Triply mixed coverings of arbitrary base curves: quasimodularity, quantum curves and a mysterious topological recursion

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Abstract. Simple Hurwitz numbers are classical invariants in enumerative geometry counting branched morphisms between Riemann surfaces with fixed ramification data. In recent years, several modifications of this notion for genus 0 base curves have appeared in the literature. Among them are so-called monotone Hurwitz numbers, which are related to the Harish-Chandra–Itzykson–Zuber integral in random matrix theory and strictly monotone Hurwitz numbers which enumerate certain Grothendieck dessins d'enfants. We generalise the notion of Hurwitz numbers to interpolations between simple, monotone and strictly monotone Hurwitz numbers for arbitrary genera and any number of arbitrary but fixed ramification profiles. This yields generalisations of several results known for Hurwitz numbers. When the target surface is of genus one, we show that the generating series of these interpolated Hurwitz numbers are quasimodular forms. In the case that all ramification is simple, we refine this result by writing this series as a sum of quasimodular forms corresponding to tropical covers weighted by Gromov–Witten invariants. Moreover, we derive a quantum curve for monotone and Grothendieck dessins d'enfants Hurwitz numbers for arbitrary genera and one arbitrary but fixed ramification profile. Thus, we obtain spectral curves via the semi-classical limit as input data for the Chekhov–Eynard–Orantin (CEO) topological recursion. Astonishingly, we find that the CEO topological recursion for the genus 1 spectral curve of the strictly monotone Hurwitz numbers computes the monotone Hurwitz numbers in genus 0. Thus, we give a new proof that monotone Hurwitz numbers satisfy CEO topological recursion. This points to an unknown relation between those enumerative invariants. Finally, specializing to target surface $\mathbb{P}^1$, we find recursions for monotone and Grothendieck dessins d'enfants double Hurwitz numbers, which enables the computation of the respective Hurwitz numbers for any genera with one arbitrary but fixed ramification profile.

Contents

1. Introduction

Hurwitz numbers are enumerations of branched morphisms between Riemann surfaces with fixed ramification data. They were first introduced by Adolf Hurwitz in the late 19th century [26] as coverings of the Riemann sphere. As observed by Hurwitz himself, these enumerations are closely related to the combinatorics of the symmetric group [27]. In particular, this yields an elegant interpretation of Hurwitz numbers in terms of factorisations in the symmetric group. In the last two decades Hurwitz numbers have branched out into several areas of mathematics, such as algebraic geometry, Gromov–Witten theory, algebraic topology, representation theory of the symmetric group, operator theory, integrable systems, random matrix models, tropical geometry and many more.

Several specifications of Hurwitz numbers with respect to the genera of the involved Riemann surfaces and the ramification profiles of the morphisms have proved to be of particular interest. A common theme is to allow a finite number of arbitrary but fixed ramification profiles and a simple ramification profile everywhere else. Hurwitz numbers obtained by such specifications are called *simple Hurwitz numbers*. Among the most important ones are *single* and *double* Hurwitz numbers.

Moreover, there are variants of Hurwitz numbers obtained by counting factorisations in the symmetric group as above, but with additional conditions. So, far, these have been studied for target surfaces of genus 0. Two of the most important cases are *monotone* and *strictly monotone* Hurwitz numbers, the latter of which are also called *Grothendieck dessins d'enfants Hurwitz numbers* [11, 13, 21–24, 32]. Monotone Hurwitz numbers appear as coefficients in the expansion of the HCIZ integral in random matrix theory [21], while strictly monotone Hurwitz numbers are equivalent to counting certain Grothendieck dessins d'enfants [1, 32].

In [24] a combinatorial interpolation between simple, monotone and strictly monotone Hurwitz numbers was introduced for genus 0 target surfaces and two arbitrary but fixed ramification profiles. This interpolation is called *triply mixed Hurwitz numbers*. In this work, we generalise the notion of triply mixed Hurwitz numbers to arbitrary genera and any number of arbitrary but fixed ramification profiles. We study these objects from several perspectives. This yields generalisations of various results known for simple Hurwitz numbers, which in particular specialise to the extremal cases of monotone and Grothendieck dessins d'enfant Hurwitz numbers.

1.1. Previous results

We summarize some of the previous results on several kinds of Hurwitz numbers, which motivated our work.

Quasimodularity. Hurwitz numbers with a target surface of genus 1 can be expressed in terms of so-called *shifted symmetric polynomials* (see Section 2.4.2). The Bloch–Okounkov theorem [4], initially proved in the special cases corresponding to Hurwitz numbers for which all ramification are simple [9, 29], implies that certain generating series associated to shifted symmetric polynomials are quasimodular. This implies that the generating series of (connected) Hurwitz numbers with a target surface of genus 1 are quasimodular, as noted in [17]. From the discussion at the end of [22, Section 3], it follows that monotone Hurwitz numbers are shifted symmetric polynomials as well.

Refined quasimodularity and tropical covers. A common theme in tropical geometry is to express geometric enumerative problems in terms of weighted graphs. In [3, 5, 6], Hurwitz numbers with only simple ramification were related to so-called *tropical covers*, i.e., piecewise linear maps between metric graphs. In particular, Hurwitz numbers were expressed as a finite sum of tropical covers. This led to the conjecture (which was proved in [19]), that each generating series obtained by considering all covers with source curves of a fixed combinatorial type are quasimodular as well. This refines the aforementioned result of [9, 29]. A tropical interpretation of monotone and strictly monotone double Hurwitz numbers for genus 0 target surfaces was first found in [11, 23] by equipping the involved tropical covers with an additional colouring and labeling. Motivated by this work, Lewanski and the first author derived a different interpretation of monotone and strictly monotone double Hurwitz numbers for genus 0 target surfaces, which is more natural in the sense that it does not require additional colouring and labeling. The involved covers are now weighted by 1-point relative Gromov–Witten invariants.

Topological recursion and quantum curves. In recent years, one of the most fruitful interactions with Hurwitz theory has been from the viewpoint of Chekhov–Eynard–Orantin (CEO) topological recursion (see [18]). CEO topological recursion associates to a spectral curve a family of differentials, which satisfies a certain recursion. Remarkably enough, for many enumerative invariants one can find spectral curves, such that the associated differentials encode these invariants as coefficients in local expansions. One says that such an enumerative problem *satisfies CEO topological recursion*. It turns out that certain kinds of Hurwitz numbers satisfy CEO topological recursion [10–12, 15, 16, 18, 30, 35]. A direct consequence of CEO topological recursion is an interpretation of the enumerative problem in terms of intersection products on the moduli space of stable curves with marked points $\overline{\mathcal{M}}_{g,n}$.

One approach to CEO topological recursion which has been very successful is in terms of so-called *quantum curves.* Given an enumerative problem, an associated quantum curves is a certain partial differential equation, which is satisfied by a generating series of the initial enumeration. This quantum curve is often time an indication for the shape of the spectral curve for which one then runs CEO topological recursion.

1.2. Results of this paper

First of all, we define our new enumerative object, i.e., *triply mixed Hurwitz numbers* for target surfaces of higher genera. We then study several specifications of these numbers from various perspectives.

Quasimodularity. When the target surface is of genus 1, we express triply mixed Hurwitz numbers in terms of shifted symmetric functions. This allows to prove that the generating series of triply mixed Hurwitz numbers are quasimodular forms of mixed weight. Our results are summarised in Theorem 4.4.

Refined Quasimodularity. We use the aforementioned result of Lewanski and the first author and derive an expression of monotone and Grothendieck dessins d'enfants Hurwitz numbers for genus 1 target surfaces and only simple ramification in terms of tropical covers. This enables us to prove that fixing the combinatorial type of the source curve of the tropical covers yields a quasimodular form. This refines the quasimodularity result for triply mixed Hurwitz numbers analogously to the simple case. The tropical correspondence theorem is stated in Theorem 5.3 and the refined quasimodularity statement in Theorem 5.11.

Quantum curves. Motivated by work of Liu, Mulase, and Sorkin [33], we derive a quantum curve for monotone and Grothendieck dessins d'enfants Hurwitz numbers for arbitrary genera and one arbitrary but fixed ramification profile. The result for the monotone enumerations can be found in Theorem 6.4.

A mysterious topological recursion. We consider the quantum curve for Grothen-dieck dessins d'enfants Hurwitz numbers for genus 1 base curves. This quantum curve yields a spectral curve via its semi-classical limit. We use this spectral curve as input data for the CEO topological recursion. Astonishingly, we prove in Section 7 that the expansion of the resulting differentials yield the monotone Hurwitz numbers for genus 0 base curves, however for a different normalisation and different spectral curve than the one in [10]. This points towards an unknown relation between the strictly monotone numbers in genus 1 and the monotone ones in genus 0.

Further recursive procedures. We prove recursions for refinements of monotone and Grothendieck dessins d'enfants double Hurwitz numbers for the target surface $\mathbb{P}^1$, which yields the Hurwitz numbers for any genera with one arbitrary but fixed ramific-ation profile. This generalises the recursion for monotone orbifold Hurwitz numbers in [11]. The explicit recursions can be found in Theorem 8.4.

Structure of the paper. In Section 2, we recall the relevant notions needed for our discussion. Triply mixed Hurwitz numbers are introduced in Section 3. Quasimodu-larity of these Hurwitz numbers is shown in Section 4 and refined quasimodularity is discussed in Section 5. In Section 6 we derive quantum curves for both mono-tone and Grothendieck dessins d'enfants Hurwitz numbers. In Section 7, we discuss the special case of the quantum curve for Grothendieck dessins d'enfants Hurwitz numbers with elliptic base curve and prove that CEO topological recursion for the semi-classical limit of this quantum curve computes monotone Hurwitz numbers with rational base curve. Finally, we derive the recursions for refinements of monotone Grothendieck dessins d'enfants double Hurwitz numbers in Section 8. We collected several examples of quasimodular generating series of triply mixed Hurwitz numbers in Appendix A.

2. Preliminaries

2.1. Hurwitz numbers

In this section, we recall some of the basic notions of Hurwitz theory. We begin by defining classical Hurwitz numbers in the most general sense. We also review some of the specifications and variations on the definition of Hurwitz numbers relevant for this paper.

2.1.1. Classical Hurwitz numbers. Let λ be a composition, i.e., a finite sequence of strictly positive integers. Denote by $|\lambda|$ the integer where λ is a composition of and let $\ell(\lambda)$ be the number of parts of λ. Write $p(\lambda)$ for the ordered composition (parti-

tion) corresponding to λ. Given a permutation $\sigma \in S_d$, denote by $c(\sigma)$ the partition which corresponds to the cycle type of σ.

Definition 2.1. Let $g', g \geq 0$ be non-negative integers, d a positive integer and $\mu = (\mu^1, \ldots, \mu^n)$ a tuple of compositions of d. In case

$$2g' - 2 = d \cdot (2g - 2) + \sum_{j=1}^{n} (|\mu^j| - \ell(\mu^j)),$$

we call $(\sigma_1, \ldots, \sigma_n, \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$ a *factorization of type* (g, g', d, μ) if the following conditions are satisfied:

(1) $\sigma_i, \alpha_i, \beta_i \in S_d$,

(2) $\sigma_1 \cdots \sigma_n = [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]$,

(3) $c(\sigma_i) = p(\mu^i)$.

If additionally we have

(4) $\langle \sigma_1, \ldots, \sigma_n, \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \rangle$ acts transitively on the set $\{1, 2, \ldots, d\}$

we call the factorization *connected*. Denote by $\mathcal{F}^\bullet(g, g', d, \mu)$ and $\mathcal{F}(g, g', d, \mu)$ the sets of factorizations respectively connected factorizations of type (g, g', d, μ). The *Hurwitz numbers* and *connected Hurwitz numbers* are defined by

$$h_{g'}^{\bullet, g}(\mu^1, \ldots, \mu^n) = \frac{1}{d!} |\mathcal{F}^\bullet(g, g', d, \mu)|,$$

respectively

$$h_{g'}^{g}(\mu^1, \ldots, \mu^n) = \frac{1}{d!} |\mathcal{F}(g, g', d, \mu)|.$$

Remark 2.2. Historically, Hurwitz numbers were first defined as an enumerative problem counting ramified morphisms between Riemann surfaces: for a fixed compact Riemann surface S of genus g, Hurwitz numbers count holomorphic maps $\pi : S' \to S$ (up to isomorphism), where S' is a compact Riemann surface of genus g', such that

- π has ramification profile $\mu^1, \ldots, \mu^n$ over n arbitrary, but fixed points on S;
- each map is weighted by $\frac{1}{|\text{Aut}(\pi)|}$.

The connection to our definition, which is due to Hurwitz, is made by considering the monodromy representations for the holomorphic maps involved (see [26, 27]).

Sometimes, we count Hurwitz numbers with labels in order to distinguish cycles of the same length. The following definition makes this more precise. We note that geometrically, this corresponds to labelling the preimages of all branch points.

Definition 2.3. A *labeled factorization* is a factorization as defined by Definition 2.1 together with a labelling:

(5) the disjoint cycles of σ_i for all i are labeled with labels $1, \ldots, \ell(\mu^i)$ such that a cycle of σ_i with label j has length $(\mu^i)_j$.

We use an arrow to indicate that we are considering labeled factorizations and Hurwitz numbers, just as we are using a dot to denote non-necessarily connected factorizations and Hurwitz numbers.

Remark 2.4. The labeled Hurwitz numbers equal the ordinary Hurwitz numbers up to a constant:

$$\vec{h}_{g'}^{\bullet;g}(\mu^1, \ldots, \mu^n) = \operatorname{Aut}\mu \cdot h_{g'}^{\bullet;g}(\mu^1, \ldots, \mu^n)$$

with $\operatorname{Aut}\mu = \prod_{m=1}^{\infty}\prod_{i=1}^{n} r_m(\mu^i)!$ and $r_m(\mu^i)$ denotes the number of parts equal to m in the partition μ^i.

2.1.2. Special instances of Hurwitz numbers. There are several special cases and variations of the above general definition of classical Hurwitz numbers. A general theme is to allow a few complicated partitions and force almost all partitions to be *simple*, i.e., equal to the partition $(2, 1, \ldots, 1)$ (which corresponds to the conjugacy class of transpositions). In the focus of this paper are the following two special cases of Hurwitz numbers.

Definition 2.5. Let $T := (2, 1, \ldots, 1)$ be the simple partition of size d. In the same setting as in Definition 2.1, we consider two cases.

(a) If $\mu^2, \ldots, \mu^n = T$, we call the resulting Hurwitz number *single base g Hurwitz number* and denote it by

$$H_{g'}^{\bullet;g}(\mu_1^1, \ldots, \mu_{\ell(\mu^1)}^1) := h_{g'}^{\bullet;g}(\mu^1, T, \ldots, T).$$

Note that the number r of simple partitions is then given by

$$r = 2g' - 2 + \ell(\mu^1) - d(2g - 1).$$

(b) If $g = 0$ and $\mu^3, \ldots, \mu^n = T$, we call the resulting number a *double Hurwitz number* and denote it by

$$h_{g'}^{\bullet}(\mu^1, \mu^2) := h_{g'}^{\bullet;0}(\mu^1, \mu^2, T, \ldots, T).$$

Note that the number r of simple partitions is then given by

$$r = 2g - 2 + \ell(\mu^1) + \ell(\mu^2).$$

2.1.3. Monotone and strictly monotone Hurwitz numbers. Two variants of Hurwitz numbers are particularly relevant for this work: monotone and strictly monotone Hurwitz numbers. In the following, we define those numbers.

Definition 2.6. Let $\mu = (\mu^1, \ldots, \mu^{k+2})$ be a tuple of compositions of d. We call a (connected/labeled) factorization of type $(0, g', d, \mu)$, where $\mu^3 = \cdots = \mu^{k+2} = (2, 1, \ldots, 1)$ a (connected/labeled) *monotone factorization of type* (g', μ^1, μ^2) if it satisfies the following additional property:

(6) the transpositions $\sigma_i = (s_i t_i)$ with $s_i < t_i$ satisfy

$$t_i \leq t_{i+1} \quad \text{for } i = 3, \ldots, k + 1.$$

We then define the monotone Hurwitz number $h^{\bullet}_{\leq, g'}(\mu^1, \mu^2)$ as the product of $\frac{1}{d!}$ and the number of monotone factorisations of type (g', μ^1, μ^2) and similarly define connected/labeled monotone double Hurwitz numbers. We define *strictly monotone double Hurwitz numbers*, also called *Grothendieck dessins d'enfant Hurwitz numbers*, by changing the monotonicity condition to a strict one:

(6′) the transpositions $\sigma_i = (s_i t_i)$ with $s_i < t_i$ satisfy

$$t_i < t_{i+1} \quad \text{for } i = 3, \ldots, k + 1.$$

We denote the strictly monotone double Hurwitz number by $h^{\bullet}_{<, g'}(\mu^1, \mu^2)$.

We note that Definition 3.1 generalizes (strictly) monotone Hurwitz numbers to target surfaces of higher genera. Further, we note that strictly monotone Hurwitz numbers are also called *Grothendieck dessins d'enfants Hurwitz numbers*, as they are equivalent to an enumeration of Grothendieck dessins d'enfants as proved in [1, Section 4.3.2].

2.2. Quantum curves

In [33], the connected single base g Hurwitz numbers were studied with a view towards topological recursion. The main results are summarized in Theorem 2.7.

Denote by $\mathbb{Z}_+$ the set of positive integers and an element $\nu \in \mathbb{Z}_+^n$ by

$$\nu = (\nu_1, \ldots, \nu_n).$$

Define the discrete Laplace transform of the single base g Hurwitz numbers by

$$F_{g'}^g(x_1, \ldots, x_n) = \sum_{\nu \in \mathbb{Z}_+^n} \vec{H}_{g'}^g(\nu_1, \ldots, \nu_n) \prod_{i=1}^{n} e^{-w_i \nu_i} \quad (x_i = e^{-w_i}),$$

which is usually referred to as the *free energy*. We further define the so-called *partition function* by

$$Z^g(x, \hbar) = \exp\left(\sum_{g'=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n!} \hbar^{2g'-2+n} F_{g'}^g(x, \ldots, x) \right).$$

Theorem 2.7 ([33, Theorem 1.1]). *The free energies $F_{g'}^g(x_1, \ldots, x_n)$, for $2g - 2 + \ell(\mu) > 0$, satisfy a cut-and-join type partial differential equation.*

The single base g Hurwitz partition function satisfies a quantum curve-like infinite order differential equation

$$\hbar x \frac{d}{dx}\left[1 - \hbar^{1-\chi} x e^{\hbar x \frac{d}{dx}} \left(\frac{d}{dx} x\right)^{1-\chi}\right] Z^g(x, \hbar) = 0,$$

where $\chi = 2 - 2g$ (which is the Euler characteristic of the base curve of the Hurwitz problem). If we introduce

$$y = \hbar x \frac{d}{dx}$$

and regard it as a commuting variable, then the total symbol of the above operator produces the following equation

$$x = y^{\chi-1} e^{-y},$$

which is commonly referred to as a Lambert curve.

Remark 2.8. A cut-and-join type partial differential equation is a certain kind of partial differential equation, which reflects the combinatorics of multiplying elements in the symmetric group. In the case of simple Hurwitz numbers, these combinatorics entail the fact that left-multiplication by a transposition either joins two cycles to one or cuts a single cycle into two.

2.3. Stirling numbers

We now define Stirling numbers of the first and second kind.

Definition 2.9. For $n, k \in \mathbb{N}$, we define *Stirling numbers of the first kind* by the recurrence relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = n \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix} \quad \text{for } k > 0$$

and

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1 \quad \text{and} \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ n \end{bmatrix} = 0 \quad \text{for } n > 0,$$

and *Stirling numbers of the second kind* by the recurrence relation

$$\begin{Bmatrix} n+1 \\ k \end{Bmatrix} = k \begin{Bmatrix} n \\ k \end{Bmatrix} + \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \quad \text{for } k > 0$$

and

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = 1 \quad \text{and} \quad \begin{Bmatrix} n \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ n \end{Bmatrix} = 0 \quad \text{for } n > 0.$$

Remark 2.10. Recall the generating functions of Stirling numbers

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{z^{n-k}}{n!} = \prod_{r=1}^{n-1} (1 - rz) \quad \text{and} \quad \sum_{n=k}^{\infty} \begin{Bmatrix} n \\ k \end{Bmatrix} x^{n-k} = \prod_{r=1}^{k} \frac{1}{1 - rx}. \tag{2.1}$$

2.4. Shifted symmetric functions and quasimodular forms

2.4.1. Central characters. Let $\mathcal{Z}_n$ be the center of the group algebra $\mathbb{C}[S_n]$ of the symmetric group. Given a partition λ, denote by $|\lambda|$ the integer where λ is a partition of and by C_λ the sum of all elements in $S_{|\lambda|}$ of cycle type λ. Let $(\rho_\lambda, V_\lambda)$ be the irreducible representation of $S_{|\lambda|}$ corresponding to λ. Note that a basis of $\mathcal{Z}_n$ is given by C_λ for all partitions λ of n. Every element $z \in \mathcal{Z}_n$ defines by Schur's lemma a constant, called the *central character* $\omega^\lambda(z)$, by which $\rho_\lambda(z)$ acts on V_λ. For example, the central character of the conjugacy class sums C_ν is given by

$$f_\nu(\lambda) := \omega^\lambda(C_\nu) = |C_\nu| \frac{\chi^\lambda(\nu)}{\dim \lambda}, \tag{2.2}$$

where (by abuse of notation) $|C_\nu|$ denotes the size of the conjugacy class of elements of cycle type ν in $S_{|\nu|}$, χ^λ denotes the character of ρ_λ and $\dim \lambda$ the dimension of ρ_λ. We extend equation (2.2) to the case when $|\nu| \neq |\lambda|$ by

$$f_\nu(\lambda) := \binom{|\lambda|}{|\nu|} |C_\nu| \frac{\chi^\lambda(\nu)}{\dim \lambda}.$$

Observe that if ν is a partition without parts equal to 1 the quantity $\binom{|\lambda|}{|\nu|}|C_\nu|$ equals the size of the conjugacy class of cycle type ν in $S_{|\lambda|}$ instead of in $S_{|\nu|}$. For a tuple of partitions $\mu = (\mu^1, \dots, \mu^n)$ let $f_\mu = \prod_{i=1}^{n} f_{\mu^i}$. Hurwitz numbers can be expressed in terms of central characters, for example classical Hurwitz numbers of torus coverings satisfy

$$h_{g,'}^{\bullet;1}(\mu) = \sum_{|\lambda|=d} f_\mu(\lambda). \tag{2.3}$$

with d – as in the definition of Hurwitz numbers – implicitly given by the size of the partitions where μ consists of. In Proposition 4.1 we generalise this expression to triply mixed Hurwitz numbers.

Let $\Xi_d = (J_1, J_2, \dots, J_d, 0, 0, \dots)$ be the sequence of Jucys–Murphy elements given by $J_k = \sum_{i=1}^{k-1} (ik)$. Although the Jucys–Murphy elements do not belong to $\mathcal{Z}_n$, symmetric polynomials in Ξ_d are elements of $\mathcal{Z}_n$. More precisely, every element of $\mathcal{Z}_n$ can be written as a symmetric polynomial in Ξ_d, see [28, 34]. Remarkably, the central character of a symmetric polynomial f evaluated at Ξ_d simply equals

$$\omega^\lambda(f(\Xi_d)) = f(\text{cont}_\lambda),$$

where cont_λ denotes the sequence of all contents of λ.

The commutator sum $\Re = \sum_{\alpha,\beta\in S_d}[\beta,\alpha]$ also is an element of $\mathcal{Z}_n$. Its central character equals

$$\omega^\lambda(\Re) = \left(\frac{d!}{\dim\lambda}\right)^2.$$

2.4.2. Central characters as shifted symmetric polynomials.

Central characters of $z \in \mathcal{Z}_n$ are examples of *shifted symmetric polynomials*, introduced by Okounkov and Olshanski [37]. A rational polynomial in m variables $x_1, \ldots, x_m$ is called *shifted symmetric* if it is invariant under the action of all $\sigma \in \mathfrak{S}_m$ given by $x_i \mapsto x_{\sigma(i)} + i - \sigma(i)$ (or, more symmetrically, $x_i - i \mapsto x_{\sigma(i)} - \sigma(i)$). Denote by $\Lambda^*(m)$ the space of shifted symmetric polynomials in m variables, which is filtered by the degree of the polynomials. We have forgetful maps $\Lambda^*(m) \to \Lambda^*(m-1)$ given by $x_m \mapsto 0$, so that we can define the space of shifted symmetric polynomials Λ^* as

$$\varprojlim_m \Lambda^*(m)$$

in the category of filtered algebras. As $\omega^\lambda(z)$ is a shifted symmetric polynomial, it can be expressed as a symmetric polynomial in $\lambda_1 - 1, \lambda_2 - 2, \ldots$. For example,

$$f_{()}(\lambda) = 1,$$

$$f_{(1)} = |\lambda| = \sum_{i=1}^\infty ((\lambda_i - i) + i),$$

$$f_{(2)}(\lambda) = \frac{1}{2}\sum_{i=1}^\infty \left(\left(\lambda_i - i + \frac{1}{2}\right)^2 - \left(-i + \frac{1}{2}\right)^2\right).$$

More precisely, considered as a vector space, the algebra of shifted symmetric functions Λ^* has a basis f_ν where ν ranges over all partitions.

2.4.3. The Bloch–Okounkov theorem.

A distinguished generating set for Λ^* is given by the renormalized shifted symmetric power sums

$$Q_0(\lambda) = 1,$$

$$Q_k(\lambda) = c_k + \frac{1}{(k-1)!}\sum_{i=1}^\infty \left(\left(\lambda_i - i + \frac{1}{2}\right)^{k-1} - \left(-i + \frac{1}{2}\right)^{k-1}\right), \qquad k \geq 1,$$

with c_k defined by

$$\frac{1}{\varsigma(z)} := \frac{1}{2\sinh(z/2)} =: \sum_{k=1}^\infty c_k z^{k-1}. \tag{2.4}$$

Define a weight grading on Λ^* by assigning to Q_k weight k.

This weight grading corresponds to the weight of quasimodular forms under the Bloch–Okounkov theorem, as follows. The graded algebra of quasimodular forms (for $SL_2(\mathbb{Z})$) is given by $\tilde{M} = \mathbb{Q}[P, Q, R]$, where $P = -24G_2$, $Q = 240G_4$, $R = -504G_6$ are Ramanujan's notation for the Eisenstein series

$$G_k(\tau) = -\frac{B_k}{2k} + \sum_{r=1}^{\infty} \sum_{m=1}^{\infty} m^{k-1} q^{mr} \quad (B_k = k\text{th Bernoulli number and } q = e^{2\pi i \tau})$$

of weight k. Given a function f on partitions, introduce the q-bracket of f, given by

$$\langle f \rangle_q = \frac{\sum_{\lambda \in \mathcal{P}} f(\lambda) q^{|\lambda|}}{\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}} \in \mathbb{C}[[q]].$$

The denominator $\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}$ equals $q^{1/24} \eta(\tau)^{-1}$ with $\eta(\tau)$ the Dedekind eta function. Then, by the celebrated Bloch–Okounkov theorem the q-bracket $\langle f \rangle_q$ of a shifted symmetric polynomial f is the q-expansion of a quasimodular form [4, Theorem 0.5]. Moreover, if f has weight k as defined above, then $\langle f \rangle_q$ is quasimodular of the same weight k.

2.4.4. Weights of central characters. We give the (mixed) weight of the central characters defined in Section 2.4.1 in terms of the weight grading on Λ^*. Define completion coefficients $q_{k,\nu}$ with $k \geq 2$ and ν a partition by

$$Q_k = \sum_{\nu} q_{k,\nu} f_{\nu}.$$

The Gromov–Witten/Hurwitz correspondence provides the following formula for these coefficients:

Proposition 2.11 ([36, Proposition 3.2]). *The completion coefficients $q_{k,\nu}$ satisfy*

$$\sum_{k=1}^{\infty} q_{k+1,\nu} z^k = \frac{1}{|\nu|!} (e^{z/2} - e^{-z/2})^{|\nu|-1} \prod_{i=1}^{\ell(\nu)} (e^{\nu_i z/2} - e^{-\nu_i z/2}).$$

In particular, $q_{k,\nu} = 0$ if $|\nu| + \ell(\nu) > k$. Correspondingly, the function f_{ν} is of (mixed) weight at most $|\nu| + \ell(\nu)$, which was proved in [31, Theorem 5].

For the weights of symmetric polynomials evaluated at the contents of a partition we have the following result. Let h_n and e_n be the complete homogeneous symmetric polynomial respectively elementary symmetric polynomial of degree n.

Proposition 2.12. *Let $d \geq 1$. The top-weight part of $h_d(\text{cont}_\lambda)$ and $e_d(\text{cont}_\lambda)$ is given by $\frac{1}{2^{d-1}} Q_3(\lambda)^d$.*

Proof. Let

$$(x)_k = x(x-1)\cdots(x-k+1).$$

Then, for the symmetric polynomial

$$\underline{p}_k(x_1, \ldots, x_n) := \sum_{i=1}^{n}(x_i)_k$$

it is known that [31, Theorem 4]

$$k\,\underline{p}_{k-1}(\mathrm{cont}_\lambda) = \sum_{i=1}^{\infty}(\lambda_i - i + 1)_k - (-i + 1)_k \quad (k \geq 1) \tag{2.5}$$

where the right-hand side is a shifted symmetric polynomial of mixed degree $\leq k + 1$.

   Given a symmetric polynomial g of degree d and with constant term equal to zero, one can write g as a polynomial in the $\underline{p}_k$ for $k \geq 1$. Assign to $\underline{p}_k$ weight $k + 2$ in accordance with equation (2.5), i.e., $\underline{p}_k(\mathrm{cont}_\lambda)$ is a shifted symmetric polynomial of mixed weight $k + 2$. Observe that the monomial $\underline{p}_1^d$ is the unique monomial of degree d and weight at least (precisely) $3d$. Hence, the central character $\omega^\lambda(f(\Xi_d))$ of g is a shifted symmetric function of weight at most $3d$ with top-degree part up to a multiplicative constant equal to $Q_3(\lambda)^d$. Specializing g to h_d and e_d the result follows by observing that in this case the coefficient of $\underline{p}_1^d$ equals $\frac{1}{2^{d-1}}$.   ∎

2.5. Gromov–Witten invariants with target $\mathbb{P}^1$

In this section, we introduce the basic notions of Gromov–Witten theory needed for this work. For a more concise introduction in the context of tropical geometry, see e.g., [7]. For a more general introduction to the topic, we recommend [40].

   We denote by $\bar{M}_{g,n}(\mathbb{P}^1, d)$ the moduli space of stable maps with n marked points, which is a Deligne–Mumford stack of virtual dimension $2g - 2 + 2d + n$. It consists of tuples $(X, x_1, \ldots, x_n, f)$, such that X is a connected, projective curve of genus g with at worst nodal singularities, $x_1, \ldots, x_n$ are non-singular points on X and $f \colon X \to \mathbb{P}^1$ is a function with $f_*([X]) = d[\mathbb{P}^1]$. Moreover, f may only have a finite automorphism group (respecting markings and singularities). In order to define enumerative invariants, we introduce

- The i-th evaluation morphism is the map $\mathrm{ev}_i \colon \bar{M}_{g,n}(\mathbb{P}^1, d) \to \mathbb{P}^1$ by mapping the tuple $(X, x_1, \ldots, x_n, f)$ to x_i.

- The i-th cotangent line bundle $\mathbb{L}_i \to \bar{M}_{g,n}(\mathbb{P}^1, d)$ is obtained by identifying the fiber of each point with the cotangent space $\mathbb{T}_{x_i}^*(X)$. The first Chern class of i-th cotangent line bundle is called a ψ-*class,* which we denote by $\psi_i = c_1(\mathbb{L}_i)$.

   This yields the following definition.

Definition 2.13. Fix g, n, d and let $k_1, \ldots, k_n$ be non-negative integers, such that

$$k_1 + \cdots + k_n = 2g + 2d - 2.$$

Then, the stationary Gromov–Witten invariant is defined by

$$\langle \tau_{k_1}(\mathrm{pt}) \cdots \tau_{k_n}(\mathrm{pt}) \rangle_{g,n}^{\mathbb{P}^1} = \int\limits_{[\bar{M}_{g,n}(\mathbb{P}^1)]^{\mathrm{vir}}} \prod \mathrm{ev}_i^*(\mathrm{pt})\psi_i^{k_i},$$

where pt denotes a point on $\mathbb{P}^1$.

Remark 2.14. Analogously, we can define Gromov–Witten invariants for more general target curves Y

$$\langle \tau_{k_1}(\mathrm{pt}) \cdots \tau_{k_n}(\mathrm{pt}) \rangle_{g,n}^{Y}.$$

The following identity, which should be compared with equation (2.3) in the previous discussion on shifted symmetric functions, was proved in [36] for elliptic curves E:

$$\langle \tau_{k_1}(\mathrm{pt}) \cdots \tau_{k_n}(\mathrm{pt}) \rangle_{g,n}^{E,d} = \sum_{|\lambda|=d} \prod_{i=1}^{n} Q_{k_i+2}(\lambda).$$

Similarly, we consider the moduli space of relative stable maps $\bar{M}_{g,n}(\mathbb{P}^1, \nu, \mu, d)$ relative to two partitions μ, ν of d and define the relative Gromov–Witten invariants by

$$\langle \nu \mid \tau_{k_1}(\mathrm{pt}) \cdots \tau_{k_n}(\mathrm{pt}) \mid \mu \rangle_{g,n}^{\mathbb{P}^1} = \int\limits_{[\bar{M}_{g,n}(\mathbb{P}^1, \nu, \mu, d)]^{\mathrm{vir}}} \prod \mathrm{ev}_i^*(\mathrm{pt})\psi_i^{k_i}.$$

We note that in the following, we add subscripts "∘" and "•", which correspond to *connected* or *not necessarily connected* (for simplicity also called *disconnected* Gromov–Witten invariants), which in turn correspond to considering connected or disconnected stable maps.

2.6. Tropical covers and monotone/Grothendieck dessins d'enfants Hurwitz numbers

A detailed introduction to tropical covers can be found in [2]. We note that all graphs considered may contain half-edges.

Definition 2.15. An *abstract tropical curve* is a connected metric graph Γ, together with a function associating a genus $g(v)$ to each vertex v. Let $V(\Gamma)$ be the set of its vertices. Let $E(\Gamma)$ be the set of its internal edges, which we require to be bounded and let $E'(\Gamma)$ its set of all edges, respectively. The set of half-edges, which we call ends is

therefore $E'(\Gamma) \setminus E(\Gamma)$, and all ends are considered to have infinite length. The genus of an abstract tropical curve Γ is $g(\Gamma) := h^1(\Gamma) + \sum_{v \in V(\Gamma)} g(v)$, where $h^1(\Gamma)$ is the first Betti number of the underlying graph. An *isomorphism* of a tropical curve is an automorphism of the underlying graph that respects edges' lengths and vertices' genera. The *combinatorial type* of a tropical curve is the graph obtained by disregarding its metric structure.

Remark 2.16. An important tropical curve for this work is defined as $\mathbb{R}$ with finitely many decorated points. These points are the vertices, the intervals between the vertices (and from the extremal vertices to $\pm\infty$) are the edges and the length of the edges is the lengths of the intervals. For any choice of points, we denote the resulting abstract tropical curve by $\mathbb{P}^1_{\text{trop}}$ to highlight it as an analogue of the tropical projective line.

We now define tropical covers. For an illustration, see, e.g., [7, Figure 1].

Definition 2.17. A *tropical cover* is a surjective harmonic map $\pi: \Gamma_1 \to \Gamma_2$ between abstract tropical curves as in [2, Section 2], i.e., it satisfies the following conditions.

(i) $\pi(V(\Gamma_1)) \subset V(\Gamma_2)$.

(ii) $\pi^{-1}(E'(\Gamma_2)) \subset E'(\Gamma_1)$.

(iii) For each edge $e \in E'(\Gamma_i)$, denote by $l(e)$ its length. Interpreting $e \in E'(\Gamma_1)$, $\pi(e) \in E'(\Gamma_2)$ as intervals $[0, l(e)]$ and $[0, l(\pi(e))]$, we require π restricted to e to be a linear map of slope $\omega_e \in \mathbb{Z}_{\geq 0}$, that is $\pi: [0, l(e)] \to [0, l(\pi(e))]$ is given by $\pi(t) = \omega_e \cdot t$. We call ω_e the *weight* of e. If $\pi(e)$ is a vertex, we have $\omega_e = 0$.

(iv) For a vertex $v \in \Gamma_1$, let $v' = \pi(v)$. We choose an edge e' adjacent to v'. We define the local degree at v as

$$d_v = \sum_{\substack{e \in \Gamma_1 \\ \pi(e) = e'}} \omega_e.$$

We require d_v to be independent of the choice of edge e' adjacent to v'. We call this fact the *balancing* or *harmonicity condition*.

We furthermore introduce the following notions.

(i) The *degree* of a tropical cover π is the sum over all local degrees of preimages of any point in Γ_2. Due to the harmonicity condition, this number is independent of the point in Γ_2.

(ii) For any end $e \in \Gamma_2$, we define μ_e as the partition of weights of the ends of Γ_1 mapping to e. We call μ_e the *ramification profile* above e.

We call two tropical covers $\pi\colon\Gamma_1\to\Gamma_2$ and $\pi'\colon\Gamma_1'\to\Gamma_2$ *isomorphic* if there exists an isomorphism of graph $f\colon\Gamma_1\to\Gamma_1'$ respecting labels and weights, such that $\pi=\pi'\circ f$. We denote the automorphism group of a tropical cover π by $\mathrm{Aut}(\pi)$.

Theorem 2.18 ([25]). *Let g be a non-negative integer, and μ,ν partitions of the same size $d>0$. We have*

$$h_{g;\mu,\nu}^{\leq,\bullet}=\sum_{\lambda\vdash b}\ \sum_{\pi\in\Gamma(\mathbb{P}_{\mathrm{trop}}^1,g;\mu,\nu,\lambda)}\frac{1}{|\operatorname{Aut}(\pi)|}\frac{1}{\ell(\lambda)!}\prod_{v\in V(\Gamma)}m_v\prod_{e\in E(\Gamma)}\omega_e,$$

$$h_{g;\mu,\nu}^{<,\bullet}=\sum_{\lambda\vdash b}\ \sum_{\pi\in\Gamma(\mathbb{P}_{\mathrm{trop}}^1,g;\mu,\nu,\lambda)}\frac{1}{|\operatorname{Aut}(\pi)|}\frac{1}{\ell(\lambda)!}\prod_{v\in V(\Gamma)}(-1)^{1+\mathrm{val}(v)}m_v\prod_{e\in E(\Gamma)}\omega_e$$

where $\Gamma(\mathbb{P}_{\mathrm{trop}}^1,g;\mu,\nu,\lambda)$ is the set of tropical covers

$$\pi\colon\Gamma\longrightarrow\mathbb{P}_{\mathrm{trop}}^1=\mathbb{R}$$

with $b=2g-2+\ell(\mu)+\ell(\nu)$ points $p_1,\ldots,p_b$ fixed on the codomain $\mathbb{P}_{\mathrm{trop}}^1$, such that the following conditions are satisfied.

(i) *The unbounded left (resp. right) pointing ends of Γ have weights given by the partition μ (resp. ν).*

(ii) *The graph Γ has $l:=\ell(\lambda)$ vertices. Let $V(\Gamma)=\{v_1,\ldots,v_l\}$ be the set of its vertices. Then $\pi(v_i)=p_i$. Moreover, let $w_i=\mathrm{val}(v_i)$ be the corresponding valences (degrees).*

(iii) *We assign an integer $g(v_i)$ as the genus to v_i and the following condition holds true:*

$$h^1(\Gamma)+\sum_{i=1}^l g(v_i)=g.$$

(iv) *We have $\lambda_i=\mathrm{val}(v_i)+2g(v_i)-2$.*

(v) *For each vertex v_i, let $\mathbf{x}^+$ (resp. $\mathbf{x}^-$) be the tuple of weights of those edges adjacent to v_i, which map to the right-hand (resp. left-hand) of p_i. The multiplicity m_{v_i} of v_i is defined to be*

$$m_{v_i}=(\lambda_i-1)!|\operatorname{Aut}(\mathbf{x}^+)||\operatorname{Aut}(\mathbf{x}^-)|$$
$$\sum_{g_1^i+g_2^i=g(v_i)}\langle\tau_{2g_2^i-2}(\mathrm{pt})\rangle_{g_2^i}^{\mathbb{P}^1,\circ}\langle\mathbf{x}^+,\tau_{2g_1^i-2+\ell(\mathbf{x}^+)+\ell(\mathbf{x}^-)}(\mathrm{pt}),\mathbf{x}^-\rangle_{g_1^i}^{\mathbb{P}^1,\circ}.$$

Remark 2.19. The involved Gromov–Witten invariants can be computed using the functions $\varsigma(z)=2\sinh(z/2)=e^{z/2}-e^{-z/2}$ and $\mathcal{S}(z)=\frac{\varsigma(z)}{z}$.

- Recall the definition of the constants c_i determined by $\varsigma(z)$ in (2.4). It is well known that

$$\langle \tau_{2l-2}(\mathrm{pt})\rangle_{l,1}^{\mathbb{P}\cdot,\circ} = c_{2l}.$$

- It was proved in [36] that

$$\langle \mathbf{x}^+, \tau_{2g-2+\ell(\mathbf{x}+)+\ell(\mathbf{x}-)}(\mathrm{pt}), \mathbf{x}^-\rangle_g^{\mathbb{P}^1,\circ}$$

$$= \frac{1}{|\operatorname{Aut}(\mathbf{x}^+)||\operatorname{Aut}(\mathbf{x}^-)|}[z^{2g}]\frac{\prod_{\mathbf{x}_i^+}\mathcal{S}(\mathbf{x}_i^+ z)\prod_{\mathbf{x}_i^-}\mathcal{S}(\mathbf{x}_i^- z)}{\mathcal{S}(z)}.$$

3. Triply mixed Hurwitz numbers

We now introduce triply mixed Hurwitz numbers by the conditions (1)–(3) of the definition of Hurwitz numbers, together with a refined version of both conditions (6) and (6′) of (strictly) monotone Hurwitz numbers.

Call two compositions/partitions μ and μ' *equivalent* when they only differ by 1-entries. Further, for a fixed positive integer d, a fixed partition μ and $\sigma \in S_d$ write $\mathcal{C}(\sigma) = \mu$ if the cycle type of σ is equivalent to μ.

Definition 3.1. Let $g', g \geq 0$ be non-negative integers, d a positive integer and $\mu = (\mu^1, \ldots, \mu^n)$ a tuple of ordered partitions (not necessarily of the same integers). Furthermore, let

$$b = b(g, g', \mu) = 2g' - 2 - d \cdot (2g - 2) + \sum_i \big(\ell(\mu^i) - |\mu^i|\big). \qquad (3.1)$$

For non-negative integers k, l, m, such that $k + l + m = b$, we define a *triply mixed factorisation of type*

$$(g, d, \mu, k, l, m)$$

to be tuple

$$(\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_b, \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g),$$

such that

- (1) $\sigma_i, \tau_i, \alpha_i, \beta_i \in S_d$,
- (2) $\sigma_1 \cdots \sigma_n \tau_1 \cdots \tau_b = [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]$,
- (3) $\mathcal{C}(\sigma_i) = \mu^i$ and the τ_j are transpositions,
- (6) for $\tau_i = (s_i t_i)$ with $s_i < t_i$, we have
  - $t_i \leq t_{i+1}$ for $i = k+1, \ldots, k+l-1$,
  - $t_i < t_{i+1}$ for $i = k+l+1, \ldots, k+l+m-1$.

If in addition, we have

(4) $\langle \sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_b, \alpha, \beta \rangle$ acts transitively on $\{1, \ldots, d\}$,

we call the factorisation *connected*. We denote by $M^{\bullet}(g, d, \mu, k, l, m)$ the set of triply mixed factorisations of type (g, d, μ, k, l, m) and by $M^{\circ}(g, d, \mu, k, l, m)$ the set of connected triply mixed factorisations of type (g, d, μ, k, l, m). Then we define the *triply mixed Hurwitz number* and the *connected triply mixed Hurwitz numbers* respectively by

$$H^{g,d;\bullet}_{g';k,l,m}(\mu) = \frac{1}{d!}|M^{\bullet}(g, d, \mu, k, l, m)|,$$

and

$$H^{g,d}_{g';k,l,m}(\mu) = \frac{1}{d!}|M^{\circ}(g, d, \mu, k, l, m)|.$$

Remark 3.2. Triply mixed Hurwitz numbers are interpolations between classical, monotone and strictly monotone Hurwitz numbers. Namely, taking $l = m = 0$ yields classical Hurwitz numbers, $k = m = 0$ yields monotone Hurwitz numbers and $k = l = 0$ yields strictly monotone Hurwitz numbers.

As mentioned before, we study several specifications of triply mixed Hurwitz numbers, for example only in Section 4 we allow k, l and m to be arbitrary. In the rest of the work we specialise to (strictly) monotone Hurwitz numbers by letting $k = m = 0$ and $k = l = 0$ respectively. For convenience, we introduce additional notation distinguishing those cases.

(a) In Section 4, we study triply mixed Hurwitz numbers for target surfaces of genus 1. We abbreviate $H^{1,d}_{g';k,l,m}(\mu)$ by $H^{d}_{g';k,l,m}(\mu)$.

(b) In Section 5, we study triply mixed Hurwitz numbers with target surfaces of genus 1 with $\mu = ()$ in the following cases:

- in the monotone case we denote $H^{d}_{\leq,g} = H^{d}_{g;0,b(g,\mu),0}(\mu)$;

- in the strictly monotone case we denote

$$H^{d}_{<,g} = H^{d}_{g;0,0,b(g,\mu)}(\mu).$$

(c) In Section 6, we study two cases of triply mixed Hurwitz numbers for target surfaces of arbitrary genus and $\mu = ((\mu_1, \ldots, \mu_n))$, i.e., one arbitrary, but fixed ramification profile:

- in the monotone case we denote

$$\vec{H}^{g;\bullet}_{\leq,g'}(\mu_1, \ldots, \mu_n) = \vec{H}^{g,d;\bullet}_{g';0,b(g,\mu),0}(\mu);$$

- In the strictly monotone case we denote

$$\vec{H}^{g;\bullet}_{<,g'}(\mu_1, \ldots, \mu_n) = \vec{H}^{g,d;\bullet}_{g';0,0,b(g,\mu)}(\mu).$$

(d) In Section 8, we again study two cases of triply mixed Hurwitz numbers for target surfaces of genus 0 and $\mu = (\mu^1, \mu^2)$, i.e., two arbitrary but fixed ramification profiles for two cases:

- in the monotone case we denote

$$h_{\bar{g}}^{\leq}(\mu^1, \mu^2) = H_{g;0,b(g,\mu),0}^{0,d}(\mu);$$

- in the strictly monotone case we denote

$$h_g^<(\mu^1, \mu^2) = H_{g;0,0,b(g,\mu)}^{0,d;\bullet}(\mu).$$

4. Quasimodularity of triply mixed Hurwitz numbers

Fix $g' \geq 2$. The generating series

$$\sum_{d \geq 1} H_{g';k,0,0}^{d;\bullet}(\mu) q^d$$

of ordinary Hurwitz numbers with $g = 1$ is known to be a quasimodular form (recall $q = e^{2\pi i \tau}$). This was observed by Dijkgraaf in the simplest case ($\mu = (\ )$), rigorously proved by Kaneko and Zagier and follows in full generality from the Bloch–Okounkov theorem as noted by Eskin and Okounkov [9, 17, 29]. In this section we extend this result to the generating series of triply mixed Hurwitz numbers with $g = 1$. We begin by expressing triply mixed Hurwitz numbers in terms of shifted symmetric functions.

Let h_n and e_n be the complete homogeneous symmetric polynomial respectively elementary symmetric polynomial of degree n.

Proposition 4.1. *Let $g, g' \geq 0$, let μ be a tuple of partitions and take $k, l, m \geq 0$ with $k + l + m = b$, where $b = b(g, g', \mu)$ is given by equation (3.1). Then,*

$$H_{g';k,l,m}^{g,d;\bullet}(\mu) = \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!}\right)^{2-2g} f_\mu(\lambda) f_{(2)}(\lambda)^k h_l(\mathrm{cont}_\lambda) e_m(\mathrm{cont}_\lambda), \qquad (4.1)$$

where the sum is over all partitions of size d.

Proof. First, we rewrite the triply mixed Hurwitz number in terms of the center of the group algebra $\mathcal{Z}_d$, see Section 2.4.1 for the notation in this proof. Observe that

$$h_k(\Xi_d) = \sum_{\substack{2 \leq t_1 \leq \cdots \leq t_k \leq d \\ s_i < t_i}} (s_1 t_1) \cdots (s_k t_k)$$

and

$$e_k(\Xi_d) = \sum_{\substack{2 \le t_1 < \cdots < t_k \le d \\ s_i < t_i}} (s_1 t_1) \cdots (s_k t_k).$$

Hence,

$$H^{d;\bullet}_{g';k,l,m}(\mu) = \frac{1}{d!}[C_e] \mathfrak{R}^g C_{\mu^1} \cdots C_{\mu^n} C^k_{(2)} h_l(\Xi_d) e_m(\Xi_d).$$

Observe that $\chi^\lambda(C_\sigma) = \chi^\lambda(e)\omega^\lambda(C_\sigma)$ for all $\sigma \in S_d$. Hence, the Schur orthogonality relation can be written in the unusual form $\sum_\lambda \chi^\lambda(e)^2 \omega^\lambda(C_\sigma) = \delta_{e\sigma}|S_d|$. We find

$$H^{d;\bullet}_{g';k,l,m}(\mu) = \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!}\right)^2 \omega^\lambda\big(\mathfrak{R}^g C_{\mu^1} \cdots C_{\mu^n} C^k_{(2)} h_l(\Xi_d) e_m(\Xi_d)\big)$$

$$= \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!}\right)^{2-2g} f_\mu(\lambda) f_{(2)}(\lambda)^k h_l(\mathrm{cont}_\lambda) e_m(\mathrm{cont}_\lambda). \qquad \blacksquare$$

Definition 4.2. Let $t_\mu = \prod_{i,j} t_{i,\mu^i_j}$ be a formal variable. Define the *Hurwitz potential* by

$$\mathfrak{H}^\bullet = \sum H^{d;\bullet}_{g';k,l,m}(\mu) t_\mu \frac{u^k}{k!} v^l w^m q^d,$$

where the sum is over all k, l, m, d, μ for which $H^{d;\bullet}_{g';k,l,m}(\mu)$ is defined. Analogously define the connected Hurwitz potential $\mathfrak{H}$.

Remark 4.3. By a standard argument the Hurwitz potential and the connected Hurwitz potential are related by

$$\exp \mathfrak{H} = 1 + \mathfrak{H}^\bullet.$$

Theorem 4.4. *Let $g' \ge 2$. Then*

$$\sum_{d=1}^\infty H^d_{g';k,l,m}(\mu) q^d \qquad (4.2)$$

is a quasimodular form of mixed weight $\le 6g' - 6 + \sum_i (4\ell(\mu^i) - 2|\mu^i|)$. Moreover, for fixed $b = k + l + m$ the top weight parts of

$$2^{l+m+\delta_{l,0}+\delta_{m,0}-2} \sum_{d=1}^\infty H^{d;}_{g';k,l,m}(\mu) q^d$$

ranging over all $k, l, m \ge 0$ are equal.

Proof. Observe that $f_\mu(\lambda) f_{(2)}(\lambda)^k h_l(\mathrm{cont}_\lambda) e_m(\mathrm{cont}_\lambda)$ is a shifted symmetric polynomial of mixed weight at most

$$\sum_i (|\mu^i| + \ell(\mu_i)) + 3(k + l + m)$$

by the results in Section 2.4.4. By equation (3.1) this weight equals

$$6g' - 6 + \sum_i \left(4\ell(\mu^i) - 2|\mu^i| \right).$$ (4.3)

Observe that evaluating f_μ, $h_l(\text{cont})$ or $e_m(\text{cont})$ at the empty partition yields 0 unless μ is empty respectively $l = 0$ or $m = 0$. In other words, using equation (4.1) to define Hurwitz numbers for $d = 0$, one obtains $H^0_{g';k,l,m}(\mu) = 1$ if $\mu = ()$ and $k = l = m = 0$ and 0 else. Hence, Proposition 4.1 and Remark 4.3 imply that

$$\mathfrak{S} = \log\left(q^{1/24} \eta(\tau)^{-1} \sum \langle f_\mu f^k_{(2)} h_l(\text{cont}) e_m(\text{cont}) \rangle_q t_\mu \frac{u^k}{k!} v^l w^m \right).$$

By the Bloch–Okounkov theorem,

$$\langle f_\mu f^k_{(2)} h_l(\text{cont}) e_m(\text{cont}) \rangle_q$$

is a quasimodular form of weight at most given by Equation (4.3). Taking a formal Taylor expansion, quasimodularity of the generating series in equation (4.2) follows. The second part of the statement follows directly from Proposition 2.12. ∎

Remark 4.5. In case $g' = 1$ the series in equation (4.2) equals $-\log(q^{-1/24}\eta(\tau))$. This is not a quasimodular form, but it is a primitive of a quasimodular form. Namely, its derivative equals up to a constant the Eisenstein series of weight 2.

5. Refined quasimodularity and tropical covers

We continue the study of the series

$$H_{\leq,g} = \sum H^d_{\leq,g} q^d \quad \text{and} \quad H_{<,g} = \sum H^d_{<,g} q^d$$

by expressing them as a finite sum of quasimodular forms, with each summand corresponding to a combinatorial type of the source curve of tropical covers.

5.1. Tropical monotone and Grothendieck dessins d'enfants elliptic covers

In this section we express the numbers $H^d_{\leq,g}(\mu)$ and $H^d_{<,g}(\mu)$ in terms of tropical covers of the tropical elliptic curve E_{trop}, which is a circle with a point p_0. As for the tropical projective line, we may add additional 2-valent vertices to E_{trop}.

Definition 5.1. We fix $g \geq 0$, an orientation on E_{trop} and points $p_1, \ldots, p_{2g-2}$ of E_{trop}, such that $p_0, p_1, \ldots, p_{2g-2}$ is ordered according to the orientation. Let $\pi: \Gamma \to E_{\text{trop}}$ be a tropical cover of genus g and degree d, such that $\pi^{-1}(p_0)$ does not contain

any vertices and where Γ has at most $2g - 2$ vertices $v_1, \ldots, v_n$, $n \leq 2g - 2$. We require $v_i \in \pi^{-1}(p_i)$ for $i \in [n]$. We set $\lambda_i = \mathrm{val}(v_i) + 2g(v_i) - 2$ and obtain a composition $\lambda(\pi) = (\lambda_1, \ldots, \lambda_n)$. If $|\lambda| = 2g - 2$, we call π a *monotone elliptic tropical cover* of type (g, d) and denote by $\Gamma(E_{\mathrm{trop}}, g, d)$ the set of all monotone elliptic tropical covers of type (g, d).

We further associate two multiplicities to each cover $\pi \in \Gamma(E_{\mathrm{trop}}, g, d)$, one corresponding to the monotone case and one to the strictly monotone case:

$$\mathrm{mult}_{\leq}(\pi) = \frac{1}{|\mathrm{Aut}(\pi)|} \frac{1}{\ell(\lambda(\pi))!} \prod_{v \in V(\Gamma)} m_v \prod_{e \in E(\Gamma)} \omega_e,$$

$$\mathrm{mult}_{<}(\pi) = \frac{1}{|\mathrm{Aut}(\pi)|} \frac{1}{\ell(\lambda(\pi))!} \prod_{v \in V(\Gamma)} (-1)^{1 + \mathrm{val}(v)} m_v \prod_{e \in E(\Gamma)} \omega_e,$$

where for each vertex v_i, let $\mathbf{x}^+$ (resp. $\mathbf{x}^-$) be the right-hand (resp. left-hand) side weights with respect to the orientation on E_{trop}. The multiplicity m_{v_i} of v_i is defined to be

$$m_{v_i} = (\lambda_i - 1)! |\mathrm{Aut}(\mathbf{x}^+)| |\mathrm{Aut}(\mathbf{x}^-)|$$
$$\sum_{g_1^i + g_2^i = g(v_i)} \langle \tau_{2g_2^i - 2}(\omega) \rangle_{g_2^i}^{\mathbb{P}^1, \circ} \langle \mathbf{x}^+, \tau_{2g_1^i - 2 + \ell(\mathbf{x}^+) + \ell(\mathbf{x}^-)}(\omega), \mathbf{x}^- \rangle_{g_1^i}^{\mathbb{P}^1, \circ}.$$

Remark 5.2. We note that the multiplicity of the vertex is defined in a similar manner as in Theorem 2.18. This is due to our construction below, which glues tropical covers of $\mathbb{P}^1_{\mathrm{trop}}$ to tropical covers of E_{trop}.

This yields the following correspondence theorem, which we prove (below) in Section 5.2.

Theorem 5.3. *Fix $g \geq 0$ and $d > 0$. Then, we have the following identities:*

$$H_{\leq, g}^d = \sum_{\pi \in \Gamma(E_{\mathrm{trop}}; g, d)} \mathrm{mult}_{\leq}(\pi), \quad H_{<, g}^d = \sum_{\pi \in \Gamma(E_{\mathrm{trop}}; g, d)} \mathrm{mult}_{<}(\pi).$$

5.2. Proof of Theorem 5.3

In this section we prove Theorem 5.3. We only work out the details for the monotone case as the strictly monotone case is completely parallel. Before starting with the proof, we make the following remark.

Remark 5.4. A similar statement was proved in [5, Theorem 2.13]. More precisely, a correspondence theorem expressing simple covers of elliptic curves (i.e., no (strict) monotonicity conditions and only simple ramification) in terms of tropical covers. We

point out that while the idea of our proof of Theorem 5.3 is similar to the idea of the proof of [5, Theorem 2.13], the proof itself is still very different in nature. This is due to an important technical subtlety: in [5], it was possible to construct a tropical elliptic cover directly from a factorisation in the symmetric group and vice versa; in our setting, this is no longer possible due to the nature of Theorem 2.18 as it is derived using the bosonic Fock space. However, we will show that the relation between factorisations and tropical covers is still close enough to derive our correspondence theorem in Theorem 5.3.

First, we recall the definition of $H^d_{\leq,g}$. We count tuples of certain permutations

$$(\tau_1, \ldots, \tau_{2g-2}, \alpha, \beta),$$

such that

$$\tau_{2g-2} \cdots \tau_1 = \alpha\beta\alpha^{-1}\beta^{-1}.$$

However, this is equivalent to

$$\tau_{2g-2} \cdots \tau_1 \beta = \alpha\beta\alpha^{-1}.$$

Thus, letting $\sigma_1 = \beta$ and $\sigma_2 = \alpha\beta\alpha^{-1}$, we see that $H^d_{\leq,g}$ is equal to $\frac{1}{d!}$ times the number of tuples $(\sigma_1, \tau_1, \ldots, \tau_{2g-2}, \sigma_2, \alpha)$, such that

(1) $\sigma_1, \sigma_2, \alpha, \tau_i \in S_d$,

(2) $\mathcal{C}(\tau_i) = (2, 1, \ldots, 1), \mathcal{C}(\sigma_1) = \mathcal{C}(\sigma_2)$,

(3) $\tau_{2g-2} \cdots \tau_1 \sigma_1 = \sigma_2$,

(4) $\alpha\sigma_1\alpha^{-1} = \sigma_2$ (we note that thus $\mathcal{C}(\sigma_1) = \mathcal{C}(\sigma_2)$),

(5) τ_i satisfy the monotonicity condition,

(6) the group $\langle\sigma_1, \tau_1, \ldots, \tau_{2g-2}, \sigma_2, \alpha\rangle$ acts transitively on $\{1, \ldots, d\}$.

Observe that we count tuples very similar to the monotone double Hurwitz numbers framework for $\mu = \nu$. We note that in the above description the group $\langle\sigma_1, \tau_1, \ldots, \tau_{2g-2}, \sigma_2\rangle$, i.e., without the generator α, might have several orbits acting on $\{1, \ldots, d\}$. By capturing this data, we can make the full transition to monotone double Hurwitz numbers. We now make this more precise.

For a monotone base 0 factorisation $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$, we consider the orbits of the action of the group $\langle\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2\rangle$ on $\{1, \ldots, d\}$. The orbits then naturally yield connected monotone base 0 factorisations $(\sigma_1^{(i)}, \tau_{i(1)}, \ldots, \tau_{i(l_k)}, \sigma_2^{(i)})$ of type $(g_i, \eta_1^i, \eta_2^i)$ for $i = 1, \ldots, n$ (for some arbitrary n), where $\sigma_j^{(i)}$ have pairwise disjoint orbits with $\prod \sigma_j^{(i)} = \sigma_j$ for $j = 1, 2$, such that

(i) $\bigcup \eta_1^i = \bigcup \eta_2^i = \mathcal{C}(\sigma_1) = \mathcal{C}(\sigma_2)$;

(ii)   the sets $\{i(1), \ldots, i(l_k)\}$ are pairwise disjoint and $\bigcup\{i(1), \ldots, i(l_k)\} = [b]$;

(iii) $\sum g_i = g + n - 1$.

We call these condition (i)–(iii) the *orbit conditions*. Now, fix an unordered tuple $\Lambda = ((g_i, \eta_1^i, \eta_2^i))$ satisfying these conditions (i) and (iii). We then define $h_g(\mu, \Lambda)$ to be $\frac{1}{d!}$ times the number of monotone base 0 factorisations whose orbits yield the data Λ.

Our strategy now is as follows. To each monotone base 0 factorisation $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$, we want to associate $\alpha \in S_d$, such that $\alpha \sigma_1 \alpha^{-1} = \sigma_2$ and enumerate those α. This number only depends on the data Λ and is actually encoded in the tropical pictures.

Cutting elliptic covers. We fix a monotone elliptic tropical cover $\pi \colon \Gamma \to E_{\text{trop}}$ of type (g, d). We consider $\pi^{-1}(p_0)$ and collect the weights in the preimage of p_0 in the partition Δ.

We cut the elliptic curve E_{trop} at p_0 open and cut the source curve in the preimage. This way, we obtain a tropical cover $\pi' \colon \Gamma' \to \mathbb{P}^1_{\text{trop}}$ in $\Gamma(\mathbb{P}^1_{\text{trop}}, g; \Delta, \Delta, \lambda(\pi))$ for some non-negative integer g, which we call the *cut-cover associated to π*. This cover may be disconnected. The connected components yield tropical covers in $\Gamma(\mathbb{P}^1_{\text{trop}}, g_i; \Delta_i, \Delta^j, \lambda^i)$, such that

(i) $\bigcup \Delta_i = \bigcup \Delta^j = \Delta$;

(ii) $\bigcup \lambda^i = \lambda$;

(iii)   the genera satisfy $\sum g_i = g + n - 1$.

We call these conditions the *component conditions*. Observe that the component conditions (i) and (iii) coincide with orbit conditions (i) and (iii) above for monotone factorisations.

We now fix an unordered tuple $\Lambda = ((g_i, \Delta_i, \Delta^i))$ satisfying conditions (i) and (iii) and denote by $\Gamma(\mathbb{P}^1_{\text{trop}}, g; \Delta, \Delta, \lambda; \Lambda)$ the set of all tropical covers in $\Gamma(\mathbb{P}^1, g; \Delta, \Delta, \lambda)$, such that their connected components yield the data Λ.

By the inclusion-exclusion principle, we obtain the following lemma, which states that the data of connected components of monotone factorisations are captured by connected components in the tropical covers of Theorem 2.18:

Lemma 5.5. *Let g be a non-negative integer, $b = 2g - 2 + 2\ell(\mu)$, and Δ a partition of some positive integer. Further fix an unordered tuple $\Lambda = ((g_i, \eta_1^i, \eta_2^i))$ satisfying the component conditions* (i) *and* (iii)*. Then we obtain*

$$h_g(\Delta, \Lambda) = \sum_{\lambda \vdash b} \sum_{\pi \in \Gamma(\mathbb{P}^1_{\text{trop}}, g; \Delta, \Delta, \lambda; \Lambda)} \text{mult}(\pi).$$

Regluing elliptic covers. The role of monotone elliptic covers is essentially to encode ways to find permutations α for monotone base 0 factorisations, which enrich them to be a factorisation contributing to $H^d_{\leq, g}$. We now make this more precise.

Let $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$ be a monotone base 0 factorisation of type (g, μ, μ). We fix a permutation α, such that $\alpha \sigma_1 \alpha^{-1} = \sigma_2$. We observe that conjugation by α maps

the cycles of σ_1 bijectively to the cycles of σ_2. In other words, when we choose a labelling of the cycles of σ_1 and σ_2 by $1, \ldots, \ell(\mu)$, then α induces a bijection

$$I_\alpha \colon \{1, \ldots, \ell(\mu)\} \to \{1, \ldots, \ell(\mu)\},$$

such that for $j \in \{1, \ldots, \ell(\mu)\}$ the length of the cycle of σ_1 labeled j coincides with the length of cycle of σ_2 labeled $I_\alpha(j)$. This corresponds to a gluing process in the tropical setting.

Let $\pi' \colon \Gamma' \to \mathbb{P}^1_{\mathrm{trop}}$ in $\Gamma(\mathbb{P}^1, g; \mu, \mu, \lambda)$ for some $\lambda \vdash b$. We label the left ends of Γ' by $1, \ldots, \ell(\mu)$ and the right ends of Γ' as well. We fix a bijection

$$I \colon \{1, \ldots, \ell(\mu)\} \to \{1, \ldots, \ell(\mu)\},$$

such that the weight of the left weight labeled j is the same as the weight of right end labeled $I(j)$. We now glue the ends of $\mathbb{P}^1_{\mathrm{trop}}$ together to obtain E_{trop}, where p_0 is gluing point. We also glue the source curve Γ' according to I, i.e., we glue the left end labeled j to the right end labeled $I(j)$. This way, we obtain a monotone elliptic tropical cover of type (g', d) for $g' = \frac{b+2}{2}$.

Definition 5.6. Let $\pi \colon \Gamma \to E_{\mathrm{trop}}$ a monotone elliptic tropical cover of type (g, d) and $\pi' \colon \Gamma' \to \mathbb{P}^1_{\mathrm{trop}}$ be the associated cut-cover, where $\pi' \in \Gamma(\mathbb{P}^1, g'; \mu, \mu, \lambda; \Lambda)$ for some g' and $\lambda \vdash b$. We fix a monotone base 0 factorisation $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$ of type (g, μ, μ). We label the left ends of Γ' by σ_1 and the right ends by σ_2. We denote by $n_{\pi, \pi'}$ the number of $\alpha \in S_d$, where $\alpha \sigma_1 \alpha^{-1} = \sigma_2$ and such that the associated gluing of π' induced by α yields π.

This situation was analysed in [5].

Proposition 5.7 ([5]). *For an elliptic cover π with tuple $\Delta = (m_1, \ldots, m_r)$ over the base point p_0 and its cut-cover π', we have*

$$n_{\pi, \pi'} = \frac{|\operatorname{Aut}(\pi')|}{|\operatorname{Aut}(\pi)|} m_1 \cdots m_r.$$

Moreover, we have the following lemma. Recall that a monotone elliptic tropical cover is always a connected cover by definition.

Lemma 5.8. *Let π be a monotone elliptic tropical cover and π' be the associated cut-cover. Furthermore, let $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$ be a monotone factorisation, whose orbits yield the same data Λ as the cut-cover π'. Let α be a permutation as in Definition 5.6, then*

$$\langle \sigma_1, \tau_1, \ldots, \tau_b, \sigma_2, \alpha \rangle$$

is a transitive subgroup of S_d.

Proof. This follows immediately from the fact that π is a connected cover and α joins the connected components. ∎

We are now ready to prove Theorem 5.3.

Proof of Theorem 5.3. In the beginning of this section, we have seen that, for fixed g, d and $b = 2g - 2$, we have $H_{\leq,g}^d$ is equal to $\frac{1}{d!}$ times $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2, \alpha)$, such that $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$ is a monotone factorisation, $\alpha \sigma_1 \alpha^{-1} = \sigma_2$ and $\langle \sigma_1, \tau_1, \ldots, \tau_b, \sigma_2, \alpha \rangle$ is a transitive subgroup of S_d. We associated to each monotone factorisation $\kappa = (\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$ the number $n(\kappa)$ of permutations α, such that $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2, \alpha)$ contributes to $H_{\leq,g}^d$. Thus, we obtain

$$H_{\leq,g}^d = \frac{1}{d!} \sum n(\kappa),$$

where we sum over all monotone factorisations with b transpositions in S_d. We have also seen that several monotone factorisations yield the same number $n(\kappa)$, i.e., those satisfying the same orbit conditions Λ. We thus denote by $n(\Lambda)$ the number $n(\kappa)$ for all monotone factorisations whose orbits yield the data Λ. We now group those together, i.e., we obtain

$$H_{\leq,g}^d = \sum h_{g'}(\mu, \Lambda) \cdot n(\Lambda),$$

where we sum over all non-negative integer $g' \leq g$, partitions μ of d and tuples Λ satisfying the orbit conditions (i) and (iii). We now analyse each summand

$$h_{g'}(\mu, \Lambda) \cdot n(\Lambda).$$

In fact, we want to prove that

$$h_{g'}(\mu, \Lambda) \cdot n(\Lambda) = \sum_{\lambda \vdash b} \sum_{\pi'} \sum_{\pi} \text{mult}(\pi') \cdot n_{\pi, \pi'},$$

where the second summand is over all cover $\pi' \in \Gamma(\mathbb{P}_{\text{trop}}^1, g'; \mu, \mu, \lambda; \Lambda)$ and the second summand over all monotone elliptic tropical covers π, such that π' is their cut-cover.

By Lemma 5.5,

$$h_g(\mu, \Lambda) \cdot n(\Lambda) = \sum_{\lambda \vdash b} \sum_{\pi' \in \Gamma(\mathbb{P}_{\text{trop}}^1, g; \mu, \mu, \lambda; \Lambda)} \text{mult}(\pi') \cdot n(\Lambda)$$

$$= \sum_{\lambda \vdash b} \sum_{\pi' \in \Gamma(\mathbb{P}_{\text{trop}}^1, g; \mu, \mu, \lambda; \Lambda)} (\text{mult}(\pi') \cdot n(\Lambda)).$$

We now observe that each α, which contributes to $n(\Lambda)$ for a given monotone factorisation, whose orbits yield Λ contributes to a gluing of π' to an monotone elliptic tropical cover. Thus, we have by Lemma 5.8

$$n(\Lambda) = \sum_\pi n_{\pi,\pi'},$$

where we sum over all monotone elliptic tropical covers π whose cut-cover is π'. Thus, by Proposition 5.7 we have

$$h_g(\mu, \Lambda) \cdot n(\Lambda) = \sum_{\lambda \vdash b} \sum_{\pi' \in \Gamma(\mathbb{P}^1_{\text{trop}}, g; \mu, \mu, \lambda; \Lambda)} \left(\text{mult}(\pi') \cdot \sum_\pi n_{\pi,\pi'} \right)$$

$$= \sum_{\lambda \vdash b} \sum_{\pi' \in \Gamma(\mathbb{P}^1_{\text{trop}}, g; \mu, \mu, \lambda; \Lambda)} \sum_\pi \text{mult}(\pi') \cdot n_{\pi,\pi'},$$

as desired. We further see that

$$\text{mult}(\pi') \cdot n_{\pi,\pi'} = \frac{1}{|\text{Aut}(\pi')|} \frac{1}{\ell(\lambda)!} \prod_{v \in V(\Gamma)} m_v \prod_{e \in E(\Gamma)} \omega_e \cdot \prod_{i=1}^{\ell(\mu)} \mu_i \cdot \frac{|\text{Aut}(\pi')|}{|\text{Aut}(\pi)|}$$

$$= \text{mult}(\pi)$$

and obtain

$$H^d_{\le,g} = \sum_\Lambda h_{g'}(\mu; \Lambda) \cdot n(\Lambda) = \sum_\Lambda \sum_{\lambda \vdash b} \sum_{\pi'} \sum_\pi \text{mult}(\pi') \cdot n_{\pi,\pi'}$$

$$= \sum_\Lambda \sum_{\lambda \vdash b} \sum_{\pi'} \sum_\pi \text{mult}(\pi).$$

As π' and Λ are determined by π, we can omit those summands and just sum over all monotone elliptic tropical covers π. This yields

$$H^d_{\le,g} = \sum_{\lambda \vdash b} \sum_\pi \text{mult}(\pi),$$

where we sum over all monotone elliptic tropical covers π of type (g, d) as desired. ∎

5.3. Refined quasimodularity

Definition 5.9. We fix a combinatorial type G of a tropical curve Γ with

$$n := |V(\Gamma)| \le 2g(\Gamma) - 2.$$

Moreover, we fix an orientation on E_{trop} and a linear ordering Ω on the vertices of G. We denote by v_i the i-th vertex according to Ω. We further choose points $p_1, \ldots, p_{2g-2}$ on E_{trop}, such that they are linearly ordered along the orientation we chose on E_{trop}. We further fix a series of integers $\underline{g}' = (g_1, \ldots, g_n)$.

We denote by $\Gamma(G, \Omega; g, \underline{g}')$ the set of all covers $\pi \in \Gamma(E_{\text{trop}}, g, d)$ for some $d \in \mathbb{N}$, such that

(1) for $\pi \colon \Gamma \to E_{\text{trop}}$ the combinatorial type of Γ is G;

(2) $\pi(v_i) = p_i$;

(3) $g(v_i) = g_i$.

Moreover, we associate two generating series to each combinatorial type G:

$$I^{G,\Omega}_{\leq,\underline{g}'} := \sum_{\pi \in \Gamma(G,\Omega;g,\underline{g}')} \text{mult}_{\leq}(\pi) q^{\deg(\pi)},$$

$$I^{G,\Omega}_{<,\underline{g}'} := \sum_{\pi \in \Gamma(G,\Omega;g,\underline{g}')} \text{mult}_{<}(\pi) q^{\deg(\pi)}.$$

Remark 5.10. We observe that by the correspondence theorem

$$H_{\leq,g} = \sum_{(G,\Omega,\underline{g}')} I^{G,\Omega}_{\leq,\underline{g}'} \quad \text{and} \quad H_{\leq,g} = \sum_{(G,\Omega,g)} I^{G,\Omega}_{<,\underline{g}'}, \tag{5.1}$$

where we sum over all combinatorial types G on at most $n \leq 2g - 2$ vertices, orders Ω on G and tuples $\underline{g}' = (g_1, \ldots, g_n)$.

Theorem 5.11. *For $g' \geq 2$, the series $I^{G,\Omega}_{\leq,\underline{g}'}$ and $I^{G,\Omega}_{<,\underline{g}'}$ are quasimodular forms of mixed weight less or equal $2(\sum_{i=1}^{n} g_i + |E(G)|)$.*

Proof. This follows from [19, Theorem 6.1 and Corollary 8.4], where it is proved that the generating series associated to a tropical cover with target curve of fixed combinatorial type, order and fixed ramification profile is a quasimodular form whenever the multiplicity of the cover is a polynomial in the edge weights. The only thing to check in our case is that the local vertex multiplicities are polynomial, which is true as proved in [19, Theorem 4.1]. ∎

Combining equation (5.1) and Theorem 5.11, we obtain the following corollary (which is a special case of Theorem 4.4):

Corollary 5.12. *The generating series*

$$\sum_{d \geq 1} H^d_{\leq,g} q^d \quad \text{and} \quad \sum_{d \geq 1} H^d_{<,g} q^d$$

are quasimodular forms of mixed weight $\leq 6g - 6$.

Proof. The only thing left to prove is the weight part of the corollary. In order to see this, we consider $2(\sum_{i=1}^{n} g_i + |E(G)|)$ and observe $2|E(G)| = \sum_{i=1}^{n} \text{val}(v_i)$, where we sum over all vertices of G. Moreover, by definition, we have $\text{val}(v_i) = \lambda_i - 2g_i + 2$. We obtain

$$2\Big(\sum_{i=1}^{n} g_i + |E(G)|\Big) = 2\sum_{i=1}^{n} g_i + \sum_{i=1}^{n}(\lambda_i - 2g_i + 2)$$

$$= \sum_{i=1}^{n} \lambda_i + 2n$$

$$= 2g - 2 + 2n \le 2g - 2 + 2(2g - 2) = 6g - 6,$$

where we used $\sum_{i=1}^{n} \lambda_i = 2g - 2$ and $n \le 2g - 2$. This yields the weight as desired. ∎

6. Quantum curve for (strictly) monotone base g Hurwitz numbers

Motivated by the successful study of base g Hurwitz numbers in [33] and monotone Hurwitz numbers [21] with a view towards topological recursion, we connect these ideas by enumerating base g Hurwitz numbers with monotonicity conditions. In particular, this section is devoted to deriving a quantum curve for this new enumerative problem.

Recall the connected labeled monotone base g Hurwitz numbers $\vec{H}^g_{\le,g'}(\mu_1, \ldots, \mu_n)$ from Section 2.1.

Definition 6.1. We define

$$F^g_{\le,g'}(x_1, \ldots, x_n) = \sum_{\mu \in \mathbb{Z}^n_+} \vec{H}^g_{\le,g'}(\mu_1, \ldots, \mu_n) x_1^{\mu_1} \ldots x_n^{\mu_n},$$

and analogously we define the generating series for the strictly monotone case

$$F^g_{<,g'}(x_1, \ldots, x_n).$$

Definition 6.2. We define the partition function of the base g monotone Hurwitz numbers, as the formal series in variables $x, \hbar$, given by

$$Z^g_{\le} = Z^g_{\le}(x, \hbar) = \exp\Big[\sum_{g'=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g'-2+n}}{n!} F^{\le,g}_{g'}(x, x, \ldots, x)\Big]$$

$$= \exp\Big[\sum_{g'=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g'-2+n}}{n!} \sum_{\mu \in \mathbb{Z}^n_+} \vec{H}^g_{g'}(\mu_1, \ldots, \mu_n) x^{|\mu|}\Big],$$

and analogously the partition function $Z_<^g = Z_<^g(x, \hbar)$ of the base g strictly monotone Hurwitz numbers.

Proposition 6.3. *For the partition functions for the monotone and strictly monotone case, we have*

$$Z_{\leq}^g = 1 + \sum_{d=1}^{\infty} \sum_{b=0}^{\infty} \left\{ \begin{matrix} d+b-1 \\ d-1 \end{matrix} \right\} (d!)^{1-\chi} x^d \hbar^{b+d(1-\chi)}$$

$$= 1 + \sum_{d=1}^{\infty} (d!)^{1-\chi} x^d \hbar^{d(1-\chi)} \prod_{j=1}^{d-1} \frac{1}{1-j\hbar}$$

and

$$Z_{<}^g = 1 + \sum_{d=1}^{\infty} \sum_{b=0}^{d-1} \left[\begin{matrix} d \\ d-b \end{matrix} \right] (d!)^{1-\chi} x^d \hbar^{b+d(1-\chi)}$$

$$= 1 + \sum_{d=1}^{\infty} (d!)^{1-\chi} x^d \hbar^{1-\chi} \prod_{j=1}^{d-1} (1-j\hbar),$$

where $\chi = 2 - 2g$ and the equalities are understood in the sense of formal power series.

Proof. Since

$$\sum_{g'=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g'-2+n}}{n!} F_{\leq,g'}^g(x, x, \ldots, x)$$

$$= \sum_{g'=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g'-2+n}}{n!} \sum_{\mu \in \mathbb{Z}_+^n} \vec{H}_{\leq,g'}^g(\mu_1, \ldots, \mu_n) x^{|\mu|}$$

counts transitive (connected) monotone base g factorisations, we can use the exponential formula and find the generating series for the not necessarily transitive factorisations

$$Z_{\leq}^g = 1 + \sum_{g'=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g'-2+n}}{n!} \sum_{\mu_1, \ldots, \mu_n=1}^{\infty} \vec{H}_{\leq,g'}^{\bullet,g}(\mu_1, \ldots, \mu_n) x^{|\mu|}.$$

Collecting all factorizations for given $d = |\mu|$, we get

$$Z_{\leq}^g = 1 + \sum_{g'=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g'-2+n}}{n!} \sum_{d=0}^{\infty} \left(\sum_{|\mu|=d} \vec{H}_{\leq,g'}^{\bullet,g}(\mu) \right) x^d.$$

Recall that

$$b = b(h, n, |\mu|) = 2g' - 2 + n - |\mu|(2g - 1),$$

so we write

$$\sum_{g'=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g'-2+n}}{n!} \sum_{d=0}^{\infty} \left(\sum_{|\mu|=d} \vec{H}_{\leq,g'}^{\bullet,g}(\mu) \right) x^d$$

$$= \sum_{g'=0}^{\infty} \sum_{n=1}^{\infty} \sum_{d=0}^{\infty} \frac{\sum_{|\mu|=d} \vec{H}_{\leq,g'}^{\bullet,g}(\mu)}{n!} \hbar^b (x\hbar^{2g-1})^d$$

$$= \sum_{g'=0}^{\infty} \sum_{n=1}^{\infty} \sum_{d=0}^{\infty} \frac{\sum_{|\mu|=d} \vec{H}_{\leq,g'}^{\bullet,g}(\mu)}{n!} \hbar^b (x\hbar^{1-\chi})^d.$$

Since $\vec{H}_{\leq,g'}^{g}(\mu)$ is non-zero if $b \geq 0$, we can rearrange the series by collecting all possible g', n for a given b and d. Viewing the partition function $Z_{\leq}^{g}$ as a series in $\mathbb{Q}[\![\hbar, x\hbar^{1-\chi}]\!]$, we find that the coefficient of $x^d \hbar^{b+d(1-\chi)}$ is precisely the number of monotone base g factorizations of length b in S_d, i.e.,

$$[x^d \hbar^{b+d(1-\chi)}] Z_{\leq}^{g}$$

$$= \frac{1}{d!} \# \left\{ (\tau_1, \ldots, \tau_b, \sigma, \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g) \, \middle| \, \begin{array}{l} \tau_i \text{ monotone transpositions,} \\ \sigma, \alpha_1, \beta_1 \ldots \alpha_g, \beta_g \in S_d, \\ \sigma \tau_1 \ldots \tau_b = [\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g] \end{array} \right\}.$$

Since the α_i, β_i run over all elements in S_d and using [10, Lemma 17], we obtain

$$[x^d \hbar^{b+d(1-\chi)}] Z^g = (d!)^{2g-1} \# \{ (\tau_1, \ldots, \tau_b) \mid \tau_i \text{ monotone transpositions} \}$$

$$= (d!)^{1-\chi} \left\{ \begin{array}{c} d+b-1 \\ d-1 \end{array} \right\}.$$

For the last equality, we find

$$Z_{\leq}^{g} = 1 + \sum_{d=1}^{\infty} \sum_{b=0}^{\infty} \left\{ \begin{array}{c} d+b-1 \\ d-1 \end{array} \right\} (d!)^{1-\chi} x^d \hbar^{b+d(1-\chi)}$$

$$= 1 + \sum_{d=1}^{\infty} (d!)^{1-\chi} x^d \hbar^{d(1-\chi)} \sum_{b=0}^{\infty} \left\{ \begin{array}{c} d+b-1 \\ d-1 \end{array} \right\} \hbar^b$$

$$= 1 + \sum_{d=1}^{\infty} (d!)^{1-\chi} x^d \hbar^{d(1-\chi)} \prod_{j=1}^{d-1} \frac{1}{1-j\hbar},$$

where we used the well-known identity in equation (2.1).

In the case of the strictly monotone Hurwitz numbers, we do a similar calculation and view $Z_<^g$ as an element of $\mathbb{Q}[\![\hbar, x\hbar^{1-\chi}]\!]$. We find

$$[x^d \hbar^{b+d(1-\chi)}]Z_<^g = (d!)^{1-\chi}\#\{(\tau_1, \ldots, \tau_b) \mid \tau_i \text{ strictly monotone transpositions}\}$$

and in particular

$$[x^d \hbar^{b+d(1-\chi)}]Z_<^g = 0 \quad \text{for } b \geq d.$$

These tuples can be expressed by evaluating the elementary symmetric polynomials in the Jucys–Murphy elements (defined in Section 2.4.1). This also yields an enumeration by evaluating the same polynomials in the number of summands of the i-th Jucys–Murphy element, which yields (see e.g., [32, eq. (4)])

$$\#\{(\tau_1, \ldots, \tau_b) \mid \tau_i \text{ strictly monotone transpositions }\} = \begin{bmatrix} d \\ d-b \end{bmatrix}$$

for $b \leq d$. Hence, we have the assertion

$$\begin{aligned}
Z_<^g &= 1 + \sum_{d=1}^{\infty} \sum_{b=0}^{d-1} \begin{bmatrix} d \\ d-b \end{bmatrix} (d!)^{1-\chi} x^d \hbar^{b+d(1-\chi)} \\
&= 1 + \sum_{d=1}^{\infty} (d!)^{1-\chi} x^d \hbar^{1-\chi} \sum_{b=0}^{d-1} \begin{bmatrix} d \\ d-b \end{bmatrix} \hbar^d \\
&= 1 + \sum_{d=1}^{\infty} (d!)^{1-\chi} x^d \hbar^{1-\chi} \prod_{j=1}^{d-1} (1 - j\hbar),
\end{aligned}$$

where the last equality follows by equation (2.1). ∎

Theorem 6.4. *The partition function $Z_\leq^g$ satisfies the differential equation*

$$[\hat{x}\hat{y}^2 + \hat{y} + (\hat{y}\hat{x})^{2g}]Z_\leq^g = 0,$$

where $\hat{x} = x$ and $\hat{y} = -\hbar\frac{\partial}{\partial x}$.

Proof. The recursion formula for the Stirling numbers of the second kind yields

$$\begin{Bmatrix} d+b-1 \\ d-1 \end{Bmatrix} = (d-1)\begin{Bmatrix} d+b-2 \\ d-1 \end{Bmatrix} + \begin{Bmatrix} d+b-2 \\ d-2 \end{Bmatrix}.$$

We multiply this equation by $\frac{(d!)^{2g}}{(d-1)!}x^d \hbar^{b+d(1-\chi)}$ and sum over $d \geq 1, b \geq 0$. For reasons of clarity, we first do the computations term by term before conflating them.

For the term on the left-hand side we have

$$\sum_{\substack{d=1 \\ b=0}}^{\infty} \left\{ \begin{matrix} d+b-1 \\ d-1 \end{matrix} \right\} \frac{(d!)^{2g}}{(d-1)!} x^d \hbar^{b+d(1-\chi)}$$

$$= \sum_{\substack{d=1 \\ b=0}}^{\infty} \left\{ \begin{matrix} d+b-1 \\ d-1 \end{matrix} \right\} d(d!)^{1-\chi} x^d \hbar^{b+d(1-\chi)}$$

$$= x \frac{\partial}{\partial x} \sum_{\substack{d=1 \\ b=0}}^{\infty} \left\{ \begin{matrix} d+b-1 \\ d-1 \end{matrix} \right\} (d!)^{1-\chi} x^d \hbar^{b+d(1-\chi)}$$

$$= x \frac{\partial}{\partial x} \left(1 + \sum_{\substack{d=1 \\ b=0}}^{\infty} \left\{ \begin{matrix} d+b-1 \\ d-1 \end{matrix} \right\} (d!)^{1-\chi} x^d \hbar^{b+d(1-\chi)} \right)$$

$$= x \frac{\partial}{\partial x} Z_{\leq}^g,$$

where we used the fact that the derivative of the constant 1 vanishes. For the first expression on the right-hand side we get

$$\sum_{\substack{d=1 \\ b=0}}^{\infty} (d-1) \left\{ \begin{matrix} d+b-2 \\ d-1 \end{matrix} \right\} \frac{(d!)^{2g}}{(d-1)!} x^d \hbar^{b+d(1-\chi)}$$

$$= \sum_{\substack{d=1 \\ b=0}}^{\infty} d(d-1) \left\{ \begin{matrix} d+b-2 \\ d-1 \end{matrix} \right\} (d!)^{1-\chi} x^d \hbar^{b+d(1-\chi)}$$

$$= x^2 \hbar \frac{\partial^2}{\partial x^2} \sum_{\substack{d=1 \\ b=0}}^{\infty} \left\{ \begin{matrix} d+b-2 \\ d-1 \end{matrix} \right\} (d!)^{1-\chi} x^d \hbar^{b-1+d(1-\chi)}.$$

Now, note that for $b = 0$ we have $\left\{ \begin{smallmatrix} d-2 \\ d-1 \end{smallmatrix} \right\} = 0$. Hence,

$$x^2 \hbar \frac{\partial^2}{\partial x^2} \sum_{\substack{d=1 \\ b=0}}^{\infty} \left\{ \begin{matrix} d+b-2 \\ d-1 \end{matrix} \right\} (d!)^{1-\chi} x^d \hbar^{b-1+d(1-\chi)}$$

$$= x^2 \hbar \frac{\partial^2}{\partial x^2} \sum_{\substack{d=1 \\ b=1}}^{\infty} \left\{ \begin{matrix} d+b-2 \\ d-1 \end{matrix} \right\} (d!)^{1-\chi} x^d \hbar^{b-1+d(1-\chi)}$$

$$= x^2 \hbar \frac{\partial^2}{\partial x^2} \sum_{\substack{d=1 \\ b=0}}^{\infty} \left\{ \begin{matrix} d+b-1 \\ d-1 \end{matrix} \right\} (d!)^{1-\chi} x^d \hbar^{b+d(1-\chi)}$$

$$= x^2 \hbar \frac{\partial^2}{\partial x^2} Z_{\leq}^g,$$

where we performed the shift $b' = b - 1$ in the second equality. For the last term we obtain

$$\sum_{\substack{d=1 \\ b=0}}^{\infty} \begin{Bmatrix} d+b-2 \\ d-2 \end{Bmatrix} \frac{(d!)^{2g}}{(d-1)!} x^d \hbar^{b+d(1-\chi)}$$

$$= \sum_{\substack{d=1 \\ b=0}}^{\infty} d^{2g} \begin{Bmatrix} d+b-2 \\ d-2 \end{Bmatrix} ((d-1)!)^{1-\chi} x^d \hbar^{b+d(1-\chi)}$$

$$= \left(x\frac{\partial}{\partial x}\right)^{2g} x\hbar^{1-\chi} \sum_{\substack{d=1 \\ b=0}}^{\infty} \begin{Bmatrix} d+b-2 \\ d-2 \end{Bmatrix} ((d-1)!)^{1-\chi} x^{d-1} \hbar^{b-(d-1)(1-\chi)}$$

$$= \left(x\frac{\partial}{\partial x}\right)^{2g} x\hbar^{1-\chi} \left(1 + \sum_{\substack{d=1 \\ b=0}}^{\infty} \begin{Bmatrix} d+b-1 \\ d-1 \end{Bmatrix} (d!)^{1-\chi} x^d \hbar^{b+d(1-\chi)}\right)$$

$$= \frac{x}{\hbar} \left(\hbar\frac{\partial}{\partial x} x\right)^{2g} \left(1 + \sum_{\substack{d=1 \\ b=0}}^{\infty} \begin{Bmatrix} d+b-1 \\ d-1 \end{Bmatrix} (d!)^{1-\chi} x^d \hbar^{b+d(1-\chi)}\right)$$

$$= \frac{x}{\hbar} \left(\hbar\frac{\partial}{\partial x} x\right)^{2g} Z_{\leq}^g.$$

Putting things together and multiplying by $\frac{\hbar}{x}$ we get

$$\left[x\hbar^2\frac{\partial}{\partial x} - \hbar\frac{\partial}{\partial x} + \left(\hbar\frac{\partial}{\partial x} x\right)^{2g}\right] Z_{\leq}^g = 0.$$

Substituting $\hat{x} = x$ and $\hat{y} = -\hbar\frac{\partial}{\partial x}$ we obtain the claim

$$[\hat{x}\hat{y}^2 + \hat{y} + (\hat{y}\hat{x})^{2g}] Z_{\leq}^g = 0. \qquad \blacksquare$$

For the strictly monotone Hurwitz number we have a similar result.

Theorem 6.5. *The partition function $Z_{<}^g$ satisfies the differential equation*

$$[\hat{y} + (1 - \hat{x}\hat{y})(\hat{y}\hat{x})^{2g}] Z_{<}^g = 0,$$

where $\hat{x} = x$ and $\hat{y} = -\hbar\frac{\partial}{\partial x}$.

Proof. The proof is in the same spirit as in Theorem 6.4 but uses the Stirling numbers of the first kind instead. $\qquad \blacksquare$

Remark 6.6. Note, that for $g = 0$, we recover the quantum curve for the usual monotone (strictly monotone) Hurwitz numbers of [10] (resp. [13]).

7. A mysterious topological recursion

In this section we consider the quantum curve of the strictly monotone Hurwitz numbers derived in Theorem 6.5. We focus on the case with elliptic base curve, i.e., on $g = 1$. Computing the semi-classical limit, we obtain the spectral curve

$$y + (1 - xy)(xy)^2 = 0$$

with parametrization

$$x(z) = \frac{(z-1)^2}{z}, \quad y(z) = \frac{z}{(z-1)^3}.$$

Surprisingly, running topological recursion for this input data yields the monotone single Hurwitz numbers with up to a combinatorial prefactor. More precisely, we obtain the cumulants of the Weingarten function. This points towards an unknown relationship between the combinatorics of strictly monotone Hurwitz numbers with elliptic base curve and monotone Hurwitz numbers with rational base curve.

As already noted in the introduction, topological recursion for monotone single Hurwitz numbers with rational base curves and a different normalisation was proved in [10], however for a different spectral curve and the exclusion of the $(0, 2)$-case. In our normalisation, the $(0, 2)$-case still encodes the relevant invariants. We begin by defining the correct normalisation of monotone single Hurwitz numbers for our purpose, which coincide with the cumulants of the Weingarten function. The latter was motivated by discussions with James Mingo on problems of higher order freeness in free probability [8]. In particular, using the following normalizations, the numbers coincide with certain values of the Möbius function on the set of partitioned permutations, this observation is still not perfectly understood and will be investigated in future work.

Definition 7.1. Let g be a non-negative integer, d, n be a positive integers and μ a partition d of length l. We denote by $m_{g,n}(\mu)$ the number of connected labeled monotone factorizations of a fixed (but arbitrary) permutation σ with $C(\sigma) = \mu$. Moreover, we put

$$C_{g,n}(\mu) = (-1)^{2g-2+n+|\mu|} m_{g,n}(\mu) = (-1)^{n+|\mu|} m_{g,n}(\mu)$$

and denote by $W_{g,n}(x_1, \ldots, x_n)$ the corresponding generating series, i.e.,

$$W_{g,n}(x_1, \ldots, x_n) = \sum_{\mu_1, \ldots, \mu_n = 1}^{\infty} \frac{C_{g,n}(\mu_1, \ldots, \mu_n)}{x_1^{\mu_1+1} \ldots x_n^{\mu_n+1}}.$$

Remark 7.2. (1) The numbers $C_{g,n}(\mu)$ agree with the monotone Hurwitz numbers up to the combinatorial factor $(-1)^{n+|\mu|} \prod_{i=1}^{l} \mu_i$.

(2) When we drop the connectivity condition in the definition of $m_{g,n}(\mu)$, obtain the disconnected analogs of $C_{g,n}(\mu)$. These numbers are the coefficients of the asymptotic expansion of the Weingarten function [21].

(3) In [20], the numbers $m_{g,n}(\mu)$ are put into a generating series via

$$M_{g,n}(y_1, \ldots, y_n) = \sum_{\mu_1, \ldots, \mu_n = 1}^{\infty} m_{g,n}(\mu_1, \ldots, \mu_n) y_1^{\mu_1 - 1} \cdots y_n^{\mu_n - 1}.$$

Thus their generating series relates to $W_{g,n}(x_1, \ldots, x_n)$ as follows:

$$W_{g,n}(x_1, \ldots, x_n) = \frac{M_{g,n}(\frac{-1}{x_1}, \ldots, \frac{-1}{x_n})}{x_1^2 \cdots x_n^2}.$$

Note also that this amounts to a change of variable $y_i = \frac{-1}{x_i}$ if we phrase things in the language of differential forms, i.e.,

$$M_{g,n}(y_1, \ldots, y_n) \, d \, y_1 \cdots d \, y_n = W_{g,n}(x_1, \ldots, x_n) \, d \, x_1 \cdots d \, x_n.$$

The following is the main theorem of this section, which we prove in Section 7.1.

Theorem 7.3. *The numbers $C_{g,n}(\mu)$ satisfy topological recursion with the spectral curve given by*

$$x(z) = \frac{(z-1)^2}{z}, \quad y(z) = \frac{z}{(z-1)^3},$$

i.e., the differentials

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{\mu_1, \ldots, \mu_n = 1}^{\infty} \frac{C_{g,n}(\mu_1, \ldots, \mu_n)}{x(z_1)^{\mu_1 + 1} \ldots x(z_n)^{\mu_n + 1}} \, d \, x(z_1) \cdots d \, x(z_n)$$

for $(g, n) \neq (0, 2)$, and

$$\omega_{0,2} = \sum_{\mu_1, \mu_2 = 1}^{\infty} \frac{C_{0,2}(\mu_1, \mu_2)}{x(z_1)^{\mu_1 + 1} x(z_2)^{\mu_2 + 1}} \, d \, x(z_1) \, d \, x(z_2) + \frac{d \, x(z_1) \, d \, x(z_2)}{(x(z_1) - x(z_2))^2}$$

satisfy the recursion

$$\omega_{g,n}(z_1, \ldots, z_n)$$
$$= \operatorname{Res}_{z = \pm 1} K(z_1, z) \Big[\omega_{g-1,n+1}(z, \sigma(z), z_2, \ldots, z_n)$$
$$+ \sum_{\substack{g_1 + g_2 = g \\ I \sqcup J = N \setminus \{1\}}}^{\prime} \omega_{g_1, |I| + 1}(z, z_I) \omega_{g_2, |J| + 1}(\sigma(z), z_J) \Big] \quad (7.1)$$

on $2g - 2 + n > 0$, where

$$K(z_1, z) = \frac{\frac{1}{2} \int_{\sigma(z)}^{z} \omega(z_1, \cdot)}{\omega_{0,1}(z) - \omega_{0,1}(\sigma(z))} = \frac{z(z-1)^3 \, \mathrm{d} z_1}{2(z+1)(z_1 - z)(z_1 z - 1)}, \quad \sigma(z) = \frac{1}{z},$$

and with the initial data given by

$$\omega_{0,1}(z) = y \, \mathrm{d} x, \quad and \quad \omega_{0,2}(z_1, z_2) = B(z_1, z_2) := \frac{\mathrm{d} z_1 \, \mathrm{d} z_2}{(z_1 - z_2)^2}.$$

Remark 7.4. Note that $\mathrm{d} x(z) = \frac{z^2 - 1}{z^2}$ has the zeroes $z = \pm 1$ and since $y(z)$ has a pole of order bigger than 1 at $z = 1$, the spectral curve (x, y) is irregular in the sense of [14, Section 2.1, item 2(b)]. Hence, the invariants $\omega_{g,n}$ agree with the invariants obtained from the local spectral curve obtained by removing the point $z = 1$. In particular, the residue at $z = 1$ in equation 7.1 does not contribute and it suffices to compute the residue at $z = -1$.

Moreover, we note that while

$$\omega_{0,2} \neq W_{0,2} \, \mathrm{d} x(z_1) \, \mathrm{d} x(z_2),$$

we have

$$\operatorname*{Res}_{z_1, z_2 \to \infty} x(z_1)^{\mu_1} x(z_2)^{\mu_2} \omega_{0,2}(z_1, z_2) = C_{0,2}(\mu_1, \mu_2)$$

since their difference is holomorphic by definition.

The starting point of our proof is the following recursion, which is a direct consequence of [20, Theorem 2.1].

Proposition 7.5. *Let g be a non-negative integer, $n \in \mathbb{N}$ and $\mu = (\mu_1, \ldots, \mu_n)$ a partition of a positive integer. Then we have the recursion*

$$-C_{g,n}(\mu) = \sum_{j=2}^{n} \mu_j C_{g,n-1}(\mu_1 + \mu_j, \mu_{N \setminus \{1\}}) + \sum_{\alpha + \beta = \mu_1} C_{g-1,n+1}(\alpha, \beta, \mu_{N \setminus \{1\}})$$

$$+ \sum_{\substack{\alpha + \beta = \mu_1}} \sum_{\substack{g_1 + g_2 = g \\ I \sqcup J = N \setminus \{1\}}} C_{g_1, 1 + |I|}(\alpha, \mu_I) C_{g_2, 1 + |J|}(\beta, \mu_J),$$

where $\mu_I = (\mu_{i_1}, \ldots, \mu_{i_k})$ for $I = \{i_1, \ldots, i_k\}$, $N = \{2, \ldots, n\}$ and the initial value $C_{0,1}(1) = 1$.

The following proposition reformulates the cut-and-join equation Proposition 7.5 as a differential equation for generating series.

Proposition 7.6. *It holds that*

$$
\begin{aligned}
&- W_{g,n}(x_1, \ldots, x_n) \\
&= \sum_{j=2}^{n} \frac{\partial}{\partial x_j} \frac{x_j}{x_1} \frac{x_1 W_{g,n-1}(X_{\{1,\ldots,n\}\setminus\{j\}}) - x_j W_{g,n-1}(X_{\{1,\ldots,n\}\setminus\{1\}})}{x_1 - x_j} \\
&\quad + x_1 W_{g-1,n+1}(x_1, X_{\{1,\ldots,n\}}) \\
&\quad + x_1 \sum_{\substack{g_1+g_2=g \\ I \sqcup J = N \setminus \{1\}}} W_{g_1,|I|+1}(x_1, X_I) W_{g_2,|J|+1}(x_1, X_J) - \frac{1}{x^2}\delta_{g,0}\delta_{n,1}.
\end{aligned}
$$

Proof. We multiply the cut-and-join equation from Proposition 7.5 by

$$
x_1^{-(\mu_1+1)} \ldots x_n^{-(\mu_n+1)}
$$

and sum over $\mu_1, \mu_2 \ldots, \mu_n \geq 1$. We start by dealing with the first term on the right-hand side. First observe the following. For fixed j we have

$$
\begin{aligned}
&\sum_{\mu_1,\mu_j=1}^{\infty} \mu_j \frac{C_{g,n-1}(\mu_1+\mu_j, \mu_{N\setminus\{\mu_j\}})}{x_1^{\mu_1+1} x_j^{\mu_j+1}} \\
&= -\frac{\partial}{\partial x_j} \frac{1}{x_1^2} \sum_{\mu_1 \geq 1, \mu_j \geq 0} \frac{C_{g,n-1}(\mu_1+\mu_j, \mu_{N\setminus\{\mu_j\}})}{x_1^{\mu_1-1} x_j^{\mu_j}} \\
&= -\frac{\partial}{\partial x_j} \frac{1}{x_1^2} \sum_{\nu=0}^{\infty} \sum_{\substack{\mu_1+\mu_j=\nu \\ \mu_1,\mu_j \geq 0}} \frac{C_{g,n-1}(\nu+1, \mu_{N\setminus\{\mu_j\}})}{x_1^{\mu_1} x_j^{\mu_j}}.
\end{aligned}
$$

We note that

$$
\sum_{\mu_1+\mu_j=\nu} \frac{1}{x_1^{\mu_1} x_j^{\mu_j}} = -x_1 x_j \frac{\frac{1}{x_1^{\nu+1}} - \frac{1}{x_j^{\nu+1}}}{x_1 - x_j}
$$

and hence we find

$$
\begin{aligned}
&-\frac{\partial}{\partial x_j} \frac{1}{x_1^2} \sum_{\nu=0}^{\infty} \sum_{\substack{\mu_1+\mu_j=\nu \\ \mu_1,\mu_j \geq 0}} \frac{C_{g,n-1}(\nu+1, \mu_{N\setminus\{\mu_j\}})}{x_1^{\mu_1} x_j^{\mu_j}} \\
&= \frac{\partial}{\partial x_j} \frac{x_j}{x_1} \sum_{\nu=0}^{\infty} \frac{C_{g,n-1}(\nu+1, \mu_{N\setminus\{\mu_j\}})(\frac{x_1}{x_1^{\nu+2}} - \frac{x_j}{x_j^{\nu+2}})}{x_1 - x_j}.
\end{aligned}
$$

Further, we can put this in the summation over all $\mu_1, \ldots, \mu_n$ and we obtain

$$\sum_{\mu_1 \ldots \mu_n = 1}^{\infty} \frac{\mu_j C_{g,n-1}(\mu_1 + \mu_j, \mu_{N\setminus\{\mu_j\}})}{x_1^{\mu_1+1} \ldots x_n^{\mu_n+1}}$$

$$= \frac{\partial}{\partial x_j} \frac{x_j}{x_1} \frac{x_1 W_{g,n-1}(X_{\{1,\ldots,n\}\setminus\{j\}}) - x_j W_{g,n-1}(X_{\{1,\ldots,n\}\setminus\{1\}})}{x_1 - x_j}$$

We proceed analogously for the second term and observe that

$$\sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \sum_{\alpha+\beta=\mu_1} \frac{C_{g-1,n+1}(\alpha, \beta, \mu_{N\setminus\{1\}})}{x_1^{\mu_1+1} \ldots x_n^{\mu_n+1}}$$

$$= \sum_{\mu_2,\ldots,\mu_n,\alpha,\beta=1}^{\infty} \frac{C_{g-1,n+1}(\alpha, \beta, \mu_{N\setminus\{1\}})}{x_1^{\alpha+\beta+1} \ldots x_n^{\mu_n+1}}$$

$$= x_1 W_{g-1,n+1}(x_1, X_{\{1,\ldots,x_n\}}).$$

Finally, for the third term we obtain

$$\sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \sum_{\alpha+\beta=\mu_1} \sum_{\substack{g_1+g_2=g \\ I\sqcup J=N\setminus\{1\}}} \frac{C_{g_1,1+|I|}(\alpha, \mu_I) C_{g_2,1+|J|}(\beta, \mu_J)}{x_1^{\mu_1+1} \ldots x_n^{\mu_n+1}}$$

$$= x_1 \sum_{\substack{g_1+g_2=g \\ I\sqcup J=N\setminus\{1\}}} \sum_{\substack{\alpha,\mu_i=1 \\ i\in I}}^{\infty} \frac{C_{g_1,1+|I|}(\alpha, \mu_I)}{x_1^{\alpha+1} \prod_{i\in I} x_i^{\mu_i+1}} \sum_{\substack{\alpha,\mu_i=1 \\ i\in J}}^{\infty} \frac{C_{g_1,1+|I|}(\beta, \mu_I)}{x_1^{\beta+1} \prod_{i\in J} x_i^{\mu_i+1}}$$

$$= x_1 \sum_{\substack{g_1+g_2=g \\ I\sqcup J=N\setminus\{1\}}} W_{g_1,|I|+1}(x_1, X_I) W_{g_2,|J|+1}(x_1, X_J).$$

Putting everything together yields the desired equation.   ∎

In the perspective of CEO topological recursion it is handy to rewrite the cut-and-join equation in way that $W_{g,n}(x)$ does not appear on the right-hand side.

Corollary 7.7. *It holds*

$$- (1 + 2x_1 W_{0,1}(x_1)) W_{g,n}(x_1, \ldots, x_n)$$

$$= \sum_{j=2}^{n} \frac{\partial}{\partial x_j} \frac{x_j}{x_1} \frac{x_1 W_{g,n-1}(X_{\{1,\ldots,n\}\setminus\{j\}}) - x_j W_{g,n-1}(X_{\{1,\ldots,n\}\setminus\{1\}})}{x_1 - x_j}$$

$$+ x_1 W_{g-1,n+1}(x_1, X_{\{1,\ldots,n\}})$$

$$+ x_1 \sum_{\substack{g_1+g_2=g \\ I\sqcup J=N\setminus\{1\}}}' W_{g_1,|I|+1}(x_1, X_I) W_{g_2,|J|+1}(x_1, X_J) - \frac{1}{x^2} \delta_{g,0}\delta_{n,1},$$

where $\sum'$ means that the cases $(g_1, I) = (0, \emptyset)$ or $(g_2, J) = (0, \emptyset)$ are excluded.

We now compute some special cases of $W_{g,n}$, which require special treatment in the CEO topological recursion. We have the following result for the first few values of (g, n).

Corollary 7.8 ([20, Theorem 1.1] and [22, Theorem 6.2]). *We have that*

$$C_{0,1}(\mu) = (-1)^{\mu-1}\frac{1}{\mu}\binom{2\mu-2}{\mu-1},$$

$$C_{0,2}(\mu_1, \mu_2) = (-1)^{\mu_1+\mu_2}\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\binom{2\mu_1-1}{\mu_1}\binom{2\mu_2-1}{\mu_2},$$

$$C_{0,3}(\mu_1, \mu_2, \mu_3) = (-1)^{\mu_1+\mu_2+\mu_3-1}8\mu_1\binom{2\mu_1-1}{\mu_1}\mu_2\binom{2\mu_2-1}{\mu_2}\mu_3\binom{2\mu_3-1}{\mu_3}.$$

A straightforward calculation shows the following lemma.

Lemma 7.9. *The following identities hold*

$$W_{0,1}(x(z)) = \frac{z}{(z-1)^3},$$

$$W_{0,2}(x(z_1), x(z_2)) = \frac{z_1^2 z_2^2}{(z_1^2-1)(z_2^2-1)(1-z_1z_2)^2},$$

$$W_{0,3}(x_1, x_2, x_3) = \frac{8}{x_1^2 x_2^2 x_3^2(1+\frac{4}{x_1})^{\frac{3}{2}}(1+\frac{4}{x_2})^{\frac{3}{2}}(1+\frac{4}{x_3})^{\frac{3}{2}}}$$

$$= \prod_{i=1}^{3}\frac{2}{x_i^2(1+\frac{4}{x_i})^{\frac{3}{2}}} = \prod_{i=1}^{3}\frac{2}{(z_i+1)^2}\frac{1}{x'(z_i)}.$$

The next lemma is a key step towards the topological recursion for the numbers $C_{g,n}(\mu)$, as determining the difference between the Bergman kernel and the $(0,2)$ free energy is important for the input data of the topological recursion.

Lemma 7.10. *We have*

$$W_{0,2}(x(z_1), x(z_2))dx(z_1)dx(z_2) = \frac{dz_1\,dz_2}{(1-z_1z_2)^2} = \frac{dz_1\,dz_2}{(z_1-z_2)^2} - \frac{dx(z_1)\,dx(z_2)}{(x(z_1)-x(z_2))^2}$$

and in particular

$$W_{0,2}(x(z_1), x(z_2))dx(z_1)dx(z_2) = -B\left(\frac{1}{z_1}, z_2\right).$$

Proof. From the last proposition and $x'(z_i) = \frac{z_i^2-1}{z_i^2}$ we obtain

$$W_{0,2}(x(z_1), x(z_2)) = \frac{z_1^2}{(z_1^2-1)}\frac{z_2^2}{(z_2^2-1)}\frac{1}{(1-z_1z_2)^2} = \frac{1}{x'(z_1)}\frac{1}{x'(z_2)}\frac{1}{(1-z_1z_2)^2},$$

from which the first equality follows immediately. For the second one, a straightforward calculation yields

$$(x(z_1) - x(z_2))^2 = \left(\frac{z_1^2 + 1}{z_1} - \frac{z_2^2 + 1}{z_2} \right)^2$$
$$= \frac{(1 - z_1 z_2)^2 (z_1 - z_2)^2}{z_1^2 z_2^2},$$

which yields

$$\frac{1}{(z_1 - z_2)^2} - \frac{x'(z_1) x'(z_2)}{(x(z_1) - x(z_2))^2} = \frac{1}{(z_1 - z_2)^2} - \frac{(z_1^2 - 1)(z_2^2 - 1)}{(1 - z_1 z_2)^2 (z_1 - z_2)^2}$$
$$= \frac{1}{(1 - z_1 z_2)^2}. \qquad \blacksquare$$

7.1. Proof of Theorem 7.3

Our proof of Theorem 7.3 is inspired by the approach in [10]. We begin by considering the case $(g, n) = (0, 3)$, which requires an independent discussion.

Lemma 7.11. *The multidifferential $\omega_{0,3}$ satisfies the recursion in equation* (7.1).

Proof. Recall that, by Lemma 7.9,

$$W_{0,3}(x_1, x_2, x_3) = \frac{8}{x_1^2 x_2^2 x_3^2 (1 + \frac{4}{x_1})^{\frac{3}{2}} (1 + \frac{4}{x_2})^{\frac{3}{2}} (1 + \frac{4}{x_3})^{\frac{3}{2}}}$$
$$= \prod_{i=1}^{3} \frac{2}{x_i^2 (1 + \frac{4}{x_i})^{\frac{3}{2}}}.$$

We find

$$W_{0,3}(z_1, z_2, z_3) = \prod_{i=1}^{3} \frac{2}{x_i(z)^2 (1 + \frac{4}{x_i(z)})^{\frac{3}{2}}}$$
$$= \prod_{i=1}^{3} \frac{2 z_i^2 (z_i - 1)^3}{(z_i - 1)^4 (z_i + 1)^3}$$
$$= \prod_{i=1}^{3} \frac{2}{(z_i + 1)^2} \frac{z_i^2}{(z_i + 1)(z_i - 1)}$$
$$= \prod_{i=1}^{3} \frac{2}{(z_i + 1)^2} \frac{1}{x'(z_i)}.$$

The recursion from topological recursion reads

$$\omega_{0,3}(z_1, z_2, z_3)$$

$$= \operatorname*{Res}_{z \to -1} K(z_1, z) \left[w_{0,2}(z, z_2) w_{0,2}\left(\frac{1}{z}, z_3\right) + w_{0,2}(z, z_3) w_{0,2}\left(\frac{1}{z}, z_2\right) \right]$$

$$= \operatorname*{Res}_{z \to -1} \frac{z(z-1)^3 \, \mathrm{d} z_1}{2(z+1)(z_1 - z)(z_1 z - 1) \, \mathrm{d} z} \left[\frac{\mathrm{d} z \, \mathrm{d} z_2}{(z - z_2)^2} \frac{\mathrm{d} \frac{1}{z} \, \mathrm{d} z_3}{(\frac{1}{z} - z_3)^2} \right.$$

$$\left. + \frac{\mathrm{d} z \, \mathrm{d} z_3}{(z - z_3)^2} \frac{\mathrm{d} \frac{1}{z} \, \mathrm{d} z_2}{(\frac{1}{z} - z_2)^2} \right]$$

$$= \operatorname*{Res}_{z \to -1} \frac{1}{z+1} \frac{-(z-1)^3 \, \mathrm{d} z_1}{2z(z_1 - z)(z_1 z - 1) \, \mathrm{d} z} \left[\frac{1}{(z - z_2)^2(\frac{1}{z} - z_3)^2} \right.$$

$$\left. + \frac{1}{(z - z_3)^2(\frac{1}{z} - z_2)^2} \right] \mathrm{d} z \, \mathrm{d} z_1 \, \mathrm{d} z_2 \, \mathrm{d} z_3,$$

which is of the form

$$\frac{f(z, z_1, z_2, z_3) \, \mathrm{d} z}{(z+1)} \, \mathrm{d} z_1 \, \mathrm{d} z_2 \, \mathrm{d} z_3$$

where f is holomorphic in z around $z = -1$. Hence, we get

$$\omega_{0,3}(z_1, z_2, z_3) = f(1, z_1, z_2, z_3) \, \mathrm{d} z_1 \, \mathrm{d} z_2 \, \mathrm{d} z_3$$

$$= \frac{8 \, \mathrm{d} z_1 \, \mathrm{d} z_2 \, \mathrm{d} z_3}{(z_1 + 1)^2 (z_2 + 1)^2 (z_3 + 1)^2},$$

which concludes the proof. ∎

Recall the polynomiality result for monotone Hurwitz numbers.

Theorem 7.12 ([22]). *There are symmetric rational functions $\vec{P}_{g,h}$ such that*

$$\vec{H}_{g,n}(\mu_1, \ldots, \mu_n) = \prod_{i=1}^{n} \binom{2\mu_i}{\mu_i} \vec{P}_{g,n}(\mu_1, \mu_2, \ldots, \mu_n).$$

Moreover, if $(g, n) \neq q(0, 1), (0, 2)$, then $\vec{P}_{g,n}$ is a polynomial with rational coefficients of degree $3g - 3 + n$.

Since $C_{g,n}$ agrees with $\vec{H}_{g,h}$ up to the factor $(-1)^b \prod_{i=1}^{n} \mu_i$, we immediately get

$$C_{g,n} = (-1)^b \prod_{i=1}^{n} \mu_i \binom{2\mu_i}{\mu_i} \vec{P}_{g,n}(\mu_1, \mu_2, \ldots, \mu_n).$$

Thus, for the polynomial

$$\vec{P}_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{\underline{a}=\underline{0}}^{\text{finite}} B_{g,n}(\underline{a}) \prod \mu_i^{a_i}$$

with coefficients $B_{g,n}(\underline{a})$, we can write our generating function as

$$W_{g,n}(x_1, \ldots, x_n) = \sum_{\underline{a}=\underline{0}}^{\text{finite}} B_{g,n}(\underline{a}) \prod_{i=1}^{n} \sum_{\mu_i=1}^{\infty} \mu_i^{a_i+1} \binom{2\mu_i}{\mu_i} \left(\frac{-1}{x_i}\right)^{\mu_i+1}$$

$$= \sum_{\underline{a}=\underline{0}}^{\text{finite}} B_{g,n}(\underline{a}) \prod_{i=1}^{n} f_{a_i}(x_i)$$

with

$$f_a(x) = \sum_{\mu=1}^{\infty} \mu^{a+1} \binom{2\mu}{\mu} \left(\frac{-1}{x}\right)^{\mu+1}.$$

A careful analysis of the functions f_a will give us the analytic properties of $W_{g,n}$.

Lemma 7.13. *Let* $(g, n) \neq (0, 1), (0, 2)$; *then the functions* $W_{g,n}(z_1, \ldots, z_n)$ *satisfy*

$$W_{g,n}(z_1, \ldots, z_n) = -W_{g,n}(\sigma(z_1), z_2, \ldots, z_n) = -W_{g,n}\left(\frac{1}{z_1}, z_2, \ldots, z_n\right).$$

Moreover, they are rational functions in each z_i *having poles at* $z_i = 1$ *and at* $z_i = -1$.

Proof. Note that the functions f_a satisfy the recursion

$$f_a(x) = \sum_{\mu=1}^{\infty} \mu^{a+1} \binom{2\mu}{\mu} \left(\frac{-1}{x}\right)^{\mu+1} = -\frac{\partial}{\partial x} x f_{a-1}(x), \tag{7.2}$$

i.e.,

$$f_a(x) = \left(-\frac{\partial}{\partial x} x\right)^a f_0(x)$$

and

$$f_0(x) = \sum_{n=1}^{\infty} \mu \binom{2\mu}{\mu} \left(\frac{-1}{x}\right)^{\mu+1} = \frac{2}{\sqrt{x}(x+4)^{\frac{3}{2}}}.$$

In the variable z we get

$$f_0(z) = \frac{2z^2}{(z-1)(z+1)^3}$$

and

$$f_0(\sigma(z)) = f_0\left(\frac{1}{z}\right) = \frac{2\frac{1}{z^2}}{(\frac{1}{z} - 1)(\frac{1}{z} + 1)^3} = -\frac{2z^2}{(z - 1)(z + 1)^3} = -f_0(z).$$

We find by induction

$$f_a(z) + f_a(\sigma(z)) = \frac{\partial}{\partial x} x(f_{a-1}(z) + f_{a-1}(\sigma(z))) = 0$$

and hence

$$W_{g,n}(z_1, z_2, \ldots, z_n) + W_{g,n}(\sigma(z_1), z_2, \ldots, z_n)$$
$$= \sum_{\underline{a}=\underline{0}}^{\text{finite}} B_{g,n}(\underline{a})[f_{a_1}(z) + f_{a_1}(\sigma(z))] \prod_{i=2}^{n} f_{a_i}(x_i) = 0.$$

Moreover, note that equation (7.2) reads

$$f_a(z) = \left(\frac{-z^2}{z^2 - 1} \frac{\partial}{\partial z} \frac{(z - 1)^2}{z}\right) f_{a-1}(z)$$

in the variable z. It follows by induction that f_a is rational and has a pole of order 1 at $z = 1$ and a pole of order $2a + 3$ of $z = -1$. ∎

The last result can be reformulated in terms of the forms $\omega_{g,n}$.

Corollary 7.14. *For $(g, n) \neq (0, 1), (0, 2)$, the forms $\omega_{g,n}(z_1, \ldots, z_n)$ are antisymmetric with respect to σ, i.e.,*

$$\omega_{g,n}(z_1, \ldots, z_n) = -\omega_{g,n}(\sigma(z_1), z_2, \ldots, z_n)$$
$$= -\omega_{g,n}\left(\frac{1}{z_1}, z_2, \ldots, z_n\right).$$

They only have poles at $z = \pm 1$, where the pole at $z = 1$ has at most order 1.

Proof. The assertions follow from the Lemma 7.13. ∎

Now, we are ready to prove Theorem 7.3.

Proof of Theorem 7.3. The initial data is given by $(g, n) = (0, 1), (0, 2)$ and the case of $(g, n) = (0, 3)$ was proved in Lemma 7.11. Thus, in the following we assume that $(g, n) \neq (0, 1), (0, 2), (0, 3)$.

The idea is to add the recursions for

$$W_{g,n}(z_1, z_2, \ldots, z_n) \quad \text{and} \quad W_{g,n}(\sigma(z_1), z_2, \ldots, z_n)$$

and proceed with a careful combinatorial analysis after substituting the identity in Lemma 7.13.

- Firstly we note that the left-hand side yields the following:

$$
\begin{aligned}
& -(1 + 2x_1 W_{0,1}(z_1)) W_{g,n}(z_1, z_2, \ldots, z_n) \\
& \quad -(1 + 2x_1 W_{0,1}(\sigma(z_1))) W_{g,n}(\sigma(z_1), z_2, \ldots, z_n) \\
& = -(1 + 2x_1 W_{0,1}(z_1)) W_{g,n}(z_1, z_2, \ldots, z_n) \\
& \quad +(1 + 2x_1 W_{0,1}(\sigma(z_1))) W_{g,n}(z_1, z_2, \ldots, z_n) \\
& = -2x_1 [W_{0,1}(z_1) - W_{0,1}(\sigma(z_1))] W_{g,n}(z_1, z_2, \ldots, z_n).
\end{aligned}
$$

- The first term on the right-hand side we treat is

$$
\sum_{j=2}^{n} \frac{\partial}{\partial x_j} \frac{x_j}{x_1} \frac{x_1 W_{g,n-1}(Z_{\{1,\ldots,n\}\setminus\{j\}}) - x_j W_{g,n-1}(Z_{\{1,\ldots,n\}\setminus\{1\}})}{x_1 - x_j}
$$

and its counterpart where z_1 is replaced by $\sigma(z_1)$. First we note that, we can rewrite the derivatives by

$$
\frac{\partial}{\partial x} = \frac{z^2}{z^2 - 1} \frac{\partial}{\partial z}.
$$

Now, we want to focus on what happens when we replace z_1 by $\sigma(z_1)$, using

$$
x(\sigma(z_1)) = x(z_1)
$$

we get terms of the form

$$
\frac{z_j^2}{z_j^2 - 1} \frac{\partial}{\partial z_j} \frac{x(z_j)}{x(z_1)} \\
\frac{x(\sigma(z_1)) W_{g,n-1}(\sigma(z_1), Z_{\{2,\ldots,n\}\setminus\{j\}}) - x_j W_{g,n-1}(Z_{\{1,\ldots,n\}\setminus\{1\}})}{x_1 - x_j}.
$$

By our observations the latter is equivalent to

$$
\frac{z_j^2}{z_j^2 - 1} \frac{\partial}{\partial z_j} \frac{x(z_j)}{x(z_1)} \frac{-x_1 W_{g,n-1}(Z_{\{1,\ldots,n\}\setminus\{j\}}) - x_j W_{g,n-1}(Z_{\{1,\ldots,n\}\setminus\{1\}})}{x_1 - x_j}.
$$

Thus $x_1 W_{g,n-1}(Z_{\{1,\ldots,n\}\setminus\{j\}})$ cancels in the sum and we end up with the term

$$
-2 \sum_{j=2}^{n} \frac{z_j^2}{z_j^2 - 1} \frac{\partial}{\partial z_j} \frac{x(z_j)^2}{x(z_1)} \frac{W_{g,n-1}(Z_{\{1,\ldots,n\}\setminus\{1\}})}{x_1 - x_j}.
$$

- The second term on the right-hand side is

$$
\begin{aligned}
& x_1 \big(W_{g-1,n+1}(z_1, z_1, z_2, \ldots, z_n) + W_{g-1,n+1}(\sigma(z_1), \sigma(z_1), z_2, \ldots, z_n) \big) \\
& = -2x_1 W_{g-1,n+1}(z_1, \sigma(z_1), z_2, \ldots, z_n).
\end{aligned}
$$

- The third term is

$$
x_1\Big[\sum_{\substack{g_1+g_2=g \\ I \sqcup J = N\setminus\{1\}}} W_{g_1,|I|+1}(z_1, Z_I) W_{g_2,|J|+1}(z_1, Z_J)
$$

$$
+ \sum_{\substack{g_1+g_2=g \\ I \sqcup J = N\setminus\{1\}}} W_{g_1,|I|+1}(\sigma(z_1), Z_I) W_{g_2,|J|+1}(\sigma(z_1), Z_J)
$$

$$
+ 2\sum_{j=2}^{n} \big(W_{0,2}(z_1, z_j) W_{g,n-1}(z_1, Z_{\{1,\ldots,n\}\setminus\{j\}})
$$

$$
+ W_{0,2}(\sigma(z_1), z_j) W_{g,n-1}(\sigma(z_1), Z_{\{1,\ldots,n\}\setminus\{j\}}))\Big].
$$

By Lemma 7.10 and Corollary 7.14 this yields

$$
-2x_1\Big[\sum_{\substack{g_1+g_2=g \\ I \sqcup J = N\setminus\{1\}}} W_{g_1,|I|+1}(z_1, Z_I) W_{g_2,|J|+1}(\sigma(z_1), Z_J)
$$

$$
+ \sum_{j=2}^{n} \Big(\frac{B(\sigma(z_1), z_j)}{\mathrm{d}\,x(z_1)\,\mathrm{d}\,x(z_j)} W_{g,n-1}(z_1, Z_{\{1,\ldots,n\}\setminus\{j\}})
$$

$$
+ \frac{B(z_1, z_j)}{\mathrm{d}\,x(\sigma(z_1))\,\mathrm{d}\,x(z_j)} W_{g,n-1}(\sigma(z_1), Z_{\{1,\ldots,n\}\setminus\{j\}}))\Big].
$$

As $(g,n) \neq (0,1), (0,2), (0,3)$ we have $(g, n-1) \neq (0,1), (0,2)$. Thus

$$
W_{g,n-1} dx(z_1) \cdots dx(z_{n-1})
$$

satisfies Corollary 7.14. Therefore, putting things together, dividing by

$$
-2x_1[W_{0,1}(x(z_1)) - W_{0,1}(x(\sigma(z_1)))]
$$

and multiplying with $\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_n$, we obtain

$$
\omega_{g,n}(z_1, \ldots, z_n)
$$

$$
= \frac{1}{(W_{0,1}(x(z_1)) - W_{0,1}(x(\sigma(z_1)))\,\mathrm{d}\,x_1}
$$

$$
\Big[\sum_{j=2}^{n} \frac{\mathrm{d}\,x_1\,\mathrm{d}\,x_1}{x_1^2} \frac{z_j^2}{z_j^2-1} \frac{\partial}{\partial z_j} x_j^2 \frac{\omega_{g,n-1}(z_2,\ldots,z_n)}{x_1-x_j}
$$

$$
+ \omega_{g-1,n+1}(z_1, \sigma(z_1), z_2, \ldots, z_n)
$$

$$
+ \overset{\circ}{\sum_{\substack{g_1+g_2=g \\ I \sqcup J = N\setminus\{1\}}}} \omega_{g_1,|I|+1}(z_1, Z_I) \omega_{g_2,|J|+1}(\sigma(z_1), Z_J)
$$

$$+ \sum_{j=2} \omega_{0,2}(\sigma(z_1), z_j)\omega_{g,n-1}(z_1, Z_{\{1,\ldots,n\}\setminus\{j\}})$$

$$+ \omega_{0,2}(z_1, z_j)\omega_{g,n-1}(\sigma(z_1), Z_{\{1,\ldots,n\}\setminus\{j\}})\Big]$$

$$= \frac{1}{\omega_{0,1}(z_1) - \omega_{0,1}(\sigma(z_1))}\Big[\sum_{j=2}^{n} \frac{\mathrm{d}\,x_1\,\mathrm{d}\,x_1}{x_1^2}\frac{z_j^2}{z_j^2 - 1}\frac{\partial}{\partial z_j}x_j^2\frac{\omega_{g,n-1}(z_2,\ldots,z_n)}{x_1 - x_j}$$

$$+ \omega_{g-1,n+1}(z_1, \sigma(z_1), z_2, \ldots, z_n)$$

$$+ \sum_{\substack{g_1+g_2=g \\ I \sqcup J = N\setminus\{1\}}}' \omega_{g_1,|I|+1}(z_1, Z_I)\omega_{g_2,|J|+1}(\sigma(z_1), Z_J)\Big].$$

The $\mathrm{d}\,x_1$ in the denominator originates from the trivial expansion $\frac{\mathrm{d}\,x_1}{\mathrm{d}\,x_1}$. The next step is to apply Cauchy's formula and use the fact that the $\omega_{g,n}$ are rational forms (in particular in z_1) having only poles at ± 1. We have

$$\omega_{g,n}(z_1,\ldots,z_n) = \operatorname*{Res}_{z \to z_1} \frac{\omega_{g,n}(z, z_2, \ldots, z_n)\,\mathrm{d}\,z_1}{z - z_1}$$

$$= \operatorname*{Res}_{z \to \pm 1} \frac{\mathrm{d}\,z_1}{z_1 - z}\omega_{g,n}(z, z_2, \ldots, z_n)$$

$$= \operatorname*{Res}_{z \to \pm 1} \frac{\mathrm{d}\,z_1}{z_1 - \frac{1}{z}}\omega_{g,n}\left(\frac{1}{z}, z_2, \ldots, z_n\right)$$

$$= -\operatorname*{Res}_{z \to \pm 1} \frac{\mathrm{d}\,z_1}{z_1 - \sigma(z)}\omega_{g,n}(z, z_2, \ldots, z_n),$$

the second equality is due to the fact that $\omega_{g,n}$ are rational differentials in each z_i, hence the sum over all residue must vanish, i.e.,

$$0 = \operatorname*{Res}_{z \to z_1} \frac{\omega_{g,n}(z, z_2, \ldots, z_n)\,\mathrm{d}\,z_1}{z - z_1}$$

$$+ \operatorname*{Res}_{z \to \pm 1} \frac{\omega_{g,n}(z, z_2, \ldots, z_n)\,\mathrm{d}\,z_1}{z - z_1}$$

where $\operatorname{Res}_{z \to \pm 1}$ denotes the sum of the residues at 1 and -1. Thus we get

$$\omega_{g,n}(z_1,\ldots,z_n) = \operatorname*{Res}_{z \to \pm 1} \frac{1}{2}\left[\frac{\mathrm{d}\,z_1}{z_1 - z} - \frac{\mathrm{d}\,z_1}{z_1 - \sigma(z)}\right]\omega_{g,n}(z, z_2, \ldots, z_n)$$

$$= \operatorname*{Res}_{z \to \pm 1}\left[\frac{1}{2}\int_{\sigma(z)}^{z} \omega_{0,2}(z_1, \cdot)\right]\omega_{g,n}(z, z_2, \ldots, z_n).$$

Now, we want to invoke the recursion for the $\omega_{g,n}$ which we established before. We get

$$
\begin{aligned}
\omega_{g,n}(z_1, \ldots, z_n) \\
= \operatorname*{Res}_{z \to \pm 1} \frac{\frac{1}{2} \int_{\sigma(z)}^z \omega_{0,2}(z_1, \cdot)}{\omega_{0,1}(z) - \omega_{0,1}(\sigma(z))} \Big[\sum_{j=2}^n \frac{\mathrm{d}x\,\mathrm{d}x}{x^2} \frac{z_j^2}{z_j^2 - 1} \frac{\partial}{\partial z_j} x_j^2 \frac{\omega_{g,n-1}(z_2, \ldots, z_n)}{x - x_j} \\
+ \omega_{g-1,n+1}(z, \sigma(z), z_2, \ldots, z_n) \\
+ \sum_{\substack{g_1 + g_2 = g \\ I \sqcup J = N \setminus \{1\}}} \omega_{g_1, |I|+1}(z, Z_I) \omega_{g_2, |J|+1}(\sigma(z), Z_J) \Big].
\end{aligned}
$$

First we need to argue that the residue at $z = 1$ does not contribute. But this is since

$$
K(z_1, z) = \frac{\frac{1}{2} \int_{\sigma(z)}^z \omega(z_1, \cdot)}{\omega_{0,1}(z) - \omega_{0,1}(\sigma(z))} = \frac{z(z-1)^3 \,\mathrm{d}z_1}{2(z+1)(z_1 - z)(z_1 z - 1) \,\mathrm{d}z},
$$

i.e., K has a zero of order 3 at $z = 1$ which cancels the poles (of order 1) of $\omega_{g,n}(z, z_2, \ldots, z_n)$. Hence, last two terms on the right-hand side vanish. For the first one, note that

$$
\frac{\mathrm{d}x\,\mathrm{d}x}{x^2} = \frac{(z+1)^2 \,\mathrm{d}z\,\mathrm{d}z}{(z-1)^2 z^2}
$$

has pole of order 2, so the zero of $K(z_1, z)$ cancels this as well. Lastly, we show that the first term on the right-hand side vanishes if we take the residue at $z = -1$. But by the last equation we see that the pole of order 1 of K is removed by the zero of order 2. Thus we finally arrive at

$$
\begin{aligned}
\omega_{g,n}(z_1, \ldots, z_n) \\
= \operatorname*{Res}_{z \to -1} \frac{\frac{1}{2} \int_{\sigma(z)}^z \omega_{0,2}(z_1, \cdot)}{\omega_{0,1}(z) - \omega_{0,1}(\sigma(z))} \Big[\omega_{g-1,n+1}(z, \sigma(z), z_2, \ldots, z_n) \\
+ \sum_{\substack{g_1 + g_2 = g \\ I \sqcup J = N \setminus \{1\}}} \omega_{g_1, |I|+1}(z, Z_I) \omega_{g_2, |J|+1}(\sigma(z), Z_J) \Big].
\end{aligned}
$$

∎

8. Recursions for coverings of an arbitrary base curve

In this section, we derive a recursion for monotone and Grothendieck dessins d'enfants coverings of arbitrary base curves. For $m \in \mathbb{N}$, we denote $[m] := \{1, \ldots, m\}$. Furthermore, for a partition μ, we denote the partition obtained by removing the entries in

position $i_1, \ldots, i_n$ (for some $n \leq \ell(\mu)$ and $i_j \in [\ell(\mu)]$) by $\mu[i_1, \ldots, i_n]$. We introduce the following notation.

Notation 8.1. For $g \geq 0$ and ν be a partition of a positive integer d of length n, we define

$$A_g(\nu) := |\{(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g) \in S_d^{2g} : C([\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]) = \nu\}|.$$

For each partition ν, we fix a permutation

$$\sigma_\nu = (1 \cdots \nu_1)(\nu_1 + 1 \cdots \nu_1 + \nu_2) \cdots \left(\sum_{i=1}^{n-1} \nu_i + 1 \cdots d \right),$$

where for $k \in [n]$ the cycle $(\sum_{i=1}^{k-1} \nu_i + 1 \cdots \sum_{i=1}^{k} \nu_i)$ is labeled k.

We now observe

$$\vec{H}_{g',\leq}^{\bullet,g}(\mu) = \sum_{\nu \vdash d} \frac{\vec{h}_{g'}^{\leq;\bullet}(\mu, \nu)}{|\operatorname{Aut}(\nu)|} \cdot \frac{A_g(\nu)}{|C_\nu|} \quad \text{and} \quad \vec{H}_{g',<}^{\bullet,g}(\mu) = \sum_{\nu \vdash d} \frac{\vec{h}_{g'}^{<;\bullet}(\mu, \nu)}{|\operatorname{Aut}(\nu)|} \cdot \frac{A_g(\nu)}{|C_\nu|}$$

for the usual (strictly) monotone double Hurwitz number $h_{g'}^{\leq;\bullet}(\mu, \nu)$ (resp. $h_{g'}^{<;\bullet}(\mu, \nu)$). Thus, by finding a recursive method for computing the (strictly) monotone double Hurwitz numbers in genus 0, we obtain a recursive method for all genera.

For the rest of this section, we derive a recursive structure for (strictly) monotone double Hurwitz numbers. We note that the disconnected and connected (strictly) monotone double Hurwitz numbers are equivalent by the usual inclusion-exclusion principle. We can therefore focus on the connected numbers, which yield more compact formulas.

While monotone single Hurwitz numbers satisfy a recursive formula, a recursive formula for monotone Hurwitz numbers with an additional non-trivial ramification remains an open problem. However, one approach to this problem was introduced in [11] in the case of monotone orbifold Hurwitz numbers. More precisely, instead of considering monotone orbifold Hurwitz numbers, one considers a combinatorial refinement, which determines the former numbers and satisfy a recursion. In this section, we follow this philosophy and introduce a combinatorial refinement of (strictly) monotone double Hurwitz numbers and derive a recursion.

Definition 8.2. Let μ, ν be ordered partitions of d, let $i \in [\ell(\mu)]$ and $l \in [\nu_{\ell(\nu)}]$. We define $N_g^{\leq;l,i}(\mu_i \mid \mu[i], \nu)$ to be the number of all tuples $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$, such that

(1) $\sigma_1 = \sigma_\nu$, (with σ_ν and its labelling as defined above),

(2) $c(\sigma_1) = \nu, c(\sigma_2) = \mu$,

(3) the cycles of σ_2, σ_2 are labeled,

(4) $\tau_b \cdots \tau_1 \sigma_1 = \sigma_2$,

(5) $\langle \sigma_1, \tau_1, \ldots, \tau_b, \sigma_2 \rangle$ is a transitive subgroup,

(6) for $\tau_i = (s_i t_i)$,where $s_i < t_i$, we have $t_i \leq t_{i+1}$,

(7) we have $\tau_b = (s_b t_b)$, where $t_b = \sum_{j=1}^{\ell(v)-1} v_j + l$ for some $l \in [\ell(v)]$,

(8) t_b is contained in the i-th cycle of σ_2.

We also define the numbers

$$N_g^{\leq;i}(\mu_i \mid \mu[i], v) = \sum_{l=1}^{v_n} N_g^{\leq;l,i}(\mu_i \mid \mu[i], v)$$

and

$$N_g^{\leq}(\mu, v) = \sum_{i=1}^{\ell(\mu)} N_g^{\leq;i}(\mu_i \mid \mu[i], v).$$

Similarly, we define the notions for the strictly monotone case and denote the respective numbers by $N_g^{<;l,i}(\mu_i \mid \mu[i], v)$, $N_g^{<;i}(\mu_i \mid \mu[i], v)$ and $N_g^{<}(\mu, v)$.

Remark 8.3. Before we state our recursion for

$$N_g^{\leq;l,i}(\mu_i \mid \mu[i], v) \quad \text{and} \quad N_g^{<;l,i}(\mu_i \mid \mu[i], v),$$

we make the following remarks.

- We note that we can compute the monotone Hurwitz numbers $\vec{h}_g^{\leq}(v, \mu)$ by considering all tuples $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$ satisfying conditions (2)–(6) in Definition 8.2 as changing the order of the cycle types in the factorisations does not alter the enumeration. We use this convention in the proof of Theorem 8.4.

- We see that
$$\vec{h}_g^{\leq}(v, \mu) = \frac{|C_v|}{d!} \sum_{i=1}^{m} N_g^{\leq}(\mu_i \mid \mu[i], v)$$

and

$$\vec{h}_g^{<}(v, \mu) = \frac{|C_v|}{d!} \sum_{i=1}^{m} N_g^{<}(\mu_i \mid \mu[i], v).$$

- As we are concerned with (strictly) monotone factorisations in Definition 8.2, the numbers $t_b, t_b + 1, t_b + 2, \ldots, |v|$ actually appear consecutively in the i-th cycle of σ_2 (see [11, 23]), i.e., this cycle is of the form

$$(\ldots t_b t_b + 1 t_b + 2 \ldots |v|).$$

- The above definition of $N_g^{\leq,l}(\mu_i \mid \mu[i], v)$ is motivated by the notions in [11, 23]. While of technical nature, they are more natural when considered in terms of

monodromy graphs, which is an approach taken in the works of Do, Karev, and Hahn. In their language, the numbers $N_g^{\leq,l}(\mu_i \mid \mu[i], \nu)$ counts all *"monodromy graphs, where the unique bold out-end is labeled i with counter l."* We note that these monodromy graphs are related to tropical covers but differ from the ones, we introduced in this work.

Theorem 8.4. *Let g be a non-negative integer and μ, ν partitions of the same positive integer. Moreover, let $i \in [\ell(\mu)]$ and $l \leq \nu_i$. Then we have*

$$N_g^{\leq;l,i}(\mu_i \mid \mu[i], \nu)$$

$$= \Theta(\mu_i + l - \nu_n - 1) \sum_{j \in [\ell(\mu)] \setminus \{i\}} \sum_{p=1}^{l} N_g^{\leq;p,\ell(\mu)-1}(\mu_i + \mu_j \mid \mu^{i,j}[\ell(\mu)-1], \nu)$$

$$+ \sum_{\alpha+\beta=\mu_i} \sum_{p=1}^{l} \beta N_{g-1}^{\leq;p,i}(\alpha \mid (\mu[i], \beta), \nu)$$

$$+ \sum_{\alpha+\beta=\mu_i} \sum_{g_1+g_2=g} \sum_{\substack{I_1 \cup I_2 = [\ell(\mu)] \setminus \{i\} \\ J \subset [n-1]}} \sum_{p=1}^{l} \beta N_{g_1}^{\leq;p,|I_1|+1}(\alpha \mid (\mu_{I_1}, \alpha)[|I_1|+1], \nu_{J^c})$$
$$N_{g_2}^{\leq;|I_2|+1}(\beta \mid (\mu_{I_2}, \beta), \nu_J)$$

and

$$N_g^{<;l,i}(\mu_i \mid \mu[i], \nu)$$

$$= \Theta(\mu_i + l - \nu_n - 1) \sum_{j \in [\ell(\mu)] \setminus \{i\}} \sum_{p=1}^{l-1} N_g^{<;p,\ell(\mu)-1}(\mu_i + \mu_j \mid \mu^{i,j}[\ell(\mu)-1], \nu)$$

$$+ \sum_{\alpha+\beta=\mu_i} \sum_{p=1}^{l-1} \beta N_{g-1}^{<;p,i}(\alpha \mid (\mu[i], \beta), \nu)$$

$$+ \sum_{\alpha+\beta=\mu_i} \sum_{g_1+g_2=g} \sum_{\substack{I_1 \cup I_2 = [\ell(\mu)] \setminus \{i\} \\ J \subset [n-1]}} \sum_{p=1}^{l-1} \beta N_{g_1}^{<;p,|I_1|+1}(\alpha \mid (\mu_{I_1}, \alpha)[|I_1|+1], \nu_{J^c})$$
$$\left(\sum_{p=1}^{\nu_{\max}(J)} (N_{g_2}^{<;p,|I_2|+1}(\beta \mid (\mu_{I_2}, \beta), \nu_J)) \right),$$

where Θ is the Heavyside step function, i.e., $\Theta(t) = 0$ for $t < 0$ and $\Theta(t) = 1$ for $t \geq 0$. Furthermore, we denote $\mu^{i,j} = \mu[i,j] \cup \{\mu_i + \mu_j\}$

Proof. As the proofs are completely parallel, we restrict our discussion to the mono-tone numbers. We prove this theorem by a cut-and-join analysis of transitive mono-tone factorisations $(\sigma_\nu, \tau_1, \ldots, \tau_b, \sigma)$ of type (g, ν, μ). Recall that $N_g^{\leq;l,i}(\mu_i \mid \mu[i], \nu)$

counts the number of all tuples $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$ of type (g, ν, μ) satisfying conditions (1)–(8) of Definition 8.2, where $\tau_i = (s_i t_i)$ with $s_i < t_i$, such that $\sigma_1 = \sigma_\nu$ and s_b is contained in the cycle of σ_2 labeled i. We fix such a factorisation, define

$$\Sigma = \tau_{b-1} \cdots \tau_1 \sigma_\nu$$

and observe that $\eta = (\sigma_\nu, \tau_1, \ldots, \tau_{b-1}, \Sigma)$ is a monotone factorisation as well. There are three cases.

(1) The transposition τ_b is a cut for Σ. That is, s_b and t_b are contained in the same cycle of Σ. Then left multiplication by τ_b, i.e., $\tau_b \Sigma$ cuts the cycle of Σ containing s_b and t_b into two cycles. Conversely, this means s_b and t_b are contained in different cycles of σ and τ_b joins those cycles by left multiplication $\tau_b \sigma_2$ to a joint cycle in Σ.

Thus, if τ_b is a cut for Σ, any such transitive monotone factorisation $(\sigma_\nu, \tau_1, \ldots, \tau_b, \sigma)$ contributing to $N_g^{\leq; l, i}(\mu_i \mid \mu[i], \nu)$ yields a transitive monotone factorisation $(\sigma_\nu, \tau_1, \ldots, \tau_{b-1}, \Sigma)$ contributing to $N_g^{\leq; p, \ell(\mu)-1}(\mu_i + \mu_j \mid \mu^{i,j}[\ell(\mu) - 1], \nu)$ for some $j \in [\ell(\mu)] \setminus \{i\}$ and $p \leq l$. The other way round, we start with a tuple $(\sigma_\nu, \tau_1, \ldots, \tau_{b-1}, \Sigma)$ contributing to $N_g^{\leq; p, i}(\mu_i + \mu_j \mid \mu[i, j], \nu)$ for some $j \in [\ell(\mu)] \setminus \{i\}$ and $p \leq l$. We analyse the number of possible transpositions τ_b which give a transitive monotone factorisation $(\sigma_\nu, \tau_1, \ldots, \tau_b, \sigma)$ with $\mathcal{C}(\sigma_2) = \mu$. The number s_b is fixed to be $\sum_{i=1}^{n-1} \nu_i + l$. As mentioned before, the cycle of σ_2 containing s_b (labeled i) must then be of shape

$$\left(\cdots + t_b t_b + 1 + \cdots + \sum_{k=1}^{n} \nu_i - 1 \sum_{k=1}^{n} \nu_i \right).$$

Thus the length μ_i of the cycle must be at least $\nu_n - l + 1$, in other words $\Theta(\mu_i + l - \nu_n - 1)$ must not vanish. Moreover, as t_b is fixed and s_b must be contained in the same cycle of Σ as t_b, the value of s_b is fixed as well. This yields the first summand.

(2) The transposition τ_b is a redundant join for Σ. That is, r_b and s_b are contained in two different cycles of Σ and η is a transitive monotone factorisation as well. Then left multiplication by τ_b, i.e., $\tau_b \Sigma$ joins the cycles of Σ containing s_b and t_b respectively to one cycle. Conversely, this means s_b and t_b are contained in the same cycle of σ_2 and τ_b cuts this cycle by left multiplication $\tau_b \sigma_2$ to two cycles of Σ.

Thus, if τ_b is a redundant join for Σ any such tuple $(\sigma_\nu, \tau_1, \ldots, \tau_b, \sigma_2)$ contributing to $N_g^{\leq; l, i}(\mu_i \mid \mu[i], \nu)$ yields a tuple $(\sigma_\nu, \tau_1, \ldots, \tau_{b-1}, \Sigma)$ contributing to $N_{g-1}^{\leq; p, i}(\alpha \mid (\mu[i], \beta), \nu)$ for some $i \in [\ell(\mu)], \alpha + \beta = \mu_i$ (note that the genus drops by 1 since the number of transpositions and the length of the second cycle type drop by 1) and $p \geq l$. The other way, we start with a tuple $(\sigma_\nu, \tau_1, \ldots, \tau_{b-1}, \Sigma)$ contributing to $N_{g-1}^{\leq; p, i}(\alpha \mid (\mu[i], \beta), \nu)$ for some $i \in [\ell(\mu)]$ and $\alpha + \beta = \mu_i$. We analyse the number of possible

transposition τ_b which yield a tuple $(\sigma_\nu, \tau_1, \ldots, \tau_b, \sigma_2)$ contributing to $N_g^{\leq;l,i}(\mu_i \mid \mu[i], \nu)$.

The number s_b is fixed to be $\sum_{i=1}^{n-1} \nu_i + l$. By definition, t_b is contained in a cycle of length α. We need to choose s_b, such that $\tau_b = (s_b t_b)$ joint α with another cycle to a new cycle of length μ_i. Thus, we need to choose r_b from a cycle of length β. Moreover, we can choose s_b arbitrarily, which yields a factor of β and we obtain the second summand.

(3) The transposition τ_b is an essential join for Σ. That is, s_b and t_b are contained in two different cycles of Σ and the group generated by η is non-transitive with two orbits. Then left multiplication by τ_b, i.e., $\tau_b \Sigma$ joins two cycles of Σ containing s_b and t_b respectively to one cycle. Conversely, this means s_b and t_b are contained in the same cycle of σ_2 and τ_b cuts this cycle by left multiplication $\tau_b \sigma_2$ to two cycles of Σ.

Thus, if τ_b is an essential join for Σ, any such tuple $(\sigma_\nu, \tau_1, \ldots, \tau_b, \sigma_2)$ contributing to $N_g^{\leq;l,i}(\mu_i \mid \mu[i], \nu)$ yields two tuples $(\sigma_{\nu_J}, \tau_{i(1)}, \ldots, \tau_{i(c)}, \sigma_2^1)$ and $(\sigma_{\nu_{J^c}}, \tau_{j(1)}, \ldots, \tau_{j(d)}, \sigma_2^2)$ such that

(1) $J \subset [n]$,

(2) $c + d = b$,

(3) σ_{ν_J} is the permutation obtained as the product of all disjoint cycles of σ_ν with label in J and

(4) σ_2^1 and σ_2^2 have disjoint orbits and $\sigma_2 = \sigma_2^1 \sigma_2^2$, where σ_2^i recovers the labels from σ_2 for $i = 1, 2$.

These tuples contribute to $\sum_{p=1}^{l} \beta N_{g_1}^{\leq;p,|I_1|+1}(\alpha \mid (\mu_{I_1}, \alpha)[|I_1| + 1], \nu_{J^c})$ for $p \leq l$ and to $N_{g_2}^{\leq;|I_2|+1}(\beta \mid (\mu_{I_2}, \beta), \nu_J)$ respectively. By the same arguments as in the previous case, we obtain a factor of β and thus obtain the third summand, which completes the proof of the theorem. ∎

A. Examples

We provide some examples of the quasimodular q-expansions of $\sum_d H_{k,l,m}^{g,d}(\mu)q^d$, computed using Sage [39]. See [38, Section 10] for an extensive list of quasimodular forms corresponding to simple Hurwitz numbers ($l = m = 0, \mu = (\,)$):

$$\sum_d H_{2,0,0}^{2,d}(\,)q^d = \frac{1}{2^6 3^4 5}(5P^3 - 3PQ - 2R)$$
$$= 2q^2 + 16q^3 + 60q^4 + 160q^5 + 360q^6 + 672q^7 + 1240q^8$$
$$+ 1920q^9 + 3180q^{10} + 4400q^{11} + 6832q^{12} + O(q^{13}),$$

$$\sum_d H_{0,2,0}^{2,d}()q^d = \frac{1}{2^7 3^4 5}(5P^3 - 3PQ - 2R + 45P^2 + 18Q + 90P - 153)$$

$$= 2q^2 + 13q^3 + 44q^4 + 109q^5 + 235q^6 + 422q^7 + 760q^8$$
$$+ 1151q^9 + 1875q^{10} + 2555q^{11} + 3927q^{12} + O(q^{13}),$$

$$\sum_d H_{0,0,2}^{2,d}()q^d = \frac{1}{2^7 3^4 5}(5P^3 - 3PQ - 2R - 45P^2 - 18Q - 90P + 153)$$

$$= 3q^3 + 16q^4 + 51q^5 + 125q^6 + 250q^7 + 480q^8 + 769q^9$$
$$+ 1305q^{10} + 1845q^{11} + 2905q^{12} + O(q^{13}),$$

$$\sum_d H_{2,0,0}^{3,d}(3)q^d = \frac{1}{2^{13} 3^5 5^2}\Big(-875P^5 + 1775P^3 Q - 10P^2 R - 894PQ^2$$
$$+ 4QR + 750P^4 - 1710P^2 Q + 60PR + 900Q^2$$
$$+ 135P^3 - 81PQ - 54R\Big)$$

$$= 36q^3 + 540q^4 + 3606q^5 + 15726q^6 + 53298q^7 + 149142q^8$$
$$+ 367920q^9 + 815886q^{10} + 1668150q^{11} + 3202374q^{12}$$
$$+ O(q^{13}),$$

$$\sum_d H_{0,2,0}^{3,d}(3)q^d = \frac{1}{2^{14} 3^5 5^2}\Big(-875P^5 + 1775P^3 Q - 10P^2 R - 894PQ^2$$
$$+ 4QR - 2625P^4 - 630P^2 Q + \frac{23460}{7}PR$$
$$+ \frac{648}{7}Q^2 + 2835P^3 + 1359PQ - 5814R$$
$$+ 3150P^2 - 4608Q + 7020P - 4131\Big)$$

$$= 27q^3 + 369q^4 + 2337q^5 + 9795q^6 + 32307q^7 + 88446q^8$$
$$+ 214536q^9 + 469230q^{10} + 948600q^{11} + 1803375q^{12}$$
$$+ O(q^{13}),$$

$$\sum_d H_{0,0,2}^{3,d}(3)q^d = \frac{1}{2^{14} 3^5 5^2}\Big(-875P^5 + 1775P^3 Q - 10P^2 R - 894PQ^2 + 4QR$$
$$+ 4125P^4 - 2790P^2 Q + \frac{22620}{7}PR + \frac{11952}{7}Q^2$$
$$- 2565P^3 - 1521PQ + 5706R - 3150P^2 + 4608Q$$
$$- 7020P + 4131\Big)$$

$$= 9q^3 + 171q^4 + 1269q^5 + 5931q^6 + 20991q^7 + 60696q^8$$
$$+ 153384q^9 + 346656q^{10} + 719550q^{11} + 1398999q^{12}$$
$$+ O(q^{13}).$$

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