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# On the minor problem and branching coefficients

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**Abstract.** The minor problem, namely the study of the spectrum of a principal submatrix of a Hermitian matrix taken at random on its orbit under conjugation, is revisited, with emphasis on the use of orbital integrals and on the connection with branching coefficients in the decomposition of an irreducible representation of $\mathrm{U}(n)$, resp. $\mathrm{SU}(n)$, into irreps of $\mathrm{U}(n-1)$, resp. $\mathrm{SU}(n-1)$. As is well known, the branching coefficients are trivial (equal to 0 or 1) for the branchings of $\mathrm{U}(n) \supset \mathrm{U}(n-1)$, while they are not for $\mathrm{SU}(n) \supset \mathrm{SU}(n-1)$, where multiplicities may appear. In the latter case, the problem is shown to be related to the distribution of *spacings* in the minor problem. An explicit expression is obtained for the multiplicities, in terms of an integral stemming from the minor problem, and an Ansatz is given for a closed form expression for arbitrary $n$.

## 1. Introduction

What we call the *minor problem* deals with the following question: given an $n$-by-$n$ Hermitian matrix of given spectrum, what can be said about the eigenvalues of one of its $(n-1) \times (n-1)$ principal submatrices? This question has been thoroughly studied and answered by many authors [1, 10, 21, 22]. As several other such questions, this problem of classical linear algebra has a counterpart in the realm of representation theory [9, 14, 16], namely the determination of *branching coefficients* of an irreducible representation (irrep) of $\mathrm{U}(n)$ into irreps of $\mathrm{U}(n-1)$. It is then natural to study the same question of branching coefficients in the case of the embedding $\mathrm{SU}(n-1) \subset \mathrm{SU}(n)$. The aim of this note is to review these questions and to make explicit the link, by use of orbital integrals. It is thus in the same vein as recent works on the Horn [3, 4, 6, 19, 26] or Schur [7] problems.

This paper is organized as follows. In Section 2, I review the classical minor problem and recall how it may be rephrased in terms of $\mathrm{U}(n)$ orbital integrals. Studying the distribution of *spacings* of the eigenvalues of the minor leads to modified expressions, that will be turn out to be natural for the case of $\mathrm{SU}(n)$. Explicit expressions

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are given for low $n = 3, 4$ and a closed form is conjectured for arbitrary $n$. Section 3 is devoted to the issue of branching coefficients for the embeddings $\mathrm{U}(n-1) \subset \mathrm{U}(n)$ and $\mathrm{SU}(n-1) \subset \mathrm{SU}(n)$. While the former is treated by means of Gelfand–Tsetlin triangles and does not give rise to multiplicities, as well known since Weyl [23], the latter requires a new technique. This is where the integral introduced in Section 2 proves useful and is shown to provide an explicit expression of branching coefficients, see Theorem 2, which is the main result of this paper. The use of that formula as for the behaviour of branching coefficients under *stretching*, i.e., dilatation of the weights, is briefly discussed in Section 3.3.

## 2. The classical problem

### 2.1. Notations and classical results

Let us fix some notations. If $A$ is an $n \times n$ Hermitian matrix with known eigenvalues $\alpha_1 \geq \cdots \geq \alpha_n$, what can be said about the eigenvalues $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n-1}$ of one of its principal $(n-1) \times (n-1)$ minor submatrix ("minor" in short[1])?

A first trivial observation is that if we are interested in the statistics of the $\beta$'s as $A$ is taken randomly on its $\mathrm{U}(n)$ orbit $\mathcal{O}_\alpha$, the choice of the minor among the $n$ possible ones is immaterial, since a permutation of rows and columns of $A$ gives another matrix of the orbit.

A second, less trivial, observation is that the $\beta$'s are constrained by the celebrated Cauchy–Rayleigh interlacing Theorem:

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \cdots \geq \beta_{n-1} \geq \alpha_n. \tag{1}$$

For proofs, see for example [10, 11, 24]

If $A$ is chosen at random on its orbit $\mathcal{O}_\alpha$, and uniformly in the sense of the $\mathrm{U}(n)$ Haar measure, what is the probability distribution (PDF) of the $\beta$'s? This question has been answered by Baryshnikov [1], see also [10, 22]. We first observe that the problem is invariant under a global shift of all $\alpha$'s and all $\beta$'s by a same constant: indeed a translation of $A$ by $a\mathbb{I}_n$ shifts by $a$ all its eigenvalues as well as all the eigenvalues of any of its principal minors.

Let $\Delta$ denote the Vandermonde determinant: $\Delta_n(\alpha) = \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)$ and likewise for $\Delta_{n-1}(\beta)$.

[1] In the literature, the word "minor" refers either to the submatrix or to its determinant. We use it here in the former sense.

**Theorem 1** (Baryshnikov [1]). *The PDF of the $\beta$'s on its support* (1) *is given by*

$$P(\beta \mid \alpha) = (n-1)!\frac{\Delta_{n-1}(\beta)}{\Delta_n(\alpha)}. \tag{2}$$

*Alternative proof.* This result may also be recovered in terms of orbital integrals. Let

$$\mathcal{H}_\alpha^{(n)}(X) = \int_{\mathrm{U}(n)} dU e^{\mathrm{tr}\, U\alpha U^\dagger X} \tag{3}$$

where $X \in H_n$, the space of $n \times n$ Hermitian matrices, $dU$ is the normalized Haar measure on $\mathrm{U}(n)$, and $\alpha$ stands here for the diagonal matrix $\mathrm{diag}(\alpha_i)$. In terms of the eigenvalues $x_i$ of $X$, we write this orbital integral as $\mathcal{H}^{(n)}(\alpha; x)$.

The orbit $\mathcal{O}_\alpha$ carries a unique probabilistic measure, the *orbital measure* $\mu_\alpha(dA)$, $A \in \mathcal{O}_\alpha$, whose Fourier transform (the characteristic function of the random variable $A$) is $\mathcal{H}_\alpha^{(n)}(\mathrm{i}X)$,

$$\phi_A(X) := \mathbb{E}(e^{\mathrm{i}\,\mathrm{tr}\, AX}) = \int_{\mathcal{O}_\alpha} e^{\mathrm{i}\,\mathrm{tr}\, AX} \mu_\alpha(dA) = \int_{\mathrm{U}(n)} dU e^{\mathrm{i}\,\mathrm{tr}\, U\alpha U^\dagger X} = \mathcal{H}_\alpha^{(n)}(\mathrm{i}X).$$

Let $\Pi$ be the projector of $H_n$ into $H_{n-1}$ that maps $A \in H_n$ onto its upper $(n-1) \times (n-1)$ minor submatrix $B$. According to the observation that the Fourier transform of the projection of the orbital measure is the restriction of the Fourier transform [10], the characteristic function of $B$ is $\phi_B(Y) = \phi_A(X_0)$, with $X_0 = \Pi(X) = \left(\begin{smallmatrix} Y & 0 \\ 0 & 0 \end{smallmatrix}\right) \in H_n$, $Y \in H_{n-1}$, from which the PDF of $B$ is obtained by inverse Fourier transform

$$P(B|A) = \frac{1}{(2\pi)^{(n-1)^2}} \int_{H_{n-1}} dY e^{-\mathrm{i}\,\mathrm{tr}\, YB} \int_{\mathrm{U}(n)} dU e^{\mathrm{i}\,\mathrm{tr}\, U\alpha U^\dagger X_0}.$$

After reduction to eigenvalues,[2]

$$\begin{aligned} P(\beta \mid \alpha) = {} & \frac{(n-1)!}{(2\pi)^{n-1}(\prod_1^{n-1} p!)^2} \Delta_{n-1}^2(\beta) \\ & \times \int_{\mathbb{R}^{n-1}} dx \Delta_{n-1}^2(x) \mathcal{H}^{(n)}(\alpha; \mathrm{i}(x,0)) \mathcal{H}^{(n-1)}(\beta; \mathrm{i}x)^*. \end{aligned} \tag{4}$$

(In physicist's parlance, this is the *overlap* of the two orbital integrals.) Here and below, for $x \in \mathbb{R}^{n-1}$, $(x, 0)$ denotes the corresponding vector in $\mathbb{R}^n$.

[2]The factor $(n-1)!$ comes from the fact that we are restricting the $\beta$'s to the dominant sector $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n-1}$.

Making use of the explicit expressions known for $\mathcal{H}^{(n)}(\alpha; x)$ (see [15, 17]), we find

$$P(\beta \mid \alpha) = C\ \Delta_{n-1}^2(\beta) \int_{\mathbb{R}^{n-1}} dx \Delta_{n-1}^2(x) \frac{\det(e^{\mathrm{i}\alpha_i (x,0)_j})_{i,j=1,\dots,n}}{\Delta_n(\alpha)\Delta_n((x,0))} \times \frac{\det(e^{-\mathrm{i}\beta_i x_j})_{i,j=1,\dots,n-1}}{\Delta_{n-1}(\beta)\Delta_{n-1}(x)} \tag{5}$$

$$= \frac{1}{(2\pi \mathrm{i})^{n-1}} \frac{\Delta_{n-1}(\beta)}{\Delta_n(\alpha)} \int_{\mathbb{R}^{n-1}} \frac{d^{n-1}x}{x_1 x_2 \cdots x_{n-1}} \det(e^{\mathrm{i}\alpha_i (x,0)_j})_{i,j=1,\dots,n} \times \det(e^{-\mathrm{i}\beta_i x_j})_{i,j=1,\dots,n-1} \tag{6}$$

since the prefactor reads

$$C = \frac{(n-1)! \prod_1^{n-1} p! \prod_1^{n-2} p! \mathrm{i}^{-n(n-1)/2+(n-1)(n-2)/2}}{(2\pi)^{n-1}(\prod_1^{n-1} p!)^2} = \frac{1}{(2\pi \mathrm{i})^{n-1}}$$

and since $\Delta_n((x,0)) = (\prod_{i=1}^{n-1} x_i)\Delta_{n-1}(x)$. Let's write

$$P(\beta \mid \alpha) = (n-1)! \frac{\Delta_{n-1}(\beta)}{\Delta_n(\alpha)} \mathcal{K}(\alpha; \beta), \tag{7}$$

hence

$$\mathcal{K}(\alpha;\beta) = \frac{1}{(2\pi)^{n-1}(\prod_1^{n-1} p!)^2} \Delta_n(\alpha)\Delta_{n-1}(\beta) \int_{\mathbb{R}^{n-1}} dx \Delta_{n-1}^2(x) \mathcal{H}^{(n)}(\alpha; \mathrm{i}(x,0)) \times \mathcal{H}^{(n-1)}(\beta; \mathrm{i}x)^*$$

$$= \frac{1}{(2\pi \mathrm{i})^{n-1}(n-1)!} \int_{\mathbb{R}^{n-1}} \frac{d^{n-1}x}{x_1 x_2 \cdots x_{n-1}} \det(e^{\mathrm{i}\alpha_i (x,0)_j})_{1\le i,j\le n} \times \det(e^{-\mathrm{i}\beta_i x_j})_{1\le i,j\le n-1} \tag{8}$$

in analogy with the introduction of the "volume functions" in the Horn and Schur problems [3, 4, 7]. The function $\mathcal{K}(\alpha; \beta)$ is then, as in these similar cases, a linear combination of products of Dirichlet integrals: PP $\int \frac{e^{\mathrm{i}\alpha t}}{t} = \mathrm{i}\pi\varepsilon(\alpha)$, with $\varepsilon$ the sign function. Thus $\mathcal{K}(\alpha; \beta)$ must be a piecewise constant function, supported by the product of intervals given by the interlacing theorem (1).

By making use of the integral form of the Binet–Cauchy formula (see [10]), namely

$$\int_{\mathbb{R}^k} \det(f_i(t_j))_{1\le i,j\le k} \det(g_i(t_j))_{1\le i,j\le k} = k! \det\Big(\int_R f_i(t) g_j(t)\Big)_{1\le i,j\le k},$$

with here $k = n-1$, $f_i(t) = \frac{1}{t}(e^{\mathrm{i}\alpha_i t} - e^{\mathrm{i}\alpha_n t})$, $g_i(t) = e^{-\mathrm{i}\beta_i t}$, we find

$$\mathcal{K}(\alpha;\beta) = \frac{1}{(2\pi\mathrm{i})^{n-1}} \det \int_{\mathbb{R}} \frac{dt}{t}(e^{\mathrm{i}(\alpha_i-\beta_j)t} - e^{\mathrm{i}(\alpha_n-\beta_j)t}) \tag{9}$$

$$= \frac{1}{2^{n-1}} \det \left(\varepsilon(\alpha_i - \beta_j) - \varepsilon(\alpha_n - \beta_j)\right)_{1\le i,j\le n-1} \tag{10}$$

$$= \frac{1}{2^{n-1}} \det \begin{pmatrix} \varepsilon(\alpha_1-\beta_1) & \cdots & \varepsilon(\alpha_1-\beta_{n-1}) & 1 \\ \varepsilon(\alpha_2-\beta_1) & \cdots & \varepsilon(\alpha_2-\beta_{n-1}) & 1 \\ \vdots & & & \vdots \\ \varepsilon(\alpha_n-\beta_1) & \cdots & \varepsilon(\alpha_n-\beta_{n-1}) & 1 \end{pmatrix}. \tag{11}$$

Equation (10) just reproduces a result by Olshanksi [22], since the difference $(\varepsilon(\alpha_i - \beta_j) - \varepsilon(\alpha_n - \beta_j))$ appearing there is nothing else than twice the characteristic function of the interval $[\alpha_n, \alpha_i]$, denoted $M_2(\beta_j; \alpha_n, \alpha_i)$ in [10]. Finally, it may be shown that the determinant in (11) equals $2^{n-1}$ times the characteristic function of (1), so that the piecewise constant function $\mathcal{K}$ is just 1 on its support, in agreement with (2), see [10, 22]. ■

**Remark.** The previous considerations extend to projections of matrix $A$ onto a smaller minor $k \times k$ submatrix, see [10, 21, 22].

### 2.2. Distribution of spacings

We now want to study the spacings $\gamma_i = \beta_i - \beta_{n-1}$, $i = 1, \dots, n-2$ of the eigenvalues of the principal minor. This implies that $\gamma_{n-1} = 0$ hereafter. Their distribution follows from that of the $\beta$'s by changing variables and integrating over $\beta_{n-1}$. It will be invariant under a global shift of all $\alpha$'s by some real $a$, since the latter translates also all $\beta$'s by $a$ and leaves the spacings $\gamma$ unchanged. We denote by $\overline{P}(\gamma|\alpha)$ and $\overline{\mathcal{K}}(\alpha;\gamma)$ the PDF of these spacings and its associated "volume function."[3] Integrating the expression (8) of $\mathcal{K}$ over $\beta_{n-1}$ yields a delta function

$$\overline{\mathcal{K}}(\alpha;\gamma) := \int d\beta_{n-1} \mathcal{K}(\alpha;\gamma+\beta_{n-1}) \tag{12}$$

$$= \frac{2\pi}{(n-1)!(2\pi\mathrm{i})^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{d^{n-1}x}{x_1 x_2 \cdots x_{n-1}} \delta\Big(\sum_1^{n-1} x_j\Big) \det(e^{\mathrm{i}\alpha_i (x,0)_j})_{1\le i,j\le n} \times \det(e^{-\mathrm{i}\gamma_i x_j})_{1\le i,j\le n-1}$$

[3] For convenience, we keep that denomination, although the interpretation of $\overline{\mathcal{K}}$ as the volume (of a polytope in some parameter space?), justified in other cases [3, 4, 7, 19], remains so far elusive here.

and

$$\bar{P}(\gamma \mid \alpha) = (n-1)! \frac{\Delta_{n-1}(\gamma)}{\Delta_n(\alpha)} \overline{\mathcal{K}}(\alpha; \gamma). \tag{13}$$

Hence, an alternative definition of $\overline{\mathcal{K}}$ is

$$\overline{\mathcal{K}}(\alpha; \gamma) = \frac{\Delta_n(\alpha)\Delta_{n-1}(\gamma)}{(2\pi)^{n-2}(\prod_1^{n-1} p!)^2} \int_{\mathbb{R}^{n-1}} dx \delta\Big(\sum_1^{n-1} x_j\Big) \Delta_{n-1}^2(x) \mathcal{H}^{(n)}(\alpha; \mathrm{i}(x, 0)) \times \mathcal{H}^{(n-1)}(\gamma; \mathrm{i}x)^*, \tag{14}$$

an expression that we use later in Section 3.2, where the integration over the hyperplane $\sum_1^{n-1} x_i = 0$ in $\mathbb{R}^{n-1}$ will naturally be interpreted as over the Lie algebra $\mathfrak{su}(n-1)$. A more explicit expression is

$$\overline{\mathcal{K}}(\alpha; \gamma) = \frac{2\pi}{(2\pi \mathrm{i})^{n-1}(n-1)!} \int_{\sum_1^{n-1} x_i = 0} \frac{dx}{x_1 x_2 \cdots x_{n-1}} \times \det(e^{\mathrm{i}\alpha_i x_1}, e^{\mathrm{i}\alpha_i x_2}, \ldots, e^{\mathrm{i}\alpha_i x_{n-1}}, 1)_{1 \le i \le n-1} \det(e^{-\mathrm{i}\gamma_i x_j})_{1 \le i,j \le n-1}. \tag{15}$$

The support of $\overline{\mathcal{K}}$ in the "dominant sector" $\gamma_{i+1} \le \gamma_i$ follows from that of $\mathcal{K}$, see (1),

$$\alpha_{i+1} - \alpha_j \le \gamma_i - \gamma_j \le \alpha_i - \alpha_{j+1}, \quad 1 \le i < j \le n-1. \tag{16}$$

Expanding the two determinants and using once again the Dirichlet integrals

$$\mathrm{PP} \int \frac{e^{\mathrm{i}at}}{t^r} = \mathrm{i}\pi \frac{(\mathrm{i}a)^{r-1}}{(r-1)!} \varepsilon(a),$$

one finds that $\overline{\mathcal{K}}(\alpha; \gamma)$, a combination of convoluted box splines, is a piece-wise linear function of differentiability class $C^0$.

The maximal value (in the dominant sector) of $\overline{\mathcal{K}}$, for fixed $\alpha$, is readily derived from (12), where we are integrating the function $\mathcal{K}$ equal to 1 on its support, over $\beta_{n-1}$, subject to the $n-1$ conditions $\alpha_{i+1} \le \beta_i = \gamma_i + \beta_{n-1} \le \alpha_i$, hence

$$\overline{\mathcal{K}}_{\max}(\alpha) := \max_\gamma \overline{\mathcal{K}}(\alpha; \gamma) = \min_{1 \le i \le n-1} (\alpha_i - \alpha_{i+1}). \tag{17}$$

We note that, because of the constraint $\sum_{i=1}^{n-1} x_i = 0$, the expressions (14) or (15) are invariant by a global shift of all $\alpha_i$, as expected. We may use that invariance to choose $\alpha_n = 0$, a choice that will be natural in the application to SU($n$) representations. We conclude that $\overline{\mathcal{K}}(\alpha, \gamma)$ is a function of two sets of variables, a $n$-plet $\alpha$ with $\alpha_n = 0$, and a $(n-1)$-plet $\gamma$ with $\gamma_{n-1} = 0$. We thus summarize the previous discussion by

**Proposition 1.** *The PDF of the spacings $\gamma$'s on its support* (16) *is given by*

$$\overline{P}(\beta \mid \alpha) = (n-1)! \frac{\Delta_{n-1}(\gamma)}{\Delta_n(\alpha)} \overline{\mathcal{K}}(\alpha; \gamma), \tag{18}$$

*with $\overline{\mathcal{K}}$ given in* (12)*,* (14)*, or* (15)*.*

As is clear from (12), $\overline{\mathcal{K}}$ may then be extended to a function of the unordered $\alpha$'s and $\gamma$'s, odd under the action of the symmetric group $\mathcal{S}_n$, i.e., the SU$(n)$ Weyl group, acting on $\alpha$ by $w(\alpha)_i = \alpha_{w(i)} - \alpha_{w(n)}$, $i = 1, \dots, n$, $w \in \mathcal{S}_n$, and likewise odd under the action of $\mathcal{S}_{n-1}$ on $\gamma_i$, $i = 1, \dots, n-1$.

**Explicit expressions of $\overline{\mathcal{K}}$.** Let $\epsilon_w$ denote the signature of permutation $w \in \mathcal{S}_n$. For $n = 3$, the function $\overline{\mathcal{K}}$ reads

$$\begin{aligned} \overline{\mathcal{K}}(\alpha; \gamma) &= \frac{1}{2}(|\alpha_1 - \gamma_1| - |\alpha_1 - \alpha_2 - \gamma_1| - |\alpha_2 - \gamma_1|) - (\gamma_1 \mapsto -\gamma_1) \\ &= \frac{1}{2} \sum_{w \in \mathcal{S}_3} \epsilon_w |w(\alpha)_1 - \gamma_1| \end{aligned} \tag{19}$$

which is an odd continuous function of $\gamma_1$, vanishing for $\gamma_1 \notin (-\alpha_1, \alpha_1)$, constant and equal to its extremum value

$$\pm \min(\alpha_1 - \alpha_2, \alpha_2)$$

for $|\gamma_1| \in [\min(\alpha_1 - \alpha_2, \alpha_2), \max(\alpha_1 - \alpha_2, \alpha_2))$, and linear in between. If we restrict to the dominant sector $0 \le \gamma_1$, this may be encapsulated in a single formula, see Figure 1,

$$\overline{\mathcal{K}}(\alpha; \gamma) = \min(\gamma_1, \alpha_1 - \gamma_1, \alpha_2, \alpha_1 - \alpha_2). \tag{20}$$

In words, the function grows linearly from the boundaries of its support, until it reaches it maximum (17).

A similar discussion applies to $n = 4$: $\overline{\mathcal{K}}(\alpha; \gamma)$ has a support in the dominant sector defined by the inequalities (taking $\alpha_4 = 0$)

$$(\alpha_2 - \alpha_3) \le \gamma_1 \le \alpha_1, \quad 0 \le \gamma_2 \le \alpha_2, \quad 0 \le \gamma_1 - \gamma_2 \le \alpha_1 - \alpha_3, \tag{21}$$

and a maximal value equal to

$$\overline{\mathcal{K}}_{\max}(\alpha) = \min(\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \alpha_3).$$

In that sector:

$$\begin{aligned} \overline{\mathcal{K}}(\alpha; \gamma) = \min\big(&\gamma_1 - \alpha_2 + \alpha_3, \alpha_1 - \gamma_1, \gamma_2, \alpha_2 - \gamma_2, \\ &\gamma_1 - \gamma_2, \alpha_1 - \alpha_3 - \gamma_1 + \gamma_2, \overline{\mathcal{K}}_{\max}(\alpha)\big). \end{aligned} \tag{22}$$

Its graph has an Aztec pyramid shape, see Figure 2.

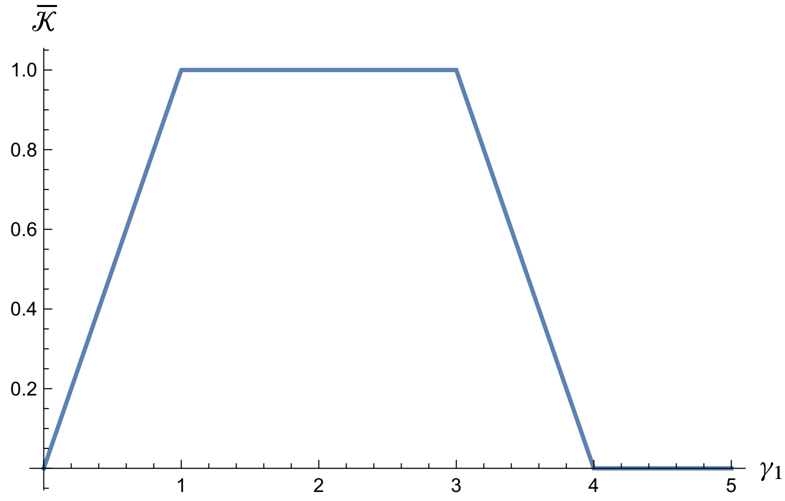


**Figure 1.** The $\overline{\mathcal{K}}$ function for $n = 3$, $\alpha = \{4, 1, 0\}$ and $\gamma_1 \geq \gamma_2 = 0$.

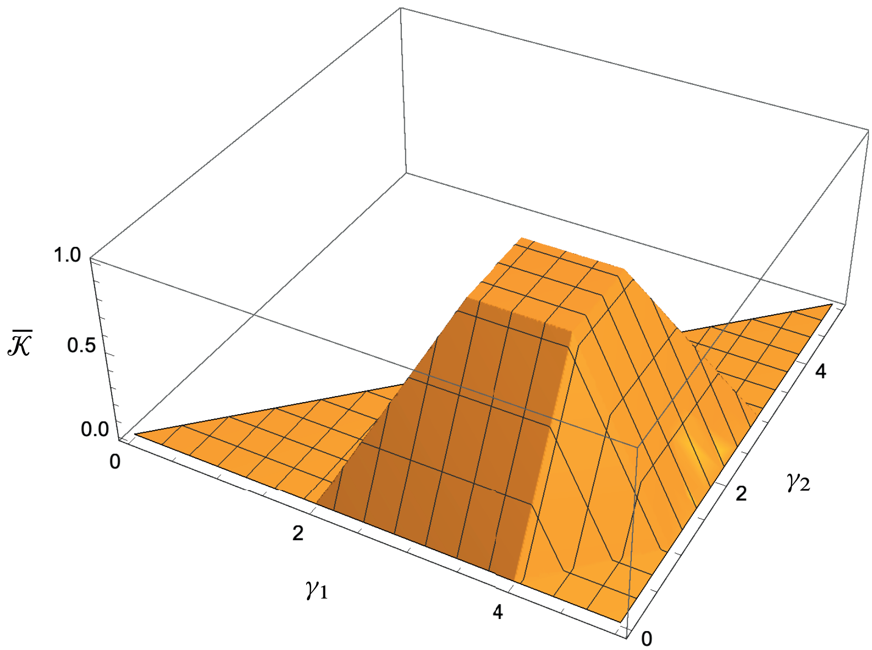


**Figure 2.** The $\overline{\mathcal{K}}$ function for $n = 4$ and $\alpha = \{5, 3, 1, 0\}$, in the $\gamma_1 \geq \gamma_2 \geq \gamma_3 = 0$ sector.

For general $n$, likewise, if we assume that the same property holds, namely that the function grows linearly with a gradient 1 in each variable $\gamma_i - \gamma_j$ from the boundary of its support (16), it is easy to infer a general formula for $\overline{\mathcal{K}}$ in the dominant sector

$$\overline{\mathcal{K}}(\alpha;\gamma) = \min\Big(\min_{1\le i<j\le n-1}(\gamma_i - \gamma_j - \alpha_{i+1} + \alpha_j, \alpha_i - \alpha_{j+1} - \gamma_i + \gamma_j), \overline{\mathcal{K}}_{\max}(\alpha)\Big) \tag{23}$$

with again, the convention that $\alpha_n = \gamma_{n-1} = 0$. This conjectured expression has been tested up to $n = 6$ and large values of $\alpha$, but clearly, a direct proof, either combinatorial or analytical, would be desirable.

## 3. The "quantum" problem

In this section, we consider the restriction of the group $\mathrm{U}(n)$, resp. $\mathrm{SU}(n)$, to its subgroup $\mathrm{U}(n-1)$, resp. $\mathrm{SU}(n-1)$, and the ensuing decomposition of their representations. For definiteness, the restriction of $\mathrm{SU}(n)$ to $\mathrm{SU}(n-1)$ we have in mind results from projecting out the simple root $\boldsymbol{\alpha}_{n-1}$ in the dual of the Lie algebra $\mathfrak{su}(n)$, and likewise for $\mathrm{U}(n)$.

### 3.1. Gelfand–Tsetlin patterns

Just like in the cases of the Horn or of the Schur problem, the minor problem is the classical counterpart of a "quantum" problem in representation theory. Given a highest weight (h.w.) irreducible representation (irrep) $V_\alpha^{(n)}$ of $\mathrm{U}(n)$, which irreps $V_\beta^{(n-1)}$ of $\mathrm{U}(n-1)$ occur and with which multiplicities, in the restriction of $\mathrm{U}(n)$ to $\mathrm{U}(n-1)$? That problem too is well known, is important in physical applications (see for example [8, 12]), and may be solved by a variety of methods. Here we first recall how to make use of Gelfand–Tsetlin triangles, i.e., triangular patterns

$$\begin{array}{ccccccccc}
x_1^{(n)} = \alpha_1 & \cdots & & & \cdots & & & \cdots & x_n^{(n)} = \alpha_n \\
& x_1^{(n-1)} & & \cdots & & \cdots & & x_{n-1}^{(n-1)} & \\
& & & \ddots & & \iddots & & & \\
& & & & x_1^{(1)} & & & &
\end{array}$$

subject to the inequalities

$$x_i^{(j+1)} \ge x_i^{(j)} \ge x_{i+1}^{(j+1)}, \quad 1 \le i, j \le n-1. \tag{24}$$

In the present context, the $\alpha$'s are integers (not necessarily positive) characterizing the irrep $V_\alpha^{(n)}$ of $\mathrm{U}(n)$. (For polynomial representations, they are non-negative and denote the lengths of the rows of the Young diagram associated with the irrep $V_\alpha^{(n)}$.)

The number of integer solutions of the inequalities (24) gives the dimension of the irrep

$$\dim(V_\alpha^{(n)}) = \#\{x_i^{(j)}\colon \text{solutions of (24)}\}.$$

The values $\beta_i = x_i^{(n-1)}$, $1 \le i \le n-1$, appearing in the second row of the triangle give the lengths of rows of the Young diagrams of the possible representations $V_\beta^{(n-1)}$ of $\mathrm{U}(n-1)$. Given those numbers, the number of solutions $x_i^{(j)}$, $1 \le i, j \le n-2$ satisfying (24) is the dimension of the representation of $\mathrm{U}(n-1)$. Thus we have the sum rule

$$\begin{aligned}\dim(V_\alpha^{(n)}) &= \#\{x_i^{(j)}\colon \text{solutions of (24)}, x^{(n)} = \alpha\}\\ &= \sum_\beta \#\{x_i^{(j)}\colon \text{solutions of (24)}, x^{(n)} = \alpha, x^{(n-1)} = \beta\}\\ &= \sum_\beta \dim(V_\beta^{(n-1)}),\end{aligned} \tag{25}$$

which is consistent with the multiplicity 1 of each $V_\beta^{(n-1)}$ appearing in the decomposition, a classical result in representation theory [20, 23, 25], see also [13, Chapter 8]. Thus one sees that the $\beta$'s satisfy the inequalities (1) and one may say that the branching coefficient, equal to 0 or 1, is given

$$\mathrm{br}_\alpha(\beta) = \mathcal{K}(\alpha; \beta) \tag{26}$$

with the convention that the discontinuous function $\mathcal{K}$ is assigned the value 1 throughout its support, including its boundaries.

The description of irreps of $\mathrm{SU}(n)$ is also well known: we have to restrict to Young diagrams with less than $n$ rows, or equivalently, to reduce Young diagrams with $n$ rows by deleting all columns of height $n$. Starting from an irrep of $\mathrm{SU}(n)$, to obtain the branching coefficients into irreps of $\mathrm{SU}(n-1)$, we apply to it the procedure above for $\mathrm{U}(n-1) \subset \mathrm{U}(n)$, and tally all irreps whose Young diagrams differ by columns of height $n-1$.

Example in SU(3). Take for $\alpha$ the adjoint representation, i.e., $\alpha = \{2, 1, 0\}$. The possible $\beta$ satisfying (1) are written in red in what follows:

$$2 \ge 2 \ge 1 \ge 1 \ge 0, \qquad \text{i.e., } \beta = \{2, 1\} \equiv \{1, 0\}, \tag{27}$$

$$2 \ge 1 \ge 1 \ge 1 \ge 0, \qquad \text{i.e., } \beta = \{1, 1\} \equiv \{0, 0\}, \tag{28}$$

$$2 \ge 2 \ge 1 \ge 0 \ge 0, \qquad \text{i.e., } \beta = \{2, 0\}, \tag{29}$$

$$2 \ge 1 \ge 1 \ge 0 \ge 0, \qquad \text{i.e., } \beta = \{1, 0\}, \tag{30}$$

where two $\beta$ are regarded as equivalent if their Young diagrams differ by a number of columns of height $n-1=2$. Hence, in $\mathrm{SU}(2) \subset \mathrm{SU}(3)$, we write

$$\overline{\mathrm{br}}_\alpha(\beta) = 1, 2, 1 \quad \text{for } \beta = \{0,0\},\ \{1,0\},\ \{2,0\} \tag{31}$$

and we check the sum rule on dimensions: $8 = 1 + 2 \times 2 + 3$. Note that removing columns of height $n-1$ in the Young diagram associated with $\beta$ amounts to changing $\beta_i$ into $\gamma_i = \beta_i - \beta_{n-1}$, i.e., focusing on spacings between the $\beta$'s, as was done in Section 2.2. More precisely, the multiplicity of weight $\gamma$ of $\mathrm{SU}(n-1)$ in the irrep of h.w. $\alpha$ of $\mathrm{SU}(n)$ equals the number of *integer* $\beta_{n-1}$ satisfying the $n-1$ inequalities

$$\alpha_{i+1} \le \gamma_i + \beta_{n-1} \le \alpha_i, \quad i = 1, \ldots, n-1.$$

We claim that number is given by the value of $\overline{\mathcal{K}}$ at shifted arguments, see below (33). It may be that a simple combinatorial argument leads directly to that result. In Section 3.2, we present a proof based on the integral representations of $\overline{\mathrm{br}}$ and $\overline{\mathcal{K}}$.

### 3.2. A $\overline{\mathcal{K}} - \overline{\mathrm{br}}$ relation

As is well known, for two compact Lie groups $H \subset G$, the branching coefficient of an irrep of $G$ into an irrep of $H$ may be expressed in terms of characters, integrated over a Cartan subgroup of $H$. In the case at hand of $\mathrm{SU}(n-1) \subset \mathrm{SU}(n)$, we denote $\overline{\mathrm{br}}$ the branching coefficient to distinguish it from the $U$-case and write

$$\overline{\mathrm{br}}_\alpha(\gamma) = \int Dt\, \chi_\alpha^{(n)}(e^{\mathrm{i}(t,0)}) (\chi_\gamma^{(n-1)}(e^{\mathrm{i}t}))^* \tag{32}$$

which computes the projection of the $\mathrm{SU}(n)$ character $\chi_\alpha^{(n)}$ restricted to the Cartan torus of $\mathrm{SU}(n-1)$ onto the $\mathrm{SU}(n-1)$ character $\chi_\gamma^{(n-1)}$. There, $Dt$ stands for the Haar measure on the Cartan torus $\mathbb{T}_{n-1}$ of $\mathrm{SU}(n-1)$

$$Dt = \frac{|\widehat{\Delta}_{n-1}(e^{\mathrm{i}t})|^2}{(2\pi)^{n-2}(n-1)!} dt$$

where we use the notations

$$\widehat{\Delta}_{n-1}(e^{\mathrm{i}t}) := \prod_{\boldsymbol{\alpha}>0} (e^{\mathrm{i}\langle \boldsymbol{\alpha}, t\rangle/2} - e^{-\mathrm{i}\langle \boldsymbol{\alpha}, t\rangle/2}) \quad \text{and} \quad \Delta_{n-1}(t) := \prod_{\boldsymbol{\alpha}>0} \langle \boldsymbol{\alpha}, t\rangle,$$

$\boldsymbol{\alpha}$ the positive roots of $\mathfrak{su}(n-1)$, and $dt$ is the Lebesgue measure on $\mathbb{T}_{n-1}$.

**Theorem 2.** *The branching coefficient, that gives the multiplicity of the irrep of* $\mathrm{SU}(n-1)$ *of h.w.* $\gamma$ *in the decomposition of the irrep of* $\mathrm{SU}(n)$ *of h.w.* $\alpha$*, is*

$$\overline{\mathrm{br}}_\alpha(\gamma) = \overline{\mathcal{K}}(\alpha + \rho_n; \gamma + \rho_{n-1}) \tag{33}$$

*with* $\rho_n$ *the Weyl vector of the algebra* $\mathfrak{su}(n)$*, and* $\rho_{n-1}$ *that of* $\mathfrak{su}(n-1)$*.*

*Proof.* We recall Kirillov's relation between a SU$(n)$ character and the orbital integral [18]:

$$\chi_\alpha^{(n)}(e^{\mathrm{i}t}) = \dim V_\alpha \frac{\Delta_n(\mathrm{i}t)}{\widehat{\Delta}_n(e^{\mathrm{i}t})} \mathcal{H}^{(n)}(\alpha+\rho_n;\mathrm{i}t) \tag{34}$$

with $\dim V_\alpha = \frac{\Delta_n(\alpha+\rho_n)}{\Delta_n(\rho)}$. Plugging in (14) the expression (34) and the analogous one for $\mathfrak{su}(n-1)$ leads to

$$\begin{aligned}
&\overline{\mathcal{K}}(\alpha+\rho_n;\gamma+\rho_{n-1})\\
&\quad= \frac{\Delta_n(\alpha+\rho_n)\Delta_{n-1}(\gamma+\rho_{n-1})}{(2\pi)^{n-2}(\prod_1^{n-1} p!)^2} \int_{\mathbb{R}^{n-2}} dt\, \Delta_{n-1}^2(t) \mathcal{H}^{(n)}(\alpha+\rho_n;\mathrm{i}(t,0))\\
&\qquad\qquad \times \mathcal{H}^{(n-1)}(\gamma+\rho_{n-1};\mathrm{i}t)^*\\
&\quad= \frac{\prod_{p=1}^{n-1} p! \prod_{p=1}^{n-2} p!}{(2\pi)^{n-2}(\prod_1^{n-1} p!)^2} \int_{\mathbb{R}^{n-2}} dt\, \Delta_{n-1}^2(t) \frac{\widehat{\Delta}_n(e^{\mathrm{i}(t,0)})\widehat{\Delta}_{n-1}(e^{\mathrm{i}t})^*}{\Delta_n(\mathrm{i}(t,0))\Delta_{n-1}(\mathrm{i}t)^*} \chi_\alpha^{(n)}(e^{\mathrm{i}(t,0)})\\
&\qquad\qquad \times \chi_\gamma^{(n-1)}(e^{\mathrm{i}t})^*\\
&\quad= \frac{\mathrm{i}^{-(n-1)}}{(2\pi)^{n-2}(n-1)!} \int_{\mathbb{R}^{n-2}} dt\, |\widehat{\Delta}_{n-1}(e^{\mathrm{i}t})|^2 \chi_\alpha^{(n)}(e^{\mathrm{i}(t,0)}) \chi_\gamma^{(n-1)}(e^{\mathrm{i}t})^* \frac{\Delta_{n-1}(t)}{\Delta_n((t,0))}\\
&\qquad\qquad \times \frac{\widehat{\Delta}_n(e^{\mathrm{i}(t,0)})}{\widehat{\Delta}_{n-1}(e^{\mathrm{i}t})}\\
&\quad= \mathrm{i}^{-(n-1)} \int_{\mathbb{T}_{n-1}} Dt \sum_{\delta\in 2\pi Q^\vee} \chi_\alpha^{(n)}(e^{\mathrm{i}(t+\delta,0)}) \chi_\gamma^{(n-1)}(e^{\mathrm{i}(t+\delta)})^* \frac{\Delta_{n-1}(t+\delta)}{\Delta_n((t+\delta,0))}\\
&\qquad\qquad \times \frac{\widehat{\Delta}_n(e^{\mathrm{i}(t+\delta,0)})}{\widehat{\Delta}_{n-1}(e^{\mathrm{i}(t+\delta)})}
\end{aligned} \tag{35}$$

where the integration is now carried out on the Cartan torus of SU$(n-1)$, $\mathbb{T}_{n-1} = \mathbb{R}^{n-2}/(2\pi Q^\vee)$, $Q^\vee$ is the $(n-2)$-dimensional coroot lattice of SU$(n-1)$, which is, in the present simply laced case, isomorphic to the root lattice. Only the ratio $\frac{\Delta_{n-1}(t+\delta)}{\Delta_n((t+\delta,0))}$ depends on $\delta$ and the summation can be carried out with the result that

$$\mathrm{i}^{-(n-1)} \frac{\widehat{\Delta}_n(e^{\mathrm{i}(t,0)})}{\widehat{\Delta}_{n-1}(e^{\mathrm{i}t})} \sum_{\delta\in 2\pi Q^\vee} \frac{\Delta_{n-1}(t+\delta)}{\Delta_n((t+\delta,0))} = 1. \tag{36}$$

Indeed, if we write $(t,0)$ in the $\mathfrak{su}(n)$ root basis: $(t,0) = \sum_{j=1}^{n-2} a_j \boldsymbol{\alpha}_j$ (with no component on $\boldsymbol{\alpha}_{n-1}$),

$$\frac{\widehat{\Delta}_n(e^{\mathrm{i}(t,0)})}{\widehat{\Delta}_{n-1}(e^{\mathrm{i}t})} = -(2\mathrm{i})^{n-1} \sin\frac{a_1}{2} \Big(\prod_{i=1}^{n-3} \sin\frac{a_{i+1}-a_i}{2}\Big) \sin\frac{a_{n-2}}{2}, \tag{37}$$

(on which it is clear that it is invariant under $a_i \mapsto a_i + p_i(2\pi)$, $p_i \in \mathbb{Z}$), while

$$\frac{\Delta_n((t,0))}{\Delta_{n-1}(t)} = -a_1 \Big(\prod_{i=1}^{n-3}(a_{i+1}-a_i)\Big) a_{n-2} \tag{38}$$

and the identity (36) follows from a repeated use of

$$\sum_{p=-\infty}^{\infty} \frac{1}{(a+2\pi p)(b-a-2\pi p)} = \frac{1}{2b} \frac{\sin\frac{b}{2}}{\sin\frac{a}{2}\sin\frac{b-a}{2}}$$

in the telescopic product (38). The right-hand side of (35) thus reduces to

$$\int_{\mathbb{T}_{n-1}} Dt\, \chi_\alpha^{(n)}(e^{\mathrm{i}(t,0)})\chi_\gamma^{(n-1)}(e^{\mathrm{i}(t)})^* = \overline{\mathrm{br}}_\alpha(\gamma)$$

by (32). ■

**Remark.** The proof above follows closely similar proofs in [4,7] that relate the classical Horn or Horn–Schur problems to the computation of Littlewood–Richardson or Kostka coefficients. However, in contrast with those cases, here the right-hand side is a single term, rather than a linear combination involving a convolution. And the equality of the multiplicity with the volume function holds true, and not only asymptotically in a semi-classical limit.[4]

Together with the results of the end of Section 2.2, Theorem 2 has immediate consequences:

**Corollary 1.** *The number of irreps of* SU$(n-1)$ *appearing in the decomposition of the irrep of* SU$(n)$ *of h.w.* $\alpha$ *(with* $\alpha_n = 0$*) is equal to the number of integer points in the polytope defined by* (16)*, where* $\alpha$ *is changed into* $\alpha + \rho_n$*, namely* $\alpha_i \to \alpha_i + n - i$*,*

$$\begin{aligned}\#\{\gamma \in \mathbb{Z}_+^{n-1} \mid \gamma_1 \geq \cdots \geq \gamma_{n-1} \geq 0,\\ \alpha_{i+1} - \alpha_j - 1 \leq \gamma_i - \gamma_j \leq \alpha_i - \alpha_{j+1} + 1,\\ 1 \leq i < j \leq n-1\}.\end{aligned}$$

Here as before, the $\alpha_i$ are the Young coordinates of the h.w. $\alpha$, i.e., the lengths of the rows of its Young diagram.

On the other hand, eq. (33) together with (17) gives the maximal value of a branching coefficient of a given $\alpha$

$$\max_\gamma \overline{\mathrm{br}}_\alpha(\gamma) = \min_i\big((\alpha+\rho_n)_i - (\alpha+\rho_n)_{i+1}\big) = \min_i(\alpha_i - \alpha_{i+1}) + 1 \qquad (39)$$

and note that $\alpha_i - \alpha_{i+1}$ is just the $i$-th Dynkin component[5] of the weight $\alpha$.

[4]See [16, Lemma 3.8] for a sufficient condition for this to happen. I am grateful to C. McSwiggen for pointing it out.

[5]Recall that the Dynkin components of a weight are its components in the fundamental weight basis. Hereafter, they are denoted by round brackets.

**Corollary 2.** *The largest multiplicity (i.e., the branching coefficient) that occurs in the branching of an irrep of* SU($n$) *of h.w.* $\alpha$ *into irreps of* SU($n-1$) *is* 1 *plus the smallest Dynkin component of* $\alpha$.

**Examples.** Take $n = 3$ and the example considered in Section 3.1. $\alpha = \{2, 1, 0\}$, i.e., $(1, 1)$ in Dynkin components,

$$\alpha + \rho_3 = \{4, 2, 0\}, \quad \gamma \in \{\{0, 0\}, \{1, 0\}, \{2, 0\}\}, \quad \gamma + \rho_2 \in \{\{1, 0\}, \{2, 0\}, \{3, 0\}\},$$

one finds with the formula (20): $\overline{\mathcal{K}}(\alpha + \rho_3; \gamma + \rho_2) = 1, 2, 1$ in agreement with (31).

For $n = 4$, take

$$\alpha = \{6, 4, 3, 0\},$$

i.e., $(2, 1, 3)$ in Dynkin components, one finds the following decomposition into 18 SU(3) weights

$$\begin{aligned}\{6, 4, 3, 0\} = {} & (1, 3)_2 \oplus (1, 2)_2 \oplus (1, 1)_2 \oplus (1, 0)_1 \oplus (0, 4)_1 \oplus (0, 3)_1 \\ & \oplus (0, 2)_1 \oplus (0, 1)_1 \oplus (2, 3)_2 \oplus (2, 2)_2 \oplus (2, 1)_2 \oplus (2, 0)_1 \\ & \oplus (1, 4)_1 \oplus (3, 3)_1 \oplus (3, 2)_1 \oplus (3, 1)_1 \oplus (3, 0)_1 \oplus (2, 4)_1 \end{aligned} \tag{40}$$

in terms of Dynkin components, and with the multiplicity appended as a subscript.

### 3.3. Stretching

The relation (33) is also well suited for the study of the behaviour of branching coefficients under “stretching.” From (39) we learn that the growth is at most linear

$$\overline{\mathrm{br}}_{s\alpha}(s\gamma) \le s \min(\alpha_i - \alpha_{i+1}) + 1.$$

For example, for $n = 3$, with Dynkin components, $\overline{\mathrm{br}}_{(s,s)}(s) = s + 1$ since

$$\begin{aligned}\overline{\mathrm{br}}_{(s,s)}(s) &= \overline{\mathcal{K}}(\{2s + 2, s + 1, 0\}; \{s + 1, 0\}) \\ &= (s + 1)\overline{\mathcal{K}}(\{2, 1, 0\}; \{1, 0\}) = s + 1,\end{aligned}$$

while for $\overline{\mathrm{br}}_{(s,s)}(s - 1)$ or $\overline{\mathrm{br}}_{(s,s)}(2s)$, we are not probing the function on its plateau and its behaviour is not always linear in $s$:

$$\overline{\mathcal{K}}(\{2s + 2, s + 1, 0\}; \{\gamma_1, 0\}) = \begin{cases} \gamma_1 & \text{if } 0 \le \gamma_1 \le s + 1, \\ 2(s + 1) - \gamma_1 & \text{if } s + 1 \le \gamma_1 \le 2(s + 1), \end{cases}$$

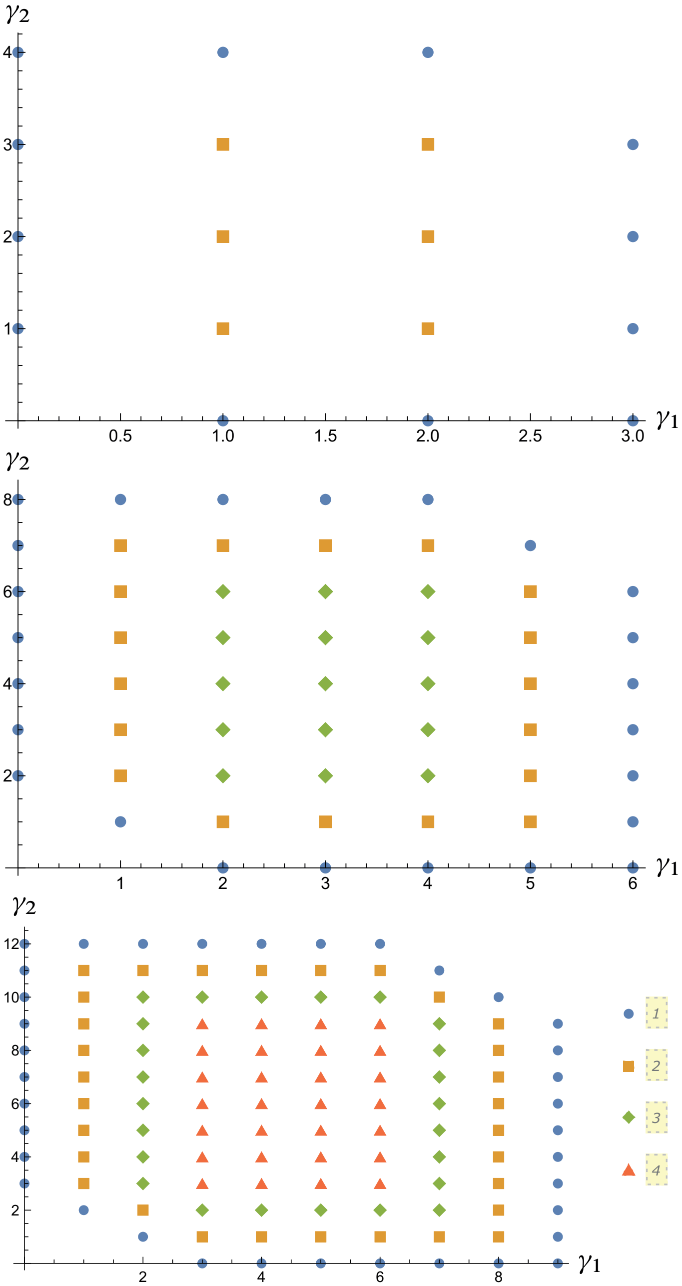


**Figure 3.** Weights in the $\gamma$-plane (in Dynkin components) appearing in the decomposition of the weight $\alpha = s\{6, 4, 3, 0\} \equiv s(2, 1, 3)$ of SU(4), for $s = 1, 2, 3$. Markers of different colours code for multiplicities from 1 to 4.

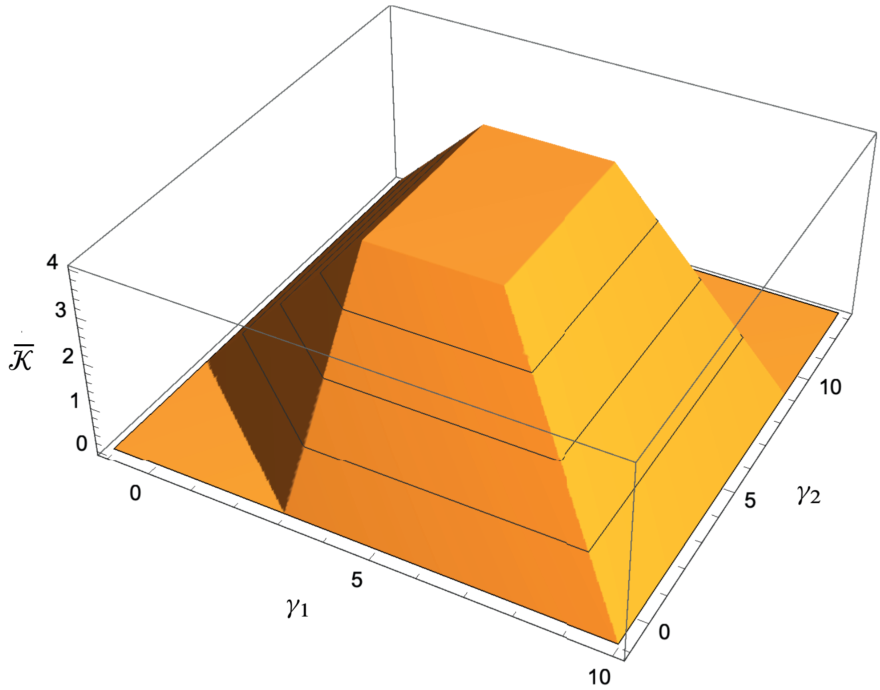


**Figure 4.** The $\overline{\mathcal{K}}$ function for $n = 4$, $\alpha + \rho_4 = \{21, 14, 10, 0\} = 3\{6, 4, 3, 0\} + \{3, 2, 1, 0\}$ and $\gamma_1, \gamma_2$ are Dynkin components. The cross-sections at altitude 1, 2, 3, 4 match the successive layers of multiplicities in Figure 3, bottom.

whence

$$\overline{\mathrm{br}}_{(s,s)}(s-1) = \overline{\mathcal{K}}(\{2s+2, s+1, 0\}; \{s, 0\}) = s$$

and

$$\overline{\mathrm{br}}_{(s,s)}(2s) = \overline{\mathcal{K}}(\{2s+2, s+1, 0\}; \{2s+1, 0\}) = 1.$$

Similar behaviours occur for branching coefficients in higher rank cases, due to the linear growth of the maximal value (17). For SU(3) $\subset$ SU(4), the points of increasing multiplicity form a matriochka pattern, see Figure 3, in a way already encountered in the Littlewood–Richardson coefficients of SU(3), see [5]. This pattern just reproduces the cross-sections of increasing altitude of the Aztec pyramid of Figure 4.

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**Note added in proof.** As already mentioned, there are many similarities – and possibly some redundancy – between the approach to the multiplicity problem developed in this paper and some previous work in the same vein, dealing with Littlewood–Richardson coefficients and Kostka numbers [3, 4, 6, 7, 19, 26]. In a recent paper [2], all these multiplicity problems were recast into a unique formalism, and general properties of the associated volume function were proved in a unified way. The present paper, on the other hand, presents explicit results and formulae for the branching coefficients.

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