 1. *For* $0 < r \leq n+1$, *the space* $W_{n+2,r}$ *is homotopy equivalent to* $(T^{n+1})^k$, *the k-skelton of* T^{n+1}, *where* $k = \left[\frac{n+1-r}{2}\right]$. *If* $r \leq 0$, $W_{n+2,r}$ *is the whole space* T^n *and* $W_{n+2,r} = \emptyset$ *if* $r > n+1$.

By using Theorem 1 and Lefschetz duality, we are able to know that the relative homotopy groups, $\pi_i(W_{n+2,r}, M_{n+2,r}) = 0$ for $i \leq n-k-1$, where k is the integer such that $n-2k-1 < r \leq n-2k+1$. Then we have the following results on the fundamental group of $M_{n,r}$ for $n \geq 4$.

Corollary 2. *Let* $n \geq 4$. *Then*

$$\pi_1(M_{n+2,r}) \simeq \begin{cases} \mathbb{Z}^{n+1} & \text{if } 0 \leq r \leq n-3 \\ F_{n+1} & \text{if } n-3 < r \leq n-1 \\ \{e\} & \text{if } n-1 < r \leq n+1. \end{cases}$$

In case $n=3$, *the inclusion* $i : M_{5,1} \to W_{5,1} \simeq (T^4)^1$ *induces surjection (but not* *isomorphism) on the fundamental groups.*

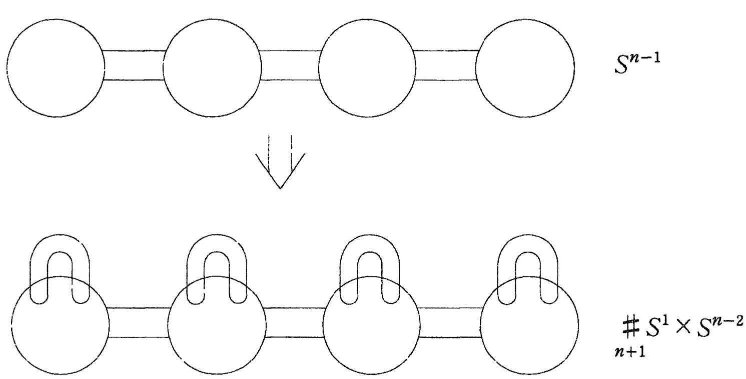
We can also calculate the homology group of $M_{n+2,r}$ easily by Theorem 1. Since manifold $M_{n+2,r}$ ($-r^2$ is regular value of f) is closed orientable codimension 1 submanifold of T^n which bounds $W_{n,r}$ it is a π-manifold and oriented cobordant to zero. The manifold $M_{n+2,r}$ is obtained from a sphere by successive surgeries, in particular we have the following from table I (in Section 2) :

Proposition 3. *For $n \geq 3$, we have*

$$M_{n+2,r} \simeq S^{n-1} \quad \text{if } n-1 < r < n+1$$
$$M_{n+2,r} \underset{n+1}{\#} (S^1 \times S^{n-2}) \text{ if } n-3 < r < n-1,$$

Where $\underset{n+1}{\#}$ denotes the connected sun $(n+1)$-times and $\simeq$ denotes homeomorphic.

In case $n=4$, more general information was obtained by M. Kapovich and J. Milson ([K-M] Theorem 3). In particular, we have $M_{5,1}$ is diffeomorphic to Σ_4, the closed orientable surface of genus 4. ([T-W], [Hav]). The homology group of $M_{n+2,1}$ is also calculated in [K-T-T].



§ 2. Critical Points of f

We study the critical points of the function

$$f(e^{ix_1}, \cdots, e^{ix_n}) = -(\cos x_1 + \cdots + \cos x_n + 1)^2 - (\sin x_1 + \cdots + \sin x_n)^2$$

on T^n. Then

$$\frac{\partial f}{\partial x_i}=2(\cos x_1+\cdots+\cos x_n+1)\sin x_i-2(\sin x_1+\cdots+\sin x_n)\cos x_i$$

and

$$\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}=\sum_{i=1}^{n}\sin x_i .$$

Hence the set of critical points of f are the union of

$$A=\{(e^{ix_1},\cdots,e^{ix_n})\in T^n:\ \sin x_1=\cdots=\sin x_n=0\}$$

and

$$B=\left\{(e^{ix_1},\cdots,e^{ix_n})\in T^n:\ \sum_{i=1}^{n}\cos x_i=-1\quad\text{and}\quad\sum_{i=1}^{n}\sin x_i=0\right\}.$$

On the set B, f attains the maximum value 0 and B is homeomorphic to $M_{n+1,1}\times S^1$. Set

$$S_k=\{(\varepsilon_1,\cdots,\varepsilon_n)\in T^n:\ \varepsilon_i=1\text{ or }-1\text{ and the cardinality of }-1\text{ is }k.\}.$$

Then

$$A\cap B^c\subset\bigcup_{k=0}^{n}S_k\text{ and }A\cup B\supset\bigcup_{k=0}^{n}S_k.$$

Lemma 2.1. *At the critical point* $x_0=(\underbrace{-1,\cdots,-1}_{k},\underbrace{1,\cdots,1}_{n-k})$ $(k=0,1,\cdots,n)$, *the characteristic polynomial of the Hessian (of f) is*:

$$(\lambda-2b)^{n-k-1}(\lambda+2b)^{k-1}(\lambda^2+2n\lambda-4b),$$

where $b=n-2k+1$. *Therefore the index of the critical point* x_0 *is* k *if* $b>0$ *and* $n-k$ *if* $b<0$. (*Note that in the case that* $b=0$ f *attains the maximum value* 0)

Proof. Since

$$\frac{\partial^2 f}{\partial x_i\partial x_j}=\begin{cases}-2\cos(x_i-x_j) & (i\neq j)\\ -2+2(C+1)\cos x_i-2S\sin x_i & (i=j)\end{cases}$$

where $C=\cos x_1+\cdots+\cos x_n$ and $S=\sin x_1+\cdots+\sin x_n$ at the point $x_0=(\underbrace{-1,\cdots,-1}_{k},\underbrace{1,\cdots,1}_{n-k})$,

$$D=\left[\begin{array}{cccc|cccc} \multicolumn{4}{c|}{\overbrace{\hphantom{-2-2(c+1)\quad -2\quad \ddots \quad -2-2(c+1)}}^{k}} & \multicolumn{4}{c}{\overbrace{\hphantom{-2+2(c+1)\quad -2\quad \ddots \quad -2+2(c+1)}}^{n-k}} \\ -2-2(c+1) & & & & & & & \\ -2 & \ddots & -2 & & & 2 & & \\ & & -2-2(c+1) & & & & & \\ & & & & -2+2(c+1) & & & \\ & 2 & & & -2 & \ddots & -2 & \\ & & & & & & -2+2(c+1) & \end{array}\right]$$

$$=2\left[\begin{array}{ccc|ccc} -1-(c+1) & & & & & \\ -1 & \ddots & -1 & & 1 & \\ & & -1-(c+1) & & & \\ & & & -1+(c+1) & & \\ & 1 & & -1 & \ddots & -1 \\ & & & & & -1+(c+1) \end{array}\right]$$

$$=2D_1$$

where $c=n-2k$. We calculate the characteristic polynomial of D_1 instead of D for simlicity.

$$|\lambda I-D_1|=\left|\begin{array}{cccccc} \lambda+1+b & & & & & \\ 1 & \ddots & 1 & & -1 & \\ & & \lambda+1+b & & & \\ & & & \lambda+1-b & & \\ & -1 & & 1 & \ddots & 1 \\ & & & & & \lambda+1-b \end{array}\right|$$

where $b=c+1=n-2k+1$. We substruct the first column from the i-th $(i=1, \cdots, k)$ and add it to the j-th $(j=k+1, \cdots, n)$. Our determinant is equal to

$$\left|\begin{array}{ccccccc} \lambda+1+b & -\lambda-b & \dots & -\lambda-b & \lambda+b & \dots & \lambda+b \\ 1 & \lambda+b & & & & & \\ \vdots & 0 & \ddots & 0 & & 0 & \\ 1 & & & \lambda+b & & & \\ -1 & & & & \lambda-b & & \\ \vdots & & 0 & & 0 & \ddots & 0 \\ -1 & & & & & & \lambda-b \end{array}\right| .$$

We then add the rows of from the second to the k-th to the first row and add $\dfrac{\lambda+b}{\lambda-b}$ times the rows of from the $k+1$-th to n-th to the first row also. This yields

$$|\lambda I-D_1|=\left|\begin{array}{ccccccc}
\lambda+k+b+\frac{\lambda+b}{\lambda-b}(n-k) & 0 & \cdots & 0 & & & \\
1 & \lambda+b & & & & & \\
\vdots & 0 & \ddots & 0 & & 0 & \\
1 & & & \lambda+b & & & \\
-1 & & & & \lambda-b & & \\
\vdots & & 0 & & 0 & \ddots & 0 \\
-1 & & & & & & \lambda-b
\end{array}\right|$$

$$=\left(\lambda+k+b+\frac{\lambda+b}{\lambda-b}(n-k)\right)(\lambda+b)^{k-1}(\lambda-b)^{n-k}$$

$$=(\lambda+b)^{k-1}(\lambda-b)^{n-k-1}(\lambda^2+n\lambda-b^2+(n-2k)b)$$

$$=(\lambda+b)^{k-1}(\lambda-b)^{n-k-1}(\lambda^2+n\lambda-b)$$

where the last equality is obtained by using $b=n-k+1$. Therefore the characteristic polynomial of $D(=2D_1)$ is:

$$(\lambda-2b)^{n-k-1}(\lambda+2b)^{k-1}(\lambda^2+2n\lambda-4b). \quad \square$$

By Lemma 2.1, we have the following table:

Critical value	index	the number of critical points
$-(n+1)^2$	0	1
$-(n-1)^2$	1	${}_{n+1}C_1$
$-(n-3)^2$	2	${}_{n+1}C_2$
$\vdots$	$\vdots$	$\vdots$
$-(n-2k+1)^2$	k	${}_{n+1}C_k$

table I

The number of critical points of index k is the cardinality of the union of S_k and S_{n-k+1}.
We denote the binomial coefficients ${}_{n+1}C_k=\binom{n+1}{k}$ by $P(k)$.

§ 3. The Homotopy Type of $W_{n+2,r}$

In this section, we study the homotopy type of

$$W_{n+2,r}=\{(e^{ix_1},\ \cdots,\ e^{ix_n})\in T^n : f\leq -r^2\}.$$

By Morse theory and Table I, $W_{n+2,r}$ is obtained from $W_{n+2,r+2}$ by attaching $P\ (k)$ handle bodies of k-dimension.

Let $n-3<r<n-1$ then W_r is homotopy equivalent to $D^n\ (\simeq W_{n+2,r+2})$ with $P(1)\ (=n+1)$ 1-cells attached. Hence $W_{n+2,r}$ is homotopy equivalent to $n+$

1 times wedge of S^1, which is homotopy equivalent to 1-skelton of T^{n+1}. $W_{n+2,n-1}$ is also homotopy equivalent to $(T^{n+1})^1$. (See Milnor [M : 1] Remark 3.4., p.20)

In order to study the case $n-5 < r \leq n-3$, we must observe the attaching maps of 2-cells. We examine at the critical point $P=(-1, -1, 1, \cdots, 1)$ of index 2. We can treat similarly at other critical points. Set

$$T^2=\{(e^{ix_1}, e^{ix_2}, 1, \cdots, 1)\in T^n : x_1, x_2\in\mathbb{R}\}$$

and let f_2 be the restriction of f on T^2. Then f_2 attains maximum value $-(n-3)^2$ at P. For a small positive number ε, we set

$$D_\varepsilon^2=\{f_2 \geq -(n-3+\varepsilon)^2\}\subset T^2.$$

Then D^2_ε is diffeomorphic to a closed 2-dimensional disk containing P as an interior point. We show that the attaching sphere of the 2-cell at P may be regarded as ∂D_ε^2.

Around the critical point P, there is a local coordinate $(y_1, \cdots, y_n)$ of T^n with $P=(0, \cdots, 0)$ such that f is expressed as

$$-y_1^2-y_2^2+y_3^2+\cdots+y_n^2-(n-3)^2 .$$

The attaching sphere S_L^1 of a 2-cell in $W_{n+2,r}$ $(n-3 < r < n-1)$ can be regarden as

$$\{(y_1, \cdots, y_n) : y_1^2+y_2^2=\delta^2, y_3=\cdots=y_n=0\}$$

(See [Mi2]). Set

$$C=\{(y_1, \cdots, y_n)\in\mathbb{R}^n : -y_1^2-y_2^2+y_3^2+\cdots+y_n^2 \geq 0\}.$$

Then we see

$$\partial D_\varepsilon^2\subset\mathbb{R}^n-C.$$

The sphere S_L^1 and ∂D_ε^2 both generate the group $\pi_1(\mathbb{R}^n-C)\simeq \pi_1(\mathbb{R}^2-\{(0, 0)\})$ $\simeq \mathbb{Z}$. Hence the two inclusions

$$i_1 : S_L^1 \longrightarrow \mathbb{R}^n - C \quad \text{and}$$
$$i_2 : \partial D_\varepsilon^2 \longrightarrow \mathbb{R}^n - C$$

are homotopic. Hence we have the following.

Lemma 3.1. *The attaching map of 2-cell at P is (free) homotopic to the composition of inclusion*

$$\partial D_\varepsilon^2 \xrightarrow{\iota_1} T^2-\mathring{D}_\varepsilon^2 \xrightarrow{\iota_2} W_{n+2,n-3+\varepsilon}$$

for some small positive number ε.

Set

$$S_j=\{(1,1,\cdots,e^{ix_j},1,\cdots,1)\in T^n:\ x_i\in\mathbb{R}\}\subset W_{n+2,n-1}\ (j=1,\cdots,n)$$

and

$$S_{n+1}=\{(e^{ix},\cdots,e^{ix})\in T^n:\ x\in\mathbb{R}\}\subset W_{n+2,n-1}.$$

Since $W_{n+2,n-1}$ is homotopy equivalent to D^n with $n+1$ 1-cells attached and each 1-cell can be regarded as the arc of S_j containing the critical point of index 1, $W_{n+2,n-1}$ is homotopy equivalent to the wedge of S^1

$$S_1\vee\cdots\vee S_{n+1}.$$

Hence $\pi_1(W_{n+2,n-1})$ is isomorphic to the free group $F(\alpha_1,\cdots,\alpha_{n+1})$ of rank $n+1$, where α_j ($j=1,\cdots,n+1$) is represented by S_j. Let β_j ($j=1,\cdots,n$) be the element of $\pi_1(T^n)$ represented by S_j, Then the inclusion i: $W_{n+2,n-1}\to T^n$ induces a homomorphism

$$i_\#:\ \pi_1(W_{n+2,n-1})\to\pi_1(T^n)$$

such that $i_\#(\alpha_j)=\beta_j$ and $i_\#(\alpha_{n+1})=\sum_{j=1}^n\beta_j$. By Lemma 3.1, the attaching map of 2-cell at the critial point $P=(-1,-1,1,\cdots,1)$ is the composition

$$\partial D_\varepsilon^2\subset T^2-\mathring{D}_\varepsilon^2\overset{j}{\subset}W_{n+2,n-3+\varepsilon}(\sim W_{n+2,n-1}).$$

The inclusion j induces homomorphism

$$\begin{array}{ccc}\pi_1(T^2-\mathring{D}_\varepsilon^2) & \xrightarrow{j_\#} & \pi_1(W_{n+2,n-3+\varepsilon})\\ \| & & \| \\ F(\gamma_1,\gamma_2) & \longrightarrow & F(\alpha_1,\cdots,\alpha_{n+1})\end{array}$$

such that $j_\#(\gamma_1)=\alpha_1$ and $j_\#(\gamma_2)=\alpha_2$, where γ_j ($j=1,2$) is represented by $S_j\subset T^2$. Since the inclusion $\partial D_\varepsilon^2\to T^2-\mathring{D}_\varepsilon^2$ represents the commutater $\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$ in the fundamental group, its image in $(T^{n+1})^2$ is trivial. The manifold $W_{n+2,n-3+\varepsilon}$ is homotopy equivalent to $(T^{n+1})^1$ and it has the same homotopy type as $\cup_{P(2)}(T^2-\mathring{D}_\varepsilon^2)$ $(\subset T^{n+1})$. Since $(T^{n+1})^2=\cup_{P(2)}T^2$ $(\subset T^{n+1})$ the homotopy equivalence h_1: $W_{n+2,n-3+\varepsilon}\to(T^{n+1})^1$ can be extended to a map

$$h_2:\ W_{n+2,n-3-\varepsilon}\longrightarrow(T^{n+1})^2$$

such that the induced map

$$h_{2*}:\ \pi_*(W_{n+2,n-3-\varepsilon},\ W_{n+2,n-3+\varepsilon}) \longrightarrow \pi_*((T^{n+1})^2,\ (T^{n+1})^1)$$

is isomorphic (both groups are isomorphic to $\mathbf{Z}^{P(2)}$). Then from the long exact sequence of homotopy groups and five Lemma, we obtain that h_2 is homotopy equivalence. We continue the above arguments for $n-2k-1 < r \leq n-2k+1$ $(k \geq 3)$.

Proposition 3.2. *Let $r < n-5$ and assume that there is a homotopy equivalence*

$$h_{r+2}:\ W_{n+2,r+2} \to (T^{n+1})^{k-1},$$

where k is the integer such that $n-2k-1 < r \leq n-2k+1$. (Note that $k \geq 3$.) Then h_{r+2} can be extended to a homotopy equivalence

$$h_r:\ W_{n+2,r} \to (T^{n+1})^k.$$

Proof. By Morse theory $W_{n+2,r}$ is homotopy equivalent to $W_{n+2,r+2}$ with $P(k)$ cells of dimension k attached. We consider at the critical point $P_k = (\underbrace{-1, \cdots, -1}_{k}, \underbrace{1, \cdots, 1}_{n-k})$ of index k. Set

$$T^k = \{(e^{ix_1}, \cdots, e^{ix_k}, 1, \cdots, 1) : x_1, \cdots, x_k \in \mathbf{R}\}$$

and let f_k be the restriction of f on T^k. For a small positive number ε, we set

$$D^k_\varepsilon = \{f_k \geq -(n-2k+1+\varepsilon)^2\}.$$

As in the previous case, the inclusion

$$\partial D^k_\varepsilon \subset T^k - \mathring{D}^k_\varepsilon \subset W_{n+2,n-2k+1+\varepsilon} (\simeq (T^{n+1})^{k-1})$$

is homotopic to the attaching map at P_k. Since $\pi_{k-1}((T^{n+1})^k) \simeq \pi_{k-1}(T^{n+1}) = 0$, for $k \geq 3$. Hence we have an extension of h_{r+2}

$$h_r:\ W_{n+2,n-2k+1+\varepsilon} \cup (\bigcup_{i=1}^{P(k)} e^k_i) \to (T^{n+1})^k$$

such that h_r induces isomorphisms on the relative homotopy groups

$$h_{r*}:\ \pi_*(W_{n+2,n-2k+1+\varepsilon} \cup (\bigcup_{i=1}^{P(k)} e^k_i) \to (T^{n+1})^k.$$

such that h_r induces isomorphisms on the relative homotopy groups

$$h_{r*}:\ \pi_*(W_{n+2,n-2k+1},\ W_{n+2,n-2k+1+\varepsilon}) \to \pi_*((T^{n+1})^k,\ (T^{n+1})^{k-1}).$$

Hence h_r is homotopy equivalece by Whitehead Theorem. ∎

Then by Proposition 3.2, proof of Theorem 1 is completed.

§ 4. Topology of $M_{n+2,r}$

By Theorem 1, we can compute the homology group of $M_{n+2,r}$. For $0 < r \leq n+1$, we choose an integer k such that $n-2k-1 < r \leq n-2k+1$. By Theorem 1, $H^j(W_{n+2,r}, \mathbb{Z})=0$ for $j \geq k+1$. We assume $n-2k-1 < r < n-2k+1$. Then $W_{n+2,r}$ and $M_{n+2,r}$ are smooth compact connected manifolds such that $\partial W_{n+2,r}=M_{n+2,r}$. By Lefschetz duality and Theorem 1,

$$(4.1) \qquad H_i(W_{n+2,r}, M_{n+2,r}) \simeq H^{n-i}(W_{n+2,r})=0$$

for $i \leq n-k-1$. When $r=n-2k+1$, $M_{n+2,r}$ has singular points and it is homotopy equivalent to

$$(4.2) \qquad M_{n+2,n-2k+1+\varepsilon} \times I \cup \left(\bigcup_{j=1}^{P(k)} e_j^k \right)$$

where I is the closed interval $[0, 1]$, e_j^k $(j=1, \cdots, P(k))$ are k-cell and ε is a small positive number. Hence by excision, we have

$$\begin{aligned} &H_i(W_{n+2,n-2k+1}, M_{n+2,n-2k+1}) \\ &\simeq H_i\Big(W_{n+2,n-2k+1+\varepsilon} \cup M_{n+2,n-2k+1+\varepsilon} \times I \cup (\bigcup_j e_j^k),\ M_{n+2,n-2k+1+\varepsilon} \times I \cup (\bigcup_j e_j^k)\Big) \\ &\simeq H_i(W_{n+2,n-2k+1+\varepsilon}, M_{n+2,n-2k+1+\varepsilon}) \\ &=0 \text{ if } i \leq n-k+1. \end{aligned}$$

Proposition 4.1. *Let n be odd. Then $M_{n+2,1}$ is an $n-1$ dimensional connected closed manifold whose homology is as follows :*

(1) $H_i(M_{n+2,1}, \mathbb{Z}) \simeq \mathbb{Z}^{P(i)}$ *for* $0 \leq i < \frac{n-1}{2}$

(2) $H_i(M_{n+2,1}, \mathbb{Z}) \simeq \mathbb{Z}^{2P(i)}$ *for* $i=\frac{n-1}{2}$

(3) $H_i(M_{n+2,1}, \mathbb{Z})=0$ *for* $i > n-1$.

Proof. By Theorem I, we have $W_{n+2,1} \simeq (T^{n+1})^{\frac{n-1}{2}}$. From the long homology exact sequence of the pair $(W_{n+2,1}, M_{n+2,1})$ and Poincaré duality we have (1). From the exact sequence

$$\begin{array}{ccccccccc}
\to H_{\frac{n+1}{2}}(W_{n+2,1}) & \to & H_{\frac{n+1}{2}}(W_{n+2,1}, M_{n+2,1}) & \to & H_{\frac{n-1}{2}}(M_{n+2,1}) & \to & H_{\frac{n-1}{2}}(W_{n+2,1}) & \to & 0 \\
\wr\| & & \wr\| \text{ Lefschetz dual} & & & & \wr\| & & \\
H_{\frac{n+1}{2}}((T^{n+1})^{\frac{n-1}{2}}) & & H^{\frac{n-1}{2}}(W_{n+2,1}) & & & & \mathbf{Z}^{P(\frac{n-1}{2})} & & \\
\| & & \wr\| & & & & & & \\
0 & & \mathbf{Z}^{P(\frac{n-1}{2})} & & & & & &
\end{array}$$

we have (2). By the dimensional reason, we have (3). ■

Proposition 4.2. *Let n be even. Then*

(1) $H_i(M_{n+2,1},\mathbf{Z}) \simeq \mathbf{Z}^{P(i)}$ *for* $0 \leq i \leq \frac{n}{2}$

(2) $H_i(M_{n+2,1},\mathbf{Z}) \simeq \mathbf{Z}^{P(n-i-1)}$ *for* $\frac{n}{2} < i \leq n-1$

(3) $H_i(M_{n+2,1},\mathbf{Z}) = 0$ *for* $i > n-1$.

Proof. By (4.3), $H_i(W_{n+2,1}, M_{n+2,1}) = 0$ for $i \leq \frac{n}{2}$. Then from the long homology exact sequence, we have (1). From (4.2) and Lefschetz duality,

$H_{i+1}(W_{n+2,1}, M_{n+2,1}) \simeq H_{i+1}(W_{n+2,1+\varepsilon}, M_{n+2,1+\varepsilon}) \simeq H^{n-i-1}(W_{n+2,1+\varepsilon}) \simeq H^{n-i-1}((T^{n+1})^{\frac{n}{2}-1})$.

For $i \geq \frac{n}{2}$, from the exact sequence,

$$\begin{array}{ccccc}
\to H_{i+1}(W_{n+2,1}) & \to & H_{i+1}(W_{n+2,1}, M_{n+2,1}) & \to H_i(M_{n+2,1}) & \to 0 \\
\| & & \wr\| & & \\
0 & & H_{n-i-1}((T^{n+1})^{\frac{n}{2}-1}) & & \\
& & \wr\| & & \\
& & \mathbf{Z}^{P(n-i-1)} & &
\end{array}$$

we have (2). (3) is dimensional reason. ■

By (4.1) and (4.2), we have $\pi_i(W_{n+2,1}, M_{n+2,1}) = 0$ for $i \leq \frac{n-1}{2}$ if n is odd and $\pi_i(W_{n+2,1}, M_{n+2,1}) = 0$ for $i \leq \frac{n}{2}$ if n is even.

Corollary 2. *Let n be odd and n $\geq$ 3. Then*

$$\pi_i(M_{n+2,1}) \simeq \pi_i((T^{n+1})^{\frac{n-1}{2}}) \quad \text{if } i \leq \frac{n-3}{2}.$$

Let n even and n $\geq$ 4. Then

$$\pi_i(M_{n+2,1}) \simeq \pi_i((T^{n+1})^{\frac{n}{2}-1}) \quad \text{if } i \leq \frac{n}{2}.$$

In particular,

Corollary 4.3.
$$\pi_i(M_{n+2,1}) \simeq \mathbb{Z}^{P(1)} \quad \text{if } n \geq 4.$$

Example. *From* (4.2), $M_{n+2,n-1}$ *is homotopy equivalent to*

$$S^{n-1} \vee \underbrace{S^1 \vee \cdots \vee S^1}_{P(1)=n+1 \text{ times}} .$$

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