Combinatorial expression of the fundamental second kind differential on an algebraic curve

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Abstract. The zero locus of a bivariate polynomial $P(x, y) = 0$ defines a compact Riemann surface Σ. The fundamental second kind differential is a symmetric $1 \otimes 1$-form on $\Sigma \times \Sigma$ that has a double pole at coinciding points and no other pole. As its name indicates, this is one of the most important geometric objects on a Riemann surface. Here we give a rational expression in terms of combinatorics of the Newton's polygon of P, involving only integer combinations of products of coefficients of P. Since the expression uses only combinatorics, the coefficients are in the same field as the coefficients of P.

1. Introduction

1.1. Example

Consider the Riemann surface Σ defined as the submanifold of $\mathbb{C} \times \mathbb{C}$ of equation $y^2 = x^2 - 1$ (and compactified by adding two points $(\infty, \infty)_+, (\infty, \infty)_-$, in the neighborhoods of which $y \sim \pm x$). A point p of Σ is a pair $p = (x, y)$ solution of the equation. Let $p = (x, y)$ and $p' = (x', y')$ be two points of Σ; then the expression (tensor product of a 1-form in p by a 1-form in p')

$$
\begin{aligned}
B(p, p') &= \frac{yy' + xx' - 1}{2yy'(x - x')^2} \, dx \otimes dx' \\
&= -\left(\frac{(y'^2 - x^2 + 1)(y^2 - x'^2 + 1)}{(x - x')^2(y - y')^2} + 1 \right) \frac{dx}{2y} \otimes \frac{dx'}{2y'}
\end{aligned}
\tag{1.1}
$$

has the following properties.
- It is symmetric $B(p, p') = B(p', p)$.
- It has a double pole at $p = p'$, i.e., at $(x, y) = (x', y')$, behaving like

$$
B(p, p') \underset{p \to p'}{\sim} \frac{dx \otimes dx'}{(x - x')^2}.
$$

• It has no other poles. Indeed, it seems to have a pole at $p = (x, y)$ and $p' = (x, -y)$ because $x' = x$, but at this point the numerator vanishes (and its derivative too), so there is no such pole. It seems to have a pole when $y = 0$, i.e., at $x = \pm 1$, but notice that $\frac{dx}{y} = \frac{dy}{x}$ has no pole at $x = \pm 1$. At $x \to \infty$, we have $y \sim \pm x(1 + O(1/x))$, so B behaves as $O(1/x^2)dx \otimes dx'$ which is analytic in a neighborhood of ∞, as can be seen in the coordinate $\tilde{x} = 1/x$, since $\frac{dx}{x^2} = -d\tilde{x}$ has no pole at $\tilde{x} \to 0$. B is called a *fundamental second kind form* on Σ (see for instance Fay's lectures [5] or [1, 3, 4, 6–9]).

1.2. General algebraic plane curves

In Theorem 3.1 below, we shall generalize the second line of (1.1) to general algebraic curves $P(x, y) = 0$ (we shall assume P irreducible and sufficiently generic). The goal is to exhibit a rational function

$$B((x, y), (x', y')) \in \mathbb{C}(x, y, x', y') \, dx \otimes dx'$$

symmetric under $(x, y) \leftrightarrow (x', y')$, and that has a double pole at $(x, y) = (x', y')$ with behaviour

$$B(p, p') \underset{p \to p'}{\sim} \frac{dx \otimes dx'}{(x - x')^2}, \tag{1.2}$$

and no other pole. Such a form is called a *fundamental second kind form.*

The existence of fundamental 2nd-kind forms is well known, B can be expressed in terms of the Green function, solution of a Dirichlet variational principle. It can also be expressed with the Theta function and involving the Abel map α, in the form

$$B(p, p') = d_p \otimes d_{p'} \log \Theta\big(\alpha(p) - \alpha(p') + \zeta\big), \tag{1.3}$$

where ζ is a zero of Θ. See [5] for details.

However, Green function, Theta function and Abel map are transcendental functions, whereas for an algebraic curve $P(x, y) = 0$, there should exist, in principle, an expression of B involving only rational functions of x and y, and it is not obvious at all how to rewrite (1.3) as a rational function of x and y.

Having a rational expression for B would be particularly useful if the coefficients of P are in a commutative field or ring $\mathbb{F}$ instead of $\mathbb{C}$, for example in $\mathbb{Z}$, in $\mathbb{Q}$ or a number field or a function field … A question is whether B is a rational expression with coefficients in the same ring or field, which is not apparent at all in expression (1.3). A rational expression of B that depends algebraically on the coefficients of P is also very useful towards the study of integrable systems, see [4].

An algorithm for finding a rational expression of B was developed by Baker and issued from Weierstrass' or Klein's works, see [1, 2]. This algorithm simplifies for

certain types of polynomials $P(x, y)$, in particular hyperelliptical cases ($\deg_y P = 2$), and the (n, s) curves [10].

Here we shall give in Theroem 3.1 a closed rational expression of the fundamental 2nd-kind differential B, from the combinatorics of the Newton's polytope, valid for any polynomial P. This expression involves only integer combination of coefficients of P, and thus elements of the same ring. We insist that this is a very simple to use, closed and simple formula, not an algorithm. We are not aware if this formula has been known before and would be grateful to readers who could point to us if it was derived before. The formula could very well have been known to Baker or mathematicians of the 19th century.

2. Algebraic plane curves

Let $P \in \mathbb{C}[x, y]$ a bivariate irreducible polynomial (not a product of polynomials)

$$P(x, y) = \sum_{i,j} P_{i,j} x^i y^j. \tag{2.1}$$

This can be generalized to Laurent polynomials, i.e., allowing negative powers of x and y, i.e., $P \in \mathbb{C}[x, 1/x, y, 1/y]$, so that the pairs (i, j) are in $\mathbb{Z}^2$.

Definition 2.1 (Newton's polygon). The finite set $N(P) = \{(i, j) \mid P_{i,j} \neq 0\} \subset \mathbb{Z}^2$ is called the Newton's *polytope* of P. Its convex envelope is called the Newton's *polygon* of P. We also define $\overset{\circ}{N}(P)$ to be the *interior* (strict) of the convex envelope of $N(P)$, and $\partial N(P)$ be the *boundary* of the convex envelope of $N(P)$.

Definition 2.2 (notation for derivatives). We shall denote the partial derivatives of P as

$$P_x(x, y) = \frac{\partial}{\partial x} P(x, y), \quad P_y(x, y) = \frac{\partial}{\partial y} P(x, y),$$

$$P_{x,x}(x, y) = \frac{\partial^2}{\partial x^2} P(x, y),$$

$$P_{x,y}(x, y) = \frac{\partial^2}{\partial x \partial y} P(x, y),$$

$$P_{y,y}(x, y) = \frac{\partial^2}{\partial y^2} P(x, y),$$

and so on.

2.1. Riemann surface

P defines an algebraic plane curve

$$\widetilde{\Sigma} = \{(x, y) \in \mathbb{C} \times \mathbb{C} \mid P(x, y) = 0\} \subset \mathbb{C} \times \mathbb{C}.$$

$\widetilde{\Sigma}$ is nearly a dimension 2 submanifold of $\mathbb{C}P^1 \times \mathbb{C}P^1$, and in fact it is the immersion in $\mathbb{C}P^1 \times \mathbb{C}P^1$ of a compact Riemann surface Σ, called its *desingularization.* The immersion is realized by two meromorphic maps x: $\Sigma \to \widetilde{\Sigma}$ and y: $\Sigma \to \widetilde{\Sigma}$, such that

$$\widetilde{\Sigma} = \{(x(p), y(p)) \mid p \in \Sigma\}.$$

For almost all points of Σ (except a finite number), at which the normal vector $(P_x(x(p), y(p)), P_y(x(p), y(p))) \neq 0$, the map $p \mapsto (x(p), y(p))$ is locally invertible, and we can identify the point $p \in \Sigma$ with its image $(x, y) = (x(p), y(p))$ in $\widetilde{\Sigma}$. The eventual exceptional points at which the vector $(P_x(x, y), P_y(x, y))$ vanishes are called *nodal points.*

2.2. Meromorphic functions and forms

It is well known that the field of meromorphic functions $\mathfrak{M}^0(\Sigma)$ on Σ, is the field of rational functions $\mathbb{C}(x, y)$ modulo $P(x, y) = 0$. Similarly, the space $\mathfrak{M}^1(\Sigma)$ of meromorphic 1-forms on Σ, is $\mathbb{C}(x, y)dx$ modulo $P(x, y) = 0$:

$$\mathfrak{M}^0(\Sigma) = \mathbb{C}(x, y) \qquad \mod P(x, y) = 0,$$
$$\mathfrak{M}^1(\Sigma) = \mathbb{C}(x, y)\, dx \quad \mod P(x, y) = 0.$$

2.3. Branchpoints and double points

At generic points $p = (x, y) \in \widetilde{\Sigma}$, such that $P_y(x, y) \neq 0$, we can locally write y as an analytic function of x, and we have

$$\frac{dy}{dx} = -\frac{P_x(x, y)}{P_y(x, y)}.$$

This fails at points such that $P_y(x, y) = 0$.

• Let (a, y_a) be a point such that $P_y(a, y_a) = 0$, and $P_x(a, y_a) \neq 0$, then we have

$$\frac{dx}{P_y(x, y)} = -\frac{dy}{P_x(x, y)}$$

whose right-hand side has no pole at (a, y_a), therefore dx vanishes at (a, y_a).

We say that (a, y_a) is a *ramification point,* and a (its image by x) is called a *branchpoint.* We say that the ramification point is *regular* if dx vanishes to the lowest

possible order, i.e., dx has a simple zero, and dy is not vanishing, and $P_y(x, y)$ has also a simple zero.

• Let (b, y_b) be a point such that $P_y(b, y_b) = 0$, and $P_x(b, y_b) = 0$. We say that (b, y_b) is a *double point*. We say that the double point is *regular* if P_y and P_x vanish to the lowest possible order, and the Hessian matrix $\begin{pmatrix} P_{x,x} & P_{x,y} \\ P_{y,x} & P_{y,y} \end{pmatrix}$ is invertible at (b, y_b). Let $\mathfrak{b}$ the set of all double points.

We say that P is *regular* if all its ramification points and double points are regular. From now on, we shall assume that P is regular. In fact, every compact Riemann surface Σ can be immersed into $\mathbb{C}P^1 \times \mathbb{C}P^1$ as the zero locus of a regular polynomial.

2.4. Punctures

Punctures are where x and/or y may tend to 0 or ∞ in the locus $P(x, y) = 0$. It is well known that punctures are in 1-1 correspondence with boundaries of the Newton's polygon.

Definition 2.3 (puncture). Let $\alpha = [(i, j) \to (i', j')] \subset \partial N(P)$ with $(i, j) \in \mathbb{Z}^2$ and $(i', j') \in \mathbb{Z}^2$, a minimal (no other integer point of $\partial N(P)$ is in α, and thus $i - i'$ and $j - j'$ are relatively prime) boundary of the Newton's polygon, oriented such that the polygon is on its right.

The puncture α is such that the locus $P(x, y) = 0$ has an asymptotic solution

$$y^{j-j'} \sim c_\alpha x^{i'-i},$$

where c_α is a root of the following polynomial of c

$$P_\alpha(c) = \sum_{r \in \mathbb{Z}} P_{i-r(i'-i), j-r(j'-j)}\, c^r \in \mathbb{C}[c, c^{-1}]$$

whose degree equals the number of integer points on the boundary of $N(P)$ containing α, and has as many roots as the number of punctures in that boundary. We define c_α the αth root (ordering the minimal oriented segments clockwise along the boundary, and the roots by their decreasing modulus).

Since $j - j'$ and $i' - i$ are relatively prime, there exist β_α and $\tilde{\beta}_\alpha$ such that

$$\beta_\alpha(j' - j) + \tilde{\beta}_\alpha(i - i') = 1.$$

We define

$$z = x^{\beta_\alpha} y^{\tilde{\beta}_\alpha}. \tag{2.2}$$

z is a local coordinate in the neighborhood of the puncture α. In that neighborhood we have

$$x \sim c_\alpha^{\tilde{\beta}_\alpha} z^{j'-j}, \quad y \sim c_\alpha^{-\beta_\alpha} z^{i-i'}.$$

We see that indeed near that point, x and/or y have a pole or a zero, i.e., this is a puncture.

Moreover, all punctures, i.e., points $\alpha \in \Sigma$ where x and/or y have a pole or zero, are of that type. Indeed, let us assume that there is a point $\alpha \in \Sigma$, and that there is a local coordinate z near α, vanishing at α, such that

$$x \sim c\, z^{-p}, \quad y \sim \tilde{c}z^{-q},$$

where $c \neq 0$ and $\tilde{c} \neq 0$, and where p and q are integers not both vanishing, i.e., x has a pole if $p > 0$ or zero if $p < 0$ or none if $p = 0$, and y has a pole if $q > 0$ or zero if $q < 0$ or none if $q = 0$. Let

$$m_{p,q} = \max_{(i,j)\in N(P)} pi + qj. \tag{2.3}$$

Define the line $D_{p,q}$ of equation $pi + qj = m_{p,q}$

$$D_{p,q} = \{(i,j) \in \mathbb{R}^2 \mid pi + qj = m_{p,q}\}.$$

By definition, $D_{p,q} \cap N(P) \neq 0$, and, due to maximality (2.3), the Newton's polytope lies to the right of that line (oriented as the vector $(q, -p)$).

We have

$$0 = P(x,y) = \sum_{(i,j)\in N(P)} P_{i,j}x^i y^j \sim z^{m_{p,q}} \sum_{(i,j)\in N(P)\cap D_{p,q}} P_{i,j}c^i \tilde{c}^j + o(z^{m_{p,q}}),$$

whose right-hand side cannot be zero if $D_{p,q}$ would contain only one point of $N(P)$, therefore $D_{p,q}$ must contain at least two points (i,j) and (i', j'). This implies that

$$p(i - i') = q(j' - j),$$

and also this implies that the line $D_{p,q}$ is tangent to the convex envelope of $N(P)$. This implies that α is a puncture as defined above.

2.5. Holomorphic forms

Let $\mathcal{O}^1(\Sigma) \subset \mathcal{M}^1(\Sigma)$ the space of "holomorphic forms": forms that have no poles at all. It is well known that if P is regular and has no double point, the space of holomorphic forms on Σ has a basis parametrized by the interior of Newton's polygon

Theorem 2.1 (holomorphic forms). *We have*

$$\mathcal{O}^1(\Sigma) = \left\{ \frac{Q(x,y)\,dx}{P_y(x,y)} \;\middle|\; Q(x,y) = \sum_{(u,v)\in \mathring{N}(P)} Q_{u,v}x^{u-1}y^{v-1},\; Q_{u,v} \in \mathbb{C} \right\}$$

and

$$\mathcal{O}^1(\Sigma) \sim \mathbb{C}^{\mathring{N}(P)}, \quad \dim \mathcal{O}^1(\Sigma) = |\mathring{N}(P)| = \mathfrak{g}.$$

Proof. This is a well-known theorem. Let us recall the proof because it will be useful for understanding the similar proof for the fundamental 2nd-kind form.

The ratio $\frac{dx}{P_y(x,y)} = -\frac{dy}{P_x(x,y)}$ has no pole at the zeros of $P_y(x, y)$ i.e., at ramification points. The only places where $\frac{x^{u-1}y^{v-1}\,dx}{P_y(x,y)}$ could have poles is at punctures. Let α a puncture, and let z a local coordinate as in (2.2). In the neighborhood of α we have

$$P_y(x, y) \sim \sum_{(i'',j'')\in D_{p,q}\cap N(P)} P_{i'',j''} j'' c^{i''} \tilde{c}^{j''} z^{-pi''-q(j''-1)}$$

$$\sim z^{-m_{p,q}+q} \sum_{(i'',j'')\in D_{p,q}\cap N(P)} P_{i'',j''} j'' c^{i''} \tilde{c}^{j''}$$

$$\sim C z^{-m_{p,q}+q}, \quad C \neq 0.$$

Moreover, we have $dx = -pc\, z^{-p-1}dz$, therefore

$$\frac{x^{u-1}y^{v-1}\,dx}{P_y(x, y)} \sim O(z^{-up-vq+m_{p,q}-1})\,dz.$$

It has no pole at α, i.e., at $z = 0$ if and only if $up + vq - m_{p,q} < 0$, i.e., the point (u, v) lies strictly to the right of the line $D_{p,q}$. If this is true for all punctures, then (u, v) must lie to the right of every tangent of the convex envelope, i.e., $(u, v) \in \mathring{N}(P)$. Reciprocally, if $(u, v) \in \mathring{N}(P)$, then the form $\frac{x^{u-1}y^{v-1}\,dx}{P_y(x,y)}$ has no pole at all, it is holomorphic.   ∎

In the proof we have seen that more generally, for $(u, v) \in \mathbb{Z}^2$, the form

$$\frac{x^{u-1}y^{v-1}\,dx}{P_y(x, y)}$$

has

- no pole at α if (u, v) is on the right of the line $D_{p,q}$,
- a simple pole at α if (u, v) is on the line $D_{p,q}$,
- a pole at α of degree $1 + up + vq - m_{p,q} \geq 2$ if (u, v) is on the left of $D_{p,q}$.

When P has double points, we would like the ratio $\frac{Q(x,y)dx}{P_y(x,y)}$ to have no pole at double points (which are zeros of $P_y(x, y)$ but not of dx), i.e., that $Q(b, y_b) = 0$ for each double point, in other words:

Proposition 2.1 (holomorphic forms, double-points). *If P is regular and has double points, we have*

$$\mathcal{O}^1(\Sigma) = \mathrm{Ker}\begin{pmatrix} \mathbb{C}^{\mathring{N}(P)} \to \mathbb{C}^{\mathfrak{b}}, \\ Q \mapsto \{Q(b, y_b)\}_{b \in \mathfrak{b}} \end{pmatrix},$$

i.e.,

$$\mathcal{O}^1(\Sigma) = \left\{ \frac{Q(x, y)\,dx}{P_y(x, y)} \;\middle|\; Q(x, y) = \sum_{(i,j) \in \mathring{N}(P)} Q_{i,j} x^{i-1} y^{j-1}, \right.$$
$$\left. \text{for all } b \in \mathfrak{b} \text{ such that } Q(b, y_b) = 0 \right\}.$$

We have the exact sequence

$$0 \to \mathcal{O}^1(\Sigma) \to \mathbb{C}^{\mathring{N}(P)} \to \mathbb{C}^{\mathfrak{b}} \to 0$$

and

$$\dim \mathcal{O}^1(\Sigma) = \mathfrak{g} = |\mathring{N}(P)| - |\mathfrak{b}|.$$

In Appendix A, we show that if the coefficients of P are in a commutative field $\mathbb{F}$ of characteristic 0, then there is a basis of $\mathcal{O}^1(\Sigma)$ with polynomials $Q \in \mathbb{F}[x, y]$ the same field as P. This is not obvious, because if $\mathbb{F}$ is not algebraically closed, the double points are in an algebraic extension of $\mathbb{F}$, and a priori the coefficients $Q_{i,j}$ would only be in that extension.

3. Fundamental 2nd-kind form

Let us look for a fundamental 2nd-kind form as follows:

$$B(x, y; x', y') = -\frac{\frac{P(x,y')P(x',y)}{(x-x')^2(y-y')^2} - Q(x, y; x', y')}{P_y(x, y) P_y(x', y')}\, dx\, dx' \tag{3.1}$$

with $Q \in \mathbb{C}[x, y, x', y']$ a polynomial of all its variables (resp. $Q \in \mathbb{C}[x, 1/x, y, 1/y, x', 1/x', y', 1/y']$ if P is a Laurent polynomial), symmetric under $(x, y) \leftrightarrow (x', y')$, chosen so that $B(x, y)$ has a pole only at $(x, y) = (x', y')$ on the curve Σ, in particular, Q must be chosen so that B has no pole at punctures.

Remark that (3.1) has a double pole at $(x, y) = (x', y')$, indeed near that point

$$P(x, y') = P(x, y') - P(x, y) \sim (y' - y) P_y(x, y),$$
$$P(x', y) = P(x', y) - P(x', y') \sim (y - y') P_y(x, y),$$

which implies behavior (1.2).

Remark that there is no pole at points $y = y'$ with $x \neq x'$, indeed at such points

$$\frac{P(x, y') P(x', y)}{(x - x')^2 (y - y')^2} \sim \frac{-P_y(x, y) P_y(x', y)}{(x - x')^2},$$

which has no pole if $x \neq x'$. Similarly, there is no pole at points $x = x'$ with $y \neq y'$.

3.1. Regular polynomial, without double points

Let us first consider the case where P is regular and has no double points. In that case, we know that $\frac{dx}{P_y(x,y)}$ has no pole at ramification points. The only possible other poles of (3.1) could be at punctures. We have to choose Q in order to kill them.

The following is the main theorem of this article:

Theorem 3.1 (fundamental 2nd-kind form). *The $1 \otimes 1$-form*

$$B((x, y); (x', y')) = -\frac{\frac{P(x,y')P(x',y)}{(x-x')^2(y-y')^2} - Q(x, y; x', y')}{P_y(x, y) P_y(x', y')} \, dx \, dx', \qquad (3.2)$$

with

$$
\begin{aligned}
Q(x, &y; x', y') \\
= \sum_{(i,j) \in \mathcal{N}} &\sum_{(i',j') \in \mathcal{N}} P_{i,j} P_{i',j'} \sum_{(u,v) \in \mathbb{Z}^2 \cap \text{ triangle } (i,j),(i',j'),(i,j)} |u - i| \, |v - j'| \\
&\Big(\delta_{(u,v) \notin \mathring{\mathcal{N}} \cup [(i,j),(i',j')]} x^{u-1} y^{v-1} x'^{i+i'-u-1} y'^{j+j'-v-1} \\
&+ \delta_{(u,v) \notin \mathring{\mathcal{N}}, \, (i+i'-u, j+j'-v) \in \mathring{\mathcal{N}}} \, x'^{u-1} y'^{v-1} x^{i+i'-u-1} y^{j+j'-v-1} \\
&+ \frac{1}{2} \delta_{(u,v) \in [(i,j),(i',j')]} x^{u-1} y^{v-1} x'^{i+i'-u-1} y'^{j+j'-v-1} \Big), \qquad (3.3)
\end{aligned}
$$

is a fundamental 2nd-kind form. Here δ_C means 1 if the condition C is fulfilled, and 0 if not.

Remark. Every term in the last line appears twice in the sum, because it is symmetric in $(i, j) \leftrightarrow (i', j')$. We could remove the $\frac{1}{2}$ factor, by summing only over distinct pairs $\{(i, j), (i', j')\}$.

It is obvious from expression (3.3), that if the coefficients $P_{i,j}$ belong to a commutative ring $\mathbb{F}$, then the coefficients of Q are in the same ring.

Proof. We have to show that Q is symmetric, and B has no pole at punctures.

Let us consider a puncture α where $x \sim c z^{-p}$ and $y \sim \tilde{z}^{-q}$, with both $p > 0$ and $q > 0$ (the other cases, can be obtained by changing $x \to 1/x$ and/or $y \to 1/y$, and remarking that expression (3.2) is unchanged under these changes. The cases where

$p = 0$ or $q = 0$ can be done similarly). Near $(x, y) \to \alpha$ we have $x \to \infty$ and $y \to \infty$, and thus

$$\frac{P(x, y') P(x', y)}{(x - x')^2 (y - y')^2}$$

$$\sim \sum_{(i,j) \in \mathcal{N}} \sum_{(i',j') \in \mathcal{N}} \sum_{k \geq 1} \sum_{l \geq 1} P_{i,j} P_{i',j'} kl x^{i-k-1} y'^{j+l-1} x'^{i'+k-1} y^{j'-l-1}$$

$$\sim \sum_{(u,v) \in \mathbb{Z}^2} x^{u-1} y^{v-1} \Big(\sum_{(i,j) \in \mathcal{N}} \sum_{(i',j') \in \mathcal{N}} \sum_{k \geq 1} \sum_{l \geq 1} P_{i,j} P_{i',j'} \, kl$$

$$\delta_{u,i-k} \delta_{v,j'-l} x'^{i'+k-1} y'^{j+l-1} \Big),$$

where the last bracket contains in fact a finite sum. All the monomials such that $(u, v) \notin \overset{\circ}{\mathcal{N}}$ and (u, v) is above the line $D_{p,q}$, gives a pole at α, and must be compensated by a term in Q.

Let us consider such an (u, v) monomial. Notice that (u, v) above $D_{p,q}$ implies that $u = i - k \geq i'$ and $v = j' - l \geq j$, which implies in particular that this can occur only if $i > i'$ and $j' > j$. Moreover, since all the segment $[(i, j), (i', j')]$ is contained in the Newton's polygon and is not left of $D_{p,q}$, we see that (u, v) must belong to the triangle $((i, j), (i', j'), (i', j))$.

Consider the point $(u', v') = (i' + k, j + l) = (i + i' - u, j + j' - v)$, which is the symmetric of (u, v) with respect to the middle of $[(i, j), (i', j')]$.

So, let us thus assume that $i > i'$ and $j' > j$ and (u, v) belongs to the triangle $((i, j), (i', j'), (i', j))$, and let us consider different cases.

- $(u, v) \notin [(i, j), (i', j')]$, i.e. it is strictly in the interior of the triangle. If $(u, v) \notin \overset{\circ}{\mathcal{N}}$, then the monomial $P_{i,j} P_{i',j'} kl x^{u-1} y^{v-1} x'^{u'-1} y'^{v'-1}$ should appear in Q and is indeed the first term in (3.3). Notice that in that case the point (u', v') can't be above $D_{p,q}$. There are then two sub-cases.

  - $(u', v') \in \overset{\circ}{\mathcal{N}}$, then we can add to Q a monomial proportional to $x^{u'-1} y^{v'-1}$ without adding a pole to B. In particular, we can add

    $$P_{i,j} P_{i',j'} kl x^{u'-1} y^{v'-1} x'^{u-1} y'^{v-1},$$

    which is the second term in (3.3). It is the term needed to make Q symmetric under the exchange $(x, y) \leftrightarrow (x', y')$, and that doesn't appear anywhere else.

  - $(u', v') \notin \overset{\circ}{\mathcal{N}}$. Notice that since $(u, v) \notin [(i, j), (i', j')]$, we also have $(u', v') \notin [(i, j), (i', j')]$. Moreover, if $(u', v') \notin \overset{\circ}{\mathcal{N}}$, this implies that (u', v') is in the triangle $(i', j'), (i, j), (i', j)$ and thus below some tangent $D_{p',q'}$ with $p' \leq 0$ and $q' \leq 0$. This means that the monomial $P_{i,j} P_{i',j'} kl x^{u'-1} y^{v'-1} x'^{u-1} y'^{v-1}$ will appear in Q in the contribution with $(i, j) \leftrightarrow (i', j')$. This guarantees that this contribution to Q is symmetric in $(x, y) \leftrightarrow (x', y')$.

- $(u, v) \in [(i, j), (i', j')]$. This implies $(u', v') \in [(i, j), (i', j')]$ as well. Remarking that if $(u, v) \in [(i, j), (i', j')]$, we have

$$kl = (i - u)(j' - v) = (u - i')(v - j)$$

and

$$P_{i,j} P_{i',j'} (i - u)(j' - v) x^{u-1} y^{v-1} x'^{u'-1} y'^{v'-1}$$
$$= P_{i',j'} P_{i,j} (i' - u)(j - v) x^{u-1} y^{v-1} x'^{u'-1} y'^{v'-1}$$

i.e., this monomial appears twice in the sum (3.3) because it also appears in the term $(i, j) \leftrightarrow (i', j')$, and this is why it has to be multiplied by $\frac{1}{2}$.

If $(u, v) \in [(i, j), (i', j')]$, we have

$$kl = (i - u)(j' - v) = (i - u')(j' - v'),$$

the monomial $P_{i,j} P_{i',j'} kl x^{u'-1} y'^{v'-1} x^{u-1} y^{v-1}$ also appears in (3.3), and this guarantees that this contribution to Q is symmetric under $(x, y) \leftrightarrow (x', y')$. Also, if $(u, v) \in \mathring{\mathcal{N}} \cap [(i, j), (i', j')]$, this implies that $(u', v') \in \mathring{\mathcal{N}} \cap [(i, j), (i', j')]$, and thus this monomial and its symmetric under $(x, y) \leftrightarrow (x', y')$ are both inside Newton's polygon, so don't contribute to poles of B.

We have shown that the polynomial of (3.3) is symmetric under $(x, y) \leftrightarrow (x', y')$, and up to monomials inside $\mathring{\mathcal{N}}$, it compensates all the terms of $\frac{P(x,y')P(x',y)}{(x-x')^2(y-y')^2}$ that could possibly diverge at punctures. This concludes the proof. ∎

Corollary 3.1 (shifted fundamental 2nd-kind form). *Every fundamental 2nd-kind form can be written as*

$$B_\kappa(x, y; x', y') = \frac{\left(\frac{P(x,y')-P(x',y)}{(x-x')(y-y')}\right)^2 - Q_\kappa(x, y; x', y')}{4 \, P_y(x, y) P_y(x', y')} \, dx \, dx'. \qquad (3.4)$$

where κ is a symmetric matrix in $\mathbb{C}^{\mathring{N} \times \mathring{N}}$, and

$$Q_\kappa(x, y; x', y') = Q(x, y; x', y')$$
$$- \sum_{(i,j),(i',j') \in \mathring{N} \times \mathring{N}} \kappa_{(i,j),(i',j')} \, x^{i-1} y^{j-1} x'^{i'-1} y'^{j'-1}. \qquad (3.5)$$

Proof. Indeed one can add to B any symmetric bilinear combination of holomorphic forms $\mathcal{O}^1(\Sigma) \otimes^{\mathrm{sym}} \mathcal{O}^1(\Sigma)$.

Conversely, if $\widetilde{B}$ is another fundamental 2nd-kind form, then $\widetilde{B} - B$ must have no pole at all, it must be a symmetric bilinear combination of holomorphic forms $\mathcal{O}^1(\Sigma) \otimes^{\mathrm{sym}} \mathcal{O}^1(\Sigma)$. ∎

3.2. Double points

Theorem 3.2 (fundamental 2nd-kind form: double points case). *If P has double points, then the following is a fundamental 2nd-kind form:*

$$B(x, y; x', y') = -\frac{\frac{P(x,y')P(x',y)}{(x-x')^2(y-y')^2} - Q(x, y; x', y') - \tilde{Q}(x, y; x', y')}{P_y(x, y)P_y(x', y')} \, dx \, dx'$$

with Q defined in (3.3)*, and $xyx'y'\tilde{Q} \in \mathbb{C}^{\mathring{N}(P)} \overset{\mathrm{sym}}{\otimes} \mathbb{C}^{\mathring{N}(P)}$ defined as follows. Let us choose $I \subset \mathring{N}(P)$ such that $|I| = |\mathfrak{b}|$. We recall that the matrix B_I defined in Lemma* A.1 *is invertible. We define, for every $(i, j), (i', j') \in I \times I$,*

$$
\begin{aligned}
\tilde{Q}_{i,j;i',j'} &= -\sum_{b \neq b' \in \mathfrak{b} \times \mathfrak{b}} (B_I^{-1})_{(i,j),b}(B_I^{-1})_{(i',j'),b'}\Big(Q(b, y_b; b', y_{b'}) \\
&\qquad\qquad\qquad\qquad\qquad - \frac{P(b, y_{b'})P(b', y_b)}{(b - b')^2(y_b - y_{b'})^2}\Big) \\
&\quad - \sum_{b \in \mathfrak{b}} (B_I^{-1})_{(i,j),b}(B_I^{-1})_{(i',j'),b}(Q(b, y_b; b, y_b) - \gamma_b),
\end{aligned}
\tag{3.6}
$$

where $\gamma_b = P_{x,y}(b, y_b)^2 - P_{x,x}(b, y_b)P_{y,y}(b, y_b)$, and finally

$$\tilde{Q}(x, y; x', y') = \sum_{(i,j),(i',j') \in I \times I} \tilde{Q}_{i,j;i',j'} \, x^{i-1}y^{j-1}x'^{i'-1}y'^{j'-1}.$$

Proof. Since $\tilde{Q}$ has its coefficients in $I \subset \mathring{N}(P)$, it does not create poles at punctures, and by construction it kills poles at double points. ∎

3.3. Hyperelliptical case

Consider the case $P(x, y) = y^2 - P(x)$, with $P(x) \in \mathbb{C}[x]$ a polynomial of even degree, whose zeros are all distinct (thus no double points). Theorem 3.1 gives

$$
\begin{aligned}
&B(x, y; x', y') \\
&= \frac{dx}{P_y(x, y)}\frac{dx'}{P_y(x', y')}\Big(\frac{2yy'}{(x - x')^2} + \frac{P(x) + P(x')}{(x - x')^2} - \sum_i iP_{2i}x^{i-1}x'^{i-1} \\
&\qquad\qquad\qquad\qquad - \sum_{i > \frac{1}{2}\deg P} P_i \sum_{\frac{1}{2}i < u < i}(i - u)(x^{u-1}x'^{i-u-1} + x^{i-u-1}x'^{u-1})\Big).
\end{aligned}
$$

In that case, we can further simplify the expression of B. See [4].

Theorem 3.3 (hyperellitical curves). *Let $U(x) = (\sqrt{P(x)})_+$ be the polynomial part (expanding near ∞) of its square-root, and let $V(x) = P(x) - U(x)^2$. Then*

$$B((x, y); (x', y')) = \frac{yy' + U(x)U(x') + \frac{1}{2}V(x) + \frac{1}{2}V(x')}{2yy'(x - x')^2} \, dx \, dx'.$$

is a fundamental 2nd-kind form.

Proof. The symmetry $(x, y) \leftrightarrow (x', y')$ is obvious. The double pole at $(x, y) = (x', y')$ is obvious. The absence of pole at $(x', y') = (x, -y)$ is also obvious, and the absence of poles at the zeros of y as well. The only possible poles could come from the two punctures $x \to \infty$, and $y \sim \pm U(x)$. Notice that $\deg V < \deg U$. Consider the $+$ sign (the minus sign is similar). At this puncture we have

$$y = U(x) + \frac{V(x)}{2U(x)} + O(1/x^2),$$

therefore

$$\frac{yy' + U(x)U(x') + \frac{1}{2}V(x) + \frac{1}{2}V(x')}{2yy'} \sim \frac{y' + U(x')}{2y'} + O(1/x)$$

and thus

$$B((x, y); (x', y')) \sim \frac{dx}{x^2}\left(\frac{y' + U(x')}{2y'} + O(1/x)\right)dx',$$

which is regular at $x \to \infty$. ∎

3.4. (n, s) curves

(n, s) curves are those whose Newton's polygon is a triangle:

$$P(x, y) = y^n - x^s - \sum_{\substack{i \geq 0, j \geq 0, \\ ni + js < ns}} P_{i,j} x^i y^j.$$

We assume that n and s are relatively prime. In that case, we get

$$
\begin{aligned}
Q(x, &y; x', y') \\
&= -\sum_{\substack{k>0, l>0 \\ kn+ls<ns}} (s - k)(n - l)(x^{s-k-1}y^{n-l-1}x'^{k-1}y'^{l-1} \\
&\qquad\qquad\qquad\qquad + x^{k-1}y^{l-1}x'^{s-k-1}y'^{n-l-1}) \\
&\quad + \sum_{\substack{0<u<s \\ 0<v<n \\ un+vs>ns}} \sum_{\substack{u'>0, v'>0 \\ u'n+v's<ns}} \sum_{k=1}^{\min(u',s-u)-1} \sum_{l=1}^{\min(v',n-v)-1} kl\, P_{u+k, v'-l} \\
&\qquad\qquad\qquad P_{u'-k, v+l}(x^{u-1}y^{v-1}x'^{u'-1}y'^{v'-1} + x^{u'-1}y^{v'-1}x'^{u-1}y'^{v-1})
\end{aligned}
$$

$$+ \sum_{(i,j)} \sum_{(u,v)\in[(i,j),(s,0)]} (u-i)v P_{i,j} x^{u-1} y^{v-1} x'^{i+s-u-1} y'^{j-v-1}$$

$$- \sum_{(i,j)} \sum_{(u,v)\in[(i,j),(0,n)]} (i-u)(n-v) P_{i,j} x^{u-1} y^{v-1} x'^{i-u-1} y'^{j+n-v-1}$$

$$+ \sum_{\text{pairs } \{(i,j),(i',j')\}} \sum_{(u,v)\in[(i,j),(i',j')]} |u-i||v-j'| P_{i,j} P_{i',j'} x^{u-1} y^{v-1}$$
$$x'^{i+i'-u-1} y'^{j+j'-v-1}.$$

Notice that the last three lines are in fact in $\overset{\circ}{N}(P)$ and could be removed, we could keep only

$$- \sum_{\substack{k>0 \\ l>0 \\ kn+ls<ns}} (s-k)(n-l)(x^{s-k-1} y^{n-l-1} x'^{k-1} y'^{l-1} + x^{k-1} y^{l-1} x'^{s-k-1} y'^{n-l-1})$$

$$+ \sum_{\substack{0<u<s \\ 0<v<n \\ un+vs>ns}} \sum_{\substack{u'>0 \\ v'>0 \\ u'n+v's<ns}} \sum_{k=1}^{\min(u',s-u)-1} \sum_{l=1}^{\min(v',n-v)-1} kl P_{u+k,v'-l} P_{u'-k,v+l} (x^{u-1} y^{v-1} x'^{u'-1} y'^{v'-1} + x^{u'-1} y'^{v'-1} x'^{u-1} y'^{v-1}).$$

See [10].

Example $(n,s) = (4,3)$:

$$\begin{aligned} Q(x,y;x',y') = & -(xy^2 + x'y'^2) - 2(xyy' + yx'y') \\ & - 2(y^2x' + xy'^2) + P_{0,3}P_{2,0}(y+y') \\ & - 2P_{2,0}yy' + 2P_{0,3}(xx' + yy') + P_{1,2}xx' \\ & + P_{0,3}P_{2,1}yy' + P_{0,2}P_{2,0}, \end{aligned}$$

where all terms in the 2nd line could be discarded.

3.5. Example of degree 3 representation of an elliptic curve

Consider

$$P(x,y) = x^3 + y^3 + 1 + Cxy.$$

Its Newton's polygon is the triangle $(0,0), (0,3), (3,0)$, it has one interior point, so this is a genus 1 curve. Each boundary has 4 integer points, i.e., 3 segments, i.e., there are 9 punctures. Let $\rho = e^{2\pi i/3}$ a cubic root of unity. The 9 punctures are at

- $x \to \infty$, $y \sim -\rho^k x$, $k = 0, 1, 2$;
- $x = 0$, $y = -\rho^k$, $k = 0, 1, 2$;
- $y = 0$, $x = -\rho^k$, $k = 0, 1, 2$.

Theorem 3.1 gives

$$Q(x,y;x',y') = xy + x'y' + 2yx' + 2xy',$$

$$\frac{P(x, y')}{y' - y} = \frac{P(x, y') - P(x, y)}{y' - y} = y^2 + yy' + y'^2 + Cx,$$

i.e.,

$$B(x, y; x', y') = \Big(\frac{(y^2 + yy' + y'^2 + Cx)(y^2 + yy' + y'^2 + Cx')}{(x - x')^2}$$
$$+ xy + x'y' + 2xy' + 2x'y \Big) \frac{dx}{3y^2 + Cx} \frac{dx'}{3y'^2 + Cx'}.$$

4. Other fields

Let us assume that all coefficients of P are in a commutative field $\mathbb{F}$ rather than $\mathbb{C}$, for example $\mathbb{Q}$ or a number field or a function field. Are the coefficients of Q (and $\widetilde{Q}$) in B in the same field?

If we have no double points, this is rather obvious, and we can replace commutative field $\mathbb{F}$ by commutative ring: the coefficients of Q are integer bilinear combinations of coefficients of P so they are always in the same ring.

If there are double points, it is more subtle to see that the polynomial $\widetilde{Q}$ in (3.6) has its coefficients in $\mathbb{F}$. Let us study this question now in greater details.

4.1. Branchpoints and double points

Assume that $\mathbb{F}$ is a commutative field.

Definition 4.1 (discriminant). We define the x-discriminants:

$$D(x) = \text{Resultant}_y (P(x, y), P_y(x, y)) \in \mathbb{F}[x],$$
$$\widetilde{D}(x) = \text{Resultant}_y (P(x, y), P_x(x, y)) \in \mathbb{F}[x].$$

And we define the discriminant Δ as the discriminant of $D(x)$, i.e., the resultant of $D(x)$ and $D'(x)$.

$$\Delta = \text{Discriminant}(D) = \text{Resultant}(D, D'), \quad \Delta \in \mathbb{F}.$$

Definition 4.2 (branchpoints). Let us consider the set of roots of $D(x)$ in an algebraic closure of $\mathbb{F}$.

• If a is a root of $D(x)$ and not a root of $\widetilde{D}(x)$, we say that a is a *branchpoint*. Let

$$\mathfrak{a} = \{a_1, \ldots, a_{|\mathfrak{a}|}\}$$

the set of all branchpoints. We shall consider the field extension

$$\mathbb{F}[\mathfrak{a}] = \mathbb{F}[a_1, \ldots, a_{|\mathfrak{a}|}].$$

• If b is a root of $D(x)$ and a root of $\widetilde{D}(x)$, we say that b is a *double point.* Let

$$\mathfrak{b} = \{b_1, \ldots, b_{|\mathfrak{b}|}\}$$

the set of all double points.

Lemma 4.1. *The symmetric polynomials (over $\mathbb{F}$) of $b_1, \ldots, b_{|\mathfrak{b}|}$ are in $\mathbb{F}$. The symmetric polynomials of $a_1, \ldots, a_{|\mathfrak{a}|}$ are in $\mathbb{F}$.*

Proof. In Appendix B. ∎

Theorem 4.1. *If $\mathbb{F}$ is a commutative field of characteristic 0, then the coefficients of $\widetilde{Q}$ in (3.6) are in $\mathbb{F}$.*

Proof. In Appendix A. ∎

Conclusion

The formula has a nice combinatorial graphical interpretation: it is a sum over oriented pairs of distinct points in the Newton's polygon, each pair (i, j), (i', j') defines a triangle (i, j), (i', j'), (i, j'), and then we sum over all points (u, v) in that triangle, yielding a monomial

$$P_{i,j} P_{i',j'} |u - i| |v - j'| x^{u-1} y^{v-1} x^{\prime i + i' - u - 1} y^{\prime j + j' - v - 1},$$

depending on whether the point (u, v) is on the segment $[(i, j), (i', j')]$, and depending whether the point (u, v) is inside $\overset{\circ}{N}(P)$ or outside, and also depending on whether the symmetric (with respect to the middle of $[(i, j), (i', j')]$) of (u, v) is inside or outside $\overset{\circ}{N}(P)$.

The factor $|u - i| |v - j'|$ is the area of the rectangle of corners (u, v), (i, j'). This is extremely easy to implement.

A. Lemma for double points

Lemma A.1. *The matrix $B = (b^i y_b^j)_{b \in \mathfrak{b}, (i,j) \in \overset{\circ}{N}(P)}$ has rank $|\mathfrak{b}|$. Let $I \subset \overset{\circ}{N}(P)$ such that $|I| = |\mathfrak{b}|$. The matrix $B_I = (b^i y_b^j)_{b \in \mathfrak{b}, (i,j) \in I}$ is invertible. Let $\bar{I} = \overset{\circ}{N}(P) - I$ the complement of I.*

Then, for each $(i', j') \in \bar{I}$,

$$Q_{i,j}^{(i',j')} = \begin{cases} \det_{b \in \mathfrak{b}, (u,v) \in I - (i,j) + (i',j')} (b^u y_b^v) / \det_{b \in \mathfrak{b}, (u,v) \in I} (b^u y_b^v) & \text{if } (i,j) \in I, \\ \delta_{i,i'} \delta_{j,j'} & \text{if } (i,j) \notin I, \end{cases}$$

$$\text{(A.1)}$$

and the one-form

$$\omega^{(i',j')} = \frac{dx}{P_y(x,y)} \sum_{(i,j)\in\mathring{N}(P)} Q_{i,j}^{(i',j')} \, x^{i-1} y^{j-1}$$

belongs to $\mathcal{O}^1(\Sigma)$ and

$$\mathcal{O}^1(\Sigma) = \mathrm{span}(\omega^{(i',j')})_{(i',j')\in\bar{I}}.$$

In particular,

$$\dim \mathcal{O}^1(\Sigma) = |\mathring{N}(P)| - |\mathfrak{b}|.$$

Proof. First we admit (well known) that $|\mathfrak{b}| < |\mathring{N}(P)|$. Let $I \subset \mathring{N}(P)$ such that $|I| = |\mathfrak{b}|$. Assume that the matrix

$$B_I = (b^i y_b^j)_{(i,j)\in I, b\in\mathfrak{b}}$$

would be non-invertible, in particular, there must exist a vecor $(R_{i,j})_{(i,j)\in I}$ such that

$$R(b, y_b) = \sum_{(i,j)\in I} R_{i,j} b^i y_b^j = 0 \quad \text{for all } b \in \mathfrak{b},$$

i.e., the polynomial R annihiliates all pairs (b, y_b). However, this polynomial with coefficients in $I \subset \mathring{N}(P)$ is of degrees too small to be in the ideal generated by $P_x \cap P_y$ mod P, which implies that the set of solutions of $R(b, y_b) = 0$ must be strictly smaller than $|\mathfrak{b}|$, which is a contradiction. Therefore, the matrix B_I is invertible, and thus B has rank $|\mathfrak{b}|$. Let us consider $Q \in \mathbb{C}^{\mathring{N}(P)}$, that we decompose as $Q = Q_I + Q_{\bar{I}}$. $\mathcal{O}^1(\Sigma)$ is the vector space of $Q \frac{dx}{P_y(x,y)}$ such that

$$0 = B.Q = B_I.Q_I + B_{\bar{I}}.Q_{\bar{I}},$$

since B_I is invertible, this implies

$$Q_I = -B_I^{-1}.B_{\bar{I}}.Q_{\bar{I}}.$$

In other words, each choice of $Q_{\bar{I}}$, i.e., $(i', j') \in \bar{I}$ yields an element of $\mathcal{O}^1(\Sigma)$.

The rest of the lemma follows by writing explicitly the inverse matrix in terms of minors. ∎

A.1. Case of other fields

Consider the case where $P \in \mathbb{F}[x, y]$ with $\mathbb{F}$ a commutative field.

Lemma A.2. *All coefficients $Q_{i,j}^{(i',j')}$ defined in (A.1) belong to the same field $\mathbb{F}$ as the coefficients of P.*

Proof. Notice that the matrix $B_I^{-1}.B_{\bar{I}}$ belongs a priori to $\mathbb{F}[\mathfrak{b}]^{I \times \bar{I}}$, however, the matrix product amounts to a symmetric sum on $b \in \mathfrak{b}$, and thus the matrix elements of $B_I^{-1}.B_{\bar{I}}$ are symmetric polynomials of the elements of $\mathfrak{b}$, and from Lemma 4.1 they belong to $\mathbb{F}$:

$$B_I^{-1}.B_{\bar{I}} \in \mathbb{F}^{I \times \bar{I}}.$$

∎

B. Proof of Lemma 4.1

Proof. If b is a double point (assumed regular), it must be a root of $D(x)$, and it thus belongs to an algebraic extension $\mathbb{F}[b]$. Moreover, b must be a common root of $D(x)$ and $\tilde{D}(x)$, therefore the vector $(b, b^2, b^3, b^4, \ldots, b^{\deg D + \deg \tilde{D}})$ must be in the kernel of the matrix

$$R = \begin{pmatrix} D_0 & D_1 & D_2 & \ldots & D_{\deg D} & & & & & & \\ & D_0 & D_1 & D_2 & & \ldots & & D_{\deg D} & & & \\ & & \ddots & & & & & & \ddots & & \\ & & & \ddots & & & & & & \ddots & \\ & & & & \ddots & & & & & & \ddots \\ & & & & & \ddots & & & & & \ddots \\ & & & & & & D_0 & D_1 & D_2 & \ldots & D_{\deg D} \\ \tilde{D}_0 & \tilde{D}_1 & \tilde{D}_2 & \ldots & & & & \tilde{D}_{\deg \tilde{D}} & & & \\ & \tilde{D}_0 & \tilde{D}_1 & \tilde{D}_2 & & \ldots & & & \tilde{D}_{\deg \tilde{D}} & & \\ & & \ddots & & & & & & & \ddots & \\ & & & \tilde{D}_0 & \tilde{D}_1 & & \tilde{D}_2 & \ldots & & & \tilde{D}_{\deg \tilde{D}} \end{pmatrix}.$$

That matrix must have a kernel of dimension $|\mathfrak{b}|$, and its rank is $\deg D + \deg \tilde{D} - |\mathfrak{b}|$. Let us decompose it by blocks as

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where D is invertible, and all matrices have their coefficients in $\mathbb{F}$. Writing that $R.(b, b^2, \ldots, b^{\deg D + \deg \tilde{D}}) = 0$ implies that

$$b^{|\mathfrak{b}|+1} = -\sum_{i=1}^{|\mathfrak{b}|} (D^{-1}C)_{|\mathfrak{b}|+1,i} b^i.$$

This implies that there is a polynomial $S(x) \in \mathbb{F}[x]$ of degree $|\mathfrak{b}|$ with all coefficients in $\mathbb{F}$, such that

$$\sum_{i=0}^{|\mathfrak{b}|} S_i b^i = 0, \quad S_{|\mathfrak{b}|} = 1, S_{i-1} = (D^{-1}C)_{|\mathfrak{b}|+1,i}.$$

This implies that integer symmetric polynomials of $b_1, \dots, b_{|\mathfrak{b}|}$ are algebraic combinations of the S_is and thus belong to $\mathbb{F}$.

The symmetric polynomials of branchpoints together with double points, i.e., all the zeros of $D(x) \in \mathbb{F}[x]$, are in $\mathbb{F}$. By subtracting those of the double points, we see that integer symmetric polynomials of branchpoints are in $\mathbb{F}$. ∎

Remark that the proof needs to compute the inverse D^{-1}, and in particular needs to divide by $\det D$. This is where we needed a field rather than a ring, and also we simplified the discussion by assuming characteristic 0.

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