

# Classifying compact 4-manifolds via generalized regular genus and G-degree

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**Abstract.**  $(d + 1)$ -colored graphs, i.e., edge-colored graphs that are  $(d + 1)$ -regular, have already been proved to be a useful representation tool for compact PL  $d$ -manifolds, thus extending the theory (known as *crystallization theory*) originally developed for the closed case. In this context, combinatorially defined PL invariants play a relevant role. The present paper focuses in particular on *generalized regular genus* and *G-degree*: the first one extending to higher dimension the classical notion of Heegaard genus for 3-manifolds, the second one arising, within theoretical physics, from the theory of random tensors as an approach to quantum gravity in dimension greater than two. We establish several general results concerning the two invariants, in relation with invariants of the boundary and with the rank of the fundamental group, as well as their behaviour with respect to connected sums. We also compute both generalized regular genus and G-degree for interesting classes of compact  $d$ -manifolds, such as handlebodies, products of closed manifolds by the interval and  $\mathbb{D}^2$ -bundles over  $\mathbb{S}^2$ . The main results of the paper concern dimension 4, where we obtain the classification of all compact PL manifolds with generalized regular genus at most one, and of all compact PL manifolds with G-degree at most 18; moreover, in case of empty or connected boundary, the classifications are extended to generalized regular genus two and to G-degree 24.

## 1. Introduction

In the PL  $d$ -dimensional setting ( $d \geq 3$ ), both the invariants *generalized regular genus* and *G-degree* have been recently introduced, making use of the possibility of representing all compact PL  $d$ -manifolds by means of regular  $(d + 1)$ -colored graphs (i.e., graphs whose vertices have degree  $d + 1$ , and so that the  $d + 1$  edges adjacent to each vertex are injectively colored by the colors  $\{0, 1, \dots, d\}$ ): see [14, 16] respectively, or the following Section 2.

This kind of representation for compact PL manifolds and the study of the above invariants has been deeply motivated by the strong connections – accurately described

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in [14] – between random tensor models and the so-called *crystallization theory*, which is a useful combinatorial tool for the topological and geometrical study of PL manifolds of arbitrary dimension (assumed to be closed, in the “classical” version of the theory) via edge-colored graphs.

Without going into details, we only recall that colored tensor models were introduced within theoretical physics as a random geometry approach to quantum gravity, following the successes of random matrix models ([25]) and, in this context, the coefficients of the  $1/N$  expansion of the correlation functions in dimension  $d$  are generating functions of regular bipartite  $(d + 1)$ -colored graphs; moreover, the quantity driving the  $1/N$  expansion is the *Gurau degree*, whose definition involves the genera of the surfaces where the considered graphs regularly embed (see Definition 2.5 for details), exactly as the strictly related notions of generalized regular genus and G-degree ([3, 4, 39]). Hence, any result obtained about generalized regular genus and/or G-degree is not only an achievement in the comprehension and possible classification of manifolds in the PL category, but may also bring insights in the interactions between geometry and physics.<sup>1</sup>

Other types of tensor models whose Feynman graphs include or are a particular kind of colored graphs have also been introduced (for instance the  $O(N)^3$ -invariant, the multi-orientable or the colored SYK models; see [5, 6, 40, 51] and references therein) and involve the definition of suitable “degrees” on colored graphs whose topological significance with respect to the dual colored triangulations can be investigated, as done for example in [29].

As far as the  $d$ -dimensional setting is concerned, the present paper proves in Section 4 several general properties of generalized regular genus and G-degree, relating them to the analogue invariants for the boundary manifold (Proposition 4.6), or for the summands of a connected sum decomposition (Proposition 4.7), and establishing an inequality between generalized regular genus and the rank of the fundamental group, in case of manifolds with empty or connected boundary (Proposition 4.2). Moreover, in Section 5, standard graphs representing some interesting classes of PL  $d$ -manifolds are obtained, yielding the computation of the associated invariants: see Proposition 5.1 concerning handlebodies, and Proposition 5.3 concerning the product between a closed  $d$ -manifold and the interval. A similar approach is then performed in Section 6, in the 4-dimensional setting, as regards the  $\mathbb{D}^2$ -bundles over  $\mathbb{S}^2$ .

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<sup>1</sup>Note that, while the so-called *colored tensor model* is usually formulated in terms of complex tensors, a version involving real tensor fields, known as the Gurau–Witten model, has gained considerable interest in the study of quantum mechanical models with random interactions and their holographic properties ([36, 52]). Its  $1/N$  expansion involves both bipartite and non-bipartite graphs, and hence the investigation concerns both orientable and non-orientable manifolds.

The focus of the paper is, indeed, on dimension  $d = 4$ , where crystallization theory already yielded classification results in the closed case both with respect to regular genus and with respect to gem-complexity (see [12, 15] and their references).

On the whole, experimental approaches to PL classification of 4-manifolds via combinatorial descriptions and associated PL-invariants are recent and still developing: as interesting examples we recall the use of Turaev’s shadows and shadow complexity (see for instance [41, 44] for the closed case, and [48] for the case of compact acyclic 4-manifolds) and that of trisections and trisection genus (see [45, 46, 50]).<sup>2</sup>

In the 4-dimensional setting, the present paper applies the general combinatorial properties of graphs representing compact  $d$ -manifold (obtained in Section 3), together with classical methods of crystallization theory and recent achievements about Dehn surgery, in order to yield classifying results for compact PL 4-manifolds  $M^4$  with respect to both their generalized regular genus  $\bar{\mathcal{G}}(M^4)$  and their G-degree  $\mathcal{D}_G(M^4)$ .

In particular, we prove in Section 7 the following statements (where  $\mathbb{S}^1 \times \mathbb{S}^3$  and  $\mathbb{S}^1 \tilde{\times} \mathbb{S}^3$  denote the orientable and non-orientable sphere bundle over  $\mathbb{S}^1$ ,  $\mathbb{Y}_m^4$  and  $\tilde{\mathbb{Y}}_m^4$  denote the orientable and non-orientable 4-handlebody of genus  $m$ ,  $\xi_c$  denotes the  $\mathbb{D}^2$ -bundle over  $\mathbb{S}^2$  with Euler class  $c$ , while  $M^4(K, d)$  denotes the compact PL 4-manifold obtained from the 4-disk by adding a 2-handle according to the framed knot  $(K, d)$ , whose boundary is the 3-manifold  $M^3(K, d)$  obtained from the 3-sphere by Dehn surgery on the same framed knot):

**Theorem 1.1.** *Let  $M^4$  be a compact PL 4-manifold with no spherical boundary components. Then*

- a.  $\bar{\mathcal{G}}(M^4) = 0$  if and only if  $M^4 \cong \mathbb{S}^4$ ;
- b.  $\bar{\mathcal{G}}(M^4) = 1$  if and only if
  - either  $M^4 \in \{\mathbb{S}^1 \times \mathbb{S}^3, \mathbb{S}^1 \tilde{\times} \mathbb{S}^3\}$ ,
  - or  $M^4 \in \{\mathbb{Y}_1^4, \tilde{\mathbb{Y}}_1^4\}$ ,
  - or  $M^4 \cong \bar{M} \times I$ , where  $\bar{M}$  is a genus one closed 3-manifold;
- c. if  $M^4$  has empty or connected boundary and  $\bar{\mathcal{G}}(M^4) = 2$ , then
  - either  $M^4 \in \{\#_2(\mathbb{S}^1 \times \mathbb{S}^3), \#_2(\mathbb{S}^1 \tilde{\times} \mathbb{S}^3), \mathbb{C}\mathbb{P}^2\}$ ,
  - or  $M^4 \in \{\mathbb{Y}_2^4, \tilde{\mathbb{Y}}_2^4, (\mathbb{S}^1 \times \mathbb{S}^3)\#\mathbb{Y}_1^4, (\mathbb{S}^1 \tilde{\times} \mathbb{S}^3)\#\mathbb{Y}_1^4, (\mathbb{S}^1 \tilde{\times} \mathbb{S}^3)\#\tilde{\mathbb{Y}}_1^4, \mathbb{S}^2 \times \mathbb{D}^2, \xi_c\}$ ,
  - or  $M^4 \cong M^4(K, d)$ ,  $(K, d)$  being a non-trivial framed knot such that  $M^3(K, d) = L(\alpha, \beta)$ , with  $\alpha \geq 3$ .

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<sup>2</sup>Generalizing [50], a recent study also performs an approach to trisection genus via crystallizations, giving rise to the notion of *gem-induced trisection genus*, which applies also to manifolds with connected boundary: see [13].

**Theorem 1.2.** *Let  $M^4$  be a compact PL 4-manifold with no spherical boundary components. Then*

- a.  $\mathcal{D}_G(M^4) = 0$  if and only if  $M^4 \cong \mathbb{S}^4$ ;
- b.  $\mathcal{D}_G(M^4) = 12$  if and only if
  - either  $M^4 \in \{\mathbb{S}^1 \times \mathbb{S}^3, \mathbb{S}^1 \tilde{\times} \mathbb{S}^3\}$ ,
  - or  $M^4 \in \{\mathbb{Y}_1^4, \tilde{\mathbb{Y}}_1^4\}$ ;
- c.  $\mathcal{D}_G(M^4) = 18$  if and only if  $M^4 \in \{L(2, 1) \times I, (\mathbb{S}^1 \times \mathbb{S}^2) \times I, (\mathbb{S}^1 \tilde{\times} \mathbb{S}^2) \times I\}$ .

*No other compact PL 4-manifold (with no spherical boundary components) exists with  $\mathcal{D}_G(M^4) \leq 23$ .*

*Moreover, if  $M^4$  has empty or connected boundary, then*

- d.  $\mathcal{D}_G(M^4) = 24$  if and only if
  - either  $M^4 \in \{\#_2(\mathbb{S}^1 \times \mathbb{S}^3), \#_2(\mathbb{S}^1 \tilde{\times} \mathbb{S}^3), \mathbb{C}\mathbb{P}^2\}$ ,
  - or  $M^4 \in \{\mathbb{Y}_2^4, \tilde{\mathbb{Y}}_2^4, (\mathbb{S}^1 \times \mathbb{S}^3)\#\mathbb{Y}_1^4, (\mathbb{S}^1 \tilde{\times} \mathbb{S}^3)\#\mathbb{Y}_1^4, (\mathbb{S}^1 \tilde{\times} \mathbb{S}^3)\#\tilde{\mathbb{Y}}_1^4, \mathbb{S}^2 \times \mathbb{D}^2, \xi_2\}$ .

A direct consequence of Theorem 1.2 is the identification of all compact orientable PL 4-manifolds (resp. compact orientable PL 4-manifolds with empty or connected boundary), represented by regular graphs involved in the first four (resp. five) most significant non-null terms of the  $1/N$  expansion of the correlation functions ([4, 18]); see Remark 7.16.

Further results of the present paper are also the characterization of 4-dimensional handlebodies as the only PL 4-manifolds with connected (non-empty) boundary whose generalized regular genus equals that of their boundary (see Theorem 7.10), while the equality between generalized regular genus and the rank of the fundamental group characterizes  $\#_\rho(\mathbb{S}^1 \times \mathbb{S}^3)$  and  $\#_\rho(\mathbb{S}^3 \tilde{\times} \mathbb{S}^1)$  in the closed case,  $\#_\alpha(\mathbb{S}^1 \times \mathbb{S}^3)\#\mathbb{Y}_\beta^4$ ,  $\#_\alpha(\mathbb{S}^1 \tilde{\times} \mathbb{S}^3)\#\mathbb{Y}_\beta^4$ , and  $\#_\alpha(\mathbb{S}^1 \tilde{\times} \mathbb{S}^3)\#\tilde{\mathbb{Y}}_\beta^4$  in the connected boundary case (see Theorem 7.11).

Note that, as a consequence of the above results, all  $\mathbb{D}^2$ -bundles over  $\mathbb{S}^2$  turn out to have generalized regular genus 2, thus proving that generalized regular genus is not finite-to-one in dimension four, and that, in the 4-dimensional case of non-empty boundary, the equality between generalized regular genus and the “classical” invariant *regular genus* does not hold, even if the boundary is assumed to be connected (Corollary 7.7).

## 2. Preliminaries

In the present section we will briefly review some basic notions of the so called *crystallization theory*, which is a representation tool for general piecewise linear (PL)

compact manifolds, without assumptions about dimension, connectedness, orientability or boundary properties (see the “classical” survey paper [28], or the more recent one [15], concerning the 4-dimensional case).

From now on, unless otherwise stated, all spaces and maps will be considered in the PL category, and all manifolds will be assumed to be compact and connected.

**Definition 2.1.** A  $(d + 1)$ -colored graph ( $d \geq 2$ ) is a pair  $(\Gamma, \gamma)$ , where  $\Gamma = (V(\Gamma), E(\Gamma))$  is a multigraph (i.e., multiple edges are allowed, but loops are forbidden) which is regular of degree  $d + 1$ , and  $\gamma$  is an *edge-coloration*, that is a map  $\gamma: E(\Gamma) \rightarrow \Delta_d = \{0, \dots, d\}$  which is injective on adjacent edges

In the following, for sake of concision, when the coloration is clearly understood, we will denote colored graphs simply by  $\Gamma$ .

For every  $\{c_1, \dots, c_h\} \subseteq \Delta_d$ , let  $\Gamma_{\{c_1, \dots, c_h\}}$  be the subgraph obtained from  $(\Gamma, \gamma)$  by deleting all the edges that are not colored by the elements of  $\{c_1, \dots, c_h\}$ . In this setting, the complementary set of  $\{c\}$  (resp.  $\{c_1, \dots, c_h\}$ ) in  $\Delta_d$  will be denoted by  $\hat{c}$  (resp.  $\hat{c}_1 \dots \hat{c}_h$ ). The connected components of  $\Gamma_{\{c_1, \dots, c_h\}}$  are called  $\{c_1, \dots, c_h\}$ -residues or  $h$ -residues of  $\Gamma$ ; their number is denoted by  $g_{\{c_1, \dots, c_h\}}$  (or, for short, by  $g_{c_1, c_2}$ ,  $g_{c_1, c_2, c_3}$  and  $g_{\hat{c}}$  if  $h = 2$ ,  $h = 3$  and  $h = d$  respectively).

A  $d$ -dimensional pseudocomplex  $K(\Gamma)$  can be associated to a  $(d + 1)$ -colored graph  $\Gamma$ :

- take a  $d$ -simplex for each vertex of  $\Gamma$  and label its vertices by the elements of  $\Delta_d$ ;
- if two vertices of  $\Gamma$  are  $c$ -adjacent ( $c \in \Delta_d$ ), glue the corresponding  $d$ -simplices along their  $(d - 1)$ -dimensional faces opposite to the  $c$ -labeled vertices, so that equally labeled vertices are identified.

In general,  $|K(\Gamma)|$  is a  $d$ -pseudomanifold and  $\Gamma$  is said to *represent* it.

Note that, by construction,  $K(\Gamma)$  is endowed with a vertex-labeling by  $\Delta_d$  that is injective on any simplex. Moreover,  $\Gamma$  turns out to be the 1-skeleton of the dual complex of  $K(\Gamma)$ . The duality establishes a bijection between the  $\{c_1, \dots, c_h\}$ -residues of  $\Gamma$  and the  $(d - h)$ -simplices of  $K(\Gamma)$  whose vertices are labeled by  $\Delta_d - \{c_1, \dots, c_h\}$ .

Given a pseudocomplex  $K$  and an  $h$ -simplex  $\sigma^h$  of  $K$ , the *disjoint star* of  $\sigma^h$  in  $K$  is the pseudocomplex obtained by taking all  $d$ -simplices of  $K$  having  $\sigma^h$  as a face and identifying only their faces that do not contain  $\sigma^h$ . The *disjoint link*,  $\text{lkd}(\sigma^h, K)$ , of  $\sigma^h$  in  $K$  is the subcomplex of the disjoint star formed by those simplices that do not intersect  $\sigma^h$ . In particular, given a  $(d + 1)$ -colored graph  $\Gamma$ , each connected component of  $\Gamma_{\hat{c}}$  ( $c \in \Delta_d$ ) is a  $d$ -colored graph representing the disjoint link of a  $c$ -labeled vertex of  $K(\Gamma)$ , that is also (PL) homeomorphic to the link of this vertex in the first barycentric subdivision of  $K(\Gamma)$ .

**Definition 2.2.** A *singular (PL)  $d$ -manifold* is a closed connected  $d$ -dimensional polyhedron admitting a simplicial triangulation where the links of vertices are closed

connected  $(d - 1)$ -manifolds, while the links of all  $h$ -simplices of the triangulation with  $h > 0$  are (PL)  $(d - h - 1)$ -spheres. Vertices whose links are not PL  $(d - 1)$ -spheres are called *singular*. Note that, in case of polyhedra arising from colored graphs, the condition about links of vertices obviously implies the one about links of  $h$ -simplices, with  $h > 0$ .

Therefore,

- $|K(\Gamma)|$  is a singular  $d$ -manifold if and only if, for each color  $c \in \Delta_d$ , all  $\hat{c}$ -residues of  $\Gamma$  represent closed connected  $(d - 1)$ -manifolds.

In particular,

- $|K(\Gamma)|$  is a closed  $d$ -manifold if and only if, for each color  $c \in \Delta_d$ , all  $\hat{c}$ -residues of  $\Gamma$  represent the  $(d - 1)$ -sphere.

**Remark 2.3.** If  $N$  is a singular  $d$ -manifold, then a compact  $d$ -manifold  $\tilde{N}$  is easily obtained by deleting small open neighborhoods of its singular vertices. Obviously,  $N = \tilde{N}$  if and only if  $N$  is a closed manifold; otherwise,  $\tilde{N}$  has non-empty boundary (without spherical components). Conversely, given a compact  $d$ -manifold  $M$ , a singular  $d$ -manifold  $\hat{M}$  can be constructed by capping off each component of  $\partial M$  by a cone over it.

Note that, by restricting ourselves to the class of compact  $d$ -manifolds with no spherical boundary components, the above correspondence is bijective and so singular  $d$ -manifolds and compact  $d$ -manifolds of this class can be associated to each other in a well-defined way.

For this reason, throughout the present work, we will restrict our attention to compact manifolds without spherical boundary components. Obviously, in this wider context, closed  $d$ -manifolds are characterized by  $M = \hat{M}$ .

In virtue of the bijection described in Remark 2.3, a  $(d + 1)$ -colored graph  $\Gamma$  is said to *represent* a compact  $d$ -manifold  $M$  with no spherical boundary components if and only if it represents the associated singular manifold  $\hat{M}$ .

The following theorem extends to the boundary case a well-known result – originally stated in [49] – founding the combinatorial representation theory for closed manifolds of arbitrary dimension via colored graphs.

**Proposition 2.4** ([16]). *Any compact orientable (resp. non-orientable)  $d$ -manifold with no spherical boundary components admits a bipartite (resp. non-bipartite)  $(d + 1)$ -colored graph representing it.*

If  $\Gamma$  represents a compact  $d$ -manifold, a  $d$ -residue of  $\Gamma$  will be called *ordinary* if it represents  $\mathbb{S}^{d-1}$ , *singular* otherwise. Similarly, a color  $c$  will be called *singular* if at least one of the  $\hat{c}$ -residues of  $\Gamma$  is singular.

The existence of a particular type of embedding of colored graphs into surfaces, is the key result in order to define the important notion of regular genus.

**Proposition 2.5** ([31]). *Let  $(\Gamma, \gamma)$  be a bipartite (resp. non-bipartite)  $(d + 1)$ -colored graph of order  $2p$ . Then for each cyclic permutation  $\varepsilon = (\varepsilon_0, \dots, \varepsilon_d)$  of  $\Delta_d$ , up to inverse, there exists a cellular embedding, called regular,<sup>3</sup> of  $(\Gamma, \gamma)$  into an orientable (resp. non-orientable) closed surface  $F_\varepsilon(\Gamma)$  whose regions are bounded by the images of the  $\{\varepsilon_j, \varepsilon_{j+1}\}$ -colored cycles, for each  $j \in \mathbb{Z}_{d+1}$ . Moreover, the genus (resp. half the genus)  $\rho_\varepsilon(\Gamma)$  of  $F_\varepsilon(\Gamma)$  satisfies*

$$2 - 2\rho_\varepsilon(\Gamma) = \sum_{j \in \mathbb{Z}_{d+1}} g_{\varepsilon_j \varepsilon_{j+1}} + (1 - d)p.$$

*No regular embeddings of  $(\Gamma, \gamma)$  exist into non-orientable (resp. orientable) surfaces.*

The *Gurau degree* (often called *degree* in the tensor models literature, see [37]) and the *regular genus* of a colored graph are defined in terms of the embeddings of Proposition 2.5.

**Definition 2.6.** Let  $(\Gamma, \gamma)$  be a  $(d + 1)$ -colored graph. If  $\{\varepsilon^{(1)}, \varepsilon^{(2)}, \dots, \varepsilon^{(\frac{d!}{2})}\}$  is the set of all cyclic permutations of  $\Delta_d$  (up to inverse),  $\rho_{\varepsilon^{(i)}}(\Gamma)$  ( $i = 1, \dots, \frac{d!}{2}$ ) is called the *regular genus of  $\Gamma$  with respect to the permutation  $\varepsilon^{(i)}$* . Then, the *Gurau degree* (or *G-degree* for short) of  $\Gamma$ , denoted by  $\omega_G(\Gamma)$ , is defined as

$$\omega_G(\Gamma) = \sum_{i=1}^{\frac{d!}{2}} \rho_{\varepsilon^{(i)}}(\Gamma),$$

and the *regular genus* of  $\Gamma$ , denoted by  $\rho(\Gamma)$ , is defined as

$$\rho(\Gamma) = \min\left\{\rho_{\varepsilon^{(i)}}(\Gamma) / i = 1, \dots, \frac{d!}{2}\right\}.$$

Note that, in dimension 2, any bipartite (resp. non-bipartite) 3-colored graph  $(\Gamma, \gamma)$  represents an orientable (resp. non-orientable) surface  $|K(\Gamma)|$  and  $\rho(\Gamma) = \omega_G(\Gamma)$  is exactly the genus (resp. half the genus) of  $|K(\Gamma)|$ . On the other hand, for  $d \geq 3$ , the G-degree of any  $(d + 1)$ -colored graph (resp. the regular genus of any  $(d + 1)$ -colored graph representing a closed PL  $d$ -manifold) is proved to be a non-negative integer, both in the bipartite and non-bipartite case: see [14, Proposition 7] (resp. [21, Proposition A]).

As a consequence of the definition of regular genus of a colored graph and of Proposition 2.4, two PL invariants for compact  $d$ -manifolds can be defined as follow

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<sup>3</sup>Regular embeddings are called *Jackets* in the tensor models context.

**Definition 2.7.** Let  $M$  be a compact (PL)  $d$ -manifold ( $d \geq 2$ ). The *generalized regular genus* of  $M$  is defined as

$$\bar{\mathcal{G}}(M) = \min\{\rho(\Gamma) \mid (\Gamma, \gamma) \text{ represents } M\},$$

and the *Gurau degree* (or *G-degree*) of  $M$  is defined as

$$\mathcal{D}_G(M) = \min\{\omega_G(\Gamma) \mid (\Gamma, \gamma) \text{ represents } M\}.$$

For any  $(d + 1)$ -colored graph  $\Gamma$ , the inequality  $\omega_G(\Gamma) \geq \frac{d!}{2} \cdot \rho(\Gamma)$  obviously holds. Hence, for any compact  $d$ -manifold  $M$ :

$$\mathcal{D}_G(M) \geq \frac{d!}{2} \cdot \bar{\mathcal{G}}(M).$$

**Remark 2.8.** Note that, in case  $M$  is a closed PL  $d$ -manifold, the generalized regular genus coincides by definition with the PL invariant *regular genus* (see Section 4), extending to higher dimension the Heegaard genus of a 3-manifold ([23]). Regular genus zero succeeds in characterizing spheres in arbitrary dimension,<sup>4</sup> and many classifying results via regular genus have been obtained, especially in dimension 4 and 5 (see [2, 10, 15, 17], and the references therein). Also G-degree zero characterizes spheres in arbitrary dimension, and some classifying results via this invariant have recently been obtained for compact 3-manifolds and for closed PL 4-manifolds: see [14, 16].

Finally, we recall that, within crystallization theory, a finite set of combinatorial moves have been defined, which translate the homeomorphism problem of the represented polyhedra.

**Definition 2.9.** An  $r$ -dipole ( $1 \leq r \leq d$ ) of colors  $c_1, \dots, c_r$  of a  $(d + 1)$ -colored graph  $(\Gamma, \gamma)$  is a subgraph of  $\Gamma$  consisting in two vertices joined by  $r$  edges, colored by  $c_1, \dots, c_r$ , such that its vertices belong to different connected components of  $\Gamma_{\hat{c}_1 \dots \hat{c}_r}$ .

The *elimination* of an  $r$ -dipole in  $\Gamma$  can be performed by deleting the subgraph and welding the remaining hanging edges according to their colors; in this way another  $(d + 1)$ -colored graph  $(\Gamma', \gamma')$  is obtained. The inverse operation is called the *addition* of the dipole to  $\Gamma'$ .

The dipole is called *proper* if  $|K(\Gamma)|$  and  $|K(\Gamma')|$  are PL homeomorphic. It is known that this happens when at least one of the two connected components of  $\Gamma_{\hat{c}_1 \dots \hat{c}_r}$  intersecting the dipole represents a  $(d - r)$ -sphere ([32, Proposition 5.3]).<sup>5</sup>

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<sup>4</sup>In Proposition 4.4 we will prove that the same characterization holds for generalized regular genus, too, within the wider class of compact  $d$ -manifolds.

<sup>5</sup>Note that, if  $|K(\Gamma)|$  is a singular  $d$ -manifold, all  $r$ -dipoles are proper, for  $1 < r \leq d$ .



**Remark 2.10.** Neither the G-degree nor the regular genus of a  $(d + 1)$ -colored graph are affected by elimination of 1-dipoles. Therefore, from any  $(d + 1)$ -colored graph  $\Gamma$  representing a compact PL  $d$ -manifold  $M$  with empty or connected boundary, by eliminating (proper) 1-dipoles, a  $(d + 1)$ -colored graph can be obtained, still representing  $M$ , with the same G-degree and regular genus as  $\Gamma$  and having only one  $\hat{i}$ -residue for each  $i \in \Delta_d$ . Such a  $(d + 1)$ -colored graph is said to be a *crystallization* of  $M$ .

### 3. Combinatorial properties of graphs representing singular $d$ -manifolds

In [7, 9, 17], interesting combinatorial formulae have been obtained, regarding both regular edge-colored graphs representing closed  $d$ -manifolds and edge-colored graphs with boundary (see [28], or the next Section 4) representing  $d$ -manifolds with non-empty boundary. Here, we will generalize them to regular edge-colored graphs representing (via singular  $d$ -manifolds) all compact (PL)  $d$ -manifolds.

In the following, let  $(\Gamma, \gamma)$  be a (possibly disconnected)  $(d + 1)$ -colored graph representing a (possibly disconnected) singular  $d$ -manifold  $N^d$ . If  $\Gamma$  (resp.  $\Gamma_{\mathcal{B}}$ , with  $\mathcal{B} \subset \Delta_d$ ) has  $g \geq 1$  (resp.  $g_{\mathcal{B}} \geq 1$ ) connected components  $\Gamma^1, \Gamma^2, \dots, \Gamma^g$  (resp.  $H^1, H^2, \dots, H^{g_{\mathcal{B}}}$ ), for each permutation  $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{d-1}, \varepsilon_d)$  of  $\Delta_d$  we define

$$\rho_{\varepsilon}(\Gamma) = \sum_{i=1}^g \rho_{\varepsilon}(\Gamma^i)$$

(resp.

$$\rho_{\varepsilon}(\Gamma_{\mathcal{B}}) = \sum_{i=1}^{g_{\mathcal{B}}} \rho_{\varepsilon}(H^i),$$

where by  $\rho_{\varepsilon}(H^i)$  we denote the regular genus of  $H^i$  with respect to the permutation induced by  $\varepsilon$  on the subset  $\mathcal{B}$  of  $\Delta_d$ ).

**Proposition 3.1.** *If  $(\Gamma, \gamma)$  is a  $(d + 1)$ -colored graph with  $g \geq 1$  connected components, representing a (possibly disconnected) singular  $d$ -manifold  $N^d$ , then*

- if  $\#\mathcal{B} = m$  and  $m \leq d - 1$ ,  $m$  odd,

$$2g_{\mathcal{B}} = (2 - m)p + \sum_{s=2}^{m-1} (-1)^s \sum_{i_1, i_2, \dots, i_s \in \mathcal{B}} g_{i_1, i_2, \dots, i_s}; \tag{1}$$

- if  $\#\mathcal{B} = m$  and  $m \leq d - 1$ ,  $m$  even,

$$0 = (2 - m)p + \sum_{s=2}^{m-1} (-1)^s \sum_{i_1, i_2, \dots, i_s \in \mathcal{B}} g_{i_1, i_2, \dots, i_s}; \quad (2)$$

- if  $\mathcal{B} = \Delta_d - \{i\}$ , with  $i$  non-singular color and  $d$  odd,

$$2g_i = (2 - d)p + \sum_{s=2}^{d-1} (-1)^s \sum_{i_1, i_2, \dots, i_s \in \Delta_d - \{i\}} g_{i_1, i_2, \dots, i_s}; \quad (3)$$

- if  $\mathcal{B} = \Delta_d - \{i\}$ , with  $i$  non-singular color and  $d$  even,

$$0 = (2 - d)p + \sum_{s=2}^{d-1} (-1)^s \sum_{i_1, i_2, \dots, i_s \in \Delta_d - \{i\}} g_{i_1, i_2, \dots, i_s}. \quad (4)$$

Moreover,

- for all  $i \in \Delta_d$ ,

$$g_{\varepsilon_{i-1}, \varepsilon_{i+1}} = g_{\varepsilon_{i-1}, \varepsilon_i, \varepsilon_{i+1}} + (\rho_\varepsilon(\Gamma) - \rho_\varepsilon(\Gamma_{\widehat{\varepsilon_i}})) - (g - g_{\widehat{\varepsilon_i}}); \quad (5)$$

- for all  $i, j$  non-consecutive in  $\Delta_d$ ,

$$g_{\widehat{\varepsilon_i}, \widehat{\varepsilon_j}} = (g_{\widehat{\varepsilon_i}} + g_{\widehat{\varepsilon_j}} - g) + \rho_\varepsilon(\Gamma) - \rho_\varepsilon(\Gamma_{\widehat{\varepsilon_i}}) - \rho_\varepsilon(\Gamma_{\widehat{\varepsilon_j}}) + \rho_\varepsilon(\Gamma_{\widehat{\varepsilon_j}, \widehat{\varepsilon_i}}); \quad (6)$$

- for all  $r \in \Delta_d - \{i - 1, i, i + 1\}$ ,

$$g_{\varepsilon_{i-1}, \varepsilon_{i+1}} = g_{\varepsilon_{i-1}, \varepsilon_i, \varepsilon_{i+1}} + g_{\varepsilon_{i-1}, \varepsilon_{i+1}, \varepsilon_r} + \rho_\varepsilon(\Gamma_{\varepsilon_{i-1}, \varepsilon_i, \varepsilon_{i+1}, \varepsilon_r}) - g_{\varepsilon_{i-1}, \varepsilon_i, \varepsilon_{i+1}, \varepsilon_r}. \quad (7)$$

*Proof.* By definition of generalized regular genus with respect to the permutation  $\varepsilon$ :

$$\sum_{j \in \mathbb{Z}_{d+1}} g_{\varepsilon_j, \varepsilon_{j+1}} + (1 - d)p = 2g - 2\rho_\varepsilon(\Gamma) \quad (8)$$

By applying the same relation to the (possibly disconnected) subgraph  $\Gamma_{\widehat{\varepsilon_i}}$  ( $i \in \Delta_d$ ), we have

$$\sum_{j \in \mathbb{Z}_{d+1} - \{i-1, i\}} g_{\varepsilon_j, \varepsilon_{j+1}} + g_{\varepsilon_{i-1}, \varepsilon_{i+1}} + (2 - d)p = 2g_{\widehat{\varepsilon_i}} - 2\rho_\varepsilon(\Gamma_{\widehat{\varepsilon_i}}) \quad (9)$$

In order to prove relations (1) and (2), recall that each connected component of  $\Gamma_{\mathcal{B}}$  represents the disjoint link of a  $(d - m)$ -simplex in the singular  $d$ -manifold  $|K(\Gamma)| = N^d$ , which – under the hypothesis  $m \leq d - 1$  – is homeomorphic to the  $(m - 1)$ -sphere. Hence, its Euler characteristic equals 2 if  $m$  is odd and 0 if  $m$  is even.

The quoted formulae simply perform the computation of the Euler characteristic from the combinatorial features of the representing graph.

As a particular case, when  $m = 3$  (with  $d \geq 4$ ), we obtain the following formula, which holds for any  $(d + 1)$ -colored graph  $(\Gamma, \gamma)$  representing a singular  $d$ -manifold, with  $d \geq 4$ :

$$2g_{r,s,t} = g_{r,s} + g_{s,t} + g_{r,t} - p. \quad (10)$$

Also relations (3) and (4) directly follow from the computation of the Euler characteristic via combinatorial elements of the representing graph: in fact, if  $\mathcal{B} = \Delta_d - \{i\}$  and all  $i$ -colored vertices of  $K(\Gamma)$  are not singular, each connected component of  $\Gamma_i$  represents the  $d$ -sphere.

The difference between relation (8) and relation (9), by making use of relation (10) applied to the subset  $\mathcal{B} = \{i - 1, i, i + 1\}$  of  $\Delta_n$ , yields (5). On the other hand, the difference between relation (5) and the same relation applied to the graph  $\Gamma_{\varepsilon_j}^{\wedge}$  (for  $j \notin \{i - 1, i, i + 1\}$ ) yields (6).

As a consequence, since  $g_{\varepsilon_i, \varepsilon_j}^{\wedge} \geq g_{\varepsilon_i}^{\wedge} + g_{\varepsilon_j}^{\wedge} - g$  trivially holds, for all  $i, j$  non-consecutive in  $\varepsilon$  we have

$$\rho_{\varepsilon}(\Gamma) - \rho_{\varepsilon}(\Gamma_{\varepsilon_i}^{\wedge}) - \rho_{\varepsilon}(\Gamma_{\varepsilon_j}^{\wedge}) + \rho_{\varepsilon}(\Gamma_{\varepsilon_j, \varepsilon_i}^{\wedge}) \geq 0. \quad (11)$$

Moreover, by applying formula (6) to the graph  $\Gamma_{\varepsilon_i}^{\wedge}$ , we obtain

$$\begin{aligned} g_{\varepsilon_i, \varepsilon_j, \varepsilon_k}^{\wedge} &= (g_{\varepsilon_i, \varepsilon_k}^{\wedge} + g_{\varepsilon_i, \varepsilon_j}^{\wedge} - g_{\varepsilon_i}^{\wedge}) + \rho_{\varepsilon}(\Gamma_{\varepsilon_i}^{\wedge}) - \rho_{\varepsilon}(\Gamma_{\varepsilon_i, \varepsilon_k}^{\wedge}) \\ &\quad - \rho_{\varepsilon}(\Gamma_{\varepsilon_i, \varepsilon_j}^{\wedge}) + \rho_{\varepsilon}(\Gamma_{\varepsilon_i, \varepsilon_j, \varepsilon_k}^{\wedge}) \end{aligned} \quad (12)$$

for all  $j, k$  non-consecutive in  $(\varepsilon_0, \dots, \varepsilon_{i-1}, \varepsilon_{i+1}, \dots, \varepsilon_d)$ .

Now, if  $j_1, j_2, \dots, j_{d-3} \notin \{i - 1, i, i + 1\}$ , the difference between relation (5) and the same relation applied to the graph  $\Gamma_{\varepsilon_{j_1}, \varepsilon_{j_2}, \dots, \varepsilon_{j_{d-3}}}^{\wedge}$  yields

$$\begin{aligned} g_{\varepsilon_i, \varepsilon_{j_1}, \varepsilon_{j_2}, \dots, \varepsilon_{j_{d-3}}}^{\wedge} &= (g_{\varepsilon_i}^{\wedge} + g_{\varepsilon_{j_1}, \varepsilon_{j_2}, \dots, \varepsilon_{j_{d-3}}}^{\wedge} - g) \\ &\quad + [\rho_{\varepsilon}(\Gamma) - \rho_{\varepsilon}(\Gamma_{\varepsilon_i}^{\wedge}) - (\rho_{\varepsilon}(\Gamma_{\varepsilon_{j_1}, \varepsilon_{j_2}, \dots, \varepsilon_{j_{d-3}}}^{\wedge}) \\ &\quad - \rho_{\varepsilon}(\Gamma_{\varepsilon_i, \varepsilon_{j_1}, \varepsilon_{j_2}, \dots, \varepsilon_{j_{d-3}}}^{\wedge}))]. \end{aligned} \quad (13)$$

Note that  $\{\widehat{\varepsilon_i}, \widehat{\varepsilon_{j_1}}, \widehat{\varepsilon_{j_2}}, \dots, \widehat{\varepsilon_{j_{d-3}}}\} = \{\varepsilon_{i-1}, \varepsilon_{i+1}, \varepsilon_r\}$  with  $r \in \Delta_d - \{i - 1, i, i + 1\}$ . Hence, the previous relation may be written as

$$\begin{aligned} g_{\varepsilon_{i-1}, \varepsilon_{i+1}, \varepsilon_r} &= (g_{\varepsilon_i}^{\wedge} + g_{\varepsilon_{i-1}, \varepsilon_{i+1}, \varepsilon_r} - g) + \rho_{\varepsilon}(\Gamma) - \rho_{\varepsilon}(\Gamma_{\varepsilon_c}^{\wedge}) \\ &\quad - \rho_{\varepsilon}(\Gamma_{\varepsilon_{c-1}, \varepsilon_c, \varepsilon_{c+1}, \varepsilon_r}) + \rho_{\varepsilon}(\Gamma_{\varepsilon_{i-1}, \varepsilon_{i+1}, \varepsilon_r}). \end{aligned}$$

Since  $\Gamma_{i-1,i+1,r}$  represents  $\mathbb{S}^2$ ,  $\rho_\varepsilon(\Gamma_{i-1,i+1,r}) = 0$  holds; hence, we may further simplify the relation as

$$g_{\varepsilon_{i-1,\varepsilon_{i+1},\varepsilon_r}} = (g_{\widehat{\varepsilon_i}} + g_{\varepsilon_{i-1,\varepsilon_i,\varepsilon_{i+1},\varepsilon_r}} - g) + \rho_\varepsilon(\Gamma) - \rho_\varepsilon(\Gamma_{\widehat{\varepsilon_i}}) - \rho_\varepsilon(\Gamma_{\varepsilon_{i-1,\varepsilon_i,\varepsilon_{i+1},\varepsilon_r}}). \tag{14}$$

Finally, by comparing relation (5) and relation (14), we obtain relation (7). ■

**Remark 3.2.** Note that relation (5) also yields

$$\rho_\varepsilon(\Gamma) - \rho_\varepsilon(\Gamma_{\widehat{\varepsilon_i}}) = (g - g_{\widehat{\varepsilon_i}}) - (g_{\varepsilon_{i-1,\varepsilon_{i+1}}} - g_{\varepsilon_{i-1,\varepsilon_i,\varepsilon_{i+1}}}) \geq 0 \quad \text{for all } i \in \Delta_d. \tag{15}$$

In [21] the inequality  $\rho_\varepsilon(\Gamma_{\widehat{\varepsilon_i}}) \leq \rho_\varepsilon(\Gamma)$  was already proved to hold for any  $(d + 1)$ -colored graph.

**Proposition 3.3.** *Let  $(\Gamma, \gamma)$  be a bipartite (resp. non-bipartite)  $(d + 1)$ -colored graph representing a singular  $d$ -manifold.*

- *If a color  $i \in \Delta_d$  exists such that  $\rho_\varepsilon(\Gamma) = \rho_\varepsilon(\Gamma_{\widehat{\varepsilon_i}})$ , then  $\rho_\varepsilon(\Gamma_{\mathcal{B}}) = \rho_\varepsilon(\Gamma_{\mathcal{B}-\{i\}})$  for each subset  $\mathcal{B} \subset \Delta_d$  with  $\{i - 1, i, i + 1\} \subset \mathcal{B}$ .*
- *In particular, if  $\mathcal{B} = \{i - 1, i, i + 1, r\}$ ,  $\rho_\varepsilon(\Gamma) = \rho_\varepsilon(\Gamma_{\widehat{\varepsilon_i}})$  implies  $\rho_\varepsilon(\Gamma_{\mathcal{B}}) = 0$ .*

*Proof.* As a consequence of relations (6) and (13), we have

$$\begin{aligned} 0 &\leq \rho_\varepsilon(\Gamma_{\widehat{\varepsilon_{j_1,\varepsilon_{j_2},\dots,\varepsilon_{j_{n-3}}}}}) - \rho_\varepsilon(\Gamma_{\widehat{\varepsilon_i,\varepsilon_{j_1,\varepsilon_{j_2},\dots,\varepsilon_{j_{n-3}}}}}) \\ &\leq \rho_\varepsilon(\Gamma_{\widehat{\varepsilon_{j_1,\varepsilon_{j_2},\dots,\varepsilon_{j_{n-2}}}}}) - \rho_\varepsilon(\Gamma_{\widehat{\varepsilon_i,\varepsilon_{j_1,\varepsilon_{j_2},\dots,\varepsilon_{j_{n-2}}}}}) \\ &\quad \vdots \\ &\leq \rho_\varepsilon(\Gamma_{\widehat{\varepsilon_{j_1,\varepsilon_{j_2}}}}) - \rho_\varepsilon(\Gamma_{\widehat{\varepsilon_i,\varepsilon_{j_1,\varepsilon_{j_2}}}}) \\ &\leq \rho_\varepsilon(\Gamma_{\widehat{\varepsilon_{j_1}}}) - \rho_\varepsilon(\Gamma_{\widehat{\varepsilon_{j_1,\varepsilon_i}}}) \end{aligned}$$

for each  $j_1, j_2, \dots, j_{d-3} \notin \{i - 1, i, i + 1\}$ .

The first statement now easily follows. As regards the second one, it is sufficient to note that, in case  $\mathcal{B} = \{i - 1, i, i + 1, r\}$ ,  $\Gamma_{\mathcal{B}-\{i\}}$  represents a 2-dimensional sphere, and hence its regular genus is zero. ■

### 4. General properties of generalized regular genus

Within crystallization theory, two standard methods are known, in order to obtain a presentation of the fundamental group of a closed manifold directly from a graph representing it. The following extensions to compact manifolds and singular manifolds hold:

**Proposition 4.1.** *Let  $(\Gamma, \gamma)$  be a  $(d + 1)$ -colored graph representing the singular  $d$ -manifold  $N$  and the associated compact  $d$ -manifold  $\check{N}$ .*

- *For each  $i, j \in \Delta_d$ , let  $X_{ij}$  (resp.  $R_{ij}$ ) be a set in bijection with the connected components of  $\Gamma_{ij}$  (resp. with the  $\{i, j\}$ -colored cycles of  $\Gamma$ ), and let  $\bar{R}_{ij}$  be a subset of  $X_{ij}$  corresponding to the a maximal tree of the subcomplex  $K_{ij}$  of  $K(\Gamma)$  (consisting only of vertices labeled  $i$  and  $j$ , and edges connecting them). Then*

- a. *if  $i, j \in \Delta_d$  are not singular in  $\Gamma$ ,*

$$\pi_1(\check{N}) = \langle X_{ij} / R_{ij} \cup \bar{R}_{ij} \rangle;$$

- a'. *if no color in  $\Delta_d - \{i, j\}$  is singular in  $\Gamma$ ,*

$$\pi_1(N) = \langle X_{ij} / R_{ij} \cup \bar{R}_{ij} \rangle.$$

- *For each  $i \in \Delta_d$ , let  $X_i$  (resp.  $R_i$ ) be a set in bijection with the  $i$ -colored edges of  $\Gamma$  (resp. with the  $\{i, j\}$ -colored cycles of  $\Gamma$ , for any  $j \in \Delta_d - \{i\}$ ) and let  $\bar{R}_i$  be a subset of  $X_i$  corresponding to a minimal set of  $i$ -colored edges of  $\Gamma$  connecting  $\Gamma_i$ . Then*

- b. *if  $i \in \Delta_d$  is not singular in  $\Gamma$ ,*

$$\pi_1(\check{N}) = \langle X_i / R_i \cup \bar{R}_i \rangle;$$

- b'. *if no color in  $\Delta_d - \{i\}$  is singular in  $\Gamma$ ,*

$$\pi_1(N) = \langle X_i / R_i \cup \bar{R}_i \rangle.$$

*Proof.* It is a direct consequence of some general results concerning the fundamental groups of pseudocomplexes associated to colored graphs; see [20]. ■

The following statement yields an interesting inequality between the generalized regular genus and the rank of the fundamental group, for any compact manifold with connected boundary. The analogous inequality for closed manifolds is well known; see [21, Proposition B].

**Proposition 4.2.** *Let  $M$  be a compact  $d$ -manifold with empty or connected boundary. Then*

$$\bar{\mathcal{G}}(M) \geq \text{rk}(\pi_1(M))$$

*Proof.* Let  $(\Gamma, \gamma)$  be a  $(d + 1)$ -colored graph realizing the generalized regular genus of  $M$ , with respect to the permutation  $\varepsilon$  of  $\Delta_d$ , i.e.,  $\rho_\varepsilon(\Gamma) = \bar{\mathcal{G}}(M)$ . Let  $i$  and  $j$  be two not singular colors that are not consecutive in the permutation  $\varepsilon$ : they certainly exist since  $M$  has empty or connected boundary and so  $\Gamma$  has at most one singular color. It is now sufficient to consider the presentation of the fundamental group of  $M$

given by Proposition 4.1 (a), with respect to colors  $i$  and  $j$  and to recall that, in virtue of formulae (6) and (15),

$$\#(X_{ij} - \bar{R}_{ij}) \leq g_{\varepsilon_i, \varepsilon_j} - (g_{\varepsilon_i} + g_{\varepsilon_j} - 1) \leq \rho_\varepsilon(\Gamma). \quad \blacksquare$$

Let us now recall that another graph-based representation theory for compact (PL) manifolds exists, making use of colored graphs which fail to be regular. More precisely, any compact  $d$ -manifold can be represented by a pair  $(\Lambda, \lambda)$ , where  $\lambda$  is still an edge-coloration on  $E(\Lambda)$  by means of  $\Delta_d$ , but  $\Lambda$  may miss some (or even all)  $d$ -colored edges: such a pair is said to be a  $(d + 1)$ -colored graph with boundary, regular with respect to color  $d$ , and vertices missing the  $d$ -colored edge are called boundary vertices (see [28]).

An easy combinatorial procedure, called *capping off*, enables to connect this representation to the one – involving only regular colored graphs – considered in Section 2.

**Proposition 4.3** ([27]). *Let  $(\Lambda, \lambda)$  be a  $(d + 1)$ -colored graph with boundary, regular with respect to color  $d$ , representing the compact  $d$ -manifold  $M$ . Chosen a color  $c \in \Delta_{d-1}$ , let  $(\Gamma, \gamma)$  be the regular  $(d + 1)$ -colored graph obtained from  $\Lambda$  by capping off with respect to color  $c$ , i.e., by joining two boundary vertices by a  $d$ -colored edge, whenever they belong to the same  $\{c, d\}$ -colored path in  $\Lambda$ . Then,  $(\Gamma, \gamma)$  represents the singular  $d$ -manifold  $\hat{M}$ , and hence  $M$ , too.*

By means of (non-regular) edge-colored graphs with boundary, together with a suitable extension of Proposition 2.5, Gagliardi introduced within crystallization theory a “classical” notion of *regular genus* for compact  $d$ -manifolds, too (see [31, 33]). The following result establishes a comparison between regular genus and generalized regular genus (as defined in Section 2: see Definitions 2.6 and 2.7) for any compact  $d$ -manifold.

**Proposition 4.4.** *Let  $M$  be a compact  $d$ -manifold, with  $d \geq 3$ , and let  $\mathcal{G}(M)$  denote the regular genus of  $M$ . Then*

$$\bar{\mathcal{G}}(M) \leq \mathcal{G}(M).$$

Moreover,

- a.  $\bar{\mathcal{G}}(M) = \mathcal{G}(M)$  if  $M$  is a closed  $d$ -manifold;
- b.  $\bar{\mathcal{G}}(M) = 0 \iff M = \mathbb{S}^d \iff \mathcal{G}(M) = 0$ ;
- c.  $\bar{\mathcal{G}}(M) = \mathcal{G}(M)$  if  $M$  is a compact 3-manifold with connected boundary;
- d. there exist compact 3-manifolds (with disconnected boundary) with the property  $\bar{\mathcal{G}}(M) < \mathcal{G}(M)$ .

*Proof.* The general inequality is a consequence of the “capping off” procedure, recalled in Proposition 4.3. In fact, let us assume the regular genus of  $M$  to be realized by the (not regular) graph with boundary  $\Lambda$  with respect to the cyclic permutation  $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{d-1}, \varepsilon_d = d)$  of  $\Delta_d$ . Then, it is not difficult to prove that, if  $c \in \{\varepsilon_0, \varepsilon_{d-1}\}$  is chosen, and  $\Gamma$  is the (regular)  $(d + 1)$ -colored graph obtained from  $\Lambda$  by capping off with respect to color  $c$ , the generalized regular genus of  $\Gamma$  with respect to  $\varepsilon$  equals the regular genus of  $\Lambda$  with respect to the same permutation:  $\rho_\varepsilon(\Gamma) = \rho_\varepsilon(\Lambda) = \mathcal{G}(M)$ .

Equality (a) is trivial by definition (as already pointed out in Remark 2.8).

Regarding statement (b), first note that, obviously,  $\bar{\mathcal{G}}(\mathbb{S}^d) = \mathcal{G}(\mathbb{S}^d) = 0$ ; moreover, the main theorem of [26] ensures that, if  $\Gamma$  represents a closed  $d$ -manifold  $M$ ,  $\rho(\Gamma) = 0$  implies  $M$  to be a PL  $d$ -sphere. In order to complete the proof of both co-implications, let us consider a (regular)  $(d + 1)$ -colored graph  $\Gamma$  such that there exists a cyclic permutation  $\varepsilon$  of  $\Delta_d$  with  $\rho_\varepsilon(\Gamma) = 0$ ; we want to prove that  $|K(\Gamma)|$  is a closed  $d$ -manifold. If  $d = 2$  then  $|K(\Gamma)| \cong \mathbb{S}^2$ , since  $\rho_\varepsilon(\Gamma)$  trivially coincides with the genus of the surface  $|K(\Gamma)|$ . Suppose now our claim to be true in each dimension  $< d$ ; given  $i \in \mathbb{Z}_{d+1}$ , let  $\Xi$  be a connected component of  $\Gamma_{\varepsilon_i}$ , which is a  $d$ -colored graph. Since  $\rho_\varepsilon(\Xi) \leq \rho_\varepsilon(\Gamma)$  (see inequality (15)) then, by induction,  $\Xi$  represents a PL  $(d - 1)$ -sphere and, therefore,  $|K(\Gamma)|$  is a closed PL  $d$ -manifold, and statement (b) is proved.

Note that the inequality  $\bar{\mathcal{G}}(M) \geq \mathcal{G}(M)$  for 3-dimensional manifolds with connected boundary, yielding relation (c), is proved in [24].

The same paper also presents examples of the strict inequality (d): if  $F$  is a closed surface of genus  $g$ , then  $\bar{\mathcal{G}}(F \times I) = g < \mathcal{G}(F \times I) = 2g$ . ■

**Remark 4.5.** In Section 7 we will prove that the equality between the two invariants does not hold for 4-manifolds with boundary, even if the boundary is assumed to be connected; see Corollary 7.7(b).

As regards the invariant regular genus, a well-known relation (that is,  $\mathcal{G}(M) \geq \mathcal{G}(\partial M)$ ) compares the regular genus of any compact manifold with the regular genus of its boundary; in the case of connected boundary, the following extensions hold, concerning both the generalized regular genus and the G-degree:

**Proposition 4.6.** *Let  $M$  be a compact  $d$ -manifold with (non-empty) connected boundary. Then*

$$\bar{\mathcal{G}}(M) \geq \mathcal{G}(\partial M) \quad \text{and} \quad \mathcal{D}_G(M) \geq d \cdot \mathcal{D}_G(\partial M).$$

*Proof.* The first inequality is an easy consequence of (15), applied to a regular graph  $\Gamma$  representing  $M$ , so that  $\bar{\mathcal{G}}(M) = \rho_\varepsilon(\Gamma)$  ( $\varepsilon$  being a cyclic permutation of  $\Delta_d$ ) and having color  $i$  as its (only) singular color.

The second inequality may be obtained in a similar way, by making use of the relation  $\omega_G(\Gamma) \geq d \cdot \omega_G(\Gamma_i)$ , proved in [38, Lemma 4.6] for each  $(d + 1)$ -colored graph and for each color  $i \in \Delta_d$ . ■

A  $d$ -dimensional extension of the construction described in [24, Proposition 5 (i)] (resp. in [24, Proposition 5 (ii)]), performed in [35, Section 7] in a general setting including graphs representing singular  $d$ -manifolds, allows to easily obtain graphs representing connected sums (resp. boundary connected sums) of compact PL  $d$ -manifolds directly from the graphs representing the summands.

We briefly recall that, if  $(\Gamma_1, \gamma_1)$  and  $(\Gamma_2, \gamma_2)$  are two disjoint  $(d + 1)$ -colored graphs and  $v_i \in V_i$  for each  $i \in \{1, 2\}$ , the *graph connected sum* of  $\Gamma_1, \Gamma_2$  with respect to vertices  $v_1, v_2$  (denoted by  $\Gamma_1 \#_{v_1 v_2} \Gamma_2$ ) is the graph obtained from  $\Gamma_1$  and  $\Gamma_2$  by deleting  $v_1$  and  $v_2$  and welding the “dangling” edges of the same color. It is not difficult to check that, if all  $d$ -residues containing  $v_1$  and  $v_2$  are ordinary (resp. if both  $v_1$  and  $v_2$  belong to exactly one singular  $d$ -residue,  $\Xi_1$  and  $\Xi_2$ , say), then  $\Gamma_1 \#_{v_1 v_2} \Gamma_2$  represents the (internal) connected sum between  $|K(\Gamma_1)|$  and  $|K(\Gamma_2)|$  (resp. represents the boundary connected sum between  $|K(\Gamma_1)|$  and  $|K(\Gamma_2)|$ , performed on the boundary components corresponding to  $\Xi_1$  and  $\Xi_2$ ).

**Proposition 4.7.** *Let  $M_1$  and  $M_2$  be compact  $d$ -manifolds. Then*

$$\bar{\mathcal{G}}(M_1 \# M_2) \leq \bar{\mathcal{G}}(M_1) + \bar{\mathcal{G}}(M_2) \quad \text{and} \quad \bar{\mathcal{G}}(M_1 \partial \# M_2) \leq \bar{\mathcal{G}}(M_1) + \bar{\mathcal{G}}(M_2);$$

$$\mathcal{D}_G(M_1 \# M_2) \leq \mathcal{D}_G(M_1) + \mathcal{D}_G(M_2)$$

and

$$\mathcal{D}_G(M_1 \partial \# M_2) \leq \mathcal{D}_G(M_1) + \mathcal{D}_G(M_2).$$

*Proof.* It is an easy consequence of the above described constructions; see the quoted papers, together with [14, Proposition 10]. ■

## 5. Representing handlebodies and products $\times I$

**Proposition 5.1.** *For any  $d \geq 4$ , a bipartite (resp. non-bipartite)  $(d + 1)$ -colored graph exists, with order  $2d$  and regular genus one with respect to any permutation of  $\Delta_d$ , representing the genus one  $d$ -dimensional handlebody  $\mathbb{Y}_1^d$  (resp.  $\tilde{\mathbb{Y}}_1^d$ ). Hence, for each  $d \geq 4$ ,*

$$\bar{\mathcal{G}}(\mathbb{Y}_1^d) = \bar{\mathcal{G}}(\tilde{\mathbb{Y}}_1^d) = 1 \quad \text{and} \quad \mathcal{D}_G(\mathbb{Y}_1^d) = \mathcal{D}_G(\tilde{\mathbb{Y}}_1^d) = \frac{d!}{2}. \quad (16)$$



Moreover, for each  $d \geq 4$  and for each  $m \geq 1$ ,

$$\bar{\mathcal{G}}(\mathbb{Y}_m^d) = \bar{\mathcal{G}}(\tilde{\mathbb{Y}}_m^d) = m \quad \text{and} \quad \mathcal{D}_G(\mathbb{Y}_m^d) = \mathcal{D}_G(\tilde{\mathbb{Y}}_m^d) = m \cdot \frac{d!}{2}. \quad (17)$$

*Proof.* For any  $d \geq 3$ , an order  $2(d + 1)$   $(d + 1)$ -colored graph with boundary  $(H, h)$  (resp.  $(H', h')$ ) is well known, which represents the genus one  $d$ -dimensional handlebody  $\mathbb{Y}_1^d$  (resp.  $\tilde{\mathbb{Y}}_1^d$ ), see [33]. By applying to  $(H, h)$  (resp.  $(H', h')$ ) the ‘‘capping off’’ procedure described in Proposition 4.3, a (regular) order  $2(d + 1)$   $(d + 1)$ -colored graph representing  $\mathbb{Y}_1^d$  (resp.  $\tilde{\mathbb{Y}}_1^d$ ) is obtained. It is easy to check that it admits a (proper) 2-dipole, whose elimination yields a (minimal) order  $2d$  regular  $(d + 1)$ -colored graph  $(\hat{H}, \hat{h})$  (resp.  $(\hat{H}', \hat{h}')$ ) representing  $\mathbb{Y}_1^d$  (resp.  $\tilde{\mathbb{Y}}_1^d$ ); see Figure 1 for the orientable 4-dimensional case. A direct computation gives  $\rho_\varepsilon(\hat{H}) = 1$  (resp.  $\rho_\varepsilon(\hat{H}') = 1$ ) for each permutation  $\varepsilon$  of  $\Delta_d$ . Hence, the classification of compact PL  $d$ -manifolds with generalized regular genus zero (and with G-degree zero) easily allows to prove (16); see Proposition 4.4 (b). Now, it is not difficult to check that, for each  $m \geq 1$ , the graph connected sum construction hinted to in Section 4, with suitable choices of the vertices, enables to obtain a bipartite (resp. non-bipartite)  $(d + 1)$ -colored graph representing the genus  $m$   $d$ -dimensional handlebody  $\mathbb{Y}_m^d = \#_m \mathbb{Y}_1^d$  (resp.  $\tilde{\mathbb{Y}}_m^d = \#_m \tilde{\mathbb{Y}}_1^d$ ): its order is  $2md - 2(m - 1)$  and its regular genus is  $m$  with respect to any permutation of  $\Delta_d$ . See Figure 2 for an example, in case  $d = 4$  and  $m = 2$ . As a consequence of this construction, together with the inequalities  $\bar{\mathcal{G}}(M) \geq \mathcal{G}(\partial M)$  and  $\mathcal{D}_G(M) \geq d \cdot \mathcal{D}_G(\partial M)$  (see Proposition 4.6), both equalities of (17) easily follow. ■

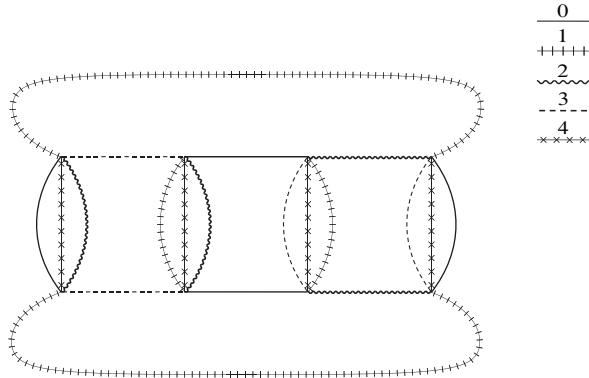
**Remark 5.2.** Note that, by a suitable application of the procedure of graph connected sum, it is easy to obtain also a (bipartite or non-bipartite)  $(d + 1)$ -colored graph representing the connected sum of  $m$  ( $m \geq 2$ ) (orientable or non-orientable)  $d$ -dimensional handlebodies. If  $\tilde{\mathbb{Y}}_r^d$  denotes either the orientable or non-orientable genus  $r$   $d$ -dimensional handlebody, then the graph representing  $\tilde{\mathbb{Y}}_{r_1}^d \# \dots \# \tilde{\mathbb{Y}}_{r_m}^d$  has order  $2d(r_1 + \dots + r_m)$  (since the procedure has to be preceded by the insertion of  $m$   $d$ -dipoles, in order to obtain  $m$  ordinary  $d$ -residues) and its regular genus is  $r_1 + \dots + r_m$  with respect to any permutation of  $\Delta_d$ . See Figure 3 for an example, in case  $d = 4$ ,  $m = 2$  and  $r_1 = r_2 = 1$ .

The following inequalities directly follow by construction:<sup>6</sup>

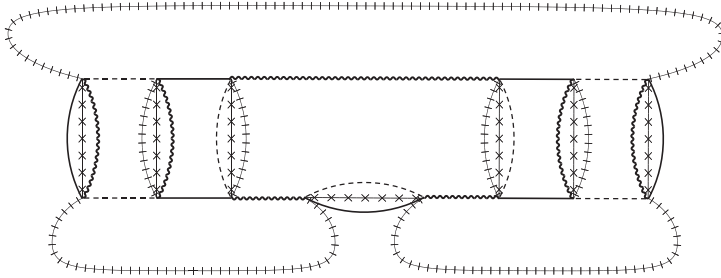
$$\bar{\mathcal{G}}(\tilde{\mathbb{Y}}_{r_1}^d \# \dots \# \tilde{\mathbb{Y}}_{r_m}^d) \leq r_1 + \dots + r_m \quad \mathcal{D}_G(\tilde{\mathbb{Y}}_{r_1}^d \# \dots \# \tilde{\mathbb{Y}}_{r_m}^d) \leq (r_1 + \dots + r_m) \cdot \frac{d!}{2}. \quad (18)$$

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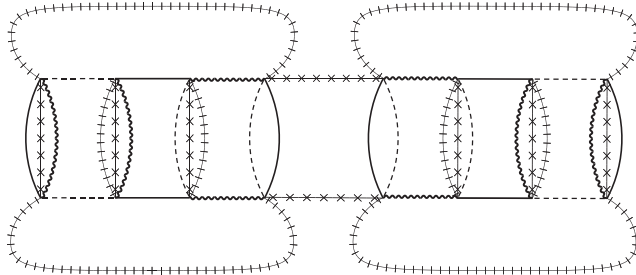
<sup>6</sup>Note that, in virtue of relation  $\mathcal{D}_G(M) \geq \frac{d!}{2} \cdot \bar{\mathcal{G}}(M)$  (see Section 2), if the equality concerning generalized regular genus holds, then the one concerning G-degree holds, too.



**Figure 1.** Regular 5-colored graph representing  $\mathbb{Y}_1^4$ .



**Figure 2.** Regular 5-colored graph representing the boundary connected sum  $\mathbb{Y}_1^4 \partial \# \mathbb{Y}_1^4 = \mathbb{Y}_2^4$ .



**Figure 3.** Regular 5-colored graph representing the connected sum  $\mathbb{Y}_1^4 \# \mathbb{Y}_1^4$ .

**Proposition 5.3.** *For any  $d \geq 3$ , given a bipartite (resp. non-bipartite)  $(d + 1)$ -colored graph  $(\Lambda, \lambda)$  representing a closed orientable (resp. non-orientable)  $d$ -manifold  $M$ , then a bipartite (resp. non-bipartite)  $(d + 2)$ -colored graph  $(\tilde{\Lambda}, \tilde{\lambda})$  representing the singular  $(d + 1)$ -manifold  $M \times I$  exists, with  $\#V(\tilde{\Lambda}) = \#V(\Lambda)$  and*

$\rho(\tilde{\Lambda}) = \rho(\Lambda)$ . Hence, for each  $d \geq 3$ ,

$$\bar{\mathcal{G}}(M \times I) = \mathcal{G}(M) \quad \text{for any closed } d\text{-manifold } M.$$

Moreover,

$$\omega_G(\tilde{\Lambda}) = \frac{d!}{2} \left[ \sum_{i \in \{1, \dots, d\}} (p - g_{\varepsilon_0 \varepsilon_i}) - (p - 1) + \frac{2}{(d - 1)!} \omega_G(\Lambda) \right],$$

$\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_d)$  being the cyclic permutation of  $\Delta_d$  so that  $\rho(\Lambda) = \rho_\varepsilon(\Lambda)$ .

*Proof.* Let  $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_d)$  be the cyclic permutation of  $\Delta_d$  so that  $\rho(\Lambda) = \rho_\varepsilon(\Lambda)$ . If  $(\tilde{\Lambda}, \tilde{\lambda})$  is obtained from  $\Lambda$  by adding a  $(d + 1)$ -colored edge between any pair of  $\varepsilon_0$ -adjacent vertices, then it is easy to check that  $\tilde{\Lambda}$  represents  $M \times I$  and  $\rho_{\varepsilon'}(\tilde{\Lambda}) = \rho(\tilde{\Lambda})$ , where  $\varepsilon' = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_d, d + 1)$ ; see [27], and Figure 4 for an example of the construction, with  $M = L(2, 1)$ . This fact proves the first part of the statement and, as a consequence, the inequality  $\bar{\mathcal{G}}(M \times I) \leq \mathcal{G}(M)$  for any closed  $d$ -manifold  $M$ . On the other hand, since  $M \times I$  has two boundary components PL-homeomorphic to  $M$ , any  $(d + 2)$ -colored graph representing  $M \times I$  as a singular  $(d + 1)$ -manifold must have a  $(d + 1)$ -residue representing  $M$ , and hence must have regular genus greater or equal to  $\mathcal{G}(M)$ . As regards the computation of the G-degree, a direct application of the formula

$$\omega_G(\Gamma) = \frac{(d - 1)!}{2} \left( d + \frac{d}{2}(d - 1)p - \sum_{r, s \in \Delta_d} g_{rs} \right)$$

(proved in [14] for any  $(d + 1)$ -colored graph) to  $(\tilde{\Lambda}, \tilde{\lambda})$  yields (in virtue of the combinatorial structure of  $\tilde{\Lambda}$ );

$$\begin{aligned} \omega_G(\tilde{\Lambda}) &= \frac{d!}{2} \left[ (d + 1) + \frac{(d + 1)dp}{2} - \sum_{r, s \in \Delta_{d+1}} g_{\varepsilon_r \varepsilon_s} \right] \\ &= \frac{d!}{2} \left[ (d + 1) + \frac{(d + 1)dp}{2} - \sum_{r, s \in \Delta_d} g_{\varepsilon_r \varepsilon_s} - \sum_{i \in \{1, \dots, d\}} g_{\varepsilon_0 \varepsilon_i} - p \right]. \end{aligned}$$

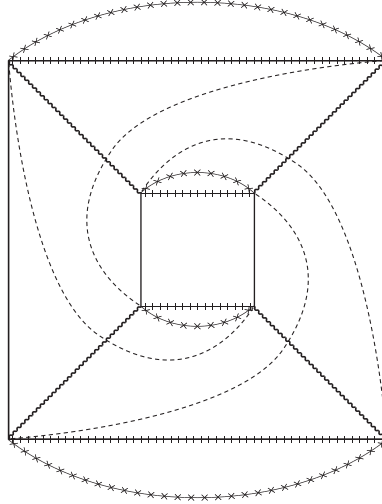
Moreover, by making use of the similar computation for  $\omega_G(\Lambda)$ , yielding

$$\sum_{r, s \in \Delta_d} g_{\varepsilon_r \varepsilon_s} = d + \frac{d}{2}(d - 1)p - \frac{2}{(d - 1)!} \omega_G(\Lambda),$$

we obtain

$$\begin{aligned} \omega_G(\tilde{\Lambda}) &= \frac{d!}{2} \left[ (d + 1) + \frac{(d + 1)dp}{2} - d - \frac{d}{2}(d - 1)p - \sum_{i \in \{1, \dots, d\}} g_{\varepsilon_0 \varepsilon_i} - p \right. \\ &\quad \left. + \frac{2}{(d - 1)!} \omega_G(\Lambda) \right], \end{aligned}$$

which the last formula of the statement follows from. ■



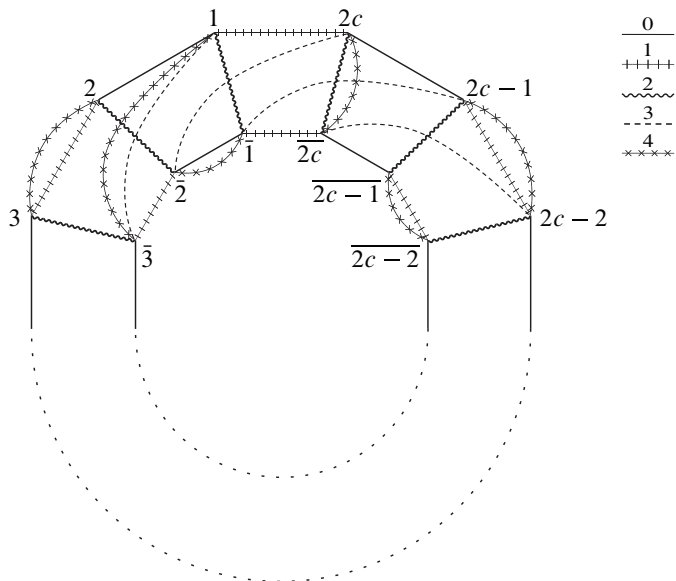
**Figure 4.** Regular 5-colored graph representing the product  $L(2, 1) \times I$ .

### 6. Representing $\mathbb{D}^2$ -bundles over $\mathbb{S}^2$

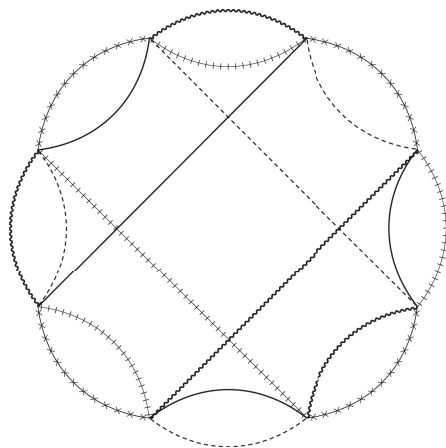
In [8], a 5-colored graph with boundary  $(\tilde{\Gamma}_c, \tilde{\gamma}_c)$  (resp.  $(\tilde{\Gamma}_0, \tilde{\gamma}_0)$ ) representing the  $\mathbb{D}^2$ -bundle over  $\mathbb{S}^2$ ,  $\xi_c$ , with Euler class  $c$  (resp. the trivial  $\mathbb{D}^2$ -bundle over  $\mathbb{S}^2$ ,  $\mathbb{S}^2 \times \mathbb{D}^2$ ) is produced, for all  $c \in \mathbb{Z}^+ - \{1\}$ ; all these graphs have regular genus equal to three. This allows to prove – by means also of some theoretical results about the “gap” between regular genus and the rank of the fundamental group of a PL 4-manifold with boundary – that  $\mathcal{G}(\xi_c) = \mathcal{G}(\mathbb{S}^2 \times \mathbb{D}^2) = 3$ .

The regular 5-colored graphs obtained from the above graphs by means of the “capping off” procedure described in Proposition 4.3 (which represent the singular 4-manifolds  $\hat{\xi}_c$  and  $\widehat{\mathbb{S}^2 \times \mathbb{D}^2}$ , and hence the compact 4-manifolds  $\xi_c$  and  $\mathbb{S}^2 \times \mathbb{D}^2$ , too) have regular genus three by construction, have the same order as the starting graphs with boundary (i.e.,  $4c + 6$  for  $\xi_c$  for all  $c \in \mathbb{Z}^+ - \{1\}$ , and 14 for  $\mathbb{S}^2 \times \mathbb{D}^2$ ), but admit a (proper) 2-dipole involving colors non-consecutive in the permutation  $\varepsilon$  realizing the minimum generalized regular genus, together with two 2-dipoles involving colors consecutive in the permutation  $\varepsilon$ .

Now, it is easy to check, via Proposition 2.5 and Definition 2.9, that the elimination of a 2-dipole involving colors non-consecutive (resp. consecutive) in the permutation  $\varepsilon$  decreases by one (resp. does not affect) the regular genus with respect to  $\varepsilon$ . Hence, the elimination of the three 2-dipoles yields a regular 5-colored graph  $(\Lambda_c, \lambda_c)$  for all  $c \in \mathbb{Z}^+ - \{1\}$  (resp.  $(\Lambda_0, \lambda_0)$ ) representing  $\xi_c$  (resp.  $\mathbb{S}^2 \times \mathbb{D}^2$ ) with the same order



**Figure 5.** The regular 5-colored graph  $\Lambda_c$  representing the compact 4-manifold  $\xi_c$ .



**Figure 6.** The regular 5-colored graph  $\Lambda_0$  representing the compact 4-manifold  $S^2 \times \mathbb{D}^2$ .

$4c$  (resp. 8) as the standard crystallization of  $L(c, 1)$  (resp. of  $S^1 \times S^2$ ); see Figure 5 (resp. Figure 6).

As a consequence, we have

$$\bar{\mathcal{G}}(\xi_c) \leq 2 \quad \text{for all } c \in \mathbb{Z}^+ - \{1\}$$

and

$$\overline{\mathcal{G}}(\mathbb{S}^2 \times \mathbb{D}^2) \leq 2.$$

Actually, in the following Corollary 7.6, we will prove that all compact 4-manifolds of this infinite class turn out to have generalized regular genus equal to two.

As far as the G-degree is concerned, we recall that [18, Proposition 5] proves, for each 5-colored graph representing a compact PL 4-manifold,

$$\omega_G(\Gamma) = 6(\rho_\varepsilon(\Gamma) + \rho_{\varepsilon'}(\Gamma)), \tag{19}$$

where  $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$  is an arbitrary permutation of  $\Delta_4$  and  $\varepsilon'$  is the “associated” permutation, i.e.,  $\varepsilon' = (\varepsilon_0, \varepsilon_2, \varepsilon_4, \varepsilon_1, \varepsilon_3)$ .

A direct computation allows to check that, if  $\varepsilon'$  denotes the permutation associated to the one realizing  $\rho(\Lambda_c) = 2$  (resp.  $\rho(\Lambda_0) = 2$ ), then the regular genus with respect to  $\varepsilon'$  is  $2c - 2$  for all  $c \in \mathbb{Z}^+ - \{1\}$  (resp. is also 2) Then, formula (19) yields

$$\omega_G(\Lambda_c) = 6[2 + (2c - 2)] = 12c \quad \text{for all } c \in \mathbb{Z}^+ - \{1\}$$

(resp.  $\omega_G(\Lambda_0) = 6(2 + 2) = 24$ ).

Hence,

$$\mathcal{D}_G(\xi_c) \leq 12c \quad \text{for all } c \in \mathbb{Z}^+ - \{1\}$$

and

$$\mathcal{D}_G(\mathbb{S}^2 \times \mathbb{D}^2) \leq 24.$$

## 7. Classifying results in dimension 4

### 7.1. Classifying with respect to generalized regular genus

In the 4-dimensional setting, formula (10) enables to prove the following useful results.

**Proposition 7.1.** *Let  $(\Gamma, \gamma)$  be a connected 5-colored graph representing a singular 4-manifold  $N^4$  (and the associated compact 4-manifold  $\check{N}^4$ ). For each cyclic permutation  $\varepsilon$  of  $\Delta_4$ ,*

$$\chi(N^4) = 2 - 2\rho_\varepsilon(\Gamma) + \sum_{i \in \Delta_4} \rho_\varepsilon(\Gamma_{\widehat{\varepsilon}_i}). \tag{20}$$

Moreover, if  $\rho_\varepsilon$  and  $\rho_{\widehat{\varepsilon}_i}$  respectively denote  $\rho_\varepsilon(\Gamma)$  and  $\rho_\varepsilon(\Gamma_{\widehat{\varepsilon}_i})$ , then

$$g_{\widehat{\varepsilon}_{i-1}, \widehat{\varepsilon}_{i+1}} = g_{\varepsilon_i, \varepsilon_{i+2}, \varepsilon_{i+3}} = (g_{\widehat{\varepsilon}_{i-1}} + g_{\widehat{\varepsilon}_{i+1}} - 1) + \rho_\varepsilon - \rho_{\widehat{\varepsilon}_{i-1}} - \rho_{\widehat{\varepsilon}_{i+1}}; \tag{21}$$

for all  $j, k$  non-consecutive in  $\Delta_4 - \{i\}$ ,

$$g_{\widehat{\varepsilon_i, \widehat{\varepsilon_j, \widehat{\varepsilon_k}}} = (g_{\widehat{\varepsilon_i, \widehat{\varepsilon_j}} + g_{\widehat{\varepsilon_i, \widehat{\varepsilon_k}}} - g_{\widehat{\varepsilon_i}}) + \rho_{\widehat{\varepsilon_i}}; \tag{22}$$

$$\sum_{i \in \Delta_4} g_{\widehat{\varepsilon_{i-1}, \widehat{\varepsilon_{i+1}}} = \left( 2 \sum_{i \in \Delta_4} g_{\widehat{\varepsilon_i}} - 5 \right) + 5\rho_\varepsilon - 2 \sum_{i \in \Delta_4} \rho_{\widehat{\varepsilon_i}}; \tag{23}$$

$$\rho_{\widehat{\varepsilon_{i-1}}} + \rho_{\widehat{\varepsilon_{i+1}}} \leq \rho_\varepsilon; \tag{24}$$

$$\sum_{i \in \Delta_4} \rho_{\widehat{\varepsilon_i}} \leq \left\lfloor \frac{5}{2} \rho_\varepsilon \right\rfloor; \tag{25}$$

$$\sum_{i \in \Delta_4} \rho_{\widehat{\varepsilon_i}} = \frac{5}{2} \rho_\varepsilon \iff \rho_{\widehat{\varepsilon_{i-1}}} + \rho_{\widehat{\varepsilon_{i+1}}} = \rho_\varepsilon \text{ for all } i \in \Delta_4. \tag{26}$$

*Proof.* Relations (21) and (22) are nothing but relations (6) and (12), in case  $\Gamma$  is assumed to be a connected graph representing a singular 4-manifold.

Summing up relations (21), for each  $i \in \Delta_4$ , yields relation (23).

Since  $g_{\widehat{\varepsilon_{i-1}, \widehat{\varepsilon_{i+1}}} \geq g_{\widehat{\varepsilon_{i-1}}} + g_{\widehat{\varepsilon_{i+1}}} - 1$  trivially holds, relations (24) and (25) follow from relations (21) and (23) respectively.

Finally, the co-implication (26) is a direct consequence. ■

**Corollary 7.2.** *Let  $(\Gamma, \gamma)$  be a connected 5-colored graph representing  $S^4$ . Then, for each cyclic permutation  $\varepsilon$  of  $\Delta_4$ ,*

$$\rho_\varepsilon(\Gamma) = \frac{1}{2} \sum_{i \in \Delta_4} \rho_\varepsilon(\Gamma_{\widehat{\varepsilon_i}}). \tag{27}$$

**Proposition 7.3.** *Let  $(\Gamma, \gamma)$  be a bipartite (resp. non-bipartite) 5-colored graph representing a compact PL 4-manifold  $M^4$  with empty or connected boundary and let  $\varepsilon$  be a cyclic permutation of  $\Delta_4$ . If there exists  $i \in \Delta_4$  so that  $\rho_\varepsilon(\Gamma_{\widehat{\varepsilon_i}}) = 0$ , then  $M^4 \cong \#_\alpha(S^1 \times S^3) \# \mathbb{Y}_\beta^4$  (resp.  $M^4 \cong \#_\alpha(S^1 \tilde{\times} S^3) \# \mathbb{Y}_\beta^4$  or  $M^4 \cong \#_\alpha(S^1 \tilde{\times} S^3) \# \tilde{\mathbb{Y}}_\beta^4$ ), with  $\alpha, \beta \geq 0$  and  $\alpha + \beta \leq \rho_\varepsilon(\Gamma)$  and  $\beta \leq \rho_\varepsilon(\Gamma_{\widehat{\varepsilon_c}})$ ,  $c \in \Delta_4$  being the singular color of  $\Gamma$  (if any<sup>7</sup>).*

*In particular,*

- a. *if  $M^4$  is a closed 4-manifold and there exists  $i \in \Delta_4$  so that  $\rho_\varepsilon(\Gamma_{\widehat{\varepsilon_i}}) = 0$ , then  $M^4 \cong \#_\alpha(S^1 \times S^3)$  (resp.  $M^4 \cong \#_\alpha(S^1 \tilde{\times} S^3)$ ), with  $\alpha \leq \rho_\varepsilon(\Gamma)$ ;*
- b. *if  $M^4$  has (non-empty) connected boundary, and  $\rho_\varepsilon(\Gamma) = \rho_\varepsilon(\Gamma_{\widehat{\varepsilon_c}})$ ,  $c \in \Delta_4$  being the only singular color of  $\Gamma$ , then  $M^4 \cong \mathbb{Y}_m^4$  (resp.  $M^4 \cong \tilde{\mathbb{Y}}_m^4$ ), with  $m \leq \rho_\varepsilon(\Gamma)$ .*

---

<sup>7</sup>If all colors are not singular, then  $\beta = 0$  as in case (a).

*Proof.* Let  $\varepsilon$  be the cyclic permutation of  $\Delta_4$  such that  $\rho(\Gamma) = \rho_\varepsilon(\Gamma)$ ; without loss of generality we may assume  $\varepsilon = (0, 1, 2, 3, 4)$  and, by Remark 2.10,  $g_{\hat{i}} = 1$  for each  $i \in \Delta_4$ . Moreover, since the cyclic permutation  $\varepsilon$  is defined up to inverse, we may further assume that  $i \in \{c+1, c+2\}$ ,  $c \in \Delta_4$  being the color of the possible singular vertex.

Now, relation (22) yields

$$g_{\widehat{c-1, c+1, c+2}} = g_{\widehat{c-1, i}} + g_{\widehat{c+1, c+2}} - g_{\hat{i}} = 1 + \rho_\varepsilon(\Gamma_{\varepsilon_i}^\wedge) = g_{\widehat{c-1, i}} + g_{\widehat{c+1, c+2}} - 1.$$

Hence,  $K(c-1, c+1, c+2)$  collapses onto  $K(c-1, j)$ , where

$$\{j\} = \{c+1, c+2\} - \{i\}.$$

Since  $c-1, c+1, c+2$  are not singular colors, in the bipartite (resp. non-bipartite) case  $N(c-1, c+1, c+2) = \mathbb{Y}_m^4$  (resp.  $N(c-1, c+1, c+2) = \tilde{\mathbb{Y}}_m^4$ ) follows, with  $m = g_{\widehat{c-1, j}} - 1$ .

On the other hand,  $K(c, c-2)$  consists of  $g_{\widehat{c, c-2}}$  edges; since  $c-2$  is not a singular color,  $N(c, c-2)$  is obtained by the cone over  $\text{lkd}(v_c)$  by adding  $g_{\widehat{c, c-2}} - 1$  1-handles.

Since  $\widehat{M}^4 = N(c-1, c+1, c+2) \cup_\phi N(c, c-2)$ ,  $\phi$  being a boundary homeomorphism, in the bipartite (resp. non-bipartite) case

$$\partial N(c, c-2) = \partial N(c-1, c+1, c+2) = \partial \mathbb{Y}_m^4 = \#_m(\mathbb{S}^1 \times \mathbb{S}^2)$$

(resp.  $\partial N(c, c-2) = \partial N(c-1, c+1, c+2) = \partial \tilde{\mathbb{Y}}_m^4 = \#_m(\mathbb{S}^1 \tilde{\times} \mathbb{S}^2)$ ) follows.

In the orientable case, we have  $\text{lkd}(v_c) \# [\#_{m'}(\mathbb{S}^1 \times \mathbb{S}^2)] = \#_m(\mathbb{S}^1 \times \mathbb{S}^2)$ , where  $m' = g_{\widehat{c, c-2}} - 1$ , i.e., in virtue of the uniqueness of the sum decomposition in dimension 3,  $\text{lkd}(v_c) = \#_{m-m'}(\mathbb{S}^1 \times \mathbb{S}^2)$ . The statement now follows from [19, Lemma 1], since

$$\begin{aligned} \widehat{M}^4 &= [\mathbb{Y}_{m'}^4 \overset{\partial}{\#} \mathbb{Y}_{m-m'}^4] \cup_\phi [\mathbb{Y}_{m'}^4 \overset{\partial}{\#} [v_c * \#_{m-m'}(\mathbb{S}^1 \times \mathbb{S}^2)]] \\ &= [\mathbb{Y}_{m'}^4 \cup_\phi \mathbb{Y}_{m'}^4] \# [\mathbb{Y}_{m-m'}^4 \cup_\phi [v_c * \#_{m-m'}(\mathbb{S}^1 \times \mathbb{S}^2)]] \\ &= \#_{m'}(\mathbb{S}^1 \times \mathbb{S}^3) \# \widehat{\mathbb{Y}}_{m-m'}^4. \end{aligned}$$

The non-orientable case may be proved in full analogy, by distinguishing the case of orientable and non-orientable boundary.

Both point (a) and point (b) of the statement are nothing but particular cases of the general statement; indeed, as regards point (b), the second part of Proposition 3.3 yields, for  $d = 4$ ,

$$\rho_\varepsilon(\Gamma) = \rho_\varepsilon(\Gamma_{\varepsilon_c}^\wedge) \implies \rho_\varepsilon(\Gamma_{c+2}^\wedge) = \rho_\varepsilon(\Gamma_{c-2}^\wedge) = 0. \quad \blacksquare$$



**Remark 7.4.** Note that point (a) of the above proposition could be independently proved simply by noting that relation (22) yields

$$g_{i,i+2} = g_{i,i+1,i+2} + g_{i-1,i,i+2} - 1,$$

which ensures – via [7, Lemma 5] – that  $K(i - 2, i - 1, i + 1)$  collapses to a graph, i.e.,  $N(i - 2, i - 1, i + 1)$  is a handlebody. Since also  $N(i, i + 2)$  is obviously a handlebody, statement (a) follows via a well-known theorem by Montesinos and Laudenbach and Poenaru (see [47] and [43]).

Also, point (b) could be independently proved by noting that, by formula (21),  $g_{\widehat{c}, \widehat{c+2}} = 1$  follows, i.e.,  $N(c, c + 2)$  is homeomorphic to the cone over  $\text{lkd}(v_c)$ . Moreover, relation (22) yields

$$g_{\widehat{c-2}, \widehat{c-1}, \widehat{c+1}} = g_{\widehat{c-2}, \widehat{c-1}} + g_{\widehat{c+1}, \widehat{c-2}} - 1,$$

which ensures – via Lemma 5 of [7] – that  $K(c - 2, c - 1, c + 1)$  collapses to a graph, i.e.,  $N(c - 2, c - 1, c + 1)$  is a handlebody; this proves statement (b).

We are now able to classify all compact 4-manifolds with generalized regular genus one.

**Proposition 7.5.** *Let  $M^4$  be a compact 4-manifold with  $\overline{\mathcal{G}}(M^4) = 1$ . Then,*

- either  $M^4 \in \{\mathbb{S}^1 \times \mathbb{S}^3, \mathbb{S}^1 \widetilde{\times} \mathbb{S}^3\}$
- or  $M^4 \in \{\mathbb{Y}_1^4, \widetilde{\mathbb{Y}}_1^4\}$
- or  $M^4 \cong \overline{M} \times I$ , where  $\overline{M}$  is a genus one closed 3-manifold.

*Proof.* Three cases occur:

- $M^4$  is a closed 4-manifold;
- $M^4$  is a compact 4-manifold with (non-empty) connected boundary;
- $M^4$  is a compact 4-manifold with disconnected boundary.

In the first and second case, if  $\Gamma$  represents  $M^4$  with  $\rho_\varepsilon(\Gamma) = \rho(\Gamma) = \overline{\mathcal{G}}(M^4) = 1$ , then  $\Gamma$  may be assumed to satisfy  $g_i = 1$  for all  $i \in \Delta_4$  (see Remark 2.10). Relation (25) directly implies the existence of at least a color  $i \in \Delta_4$  such that  $\rho_\varepsilon(\Gamma_{\varepsilon_i}^\wedge) = 0$ . Hence, Proposition 7.3 proves the statement; see in particular points (a) and (b).

Let us take into account the third case, i.e.,  $K(\Gamma)$ , with  $|K(\Gamma)| = \widehat{M}^4$ , contains more than one singular vertex. By using the notations of Proposition 7.1, if  $c$  is one of the colors of singular vertices,  $\rho_c \leq \rho = 1$  obviously implies that  $g_c = 1$  may be assumed to hold, i.e.,  $K(\Gamma)$  contains only one  $c$ -colored singular vertex. Let now  $d$  be the color of another singular vertex; by relation (24),  $d \in \{c - 1, c + 1\}$  follows, while relation (25) implies  $\rho_i = 0$  for all  $i \in \Delta_4 - \{c, d\}$  (i.e.,  $K(\Gamma)$  contains exactly

two singular vertices). Without loss of generality, let us assume  $\rho_{\widehat{c}} = \rho_{\widehat{c+1}} = \rho = 1$  and  $\rho_i = 0$  for all  $i \in \Delta_4 - \{c, c + 1\}$ .

Since  $g_{\widehat{c}, \widehat{c+2}} = 1 + \rho - \rho_{\widehat{c}} - \rho_{\widehat{c+2}} = 1 + 1 - 1 - 0 = 1$ ,  $K(c, c + 2)$  consists of one only edge, i.e.,  $N(c, c + 2)$  is homeomorphic to the cone over  $\text{lkd}(v_c)$ .

On the other hand, by relation (22),  $\rho_{\widehat{c-1}} = 0$  yields  $g_{\widehat{c-2}, \widehat{c-1}, \widehat{c+1}} = g_{\widehat{c-2}, \widehat{c-1}} + g_{\widehat{c+1}, \widehat{c-2}} - 1$ ; hence – via [7, Lemma 5] –  $K(c - 2, c - 1, c + 1)$  is proved to collapse onto  $K(c - 2, c + 1)$ , which consists of exactly one edge (since  $g_{\widehat{c+1}, \widehat{c-2}} = 1 + (\rho - \rho_{\widehat{c-2}}) - \rho_{\widehat{c+1}} = 1 + 1 - 0 - 1 = 1$ ). This proves that  $N(c - 2, c - 1, c + 1)$  is homeomorphic to the cone over  $\text{lkd}(v_{c+1})$ , too.

Thus,  $M^4 \cong \bar{M} \times I$ , where  $\bar{M}$  is a genus one closed 3-manifold (homeomorphic to both  $\text{lkd}(v_c)$  and  $\text{lkd}(v_{c+1})$ ), now easily follows. ■

**Corollary 7.6.** *Let  $\xi_c$  be the  $\mathbb{D}^2$ -bundle over  $S^2$  with Euler class  $c$ , for all  $c \in \mathbb{Z}^+ - \{1\}$ . Then,*

$$\bar{\mathcal{G}}(\xi_c) = \bar{\mathcal{G}}(S^2 \times \mathbb{D}^2) = 2.$$

Moreover,

$$\bar{\mathcal{G}}(\mathbb{Y}_1^4 \# \mathbb{Y}_1^4) = \bar{\mathcal{G}}(\mathbb{Y}_1^4 \# \tilde{\mathbb{Y}}_1^4) = \bar{\mathcal{G}}(\tilde{\mathbb{Y}}_1^4 \# \tilde{\mathbb{Y}}_1^4) = 2.$$

*Proof.* By Proposition 7.5,  $\bar{\mathcal{G}}(\xi_c) \geq 2$ ,  $\bar{\mathcal{G}}(S^2 \times \mathbb{D}^2) \geq 2$ ,  $\bar{\mathcal{G}}(\mathbb{Y}_1^4 \# \mathbb{Y}_1^4) \geq 2$ ,  $\bar{\mathcal{G}}(\mathbb{Y}_1^4 \# \tilde{\mathbb{Y}}_1^4) \geq 2$  and  $\bar{\mathcal{G}}(\tilde{\mathbb{Y}}_1^4 \# \tilde{\mathbb{Y}}_1^4) \geq 2$  trivially follow.

On the other hand, in Section 6 (resp. in Section 5), we have obtained 5-colored graphs with generalized regular genus two representing  $S^2 \times \mathbb{D}^2$  and  $\xi_c$ , for all  $c \in \mathbb{Z}^+ - \{1\}$  (resp. representing  $\mathbb{Y}_1^4 \# \mathbb{Y}_1^4$ ,  $\mathbb{Y}_1^4 \# \tilde{\mathbb{Y}}_1^4$  and  $\tilde{\mathbb{Y}}_1^4 \# \tilde{\mathbb{Y}}_1^4$ ); see Figures 6 and 5 (resp. see Figure 3). Hence, the statement is proved. ■

The results about non-finiteness-to-one of generalized regular genus (already pointed out in Section 1 and in Remark 4.5) now easily follow:

**Corollary 7.7.** (a) *Generalized regular genus is not finite-to-one in dimension four.*

(b) *In dimension four, the equality between regular genus and generalized regular genus of manifolds with boundary does not hold, even if the boundary is assumed to be connected.*

Further results are obtained, concerning compact PL 4-manifolds with generalized regular genus two.

**Proposition 7.8.** *Let  $M^4$  be a compact 4-manifold with empty or connected boundary, with  $\bar{\mathcal{G}}(M^4) = 2$ . Then*

- either  $M^4 \in \{\#_2(S^1 \times S^3), \#_2(S^1 \tilde{\times} S^3), \mathbb{C}P^2\}$ ,
- or  $M^4 \in \{\mathbb{Y}_2^4, \tilde{\mathbb{Y}}_2^4, (S^1 \times S^3) \# \mathbb{Y}_1^4, (S^1 \tilde{\times} S^3) \# \mathbb{Y}_1^4, (S^1 \tilde{\times} S^3) \# \tilde{\mathbb{Y}}_1^4, S^2 \times \mathbb{D}^2, \xi_2\}$ ,

- or  $M^4 \cong M^4(K, d)$ ,  $(K, d)$  being a framed knot such that  $M^3(K, d) = L(\alpha, \beta)$  with  $\alpha \geq 3$ .

*Proof.* Let  $(\Gamma, \gamma)$  be a 5-colored graph representing  $M^4$ , with  $\rho(\Gamma) = \bar{\mathcal{G}}(M^4) = 2$ . Without loss of generality, for sake of simplicity we may assume that the cyclic permutation  $\varepsilon$  of  $\Delta_4$  such that  $\rho(\Gamma) = \rho_\varepsilon(\Gamma)$  is  $\varepsilon = (0, 1, 2, 3, 4)$  and that  $g_{\hat{i}} = 1$  holds for each  $i \in \Delta_4$  (see Remark 2.10).

In virtue of the inequality (24),  $\rho(\Gamma) = 2$  implies that the only possible cases are

- there exists  $i \in \Delta_4$  so that  $\rho_{\hat{i}} = 0$ ;
- $\rho_{\hat{i}} = 1$  for all  $i \in \Delta_4$ .

In the first case, by Proposition 7.3 either  $M^4 \cong \#_\alpha(\mathbb{S}^1 \times \mathbb{S}^3) \# \mathbb{Y}_\beta^4$  or  $M^4 \cong \#_\alpha(\mathbb{S}^1 \tilde{\times} \mathbb{S}^3) \# \mathbb{Y}_\beta^4$  or  $M^4 \cong \#_\alpha(\mathbb{S}^1 \tilde{\times} \mathbb{S}^3) \# \tilde{\mathbb{Y}}_\beta^4$  hold, with  $\alpha, \beta \geq 0$  and  $\alpha + \beta = 2$ . Hence, if  $M^4$  is a closed 4-manifold,  $M^4 \in \{\#_2(\mathbb{S}^1 \times \mathbb{S}^3), \#_2(\mathbb{S}^1 \tilde{\times} \mathbb{S}^3)\}$  follows, while  $M^4 \in \{\mathbb{Y}_2^4, \tilde{\mathbb{Y}}_2^4, (\mathbb{S}^1 \times \mathbb{S}^3) \# \mathbb{Y}_1^4, (\mathbb{S}^1 \tilde{\times} \mathbb{S}^3) \# \mathbb{Y}_1^4, (\mathbb{S}^1 \tilde{\times} \mathbb{S}^3) \# \tilde{\mathbb{Y}}_1^4\}$  follows if  $M^4$  has connected boundary.

Let us now take into account the second case. If  $c$  is the color of the only (possible) singular vertex  $v_c$  of  $K(\Gamma)$  or any color if  $M^4$  is closed, relations (21) and (22) yield

$$g_{\widehat{i-1}, \widehat{i+1}} = 1 + \rho - \rho_{\widehat{i-1}} - \rho_{\widehat{i+1}} = 1 \quad \text{for all } i \in \Delta_4,$$

in particular

$$g_{\widehat{c}, \widehat{c-2}} = 1;$$

and

$$\begin{aligned} g_{\widehat{c-1}, \widehat{c+1}, \widehat{c+2}} &= g_{\widehat{c-1}, \widehat{c+1}} + g_{\widehat{c+1}, \widehat{c+2}} + \rho_{\widehat{c+1}} - 1 \\ &= g_{\widehat{c-1}, \widehat{c+1}} + g_{\widehat{c+1}, \widehat{c+2}} \\ &= 1 + g_{\widehat{c+1}, \widehat{c+2}}. \end{aligned}$$

Hence,  $M^4$  is simply-connected (in virtue of Proposition 4.1) and  $N(c, c-2)$  is the cone over  $\text{lkd}(v_c)$ , while  $K(c-1, c+1, c+2)$  collapses – by arguments already used in [8, Lemma 2] – to two 2-simplices with common boundary, i.e.,  $N(c-1, c+1, c+2)$  is obtained from a 0-handle  $H^{(0)} = \mathbb{D}^4$  by addition of one 2-handle  $H^{(2)}$ , according to a framed knot  $(K, d)$ .

Now, if  $v_c$  is not singular (that is, if  $M^4$  is a closed 4-manifold),  $N(c, c-2)$  is a 4-dimensional disk, and hence – by a well-known theorem in [34] –  $(K, d)$  turns out to be the trivial knot with framing 1. So,  $M^4 \cong \mathbb{C}\mathbb{P}^2$  directly follows.

On the other hand, if  $v_c$  is a singular vertex,  $M^4 \cong M^4(K, d)$  holds,  $(K, d)$  being a framed knot such that  $M^3(K, d)$  has genus one (equal to  $\rho_{\widehat{c}}$ ).

If  $M^3(K, d) \cong \mathbb{S}^1 \times \mathbb{S}^2$ , a classic result of Dehn surgery ensures  $(K, d)$  to be the 0-framed trivial knot (see [30]), i.e.,  $M^4 \cong \mathbb{S}^2 \times \mathbb{D}^2$ . Further, if  $M^3(K, d) \cong L(2, 1)$ , another, more recent, result of Dehn surgery ensures  $(K, d)$  to be the 2-framed trivial knot (see [42]), i.e.,  $M^4 \cong \xi_2$ .

The only remaining cases concern simply-connected 4-manifolds  $M^4(K, d)$  having lens spaces  $L(\alpha, \beta)$ , with  $\alpha \geq 3$ , as boundary. ■

We are now able to prove the theorem, already stated in Section 1, that summarizes the obtained classification results for compact 4-manifolds according to their generalized regular genus.

*Proof of Theorem 1.1.* (a) is nothing but the case  $d = 4$  of Proposition 4.4 (b).

(b) is a direct consequence of Propositions 5.1, 5.3, and 7.5, together with the well-known existence of 5-colored graphs of regular genus one representing the two  $\mathbb{S}^3$ -bundles over  $\mathbb{S}^1$ .

With regard to (c), the result comes directly from Proposition 7.8, since for each  $c \in \mathbb{Z}$ ,  $\xi_c$ , the  $\mathbb{D}^2$ -bundle over  $\mathbb{S}^2$  with Euler class  $c$ , is exactly  $M^4(K_0, c)$ ,  $(K_0, c)$  being the  $c$ -framed trivial knot. ■

**Proposition 7.9.** *Let  $(\Gamma, \gamma)$  be a 5-colored graph with exactly one singular color and with  $\rho(\Gamma) = 2$ . Then, either  $K(\Gamma)$  has exactly one singular vertex (and therefore  $\Gamma$  represents one of the compact 4-manifolds detected in Proposition 7.8), or  $M^4 \in \{\mathbb{Y}_1^4 \# \mathbb{Y}_1^4, \mathbb{Y}_1^4 \# \tilde{\mathbb{Y}}_1^4, \tilde{\mathbb{Y}}_1^4 \# \tilde{\mathbb{Y}}_1^4\}$ .*

*Proof.* Let  $c$  be the singular color of  $\Gamma$  and let  $\varepsilon$  be the cyclic permutation of  $\Delta_4$  such that  $\rho(\Gamma) = \rho_\varepsilon(\Gamma) = 2$ ; further, let us assume (without loss of generality)  $\varepsilon = (0, 1, 2, 3, 4)$  and  $g_i = 1$  for each  $i \in \Delta_4 - \{c\}$  (see Remark 2.10). It is easy to check that, if  $K(\Gamma)$  has more than one singular vertex,  $\Gamma$  may be assumed to have exactly two  $\hat{c}$ -residues, both with regular genus one with respect to the permutation induced by  $\varepsilon$ . Hence,  $\rho_{\hat{c}} = 2$ . Arguments similar to those used in the proof of Proposition 7.3 (b) ensure that  $K(c, c + 2)$  consists of two edges, with a common end-point (i.e., the  $(c + 2)$ -labeled vertex) and with the other end-points consisting in the two singular  $c$ -labeled vertices of  $K(\Gamma)$ ,  $v'_c$  and  $v''_c$ , say. This easily implies that  $N(c, c + 2)$  is homeomorphic to the boundary connected sum of a 4-disk, the cone  $v'_c * \text{lkd}(v'_c)$  and the cone  $v''_c * \text{lkd}(v''_c)$ ; hence, the boundary of  $N(c, c + 2)$  is  $\text{lkd}(v'_c) \# \text{lkd}(v''_c)$ .

On the other hand, formula (21) yields  $\rho_{\widehat{c-2}} = 0$ , and hence  $g_{\widehat{c-2}, \widehat{c-1}, \widehat{c+1}} = g_{\widehat{c-2}, \widehat{c-1}} + g_{\widehat{c+1}, \widehat{c-2}} - 1$ , which implies that  $K(c - 2, c - 1, c + 1)$  collapses to a graph, i.e.,  $N(c - 2, c - 1, c + 1)$  is a handlebody of genus  $m = g_{\widehat{c-1}, \widehat{c+1}} - 1 = 2 - \rho_{\widehat{c-1}} - \rho_{\widehat{c+1}} \leq 2$ , whose boundary is a connected sum of  $m \leq 2$  (orientable or non-orientable) sphere bundles over  $\mathbb{S}^1$ .

Since the boundaries of  $N(c, c + 2)$  and  $N(c - 2, c - 1, c + 1)$  have to be identified, then  $m = 2$  and both  $\text{lkd}(v'_c)$  and  $\text{lkd}(v''_c)$  must be homeomorphic to an orientable or non-orientable sphere bundle over  $S^1$ .

This easily proves that  $M^4 \in \{\mathbb{Y}_1^4 \# \mathbb{Y}_1^4, \mathbb{Y}_1^4 \# \tilde{\mathbb{Y}}_1^4, \tilde{\mathbb{Y}}_1^4 \# \tilde{\mathbb{Y}}_1^4\}$ , according to the orientability of the connected components of the boundary (i.e., of the singularities  $\text{lkd}(v'_c)$  and  $\text{lkd}(v''_c)$ ). ■

**Theorem 7.10.** *Let  $M^4$  be a compact 4-manifold with (non-empty) connected boundary. Then*

$$\bar{\mathcal{G}}(M^4) = \mathcal{G}(\partial M^4) = m \geq 1 \iff M^4 \in \{\mathbb{Y}_m^4, \tilde{\mathbb{Y}}_m^4\}$$

*Proof.* Let  $(\Gamma, \gamma)$  be a 5-colored graph realizing the generalized regular genus of  $M^4$ , with respect to the permutation  $\varepsilon$  of  $\Delta_4$ ; then  $\rho(\Gamma) = \rho_\varepsilon(\Gamma) = m$ . If  $v_c$  is the only singular vertex of  $|K(\Gamma)| = \hat{M}^4$ , then the subgraph  $\Gamma_{\hat{c}}$  represents  $\partial M^4$ , and hence  $\rho_\varepsilon(\Gamma_{\hat{c}}) \geq \mathcal{G}(\partial M^4) = m$ . Since  $\rho_\varepsilon(\Gamma_{\hat{c}}) \leq \rho_\varepsilon(\Gamma)$  holds for any colored graph and for any color  $c$ , the equality  $\rho(\Gamma) = \rho_\varepsilon(\Gamma_{\hat{c}})$  follows.

$M^4 \in \{\mathbb{Y}_m^4, \tilde{\mathbb{Y}}_m^4\}$  is now a direct consequence of Proposition 7.3(b). Proposition 5.1 yields the converse implication of the statement. ■

**Theorem 7.11.** *Let  $M^4$  be a compact 4-manifold with empty or connected boundary. Then*

- a.  $\bar{\mathcal{G}}(M^4) \geq \text{rk}(\pi_1(M^4))$ ;
- b.  $\bar{\mathcal{G}}(M^4) = \text{rk}(\pi_1(M^4)) = \rho$  if and only if
  - if  $M^4$  is orientable,
    - either  $M^4 \cong \#_\rho(S^1 \times S^3)$ ,
    - or  $M^4 \cong \#_\alpha(S^1 \times S^3) \# \mathbb{Y}_\beta^4$  with  $\alpha + \beta = \rho$ ,
  - if  $M^4$  is non-orientable,
    - either  $M^4 \cong \#_\rho(S^1 \tilde{\times} S^3)$ ,
    - or  $M^4 \cong \#_\alpha(S^1 \tilde{\times} S^3) \# \tilde{\mathbb{Y}}_\beta^4$ , or  $M^4 \cong \#_\alpha(S^1 \tilde{\times} S^3) \# \mathbb{Y}_\beta^4$ , again with  $\alpha + \beta = \rho$ ;
- c.  $\bar{\mathcal{G}}(M^4) \neq \text{rk}(\pi_1(M^4)) \implies \bar{\mathcal{G}}(M^4) - \text{rk}(\pi_1(M^4)) \geq 2$ .

*Proof.* The first statement is nothing but a particular case of Proposition 4.2. The proof of that result, in the 4-dimensional setting, yields (via formula (21))

$$\#(X_{ij} - \bar{R}_{ij}) \leq g_{i,j} - (g_i + g_j - 1) = \rho - \rho_i - \rho_j,$$

for any pair  $i, j$  of colors non-consecutive in  $\varepsilon$  and such that both  $\Gamma_i$  and  $\Gamma_j$  represent spheres. Hence, if  $(\Gamma, \gamma)$  is a 5-colored graph realizing the generalized regular genus of  $M^4$ , with respect to the permutation  $\varepsilon$  of  $\Delta_4$  (i.e.,  $\rho_\varepsilon(\Gamma) = \rho = \bar{\mathcal{G}}(M^4)$ ),  $\bar{\mathcal{G}}(M^4) =$

$\text{rk}(\pi_1(M^4))$  trivially implies  $\rho_i = \rho_j = 0$ , while  $\bar{\mathcal{G}}(M^4) - \text{rk}(\pi_1(M^4)) = 1$  trivially implies  $\rho_i + \rho_j \leq 1$ .

The second and third statements now easily follow from Proposition 7.3, since all the represented compact 4-manifolds actually satisfy the equality between the generalized regular genus and the rank of the fundamental group. ■

### 7.2. Classifying with respect to G-degree

In order to face the classification problem for compact 4-manifolds with respect to G-degree, we need a further definition<sup>8</sup> and some preliminary results.

**Definition 7.12.** For each compact  $d$ -manifold  $M$ , its (*generalized*) *gem-complexity* is the non-negative integer  $k(M) = p - 1$ , where  $2p$  is the minimum order of a (regular)  $(d + 1)$ -colored graph representing  $M$ .

**Lemma 7.13.** Let  $(\Gamma, \gamma)$  be an order  $2p$  5-colored graph representing a compact 4-manifold  $M^4$ . Then, for each cyclic permutation  $\varepsilon$  of  $\Delta_4$ ,

$$p = \frac{1}{6}\omega_G(\Gamma) + 2\rho_\varepsilon(\Gamma) - \sum_{i \in \Delta_4} \rho_\varepsilon(\Gamma_{\varepsilon_i}^\wedge) + \sum_{i \in \Delta_4} (g_i - 1) + 1.$$

*Proof.* [14, Theorem 22] gives the following formula for the G-degree of  $\Gamma$ :

$$\omega_G(\Gamma) = 6\left((p - 1) - \sum_{i \in \Delta_4} (g_i - 1) + (\chi(K(\Gamma)) - 2)\right).$$

Hence, the result comes by comparing it with formula (20). ■

**Lemma 7.14.** Let  $(\Gamma, \gamma)$  be an order  $2p$  5-colored graph representing a compact 4-manifold  $M^4$  and satisfying  $g_i = 1$  for each  $i \in \Delta_4$ . Then

$$p = 1 + \frac{\omega_G(\Gamma) - \sum_{i \in \Delta_4} \omega_G(\Gamma_i)}{3}.$$

In particular, if  $M^4$  is the product  $\bar{M} \times I$ ,  $\bar{M}$  being a closed 3-manifold, then

$$\omega_G(\Gamma) \geq 5k(\bar{M}).$$

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<sup>8</sup>Note that Definition 7.12 naturally extends – via graphs representing singular  $d$ -manifolds – an important PL invariant originally defined for closed manifolds. Many significant classification results have been obtained within crystallization theory with respect to it; see, for example, [1, 11] for the dimension 3, [12, 15] for the dimension 4. A classification according to gem-complexity for compact orientable 3-manifolds with toric boundary is obtained in [22].

*Proof.* Since  $g_i = 1$  for all  $i \in \Delta_4$ , [14, Lemma 13] directly yields

$$\omega_G(\Gamma) = 3(p-1) + \sum_{i \in \Delta_4} \omega_G(\Gamma_i),$$

which proves the general statement.

On the other hand, If  $|K(\Gamma)| = \widehat{M}^4$  contains  $h$  ( $1 \leq h \leq 5$ ) singular vertices, labeled with  $h$  different colors,  $\omega_G(\Gamma_{j_s}) \geq 3$  holds for (at least)  $h$  colors  $j_1, \dots, j_h \in \Delta_4$ . Hence,

$$p \leq \frac{\omega_G(\Gamma)}{3} - (h-1)$$

easily follows.

In particular, if  $\Gamma$  represents  $\bar{M} \times I$ , the hypothesis  $g_i = 1$  (for all  $i \in \Delta_4$ ) implies the existence of two colors  $c_1, c_2 \in \Delta_4$  so that both the 4-residues  $\Gamma_{c_1}^{\wedge}$  and  $\Gamma_{c_2}^{\wedge}$  of  $\Gamma$  represent  $\bar{M}$ . Then, for each  $i \in \{1, 2\}$ ,  $\omega_G(\Gamma_{c_i}^{\wedge}) \geq \mathcal{D}_G(\bar{M}) = k(\bar{M}) = \bar{p} - 1$ , with  $\#V(\Gamma) = \#V(\Gamma_{c_i}^{\wedge}) = 2p \geq 2\bar{p}$ . Hence,

$$\omega_G(\Gamma) \geq 3(p-1) + 2\mathcal{D}_G(\bar{M}) \geq 5(\bar{p}-1) = 5\mathcal{D}_G(\bar{M}) = 5k(\bar{M}). \quad \blacksquare$$

As a consequence, we are now able to classify all compact 4-manifolds up to G-degree 18 and – under a suitable condition – up to G-degree 24.

**Proposition 7.15.** *Let  $(\Gamma, \gamma)$  be a 5-colored graph representing a compact 4-manifold  $M^4$ .*

- a. *If  $\omega_G(\Gamma) \in \{0, 6\}$ , then  $M^4 \cong \mathbb{S}^4$ ;*
- b. *if  $\omega_G(\Gamma) = 12$ , then*
  - *either  $M^4 \in \{\mathbb{S}^4, \mathbb{S}^1 \times \mathbb{S}^3, \mathbb{S}^1 \tilde{\times} \mathbb{S}^3\}$*
  - *or  $M^4 \in \{\mathbb{Y}_1^4, \tilde{\mathbb{Y}}_1^4\}$ ;*
- c. *if  $\omega_G(\Gamma) = 18$ , then*
  - *either  $M^4 \in \{\mathbb{S}^4, \mathbb{S}^1 \times \mathbb{S}^3, \mathbb{S}^1 \tilde{\times} \mathbb{S}^3\}$*
  - *or  $M^4 \in \{\mathbb{Y}_1^4, \tilde{\mathbb{Y}}_1^4\}$*
  - *or  $M^4 \in \{L(2, 1) \times I, (\mathbb{S}^1 \times \mathbb{S}^2) \times I, (\mathbb{S}^1 \tilde{\times} \mathbb{S}^2) \times I\}$ .*

*No other 5-colored graph representing a compact 4-manifold exists with  $\omega_G(\Gamma) \leq 23$ . Moreover, if  $(\Gamma, \gamma)$  has one singular color at most*

- d. *if  $\omega_G(\Gamma) = 24$ , then*
  - *either  $M^4 \in \{\mathbb{S}^4, \mathbb{S}^1 \times \mathbb{S}^3, \mathbb{S}^1 \tilde{\times} \mathbb{S}^3, \#_2(\mathbb{S}^1 \times \mathbb{S}^3), \#_2(\mathbb{S}^1 \tilde{\times} \mathbb{S}^3), \mathbb{C}\mathbb{P}^2\}$*
  - *or  $M^4 \in \{\mathbb{Y}_1^4, \tilde{\mathbb{Y}}_1^4, \mathbb{Y}_1^4 \# \mathbb{Y}_1^4, \mathbb{Y}_1^4 \# \tilde{\mathbb{Y}}_1^4, \tilde{\mathbb{Y}}_1^4 \# \tilde{\mathbb{Y}}_1^4, \mathbb{Y}_2^4, \tilde{\mathbb{Y}}_2^4, (\mathbb{S}^1 \times \mathbb{S}^3) \# \mathbb{Y}_1^4, (\mathbb{S}^1 \tilde{\times} \mathbb{S}^3) \# \mathbb{Y}_1^4, (\mathbb{S}^1 \tilde{\times} \mathbb{S}^3) \# \tilde{\mathbb{Y}}_1^4, \mathbb{S}^2 \times \mathbb{D}^2, \xi_2\}$ .*

*Proof.* If  $\Gamma$  is a 5-colored graph representing a compact 4-manifold, then  $\omega_G(\Gamma) \leq 6$  (resp.  $\omega_G(\Gamma) \leq 18$ ) (resp.  $\omega_G(\Gamma) \leq 24$ ) implies, via formula (19) (i.e.,  $\omega_G(\Gamma) = 6(\rho_\varepsilon(\Gamma) + \rho_{\varepsilon'}(\Gamma))$ ),  $(\varepsilon, \varepsilon')$  being an arbitrary pair of associated permutations of  $\Delta_4$ , the existence of a permutation  $\varepsilon$  of  $\Delta_4$  such that  $\rho_\varepsilon(\Gamma) = 0$  (resp.  $\rho_\varepsilon(\Gamma) \leq 1$ ) (resp.  $\rho_\varepsilon(\Gamma) \leq 2$ ).

In case  $\rho(\Gamma) = \min_\varepsilon \rho_\varepsilon(\Gamma) = 0$ ,  $M^4 \cong \mathbb{S}^4$  follows from Proposition 4.4 (b). This proves statement (a).

In case  $\rho(\Gamma) = \min_\varepsilon \rho_\varepsilon(\Gamma) = 1$ , Proposition 7.5 ensures  $\Gamma$  to represent either a closed 4-manifold  $M^4$ , with  $M^4 \in \{\mathbb{S}^4, \mathbb{S}^1 \times \mathbb{S}^3, \mathbb{S}^1 \tilde{\times} \mathbb{S}^3\}$ , or a compact 4-manifold with non-empty boundary  $M^4$ , with  $M^4 \in \{\mathbb{Y}_1^4, \tilde{\mathbb{Y}}_1^4, \bar{M} \times I\}$ ,  $\bar{M}$  being a closed genus one 3-manifold. On the other hand,  $\rho_\varepsilon(\Gamma) \leq 1$  implies, via inequality (15),  $\rho_\varepsilon(\Gamma_{\varepsilon_i}) \leq \rho_\varepsilon(\Gamma) \leq 1$  for each  $i \in \Delta_4$ ; hence, a (possible) sequence of proper 1-dipoles allows to consider the additional assumption  $g_i = 1$  for all  $i \in \Delta_4$  (see Remark 2.10). Now, Lemma 7.14 ensures that, if  $M^4 \cong \bar{M} \times I$ ,  $\bar{M} (\neq \mathbb{S}^4)$  being a closed 3-manifold, then  $\omega_G(\Gamma) \geq 15$  holds. This fact completes the proof of statement (b).

Moreover, Lemma 7.14 proves that, if  $M^4 \cong \bar{M} \times I$  and  $\omega_G(\Gamma) = 18$ , the closed 3-manifold  $\bar{M}$  has gem-complexity  $k(\bar{M}) \leq 3$ . The existing classification of closed 3-manifolds via gem-complexity (see [11]) implies  $\bar{M}$  to be either  $L(2, 1)$  or  $\mathbb{S}^1 \times \mathbb{S}^2$  or  $\mathbb{S}^1 \tilde{\times} \mathbb{S}^2$ . This completes the proof of statement (c).

Let us now take into account the last case  $\omega_G(\Gamma) = 24$ , with  $\rho_\varepsilon(\Gamma) = 2$  for each permutation  $\varepsilon$  of  $\Delta_4$ , and with the additional hypothesis that  $\Gamma$  has one singular color at most. Some subcases occur.

- There exist  $c \in \Delta_4$  and a cyclic permutation  $\varepsilon$  of  $\Delta_4$  such that  $\rho_\varepsilon(\Gamma_{\hat{c}}) = 2$ .

In this case, the identification of the represented compact 4-manifold follows from  $\rho_\varepsilon(\Gamma) = \rho_\varepsilon(\Gamma_{\hat{c}}) = 2$ , via Proposition 7.3 (b) and Proposition 7.9: either  $M^4 \in \{\mathbb{S}^4, \mathbb{S}^1 \times \mathbb{S}^3, \mathbb{S}^1 \tilde{\times} \mathbb{S}^3, \#_2(\mathbb{S}^1 \times \mathbb{S}^3), \#_2(\mathbb{S}^1 \tilde{\times} \mathbb{S}^3)\}$  (in case  $K(\Gamma)$  has no singular vertex), or  $M^4 \in \{\mathbb{Y}_1^4, \tilde{\mathbb{Y}}_1^4, (\mathbb{S}^1 \times \mathbb{S}^3)\# \mathbb{Y}_1^4, (\mathbb{S}^1 \tilde{\times} \mathbb{S}^3)\# \mathbb{Y}_1^4, (\mathbb{S}^1 \tilde{\times} \mathbb{S}^3)\# \tilde{\mathbb{Y}}_1^4, \mathbb{Y}_2^4, \tilde{\mathbb{Y}}_2^4\}$  (in case  $K(\Gamma)$  has exactly one singular vertex), or  $M^4 \in \{\mathbb{Y}_1^4\# \mathbb{Y}_1^4, \mathbb{Y}_1^4\# \tilde{\mathbb{Y}}_1^4, \tilde{\mathbb{Y}}_1^4\# \tilde{\mathbb{Y}}_1^4\}$  (in case  $K(\Gamma)$  has exactly two singular vertices, labeled by the same color).

- For each permutation  $\varepsilon$  of  $\Delta_4$ ,  $\rho_\varepsilon(\Gamma_{\varepsilon_i}) \leq 1$  for all  $i \in \Delta_4$ , and, therefore – via a (possible) sequence of proper 1-dipoles – we may assume  $g_i = 1$  for each  $i \in \Delta_4$ .

If  $M^4$  is closed, the result comes from [14, Propositions 33 and 35].

Otherwise, note that, by Lemma 7.13,  $4 \leq p \leq 8$ ; hence the disjoint links of the singular vertices of  $K(\Gamma)$  can only represent  $L(2, 1)$ ,  $\mathbb{S}^1 \times \mathbb{S}^2$  or  $\mathbb{S}^1 \tilde{\times} \mathbb{S}^2$ , since all other 4-colored graphs with  $p \leq 8$ , not representing  $\mathbb{S}^3$ , admit at least one regular embedding into a surface of genus greater than one. As a consequence, since  $K(\Gamma)$  has exactly one singular vertex (in virtue of the assumption  $\rho_\varepsilon(\Gamma_{\varepsilon_i}) \leq 1$  for all  $i \in \Delta_4$ ),



Proposition 7.8 yields  $M^4 \in \{\mathbb{Y}_1^4, \tilde{\mathbb{Y}}_1^4, \mathbb{Y}_2^4, \tilde{\mathbb{Y}}_2^4, (\mathbb{S}^1 \times \mathbb{S}^3)\#\mathbb{Y}_1^4, (\mathbb{S}^1 \tilde{\times} \mathbb{S}^3)\#\mathbb{Y}_1^4, (\mathbb{S}^1 \tilde{\times} \mathbb{S}^3)\#\tilde{\mathbb{Y}}_1^4, \mathbb{S}^2 \times \mathbb{D}^2, \xi_2\}$ . ■

**Remark 7.16.** As hinted to in Section 1, the G-degree  $\omega_G$  of colored graphs plays an important role within colored tensor models theory, in virtue of the following  $1/N$  expansion of the correlation functions

$$\frac{1}{N^d} \log Z[N, \{t_B\}] = \sum_{\omega_G \geq 0} N^{-\frac{2}{(d-1)!} \omega_G} F_{\omega_G}[\{t_B\}] \in \mathbb{C}[[N^{-1}, \{t_B\}]], \quad (28)$$

where the coefficients  $F_{\omega_G}[\{t_B\}]$  are generating functions of connected bipartite  $(d + 1)$ -colored graphs with fixed G-degree  $\omega_G$ . Moreover, [18, Theorem 1] proves that, if  $d$  even,  $d \geq 4$ , the only non-null terms in the above formula are the ones corresponding to even (integer) powers of  $1/N$ .

Hence, for  $d = 4$ , Proposition 7.15 yields the identification of all compact orientable PL 4-manifolds (resp. compact orientable PL 4-manifolds with empty or connected boundary), represented by regular graphs involved in the first four (resp. five) most significant non-null terms of the  $1/N$  expansion of formula (28).

Proposition 7.15 allows us to prove the second main result of the present paper (already stated in Section 1).

*Proof of Theorem 1.2.* It is well known that the minimal (order ten) crystallizations of  $\mathbb{S}^1 \times \mathbb{S}^3$  and  $\mathbb{S}^1 \tilde{\times} \mathbb{S}^3$  have G-degree 12 (see [14, Corollary 26]). Moreover, in Proposition 5.1  $\mathbb{Y}_1^4$  and  $\tilde{\mathbb{Y}}_1^4$  are proved to have G-degree 12, too.

Then it is easy to check that the 5-colored graphs obtained from the minimal (order eight) crystallizations of  $L(2, 1)$ ,  $\mathbb{S}^1 \times \mathbb{S}^2$  and  $\mathbb{S}^1 \tilde{\times} \mathbb{S}^2$  by applying the procedure described in Proposition 5.3 have G-degree equal to 18 and represent  $L(2, 1) \times I$ ,  $(\mathbb{S}^1 \times \mathbb{S}^2) \times I$  and  $(\mathbb{S}^1 \tilde{\times} \mathbb{S}^2) \times I$  respectively.

Further, the well-known 5-colored graph representing the closed 4-manifolds  $\#_2(\mathbb{S}^1 \times \mathbb{S}^3)$ ,  $\#_2(\mathbb{S}^1 \tilde{\times} \mathbb{S}^3)$  and  $\mathbb{C}\mathbb{P}^2$  have G-degree 24, as well as the 5-colored graphs representing  $\mathbb{Y}_2^4$ ,  $\tilde{\mathbb{Y}}_2^4$ ,  $(\mathbb{S}^1 \times \mathbb{S}^3)\#\mathbb{Y}_1^4$ ,  $(\mathbb{S}^1 \tilde{\times} \mathbb{S}^3)\#\mathbb{Y}_1^4$ , and  $(\mathbb{S}^1 \tilde{\times} \mathbb{S}^3)\#\tilde{\mathbb{Y}}_1^4$  obtained by graph connected sums (see Section 4).

Finally, in Section 6, we have obtained a 5-colored graph  $(\Lambda_0, \lambda_0)$  (resp.  $(\Lambda_2, \lambda_2)$ ) with  $\rho_\varepsilon(\Lambda_0) = \rho_{\varepsilon'}(\Lambda_0) = 2$  (resp.  $\rho_\varepsilon(\Lambda_2) = \rho_{\varepsilon'}(\Lambda_2) = 2$ ), where  $(\varepsilon, \varepsilon')$  is a pair of associated permutations of  $\Delta_4$ ; see Figure 5 (resp. Figure 6). Hence, by formula (19),  $\omega_G(\Lambda_0) = \omega_G(\Lambda_2) = 6(2 + 2) = 24$  follows.

The statement is now a direct consequence of Proposition 7.15. ■

**Corollary 7.17.** *Let  $\xi_c$  be the  $\mathbb{D}^2$ -bundle over  $\mathbb{S}^2$  with Euler class  $c$ , for all  $c \in \mathbb{Z}^+ - \{1\}$ . Then*

$$\mathcal{D}_G(\xi_2) = \mathcal{D}_G(\mathbb{S}^2 \times \mathbb{D}^2) = 24,$$

while

$$30 \leq \mathcal{D}_G(\xi_c) \leq 12c \quad \text{for all } c \in \mathbb{Z}^+ - \{1, 2\}.$$

Moreover,

$$\mathcal{D}_G(\mathbb{Y}_1^4 \# \mathbb{Y}_1^4) = \mathcal{D}_G(\mathbb{Y}_1^4 \# \tilde{\mathbb{Y}}_1^4) = \mathcal{D}_G(\tilde{\mathbb{Y}}_1^4 \# \tilde{\mathbb{Y}}_1^4) = 24.$$

*Proof.* The statements concerning  $\xi_c$ , for all  $c \in \mathbb{Z}^+ - \{1\}$ , and  $\mathbb{S}^2 \times \mathbb{D}^2$  (resp.  $\mathbb{Y}_1^4 \# \mathbb{Y}_1^4$ ,  $\mathbb{Y}_1^4 \# \tilde{\mathbb{Y}}_1^4$  and  $\tilde{\mathbb{Y}}_1^4 \# \tilde{\mathbb{Y}}_1^4$ ) are trivial consequence of Theorem 1.2, together with the constructions presented in Section 6 (resp. in Section 5).

Alternatively,

$$\mathcal{D}_G(\xi_2) = \mathcal{D}_G(\mathbb{S}^2 \times \mathbb{D}^2) = \mathcal{D}_G(\mathbb{Y}_1^4 \# \mathbb{Y}_1^4) = \mathcal{D}_G(\mathbb{Y}_1^4 \# \tilde{\mathbb{Y}}_1^4) = \mathcal{D}_G(\tilde{\mathbb{Y}}_1^4 \# \tilde{\mathbb{Y}}_1^4) = 24$$

could also be proved directly from the computation of their generalized regular genus, performed in Corollary 7.6, by making use of the relation  $\mathcal{D}_G(N) \geq \frac{d!}{2} \cdot \bar{\mathcal{G}}(N)$  (recalled in Section 2). ■

**Conjecture 7.18.** *Let  $\xi_c$  be the  $\mathbb{D}^2$ -bundle over  $\mathbb{S}^2$  with Euler class  $c$ . Then*

$$\mathcal{D}_G(\xi_c) = 12c \quad \text{for all } c \in \mathbb{Z}^+ - \{1, 2\}.$$

The general computation of the G-degree of the products with the interval, performed in Section 4, gives rise in dimension 4 to the following result.

**Proposition 7.19.** *For each closed 3-manifold  $M$ ,*

$$\mathcal{D}_G(M \times I) \leq 6 \cdot \mathcal{D}_G(M).$$

*Proof.* If  $(\Gamma, \gamma)$  is a crystallization of  $M$  realizing gem-complexity (i.e.,  $\#V(\Gamma) = 2\bar{p}$ , where  $k(M) = \bar{p} - 1$ ), it is well known that  $\mathcal{D}_G(M) = \omega_G(\Gamma) = \bar{p} - 1$  ([14]). If  $\tilde{\Gamma}$  is the 5-colored graph representing  $M \times I$  considered in Proposition 5.3, the last formula of Proposition 5.3 becomes

$$\omega_G(\tilde{\Gamma}) = 3 \left[ \sum_{i \in \{1, 2, 3\}} (\bar{p} - g_{\varepsilon_0 \varepsilon_i}) - (\bar{p} - 1) + \omega_G(\Gamma) \right],$$

$\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)$  being the cyclic permutation of  $\Delta_3$  so that  $\rho(\Gamma) = \rho_\varepsilon(\Gamma)$ . On the other hand,  $\sum_{i \in \{1, 2, 3\}} g_{\varepsilon_0 \varepsilon_i} = g_{\varepsilon_0 \varepsilon_1} + g_{\varepsilon_0 \varepsilon_2} + g_{\varepsilon_0 \varepsilon_3} = g_{\varepsilon_0 \varepsilon_1} + g_{\varepsilon_0 \varepsilon_2} + g_{\varepsilon_1 \varepsilon_2} = 2 - 2\rho(\Gamma_{\varepsilon_3}) + \bar{p} = 2 + \bar{p}$ , since both  $g_{i,j} = g_{\hat{i}\hat{j}}$  and  $\rho(\Gamma_{\hat{i}}) = 0$  (for all  $i, j \in \Delta_3$ ) hold in any crystallization of a closed 3-manifold. Hence,

$$\omega_G(\tilde{\Gamma}) = 3[3\bar{p} - (2 + \bar{p}) - (\bar{p} - 1) + \omega_G(\Gamma)] = 6 \cdot (\bar{p} - 1) = 6 \cdot \mathcal{D}_G(M).$$

The claim now trivially follows. ■

**Corollary 7.20.**  $\mathcal{D}_G(L(3, 1) \times I) = 30$ .

*Proof.* Since  $\mathcal{D}_G(L(3, 1)) = k(L(3, 1)) = 5$  (see [11] and [14, Theorem 16]) and  $\mathcal{D}_G(N) \equiv 0 \pmod{6}$  for each singular 4-manifold (see [18] or formula (19)), the statement is a trivial consequence of Proposition 7.19, together with Theorem 1.2 (or Lemma 7.14). ■

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