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Lozenge tilings of hexagons with removed core and satellites

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Abstract. We consider regions obtained from 120 degree rotationally invariant hexagons by removing a core and three equal satellites (all equilateral triangles) so that the resulting region is both vertically symmetric and 120 degree rotationally invariant, and give simple product formulas for the number of their lozenge tilings. We describe a new method of approach for proving these formulas, and give the full details for an illustrative special case. As a byproduct, we are also able to generalize this special case in a different direction, by finding a natural counterpart of a twenty year old formula due to Ciucu, Eisenkölbl, Krattenthaler, and Zare, which went unnoticed until now. The general case of the original problem will be treated in a subsequent paper. We then work out consequences for the correlation of holes, which were the original motivation for this study.

Contents

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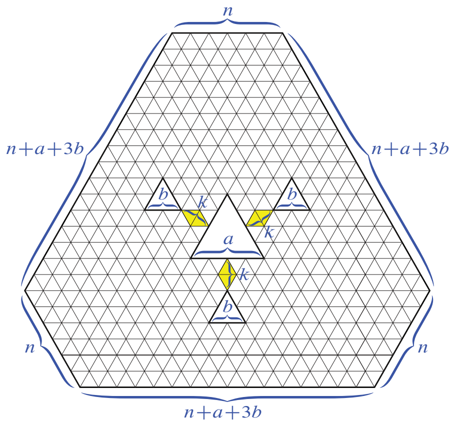

Figure 1. The region $S_{n,a,b,k}$ (left) and $S'_{n,a,b,k}$ (right) for $n = 6$, $a = 4$, $b = 2$, $k = 1$.

1. Introduction

The fact that not only the number of plane partitions that fit in a box (equivalently, lozenge tilings[1] of a hexagon), but also all the symmetry classes (a total of ten) are given by simple product formulas, is of singular beauty in enumerative combinatorics.[2] This has been a rich source of inspiration for many researchers over the last four decades. Just to skim the surface, we mention [1, 4, 36, 42, 47, 48] and the survey [39] for more recent developments. Works of the first author inspired by this include [7–11, 15, 20, 23]. Probabilistic aspects were studied by Cohn, Larsen, and Propp [25], Borodin, Gorin and Rains [3], and Bodini, Fusy, and Pivoteau [2]. Another extension was given by Vuletić [49].

In this paper we consider regions obtained from 120 degree rotationally invariant hexagons by removing a core and three equal satellites (all equilateral triangles) so that the resulting region is both vertically symmetric and 120 degree rotationally

[1] A lozenge is the union of two adjacent unit triangles on the triangular lattice; a lozenge tiling of a lattice region R is a covering of R by lozenges that has no gaps or overlaps.

[2] To specify just one of them, MacMahon proved [45] – in an equivalent formulation – that the number of lozenge tilings of a hexagon of side lengths a, b, c, a, b, c (in cyclic order) on the triangular lattice is equal to

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}.$$

The other nine are only somewhat more complicated.

invariant (Figure 1 shows the two types of regions that are obtained; see Section 2 for the precise definitions), and give simple product formulas for the number of their lozenge tilings.

The reader may find interesting the account of how these regions were found.

The special case of these regions when the core is empty was discovered by the first author in 1999, when he noticed that the number of its lozenge tilings seems to always factor fully into relatively small prime factors (such integers are sometimes referred to as "round").

This seemed a very hard result to prove (indeed, even guessing the precise product formula seemed exceedingly hard). Using the Lindström–Gessel–Viennot theorem [30, 44] it is clearly possible to derive a determinant for the number of lozenge tilings in the case of even size satellites, but Krattenthaler's identification of factors method for evaluating determinants (see a brief description in the fourth to last paragraph of Section 2), which had proved successful on many occasions before (see, e.g., [18, 19, 21, 22, 39]) was not applicable due to the lack of a polynomial parameter. Furthermore, it was the odd size that interested the first author most. The reason had to do with [8], where he discovered that the distribution of gaps in random lozenge tilings is governed by Coulomb's law of two-dimensional electrostatics: [8] could handle a multitude of even holes but only a single odd hole, and in order to support the conjecture that 2D Coulomb governs the distribution of holes for arbitrary holes (this conjecture was published in [9]) it was desirable to have an example with three odd holes; the fact that they were not collinear made this instance especially interesting. Having an exact, simple product formula for the number of tilings of the hexagon with three holes, the correlation of the holes can be worked out and its asymptotics determined, confirming thus the above mentioned "electrostatic conjecture."

The first author mentioned this observation to Christian Krattenthaler in 2003, and considered briefly a project to attempt proving it, but the project was abandoned due to the above mentioned complications and limitations.

An important step ahead was achieved in 2010, when the first author noticed that the round factorization persists if a fourth hole is added at the center. The reason this is so helpful is because it introduces a new parameter in the data, and the counts can be proved to be polynomials in this parameter. Then the data is not just integers that factor into relatively small (but otherwise mysterious) prime factors, but polynomials in the new parameter that factor fully into linear factors. This also gives a more objective measure of "roundness" than just factorization of integers into relatively small prime factors.

While the first author was working on enumerating the tilings of these regions in 2014, he showed them to Tri Lai (he was the first author's Ph.D. student at the time), who then in 2017 co-wrote the paper [43] which involves these regions. To be precise, [43] focuses on counting the lozenge tilings of these regions which are invariant under rotation by 120°, and also of those which are both vertically symmetric and invariant under this rotation (these follow, after a considerable amount of work, by applying the factorization theorem [6] and Kuo condensation [40, 41]; see [16, 17] for earlier examples). The straight count of lozenge tilings is not mentioned in [43]. However, it turns out (see Conjecture 2.1 below) that the straight count and the 120°-rotationally-invariant count are very closely related!

We are now finally presenting ourselves these regions found many years ago by the first author, and our work on the problem of counting their lozenge tilings, a question that seems singularly hard in the circle of lozenge tiling problems. This is due to a large extent to the fact that it does not seem possible to extend this family so as to obtain a proof by applying Kuo's graphical condensation method, and also that it does not seem possible to deduce it from other results using standard combinatorial arguments.

It seemed very difficult even to find an explicit conjectural formula for the number of lozenge tilings of these regions, even with the great help that the extra parameter (the size of the core) brought in. The second author succeeded in finding one in 2016, and this is how this collaboration began.

2. Statement of main results and conjectures

The regions we present in this paper are hexagons on the triangular lattice[3] with one central and three satellite up-pointing triangular holes[4] so that

i. the hexagon with holes is both vertically symmetric and 120 degree rotationally invariant, and
ii. the gap between each satellite and the core can be bridged by a string of whole lozenges lined up along their long diagonals.

This common description leads to two families of different-looking regions, depending on whether the satellites point towards or away from the core. In the former case, condition (ii) above amounts to the requirement that the side-length of the core is even (see the picture on the left in Figure 1 for an illustration), while in the latter

[3]Throughout this paper, with the exception of Section 3, we draw the triangular lattice so that one family of lattice lines is horizontal.

[4]The case when the central hole is empty comes up several times in the paper. Besides being the first case that was discovered (see the first paragraph on the history of the problem in the Introduction), it also yields a hole arrangement that is a special case of the bowtie triads studied in [24]. This allows us to extend results we obtain for equal triangular satellites to bowtie shaped satellites that are not necessarily equal (see Theorems 2.5 and 2.6).

the side-length of the satellites is required to be even (an example of this is shown on the right in Figure 1).

Assume therefore that n, a, b and k are non-negative integers with a even, and define $S_{n,a,b,k}$ to be the region obtained from the hexagon $H_{n,n+a+3b}$ of side-lengths[5] n, $n+a+3b$, n, $n+a+3b$, n, $n+a+3b$ (clockwise from top) by removing a triangle of side a from its center and three satellite triangular holes, each of side b, as indicated on the left in Figure 1 (we emphasize that k is the length of a chain of lozenges that would bridge the gap between each satellite and the core; there are $2k$ lattice spacings between a satellite and the core). For non-negative integers n, a, b and k with b even, define $S'_{n,a,b,k}$ to be the region obtained from the same hexagon $H_{n,n+a+3b}$ by removing a triangle of side a from its center and three satellite triangular holes of side b as indicated on the right in Figure 1 (k has the same significance as in the picture on the left in that figure). For the first case, one must have $k \leq n/2$ in order for the satellites to be contained in the region.

Our original interest in these regions (and indeed the reason we found them) came from discovering (see [8]) that for quite general distributions of even[6] triangular holes around the center of a very large hexagon, the number of lozenge tilings of the hexagon with holes varies with the position of the holes precisely[7] as the exponential of the negative of the 2D electrostatic potential of a naturally corresponding system of electrical charges. This striking observation lended itself to generalization. We needed an example involving non-collinear holes of odd side-lengths, for which we could work out the needed asymptotics.

From this point of view, the more interesting family for us is $S_{n,a,b,k}$, as it can have three non-collinear odd charges ($S'_{n,a,b,k}$ can have at most one odd charge, a case already covered by [8]). The formula for the number of tilings of $S_{n,a,b,k}$ can then be used to determine the asymptotics of the correlation (see (2.12) for its definition) of the system of its four holes, providing thus the first example in the literature involving large non-collinear odd holes; we work this out in Theorem 2.2 below (collinear holes of arbitrary size on the square lattice were treated in [12, 13], and unit holes of arbitrary positions on the square lattice in [26]; see also [10] for arbitrary holes

[5]The form of these lengths is required in order for the obtained region to be balanced (i.e., have the same number of up- and down-pointing unit triangles), a necessary condition for the existence of lozenge tilings. Indeed, in a lattice hexagon of side-lengths n, $n+\alpha$, n, $n+\alpha$, n, $n+\alpha$ (clockwise from top) there are α more up-pointing than down-pointing unit triangles, and in the union of the four holes this excess is $a+3b$.

[6]With the exception of one, which could be odd.

[7]In the double limit as first the enclosing hexagon becomes infinite, and then the separation between the holes approaches infinity.

of side two on the triangular lattice, and the extension [14] to weighted doubly periodic planar bipartite lattices in the liquid phase of the Kenyon–Okounkov–Sheffield classification [35] of the dimer models).

We therefore focus in this paper on the regions $S_{n,a,b,k}$. Analogous results to the ones we present below exist also for the regions $S'_{n,a,b,k}$, but due to the involved nature of the arguments and the fact that the $S_{n,a,b,k}$'s already provide us with the asymptotics we were after in the first place, we do not present them here.

Throughout this paper we define products according to the convention

$$\prod_{k=m}^{n-1} \mathrm{Expr}(k) = \begin{cases} \prod_{k=m}^{n-1} \mathrm{Expr}(k) & \text{if } n > m, \\ 1 & \text{if } n = m, \\ \dfrac{1}{\prod_{k=n}^{m-1} \mathrm{Expr}(k)} & \text{if } n < m. \end{cases} \tag{2.1}$$

We recall that the Pochhammer symbol $(\alpha)_k$ is defined for any integer k to be

$$(\alpha)_k := \prod_{i=0}^{k-1} (\alpha + i),$$

thus according to (2.1)

$$(\alpha)_k := \begin{cases} \alpha(\alpha+1)\ldots(\alpha+k-1) & \text{if } k > 0, \\ 1 & \text{if } k = 0, \\ 1/((\alpha-1)(\alpha-2)\ldots(\alpha+k)) & \text{if } k < 0. \end{cases} \tag{2.2}$$

For half-integers k, define the Pochhammer symbol $(\alpha)_k$ by

$$(\alpha)_k := \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}. \tag{2.3}$$

Denote by $\mathrm{M}(R)$ the number of lozenge tilings of the region R on the triangular lattice, and by $\mathrm{M}_r(R)$ the number of its lozenge tilings that are invariant under rotation by 120 degrees.

Our main goal in this paper is to find a formula for $\mathrm{M}(S_{n,a,b,k})$. When $k = 0$, the three satellites touch the core, and due to forced lozenges, removing the core and the three satellites is equivalent (as far as counting lozenge tilings of the leftover region) to removing just a larger core, of side $a + 3b$. Therefore, the case $k = 0$ follows by the main result of [18, Theorem 1].

There is a simple relationship between $\mathrm{M}(S_{n,a,b,k})$ and $\mathrm{M}_r(S_{n,a,b,k})$, the number of lozenge tilings of the region $S_{n,a,b,k}$ which are invariant under rotation by 120 degrees. A formula for the latter was proved by Lai and Rohatgi in [43]. However, in Theorem 2.3 we provide a (rather radical) rewriting of their formula,[8] which has

[8]In the case when n is even; a similar rewriting holds for n odd, but we do not need it in this paper.

the advantage that it works for both even and odd satellite sizes (the original formulas were very different in the two cases; compare [43, (2.9) and (2.11)]).

The simplest way to express our formula for $\mathrm{M}(S_{n,a,b,k})$ is to introduce the *normalized counts* $\overline{\mathrm{M}}(S_{n,a,b,k})$ and $\overline{\mathrm{M}}_r(S_{n,a,b,k})$ as follows:

$$\overline{\mathrm{M}}(S_{n,a,b,k}) := \frac{\mathrm{M}(S_{n,a,b,k})}{\mathrm{M}(S_{n,a,b,0})}, \tag{2.4}$$

$$\overline{\mathrm{M}}_r(S_{n,a,b,k}) := \frac{\mathrm{M}_r(S_{n,a,b,k})}{\mathrm{M}_r(S_{n,a,b,0})}. \tag{2.5}$$

Then our formula for $\mathrm{M}(S_{n,a,b,k})$ follows (using also the three paragraphs above) from the following conjecture.

Conjecture 2.1. *For non-negative integers n, a, b and k with a even we have*

$$\frac{\overline{\mathrm{M}}(S_{n,a,b,k})}{\overline{\mathrm{M}}_r(S_{n,a,b,k})^3} = \left[\prod_{i=1}^{k} \frac{(a+6i-4)(a+3b+6i-2)}{(a+6i-2)(a+3b+6i-4)}\right]^2. \tag{2.6}$$

While this is still strictly speaking a conjecture, we mention that we do have a new approach to tackle it, which we are confident that will lead to a proof. We describe in Sections 4 and 5 this new method, which uses the identification of factors method on a particularly convenient determinant, and give the details of the proof for the special case of Conjecture 2.1 when $b = 0$. This case corresponds to the cored hexagons treated in [18]. As a byproduct, we are able to deal with a different generalization of this special case, which leads us to a new family of regions whose number of lozenge tilings is expressible by a product formula; see Theorem 5.1 in Section 5 (in fact, using results from [24], this can be generalized; see Remark 5.1 in Section 5). The relative briefness of the proof we present here compared to the original proof in [18] illustrates the advantages of our new method. Some details still need to be worked out for the proof of the general case (when b is not necessarily zero), which will be presented in a subsequent paper.

Remark 2.1. It is amusing to note that the product on the right-hand side of (2.6) can be written as

$$\frac{\dfrac{\left(\frac{a}{6}+\frac{1}{3}\right)_k}{\left(\frac{a}{6}+\frac{2}{3}\right)_k}}{\dfrac{\left(\frac{a+3b}{6}+\frac{1}{3}\right)_k}{\left(\frac{a+3b}{6}+\frac{2}{3}\right)_k}},$$

and that a is the size of the core, while $a + 3b$ is the size of the enlarged core arising in the special case $k = 0$. It is remarkable that this ratio does not depend on n.

The relationship between the un-normalized counts $\mathrm{M}(S_{n,a,b,k})$ and $\mathrm{M}_r(S_{n,a,b,k})$ is detailed in the following equivalent restatement of Conjecture 2.1.

Conjecture 2.2. *For non-negative integers n, a, b and k with a even we have*

$$\frac{\mathrm{M}(S_{n,a,b,k})}{\mathrm{M}_r(S_{n,a,b,k})^3} = \prod_{i=1}^{k}\left[\frac{(a+6i-4)(a+3b+6i-2)}{(a+6i-2)(a+3b+6i-4)}\right]^2 \times \begin{cases} \mathcal{E} & \text{if } n \text{ is even,} \\ \mathcal{O} & \text{if } n \text{ is odd,} \end{cases} \tag{2.7}$$

where

$$\mathcal{E} := \left[\frac{(a+3b+2)_{n/4,6}(a+3b+3n/2+1)_{n/4,6}}{(a+3b+4)_{n/4,6}(a+3b+3n/2+5)_{n/4,6}}\right]^2,$$

$$\mathcal{O} := \frac{(a+3b+3n+2)^2}{4(a+3b+3(n+1)/2-1)^2} \times \left[\frac{(a+3b+2)_{(n+1)/4,6}(a+3b+3(n+1)/2+1)_{(n+1)/4,6}}{(a+3b+4)_{(n+1)/4,6}(a+3b+3(n+1)/2+5)_{(n+1)/4,6}}\right]^2,$$

$$(\alpha)_{k,m} := m^k\left(\frac{\alpha}{m}\right)_k,$$

and the half-integer index Pochhammer symbols are defined by (2.3).

We note that when the parameter n is even in the above formula, it simplifies (writing the n- and a-parameters as $2n$ and $2a$, to spell out their evenness) to

$$\frac{\mathrm{M}(S_{2n,2a,b,k})}{\mathrm{M}_r(S_{2n,2a,b,k})^3} = \left[\frac{\left(\frac{a}{3}+\frac{1}{3}\right)_k\left(\frac{a}{3}+\frac{b}{2}+k+\frac{1}{3}\right)_{n/2-k}\left(\frac{a}{3}+\frac{b}{2}+\frac{n}{2}+\frac{1}{6}\right)_{n/2}}{\left(\frac{a}{3}+\frac{2}{3}\right)_k\left(\frac{a}{3}+\frac{b}{2}+k+\frac{2}{3}\right)_{n/2-k}\left(\frac{a}{3}+\frac{b}{2}+\frac{n}{2}+\frac{5}{6}\right)_{n/2}}\right]^2. \tag{2.8}$$

Remark 2.2. In the special case when $a=b=0$ – when our region becomes a regular hexagon H_n of side n – the two branches of the formula in Conjecture 2.2 unify to give

$$\frac{\mathrm{M}(H_n)}{\mathrm{M}_r(H_n)^3} = \left[\frac{\left(\frac{1}{3}\right)_n}{\left(\frac{2}{3}\right)_n}\right]^2, \tag{2.9}$$

a result that readily follows from the well-known formulas for symmetry classes of plane partitions (compare [47, Cases 1 and 3]).

Throughout the asymptotic analysis, we will focus on the case when the parameter n is even. This will help keep its length manageable, while capturing the details of the asymptotics of our formulas. Analogous results exist for odd n.

We present now our (rather radical) rewriting of the formulas for $\mathrm{M}_r(S_{n,a,b,k})$ found by Lai and Rohatgi [43] (in line with the previous paragraph, we only treat here the case when the n-parameter is even; see also footnote 8). The new form has the advantage that it works for both even and odd satellite sizes (the original formulas were quite different in the two cases;[9] compare the form of [43, equations (2.9) and (2.11)]).

We emphasize that products with index limits out of order are to be interpreted according to the formula we presented at the beginning of Section 2.

Theorem 2.3 ([43]). *Let n, a, b and k be non-negative integers. For even n we have*

$$\begin{aligned}\mathrm{M}_r(S_{2n,2a,b,k}) = {} & \frac{\left(\frac{a}{2}+\frac{k}{2}+\frac{1}{2}\right)_k\left(a+2n+\frac{3b}{2}+\frac{1}{2}\right)_n}{2^{n^2-n-k^2-k}\left(\frac{b}{2}+n-k+\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_{n-k}}\\ & \times\left(\frac{a}{2}+\frac{b}{2}+\frac{k}{2}+\frac{1}{2}\right)_k\left(\frac{a}{2}+b+n-\frac{k}{2}+\frac{1}{2}\right)_k\\ & \times\left(\frac{a}{2}+\frac{b}{2}+n-\frac{k}{2}+\frac{1}{2}\right)_k\\ & \times\Bigg[\prod_{i=1}^{n/2-k}\left(a+\frac{3b}{2}+3k+2i\right)_i\left(a+\frac{3b}{2}+3k+2i-1\right)_{i-1}\\ & \quad\times\prod_{i=1}^{k}\left(\frac{a}{2}+\frac{i}{2}\right)_{i-1}\prod_{i=1}^{n/2-1}\left(a+\frac{3b}{2}+\frac{3n}{2}+i+\frac{1}{2}\right)_{2i}\\ & \quad\times\prod_{i=1}^{n-k-1}\frac{1}{\left(\frac{1}{2}\right)_i}\prod_{i=1}^{k}\mathcal{F}_i\Bigg]^2,\end{aligned} \tag{2.10}$$

where

$$\mathcal{F}_i := \frac{(a+b+2i+k)_{n-i-k}(a+b+2n+k-2i+2)_{b-2k+4i-3}}{\left(\frac{1}{2}\right)_i(2i)_{b-1}\left(i+\frac{b-1}{2}\right)_{n-k}},$$

[9] It is true that for even (resp., odd) satellite size, [43, (2.9) (resp., (2.11))] holds for all values of n in $S_{2n,2a,b,k}$, while the forms given here in equations (2.10) and (2.11) for n even or odd, although nearly identical, are slightly different.

while for odd n we have

$$\mathrm{M}_r(S_{2n,2a,b,k}) = \frac{\left(\frac{a}{2}+\frac{k}{2}+\frac{1}{2}\right)_k \left(a+2n+\frac{3b}{2}+\frac{1}{2}\right)_n}{2^{n^2-n-k^2-k}\left(\frac{b}{2}+n-k+\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{n-k}}$$
$$\times \left(\frac{a}{2}+\frac{b}{2}+\frac{k}{2}+\frac{1}{2}\right)_k \left(\frac{a}{2}+b+n-\frac{k}{2}+\frac{1}{2}\right)_k$$
$$\times \left(\frac{a}{2}+\frac{b}{2}+n-\frac{k}{2}+\frac{1}{2}\right)_k$$
$$\times \Bigg[\prod_{i=1}^{(n-1)/2-k} \left(a+\frac{3b}{2}+3k+2i\right)_i \left(a+\frac{3b}{2}+3k+2i+1\right)_i$$
$$\times \prod_{i=1}^{k}\left(\frac{a}{2}+\frac{i}{2}\right)_{i-1} \prod_{i=0}^{(n-1)/2-1} \left(a+\frac{3b}{2}+\frac{3n}{2}+i+1\right)_{2i+1}$$
$$\times \prod_{i=1}^{n-k-1}\frac{1}{\left(\frac{1}{2}\right)_i}\prod_{i=1}^{k}\mathfrak{F}_i\Bigg]^2. \tag{2.11}$$

Proof. Our region $S_{2n,2a,b,k}$ is, in the notation of [43], the region $\mathcal{H}_{2n,k}(b,2a)$. Formula [43, (2.9)] expresses $\mathrm{M}_r(\mathcal{H}_{2t,y}(2a,2x))$ as a power of two times the product of two specific products of linear factors (see the expressions for P_1 and P_2 in [43, (2.4)–(2.5)]). For even b, this supplies an expression for $\mathrm{M}_r(S_{2n,2a,b,k}) = \mathrm{M}_r(\mathcal{H}_{2n,k}(b,2a))$ as a ratio of products of linear factors. In order to prove part (a), we need to check that, for even n, the resulting expression agrees with (2.10), and for odd n it agrees with (2.11) – both of which are also ratios of products of linear factors. A straightforward (if lengthy) manipulation verifies this.

The case of odd b follows similarly, using formula [43, (2.11)] and the product expressions F_1 and F_2 in [43, (2.6)–(2.7)]. ■

Remark 2.3. It is worth mentioning that originally we worked out the above formulas in the case when b is odd, patterned on the product formula for $\mathrm{M}(S_{n,a,b,k})$ that we discovered; it was this that led us to the expressions in (2.10) and (2.11). An interesting feature of these formulas (which [43, (2.9) and (2.11)] do not possess) is that the expressions on their right-hand sides are defined also for even b. It is most remarkable – given how different equation [43, (2.9)] (which corresponds to even b) is from [43, (2.11)] (which corresponds to odd b) – that for even values of b, the expressions on the right-hand sides of (2.10) and (2.11) above still give the correct number of 120°-rotationally-invariant tilings of $S_{2n,a,b,k}$.

To state our next result, we need to define the correlation of holes in a sea of dimers. We define the correlation of the core and the three satellites to be

$$\omega(a,b,k) := \lim_{n\to\infty} \overline{\mathrm{M}}(S_{2n,a,b,k}) = \lim_{n\to\infty} \frac{\mathrm{M}(S_{2n,a,b,k})}{\mathrm{M}(S_{2n,a,b,0})}. \tag{2.12}$$

The original motivation for our work on the correlation of gaps in dimer systems was Fisher and Stephenson's [28] conjecture that the correlation of two monomers on the square lattice is rotationally invariant in the scaling limit. In earlier work [8, 10] we phrased this problem on the hexagonal lattice and generalized it, allowing any finite number number of gaps of any size, and proved that for fairly general types of gaps the asymptotics of their correlation is governed in the scaling limit by the Coulomb energy of a two-dimensional system of electric charges that corresponds naturally to the holes. In fact, we conjectured [9] that this holds for *all* types of gaps. More precisely, for any gaps $O_1, \dots, O_n$, we conjectured that their joint correlation $\omega(O_1, \dots, O_n)$ (a variant of definition (2.12), see Section 3 for details) has asymptotics

$$\omega(O_1, \dots, O_n) \sim C \prod_{1\le i<j\le n} \mathrm{d}(O_i, O_j)^{\frac{1}{2}\mathrm{q}(O_i)\mathrm{q}(O_j)} \tag{2.13}$$

in the limit of large mutual distances between the gaps, where the *charge* $\mathrm{q}(O)$ of the gap O equals the number of up-pointing unit holes in O minus the number of down-pointing unit holes in O, and the multiplicative constant C depends only on the shapes of the gaps. We refer to (2.13) as the *electrostatic conjecture*; it is made precise in Conjecture 3.1 in Section 3.

We recall that the Barnes G-function $G(z)$ is defined for complex z to be

$$G(z+1) = (2\pi)^{z/2} \exp\Big(-\frac{z + z^2(1+\gamma)}{2}\Big) \prod_{k=1}^{\infty} \Big\{\Big(1+\frac{z}{k}\Big)^k \exp\Big(\frac{z^2}{2k} - z\Big)\Big\}, \tag{2.14}$$

where γ is Euler's constant.

In fact, since in Theorem 2.4 below the argument of $G(z)$ is always either a non-negative integer or a non-negative half-integer, it will be enough for us to know the values of G at such values.

The function $G(z)$ satisfies the recurrence

$$G(z+1) = \Gamma(z)G(z), \tag{2.15}$$

and thus for non-negative integers n it is given by

$$G(n) = \prod_{i=0}^{n-2} i! \tag{2.16}$$

(we note that for $0 \leq n \leq 1$, when the product limits are out of order, we use the general convention (2.1) to obtain $G(0) = 0$ and $G(1) = 1$).

On the other hand, by the recurrence (2.15), we have

$$G\left(n + \frac{1}{2}\right) = G\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)\dots\Gamma\left(n - \frac{1}{2}\right). \tag{2.17}$$

All the values we need are then specified by the known fact (see, e.g., [27, Section 2.15, p. 136]) that

$$G\left(\frac{1}{2}\right) = \frac{e^{1/8}2^{1/24}}{A^{3/2}\pi^{1/4}}, \tag{2.18}$$

where A is the Glaisher–Kinkelin constant.[10] It is interesting that our results would lead one to guess this very value for $G(1/2)$, had it not already been known (see Remark 2.5 for a detailed explanation).

We are now ready to state the asymptotic result for which our hexagonal regions with four holes were designed.

Theorem 2.4. *Assuming that Conjecture 2.1 holds, for non-negative integers a, b, and k with a even we have*

$$\omega(a,b,k) \sim \left[\frac{G\left(\frac{a}{2}+1\right)}{G\left(\frac{a}{2}+\frac{3b}{2}+1\right)}\right]^2 \left\{3^{b^2/4}G\left(\frac{b}{2}+1\right)^2 k^{b(a+b)/2}\right\}^3, \quad k \to \infty, \tag{2.20}$$

where G is the Barnes G-function.

Remark 2.4. This proves the electrostatic conjecture (2.13) mentioned above for the system of holes consisting of the core and the three satellites, achieving this way the original motivating goal of this work (see also the equivalent form (3.7) of (2.20)). We discuss in detail in Section 3 the strong evidence (2.20) provides for Conjecture 3.1 in this special case.

Remark 2.5. We note that since a is even, when b is also even the values of the Barnes G-function in (2.20) are simply given by (2.16). It is most remarkable that formula (2.20) holds also for odd b, when the right-hand side involves the fourth power of the complicated constant (2.18).

---

[10]The Glaisher–Kinkelin constant (see [31]) is the value A for which

$$\lim_{n\to\infty} \frac{0!1!\dots(n-1)!}{n^{\frac{n^2}{2}-\frac{1}{12}}(2\pi)^{\frac{n}{2}}e^{-\frac{3n^2}{4}}} = \frac{e^{\frac{1}{12}}}{A}. \tag{2.19}$$

In fact, we could have guessed the value of $G(1/2)$ (had it not been known already) from the natural assumption that (2.20) holds also for odd b. Indeed, set $a = 0, b = 1$ in (2.20), and compare the leading coefficient in k on the left-hand side (which we obtain explicitly from the asymptotic analysis of our formulas) with the coefficient of the power of k on the right-hand side of the thus specialized (2.20). Using (2.17), this gives a linear equation for $G(1/2)^4$, which leads us precisely to the value in (2.18)!

Let $T_{n,k,B,a,b,c}$ be the region obtained from the hexagon whose side-lengths alternate between $n + a + b + c$ and $n + 3B - a - b - c$ (with the top side of length $n + a + b + c$) by removing from its center three bowties in a triad formation as indicated in Figure 2, where the outer lobe sizes are a, b, c, the inner lobe sizes $B - a$, $B - b$, $B - c$ (counterclockwise from top), the distance between two bowtie nodes is $3k + 3B - a - b - c$, and the distance between the outer lobes and the facing hexagon sides is $n - 2k$.

It turns out that the number of lozenge tilings of the region $T_{n,k,B,a,b,c}$ can be expressed in terms of the number of lozenge tilings of a hexagon with three satellites removed and empty core, using the main result of [24]. The connection is based on the fact that $T_{n,k,B,a,b,c}$ can be obtained from our region $S_{n,0,k,B}$ by applying three times the bowtie squeezing operation described in [24], and that by [24, Theorem 1] the ratio of the number of tilings of any two regions related by bowtie squeezings is given by a simple, conceptual product formula (see Figure 13 and the proof of Theorem 2.5 in Section 9).

We obtain the following result.

Theorem 2.5. *Writing* $\langle\alpha\rangle = G(\alpha + 1)$ *for short, we have*

$$\begin{aligned}
&\frac{\mathrm{M}(T_{n,k,B,a,b,c})}{\mathrm{M}(S_{n,0,B,k})}\\
&\quad = \frac{\langle 3k+B\rangle^3}{\langle 3k\rangle\langle B\rangle^3}\frac{\langle n+k+B\rangle^3\langle n-k+2B\rangle^3}{\langle n-2k+B\rangle^3\langle n+2k+2B\rangle^3}\\
&\qquad \times \frac{\langle 3k+3B-a-b-c\rangle^4\langle a\rangle\langle b\rangle\langle c\rangle}{\langle 3k+3B-a-b\rangle\langle 3k+3B-a-c\rangle\langle 3k+3B-b-c\rangle}\\
&\qquad \times \frac{\langle B-a\rangle\langle B-b\rangle\langle B-c\rangle}{\langle 3k+2B-a-b\rangle\langle 3k+2B-a-c\rangle\langle 3k+2B-b-c\rangle}\\
&\qquad \times \frac{\langle n-2k+a\rangle\langle n+2k+3B-a\rangle}{\langle n+k+3B-b-c\rangle\langle n-k+b+c\rangle}\\
&\qquad \times \frac{\langle n-2k+b\rangle\langle n+2k+3B-b\rangle}{\langle n+k+3B-a-c\rangle\langle n-k+a+c\rangle}\\
&\qquad \times \frac{\langle n-2k+c\rangle\langle n+2k+3B-c\rangle}{\langle n+k+3B-a-b\rangle\langle n-k+a+b\rangle}.
\end{aligned} \tag{2.21}$$

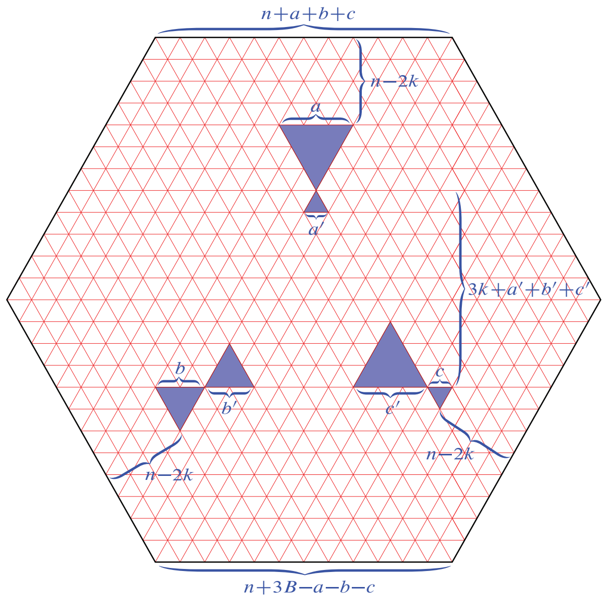

Figure 2. The region $T_{n,k,B,a,b,c}$ (a triad of bowties at the center of a hexagon); $a' = B - a$, $b' = B - b$, $c' = B - c$ (here $n = 6$, $k = 1$, $B = 4$, $a = 3$, $b = 2$, $c = 1$).

Remark 2.6. Note that the above result allows in particular to squeeze in completely the outer lobe of any of the three bowties independently, obtaining a triangular satellite of opposite orientation compared to $S_{n,a,b,k}$. This includes the case $a = 0$ of the regions $S'_{n,a,b,k}$ shown on the right in Figure 1!

The special case of Conjecture 2.1 when the core is empty, combined with Theorem 2.5, affords a product formula for the number of lozenge tilings of the regions $T_{n,k,B,a,b,c}$ of Figure 2. Asymptotic analysis of this formula for $\mathrm{M}(T_{n,k,B,a,b,c})$ lets us deduce the following result. We present the proofs of Theorems 2.5 and 2.6 in Section 9.

Theorem 2.6. *Consider three bowties X_1, X_2 and X_3 in a triad formation, as shown in Figure 2. Their outer lobes have sizes a, b and c, and their inner lobes have sizes*

$$a' = B - a, \quad b' = B - b, \quad c' = B - c,$$

respectively. The distance between the nodes of two bowties is

$$3k + a' + b' + c'.$$

Assume $a+b+c=a'+b'+c'$, *and define the correlation* $\bar{\omega}(X_1,X_2,X_3)$ *by*

$$\bar{\omega}(X_1,X_2,X_3;k)=\lim_{n\to\infty}\frac{\mathrm{M}(T_{n,k,B,a,b,c})}{\mathrm{M}(H_{n+a+b+c})}, \tag{2.22}$$

where $H_{n+a+b+c}$ *is the regular hexagon*[11] *of side* $n+a+b+c$.

Then if the special case of Conjecture 2.1 *when the core is empty holds, writing* $\langle\alpha\rangle=G(\alpha+1)$ *as before, we have*

$$\begin{aligned}&\bar{\omega}(X_1,X_2,X_3;k)\\&\quad\sim\frac{3^{B^2/8}}{(2\pi)^{B/2}}\frac{\langle\frac{B}{2}\rangle^2\langle a\rangle\langle a'\rangle}{\langle B\rangle}\frac{3^{B^2/8}}{(2\pi)^{B/2}}\frac{\langle\frac{B}{2}\rangle^2\langle b\rangle\langle b'\rangle}{\langle B\rangle}\frac{3^{B^2/8}}{(2\pi)^{B/2}}\frac{\langle\frac{B}{2}\rangle^2\langle c\rangle\langle c'\rangle}{\langle B\rangle}\\&\qquad\times(3k)^{\frac{1}{2}[(a-a')(b-b')+(a-a')(c-c')+(b-b')(c-c')]},\quad k\to\infty.\end{aligned} \tag{2.23}$$

Remark 2.7. As $3k$ is the distance between each pair of bowties, and their charges are $a-a'$, $b-b'$, and $c-c'$, this proves the electrostatic conjecture (2.13) for a system of three bowties arranged in a triad when $a+a'=b+b'=c+c'$ and $a+b+c=a'+b'+c'$. In fact, the electrostatic conjecture follows even without assuming $a+b+c=a'+b'+c'$. The reason we assumed this condition is because it allows us to compute the multiplicative constant in (2.23) explicitly. We will use it in Section 3 (see Remark 3.1).

The rest of this paper is organized as follows. In Section 3, we present consequences of the formulas presented in this section for the correlation of holes, our original motivation that led to discovering the satellite regions $S_{n,a,b,k}$. First, we show how an earlier conjecture of the first author (a stronger version of the electrostatic conjecture (2.13); see Conjecture 3.1) can be "bootstrapped" into displaying the multiplicative constant explicitly, in the case when the holes are arbitrary triangles (see Conjecture 3.2). This amounts to having explicit expressions for the correlations of single triangular holes. We present in detail the derivation of these from the formulas of Section 2 and two additional assumptions (see Conjectures I and II). Second, we show how these arguments can be extended to derive the correlation of a shamrock-shaped hole (see equation (3.12)) and of a fern-shaped hole (see equations (3.13) and (3.14)). And third, we present a surprising relation between the square and hexagonal lattice that makes the correlation of two monomers on each of them decay to zero at precisely the same rate.

In Sections 4 through 6 we develop our new approach to prove Conjecture 2.1.

[11]This is the outer boundary of the region $T_{n,k,B,a,b,c}$. As we will see in the proof of this theorem in Section 9, provided the special case of Conjecture 2.1 when the core is empty holds (an assumption we make in the statement), the limit (2.22) exists.

One feature of this is that we need to consider regions that are more general than the satellite regions $S_{n,a,b,k}$ defined at the beginning of this section (see Figure 1). The new regions extend the $S_{n,a,b,k}$'s in three different ways (see Figure 6): (1) the top, lower left and lower right side-lengths of the outer hexagon can be arbitrary non-negative integers n_1, n_2, n_3 satisfying the triangle inequality,[12] (2) the satellite sizes can be arbitrary non-negative integers b_1, b_2, b_3, and (3) the distances of the satellites to the core can be arbitrary integers[13] k_1, k_2, k_3. We denote the resulting region by $S_{n_1,n_2,n_3,a,b_1,b_2,b_3,k_1,k_2,k_3}$.

Section 4 provides a determinant formula for the number of lozenge tilings of these more general regions (the increased generality is important for our arguments to work) that holds for both even and odd[14] non-negative integer sizes b_1, b_2, b_3 of the satellites, so that the order of the determinant is independent of a and the entries are polynomials in a (see Theorem 4.5).

Section 5 shows the details of how to evaluate explicitly the determinant in Theorem 4.5 in the special case when the satellite sizes are zero. This yields Theorem 5.1, which is a counterpart of a twenty year old result of Ciucu, Krattenthaler, Eisenkölbl, and Zare (see [18, Theorems 1 and 2]) which has been previously overlooked. We use Krattenthaler's method of identification of factors (or factor exhaustion method), which consists of finding row or column linear combinations that vanish for certain values of a convenient parameter (in our case, a), to prove divisibility of the determinant by various linear factors, and eventually deduce the full expression for the determinant. The resulting linear combinations are especially simple, in contrast with the ones in [18]. In fact, the simplicity of the linear combinations of rows and columns holds also for the general case of non-zero satellite size of the regions[15] $S_{n,a,b,k}$; the reason that case is more involved is due to the more complex block structure of the determinant. The details of the general case will be presented in a forthcoming paper.

[12]The remaining side-lengths are then determined by the sizes of the core and satellites, and the condition that the region is balanced (has the same number of up- and down-pointing unit triangles).

[13]As long as the satellites are still within the outer hexagon. Note that the position of the core at the "center" of the general outer hexagon needs to be defined; we do this in Section 6 (it can also be read off from Figure 6); the satellites are still required to be along the medians of the core sides.

[14]In general, enumerating lozenge tilings of regions with odd size holes is more difficult; in this instance also, the case of even satellites is easier, and we work it out first (see Proposition 4.2); we then extend it to cover the odd size satellite case as well (see Theorem 4.4).

[15]For the determinant to fully factor into linear factors – and thus for the method of factor exhaustion to be applicable, not to mention the guessing of the formula to be tractable – we need $n_1 = n_2 = n_3$, $b_1 = b_2 = b_3$, and $k_1 = k_2 = k_3$.

We also show how the results of [24] can be used to extend Theorem 5.1 to the case when the removed structure is a shamrock, obtaining a new counterpart of a result of Ciucu and Krattenthaler (see Theorem 5.6).

In Section 6 we present another determinant formula (see Theorem 6.1) for the number of tilings of the regions $S_{n_1,n_2,n_3,a,b_1,b_2,b_3,k_1,k_2,k_3}$, which works for even satellite sizes and has the convenient property that the order of the determinant is independent of n (namely, is equal to the sum of the side-lengths of the four holes, $a + 3b$ for the region $S_{n,a,b,k}$). The entries of this determinant are explicit hypergeometric series. This has the important consequence that for concrete values of a and b (with b even; a is even by assumption), Conjecture 2.1 can be proven, at least in principle, by showing that the determinant resulting from Theorem 6.1 (whose entries are explicit hypergeometric series) evaluates to the product expression for $\mathrm{M}(S_{n,a,b,k})$ resulting from Conjecture 2.2 and Theorem 2.3. Indeed, if for instance $a = 6$ and $b = 4$, this amounts to proving that a certain determinant of order 18, whose entries are explicit hypergeometric series, is equal to the corresponding explicit product formula resulting from Conjecture 2.2 and Theorem 2.3 – an interesting identity!

The purpose of Section 7 is to show that, if b is even, when regarded as polynomials in a, $\mathrm{M}(S_{n,a,b,k})$ and the product formula for it implied by Conjecture 2.2 and Theorem 2.3 have the same degree and the same leading coefficient. This is done in Proposition 7.2, which can be regarded as a step towards proving our conjectured formula for $\mathrm{M}(S_{2n,2a,b,k})$. Section 8 presents the proof of Theorem 2.4, and in Section 9 we prove Theorems 2.5 and 2.6.

Section 3 connects closely with Section 2. Sections 4 through 6 form a somewhat independent part in terms of how our new method of approach is constructed, but then connect with Section 2 in Theorems 5.1 and 6.1. The arguments in Section 7 are mostly self contained, and so is the asymptotic analysis in Sections 8 and 9. We end the paper with some concluding remarks in Section 10.

3. Consequences for the correlation of holes

In this section we show how Theorem 2.4 can be used to “bootstrap” an earlier conjecture of the first author [9] (stated below as Conjecture 3.1) on the asymptotics of the correlation $\tilde{\omega}$ of any finite collection $O_1, \dots, O_n$ of triangular holes, by specifying explicitly the involved multiplicative constant (see Conjecture 3.2 in this section).

To achieve this, we need to discuss some more subtle points involving two other definitions of the correlation of holes, which the first author introduced in [9]. For convenience we reproduce their definitions below.

Denote the triangular lattice by $\mathcal{T}$, and draw it (only in this section) so that one family of lattice lines is vertical. Think of the hexagonal lattice $\mathcal{H}$ as the dual of $\mathcal{T}$.

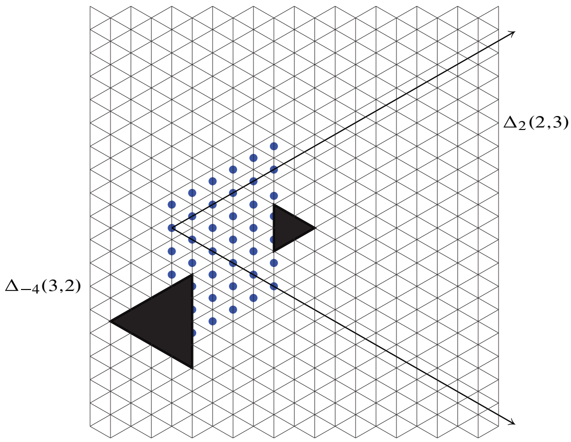

Figure 3. Marked points in our 60° coordinate system; the right 2-triangular hole $\triangleright_2(2,3) = \triangle_2(2,3)$ and the left 4-triangular hole $\triangleleft_4(3,-2) = \triangle_{-4}(3,-2)$; see text.

Then the vertices of $\mathscr{H}$ are the unit triangles of $\mathcal{T}$, and a dimer on $\mathscr{H}$ is a lozenge. Monomers on $\mathscr{H}$ are unit triangles of $\mathcal{T}$; we call them right-monomers and left-monomers according to the direction they point to. Allow holes in $\mathscr{H}$ to be arbitrary finite (not necessarily connected) unions of monomers.

Call the midpoints of vertical lattice segments in $\mathcal{T}$ *marked points*, and coordinatize them by pairs of integers in a 60° coordinate system (see Figure 3), by picking one of them to be the origin, and taking the x- and y-axes in the polar directions $-\pi/6$ and $\pi/6$, respectively. Then each right-monomer is specified by a pair of integer coordinates, and so is each left-monomer.

Define the *right k-triangular hole* $\triangleright_k(x,y)$ to be the right-pointing triangular hole with a side of length k (the unit being the side-length of a unit triangle) whose topmost marked point (those on its boundary included) has coordinates (x,y); the *left k-triangular hole* $\triangleleft_k(x,y)$ is defined to be the analogous left-pointing triangular hole. In some instances we will find it convenient to have a unifying notation for these two types of holes. To this end, for $k \in \mathbb{Z}$ we define the *k-triangular hole* $\triangle_k(x,y)$ by

$$\triangle_k(x,y) := \begin{cases} \triangleright_k(x,y) & \text{if } k > 0, \\ \triangleleft_k(x,y) & \text{if } k < 0 \end{cases} \tag{3.1}$$

(see Figure 3 for two illustrations).

We define the correlation $\tilde{\omega}$ of any finite collection $O_1,\dots,O_n$ of holes as follows.

For any positive integer N, let T_N be the torus obtained from the rhombus

$$\{(x,y)\colon |x|,|y| \le N - 1/2\}$$

on $\mathcal{T}$ by identifying opposite sides. Recall that the *charge* $\mathrm{q}(O)$ of the hole O is the difference between the number of right- and left-monomers in O. By performing a reflection across a vertical lattice line, it suffices to define the correlation when $\sum_{i=1}^{n} \mathrm{q}(O_i) \geq 0$. Define $\tilde{\omega}$ inductively as follows.

i. If $\sum_{i=1}^{n} \mathrm{q}(O_i) = 0$, set

$$\tilde{\omega}(O_1, \ldots, O_n) := \lim_{N\to\infty} \frac{\mathrm{M}(T_N \setminus O_1 \cup \cdots \cup O_n)}{M(T_N)}. \tag{3.2}$$

(Kenyon's determination of the coupling function of the hexagonal lattice [34] and the fact that the correlation of holes can be written as a determinant whose entries are values of the inverse Kasteleyn matrix – see, e.g., Fisher and Stephenson's work [28] – imply that this limit exists.)

ii. If $\sum_{i=1}^{n} \mathrm{q}(O_i) = s > 0$, set

$$\tilde{\omega}(O_1, \ldots, O_n) := \lim_{R\to\infty} \frac{R^{s/2}\tilde{\omega}(O_1, \ldots, O_n, \triangleleft_1(R, 0))}{\sqrt{C}} \tag{3.3}$$

(assuming that the limit exists), where the constant C is determined by

$$\tilde{\omega}(\triangleright_1(0,0), \triangleleft_1(R,0)) \sim C\, R^{-1/2}, \quad R \to \infty$$

(the existence of C for the square lattice version of this follows from Dubédat's work [26]).

In other words, if the total charge of the holes is strictly positive, we repeatedly send to infinity negative charges of unit magnitude until the total charge is reduced to zero, so that part (i) can be used.

Given a hole O and integers x and y, denote by $O(x, y)$ the translation of O under which its topmost (and leftmost, if there are ties) marked point is brought to the point (x, y). In [9] we presented the following generalization of the Fisher–Stephenson conjecture [28] (the latter was recently proved by Dubédat [26]), which stated that the monomer–monomer correlation on the square lattice is rotationally invariant in the scaling limit, and decays to zero as one over the square root of the distance between the monomers.

Conjecture 3.1 ([9]). *For any hole types $O_1, \ldots, O_n$ and any distinct pairs of integers (x_1, y_1), …, (x_n, y_n) we have as $R \to \infty$ that*

$$\tilde{\omega}(O_1(Rx_1, Ry_1), \ldots, O_n(Rx_n, Ry_n)) \sim \prod_{i=1}^{n} \tilde{\omega}(O_i) \prod_{1\leq i<j\leq n} \mathrm{d}((Rx_i, Ry_i), (Rx_j, Ry_j))^{\frac{1}{2}\mathrm{q}(O_i)\mathrm{q}(O_j)}, \tag{3.4}$$

where d *is the Euclidean distance expressed in units equal to a unit triangle side.*

The second correlation we need is denoted by $\hat{\omega}$. It is a variant of $\tilde{\omega}$, but defined only for those collections of holes whose total charge is even. The correlation $\hat{\omega}$ is defined inductively using (i) above, and the modification of (ii) in which $\triangleleft_1(R,0)$ is replaced by $\triangleleft_2(R,0)$ (note that this causes the constant C to be replaced by the leading coefficient C' in the asymptotics of $\hat{\omega}(\triangleright_2(0,0),\triangleleft_2(R,0))$, $R\to\infty$; it turns out that $\hat{\omega}(\triangleright_2(0,0),\triangleleft_2(R,0))\sim\frac{3}{4\pi^2}R^{-2}$, $R\to\infty$, and therefore $C'=\frac{3}{4\pi^2}$).

The special case $q=1$ of [10, Proposition 2.2], stated in terms of the correlation $\hat{\omega}$ (in [10] it is phrased in terms of a variant of $\hat{\omega}$, denoted there by $\hat{\omega}$) implies that for non-negative integers s we have

$$\hat{\omega}(\triangleright_{2s})=\frac{3^{s^2/2}}{(2\pi)^s}[0!1!\dots(s-1)!]^2. \tag{3.5}$$

Based on physical intuition, it is expected that $\tilde{\omega}$ agrees with $\hat{\omega}$, and therefore (3.5) is expected to hold with $\hat{\omega}$ replaced by $\tilde{\omega}$. If we would also know – at least conjecturally – the values of the $\tilde{\omega}(\triangleright_{2s+1})$'s, then we could write down explicitly the multiplicative constant on the right-hand side of (3.4) in the (quite general) special case when O_i is an arbitrary triangular hole (of even or odd side-length, pointing either to the right or to the left), for $i=1,\dots,n$.

Based on the experience with Theorem 2.3 (see Remark 3), we could make the daring guess that (3.5) holds with $\hat{\omega}$ replaced by $\tilde{\omega}$ *also for odd side triangular holes:* as $0!1!\dots(s-1)!=G(s+1)$, this leads to guessing that

$$\tilde{\omega}(\triangleright_k)=\frac{3^{k^2/8}}{(2\pi)^{k/2}}\left[G\left(\frac{k}{2}+1\right)\right]^2 \quad \text{for all } k\geq 0. \tag{3.6}$$

As it turns out, this daring guess is strongly supported by Theorem 2.4, as we explain in this section. We therefore formulate the following strengthening of Conjecture 3.1 in the case when the holes are arbitrary triangles.

Conjecture 3.2. *For arbitrary integers $k_1,\dots,k_n$, and any distinct pairs of integers (x_1,y_1), …, (x_n,y_n), we have*

$$\begin{aligned}&\tilde{\omega}(\triangle_{k_1}(Rx_1,Ry_1),\dots,\triangle_{k_n}(Rx_n,Ry_n))\\&\quad\sim\prod_{i=1}^{n}\frac{3^{k_i^2/8}}{(2\pi)^{|k_i|/2}}\left[G\left(\frac{|k_i|}{2}+1\right)\right]^2\\&\qquad\times\prod_{1\leq i<j\leq n}\mathrm{d}((Rx_i,Ry_i),(Rx_j,Ry_j))^{\frac{1}{2}k_ik_j},\quad R\to\infty.\end{aligned}$$

We now discuss the supporting evidence for equation (3.6). We start by rewriting the statement of Theorem 2.4 in terms of the Euclidean distance between the holes, expressed in units equal to the side-length of a unit triangle. In these units, asymptotically as $k \to \infty$, the distance between the core and each satellite is $k\sqrt{3}$, and the distance between each pair of satellites is $3k$. Denoting the core by S_0 and the satellites by $S_i = S_i(k)$, $i = 1, 2, 3$, one readily checks that the statement of Theorem 2.4 can be rewritten as

$$\omega(S_0, S_1, S_2, S_3) \\ \sim \frac{3^{a^2/8} G\left(\frac{a}{2}+1\right)^2 \left[3^{b^2/8} G\left(\frac{b}{2}+1\right)^2\right]^3}{3^{(a+3b)^2/8} G\left(\frac{a+3b}{2}+1\right)^2} \prod_{0 \le i < j \le 3} \mathrm{d}(S_i, S_j)^{\frac{1}{2}\mathrm{q}(S_i)\,\mathrm{q}(S_j)}, \tag{3.7}$$

in the limit as the satellites recede away from the core at the same rate, where d is the Euclidean distance expressed in units equal to the side-length of a unit triangle.

In order to make our arguments, we will need two additional conjectures (Conjectures I and II below). Their nature is different from the nature of the other conjectures in this article, as they simply state that various correlations of special gap systems are equal, or equal up to an overall, unspecified multiplicative constant. Because of this, we denote them by Roman numerals.

The first one consists of two special cases of the conjecture that the correlations ω and $\tilde{\omega}$ are equal up to a multiplicative factor depending only on the shapes and sizes of the holes, and not on their relative positions.

The first special case we need is that when the collection of holes consists of the core and the three satellites, when it is equivalent to the statement in the first part below. The second is described in the second part.

Conjecture I. (a) *The ratio*

$$\frac{\omega(S_0, S_1(k), S_2(k), S_3(k))}{\tilde{\omega}(S_0, S_1(k), S_2(k), S_3(k))}$$

does not depend on k.

(b) *Let* m *be a non-negative integer, and let* $\mathcal{O}$ *be the collection consisting of one* $\triangleright_1$ *and* m $\triangleright_2$ *collinear holes lined up along a horizontal axis, so that the leftmost of them is the* $\triangleright_1$. *Then*[16] $\frac{\omega(\mathcal{O})}{\tilde{\omega}(\mathcal{O})}$ *does not depend on the relative distances between the holes in the collection* $\mathcal{O}$.

This is a reasonable conjecture, as ω is defined by placing the holes at the very center of the enclosing hexagons (in the fine mesh limit as the lattice spacing approaches zero, the enclosing hexagon approaches a regular hexagon, with the core and the

[16] Here $\omega(\mathcal{O})$ is defined by a limit analogous to (2.12); see [8, (2.2)].

satellites shrinking to its center), a place where dimer statistics is governed by the translation invariant Gibbs measure of maximal entropy, which is equal to the topological entropy (see [25, 34, 35]). Note also that the denominator in equation (2.12) (resp., for part (b), the denominator in [8, (2.2)]) is a natural choice, but other choices would clearly work, so in the very definition of ω there is a residing and somewhat arbitrary multiplicative constant.

When taking ratios of correlations, the multiplicative constants cancel out. Therefore, assuming Conjecture I (a) holds we get

$$\frac{\omega(S_0, S_1(k), S_2(k), S_3(k))}{\omega(S_0, S_1(0), S_2(0), S_3(0))} = \frac{\tilde{\omega}(S_0, S_1(k), S_2(k), S_3(k))}{\tilde{\omega}(S_0, S_1(0), S_2(0), S_3(0))}. \tag{3.8}$$

By definition (2.12), the denominator on the left-hand side above is equal to 1. Thus, equation (3.8) combined with (3.7) gives

$$\frac{\tilde{\omega}(S_0, S_1(k), S_2(k), S_3(k))}{\tilde{\omega}(S_0, S_1(0), S_2(0), S_3(0))} \sim \frac{3^{a^2/8} G\left(\frac{a}{2}+1\right)^2 \left[3^{b^2/8} G\left(\frac{b}{2}+1\right)^2\right]^3}{3^{(a+3b)^2/8} G\left(\frac{a+3b}{2}+1\right)^2} \prod_{0 \le i < j \le 3} \mathrm{d}(S_i, S_j)^{\frac{1}{2}\mathrm{q}(S_i)\mathrm{q}(S_j)}. \tag{3.9}$$

However, due to forced lozenges at the points of contact of $S_1(0)$, $S_2(0)$, and $S_3(0)$ with the core S_0, the denominator on the left-hand side above is equal to $\tilde{\omega}(\triangleright_{a+3b})$. Therefore, if Conjecture 3.1 holds, (3.9) implies that, for all $0 \le a, b \in \mathbb{Z}$, a even,

$$\begin{aligned}\frac{\tilde{\omega}(\triangleright_a)\tilde{\omega}(\triangleright_b)^3}{\tilde{\omega}(\triangleright_{a+3b})} &= \frac{3^{a^2/8} G\left(\frac{a}{2}+1\right)^2 \left[3^{b^2/8} G\left(\frac{b}{2}+1\right)^2\right]^3}{3^{(a+3b)^2/8} G\left(\frac{a+3b}{2}+1\right)^2} \\ &= \frac{\frac{3^{a^2/8}}{(2\pi)^{a/2}} G\left(\frac{a}{2}+1\right)^2 \left[\frac{3^{b^2/8}}{(2\pi)^{b/2}} G\left(\frac{b}{2}+1\right)^2\right]^3}{\frac{3^{(a+3b)^2/8}}{(2\pi)^{(a+3b)/2}} G\left(\frac{a+3b}{2}+1\right)^2}.\end{aligned} \tag{3.10}$$

While strictly speaking not implying (3.6), the above equation does strikingly support it.

In fact, it turns out that equation (3.6) is implied by Conjecture 3.1 and Conjecture I, provided we make one additional assumption (see Conjecture II below). We explain this in the three paragraphs following the statement of Conjecture II.

Remark 3.1. It turns out that Theorem 2.6, Conjecture 3.1, equation (3.6) and the expected fact that the correlations $\bar{\omega}$ and $\tilde{\omega}$ agree for collections of holes of total charge zero[17] imply the following generalization of (3.6): for a bowtie $X_{a,a'}$ with lobe sizes a and a', its correlation is given by

$$\tilde{\omega}(X_{a,a'}) = \frac{3^{(a+a')^2/8}}{(2\pi)^{(a+a')/2}} \frac{G\left(\frac{a+a'}{2}+1\right)^2 G(a+1)G(a'+1)}{G(a+a'+1)}. \tag{3.11}$$

In turn, the above equation, when combined with [23, (1.4)], yields more generally the correlation of the shamrock $S(a,b,c,m)$ (the structure consisting of an up-pointing triangular core of side m and three down-pointing triangular lobes of sides a, b, and c touching it at the vertices). We obtain

$$\begin{aligned}\tilde{\omega}(S(a,b,c,m)) &= \frac{3^{(a+b+c+m)^2/8}}{(2\pi)^{(a+b+c+m)/2}} \\ &\quad \times \frac{G\left(\frac{a+b+c+m}{2}+1\right)^2 G(m+1)^3 G(a+1)G(b+1)G(c+1)}{G(a+m+1)G(b+m+1)G(c+m+1)}.\end{aligned} \tag{3.12}$$

Similarly, combining equation (3.11) above with [15, (1.5)], we can find the correlation of the fern $F(a_1,\dots,a_k)$ (a string of contiguous triangular lobes of sizes $a_1,\dots,a_k$ lined up along a lattice line, alternately oriented up and down). With $a = a_1 + \cdots + a_k$, $o = a_1 + a_3 + \cdots$, and $e = a_2 + a_4 + \cdots$, we obtain

$$\begin{aligned}\tilde{\omega}(F(a_1,\dots,a_k)) &= \frac{3^{a^2/8}}{(2\pi)^{a/2}} \frac{G\left(\frac{a}{2}+1\right)^2 G(o+1)G(e+1)}{G(a+1)} s(a_1,\dots,a_k)s(a_2,\dots,a_k),\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}s(b_1,b_2,\dots,b_{2l}) = s(b_1,b_2,\dots,b_{2l-1}) &= \frac{\prod_{1\le i\le j\le 2l-1,\ j-i+1 \text{ odd}} G(b_i+b_{i+1}+\cdots+b_j+1)}{\prod_{1\le i\le j\le 2l-1,\ j-i+1 \text{ even}} G(b_i+b_{i+1}+\cdots+b_j+1)} \\ &\quad \times \frac{1}{G(b_1+b_3+\cdots+b_{2l-1}+1)}.\end{aligned} \tag{3.14}$$

[17] When the total charge is zero, $\tilde{\omega}$ is defined by equation (3.2), and $\bar{\omega}$ by equation (2.22). So, the former is defined by including the configuration of holes in large tori, and the latter by including them (in the scaling limit) at the center of large regular hexagons. Since the dimer statistics is not distorted in the scaling limit at the center of large regular hexagons (see the first paragraph after the statement of Conjecture I), we expect the two correlations to agree.

Equations (3.12) and (3.13) then naturally extend Conjecture 3.2 to arbitrary collections of shamrocks and ferns. The details will be presented in a subsequent paper.

We point out that part (i) of the definition (3.2)–(3.3) of the correlation $\tilde{\omega}$ is most natural, but in part (ii) a very specific choice was made about how to handle collections of holes of strictly positive total charge: namely, to repeatedly send to infinity negative charges of unit magnitude[18] until the total charge is reduced to zero, so that part (i) can be used.

Once a decision is made upon how exactly to balance the total charge (e.g., for a collection of holes of total charge $2k > 0$, one way to do the balancing – the way done in the definition of $\tilde{\omega}$ – is to repeatedly send a negative monomer $\triangleleft_1$ to infinity $2k$ times; another way – corresponding to the definition of $\hat{\omega}$ – is to repeatedly send a $\triangleleft_2$ hole to infinity k times), we claim that there is a unique choice for the value of C at the denominator on the right-hand side of (3.3) that gives a chance for Conjecture 3.1 to hold.

We justify this claim for the two cases of a $\triangleleft_1$ or a $\triangleleft_2$ being sent to infinity (these are the only instances we need in our arguments below; the general case is handled the same way). For the case of a $\triangleleft_1$, the claim follows by considering in (ii) the special case when $n = 1$ and $O_1 = \triangleright_1$. Indeed, then (3.3) becomes

$$\tilde{\omega}(\triangleright_1) = \tilde{\omega}(\triangleright_1(0,0)) := \lim_{R\to\infty} \frac{R^{1/2}\tilde{\omega}\,(\triangleright_1(0,0), \triangleleft_1(R,0))}{\sqrt{C}}. \tag{3.15}$$

If we want to end up with a correlation for which Conjecture 3.1 holds, then we must have

$$\tilde{\omega}(\triangleright_1(0,0), \triangleleft_1(R,0)) \sim \tilde{\omega}(\triangleright_1(0,0))\tilde{\omega}\,(\triangleleft_1(R,0))\, R^{-1/2}, \quad R \to \infty. \tag{3.16}$$

Clearly, (3.15) and (3.16) give (using also $\tilde{\omega}(\triangleleft_1) = \tilde{\omega}(\triangleright_1)$) that $\tilde{\omega}(\triangleright_1) = \sqrt{C}$. Combined with (3.16), this gives

$$\tilde{\omega}\,(\triangleright_1(0,0), \triangleleft_1(R,0)) \sim CR^{-1/2}, \quad R \to \infty, \tag{3.17}$$

which determines C uniquely, as claimed, to be the value we used in the definition of $\tilde{\omega}$. The case of $\hat{\omega}(\triangleright_2)$ is justified the same way, leading to the unique choice of C' used in the definition of $\hat{\omega}$.

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[18]Furthermore, in the definition of $\tilde{\omega}$ these auxiliary negative unit charges are always sent to infinity along the polar direction $-\pi/6$. This was chosen for technical reasons, to aid the computations. Due to the expected rotational invariance, the obtained values should be independent of the direction.

Our second conjecture is a special case of what we could call self-consistency: that all the different possible ways to balance a given collection of holes in part (ii) of the definition lead to the same value of the correlation, provided the denominator on the right-hand side of (3.3) is always chosen to have the unique value determined by the statement of Conjecture 3.1. In fact, we only need this for our two correlations $\tilde{\omega}$ and $\hat{\omega}$, and only for a single triangular hole of side two.

Conjecture II. $\hat{\omega}(\triangleright_2) = \tilde{\omega}(\triangleright_2)$.

There is one more result on the asymptotics of the correlation ω that we need, which follows from the product formula of [7, Theorem 1.1] by the same reasoning that derived Theorem 2.4 from the formula in Conjecture 2.1. Making the same arguments that led to (3.10) (i.e., assuming that Conjecture 3.1 and Conjecture I (b) hold), we obtain

$$\frac{\tilde{\omega}(\triangleright_1)\tilde{\omega}(\triangleright_2)^m}{\tilde{\omega}(\triangleright_{2m+1})} = \frac{3^{1/8}G\left(\frac{3}{2}\right)^2 [3^{1/2}G(3)^2]^m}{3^{(2m+1)^2/8}G\left(\frac{2m+1}{2}+1\right)^2}, \quad \text{for all } m \geq 0. \tag{3.18}$$

Deducing the value of $\tilde{\omega}(\triangleright_1)$. Consider equation (3.10) (which recall follows from Theorem 2.4, provided Conjecture 3.1 and Conjecture I (a) hold), and set $a = 0$ and $b = 1$. Using $\tilde{\omega}(\triangleright_0) = 1$ (which follows from the definition of $\tilde{\omega}$), the recurrence (2.15) and the fact that $\Gamma(3/2) = \sqrt{\pi}/2$, we obtain

$$\frac{[\tilde{\omega}(\triangleright_1)]^3}{\tilde{\omega}(\triangleright_3)} = \frac{4}{3^{3/4}\pi}G\left(\frac{3}{2}\right)^4. \tag{3.19}$$

On the other hand, setting $m = 1$ in (3.18), we get

$$\frac{\tilde{\omega}(\triangleright_1)\tilde{\omega}(\triangleright_2)}{\tilde{\omega}(\triangleright_3)} = \frac{4}{3^{1/2}\pi}. \tag{3.20}$$

By (3.5) and Conjecture II, $\tilde{\omega}(\triangleright_2) = \sqrt{3}/(2\pi)$. Thus, combining equations (3.19) and (3.20), we get

$$\tilde{\omega}(\triangleright_1) = \frac{3^{1/8}}{\sqrt{2\pi}}G\left(\frac{3}{2}\right)^2. \tag{3.21}$$

Deducing the values $\tilde{\omega}(\triangleright_{2m+1})$. Having determined the value of $\triangleright_1$, the value of $\tilde{\omega}(\triangleright_{2m+1})$ for any positive integer m follows directly from (3.18), using again that (by (3.5) and Conjecture II) $\tilde{\omega}(\triangleright_2) = \sqrt{3}/(2\pi)$. This leads to (3.6), and thus to the explicit multiplicative constant in Conjecture 3.2.

We end this section with a pretty astounding way of relating the hexagonal and square lattices from the point of view of the rate of decay to zero of the monomer–monomer correlation. This is afforded by comparing the value of $\tilde{\omega}(\triangleright_1)$ derived above

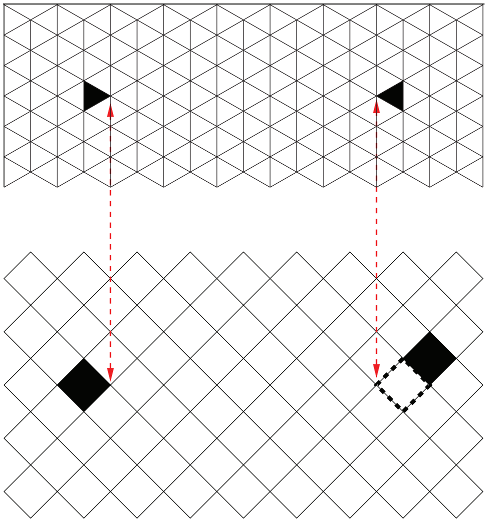

Figure 4. Squaring the hexagonal lattice: If the removed unit triangles and the removed unit squares are lined up as shown, having d long lozenge diagonals, respectively d unit square diagonals in between, then their correlations decay to zero asymptotically like $c/\sqrt{d}$ with the same value of c (namely $c = e^{1/2}2^{-5/6}A^{-6}$, A being the Glaisher–Kinkelin constant), where d is the Euclidean distance between the removed monomers, measured on the triangular lattice in units equal to a long lozenge diagonal, and on the square lattice in units equal to a unit square diagonal. Phrased in terms of monomer correlations in a sea of dimers, as the dual of the triangular lattice is the hexagonal lattice (while the square lattice is self-dual), this shows how to calibrate the size of the hexagonal lattice against the square lattice so that the monomer–monomer correlations decay identically.

to the analogous constant for the square lattice, which was determined by Hartwig [32] in 1966.

Hartwig showed in [32] that

$$\omega(\square_{0,0}, \square_{d,d+1}) \sim \frac{e^{1/2}}{2^{\frac{5}{6}} A^6} d^{-1/2}, \quad d \to \infty, \tag{3.22}$$

where $\square_{p,q}$ denotes the unit square on the square lattice whose bottom left corner has coordinates (p, q), and the correlation ω on the square lattice is defined in analogy to (2.22), using large squares centered at the origin to enclose the monomers.

On the other hand, equation (3.17), together with $\tilde{\omega}(\triangleright_1) = \sqrt{C}$ and the value for $\tilde{\omega}(\triangleright_1)$ derived in (3.21), gives

$$\tilde{\omega}(\triangleright_1(0,0), \triangleleft_1(d,0)) \sim \frac{3^{1/4}}{2\pi} G\left(\frac{3}{2}\right)^4 d^{-1/2}, \quad d \to \infty. \tag{3.23}$$

By equation (2.17) and the value (2.18) of $G(1/2)$, we have

$$G\Big(\frac{3}{2}\Big) = \frac{2^{1/24} e^{1/8} \pi^{1/4}}{A^{3/2}}, \tag{3.24}$$

and (3.23) becomes

$$\tilde{\omega}(\triangleright_1(0,0), \triangleleft_1(d,0)) \sim \frac{3^{1/4} e^{1/2}}{2^{5/6} A^6} d^{-1/2}, \quad d \to \infty. \tag{3.25}$$

By Conjecture 3.1, we should have

$$\tilde{\omega}(\triangleright_1(0,0), \triangleleft_1(d,d)) \sim \frac{3^{1/4} e^{1/2}}{2^{5/6} A^6} (d\sqrt{3})^{-1/2} = \frac{e^{1/2}}{2^{5/6} A^6} d^{-1/2}, \quad d \to \infty, \tag{3.26}$$

because the Euclidean distance between $\triangleright_1(0,0)$ and $\triangleleft_1(d,d)$ is $d\sqrt{3}$, expressed in units equal to a unit triangle side.

The agreement of the multiplicative constants in (3.22) and (3.26) is most unexpected. Note that in (3.22) the distance between the removed unit squares is d unit square diagonals, and the distance between the removed unit triangles in (3.26) is d long lozenge diagonals. Therefore, the agreement of the right-hand sides in (3.22) and (3.26) has the following interpretation: if the triangular lattice is scaled so that the lengths of a long lozenge diagonal matches the length of a unit square diagonal on the square lattice (see Figure 4), then the monomer–monomer correlations on these two lattices decay to zero at precisely the same rate. Since unit holes in lozenge tilings are equivalent to monomers in dimer systems on the hexagonal lattice, we can view this agreement as specifying how to scale the hexagonal lattice against the square lattice in order to get precisely the same decay – squaring the hexagonal lattice, as it were.

This is reminiscent of magic angle graphene superlattices [5] – except there two identical hexagonal lattices are superimposed, while here one hexagonal and one square lattice, carefully scaled.

4. Determinantal formulas for $\mathrm{M}(S_{n,a,b,k})$

The purpose of this section is to derive some convenient determinantal formulas for $\mathrm{M}(S_{n,a,b,k})$ (see Theorems 4.4 and 4.5). This derivation is divided into the following steps according to Sections 4.1–4.4.

1. First we use the Lindström–Gessel–Viennot theorem [30,44] to derive a determinantal formula for the number of lozenge tilings of $S_{n,a,b,k}$ *assuming that* b *is even*. This is standard, however, we introduce a notation that will be useful in the following. Also, for what follows, we need a more general setting, where the sizes of the three satellites are independent integers b_1, b_2, b_3.

2. Next we show that the number of lozenge tilings in this more general setting is for each $i \in \{1,2,3\}$ a polynomial in b_i when fixing the other b_j's, n, a, and k. Here we employ arguments that have been used in [18, Section 6].
3. Then we use this polynomiality to modify the determinantal formula from the first step so that it gives the correct values also if b is odd.
4. Finally, we modify the determinant further such that it reveals the polynomiality in a (so that a does not appear in the size of the matrix and all matrix entries are polynomials in a). This is necessary to be able to apply the identification of factors method, see [38, Section 2.4].

4.1. Trapezoids with triangular holes

For positive integers n, l, we refer to the isosceles trapezoid whose longer base is of length l, whose legs are of length n and with lower base angles 60° as an (n, l)*-trapezoid.* The $(11, 16)$-trapezoid is given in Figure 5. If we draw such a trapezoid on the triangular lattice in the usual way so that the longer base is horizontal and below the shorter base, and the vertices are lattice points, then the trapezoid has n more up-pointing unit triangles than down-pointing unit triangles. Hence, such a trapezoid does not have a lozenge tiling, but may have one if we remove n up-pointing unit triangles from it.

As indicated in Figure 5, such lozenge tilings correspond to families of non-intersecting lattice paths where the starting points are arranged along the left leg of the trapezoid, while the end points are situated at the centers of the /-sides of the removed triangles (which are the black triangles in our example). We number the starting points from bottom to top with 1 to n, and the removed triangles also from bottom to top and within each row from left to right with 1 to n. Then such a family of non-intersecting lattice paths induces in a natural way a permutation of $1, 2, \ldots, n$. The sign of this permutation is said to be the *sign* of the lozenge tiling. In our example, numbering the removed triangles from bottom to top and within a row from left to right gives the permutation 1 2 3 4 6 7 8 9 5 10 11.

We say that the set of removed triangles is *even* if each such triangle that is not situated in the bottom row is contained in a maximal (connected) horizontal chain of removed triangles that is of even length. The set of removed triangles in Figure 5 is even. As the lattice paths are non-intersecting, the starting points of the paths that end in a particular chain must be numbered with consecutive integers that increase as we go from left to right in the chain. Thus, the assumption guarantees that all permutations can be obtained from the identity by applying an even number of transpositions, and therefore all lozenge tilings have sign $+1$. Finally, note that removing a horizontal chain of, say, n unit triangles is equivalent (due to forced lozenges) to removing an up-pointing triangle of size n.

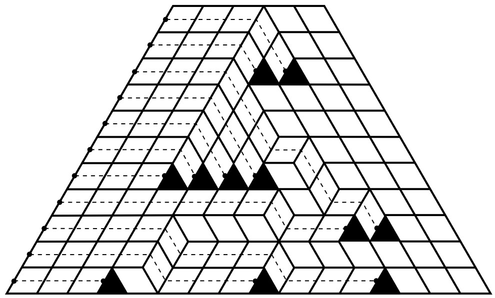

Figure 5. A lozenge tiling of an $(11, 16)$-trapezoid along with the corresponding family of non-intersecting lattice paths.

The following lemma, which follows immediately from the Lindström–Gessel–Viennot theorem, allows us to compute the number of lozenge tilings of a trapezoid with an even set of up-pointing unit triangles removed. In order to formulate it, we need to define the forward difference operator: suppose $p\colon \mathbb{Z} \to \mathbb{C}$ is a function, then Δ is defined as

$$\Delta p(x) = p(x+1) - p(x).$$

In our applications, we will usually deal with multivariate functions which can of course be viewed as univariate functions when fixing all but one variable, say, x, and in such a situation we use Δ_x to identify to which variable the forward difference is applied. Moreover, we set

$$\binom{n}{k} = \begin{cases} \dfrac{n(n-1)\dots(n-k+1)}{k!} & \text{for } k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.1. *Consider an (n,l)-trapezoid with n up-pointing unit triangles R_1, $R_2, \dots, R_n$ removed. For each i, let r_i be the row of R_i, counted from the bottom starting with 1, and c_i be the position of R_i in its row, counted from the left starting with 1. Then the signed enumeration of lozenge tilings*[19] *of the (n,l)-trapezoid where the triangles $R_1, R_2, \dots, R_n$ have been removed is*

$$\left[\prod_{i=1}^{n} \Delta_{c_i}^{r_i-1}\right] \prod_{1\leq i<j\leq n} \frac{c_j - c_i}{j-i} = \left[\prod_{i=1}^{n} \Delta_{c_i}^{r_i-1}\right] \det_{1\leq i,j\leq n}\left(\binom{c_i - d}{j-1}\right), \tag{4.1}$$

for any integer d. If $R_1, \dots, R_n$ is even, then the absolute value of this expression is the number of lozenge tilings.

[19]Each tiling being counted with a sign equal to the sign of the permutation induced by the paths of lozenges encoding the tiling; see Figure 5.

Proof. We use the bijection between lozenge tilings and families of non-intersecting lattice paths as indicated in Figure 5. The starting points of the lattice paths on the left leg of the trapezoid can be parametrized by $(1,1),(2,2),\ldots,(n,n)$, from bottom to top, while the endpoint on the $/$-side of R_i is then (c_i+r_i,r_i), and we allow unit steps $(1,0)$ and $(0,-1)$ in our paths. In general, the number of lattice paths from (a,b) to (c,d) is $\binom{c-a+b-d}{b-d}$, and so the number of paths from (j,j) to (c_i+r_i,r_i) is

$$\binom{c_i}{j-r_i} = \Delta_{c_i}^{r_i-1}\binom{c_i}{j-1},$$

which is needed to apply the Lindström–Gessel–Viennot theorem [30, 44]. This theorem can be used to express the signed enumeration of families of non-intersecting lattice paths as a determinant. In our particular case, we obtain for this signed enumeration

$$\det_{1\le i,j\le n}\left(\Delta_{c_i}^{r_i-1}\binom{c_i}{j-1}\right) = \prod_{i=1}^{n}\Delta_{c_i}^{r_i-1}\det_{1\le i,j\le n}\left(\binom{c_i}{j-1}\right), \tag{4.2}$$

where we have used the linearity in the rows of the determinant to show the equality of the expressions. On the other hand, suppose $p_j(c)$ is a sequence of monic polynomials for $j=1,\ldots,n$ with $\deg_c p_j(c)=j-1$. Then, by elementary column operations, we have

$$\det_{1\le i,j\le n}(p_j(c_i)) = \prod_{1\le i<j\le n}(c_j-c_i).$$

Choosing $p_j(c)=(j-1)!\binom{c-d}{j-1}$ as well as dividing the identity by $\prod_{j=1}^{n}(j-1)!$ shows

$$\det_{1\le i,j\le n}\left(\binom{c_i-d}{j-1}\right) = \prod_{1\le i<j\le n}\frac{c_j-c_i}{j-i},$$

in particular we see that the left-hand side is independent of d. The assertion now follows by combining this observation with (4.2). ■

As it was used in the proof, we may apply the powers of the forward difference operators also "inside" the determinant in (4.1) (by the linearity of the determinant in the rows). That way we obtain a determinant in which each row corresponds to a removed triangle. Horizontal (connected) chains of removed triangles then correspond to sets of consecutive rows (if the numbering of the removed triangles was chosen accordingly) in the matrix; these are referred to as *blocks* in the following. The parameter d will play a crucial role and it is the reason why we write the formula in (4.1) in this particular form: We will see that, for any such block, we can choose d appropriately in such a way that this block can be "eliminated." We will find it useful to eliminate certain parameters (typically the length of a chain of removed unit triangles) from the matrices underlying our determinants, as this will help us obtain expressions that are polynomials in these parameters. It is this somewhat simple observation that is applied in the following repeatedly to derive two useful formulas for our concrete problem.

Next, we apply this lemma to our setting. However, in order to be able to extend the determinental formula for even b to odd b, we need to work with satellites of independent sizes. It is not more difficult to consider a multivariate generalization, where also the three sides – originally of length n – are allowed to have independent lengths, and so are the distances between the core and the satellites.

For non-negative integers $n_1, n_2, n_3, b_1, b_2, b_3, k_1, k_2, k_3$ and non-negative even a, we denote the hexagon with side lengths

$$\begin{aligned} &n_1 + a + b_1 + b_2 + b_3, \quad n_3, \\ &n_2 + a + b_1 + b_2 + b_3, \quad n_1, \\ &n_3 + a + b_1 + b_2 + b_3, \quad n_2 \end{aligned}$$

(clockwise from the northwestern side) that has four triangular holes with side lengths a, b_1, b_2, b_3, respectively, as indicated in Figure 6 by $S_{n_1,n_2,n_3,a,b_1,b_2,b_3,k_1,k_2,k_3}$: the hole of size a (the *core*) has distance

$$(n_1 + b_1) \cdot \frac{\sqrt{3}}{2}, \quad (n_2 + b_2) \cdot \frac{\sqrt{3}}{2}, \quad (n_3 + b_3) \cdot \frac{\sqrt{3}}{2}$$

from the three sides of length

$$n_1 + a + b_1 + b_2 + b_3, \quad n_2 + a + b_1 + b_2 + b_3, \quad n_3 + a + b_1 + b_2 + b_3,$$

respectively. The three holes of size b_1, b_2, b_3 (the *satellites*) point towards the center of the core and have distance

$$2k_1 \cdot \frac{\sqrt{3}}{2}, \quad 2k_2 \cdot \frac{\sqrt{3}}{2}, \quad 2k_3 \cdot \frac{\sqrt{3}}{2}$$

from the core, respectively, where the satellite of size b_i is situated between the core and the long side of the hexagon that has distance $n_i + b_i$ from the core.

Note that the geometry of the configuration implies

$$n_1 \le a + n_2 + b_2 + n_3 + b_3. \tag{4.3}$$

This can be seen as follows. Consider the line that includes the "/"-side of the core. The length of the portion of this line included in the wedge obtained by extending the sides of length $n_2 + a + b_1 + b_2 + b_3$ and $n_3 + a + b_1 + b_2 + b_3$ until they meet, which is $a + n_2 + b_2 + n_3 + b_3$, needs to be at least as large as the length

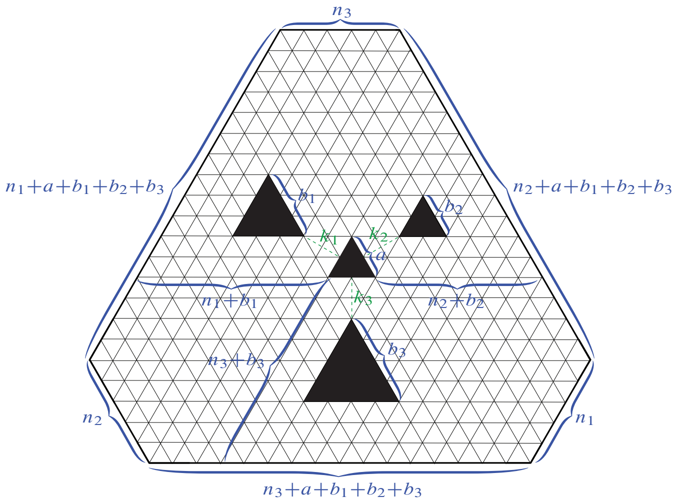

Figure 6. Independent satellites.

of the southeastern edge of the hexagon, which is n_1. By symmetry, we also have $n_2 \le a + n_1 + b_1 + n_3 + b_3$ and $n_3 \le a + n_1 + b_1 + n_2 + b_2$.

If b_1, b_2, b_3 are even, Lemma 4.1 can be applied to compute the number of lozenge tilings of this region. Indeed, set $n = n_1 + n_2 + a + b_1 + b_2 + b_3$. In order to start from an $(n, n + n_3)$-trapezoid, we add a triangle of size n_2 at the bottom left corner of the hexagon, while we add a triangle of size n_1 at the bottom right corner. We have six chains of triangles to be removed as follows:

1. at height[20] 0 of length n_2 in positions

$$1, \quad \ldots, \quad n_2;$$

2. at height 0 of length n_1 in positions

$$n_2 + n_3 + a + b_1 + b_2 + b_3 + 1, \quad \ldots, \quad n_1 + n_2 + n_3 + a + b_1 + b_2 + b_3;$$

3. at height $n_3 - 2k_3$ of length b_3 in positions

$$n_1 + \frac{a}{2} + b_1 + k_3 + 1, \quad \ldots, \quad n_1 + \frac{a}{2} + b_1 + b_3 + k_3;$$

4. at height $n_3 + b_3$ of length a in positions

$$n_1 + b_1 + 1, \quad \ldots, \quad n_1 + a + b_1;$$

[20]The height of a removed triangle is one less than its row number.

5. at height $n_3 + \frac{a}{2} + b_3 + k_1$ of length b_1 in positions

$$n_1 - 2k_1 + 1, \quad \dots, \quad n_1 - 2k_1 + b_1;$$

6. at height $n_3 + \frac{a}{2} + b_3 + k_2$ of length b_2 in positions

$$n_1 + \frac{a}{2} + b_1 + k_2 + 1, \quad \dots, \quad n_1 + \frac{a}{2} + b_1 + b_2 + k_2.$$

Using Lemma 4.1, it follows that the number of lozenge tilings of

$$S_{n_1,n_2,n_3,a,b_1,b_2,b_3,k_1,k_2,k_3}$$

is

$$\prod_{i=1}^{b_3} \Delta_{c_{3,i}}^{n_3-2k_3} \prod_{i=1}^{a} \Delta_{c_{4,i}}^{n_3+b_3} \prod_{i=1}^{b_1} \Delta_{c_{5,i}}^{n_3+\frac{a}{2}+b_3+k_1} \prod_{i=1}^{b_2} \Delta_{c_{6,i}}^{n_3+\frac{a}{2}+b_3+k_2}$$
$$\times \det \begin{pmatrix} \binom{c_{1,i}-d}{j-1} & 1 \le i \le n_2 \\ \binom{c_{2,i}-d}{j-1} & 1 \le i \le n_1 \\ \binom{c_{3,i}-d}{j-1} & 1 \le i \le b_3 \\ \binom{c_{4,i}-d}{j-1} & 1 \le i \le a \\ \binom{c_{5,i}-d}{j-1} & 1 \le i \le b_1 \\ \binom{c_{6,i}-d}{j-1} & 1 \le i \le b_2 \end{pmatrix}_{1 \le j \le n}, \tag{4.4}$$

evaluated at

$$\begin{aligned} c_{1,i} &= i, \\ c_{2,i} &= n_2 + n_3 + a + b_1 + b_2 + b_3 + i, \\ c_{3,i} &= n_1 + \frac{a}{2} + b_1 + k_3 + i, \\ c_{4,i} &= n_1 + b_1 + i, \\ c_{5,i} &= n_1 - 2k_1 + i, \\ c_{6,i} &= n_1 + \frac{a}{2} + b_1 + k_2 + i. \end{aligned}$$

We obtain the following result.

Proposition 4.2. *For even b_1, b_2, b_3, we have*

$$\mathrm{M}(S_{n_1,n_2,n_3,a,b_1,b_2,b_3,k_1,k_2,k_3}) = \det \begin{pmatrix} \binom{i-d}{j-1} & 1 \le i \le n_2 \\ \binom{n_2+n_3+a+b_1+b_2+b_3+i-d}{j-1} & 1 \le i \le n_1 \\ \binom{n_1+\frac{a}{2}+b_1+k_3+i-d}{j-1-n_3+2k_3} & 1 \le i \le b_3 \\ \binom{n_1+b_1+i-d}{j-1-n_3-b_3} & 1 \le i \le a \\ \binom{n_1-2k_1+i-d}{j-1-n_3-\frac{a}{2}-b_3-k_1} & 1 \le i \le b_1 \\ \binom{n_1+\frac{a}{2}+b_1+k_2+i-d}{j-1-n_3-\frac{a}{2}-b_3-k_2} & 1 \le i \le b_2 \end{pmatrix}_{1 \le j \le n}, \quad (4.5)$$

where d can be chosen arbitrarily.

4.2. Polynomiality in the sizes of the satellites

The technique we are using to deal with odd-sized satellites is based on the following crucial observation.

Lemma 4.3. *For any $i \in \{1, 2, 3\}$, the quantity $\mathrm{M}(S_{n_1,n_2,n_3,a,b_1,b_2,b_3,k_1,k_2,k_3})$ is a polynomial in b_i when fixing the n_i's, the k_i's, the core size a, and the two b_j's with $j \neq i$.*

Proof. We follow the ideas provided in [18, Section 6], which were used there to show the polynomiality of $\mathrm{M}(S_{n_1,n_2,n_3,a,0,0,0,0,0,0})$ in a. By symmetry, it suffices to consider the case $i = 2$.

Set $S = S_{n_1,n_2,n_3,a,b_1,b_2,b_3,k_1,k_2,k_3}$. Let R be the smallest lattice hexagon that contains the southwestern side of S, the core, and the satellites of side-lengths b_1 and b_3. For the region in Figure 7, the resulting region R is shown on the lower left in Figure 8 (delimited by the dashed line).

As b_2 varies over the non-negative integers (with all the other parameters having fixed values), the region R does not change. In particular, for any fixed choice of lozenges protruding through the dashed portion of the boundary, the number of lozenge tilings of R is a fixed number, independent of b_2.

One instance of such a tiling (for focus, only the lozenges straddling the dashed line) is shown in Figure 8. By the observation in the previous paragraph, it suffices to show that for any such fixed choice of lozenges straddling the dashed line, the complement R' of R in S – which *does* change as b_2 varies – has a number of tilings that is a polynomial in b_2.

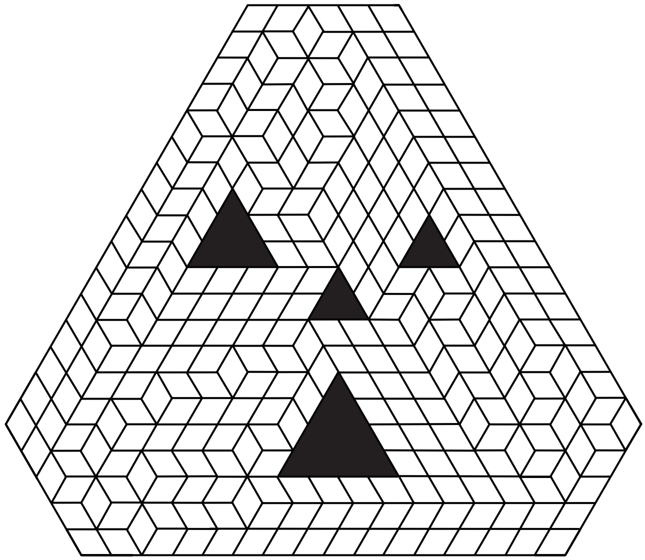

Figure 7. An example of a lozenge tiling of $S_{5,5,5,2,3,2,4,1,1,1}$.

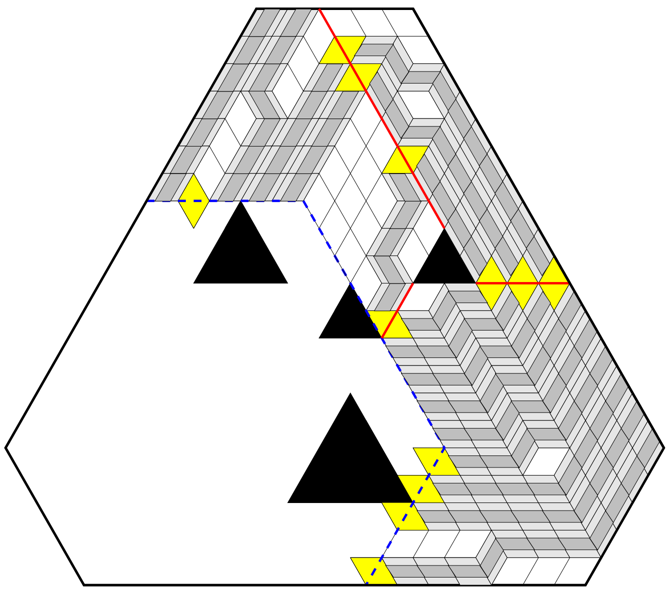

Figure 8. Proving polynomiality in b_2. The region R on the lower left delimited by the dashed line does not change as b_2 varies; the lozenges that straddle the dashed line in one of its fixed number (independent of b_2) of tilings are indicated. The number of ways each such tiling of R can be extended to a tiling of $S_{n_1,n_2,n_3,a,b_1,b_2,b_3,k_1,k_2,k_3}$ is polynomial in b_2 for fixed $n_1, n_2, n_3, a, b_1, b_3, k_1, k_2, k_3$, by the argument in [18].

This follows by the very same arguments we used in [18, Section 6]. Indeed, extend rays from the satellite of side b_2 as indicated by the red lines in Figure 8. Depending on the actual values of the fixed parameters $n_1, n_2, n_3, a, b_1, b_3, k_1, k_2, k_3$, the ray going southwest may intersect either the northeast side of R or the south side of S. Similarly, the ray going east intersects either the northeast or the southeast side of S, and the ray going northwest either the north or the northwest side of S. Figure 8 shows one of these possibilities. We prove polynomiality of $\mathrm{M}(R')$ in b_2 in this case; the others follow the same way.

As we did in [18], we fix a set $\mathcal{L}$ of lozenges straddling the blue dashed line and three sets of lozenges $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ that straddle the three red rays, respectively, and consider the number of lozenge tilings that contain the lozenges in these four sets, but no other lozengue that straddles the blue dashed line or the red rays. We show that this refined counting function is a polynomial in b_2. The crucial observation is that the lengths of the rays in R' are independent of b_2: the lengths are $\frac{a}{2} + b_1 - k_2 + n_1 + n_2 - n_3, n_2 - 2k_2, 2k_2$.

Clearly, this number is the product of the number of corresponding tilings of the three regions that the rays divide R' into. For each of these three regions, encode their tilings as families of paths of lozenges (equivalently, lattice paths on $\mathbb{Z}^2$) as indicated in Figure 8. Then by the Lindström–Gessel–Viennot theorem [30, 44], the number of tilings of each of these three regions is equal to a determinant whose order is independent of b_2, and all of whose entries are – as can be easily checked – either independent of b_2, or of the form $\binom{b_2+c}{d}$, with c and d independent of b_2. This implies that each of them is a polynomial in b_2, and the proof is complete. ■

4.3. A determinantal formula for general b_1, b_2, b_3 assuming $k_2 = k_3$

The goal of this section is to derive the following determinantal formula for

$$\mathrm{M}(S_{n_1,n_2,n_3,a,b_1,b_2,b_3,k_1,k_2,k_2})$$

that holds for general b_1, b_2, b_3. Note that we assume $k_2 = k_3$ because the situation is simpler then, but the procedure can be adapted so that it works also if $k_2 \neq k_3$. In this formula, we use the convention

$$\sum_{i=a}^{b} p(i) = -\sum_{i=b+1}^{a-1} p(i)$$

if $b < a$. Note that this implies

$$\sum_{i=a}^{a-1} p(i) = 0.$$

Theorem 4.4. *For all non-negative integers $n_1, n_2, n_3, b_1, b_2, b_3, k_1, k_2$, and even a, we have*

$$\mathrm{M}(S_{n_1,n_2,n_3,a,b_1,b_2,b_3,k_1,k_2,k_2}) = \left|\det \begin{pmatrix} M_1 & 1 \le i \le n_2 \\ M_2 & 1 \le i \le n_1 \\ \binom{i-1}{j-1-n_3+2k_2} & 1 \le i \le b_3 \\ \binom{i-\frac{a}{2}-k_2-1)}{j-1-n_3-b_3} & 1 \le i \le a \\ \binom{-\frac{a}{2}-b_1-2k_1-k_2+i-1}{j-1-n_3-\frac{a}{2}-b_3-k_1} & 1 \le i \le b_1 \\ \binom{i-1}{j-1-n_3-\frac{a}{2}-b_3-k_2} & 1 \le i \le b_2 \end{pmatrix}_{1 \le j \le n}\right|,$$

where

$$M_1 := (-1)^{[j \ge n_3-2k_2+1]b_3} \left(\binom{i-n_1-\frac{a}{2}-b_1-k_2-1}{j-1} + ((-1)^{b_1}-1) \sum_{q=n_3+\frac{a}{2}+b_3+k_1+1}^{j} \binom{i-n_1+2k_1-1}{q-1} \times \binom{-2k_1-\frac{a}{2}-b_1-k_2}{j-q} \right),$$

$$M_2 := (-1)^{[j \ge n_3+\frac{a}{2}+b_3+k_2+1]b_2} \times \binom{-n_1+n_2+n_3+\frac{a}{2}+b_2+b_3-k_2+i-1}{j-1},$$

and where we use the Iverson bracket which is defined as

$$[statement] = \begin{cases} 1 & \text{if the statement is true,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We consider the determinant in (4.5) now for arbitrary (not necessarily even) integers b_i's and set

$$d = d_1 = n_1 - 2k_1 + 1.$$

Then all entries in the first row of the fifth block (which is the one corresponding to the satellite of size b_1) are zero except for the one in column

$$j = 1 + n_3 + \frac{a}{2} + b_3 + k_1.$$

We expand with respect to this row. The new top row of the fifth block has again only a non-zero entry in column $j = 1 + n_3 + \frac{a}{2} + b_3 + k_1$, and so we expand with respect to this row now. We keep doing this until the fifth block has vanished and obtain

$$(-1)^{(n_1+n_2+n_3+\frac{3a}{2}+k_1)b_1}$$

$$\times \det \begin{pmatrix} \binom{i-d_1}{j-1} & 1 \le i \le n_2 \\ \binom{n_2+n_3+a+b_1+b_2+b_3+i-d_1}{j-1} & 1 \le i \le n_1 \\ \binom{n_1+\frac{a}{2}+b_1+k_3+i-d_1}{j-1-n_3+2k_2} & 1 \le i \le b_3 \\ \binom{n_1+b_1+i-d_1}{j-1-n_3-b_3} & 1 \le i \le a \\ \binom{n_1+\frac{a}{2}+b_1+k_2+i-d_1}{j-1-n_3-\frac{a}{2}-b_3-k_2} & 1 \le i \le b_2 \end{pmatrix}_{\substack{1 \le j \le n_3+\frac{a}{2}+b_3+k_1 \\ n_3+\frac{a}{2}+b_1+b_3+k_1+1 \le j \le n}} .$$

This can also be written as follows (omitting now the ranges for the rows i):

$$(-1)^{(n_1+n_2+n_3+\frac{3a}{2}+k_1)b_1} \det \begin{pmatrix} \mathfrak{M}_{1,1} & \mathfrak{M}_{1,2} \\ \mathfrak{M}_{2,1} & \mathfrak{M}_{2,2} \\ \mathfrak{M}_{3,1} & \mathfrak{M}_{3,2} \\ \mathfrak{M}_{4,1} & \mathfrak{M}_{4,2} \\ \mathfrak{M}_{5,1} & \mathfrak{M}_{5,2} \end{pmatrix}, \tag{4.6}$$

where

$$\mathfrak{M}_{1,1} := \binom{i-d_1}{j-1}_{1 \le j \le n_3+\frac{a}{2}+b_3+k_1},$$

$$\mathfrak{M}_{1,2} := \binom{i-d_1}{j+n_3+\frac{a}{2}+b_1+b_3+k_1-1}_{1 \le j \le n_1+n_2-n_3+\frac{a}{2}+b_2-k_1},$$

$$\mathfrak{M}_{2,1} := \binom{n_2+n_3+a+b_1+b_2+b_3+i-d_1}{j-1}_{1 \le j \le n_3+\frac{a}{2}+b_3+k_1},$$

$$\mathfrak{M}_{2,2} := \binom{n_2+n_3+a+b_1+b_2+b_3+i-d_1}{j+n_3+\frac{a}{2}+b_1+b_3+k_1-1}_{1 \le j \le n_1+n_2-n_3+\frac{a}{2}+b_2-k_1},$$

$$\mathfrak{M}_{3,1} := \binom{n_1+\frac{a}{2}+b_1+k_3+i-d_1}{j-1-n_3+2k_2}_{1 \le j \le n_3+\frac{a}{2}+b_3+k_1},$$

$$\mathfrak{M}_{3,2} := \binom{n_1+\frac{a}{2}+b_1+k_3+i-d_1}{j+n_3+\frac{a}{2}+b_1+b_3+k_1-1-n_3+2k_2}_{1 \le j \le n_1+n_2-n_3+\frac{a}{2}+b_2-k_1},$$

$$\mathfrak{M}_{4,1} := \binom{n_1+b_1+i-d_1}{j-1-n_3-b_3}_{1 \le j \le n_3+\frac{a}{2}+b_3+k_1},$$

$$\mathfrak{M}_{4,2} := \binom{n_1+b_1+i-d_1}{j+n_3+\frac{a}{2}+b_1+b_3+k_1-1-n_3-b_3}_{1 \le j \le n_1+n_2-n_3+\frac{a}{2}+b_2-k_1},$$

$$\mathfrak{M}_{5,1} := \binom{n_1+\frac{a}{2}+b_1+k_2+i-d_1}{j-1-n_3-\frac{a}{2}-b_3-k_2}_{1 \le j \le n_3+\frac{a}{2}+b_3+k_1},$$

$$\mathfrak{M}_{5,2} := \binom{n_1 + \frac{a}{2} + b_1 + k_2 + i - d_1}{\mathfrak{A}}_{1 \le j \le n_1+n_2-n_3+\frac{a}{2}+b_2-k_1}$$

and

$$\mathfrak{A} := j + n_3 + \frac{a}{2} + b_1 + b_3 + k_1 - 1 - n_3 - \frac{a}{2} - b_3 - k_2.$$

Concerning the range of the right block, note that

$$n_1 + n_2 - n_3 + \frac{a}{2} + b_2 - k_1 \ge 0$$

because of the following: consider the wedge of the line containing the $(n_1 + a + b_1 + b_2 + b_3)$-side of the hexagon and of the line containing the $(n_2 + a + b_1 + b_2 + b_3)$-side of the hexagon. Then the length of the section of the horizontal line containing the top vertex of the satellite of size b_1 in this wedge is

$$n_1 + n_2 + \frac{a}{2} + b_2 - k_1$$

which can be seen as follows (the reader is advised to look at Figure 6):

- the length of the section between the NE side of the hexagon and the line extending the NE side of the core is $n_2 + b_2$;
- the length of the section between the line extending the NE side of the core and the top vertex of the satellite of size b_1 is $\frac{a}{2} + k_1$;
- the length of section between the top vertex of the satellite of size b_1 and the NW side of the hexagon is $n_1 - 2k_1$.

This length is obviously greater than or equal to n_3, the length of the top side of the hexagon.

Next we will show that we can modify this formula by introducing $(-1)^{b_1}$ at various places such that the result is a polynomial function in b_1. Since this has of course no effect if b_1 is even, the modified formula will give the number of lozenge tilings for even b_1. However, using Lemma 4.3 and the fact that a polynomial is uniquely determined by its evaluation on even integers, the modified formula gives the number of lozenge tilings for all non-negative integers b_1 (however still assuming that b_2 and b_3 are even).

The following observations are crucial.

1. The entries in the first $n_3 + \frac{a}{2} + b_3 + k_1$ columns are all polynomials in b_1, since b_1 appears at most in the upper parameter of the binomial coefficient.

2. The entries that are right of column $n_3 + \frac{a}{2} + b_3 + k_1$ and below row n_2 are also polynomials in b_1: these entries are binomial coefficients of the form $\binom{b_1+s}{b_1+t}$ for some integers s and t. We have $b_1 + s \ge 0$ which follows basically because the

satellite of size b_1 is the leftmost removed (big) triangle except for the triangle of size n_2, which however corresponds to the top block (see also (4.3)). Thus, we can apply the symmetry of the binomial coefficient, i.e.,

$$\binom{n}{k} = \binom{n}{n-k} \quad \text{if } n \geq 0, \tag{4.7}$$

to obtain binomial coefficients where b_1 only appears in the top parameter.

3. As for the remaining entries in row 1 to n_2, they are binomial coefficients of the form $\binom{s}{b_1+t}$ where $s < b_1 + t$. In order to see this, observe that the extreme case with regard to this inequality is when $i = n_2$ and $j = 1$. In this case we need to show that

$$n_2 \leq \left(n_3 + \frac{a}{2} + b_3 + k_1\right) + b_1 + (n_1 - 2k_1).$$

However, this is obvious: $n_3 + \frac{a}{2} + b_3 + k_1$ is the "lattice" distance of the satellite of size b_1 from the bottom of the hexagon, thus $(n_3 + \frac{a}{2} + b_3 + k_1) + b_1$ is the "lattice" distance between the top of this satellite to the bottom of the hexagon and going $n_1 - 2k_1$ unit steps from this top into $\nwarrow$-direction will bring us to a point on the side of length $n_1 + a + b_1 + b_2 + b_3$, which is thus surely above the side of length n_2.

We claim that this implies that $(-1)^{b_1}\binom{s}{b_1+t}$ is polynomial in b_1: We use the second elementary transformation for binomial coefficients, i.e.,

$$\binom{n}{k} = (-1)^k \binom{k-n-1}{k} \tag{4.8}$$

to see that

$$\begin{aligned}\binom{s}{b_1+t} &= (-1)^{b_1+t}\binom{b_1+t-s-1}{b_1+t} \\ &= (-1)^{b_1+t}\binom{b_1+t-s-1}{-s-1},\end{aligned}$$

where the last step follows from the symmetry (4.7) which can be applied since

$$b_1 + t - s - 1 \geq 0.$$

It follows that we obtain a formula that is a polynomial function in b_1 and coincides with the original formula for even b_1 if we do the following:

- multiply (4.6) with $(-1)^{(n_1+n_2+n_3+\frac{3a}{2}+k_1)b_1}$;
- multiply the entries in the first n_2 rows and right of column $n_3 + \frac{a}{2} + b_3 + k_1$ with $(-1)^{b_1}$.

If we "reverse" after this modification our calculation so that we again have a block that corresponds to the satellite of size b_1, we obtain

$$(-1)^{(n_1+n_2+n_3+\frac{3a}{2}+k_1)b_1}\prod_{i=1}^{b_3}\Delta_{c_{3,i}}^{n_3-2k_3}\prod_{i=1}^{a}\Delta_{c_{4,i}}^{n_3+b_3}$$
$$\times\prod_{i=1}^{b_1}\Delta_{c_{5,i}}^{n_3+\frac{a}{2}+b_3+k_1}\prod_{i=1}^{b_2}\Delta_{c_{6,i}}^{n_3+\frac{a}{2}+b_3+k_2}$$
$$\times\det\begin{pmatrix}\binom{c_{1,i}-d_1}{j-1}(-1)^{[j\geq n_3+\frac{a}{2}+b_1+b_3+k_1+1]b_1} & 1\leq i\leq n_2\\ \binom{c_{2,i}-d_1}{j-1} & 1\leq i\leq n_1\\ \binom{c_{3,i}-d_1}{j-1} & 1\leq i\leq b_3\\ \binom{c_{4,i}-d_1}{j-1} & 1\leq i\leq a\\ \binom{c_{5,i}-d_1}{j-1} & 1\leq i\leq b_1\\ \binom{c_{6,i}-d_1}{j-1} & 1\leq i\leq b_2\end{pmatrix}_{1\leq j\leq n}$$

at

$$\begin{aligned}c_{1,i}&=i,\\ c_{2,i}&=n_2+n_3+a+b_1+b_2+b_3+i,\\ c_{3,i}&=n_1+\frac{a}{2}+b_1+k_2+i,\\ c_{4,i}&=n_1+b_1+i,\\ c_{5,i}&=n_1-2k_1+i,\\ c_{6,i}&=n_1+\frac{a}{2}+b_1+k_2+i,\end{aligned}$$

provided that $d_1=n_1-2k_1+1$. Note that $(-1)^{[j\geq n_3+\frac{a}{2}+b_1+b_3+k_1+1]b_1}$ can actually be replaced by any $(-1)^{[j\geq n_3+\frac{a}{2}+l+b_3+k_1+1]b_1}$ with $0\leq l\leq b_1$: when "eliminating" a block (in our case the fifth block) it becomes apparent that the values of certain entries, in our case the entries in columns

$$\begin{gathered}n_3+\frac{a}{2}+b_3+k_1+1,\\ n_3+\frac{a}{2}+b_3+k_1+2,\\ \vdots\\ n_3+\frac{a}{2}+b_3+k_1+b_1,\end{gathered}$$

do not play a role at all. When we reverse the procedure, we are free to choose the values conveniently. We choose $l = 0$ in the following.

Now, observe that, by the Chu–Vandermonde summation,

$$\binom{c-d_2}{j-1} = \sum_{q=1}^{j} \binom{c-d_1}{q-1}\binom{d_1-d_2}{j-q},$$

and multiply the matrix underlying the determinant from the right with the upper triangular matrix $\left(\binom{d_1-d_2}{j-i}\right)_{1\le i,j\le n}$ with determinant 1. This gives the following matrix:

$$\begin{pmatrix} M_3 & 1 \le i \le n_2 \\ \binom{c_{2,i}-d_2}{j-1} & 1 \le i \le n_1 \\ \binom{c_{3,i}-d_2}{j-1} & 1 \le i \le b_3 \\ \binom{c_{4,i}-d_2}{j-1} & 1 \le i \le a \\ \binom{c_{5,i}-d_2}{j-1} & 1 \le i \le b_1 \\ \binom{c_{6,i}-d_2}{j-1} & 1 \le i \le b_2 \end{pmatrix}_{1\le j\le n}$$

where

$$M_3 := \binom{c_{1,i}-d_2}{j-1} + ((-1)^{b_1}-1)\sum_{q=n_3+\frac{a}{2}+b_3+k_1+1}^{j} \binom{c_{1,i}-d_1}{q-1}\binom{d_1-d_2}{j-q}$$

Specializing the $c_{l,i}$, we obtain the following expression:

$$(-1)^{(n_1+n_2+n_3+\frac{3a}{2}+k_1)b_1}$$

$$\times \det \begin{pmatrix} M_4 & 1 \le i \le n_2 \\ \binom{n_2+n_3+a+b_1+b_2+b_3+i-d_2}{j-1} & 1 \le i \le n_1 \\ \binom{n_1+\frac{a}{2}+b_1+k_2+i-d_2}{j-1-n_3+2k_2} & 1 \le i \le b_3 \\ \binom{n_1+b_1+i-d_2}{j-1-n_3-b_3} & 1 \le i \le a \\ \binom{n_1-2k_1+i-d_2}{j-1-n_3-\frac{a}{2}-b_3-k_1} & 1 \le i \le b_1 \\ \binom{n_1+\frac{a}{2}+b_1+k_2+i-d_2}{j-1-n_3-\frac{a}{2}-b_3-k_2} & 1 \le i \le b_2 \end{pmatrix}_{1\le j\le n}$$

where

$$M_4 := \binom{i-d_2}{j-1} + ((-1)^{b_1} - 1) \sum_{q=n_3+\frac{a}{2}+b_3+k_1+1}^{j} \binom{i-d_1}{q-1}\binom{d_1-d_2}{j-q}.$$

We set

$$d_2 = n_1 + \frac{a}{2} + b_1 + k_2 + 1.$$

(The assumption $k_2 = k_3$ is now useful because it allows us to eliminate the blocks of the satellites of sizes b_2 and b_3 simultaneously.) With this, all entries in first row of the bottom block are zero except for the one in column $j = 1 + n_3 + \frac{a}{2} + b_3 + k_2$, and so we expand with respect to this row. We can keep doing this until the bottom block vanishes and obtain the following:

$$(-1)^{(n_1+n_2+n_3+\frac{3a}{2}+k_1)b_1+(n_1+n_2+n_3+\frac{3a}{2}+b_1+k_2)b_2}$$
$$\times \det \begin{pmatrix} M_4 & 1 \le i \le n_2 \\ \binom{n_2+n_3+a+b_1+b_2+b_3+i-d_2}{j-1} & 1 \le i \le n_1 \\ \binom{n_1+\frac{a}{2}+b_1+k_2+i-d_2}{j-1-n_3+2k_2} & 1 \le i \le b_3 \\ \binom{n_1+b_1+i-d_2}{j-1-n_3-b_3} & 1 \le i \le a \\ \binom{n_1-2k_1+i-d_2}{j-1-n_3-\frac{a}{2}-b_3-k_1} & 1 \le i \le b_1 \end{pmatrix}_{\substack{1\le j\le n_3+\frac{a}{2}+b_3+k_2 \\ n_3+\frac{a}{2}+b_2+b_3+k_2+1\le j\le n}}.$$

The first $n_3 + \frac{a}{2} + b_3 + k_2$ columns of the matrix underlying the determinant are

$$\begin{pmatrix} M_4 & 1 \le i \le n_2 \\ \binom{n_2+n_3+a+b_1+b_2+b_3+i-d_2}{j-1} & 1 \le i \le n_1 \\ \binom{n_1+\frac{a}{2}+b_1+k_2+i-d_2}{j-1-n_3+2k_2} & 1 \le i \le b_3 \\ \binom{n_1+b_1+i-d_2}{j-1-n_3-b_3} & 1 \le i \le a \\ \binom{n_1-2k_1+i-d_2}{j-1-n_3-\frac{a}{2}-b_3-k_1} & 1 \le i \le b_1 \end{pmatrix}_{1\le j\le n_3+\frac{a}{2}+b_3+k_2}.$$

Only the entries in the second block depend on b_2, and, since b_2 appears only in the upper parameter of the binomial coefficient, these entries are polynomials in b_2. The matrix consisting of the remaining columns can be written as follows:

$$\begin{pmatrix} M_5 & 1 \le i \le n_2 \\ \binom{n_2+n_3+a+b_1+b_2+b_3+i-d_2}{j+n_3+\frac{a}{2}+b_2+b_3+k_2-1} & 1 \le i \le n_1 \\ \binom{n_1+\frac{a}{2}+b_1+k_2+i-d_2}{j+\frac{a}{2}+b_2+b_3+3k_2-1} & 1 \le i \le b_3 \\ \binom{n_1+b_1+i-d_2}{j+\frac{a}{2}+b_2+k_2-1} & 1 \le i \le a \\ \binom{n_1-2k_1+i-d_2}{j+b_2+k_2-k_1-1} & 1 \le i \le b_1 \end{pmatrix},$$

where

$$\begin{aligned} M_5 := &\binom{i-d_2}{j+n_3+\frac{a}{2}+b_2+b_3+k_2-1} \\ &+ ((-1)^{b_1}-1) \sum_{q=n_3+\frac{a}{2}+b_3+k_1+1}^{j+n_3+\frac{a}{2}+b_2+b_3+k_2} \binom{i-d_1}{q-1} \\ &\qquad\qquad \times \binom{d_1-d_2}{j+n_3+\frac{a}{2}+b_2+b_3+k_2-q} \end{aligned}$$

and

$$1 \le j \le n_1+n_2-n_3+\frac{a}{2}+b_1-k_2.$$

We analyze the different blocks of the matrix.

1. Top block. First note that in

$$\binom{i-d_2}{j+n_3+\frac{a}{2}+b_2+b_3+k_2-1},$$

the upper parameter is always less than the lower parameter: this follows from

$$n_2-2k_2 \le n_1+n_3+a+b_1+b_2+b_3,$$

which is true since $(n_1+n_3+a+b_1+b_2+b_3)\cdot\frac{\sqrt{3}}{2}$ is the distance of the SW side and the NE side of the hexagon and $(n_2-2k_2)\cdot\frac{\sqrt{3}}{2}$ is the distance of the NW side of the hexagon and the satellite of size b_1. Therefore, in analogy to a situation for b_1, this binomial coefficient is a polynomial function in b_2 after multiplication with $(-1)^{b_2}$. Now, as $d_1-d_2<0$ (unless $b_1=0$ in which case the entry simplifies to the binomial coefficient that was already discussed), $\binom{d_1-d_2}{j+n_3+\frac{a}{2}+b_2+b_3+k_2-q}$ is a polynomial function in b_2 and q when multiplied with $(-1)^{b_2+q}$. In case $i-d_1$ is non-negative, we can sum over all q less than or equal to $i-d_1+1$ (because otherwise the binomial coefficient $\binom{i-d_1}{q-1}$ is zero), and, since b_2 has now disappeared from the upper bound in the summation, the entry is seen to be a polynomial function in b_2 after multiplication with $(-1)^{b_2}$. If, however $i-d_1$ is negative, then $\binom{i-d_1}{q-1}$ is a polynomial in q after multiplication with $(-1)^q$, and so the summand

$$\binom{i-d_1}{q-1}\binom{d_1-d_2}{j+n_3+\frac{a}{2}+b_2+b_3+k_2-q}$$

is a polynomial function in q. Using the fact that $\sum_{i=a}^{b} p(i)$ is a polynomial function in a and b if $p(i)$ is a polynomial in i, it follows that also in this case, the entry is a polynomial function in b_2 after multiplication with $(-1)^{b_2}$.

2. Second block. b_2 appears in the upper parameter as well as in the lower parameter of the binomial coefficient. As the upper parameter is non-negative, the symmetry can be applied in order to remove b_2 from the lower parameter.

3. As for the remaining blocks, the entries are always of the form $\binom{s}{b_2+t}$ where $s < b_2+t$, which implies that these entries are polynomial functions after multiplication with $(-1)^{b_2}$.

Summarizing we see that, in order to transform the determinant formula into a polynomial function in b_2, we need to do the following:

- multiply with $(-1)^{(n_1+n_2+n_3+\frac{3a}{2}+b_1+k_2)b_2}$;
- multiply the entries in the columns right of the column $n_3+\frac{a}{2}+b_3+k_2$ with $(-1)^{b_2}$, except for those in the second block.

Since there are $n_1+n_2-n_3+\frac{a}{2}+b_1-k_2$ columns right of the column $n_3+\frac{a}{2}+b_3+k_2$, this is equivalent to the following:

- multiply only the entries in the second block right of the column $n_3+\frac{a}{2}+b_3+k_2$ with $(-1)^{b_2}$.

Going back in our calculation and reintroducing a block with b_2 rows, we obtain

$$(-1)^{(n_1+n_2+n_3+\frac{3a}{2}+k_1)b_1}\det\begin{pmatrix} M_4 & 1\le i\le n_2\\ M_6 & 1\le i\le n_1\\ \binom{n_1+\frac{a}{2}+b_1+k_2+i-d_2}{j-1-n_3+2k_2} & 1\le i\le b_3\\ \binom{n_1+b_1+i-d_2}{j-1-n_3-b_3} & 1\le i\le a\\ \binom{n_1-2k_1+i-d_2}{j-1-n_3-\frac{a}{2}-b_3-k_1} & 1\le i\le b_1\\ \binom{n_1+\frac{a}{2}+b_1+k_2+i-d_2}{j-1-n_3-\frac{a}{2}-b_3-k_2} & 1\le i\le b_2 \end{pmatrix}_{1\le j\le n},$$

where

$$M_6 := (-1)^{[j \geq n_3 + \frac{a}{2} + b_3 + k_2 + 1] b_2} \binom{n_2 + n_3 + a + b_1 + b_3 + i - d_2}{j - 1}.$$

As for b_3, a similar argument shows that we need to make the adjustment only in the top block. This concludes the proof of the theorem. ■

4.4. Polynomiality in *a*

The purpose of this section is to modify the formula in Theorem 4.4 to reveal the polynomiality of the underlying determinant in a. More specifically, we prove the following theorem.

Theorem 4.5. *Let*

$$A = \begin{pmatrix} M_7 & 1 \leq i \leq n_2 \\ \binom{-n_1+n_2+n_3+a+b_2+b_3+i-1}{j-1} & 1 \leq i \leq n_1 \\ \binom{\frac{a}{2}+k_2+i-1}{j-1-n_3+2k_2} & 1 \leq i \leq b_3 \\ 0 & 1 \leq i \leq b_1 \\ 0 & 1 \leq i \leq b_2 \end{pmatrix}_{1 \leq j \leq n_3+b_3}$$

where

$$M_7 := \binom{i - n_1 - b_1 - 1}{j - 1} + ((-1)^{b_3} - 1) \sum_{p=1+n_3-2k_2}^{j} \binom{i - n_1 - \frac{a}{2} - b_1 - k_2 - 1}{p - 1} \binom{\frac{a}{2} + k_2}{j - p},$$

and

$$B' = \begin{pmatrix} M_8 & 1 \leq i \leq n_2 \\ M_9 & 1 \leq i \leq n_1 \\ 0 & 1 \leq i \leq b_3 \\ (-1)^j \binom{\frac{a}{2}+b_1+k_1-i+j-1}{b_1+2k_1-i} & 1 \leq i \leq b_1 \\ \binom{\frac{a}{2}+k_2+i-1}{2k_2+i-j} & 1 \leq i \leq b_2 \end{pmatrix},$$

where

$$M_8 := (-1)^{j+n_3-1}\binom{n_1+n_3+a+b_1+b_3-i+j-1}{n_1+b_1-i}$$
$$+((-1)^{b_1}-1)(-1)^{j+n_3-1}\sum_{q=n_3+\frac{a}{2}+b_3+k_1+1}^{j+n_3+a+b_3}\binom{n_1-2k_1-i+q-1}{n_1-2k_1-i}$$
$$\times\binom{j+n_3+a+b_1+b_3+2k_1-q-1}{b_1+2k_1-1}$$

and

$$M_9 := \binom{-n_1+n_2+n_3+a+b_2+b_3+i-1}{-n_1+n_2+b_2+i-j}$$
$$+((-1)^{b_2}-1)\sum_{p=1}^{-n_1+n_2+b_2-2k_2+i}\binom{-n_1+n_2+n_3+\frac{a}{2}+b_2+b_3-k_2+i-1}{-n_1+n_2+b_2-2k_2+i-p}$$
$$\times\binom{\frac{a}{2}+k_2}{2k_2-j+p},$$

and where the range of j in B' is $1 \le j \le n_1+n_2-n_3+b_1+b_2$. Then the number of lozenge tilings of $S_{n_1,n_2,n_3,a,b_1,b_2,b_3,k_1,k_2,k_2}$ is the absolute value of $\det(A \mid B')$. *The determinant is obviously a polynomial in a since all matrix entries are polynomials in a.*

Proof. We need to eliminate the fourth block in the formula in Theorem 4.4. Note that this formula can also be written up to sign as follows:

$$\prod_{i=1}^{b_3}\Delta_{c_{3,i}}^{n_3-2k_2}\prod_{i=1}^{a}\Delta_{c_{4,i}}^{n_3+b_3}\prod_{i=1}^{b_1}\Delta_{c_{5,i}}^{n_3+\frac{a}{2}+b_3+k_1}\prod_{i=1}^{b_2}\Delta_{c_{6,i}}^{n_3+\frac{a}{2}+b_3+k_2}$$
$$\times\det\begin{pmatrix}(-1)^{[j\ge 1+n_3-2b_2]b_3}M_3 & 1\le i\le n_2\\ (-1)^{[j\ge n_3+\frac{a}{2}+b_3+k_2+1]b_2}\binom{c_{2,i}-d_2}{j-1} & 1\le i\le n_1\\ \binom{c_{3,i}-d_2}{j-1} & 1\le i\le b_3\\ \binom{c_{4,i}-d_2}{j-1} & 1\le i\le a\\ \binom{c_{5,i}-d_2}{j-1} & 1\le i\le b_1\\ \binom{c_{6,i}-d_2}{j-1} & 1\le i\le b_2\end{pmatrix}_{1\le j\le n}$$

evaluated at

$$\begin{aligned}
c_{1,i} &= i, \\
c_{2,i} &= n_2 + n_3 + a + b_1 + b_2 + b_3 + i, \\
c_{3,i} &= n_1 + \frac{a}{2} + b_1 + k_2 + i, \\
c_{4,i} &= n_1 + b_1 + i, \\
c_{5,i} &= n_1 - 2k_1 + i, \\
c_{6,i} &= n_1 + \frac{a}{2} + b_1 + k_2 + i,
\end{aligned}$$

and

$$\begin{aligned}
d_1 &= n_1 - 2k_1 + 1, \\
d_2 &= n_1 + \frac{a}{2} + b_1 + k_2 + 1.
\end{aligned}$$

We multiply the matrix underlying the determinant from the right with the upper triangular matrix $\left(\binom{d_2-d_3}{j-i}\right)_{1\le i,j\le n}$ with determinant 1. In block l, $3 \le l \le 6$, the entry is then replaced by $\binom{c_{l,i}-d_3}{j-1}$. In the second block, we have

$$\sum_{p=1}^{n} (-1)^{[p\ge n_3+\frac{a}{2}+b_3+k_2+1]b_2} \binom{c_{2,i}-d_2}{p-1}\binom{d_2-d_3}{j-p}$$
$$= \binom{c_{2,i}-d_3}{j-1} + ((-1)^{b_2}-1) \sum_{p=n_3+\frac{a}{2}+b_3+k_2+1}^{j} \binom{c_{2,i}-d_2}{p-1}\binom{d_2-d_3}{j-p}.$$

As for the top block, using

$$n_3 + \frac{a}{2} + b_3 + k_2 + 1 \ge n_3 - 2k_2 + 1$$

as well as the Chu–Vandermonde summation, we have

$$\sum_{p=1}^{n} (-1)^{[p\ge 1+n_3-2k_2+b_3]b_3} \binom{c_{1,i}-d_2}{p-1}\binom{d_2-d_3}{j-p}$$
$$+ ((-1)^{b_1+b_3} + (-1)^{1+b_3}) \sum_{p\ge 1, q\ge n_3+\frac{a}{2}+b_3+k_1+1} \binom{c_{1,i}-d_1}{q-1}\binom{d_1-d_2}{p-q}\binom{d_2-d_3}{j-p}$$
$$= \binom{c_{1,i}-d_3}{j-1} + ((-1)^{b_3}-1) \sum_{p=1+n_3-2k_2}^{j} \binom{c_{1,i}-d_2}{p-1}\binom{d_2-d_3}{j-p}$$
$$+ ((-1)^{b_1+b_3} - (-1)^{b_3}) \sum_{q=n_3+\frac{a}{2}+b_3+k_1+1}^{j} \binom{c_{1,i}-d_1}{q-1}\binom{d_1-d_3}{j-q}.$$

We obtain up to sign the following:

$$\det\begin{pmatrix} M_{10} & 1\le i\le n_2\\ M_{11} & 1\le i\le n_1\\ \binom{c_{3,i}-d_3}{j-1-n_3+2k_2} & 1\le i\le b_3\\ \binom{c_{4,i}-d_3}{j-1-n_3-b_3} & 1\le i\le a\\ \binom{c_{5,i}-d_3}{j-1-n_3-\frac{a}{2}-b_3-k_1} & 1\le i\le b_1\\ \binom{c_{6,i}-d_3}{j-1-n_3-\frac{a}{2}-b_3-k_2} & 1\le i\le b_2 \end{pmatrix}_{1\le j\le n},$$

where

$$M_{10}:=\binom{c_{1,i}-d_3}{j-1}+((-1)^{b_3}-1)\sum_{p=1+n_3-2k_2}^{j}\binom{c_{1,i}-d_2}{p-1}\binom{d_2-d_3}{j-p}$$
$$+((-1)^{b_1+b_3}-(-1)^{b_3})\sum_{q=n_3+\frac{a}{2}+b_3+k_1+1}^{j}\binom{c_{1,i}-d_1}{q-1}\binom{d_1-d_3}{j-q}$$

and

$$M_{11}:=\binom{c_{2,i}-d_3}{j-1}+((-1)^{b_2}-1)\sum_{p=n_3+\frac{a}{2}+b_3+k_2+1}^{j}\binom{c_{2,i}-d_2}{p-1}\binom{d_2-d_3}{j-p}.$$

Evaluating at

$$\begin{aligned} c_{1,i}&=i, & c_{2,i}&=n_2+n_3+a+b_1+b_2+b_3+i,\\ c_{3,i}&=n_1+\frac{a}{2}+b_1+k_2+i, & c_{4,i}&=n_1+b_1+i,\\ c_{5,i}&=n_1-2k_1+i, & c_{6,i}&=n_1+\frac{a}{2}+b_1+k_2+i \end{aligned}$$

gives

$$\det\begin{pmatrix} M_{12} & 1\le i\le n_2\\ M_{13} & 1\le i\le n_1\\ \binom{n_1+\frac{a}{2}+b_1+k_2+i-d_3}{j-1-n_3+2k_2} & 1\le i\le b_3\\ \binom{n_1+b_1+i-d_3}{j-1-n_3-b_3} & 1\le i\le a\\ \binom{n_1-2k_1+i-d_3}{j-1-n_3-\frac{a}{2}-b_3-k_1} & 1\le i\le b_1\\ \binom{n_1+\frac{a}{2}+b_1+k_2+i-d_3}{j-1-n_3-\frac{a}{2}-b_3-k_2} & 1\le i\le b_2 \end{pmatrix}_{1\le j\le n},$$

where

$$M_{12} := \binom{i-d_3}{j-1} + ((-1)^{b_3}-1)\sum_{p=1+n_3-2k_2}^{j}\binom{i-d_2}{p-1}\binom{d_2-d_3}{j-p} + ((-1)^{b_1+b_3}-(-1)^{b_3})\sum_{q=n_3+\frac{a}{2}+b_3+k_1+1}^{j}\binom{i-d_1}{q-1}\binom{d_1-d_3}{j-q}$$

and

$$M_{13} := \binom{n_2+n_3+a+b_1+b_2+b_3+i-d_3}{j-1} + ((-1)^{b_2}-1)\sum_{p=n_3+\frac{a}{2}+b_3+k_2+1}^{j}\binom{n_2+n_3+a+b_1+b_2+b_3+i-d_2}{p-1} \times \binom{d_2-d_3}{j-p}.$$

Now, we perform the replacement

$$d_1 = n_1 - 2k_1 + 1, \quad d_2 = n_1 + \frac{a}{2} + b_1 + k_2 + 1,$$

and specify furthermore

$$d_3 = n_1 + b_1 + 1.$$

We get

$$\det\begin{pmatrix} M_{14} & 1 \le i \le n_2 \\ M_{15} & 1 \le i \le n_1 \\ \binom{\frac{a}{2}+k_2+i-1}{j-1-n_3+2k_2} & 1 \le i \le b_3 \\ \binom{i-1}{j-1-n_3-b_3} & 1 \le i \le a \\ \binom{-b_1-2k_1+i-1}{j-1-n_3-\frac{a}{2}-b_3-k_1} & 1 \le i \le b_1 \\ \binom{\frac{a}{2}+k_2+i-1}{j-1-n_3-\frac{a}{2}-b_3-k_2} & 1 \le i \le b_2 \end{pmatrix}_{1\le j\le n},$$

where

$$M_{14} := \binom{i-n_1-b_1-1}{j-1} + ((-1)^{b_3}-1)\sum_{p=1+n_3-2k_2}^{j}\binom{i-n_1-\frac{a}{2}-b_1-k_2-1}{p-1}\binom{\frac{a}{2}+k_2}{j-p} + ((-1)^{b_1+b_3}-(-1)^{b_3})\sum_{q=n_3+\frac{a}{2}+b_3+k_1+1}^{j}\binom{i-n_1+2k_1-1}{q-1}\binom{-b_1-2k_1}{j-q}$$

and

$$M_{15} := \binom{-n_1 + n_2 + n_3 + a + b_2 + b_3 + i - 1}{j - 1} + ((-1)^{b_2} - 1) \sum_{p=n_3+\frac{a}{2}+b_3+k_2+1}^{j} \binom{-n_1 + n_2 + n_3 + \frac{a}{2} + b_2 + b_3 - k_2 + i - 1}{p - 1} \times \binom{\frac{a}{2} + k_2}{j - p}.$$

We can now eliminate the fourth block, and obtain up to sign

$$\det \begin{pmatrix} M_{14} & 1 \le i \le n_2 \\ M_{15} & 1 \le i \le n_1 \\ \binom{\frac{a}{2}+k_2+i-1}{j-1-n_3+2k_2} & 1 \le i \le b_3 \\ \binom{-b_1-2k_1+i-1}{j-1-n_3-\frac{a}{2}-b_3-k_1} & 1 \le i \le b_1 \\ \binom{\frac{a}{2}+k_2+i-1}{j-1-n_3-\frac{a}{2}-b_3-k_2} & 1 \le i \le b_2 \end{pmatrix},$$

where the range for j is

$$1 \le j \le n_3 + b_3$$

and

$$n_3 + a + b_3 + 1 \le j \le n.$$

Note that the entries vanish for $1 \le j \le n_3 + b_3$ in blocks 4 and 5, as well as for $n_3 + a + b_3 + 1 \le j \le n$ in block 3. Also, for $1 \le j \le n_3 + b_3$, the last sums for the entries in block 1 and 2 vanish, since the upper parameter in the summation is less than the lower parameter and therefore the lower parameter in the binomial coefficients

$$\binom{-b_1 - 2k_1}{j - q} \quad \text{and} \quad \binom{\frac{a}{2} + k_2}{j - p}$$

are negative. Now, note that the $n_3 + b_3$ leftmost columns constitute the matrix A in the statement of the theorem. The entries of A are obviously polynomials in a because a appears only in the upper parameter of the binomial coefficients. We define

$$B = \begin{pmatrix} M_{16} & 1 \le i \le n_2 \\ M_{17} & 1 \le i \le n_1 \\ 0 & 1 \le i \le b_3 \\ \binom{-b_1-2k_1+i-1}{\frac{a}{2}-k_1+j-1} & 1 \le i \le b_1 \\ \binom{\frac{a}{2}+k_2+i-1}{\frac{a}{2}-k_2+j-1} & 1 \le i \le b_2 \end{pmatrix},$$

where

$$M_{16} := \binom{i - n_1 - b_1 - 1}{j + n_3 + a + b_3 - 1}$$
$$+ ((-1)^{b_3} - 1) \sum_{p=1+n_3-2k_2}^{j+n_3+a+b_3} \binom{i - n_1 - \frac{a}{2} - b_1 - k_2 - 1}{p - 1}$$
$$\times \binom{\frac{a}{2} + k_2}{j + n_3 + a + b_3 - p}$$
$$+ ((-1)^{b_1+b_3} - (-1)^{b_3}) \sum_{q=n_3+\frac{a}{2}+b_3+k_1+1}^{j+n_3+a+b_3} \binom{i - n_1 + 2k_1 - 1}{q - 1}$$
$$\times \binom{-b_1 - 2k_1}{j + n_3 + a + b_3 - q}$$

and

$$M_{17} := \binom{-n_1 + n_2 + n_3 + a + b_2 + b_3 + i - 1}{j + n_3 + a + b_3 - 1}$$
$$+ ((-1)^{b_2} - 1)$$
$$\times \sum_{p=n_3+\frac{a}{2}+b_3+k_2+1}^{j+n_3+a+b_3} \binom{-n_1 + n_2 + n_3 + \frac{a}{2} + b_2 + b_3 - k_2 + i - 1}{p - 1}$$
$$\times \binom{\frac{a}{2} + k_2}{j + n_3 + a + b_3 - p}$$

and where $1 \le j \le n_1 + n_2 - n_3 + b_1 + b_2$. We know that

$$\mathrm{M}(H_{n_1,n_2,n_3,a,b_1,b_2,b_3,k_1,k_2,k_2}) = |\det(A|B)|.$$

The entry in the first block can be simplified as follows. We can extend the first sum to all positive p as

$$\binom{\frac{a}{2} + k_2}{j + n_3 + a + b_3 - p}$$

vanishes for $1 \le p \le n_3 - 2k_2 + b_3$. Hence, by the Chu–Vandermonde summation, the first sum evaluates to

$$\binom{i - n_1 - b_1 - 1}{j + n_3 + a + b_3 - 1},$$

which can then be combined with the first term. We obtain

$$B = \begin{pmatrix} M_{18} & 1 \le i \le n_2 \\ M_{17} & 1 \le i \le n_1 \\ 0 & 1 \le i \le b_3 \\ \binom{-b_1-2k_1+i-1}{\frac{a}{2}-k_1+j-1} & 1 \le i \le b_1 \\ \binom{\frac{a}{2}+k_2+i-1}{\frac{a}{2}-k_2+j-1} & 1 \le i \le b_2 \end{pmatrix},$$

where

$$M_{18} := (-1)^{b_3}\binom{i-n_1-b_1-1}{j+n_3+a+b_3-1} + ((-1)^{b_1+b_3}-(-1)^{b_3}) \sum_{q=n_3+\frac{a}{2}+b_3+k_1+1}^{j+n_3+a+b_3} \binom{i-n_1+2k_1-1)}{q-1} \times \binom{-b_1-2k_1}{j+n_3+a+b_3-q}.$$

However, using (4.7) as well as (4.8), B can also be written as follows

$$B = \begin{pmatrix} M_{19} & 1 \le i \le n_2 \\ M_{20} & 1 \le i \le n_1 \\ 0 & 1 \le i \le b_3 \\ (-1)^{\frac{a}{2}-k_1+j-1}\binom{\frac{a}{2}+b_1+k_1-i+j-1}{b_1+2k_1-i} & 1 \le i \le b_1 \\ \binom{\frac{a}{2}+k_2+i-1}{2k_2+i-j} & 1 \le i \le b_2 \end{pmatrix},$$

where

$$M_{19} := (-1)^{j+a+n_3-1}\binom{n_1+n_3+a+b_1+b_3-i+j-1}{n_1+b_1-i} + ((-1)^{b_1}-1)(-1)^{j+n_3+a-1} \sum_{q=n_3+\frac{a}{2}+b_3+k_1+1}^{j+n_3+a+b_3} \binom{n_1-2k_1-i+q-1}{n_1-2k_1-i} \times \binom{j+n_3+a+b_1+b_3+2k_1-q-1}{b_1+2k_1-1}$$

and

$$M_{20} := \binom{-n_1+n_2+n_3+a+b_2+b_3+i-1}{-n_1+n_2+b_2+i-j}$$
$$+ ((-1)^{b_2}-1)$$
$$\times \sum_{p=1}^{-n_1+n_2+b_2-2k_2+i} \binom{-n_1+n_2+n_3+\frac{a}{2}+b_2+b_3-k_2+i-1}{-n_1+n_2+b_2-2k_2+i-p}$$
$$\times \binom{\frac{a}{2}+k_2}{2k_2-j+p}.$$

Now, it can be seen that the entries are – up to some signs – polynomials in a. Since a is even and we are only interested in the determinant up to sign, we can replace B by B' from the statement of the theorem. ■

5. The case $b_1 = b_2 = b_3 = 0$ for general n_1, n_2, n_3

In this section we demonstrate how to compute the number of lozenge tilings of $S_{n_1,n_2,n_3,a,0,0,0,0,0,0}$ using the determinant from Theorem 4.5. This establishes a new result – it gives the number of lozenge tilings of an arbitrary hexagon with a triangular hole of the suitable size[21] removed from a different position than in [18]. To be precise, in the latter a triangular hole of side a was removed from the center of the hexagon H of side-lengths $n_1, n_2+a, n_3, n_1+a, n_2, n_3+a$ (counterclockwise from the southeastern edge), while in our result below the distances from the sides of the triangular hole to the NW, NE and S sides of the hexagon are n_1, n_2 and n_3, respectively. One readily sees that this places the triangular hole inside the hexagon if and only if $n_1 \le n_2+n_3$, $n_2 \le n_1+n_3$, and $n_3 \le n_1+n_2$. The two positions agree only if $n_1 = n_2 = n_3$. For the formulation of the statement, recall that

$$\mathrm{M}(S_{n_1,n_2,n_3,0,0,0,0,0,0,0}) = \prod_{i_1=1}^{n_1}\prod_{i_2=1}^{n_2}\prod_{i_3=1}^{n_3} \frac{i_1+i_2+i_3-1}{i_1+i_2+i_3-2} =: B(n_1,n_2,n_3)$$

by MacMahon's box formula [45].

Theorem 5.1. *Let n_1, n_2, n_3 be non-negative integers with $n_1 \le n_2 \le n_3$ and $n_3 \le n_1+n_2$ and define the following monic polynomial $Q_{n_1,n_2,n_3}(a)$ in a, given in its factorized forms with only linear factors, with all the roots being negative integers,*

$$Q_{n_1,n_2,n_3}(a) = \prod_{i=\lceil (n_1+n_2-1)/2\rceil}^{\lfloor (n_1+n_3-1)/2\rfloor} (a+2i+1)^{2i+1-n_3} \prod_{i=\lfloor (n_1+n_3-1)/2\rfloor+1}^{\lfloor (n_2+n_3-1)/2\rfloor} (a+2i+1)^{n_1}$$

[21] So, that the resulting region has the same number of up- and down-pointing unit triangles.

$$\times \prod_{i=\lfloor (n_2+n_3-1)/2\rfloor+1}^{\lfloor (n_1+n_2+n_3-1)/2\rfloor} (a+2i+1)^{n_1+n_2+n_3-2i-1}$$

$$\times \prod_{i=\lceil (n_1+n_2+n_3-2)/4\rceil}^{\lfloor (n_1+n_2-2)/2\rfloor} (a+2i+1)^{4i+2-n_1-n_2-n_3}$$

$$\times \prod_{i=\lceil (n_1+n_2)/2\rceil}^{\min(\lfloor (n_2+n_3-n_1-1)/2\rfloor, \lfloor (n_1+n_3)/2\rfloor)} (a+2i)^{2i-n_3}$$

$$\times \prod_{i=\max(\lceil (n_1+n_2)/2\rceil, \lceil (n_2+n_3-n_1)/2\rceil)}^{\lfloor (n_1+n_3)/2\rfloor} (a+2i)^{n_2-n_1}$$

$$\times \prod_{i=\max(\lceil (n_2+n_3-n_1)/2\rceil, \lceil (n_1+n_3+1)/2\rceil)}^{\lfloor (n_2+n_3)/2\rfloor} (a+2i)^{n_2+n_3-2i}$$

$$\times \prod_{i=\lceil (n_1+n_3+1)/2\rceil}^{\lfloor (n_2+n_3-n_1-1)/2\rfloor} (a+2i)^{n_1}$$

$$\times \prod_{i=\lceil (n_1+n_2+n_3)/4\rceil}^{\min(\lfloor (n_1+n_2-1)/2\rfloor, \lfloor (n_2+n_3-n_1-1)/2\rfloor)} (a+2i)^{4i-n_1-n_2-n_3}$$

$$\times \prod_{i=\max(\lceil (n_2+n_3-n_1)/2\rceil, n_1)}^{\lfloor (n_1+n_2-1)/2\rfloor} (a+2i)^{2i-2n_1} \prod_{i=n_2}^{\lfloor (n_2+n_3-1)/2\rfloor} (a+2i)^{2i-2n_2}$$

$$\times \prod_{i=\lceil (n_2+n_3)/2\rceil}^{n_3} (a+2i)^{2n_3-2i} \prod_{i=1}^{\lfloor (n_1+n_2-n_3)/2\rfloor} (a+2i)^{2i}$$

$$\times \prod_{i=\lfloor (n_1+n_2-n_3)/2\rfloor+1}^{\lfloor (n_1+n_3-n_2)/2\rfloor} (a+2i)^{n_1+n_2-n_3}$$

$$\times \prod_{i=\lfloor (n_1+n_3-n_2)/2\rfloor+1}^{\min(\lfloor (n_2+n_3-n_1)/2\rfloor, n_1)} (a+2i)^{2n_1-2i}$$

$$\times \prod_{i=\lfloor (n_2+n_3-n_1)/2\rfloor+1}^{\lfloor (n_1+n_2+n_3)/4\rfloor} (a+2i)^{n_1+n_2+n_3-4i}, \tag{5.1}$$

where unlike in (2.1) *products are* 1 *if the range limits are out of order. Then*

$$\mathrm{M}(S_{n_1,n_2,n_3,a,0,0,0,0,0,0}) = B_{n_1,n_2,n_3} \frac{Q_{n_1,n_2,n_3}(a)}{Q_{n_1,n_2,n_3}(0)}.$$

Our proof approach is to apply Krattenthaler's *"identification of factors"* method [38, Section 2.4], which is in this case not complicated as the linear combinations that prove the zeros[22] turn out to be quite simple. The situation is somewhat similar for $\mathrm{M}(S_{n,n,n,a,b,b,b,k,k,k})$: the linear combinations are as simple as those used in this section. There the only additional complication lies in the more elaborate (block) structure of the matrix. One important purpose of this section is to demonstrate on a simpler example the procedure that will be used in a forthcoming paper to compute $\mathrm{M}(S_{n,n,n,a,b,b,b,k,k,k})$ in general.

In the special case $b_1 = b_2 = b_3 = 0$, Theorem 4.5 provides the following matrix, whose determinant we need to compute:

$$M_{n_1,n_2,n_3} = \begin{pmatrix} \mathfrak{M}'_{1,1} & \mathfrak{M}'_{1,2} \\ \mathfrak{M}'_{2,1} & \mathfrak{M}'_{2,2} \end{pmatrix},$$

where

$$\mathfrak{M}'_{1,1} := \binom{i-n_1-1}{j-1}_{1\le i\le n_2, 1\le j\le n_3},$$

$$\mathfrak{M}'_{1,2} := (-1)^{j+n_3-1}\binom{n_1+n_3+a-i+j-1}{n_1-i}_{1\le i\le n_2, 1\le j\le n_1+n_2-n_3},$$

$$\mathfrak{M}'_{2,1} := \binom{-n_1+n_2+n_3+a+i-1}{j-1}_{1\le i\le n_1, 1\le j\le n_3},$$

$$\mathfrak{M}'_{2,2} := \binom{-n_1+n_2+n_3+a+i-1}{-n_1+n_2+i-j}_{1\le i\le n_1, 1\le j\le n_1+n_2-n_3}.$$

We set

$$P_{n_1,n_2,n_3}(a) := \det(M_{n_1,n_2,n_3}),$$

which is obviously a polynomial in a. In the next lemma, we compute an upper bound for the degree of this polynomial. As we will see later, this will turn out to be in fact the actual degree.

Lemma 5.2. *Let n_1, n_2, n_3 be non-negative integers. Then*

$$\deg_a P_{n_1,n_2,n_3}(a) \le \left\lfloor \frac{2n_1n_2 + 2n_1n_3 + 2n_2n_3 - n_1^2 - n_2^2 - n_3^2}{4} \right\rfloor.$$

Proof. We start by modifying the matrix applying a set of elementary row and column operations. First we transform the bottom block consisting of the bottom n_1 rows: we subtract the second-to-last row from the last row, then the then the third to last row

[22] When the left-hand side is regarded as a polynomial in a.

from the (n_1+n_2-1)-st row etc. until we subtract the (n_2+1)-st row from the (n_2+2)-nd row. We repeat this, but terminate with the subtraction of (n_2+2)-nd row from the (n_2+3)-rd row. We repeat this loop n_1-1 times where in every step we perform one subtraction less than in the previous step. This way we arrive at the matrix

$$M_{n_1,n_2,n_3} = \begin{pmatrix} \mathfrak{M}''_{1,1} & \mathfrak{M}''_{1,2} \\ \mathfrak{M}''_{2,1} & \mathfrak{M}''_{2,2} \end{pmatrix},$$

where

$$\begin{aligned}
\mathfrak{M}''_{1,1} &:= \binom{i-n_1-1}{j-1}_{1\le i\le n_2, 1\le j\le n_3},\\
\mathfrak{M}''_{1,2} &:= (-1)^{j+n_3-1}\binom{n_1+n_3+a-i+j-1}{n_1-i}_{1\le i\le n_2, 1\le j\le n_1+n_2-n_3},\\
\mathfrak{M}''_{2,1} &:= \binom{-n_1+n_2+n_3+a}{j-i}_{1\le i\le n_1, 1\le j\le n_3},\\
\mathfrak{M}''_{2,2} &:= \binom{-n_1+n_2+n_3+a}{-n_1+n_2+i-j}_{1\le i\le n_1, 1\le j\le n_1+n_2-n_3}.
\end{aligned}$$

Second we modify the right block consisting of the $n_1+n_2-n_3$ rightmost columns. We add the (n_3+2)-nd column to the (n_3+1)-st column, the (n_3+3)-rd column to the (n_3+2)-nd column etc. until we add the (n_1+n_2)-nd column to the (n_1+n_2-1)-st column. We repeat this, but terminate with the addition of the (n_1+n_2-1)-st column to the (n_1+n_2-2)-nd column. We repeat this loop $n_1+n_2-n_3-1$ times where in every step we perform one addition less than in the previous step. The result is the following matrix:

$$M_{n_1,n_2,n_3} = \begin{pmatrix} \mathfrak{M}'''_{1,1} & \mathfrak{M}'''_{1,2} \\ \mathfrak{M}'''_{2,1} & \mathfrak{M}'''_{2,2} \end{pmatrix},$$

$$\begin{aligned}
\mathfrak{M}'''_{1,1} &:= \binom{i-n_1-1}{j-1}_{1\le i\le n_2, 1\le j\le n_3},\\
\mathfrak{M}'''_{1,2} &:= (-1)^{n_1+n_2-1}\binom{n_1+n_3+a-i+j-1}{-n_2+n_3-i+j}_{1\le i\le n_2, 1\le j\le n_1+n_2-n_3},\\
\mathfrak{M}'''_{2,1} &:= \binom{-n_1+n_2+n_3+a}{j-i}_{1\le i\le n_1, 1\le j\le n_3},\\
\mathfrak{M}'''_{2,2} &:= \binom{2n_2+a-j}{-n_1+n_2+i-j}_{1\le i\le n_1, 1\le j\le n_1+n_2-n_3}.
\end{aligned}$$

Now, we find the maximal degree in a of the summands in the Leibniz formula of the determinant: if we define the degree of the zero polynomial to be $-\infty$, then, in the top left block of the matrix the degree of an entry is at most 0, in the top right block it is at most $-n_2+n_3-i+j$, in the bottom left block it is at most $j-i$, while in the bottom right block it is at most $-n_1+n_2+i-j$.

Let σ be a permutation that maximizes the degree in a of the corresponding summand in the Leibniz formula of the determinant and suppose that this summand has k entries from the bottom left block. As the degree is maximal in the top right corner of this block, while in the bottom right block the degree is maximal in the bottom left corner and in the top right block the degree is maximal in the top right corner, the k entries coming from the bottom left block are situated in the top right square of size k of this block. Furthermore, there are n_1-k entries from the bottom right block, and they are situated in the bottom left square of size n_1-k in this block. Similarly, the $k+n_2-n_3$ entries from the top right block are situated in the top right square of size $k+n_2-n_3$. The upper bound for the degrees coming from these squares of size k, n_1-k and $k+n_2-n_3$, respectively, are the summands of the following expression:

$$\sum_{i=1}^{k}(n_3+1-2i)+\sum_{i=1}^{n_1-k}(n_2+1-2i)+\sum_{i=1}^{k+n_2-n_3}(n_1+1-2i)$$

This expression is equal to

$$-3k^2+3kn_1-n_1^2-3kn_2+2n_1n_2-n_2^2+3kn_3-n_1n_3+2n_2n_3-n_3^2.$$

The maximum of this expression is at $k=\frac{n_1-n_2+n_3}{2}$. Note that we need to require $k\le n_1,n_3,n_1-k\le n_1,n_1+n_2-n_3$ and $k+n_2-n_3\le n_2,n_1+n_2-n_3$, which in summary gives $n_3-n_2\le k\le \min(n_1,n_3)$. As

$$n_3-n_2\le\frac{n_1-n_2+n_3}{2}\le\min(n_1,n_3),$$

which basically follows from $n_x\le n_y+n_z$ if $\{x,y,z\}=\{1,2,3\}$ (these are necessary conditions for the removed triangle to be inside of the hexagon, see Figure 6), the maximum is attained at $\lfloor\frac{n_1-n_2+n_3}{2}\rfloor$ and at $\lceil\frac{n_1-n_2+n_3}{2}\rceil$. The maximum is then

$$\left\lfloor\frac{2n_1n_2+2n_1n_3+2n_2n_3-n_1^2-n_2^2-n_3^2}{4}\right\rfloor \\ =\begin{cases}\dfrac{2n_1n_2+2n_1n_3+2n_2n_3-n_1^2-n_2^2-n_3^2}{4} & \text{if } n_1+n_2+n_3\equiv 0 \bmod 2,\\ \dfrac{2n_1n_2+2n_1n_3+2n_2n_3-n_1^2-n_2^2-n_3^2-3}{4} & \text{if } n_1+n_2+n_3\equiv 1 \bmod 2.\end{cases}$$

■

The identification of factors method uses the following principle (for details consult [38, Section 2.4]). In order to prove that $P_{n_1,n_2,n_3}(a)$ has a zero at $a = i$ of multiplicity at least m, we need to find m independent linear combinations of the rows (or columns) that vanish of the $a = i$ specialization of M_{n_1,n_2,n_3}. Equivalently, we need to find m independent vectors in the left (resp. right) kernel of this specialization of M_{n_1,n_2,n_3}.

The odd zeros (i.e., the linear factors in (5.1) that become zero for odd values of a) are dealt with in the following lemma. If $A = (a_{i,j})$ is an $m \times n$ matrix, then the d-th (forward) difference with respect to the rows is defined to be the following $(m-d) \times n$ matrix:

$$(\Delta_i^d a_{i,j})_{1 \le i \le m-d, 1 \le j \le n}$$

Clearly, the rows of this matrix are linear combinations of rows of the original matrix. The definition is analogous for columns or operators different from Δ. Note that by the binomial theorem, $(\Delta_i^d a_{i,j})_{1 \le i \le m-d, 1 \le j \le n}$ is equal to the following product:

$$\left(\binom{d}{j-i} \right)_{\substack{1 \le i \le m-d \\ 1 \le j \le m}} \cdot (a_{i,j})_{\substack{1 \le i \le m \\ 1 \le j \le n}}. \tag{5.2}$$

Lemma 5.3. *Let $d \ge 0$, and choose row indices $1 \le i_1 \le n_2 - d$ (top block) and $1 \le i_2 \le n_1 - d$ (bottom block). Assuming $a = i_1 - i_2 - n_2 - n_3$, $i_1 + i_2 \equiv n_2 + n_3 + 1 \pmod 2$, and $i_2 + n_3 + d \ge i_1 + n_1$, the i_1-th row of the d-th difference with respect to the rows in the top block is equal to the i_2-th row of the d-th difference in the bottom block.*

Proof. This is obvious for the left block because if $a = i_1 - i_2 - n_2 - n_3$, as the i_1-th row of the top left block is equal to the i_2-th row of the bottom left block, for all $i_1 \in \{1, 2, \ldots, n_2\}$ and $i_2 \in \{1, 2, \ldots, n_1\}$.

The general entry of the d-th difference of the right top block is

$$\Delta_i^d \binom{n_1 + n_3 + a - i + j - 1}{n_1 - i} (-1)^{j+n_3+1}$$
$$= (-1)^{j+n_3+d+1} \binom{n_1 + n_3 + a - i + j - 1 - d}{n_1 - i},$$

while the general entry of the d-th difference of the right bottom block is

$$\Delta_i^d \binom{-n_1 + n_2 + n_3 + a + i - 1}{-n_1 + n_2 + i - j} = \binom{-n_1 + n_2 + n_3 + a + i - 1}{-n_1 + n_2 + i - j + d}.$$

After setting $a = i_1 - i_2 - n_2 - n_3$, the entry in row i_1 of the d-th difference in the top-right block is equal to

$$(-1)^{j+n_3+d+1}\binom{n_1 - i_2 - n_2 + j - 1 - d}{n_1 - i_1}$$
$$= (-1)^{j+n_3+d+1+n_1+i_1}\binom{-i_1 + i_2 + n_2 - j + d}{n_1 - i_1}$$

assuming $i_1 \leq n_2 - d$ and using (4.8), while the entry in row i_2 of the d-th difference in the bottom-right block is equal to

$$\binom{-n_1 + i_1 - 1}{-n_1 + n_2 + i_2 - j + d} = (-1)^{-n_1+n_2+i_2-j+d}\binom{-i_1 + i_2 + n_2 - j + d}{-n_1 + n_2 + i_2 - j + d}.$$

The assertion now follows from the symmetry of the binomial coefficient (4.7). Note that here we need the assumptions $i_2 + n_3 + d \geq i_1 + n_1$ (as the symmetry is only true if the upper parameter of the binomial coefficient is non-negative) as well as $i_1 + i_2 \equiv n_2 + n_3 + 1 \pmod 2$ to have the right sign. ■

For $d = 0$, this lemma provides the following linear factors:

$$\prod_{\substack{1\leq i_1 \leq n_2, 1 \leq i_2 \leq n_1 \\ i_1+i_2 \equiv n_2+n_3+1 \pmod 2, i_2+n_3 \geq i_1+n_1}} (a - i_1 + i_2 + n_2 + n_3)$$
$$= \prod_{i=\lceil (n_1+n_2-1)/2 \rceil}^{\lfloor (\min(n_1,n_2)+n_3-1)/2 \rfloor} (a + 2i + 1)^{2i+1-n_3}$$
$$\times \prod_{i=\max(\lceil (n_1+n_2-1)/2 \rceil, \lfloor (\min(n_1,n_2)+n_3-1)/2 \rfloor + 1)}^{\lfloor (\max(n_1,n_2)+n_3-1)/2 \rfloor} (a + 2i + 1)^{\min(n_1,n_2)}$$
$$\times \prod_{i=\max(\lceil (n_1+n_2-1)/2 \rceil, \lfloor (\max(n_1,n_2)+n_3-1)/2 \rfloor + 1)}^{\lfloor (n_1+n_2+n_3-1)/2 \rfloor} (a + 2i + 1)^{n_1+n_2+n_3-2i-1}.$$

Note that, for a fixed zero, the various linear combinations are linearly independent because each of them involves only a pair of rows and a particular row appears in at most one such linear combination.

Using our assumption $n_1 \leq n_2 \leq n_3$, the above expression simplifies to

$$\prod_{i=\lceil (n_1+n_2-1)/2 \rceil}^{\lfloor (n_1+n_3-1)/2 \rfloor} (a + 2i + 1)^{2i+1-n_3} \prod_{i=\lfloor (n_1+n_3-1)/2 \rfloor + 1}^{\lfloor (n_2+n_3-1)/2 \rfloor} (a + 2i + 1)^{n_1}$$
$$\times \prod_{i=\lfloor (n_2+n_3-1)/2 \rfloor + 1}^{\lfloor (n_1+n_2+n_3-1)/2 \rfloor} (a + 2i + 1)^{n_1+n_2+n_3-2i-1}.$$

The degree of this product is

$$\Big(-\Big\lceil\frac{n_1+n_2-1}{2}\Big\rceil+\Big\lfloor\frac{n_1+n_3-1}{2}\Big\rfloor+1\Big)$$
$$\times\Big(\Big\lceil\frac{n_1+n_2-1}{2}\Big\rceil+\Big\lfloor\frac{n_1+n_3-1}{2}\Big\rfloor-n_3+1\Big)$$
$$+n_1\Big(\Big\lfloor\frac{n_2+n_3-1}{2}\Big\rfloor-\Big\lfloor\frac{n_1+n_3-1}{2}\Big\rfloor\Big)$$
$$+\Big(\Big\lfloor\frac{n_2+n_3-1}{2}\Big\rfloor-\Big\lfloor\frac{n_1+n_2+n_3-1}{2}\Big\rfloor\Big)$$
$$\times\Big(\Big\lfloor\frac{n_1+n_2+n_3-1}{2}\Big\rfloor+\Big\lfloor\frac{n_2+n_3-1}{2}\Big\rfloor-n_1-n_2-n_3+2\Big).$$

Now, we consider the case $i_2+n_3>i_1+n_1$ in Lemma 5.3. To ensure no double count, we assume that $i_2n_3+d\geq i_1+n_1$ is fulfilled with equality, so that the relevant zero is $d-n_1-n_2$. The parity condition is $d\equiv n_1+n_2+1 (\mathrm{mod}\, 2)$. As $1\leq i_1\leq n_2-d$ and $1\leq i_2\leq n_1-d$, we have the following linear factors:

$$\prod_{\substack{d\geq 1\\ d\equiv n_1+n_2+1(2)}}(a-d+n_1+n_2)^{\max(\min(n_2-d,n_3)-\max(1,1-n_1+n_3+d)+1,0)}$$
$$=\prod_{i=\lceil (n_1+n_2+n_3-2)/4\rceil}^{\lfloor (n_1+n_2-2)/2\rfloor}(a+2i+1)^{4i+2-n_1-n_2-n_3}.$$

Also here it can be seen that, for a fixed zero, the various linear combinations are linearly independent because, in addition to the argument given for $d=0$, we make use of the fact that the transformation matrix in (5.2) has full rank $m-d$. The degree of the above product is

$$\Big(-\Big\lceil\frac{n_1+n_2+n_3-2}{4}\Big\rceil+\Big\lfloor\frac{n_1+n_2-2}{2}\Big\rfloor+1\Big)$$
$$\times\Big(2\Big\lceil\frac{n_1+n_2+n_3-2}{4}\Big\rceil+2\Big\lfloor\frac{n_1+n_2-2}{2}\Big\rfloor-n_1-n_2-n_3+2\Big).$$

In the following lemma, we identify a set of even zeros.

Lemma 5.4. *Let $d\geq 0$, and choose row indices $1\leq i_1\leq n_2-d$ (top block) and $1\leq i_2\leq n_1-d$ (bottom block). Assuming $a=i_1-i_2-n_2-n_3, i_1+i_2\equiv n_2+n_3 (\mathrm{mod}\, 2)$, $n_1+1\leq i_1$ and $-i_1+i_2-n_1+n_3+d\geq 0$, the i_1-th row of the d-th difference in the top block is equal to the i_2-th row of the d-th difference in the bottom block.*

Proof. The proof follows the proof of Lemma 5.3 up to some point: for the left block, the assertion can be deduced in the same way because we have all assumptions needed also in this lemma. Further, we have seen that, after setting $a = i_1 - i_2 - n_2 - n_3$, the entry in row i_1 of the d-th difference in top-right block is equal to

$$(-1)^{j+n_3+d+1+n_1+i_1}\binom{-i_1+i_2+n_2-j+d}{n_1-i_1},$$

which is zero since we assume $n_1 + 1 \le i_1$, while the entry in row i_2 of the d-th difference in the bottom right block is equal to

$$(-1)^{-n_1+n_2+i_2-j+d}\binom{-i_1+i_2+n_2-j+d}{-n_1+n_2+i_2-j+d},$$

which is also zero because, by $n_1 + 1 \le i_1$, the upper parameter is less than the lower parameter, and, by $-i_1 + i_2 - n_1 + n_3 + d \ge 0$, the upper parameter is positive for all $j \in \{1, 2, \ldots, n_1 + n_2 - n_3\}$. ■

For $d = 0$, we obtain the following factors:

$$\prod_{i=\lceil (n_1+n_2)/2\rceil}^{\min(\lfloor (n_2+n_3-n_1-1)/2\rfloor,\lfloor (n_1+n_3)/2\rfloor)} (a+2i)^{2i-n_3} \prod_{i=\max(\lceil (n_1+n_2)/2\rceil,\lceil (n_2+n_3-n_1)/2\rceil)}^{\lfloor (n_1+n_3)/2\rfloor} (a+2i)^{n_2-n_1}$$
$$\times \prod_{i=\max(\lceil (n_2+n_3-n_1)/2\rceil,\lceil (n_1+n_3+1)/2\rceil)}^{\lfloor (n_2+n_3)/2\rfloor} (a+2i)^{n_2+n_3-2i} \prod_{i=\lceil (n_1+n_3+1)/2\rceil}^{\lfloor (n_2+n_3-n_1-1)/2\rfloor} (a+2i)^{n_1}$$

The degree of this product is

$$\begin{aligned}
&\left(\max\left(\left\lceil\frac{n_1+n_3+1}{2}\right\rceil,\left\lceil\frac{-n_1+n_2+n_3}{2}\right\rceil\right)-\left\lfloor\frac{n_2+n_3+2}{2}\right\rfloor\right)\\
&\qquad\times\left(\max\left(\left\lceil\frac{n_1+n_3+1}{2}\right\rceil,\left\lceil\frac{-n_1+n_2+n_3}{2}\right\rceil\right)-\left\lceil\frac{n_2+n_3}{2}\right\rceil\right)\\
&\quad+\left(\min\left(\left\lfloor\frac{n_1+n_3}{2}\right\rfloor,\left\lfloor\frac{-n_1+n_2+n_3-1}{2}\right\rfloor\right)\right.\\
&\qquad\left.-\min\left(\left\lceil\frac{n_1+n_2-2}{2}\right\rceil,\left\lfloor\frac{-n_1+n_2+n_3-1}{2}\right\rfloor\right)\right)\\
&\qquad\times\left(\min\left(\left\lceil\frac{n_1+n_2}{2}\right\rceil,\left\lfloor\frac{-n_1+n_2+n_3+1}{2}\right\rfloor\right)\right.\\
&\qquad\qquad\left.-\max\left(\left\lceil\frac{n_3-n_1}{2}\right\rceil,\left\lceil\frac{n_1-n_2+n_3+1}{2}\right\rceil\right)\right)\\
&\quad+(n_2-n_1)\left(\max\left(\left\lceil\frac{-n_1+n_2+n_3}{2}\right\rceil,\left\lfloor\frac{n_1+n_3+2}{2}\right\rfloor\right)\right.\\
&\qquad\qquad\left.-\max\left(\left\lceil\frac{n_1+n_2}{2}\right\rceil,\left\lceil\frac{-n_1+n_2+n_3}{2}\right\rceil\right)\right)\\
&\quad+n_1\left(\left\lfloor\frac{-n_1+n_2+n_3+1}{2}\right\rfloor-\min\left(\left\lceil\frac{n_1+n_3+1}{2}\right\rceil,\left\lfloor\frac{-n_1+n_2+n_3+1}{2}\right\rfloor\right)\right).
\end{aligned}$$

For $d > 0$, we obtain

$$\prod_{i=\lceil (n_1+n_2+n_3)/4\rceil}^{\min(\lfloor (n_1+n_2-1)/2\rfloor, \lfloor (n_2+n_3-n_1-1)/2\rfloor)} (a+2i)^{4i-n_1-n_2-n_3} \prod_{i=\max(\lceil (n_2+n_3-n_1)/2\rceil, n_1)}^{\lfloor (n_1+n_2-1)/2\rfloor} (a+2i)^{2i-2n_1}.$$

The degree is

$$\begin{aligned}
&\Big(\max\Big(\Big\lceil \frac{-n_1+n_2+n_3-2}{2}\Big\rceil, \Big\lfloor \frac{n_1+n_2-1}{2}\Big\rfloor\Big)\\
&\quad + \max\Big(-n_1, \Big\lceil \frac{-5n_1+n_2+n_3}{2}\Big\rceil\Big)\Big)\\
&\quad \times \Big(\max\Big(\Big\lceil \frac{-n_1+n_2+n_3}{2}\Big\rceil, \Big\lfloor \frac{n_1+n_2+1}{2}\Big\rfloor\Big)\\
&\qquad - \max\Big(n_1, \Big\lceil \frac{-n_1+n_2+n_3}{2}\Big\rceil\Big)\Big)\\
&\quad + \Big(\min\Big(\Big\lfloor \frac{n_1+n_2+1}{2}\Big\rfloor, \Big\lfloor \frac{-n_1+n_2+n_3+1}{2}\Big\rfloor\Big)\\
&\qquad - \min\Big(\Big\lceil \frac{n_1+n_2+n_3}{4}\Big\rceil, \Big\lfloor \frac{-n_1+n_2+n_3+1}{2}\Big\rfloor\Big)\Big)\\
&\quad \times \Big(2\min\Big(\Big\lceil \frac{n_1+n_2+n_3}{4}\Big\rceil, \Big\lfloor \frac{-n_1+n_2+n_3+1}{2}\Big\rfloor\Big)\\
&\qquad + 2\min\Big(\Big\lfloor \frac{n_1+n_2-1}{2}\Big\rfloor, \Big\lfloor \frac{-n_1+n_2+n_3-1}{2}\Big\rfloor\Big)\\
&\qquad - n_1 - n_2 - n_3\Big).
\end{aligned}$$

A related set of even zeros can be obtained as follows. We fix an integer t with $t \equiv n_2 + n_3 \pmod 2$, and consider the zero $a = t - n_2 - n_3$. From the proof of Lemma 5.3, we know that the i_1-th row in the top left block is equal to the i_2-th row in the bottom left block if we assume and $t = i_1 - i_2$ and $a = t - n_2 - n_3$. For each such pair (i_1, i_2), we produce a zero row in the left bottom block by subtracting the i_1-th row from the top block from the i_2-th row in the bottom block. We consider the submatrix of this new matrix that consists of the rows i_2 that come from such pairs (i_1, i_2). From this matrix, we further exclude those rows i_2 that come from pairs (i_1, i_2) that were already dealt with in Lemma 5.4. The dimension of the kernel of this submatrix is a lower bound for the additional multiplicity of the zero (i.e., the multiplicity not already covered by Lemma 5.4). If m is the number of rows, then $m - (n_1 + n_2 - n_3)$ is a lower bound for this dimension because $n_1 + n_2 - n_3$ is the number of columns in the right block and the left block is zero. It is a straightforward to check that this results in the following factors:

$$\prod_{i=n_2}^{\lfloor (n_2+n_3-1)/2\rfloor} (a+2i)^{2i-2n_2} \prod_{i=\lceil (n_2+n_3)/2\rfloor}^{n_3} (a+2i)^{2n_3-2i}.$$

The degree of this product is

$$\left\lceil \frac{n_2 - n_3 - 2}{2} \right\rceil \left\lceil \frac{n_2 - n_3}{2} \right\rceil + \left\lfloor \frac{-n_2 + n_3 - 1}{2} \right\rfloor \left\lfloor \frac{-n_2 + n_3 + 1}{2} \right\rfloor.$$

As for the remaining even zeros, we need the following lemma. It is useful to define the following operator.

$$\sigma_x p(x) = p(x+1) + p(x)$$

We refer to it as the anti-difference.

Lemma 5.5. *Let $d \geq 0$. Assuming $a = j_1 - j_2 - n_3$, $j_1 + j_2 + n_3 \equiv 0 \pmod 2$, $j_1 \geq n_2 - n_1 + 1$ and $n_2 - n_1 + j_1 + d \geq j_2$, the j_1-th column of the d-th anti-difference with respect to the columns in the left block is equal to the j_2-th column of the d-th anti-difference in the right block, provided that $1 \leq j_1 \leq n_3 - d$ and $1 \leq j_2 \leq n_1 + n_2 - n_3 - d$.*

Proof. The d-th anti-difference in the bottom left block is

$$\sigma_j^d \binom{-n_1 + n_2 + n_3 + a + i - 1}{j - 1} = \binom{-n_1 + n_2 + n_3 + a + i - 1 + d}{j - 1 + d},$$

while it is

$$\binom{-n_1 + n_2 + n_3 + a + i - 1 + d}{-n_1 + n_2 + i - j}$$

in the right bottom block. If we plug in $a = j_1 - j_2 - n_3$, then we need to employ the symmetry to show that the expressions are equal, which is possible as long as $n_2 - n_1 + j_1 + d \geq j_2$. On the other hand, the d-th anti-difference in the top left block is

$$\sigma_j^d \binom{i - n_1 - 1}{j - 1} = \binom{i - n_1 - 1 + d}{j - 1 + d},$$

while the d-th anti-difference in the top right block is

$$\begin{aligned} &\sigma_j^d (-1)^{j+n_3-1} \binom{n_1 + n_3 + a - i + j - 1}{n_1 - i} \\ &\quad = (-1)^{j+n_3+d-1} \binom{n_1 + n_3 + a - i + j - 1}{n_1 - i - d}. \end{aligned}$$

After plugging in $a = j_1 - j_2 - n_3$ and applying (4.8) to the entry in the top left block, we see that we need to have

$$(-1)^{j_1+1+d} \binom{n_1 + j_1 - i - 1}{j_1 - 1 + d} = (-1)^{j_2+n_3+d-1} \binom{n_1 + j_1 - i - 1}{n_1 - i - d}.$$

By the symmetry, this is true if $j_1 \geq n_2 - n_1 + 1$ and $j_1 + j_2 + n_3 \equiv 0 \pmod 2$. ∎

From this lemma, we can deduce the following linear factors if $d = 0$.

$$\prod_{j_1=\max(1,n_2-n_1+1)}^{n_3} \prod_{\substack{j_2=1\\ j_1+j_2+n_3\equiv 0\ (2)}}^{\min(n_2-n_1+j_1,n_1+n_2-n_3)} (a-j_1+j_2+n_3)$$
$$= \prod_{i=1}^{\lfloor (n_1+n_2-n_3)/2\rfloor} (a+2i)^{2i} \prod_{i=\lfloor (n_1+n_2-n_3)/2\rfloor+1}^{\lfloor (n_1+n_3-n_2)/2\rfloor} (a+2i)^{n_1+n_2-n_3}$$
$$\times \prod_{i=\lfloor (n_1+n_3-n_2)/2\rfloor+1}^{\min(\lfloor (n_2+n_3-n_1)/2\rfloor,n_1)} (a+2i)^{2n_1-2i} \tag{5.3}$$

The degree of this product is

$$\Big(\min\Big(0,\Big\lfloor\frac{-3n_1+n_2+n_3}{2}\Big\rfloor\Big)+\Big\lfloor\frac{-n_1-n_2+n_3+2}{2}\Big\rfloor\Big)$$
$$\times\Big(\Big\lfloor\frac{n_1-n_2+n_3}{2}\Big\rfloor-\min\Big(n_1,\Big\lfloor\frac{-n_1+n_2+n_3}{2}\Big\rfloor\Big)\Big)$$
$$+\Big\lfloor\frac{n_1+n_2-n_3}{2}\Big\rfloor\Big\lfloor\frac{n_1+n_2-n_3+2}{2}\Big\rfloor$$
$$+(n_1+n_2-n_3)\Big(\Big\lfloor\frac{n_1-n_2+n_3}{2}\Big\rfloor-\Big\lfloor\frac{n_1+n_2-n_3}{2}\Big\rfloor\Big).$$

For $d > 0$, we obtain

$$\prod_{i=\lfloor (n_2+n_3-n_1)/2\rfloor+1}^{\lfloor (n_1+n_2+n_3)/4\rfloor} (a+2i)^{n_1+n_2+n_3-4i}. \tag{5.4}$$

The degree of this product is

$$\Big(\Big\lfloor\frac{-n_1+n_2+n_3}{2}\Big\rfloor-\max\Big(\Big\lfloor\frac{-n_1+n_2+n_3}{2}\Big\rfloor,\Big\lfloor\frac{n_1+n_2+n_3}{4}\Big\rfloor\Big)\Big)$$
$$\times\Big(2\max\Big(\Big\lfloor\frac{-n_1+n_2+n_3}{2}\Big\rfloor,\Big\lfloor\frac{n_1+n_2+n_3}{4}\Big\rfloor\Big)$$
$$+2\Big\lfloor\frac{-n_1+n_2+n_3+2}{2}\Big\rfloor-n_1-n_2-n_3\Big).$$

The even zeros coming from Lemma 5.5 are distinct from those that were identified before. It can be checked that the former factors are of the form $(a+2i)$ with $i \le \min(\lfloor\frac{n_1+n_2+n_3}{4}\rfloor, n_1)$,[23] while the latter are of the form $(a+2i)$ again with

[23] To see that $i \le n_1$ in (5.3), recall that $n_1 \le n_2 \le n_3$ and $n_3 \le n_1 + n_2$. As for (5.4), $n_1 > \lfloor (n_1+n_2+n_3)/4\rfloor$, would imply $\lfloor (n_2+n_3-n_1)/2\rfloor + 1 > \lfloor (n_1+n_2+n_3)/4\rfloor$, and so the product is empty.

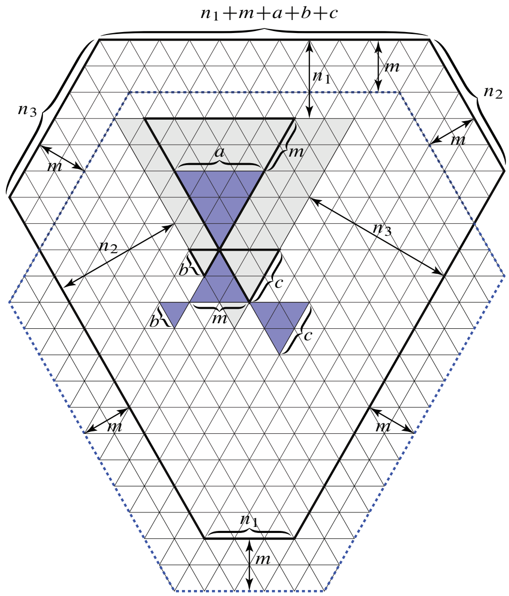

Figure 9. The region $Q_{n_1,n_2,n_3}(m+a+b+c)$ (solid contour, three thicker contour lobes – equivalently, big light shaded triangle – removed) and the region $SC'_{n_1,n_2,n_3}(a,b,c,m)$ (dotted contour, shaded four-lobed shamrock removed) for $n_1 = 3$, $n_2 = 5$, $n_3 = 6$, $m = 2$, $a = 3$, $b = 1$ and $c = 2$. The latter is obtained from the former by regarding the former region as $S_{n_1,n_2,n_3,a+b+c+m,0,0,0,0,0,0,0}$, and squeezing in the top lobe m units.

$i > \min(\lfloor \frac{n_1+n_2+n_3}{4} \rfloor, n_1)$. Now, it remains to show that the degrees of the various factors add up to the upper bound for the total degree as computed in Lemma 5.2 and to provide one additional evaluation. The former is a tedious but straightforward computation distinguishing several cases taking into the remainder of the n_i's modulo 4 and certain linear inequalities (which can be assisted by a computer algebra system). The additional evaluation is provided by $a = 0$ because then the result is equivalent to MacMahon's formula for the number of plane partitions in an $n_1 \times n_2 \times n_3$-box. It is straightforward to check that this leads to the leading coefficient in a displayed in formula (5.1). This completes the proof of Theorem 5.1.

Remark 5.1. As mentioned above, Theorem 5.1 establishes a counterpart of the main result of [18], in which the removed triangle is not in the center of the hexagon, but in a new position (the two positions agree only if $n_1 = n_2 = n_3$). Using [24, Theorem 1], one can deduce from Theorem 5.1 above a more general result. Namely, we can allow, instead of just the triangular hole, to have a more general hole shape, consisting of a triangle with three other triangles of the opposite orientation touching its vertices (what is called a shamrock in [23]).

Theorem 5.6. *Let $SC'_{n_1,n_2,n_3}(a,b,c,m)$ be the region obtained from the hexagon of sides $n_1+a+b+c$, n_2+m, $n_3+a+b+c$, n_1+m, $n_2+a+b+c$, n_3+m (clockwise from top) by removing a shamrock of core size m and lobe sizes a, b, and c (counterclockwise from top; see* [15, Figure 2.1] *for an illustration of a shamrock), placed in such a way that the top, left and right lobes are at distances n_1, n_2 and n_3 from the top, southwestern and southeastern sides of the hexagon, respectively. Then we have*

$$\frac{\mathrm{M}(SC'_{n_1,n_2,n_3}(a,b,c,m))}{\mathrm{M}(S_{n_1,n_2,n_3,a+b+c+m,0,0,0,0,0,0})} = \frac{\mathrm{H}(m)^3\,\mathrm{H}(a)\,\mathrm{H}(b)\,\mathrm{H}(c)}{\mathrm{H}(m+a)\,\mathrm{H}(m+b)\,\mathrm{H}(m+c)}\frac{\mathrm{H}(n_1+a)\,\mathrm{H}(n_2+n_3-n_1+b+c+m)}{\mathrm{H}(n_1+a+m)\,\mathrm{H}(n_2+n_3-n_1+b+c)} \times\frac{\mathrm{H}(n_2+b)\,\mathrm{H}(n_1+n_3-n_2+a+c+m)}{\mathrm{H}(n_2+b+m)\,\mathrm{H}(n_1+n_3-n_2+a+c)} \times\frac{\mathrm{H}(n_3+c)\,\mathrm{H}(n_1+n_2-n_3+a+b+m)}{\mathrm{H}(n_3+c+m)\,\mathrm{H}(n_1+n_2-n_3+a+b)}, \tag{5.5}$$

where $\mathrm{H}(n)=0!1!\dots(n-1)!$, and the denominator is given by Theorem 5.1.

For the special case of a bowtie this was also conjectured by Won Hyok Kim. This will be presented in his master thesis prepared under the supervision of the second author, along with a proof that reduces everything to proving some hypergeometric identities.

Proof. Consider the region $S_{n_1,n_2,n_3,a+b+c+m,0,0,0,0,0,0}$ (in Figure 9, it is the hexagon in the solid contour, with the lightly shaded equilateral triangle removed). Note that the number of lozenge tilings does not change if instead of the equilateral triangle of side-length $a+b+c+m$ we remove three equilateral triangles of side-lengths $a+m$, b and c touching at a common point, as shown in Figure 9. Indeed, the removal of these three triangles forces several lozenges to be part of every tiling; the union of the three holes and the forced lozenges is precisely the original equilateral triangle of side-length $m+a+b+c$.

Regard $S_{n_1,n_2,n_3,a+b+c+m,0,0,0,0,0,0}$ as a triad hexagon region of the kind considered in [24] (each of the three bowties consists in this case just of their outside lobe, and they touch at the common point). Apply the bowtie squeezing operation of [24] to the top lobe, squeezing it in m units. The resulting region is readily seen to be precisely $SC'_{n_1,n_2,n_3}(a,b,c,m)$. [24, Theorem 1] expresses then the ratio of $\mathrm{M}(SC'_{n_1,n_2,n_3}(a,b,c,m))$ and $\mathrm{M}(S_{n_1,n_2,n_3,a+b+c+m,0,0,0,0,0,0})$ as a conceptual product expression involving hyperfactorials evaluated at various natural distances within these regions. All the relevant distances can be read off from Figure 9. One is lead to formula (5.5). ∎

6. A determinantal formula of dimension $a + b_1 + b_2 + b_3$ for even b_i

The purpose of this section is to employ an idea that has been used in Section 4 to derive a determinantal formula for $\mathrm{M}(S_{n_1,n_2,n_3,a,b_1,b_2,b_3,k_1,k_2,k_3})$ when b_1, b_2, b_3 are even, such that the underlying matrix is of size $a + b_1 + b_2 + b_3$. This allows us to reduce the proof of our formula for $\mathrm{M}(S_{n,n,n,a,b,b,b,k,k,k})$ for any concrete even values of a and b to verifying certain hypergeometric identities. This shows, from a different point of view, the advantage of our new method.

More specifically, the entries of the determinants are hypergeometric ${}_3F_2$ series. Verifying the hypergeometric identities can be accomplished by computer algebra packages such as HYP [37]. The matrix entries of neighboring columns are related by so-called contiguous relations for hypergeometric series. The same applies to neighboring rows within the same "homogeneous block," so the verification of the hypergeometric identities amounts to applying such relations.

Theorem 6.1. *If a, b_1, b_2 and b_3 are even, then*

$$\begin{aligned}
&\mathrm{M}(S_{n_1,n_2,n_3,a,b_1,b_2,b_3,k_1,k_2,k_3})\\
&\quad=\prod_{i=1}^{n_1}\binom{n_2+n_3+a+b_1+b_2+b_3+i-1}{n_2}\prod_{j=1}^{a+b_1+b_2+b_3+n_1}\binom{j+n_2-1}{j-1}^{-1}\\
&\quad\times\left|\det\begin{pmatrix}\mathfrak{S}_1 & 1\le i\le b_3\\ \mathfrak{S}_2 & 1\le i\le a\\ \mathfrak{S}_3 & 1\le i\le b_1\\ \mathfrak{S}_4 & 1\le i\le b_2\end{pmatrix}_{1\le j\le a+b_1+b_2+b_3}\right|,
\end{aligned}\tag{6.1}$$

where

$$\mathfrak{S}_1 := \sum_{l\ge1}\binom{n_1+\frac{a}{2}+b_1+k_3+i-1}{l+n_2-n_3+2k_3-1}\binom{l+n_2-1}{l-1}\binom{-n_3-a-b_1-b_2-b_3}{j+n_1-l},$$

$$\mathfrak{S}_2 := \sum_{l\ge1}\binom{n_1+b_1+i-1}{l+n_2-n_3-b_3-1}\binom{l+n_2-1}{l-1}\binom{-n_3-a-b_1-b_2-b_3}{j+n_1-l},$$

$$\begin{aligned}\mathfrak{S}_3 := \sum_{l\ge1}&\binom{n_1-2k_1+i-1}{l+n_2-n_3-\frac{a}{2}-b_3-k_1-1}\binom{l+n_2-1}{l-1}\\
&\times\binom{-n_3-a-b_1-b_2-b_3}{j+n_1-l},\end{aligned}$$

$$\begin{aligned}\mathfrak{S}_4 := \sum_{l\ge1}&\binom{n_1+\frac{a}{2}+b_1+k_2+i-1}{l+n_2-n_3-\frac{a}{2}-b_3-k_2-1}\binom{l+n_2-1}{l-1}\\
&\times\binom{-n_3-a-b_1-b_2-b_3}{j+n_1-l}.\end{aligned}$$

Proof. The starting point is again (4.4), where we now eliminate the top two blocks. By setting $d = 1$ and plugging in $c_{1,i} = i$, we see that $\mathrm{M}(S_{n_1,n_2,n_3,a,b_1,b_2,b_3,k_1,k_2,k_3})$ is equal to

$$\left|\prod_{i=1}^{b_3} \Delta_{c_{3,i}}^{n_3-2k_3} \prod_{i=1}^{a} \Delta_{c_{4,i}}^{n_3+b_3} \prod_{i=1}^{b_1} \Delta_{c_{5,i}}^{n_3+\frac{a}{2}+b_3+k_1} \prod_{i=1}^{b_2} \Delta_{c_{6,i}}^{n_3+\frac{a}{2}+b_3+k_2} \det \mathfrak{B}\right|, \tag{6.2}$$

where

$$\mathfrak{B} := \begin{pmatrix} \binom{i-1}{j-1} & 1 \le i \le n_2 \\ \binom{c_{2,i}-1}{j-1} & 1 \le i \le n_1 \\ \binom{c_{3,i}-1}{j-1} & 1 \le i \le b_3 \\ \binom{c_{4,i}-1}{j-1} & 1 \le i \le a \\ \binom{c_{5,i}-1}{j-1} & 1 \le i \le b_1 \\ \binom{c_{6,i}-1}{j-1} & 1 \le i \le b_2 \end{pmatrix}_{1 \le j \le n},$$

and where

$$\begin{aligned} c_{2,i} &= n_2 + n_3 + a + b_1 + b_2 + b_3 + i, \\ c_{3,i} &= n_1 + \frac{a}{2} + b_1 + k_3 + i, \\ c_{4,i} &= n_1 + b_1 + i, \\ c_{5,i} &= n_1 - 2k_1 + i, \\ c_{6,i} &= n_1 + \frac{a}{2} + b_1 + k_2 + i \end{aligned}$$

and

$$n = n_1 + n_2 + a + b_1 + b_2 + b_3.$$

The only non-zero entry in the first row is in the first column, and so we expand with respect to this row. After having performed this reduction, the new first row has the same property. We can keep expanding until we have deleted the top block. We then set

$$c_{2,i} = n_2 + n_3 + a + b_1 + b_2 + b_3 + i$$

and obtain that the expression whose absolute value is taken in (6.2) equals

$$\prod_{i=1}^{b_3} \Delta_{c_{3,i}}^{n_3-2k_3} \prod_{i=1}^{a} \Delta_{c_{4,i}}^{n_3+b_3} \prod_{i=1}^{b_1} \Delta_{c_{5,i}}^{n_3+\frac{a}{2}+b_3+k_1} \prod_{i=1}^{b_2} \Delta_{c_{6,i}}^{n_3+\frac{a}{2}+b_3+k_2} \det \mathfrak{C},$$

where

$$\mathfrak{C} := \begin{pmatrix} \binom{n_2+n_3+a+b_1+b_2+b_3+i-1}{j-1} & 1 \le i \le n_1 \\ \binom{c_{3,i}-1}{j-1} & 1 \le i \le b_3 \\ \binom{c_{4,i}-1}{j-1} & 1 \le i \le a \\ \binom{c_{5,i}-1}{j-1} & 1 \le i \le b_1 \\ \binom{c_{6,i}-1}{j-1} & 1 \le i \le b_2 \end{pmatrix}_{n_2+1 \le j \le n}.$$

Shifting j, this becomes

$$\prod_{i=1}^{b_3} \Delta_{c_{3,i}}^{n_3-2k_3} \prod_{i=1}^{a} \Delta_{c_{4,i}}^{n_3+b_3} \prod_{i=1}^{b_1} \Delta_{c_{5,i}}^{n_3+\frac{a}{2}+b_3+k_1} \prod_{i=1}^{b_2} \Delta_{c_{6,i}}^{n_3+\frac{a}{2}+b_3+k_2} \det \mathfrak{D},$$

where

$$\mathfrak{D} := \begin{pmatrix} \binom{n_2+n_3+a+b_1+b_2+b_3+i-1}{j+n_2-1} & 1 \le i \le n_1 \\ \binom{c_{3,i}-1}{j+n_2-1} & 1 \le i \le b_3 \\ \binom{c_{4,i}-1}{j+n_2-1} & 1 \le i \le a \\ \binom{c_{5,i}-1}{j+n_2-1} & 1 \le i \le b_1 \\ \binom{c_{6,i}-1}{j+n_2-1} & 1 \le i \le b_2 \end{pmatrix}_{1 \le j \le n-n_2}$$

Taking out the factor $\binom{n_2+n_3+a+b_1+b_2+b_3+i-1}{n_2}$ from row i, $1 \le i \le n_1$, as well as the factor $\binom{j+n_2-1}{j-1}^{-1}$ from column j, $1 \le j \le n-n_2$, this is equal to

$$\prod_{i=1}^{n_1} \binom{n_2+n_3+a+b_1+b_2+b_3+i-1}{n_2} \prod_{j=1}^{n-n_2} \binom{j+n_2-1}{j-1}^{-1}$$
$$\times \prod_{i=1}^{b_3} \Delta_{c_{3,i}}^{n_3-2k_3} \prod_{i=1}^{a} \Delta_{c_{4,i}}^{n_3+b_3} \prod_{i=1}^{b_1} \Delta_{c_{5,i}}^{n_3+\frac{a}{2}+b_3+k_1} \prod_{i=1}^{b_2} \Delta_{c_{6,i}}^{n_3+\frac{a}{2}+b_3+k_2} \det \mathfrak{E}, \quad (6.3)$$

where

$$\mathfrak{E} := \begin{pmatrix} \binom{n_3+a+b_1+b_2+b_3+i-1}{j-1} & 1 \le i \le n_1 \\ \binom{c_{3,i}-1}{j+n_2-1}\binom{j+n_2-1}{j-1} & 1 \le i \le b_3 \\ \binom{c_{4,i}-1}{j+n_2-1}\binom{j+n_2-1}{j-1} & 1 \le i \le a \\ \binom{c_{5,i}-1}{j+n_2-1}\binom{j+n_2-1}{j-1} & 1 \le i \le b_1 \\ \binom{c_{6,i}-1}{j+n_2-1}\binom{j+n_2-1}{j-1} & 1 \le i \le b_2 \end{pmatrix}_{1 \le j \le n-n_2}.$$

Now, we multiply the matrix underlying the determinant in (6.3) on the right by

$$\left(\binom{-n_3-a-b_1-b_2-b_3}{j-i}\right)_{1\le i,j\le n-n_2},$$

which is a matrix that has determinant 1. We obtain that expression (6.3) is equal to

$$\prod_{i=1}^{n_1}\binom{n_2+n_3+a+b_1+b_2+b_3+i-1}{n_2}\prod_{j=1}^{n-n_2}\binom{j+n_2-1}{j-1}^{-1}$$
$$\times\prod_{i=1}^{b_3}\Delta_{c_{3,i}}^{n_3-2k_3}\prod_{i=1}^{a}\Delta_{c_{4,i}}^{n_3+b_3}\prod_{i=1}^{b_1}\Delta_{c_{5,i}}^{n_3+\frac{a}{2}+b_3+k_1}\prod_{i=1}^{b_2}\Delta_{c_{6,i}}^{n_3+\frac{a}{2}+b_3+k_2}\det\mathfrak{F},\qquad(6.4)$$

where

$$\mathfrak{F}:=\begin{pmatrix}\binom{i-1}{j-1} & 1\le i\le n_1\\ \sum_{l\ge1}\binom{c_{3,i}-1}{l+n_2-1}\binom{l+n_2-1}{l-1}\binom{-n_3-a-b_1-b_2-b_3}{j-l} & 1\le i\le b_3\\ \sum_{l\ge1}\binom{c_{4,i}-1}{l+n_2-1}\binom{l+n_2-1}{l-1}\binom{-n_3-a-b_1-b_2-b_3}{j-l} & 1\le i\le a\\ \sum_{l\ge1}\binom{c_{5,i}-1}{l+n_2-1}\binom{l+n_2-1}{l-1}\binom{-n_3-a-b_1-b_2-b_3}{j-l} & 1\le i\le b_1\\ \sum_{l\ge1}\binom{c_{6,i}-1}{l+n_2-1}\binom{l+n_2-1}{l-1}\binom{-n_3-a-b_1-b_2-b_3}{j-l} & 1\le i\le b_2\end{pmatrix}_{1\le j\le n-n_2}.$$

We now eliminate the top block as before, apply the difference operators and specialize the $c_{t,i}$'s. Then expression (6.4) becomes

$$\prod_{i=1}^{n_1}\binom{n_2+n_3+a+b_1+b_2+b_3+i-1}{n_2}\prod_{j=1}^{n-n_2}\binom{j+n_2-1}{j-1}^{-1}\det\mathfrak{G},$$

where

$$\mathfrak{G}:=\begin{pmatrix}\mathfrak{G}_1 & 1\le i\le b_3\\ \mathfrak{G}_2 & 1\le i\le a\\ \mathfrak{G}_3 & 1\le i\le b_1\\ \mathfrak{G}_4 & 1\le i\le b_2\end{pmatrix}_{n_1+1\le j\le n-n_2},$$

and

$$\mathfrak{G}_1:=\sum_{l\ge1}\binom{n_1+\frac{a}{2}+b_1+k_3+i-1}{l+n_2-n_3+2k_3-1}\binom{l+n_2-1}{l-1}\binom{-n_3-a-b_1-b_2-b_3}{j-l},$$

$$\mathfrak{G}_2:=\sum_{l\ge1}\binom{n_1+b_1+i-1}{l+n_2-n_3-b_3-1}\binom{l+n_2-1}{l-1}\binom{-n_3-a-b_1-b_2-b_3}{j-l},$$

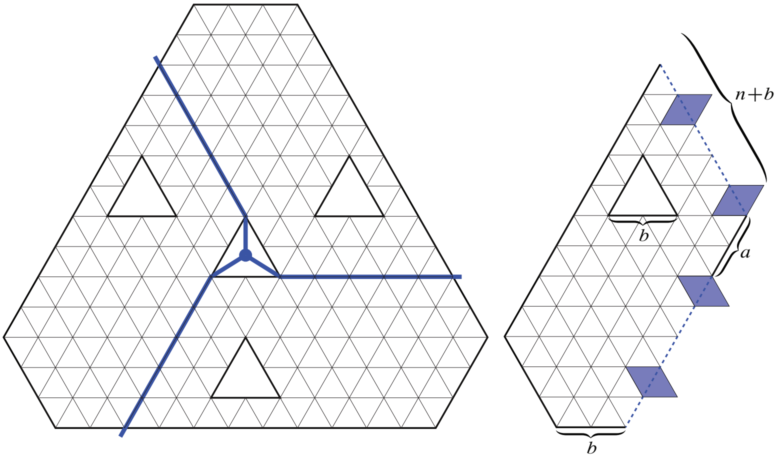

Figure 10. Applying equation (7.4) to the region $S_{n,a,b,k}$ (left; here $n = 3$, $a = 2$, $b = 2$ and $k = 1$). The region whose planar dual is G_1 (right; the dotted portions of the boundary are glued together).

$$\mathfrak{S}_3 := \sum_{l\geq 1} \binom{n_1 - 2k_1 + i - 1}{l + n_2 - n_3 - \frac{a}{2} - b_3 - k_1 - 1}\binom{l + n_2 - 1}{l - 1} \times \binom{-n_3 - a - b_1 - b_2 - b_3}{j - l},$$

$$\mathfrak{S}_4 := \sum_{l\geq 1} \binom{n_1 + \frac{a}{2} + b_1 + k_2 + i - 1}{l + n_2 - n_3 - \frac{a}{2} - b_3 - k_2 - 1}\binom{l + n_2 - 1}{l - 1} \times \binom{-n_3 - a - b_1 - b_2 - b_3}{j - l}.$$

Shifting j we arrive at formula (6.1) in the statement. ■

7. The leading coefficient in *a* for even *b*

Lemma 7.1. *For a and b even we have*

$$\mathrm{M}(S_{n,a,b,k}) = \det(\widetilde{U} + \widetilde{B})\det(\zeta\widetilde{U} + \widetilde{B})\det(\zeta^2\widetilde{U} + \widetilde{B}), \tag{7.1}$$

where

$$\zeta = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$$

is a cubic root of unity, and $\widetilde{U}$ *and* $\widetilde{B}$ *are* $(n+2b)\times(n+2b)$ *matrices given by*[24]

$$\widetilde{U} = \left[\begin{array}{c|c} I_{n+b} & O_{n+b,b} \\ \hline O_{b,n+b} & O_{b,b} \end{array}\right] \tag{7.2}$$

(where I_m *stands for the order* m *identity matrix, and* $O_{m,p}$ *for the* $m\times p$ *zero matrix) and*

$$\widetilde{B} = \left[\begin{array}{c|c} \left(\binom{a+i+j-2}{j-1}\right)_{1\le i,j\le n+b} & \left(\binom{\frac{a}{2}+k+i-1}{2k+j-1}\right)_{\substack{1\le i\le n+b\\ 1\le j\le b}} \\ \hline \left(\binom{n+a+b+j-1}{j-i}\right)_{\substack{1\le i\le b\\ 1\le j\le n+b}} & \left(\binom{n+\frac{a}{2}+b+k}{2k+j-i}\right)_{1\le i,j\le b} \end{array}\right]. \tag{7.3}$$

Proof. We use Jockusch's factorization theorem [33, Theorem 3] for plane bipartite graphs that are invariant under rotation by $2\pi/k$, for some positive integer k. For $k=3$, it states the following fact. Let r be the rotation of the plane by 120° around the origin, and let G be a plane bipartite graph invariant under r. Then, if r is color preserving (maps vertices in a bipartition class to the same class) and does not fix any vertex of G, we have

$$\mathrm{M}(G) = \mathrm{M}(G_1)\,\mathrm{M}_{\zeta}(G_1)\,\mathrm{M}_{\zeta^2}(G_1), \tag{7.4}$$

where G_1 is the orbit graph of G under the action of the rotation r, and $\mathrm{M}_u(G_1)$ is the weighted count[25] of the perfect matchings of the graph G_1 with each edge e weighted by $u^{\mathrm{cr}(e)}$ ($\mathrm{cr}(e)$, the crossing number of the edge e, is the signed count[26] of the number of times the edge crosses one of B, $r(B)$, $r^2(B)$, where B is a branch cut in the plane).

Choose G to be the planar dual[27] graph of the satellite region $S_{n,a,b,k}$, and choose the branch cut B as indicated on the left in Figure 10. Apply (7.4). To finish the proof,

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[24] We use $\widetilde{U}$ instead of the more natural $\widetilde{I}$ in order to avoid having the notation for the matrix look too much like the subsets I of $[n]$, which appear frequently in this section.

[25] I.e., the sum of the weights of all the perfect matchings of G_1, where the weight of a matching is the product of the weights of its edges.

[26] Count $+1$ each time the path traversing e from its white endpoint to its black endpoint crosses one of B, $r(B)$, $r^2(B)$ in the counterclockwise direction, and -1 each time a crossing in the clockwise direction occurs.

[27] The planar dual graph of a lattice region R on the triangular lattice has a vertex for each unit triangle inside R, and two vertices are connected by an edge if and only if the corresponding unit triangles share an edge.

it suffices to show that for the resulting orbit graph G_1 and weighted counts M_ζ and M_{ζ^2} we have

$$\mathrm{M}(G_1) = \det(\widetilde{U} + \widetilde{B}), \tag{7.5}$$

$$\mathrm{M}_\zeta(G_1) = \zeta^{n+b} \det(\zeta^2 \widetilde{U} + \widetilde{B}), \tag{7.6}$$

$$\mathrm{M}_{\zeta^2}(G_1) = \zeta^{2(n+b)} \det(\zeta \widetilde{U} + \widetilde{B}). \tag{7.7}$$

Indeed, equation (7.1) follows then from equation (7.4).

To prove (7.5), note that $\widetilde{B}$ is the Gessel–Viennot matrix for the fundamental region of $S_{n,a,b,k}$ shown on the right in Figure 10 when one encodes its lozenge tilings by paths of lozenges connecting horizontal unit segments on its boundary, and one lets each of the $n + b$ lozenges straddling the lower (resp., upper) dotted line contribute a starting (resp., ending) unit segment.

More precisely, the upper horizontal edge for a lozenge straddling the lower dotted line is a starting unit segment, and the lower horizontal edge of a lozenge straddling the upper dotted line is an ending unit segment for a path of lozenges. List the starting unit segments starting with the ones along the lower dotted line, ordered from top down, and continuing with the ones along the base, ordered from right to left. List the ending unit segments starting with the ones along the upper dotted line, ordered from bottom up, and continuing with the ones along the base of the satellite, ordered from right to left. Then one readily checks that the resulting Gessel–Viennot matrix is precisely $\widetilde{B}$.

The perfect matchings of G_1 can be viewed as lozenge tilings of the fundamental region of $S_{n,a,b,k}$ (under the action of r; see the picture on the right in Figure 10) in which the sets of lozenges straddling the two dotted lines are images of one another under r. By the Lindström–Gessel–Viennot theorem (the form in [18, Lemma 14] is most useful here; since b is even, the permutations determined by families of non-intersecting lattice paths connecting the starting points to the ending points all have the same sign), the number of such lozenge tilings in which the straddling lozenges are in positions $i_1, \dots, i_s$ is equal to the principal minor of $\widetilde{B}$ containing the rows and columns of indices $\{i_1, \dots, i_s\} \cup \{n + b + 1, \dots, n + 2b\}$. It follows that[28]

$$\mathrm{M}(G_1) = \sum_{I \subset [n+b]} \det \widetilde{B}^{I \cup \{n+b+1,\dots,n+2b\}}_{I \cup \{n+b+1,\dots,n+2b\}}. \tag{7.8}$$

However, the right-hand side above is equal to $\det(\widetilde{U} + \widetilde{B})$ by equation (7.10) below. This proves (7.5).

[28] As is customary, we write $[n]$ for the set $\{1, 2, \dots, n\}$, and A^J_I for the submatrix of A obtained by selecting the rows with indices in I and the columns with indices in J.

The weighted perfect matchings counted by $\mathrm{M}_\zeta(G_1)$ correspond to lozenge tilings of the fundamental region of $S_{n,a,b,k}$ in which each tiling is weighted by ζ^l, where l is the number of lozenges straddling one of the dotted lines. The same argument deduces then (7.6) from (7.12). Equation (7.7) follows the same way from (7.11). ■

Since all entries of the matrix $\widetilde{B}$ are polynomials in a, it follows by Lemma 7.1 that, for any fixed n, b and k with b even, $\mathrm{M}(S_{n,a,b,k})$ is the polynomial in a given by the right-hand side of (7.1). The purpose of this section is to determine the degree and the leading coefficient of this polynomial. This is accomplished in the following result.

Proposition 7.2. *For b even, the degree of $\mathrm{M}(S_{2n,2a,b,k})$ regarded as a polynomial in a is $3(n^2+2bk)$, and its leading coefficient is*

$$\left\{\frac{1}{2^{n^2-n+2k}}\frac{1}{\left(\frac{b}{2}+n-k+\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_{n-k}}\right.$$
$$\left.\times\left[\prod_{i=1}^{n-k-1}\frac{1}{\left(\frac{1}{2}\right)_i}\prod_{i=1}^{k}\frac{1}{\left(\frac{1}{2}\right)_i(2i)_{b-1}\left(i+\frac{b-1}{2}\right)_{n-k}}\right]^2\right\}^3. \tag{7.9}$$

Therefore, when regarded as polynomials in a, $\mathrm{M}(S_{2n,2a,b,k})$ and the formula for it that follows from Conjecture 2.2 and Theorem 2.3 have the same degree and the same leading coefficient.

Proof. The first factor on the right-hand side of (7.1) can be written as

$$\det(\widetilde{U}+\widetilde{B})=\sum_{I\subset[n+b]}\det\widetilde{B}^{I\cup\{n+b+1,\dots,n+2b\}}_{I\cup\{n+b+1,\dots,n+2b\}} \tag{7.10}$$

(in other words, $\det(\widetilde{U}+\widetilde{B})$ is equal to the sum of the determinants of all principal minors of $\widetilde{B}$ which contain the last b rows and columns). To see this, regard each of the first $n+b$ columns of $\widetilde{U}+\widetilde{B}$ as being the sum of the corresponding column of $\widetilde{U}$ with the corresponding column of $\widetilde{B}$, and use the fact that the determinant is a linear function.

By the same argument, we also have

$$\det(\zeta\widetilde{U}+\widetilde{B})=\sum_{I\subset[n+b]}\zeta^{n+b-|I|}\det\widetilde{B}^{I\cup\{n+b+1,\dots,n+2b\}}_{I\cup\{n+b+1,\dots,n+2b\}} \tag{7.11}$$

and

$$\det(\zeta^2\widetilde{U}+\widetilde{B})=\sum_{I\subset[n+b]}\zeta^{2(n+b-|I|)}\det\widetilde{B}^{I\cup\{n+b+1,\dots,n+2b\}}_{I\cup\{n+b+1,\dots,n+2b\}}. \tag{7.12}$$

Since all entries of $\widetilde{B}$ are polynomials in a, it follows that each summand in (7.10) is also so. By Lemma 7.3 below, for the summand corresponding to the index set $I = \{i_1, \ldots, i_s\} \subset [n+b]$, the degree in a satisfies

$$\deg_a \det \widetilde{B}_{I\cup\{n+b+1,\ldots,n+2b\}}^{I\cup\{n+b+1,\ldots,n+2b\}} \leq (i_1 - 1) + (i_2 - 2) + \cdots + (i_s - s) + 2bk - bs. \tag{7.13}$$

For fixed s, the bound on the right-hand side of (7.13) attains its maximum only for

$$\{i_1, \ldots, i_s\} = \{n+b-s+1, \ldots, n+b\},$$

when it is readily seen to equal $2bk + s(n-s)$. In turn, for even n (note that the n-parameter in the statement of Proposition 7.2 is even), this is maximum only for $s = n/2$. Therefore, for all subsets $\{i_1, \ldots, i_s\} \subset [n+b]$, we have

$$(i_1 - 1) + (i_2 - 2) + \cdots + (i_s - s) + 2bk - bs \leq 2bk + \left(\frac{n}{2}\right)^2, \tag{7.14}$$

with equality attained only if $s = n/2$ and

$$\{i_1, \ldots, i_{n/2}\} = \{n/2 + b + 1, \ldots, n+b\}.$$

By equation (7.10), it follows that $\det(\widetilde{U} + \widetilde{B})$ has degree at most $2bk + (\frac{n}{2})^2$.

However, since $\mathrm{M}_r(S_{n,a,b,k}) = \mathrm{M}(G_1)$, where G_1 is the orbit graph in the proof of Lemma 7.1, equation (7.5) implies

$$\mathrm{M}_r(S_{n,a,b,k}) = \det(\widetilde{U} + \widetilde{B}). \tag{7.15}$$

Thus, as a simple calculation shows, it follows from Theorem 2.3 that the degree of $\det(\widetilde{U} + \widetilde{B})$ is actually *equal* to $2bk + (\frac{n}{2})^2$. This can only happen if, for the unique index set I_0 for which equality is attained in (7.14), we have in fact that the degree in a of $\det B_{I_0\cup\{n+b+1,\ldots,n+2b\}}^{I_0\cup\{n+b+1,\ldots,n+2b\}}$ is equal to $2bk + (\frac{n}{2})^2$. In each of the sums on the right-hand side in (7.10)–(7.12), the term corresponding to this index I_0 has degree $2bk + (\frac{n}{2})^2$, while the degree in a of all remaining terms is strictly smaller. This implies that $\det(\zeta^i \widetilde{I}_0 + \widetilde{B})$ has degree equal to $2bk + (\frac{n}{2})^2$, for $i = 1, 2, 3$. The claim about the degree of the leading term in Proposition 7.2 follows then from the factorization of Lemma 7.1.

By Lemma 7.1, for a and b even, the leading coefficient of $\mathrm{M}(S_{n,a,b,k})$ (when regarded as a polynomial in a) is equal to the product of the leading coefficients of the three factors. By formulas (7.11)–(7.12), for $i = 1, 2$, the leading coefficient in $\det(\zeta^i \widetilde{U} + \widetilde{B})$ is equal to $\zeta^{i(n+b-|I_0|)}$ times the leading coefficient in $\det(\widetilde{U} + \widetilde{B})$. The latter can be read off directly from Theorem 2.3, and since $\zeta\zeta^2 = 1$, the claim about the leading coefficient in Proposition 7.2 follows.

The last claim in the statement follows readily by comparing (7.9) with the product formula for $\mathrm{M}(S_{2n,2a,b,k})$ implied by Conjecture 2.2 and Theorem 2.3. ■

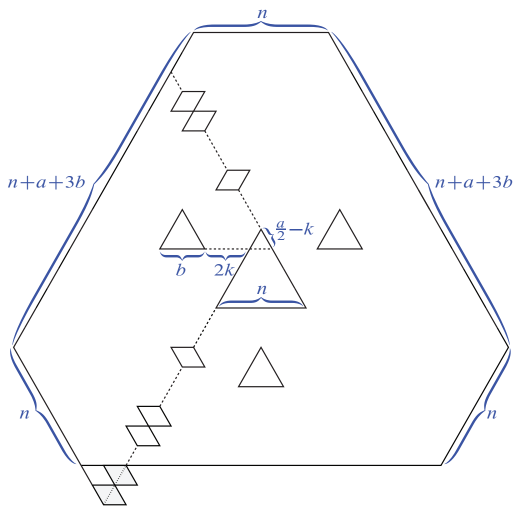

Figure 11. Obtaining equations (7.17)–(7.19) (illustrated on $S_{n,a,b,k}$ for $n=6$, $a=4$, $b=2$, $k=1$).

Lemma 7.3. *Let b be even. If $\widetilde{U}$ and $\widetilde{B}$ are given by* (7.2) *and* (7.3)*, for any index set $I=\{i_1,\dots,i_s\}\subset[n+b]$, the degree in a of the polynomial* $\det \widetilde{B}^{I\cup\{n+b+1,\dots,n+2b\}}_{I\cup\{n+b+1,\dots,n+2b\}}$ *satisfies the inequality*

$$\deg_a \det \widetilde{B}^{I\cup\{n+b+1,\dots,n+2b\}}_{I\cup\{n+b+1,\dots,n+2b\}} \\ \le (i_1-1)+(i_2-2)+\cdots+(i_s-s)+2bk-bs. \tag{7.16}$$

Proof. Let R_I be the region obtained from the fundamental region[29] of $S_{n,a,b,k}$ determined by the dashed rays in Figure 11 by removing the lozenges that straddle those rays and are at distances $i_1,\dots,i_s$ from the core (Figure 11 illustrates this for $I=\{3,6,7\}$). Encoding the lozenge tilings of R_I by families of non-intersecting paths of lozenges that connect the horizontal unit segments on the boundary of R_I (including the b such unit segments on the bottom of the left satellite), we obtain by applying the Lindström–Gessel–Viennot theorem (and using that b is even) that

$$\det \widetilde{B}^{I\cup\{n+b+1,\dots,n+2b\}}_{I\cup\{n+b+1,\dots,n+2b\}} = \mathrm{M}(R_I). \tag{7.17}$$

If we instead encode the lozenge tilings of R_I by families of non-intersecting paths of lozenges that connect the *northwest/southeast going* unit segments of the boundary of R_I, and perform Laplace expansion in the determinant of the resulting

[29]Under the action of rotation by 120 degrees.

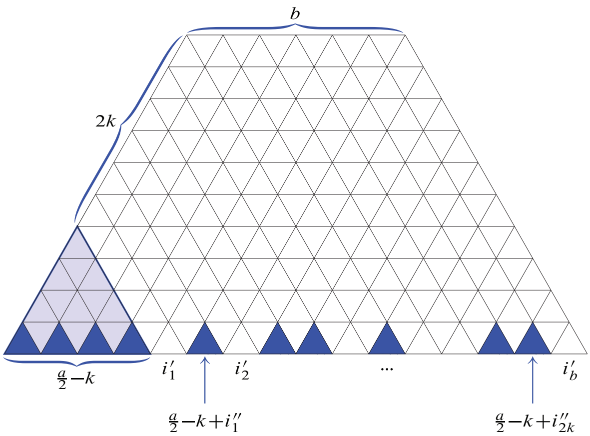

Figure 12. The region $H_{I'}$ (rotated clockwise by 120° compared to Figure 11) for $a = 14$, $b = 6$, $k = 3$ and $I' = \{1, 2, 6, 8, 9, 12\}$.

Gessel–Viennot matrix G along the rows corresponding to the b starting segments on the northeastern side of the satellite, we claim that we obtain

$$\mathrm{M}(R_I) = \pm \sum_{\{i'_1 < \cdots < i'_b\} \subset [n+b] \setminus I} (-1)^{i'_1 + \cdots + i'_b} \, \mathrm{M}(H_{I'}) \, \mathrm{M}(R_{I,I'}), \tag{7.18}$$

where $I' = \{i'_1, \ldots, i'_b\}$, and $H_{I'}$ and $R_{I,I'}$ are the regions described as follows. $H_{I'}$ is the region "spanned" by the b unit segments on the northeastern side of the left satellite and the b unit segments of the top dashed ray that are at distances $i'_1, \ldots, i'_b$ from the core – i.e., the region consisting of the union of all possible paths of lozenges that start at the former and end at the latter b unit segments. $R_{I,I'}$ is the region obtained from R_I by filling back in the satellite hole, and making b more dents along the top dashed ray, at distances $i'_1, \ldots, i'_b$ from the core (so $R_{I,I'}$ has s dents along the bottom dashed ray, and $s + b$ dents along the top dashed ray).

Indeed, the described Laplace expansion yields first an equality like (7.18) with the two tiling counts in the summand replaced by the determinants of two complementary submatrices of G. However, these submatrices are in their turn Gessel–Viennot matrices, and it is not hard to see that they correspond precisely to the above defined regions $H_{I'}$ and $R_{I,I'}$, when their tilings are encoded by families of non-intersecting paths of lozenges that connect the southwest/northeast facing unit segments of their boundary. Therefore, by the Lindström–Gessel–Viennot theorem, each of the two determinants is equal to the corresponding tiling count, yielding (7.18).

Now, switch again the direction of the paths, and encode the tilings of $R_{I,I'}$ by paths of lozenges that connect the horizontal unit segments of its boundary. Things simplify if we replace $R_{I,I'}$ by $R'_{I,I'}$, the region obtained from $R_{I,I'}$ by adding a down-pointing dented triangle of side b along its base, with b dents along its south-eastern side (see Figure 11). Then clearly $\mathrm{M}(R_{I,I'}) = \mathrm{M}(R'_{I,I'})$, and the mentioned encoding gives

$$\mathrm{M}(R_{I,I'}) = \mathrm{M}(R'_{I,I'}) = \det B(n+2b)^{I\cup I'}_{I\cup\{n+b+1,\dots,n+2b\}}, \tag{7.19}$$

where $B(n)$ is the matrix

$$B(n) := \left(\binom{a+i+j-2}{j-1} \right)_{1\le i,j\le n}. \tag{7.20}$$

A detailed picture of the region $H_{I'}$ – provided $a/2 \ge k$ – is showed in Figure 12. By Lemma 7.4, if we set[30] $\{i''_1, \dots, i''_{2k}\} := [2k+b] \setminus \{i'_1, \dots, i'_b\}$, then we have

$$\deg_a \mathrm{M}(H_{I'}) = (i''_1 - 1) + (i''_2 - 2) + \cdots + (i''_{2k} - 2k). \tag{7.21}$$

On the other hand, by Lemma 7.5 we have

$$\begin{aligned} &\deg_a \det B(n+2b)^{I\cup I'}_{I\cup\{n+b+1,\dots,n+2b\}} \\ &\quad \le (i_1 - 1) + \cdots + (i_s - s) + (i'_1 - (s+1)) + \cdots + (i'_b - (s+b)). \end{aligned} \tag{7.22}$$

By equations (7.17)–(7.19), (7.21), and (7.22), we obtain that for integers a with $a/2 \ge k$, the values of $\det \widetilde{B}^{I\cup\{n+b+1,\dots,n+2b\}}_{I\cup\{n+b+1,\dots,n+2b\}}$ depend polynomially on a, as a polynomial of degree less or equal than

$$\begin{aligned} &(i''_1 - 1) + (i''_2 - 2) + \cdots + (i''_{2k} - 2k) + (i_1 - 1) + \cdots + (i_s - s) \\ &\quad + (i'_1 - (s+1)) + \cdots + (i'_b - (s+b)). \end{aligned} \tag{7.23}$$

Since by definition $\{i'_1, \dots, i'_s, i''_1, \dots, i''_{2k}\} = \{1, \dots, 2k+b\}$, the sum on the right-hand side above is readily seen to be equal to $(i_1 - 1) + \cdots + (i_s - s) + 2bk - bs$. However, as the above mentioned polynomial agrees with $\det \widetilde{B}^{I\cup\{n+b+1,\dots,n+2b\}}_{I\cup\{n+b+1,\dots,n+2b\}}$ (which is a polynomial in a due to the fact that all its entries are so) on an infinite set of values (namely, all integers a with $a/2 \ge k$), it follows that they are identical, and thus (7.16) holds. ∎

Lemma 7.4. *Let $I' = \{i'_1, \dots, i'_b\}$, $1 \le i'_1 < \cdots < i'_b \le 2k+b$, and set $\{i''_1, \dots, i''_{2k}\} := [2k+b] \setminus \{i'_1, \dots, i'_b\}$. Then the number of lozenge tilings of the region $H_{I'}$ shown in Figure 12 is a polynomial in a of degree*

$$\deg_a \mathrm{M}(H_{I'}) = (i''_1 - 1) + (i''_2 - 2) + \cdots + (i''_{2k} - 2k). \tag{7.24}$$

[30] We are using here the fact that $i'_b \le 2k+b$. This is so because the unit segment at which a path of lozenges starting from the northeastern side of the left satellite crosses the top dashed ray is at distance at most $2k+b$ from the core.

Proof. We use the classical fact [25, 29] that the number of lozenge tilings of the trapezoid $T_m(x_1,\dots,x_n)$ of base length m, sides of length n, and with unit triangular dents on its base at positions $1 \le x_1 < \cdots < x_n \le m$, is given by

$$\mathrm{M}(T_m(x_1,\dots,x_n)) = \frac{\Delta(x_1,\dots,x_n)}{\Delta(1,\dots,n)}, \tag{7.25}$$

where

$$\Delta(x_1,\dots,x_n) := \prod_{1\le i<j\le n}(x_j - x_i). \tag{7.26}$$

Our region $H_{I'}$ (see Figure 12) is obtained from the region

$$T_{a/2+k+b}(1,\dots,a/2-k,a/2-k+i''_1,\dots,a/2-k+i''_{2k})$$

by removing the lozenges forced by the $a/2-k$ initial contiguous dents (this effectively removes a triangle of side $a/2-k$ from the left corner of the trapezoid).

Therefore, by (7.25) we have

$$\mathrm{M}(H_{I'}) = \frac{\Delta(1,\dots,a/2-k,a/2-k+i''_1,\dots,a/2-k+i''_{2k})}{\Delta(1,\dots,a/2-k,a/2-k+1,\dots,a/2-k+2k)}. \tag{7.27}$$

Clearly, one can write[31]

$$\frac{\Delta([n],n+i_1,\dots,n+i_l)}{\Delta([n],n+1,\dots,n+l)} = \frac{\Delta([n],n+i_1)}{\Delta([n],n+1)}\frac{\Delta([n],n+i_2)}{\Delta([n],n+2)}\cdots\frac{\Delta([n],n+i_l)}{\Delta([n],n+l)} \times \frac{\Delta(n+i_1,\dots,n+i_l)}{\Delta(n+1,\dots,n+l)}. \tag{7.28}$$

One readily verifies that

$$\frac{\Delta([n],n+t)}{\Delta([n],n+i)} = \frac{(n+i)_{t-i}}{(i)_{t-i}}. \tag{7.29}$$

Apply (7.28) to the right-hand side of (7.27), replacing n by $a/2-k$, l by $2k$ and i_j by i''_j. By (7.29), for $j = 1,\dots,2k$, the jth resulting fraction in the product is

$$\frac{(a/2-k+j)_{i''_j-j}}{(j)_{i''_j-j}},$$

and has therefore degree $i''_j - j$ in a. Since the last fraction in the product – which comes from the last fraction on the right-hand side of (7.28) – is a constant (as a polynomial in a), we obtain for $\mathrm{M}(H_{I'})$ an explicit expression as a product of linear factors in a, having the degree specified on the right-hand side of (7.24). This completes the proof. ∎

[31] Here $[n]$ denotes the sequence of integers $1,2,\dots,n$.

Lemma 7.5. *Let $B(n)$ be the matrix given by (7.20). Then for any $I, J \subset [n]$, $|I| = |J| = s$, $J = \{j_1 < j_2 < \cdots < j_s\}$, the degree of* $\det B(n)_I^J$ *as a polynomial in a satisfies*

$$\deg_a \det B(n)_I^J \le (j_1 - 1) + (j_2 - 2) + \cdots + (j_s - s). \tag{7.30}$$

Proof. Consider the multivariate generalization $\widetilde{B}(n)$ obtained from $B(n)$ by replacing a by a_i in all entries of row i, for $i = 1, \dots, n$. Then we have

$$\widetilde{B}(n)_I^J = \begin{pmatrix} \binom{a_{i_1}+i_1+j_1-2}{j_1-1} & \binom{a_{i_1}+i_1+j_2-2}{j_2-1} & \cdots & \binom{a_{i_1}+i_1+j_s-2}{j_s-1} \\ \binom{a_{i_2}+i_2+j_1-2}{j_1-1} & \binom{a_{i_2}+i_2+j_2-2}{j_2-1} & \cdots & \binom{a_{i_2}+i_2+j_s-2}{j_s-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{a_{i_s}+i_s+j_1-2}{j_1-1} & \binom{a_{i_s}+i_s+j_2-2}{j_2-1} & \cdots & \binom{a_{i_s}+i_s+j_s-2}{j_s-1} \end{pmatrix}.$$

All entries in column k are polynomials in $a_{i_1}, a_{i_2}, \dots, a_{i_s}$ of total degree $j_k - 1$. It follows that in the expansion of $\det \widetilde{B}(n)_I^J$ as a sum over permutations, each term, regarded as a polynomial in $a_{i_1}, a_{i_2}, \dots, a_{i_s}$, has total degree $(j_1 - 1) + \cdots + (j_s - 1)$. Therefore, the total degree of $\det \widetilde{B}(n)_I^J$ is at most $(j_1 - 1) + \cdots + (j_s - 1)$.

However, note that when any two of $a_{i_1} + i_1, a_{i_2} + i_2, \dots, a_{i_s} + i_s$ are equal, there are two identical rows in $\widetilde{B}(n)_I^J$, so its determinant is zero. This means that

$$\det \widetilde{B}(n)_I^J = P(a_{i_1}, \dots, a_{i_s}) \prod_{1 \le u < v \le s} [(a_{i_v} + i_v) - (a_{i_u} + i_u)], \tag{7.31}$$

where P is a polynomial of total degree at most $(j_1 - 1) + \cdots + (j_s - 1) - \binom{s}{2} = (j_1 - 1) + \cdots + (j_s - s)$. When specializing back all a_i's to a, the degree in a of the product on the right-hand side of (7.31) becomes zero. This completes the proof. ∎

8. Proof of Theorem 2.4

The correlation of the core and its three satellites could also be measured exclusively via rotationally symmetric tilings, by defining

$$\omega_r(a, b, k) := \lim_{n \to \infty} \frac{\mathrm{M}_r(S_{2n,a,b,k})}{\mathrm{M}_r(S_{2n,a,b,0})}. \tag{8.1}$$

The asymptotics of this correlation is given in the following result.

Proposition 8.1. *For non-negative a, b and k with a even, we have*

$$\omega_r(a,b,k) \sim 3^{b^2/4} \frac{G\left(\frac{b}{2}+1\right)^2}{\left\{\frac{\Gamma(\frac{a}{6}+\frac{b}{2}+\frac{1}{3})}{\Gamma(\frac{a}{6}+\frac{b}{2}+\frac{2}{3})}\frac{\Gamma(\frac{a}{6}+\frac{2}{3})}{\Gamma(\frac{a}{6}+\frac{1}{3})}\frac{G(\frac{a}{2}+\frac{3b}{2}+1)}{G(\frac{a}{2}+1)}\right\}^{2/3}} k^{b(a+b)/2}, \quad k\to\infty. \tag{8.2}$$

Proof. We use the formulas for $\mathrm{M}_r(S_{n,a,b,k})$ provided in [43]. Since these are quite different for even and odd b, we distinguish between these two cases. Throughout this proof, n is even (this can be assumed without loss of generality, as (8.1) only involves even values of the n-parameter of $S_{n,a,b,k}$).

Case 1: *b* even. The 120°-rotationally invariant tilings of $S_{n,a,b,k}$ can be identified with the perfect matchings of the quotient graph G of its planar dual under the action of the group generated by a 120° rotation (see, e.g., [20] for a detailed discussion of the case $b=0$, which readily adapts to the case of general b). This quotient graph G is a bipartite planar graph that can be drawn in the plane so that it is symmetric about an axis. Thus, the factorization theorem of [6] can be applied to it. The resulting two "halves" are planar duals of regions whose lozenge tilings were enumerated by Lai and Rohatgi in [43]. The statement of the factorization theorem of [6] then yields

$$\mathrm{M}_r(S_{n,a,b,k}) = 2^{n+b} P_1\left(\frac{a}{2}+1,k,\frac{n}{2}-k-1,\frac{b}{2}\right) P_2\left(\frac{a}{2}+1,k,\frac{n}{2}-k,\frac{b}{2}\right), \tag{8.3}$$

where, cf. [43, (2.4) and (2.5)], P_1 and P_2 are given by

$$P_1(x,y,z,a) := \frac{1}{2^{y+z}} \prod_{i=1}^{y+z} \frac{(2x+6a+2i)_i(x+3a+2i+\frac{1}{2})_{i-1}}{(i)_i(x+3a+i+\frac{1}{2})_{i-1}} \prod_{i=1}^{a} \frac{T_1(i)}{B_1(i)}, \tag{8.4}$$

where

$$T_1(i) := (z+i)_{y+a-2i+1}(x+y+2z+2a+2i)_{2y+2a-4i+2} \times (x+3i-2)_{y-i+1}(x+3y+2i-1)_{i-1},$$

$$B_1(i) := (i)_y(y+2z+2i-1)_{y+2a-4i+3}(2z+2i)_{y+2a-4i+1} \times (x+y+z+2a+i)_{y+a-2i+1},$$

and

$$P_2(x,y,z,a) := \frac{(\frac{x}{2}+\frac{3y}{2})_a(x+2y+z+2a)_a}{2^{2a}(\frac{x}{2}+\frac{3y}{2}+z+a+\frac{1}{2})_a}\frac{1}{2^{y+z}} \times \prod_{i=1}^{y+z}\frac{(2x+6a+2i-2)_{i-1}(x+3a+2i-\frac{1}{2})_i}{(i)_i(x+3a+i-\frac{1}{2})_{i-1}}\prod_{i=1}^{a}\frac{T_2(i)}{B_2(i)}, \tag{8.5}$$

where

$$\begin{aligned}T_2(i) &:= (z+i)_{y+a-2i+1}(x+y+2z+2a+2i-1)_{2y+2a-4i+3}\\ &\quad\times(x+3i-2)_{y-i}(x+3y+2i-1)_{i-1},\\ B_2(i) &:= (i)_y(y+2z+2i-1)_{y+2a-4i+3}(2z+2i)_{y+2a-4i+1}\\ &\quad\times(x+y+z+2a+i-1)_{y+a-2i+2}.\end{aligned}$$

Combining (8.3) with its $k=0$ specialization we get

$$\frac{\mathrm{M}_r(S_{n,a,b,k})}{\mathrm{M}_r(S_{n,a,b,0})} = \frac{P_1(\frac{a}{2}+1,k,\frac{n}{2}-k-1,\frac{b}{2})}{P_1(\frac{a}{2}+1,0,\frac{n}{2}-1,\frac{b}{2})}\frac{P_2(\frac{a}{2}+1,k,\frac{n}{2}-k,\frac{b}{2})}{P_2(\frac{a}{2}+1,0,\frac{n}{2},\frac{b}{2})}. \tag{8.6}$$

It is easy to see that, for fixed a, b and k, as $n\to\infty$ we have

$$\lim_{n\to\infty}\frac{P_1(\frac{a}{2}+1,k,\frac{n}{2}-k-1,\frac{b}{2})}{P_1(\frac{a}{2}+1,0,\frac{n}{2}-1,\frac{b}{2})} = \prod_{i=1}^{b/2}\frac{1}{(i)_k}\frac{(\frac{a}{2}+3i-1)_{k-i+1}}{(\frac{a}{2}+3i-1)_{-i+1}}\frac{(\frac{a}{2}+3k+2i)_{i-1}}{(\frac{a}{2}+2i)_{i-1}} \tag{8.7}$$

and

$$\lim_{n\to\infty}\frac{P_2(\frac{a}{2}+1,k,\frac{n}{2}-k,\frac{b}{2})}{P_2(\frac{a}{2}+1,0,\frac{n}{2},\frac{b}{2})} = \frac{(\frac{a}{4}+\frac{3k}{2}+\frac{1}{2})_{b/2}}{(\frac{a}{4}+\frac{1}{2})_{b/2}}\prod_{i=1}^{b/2}\frac{1}{(i)_k}\frac{(\frac{a}{2}+3i-1)_{k-i}}{(\frac{a}{2}+3i-1)_{-i}}\frac{(\frac{a}{2}+3k+2i)_{i-1}}{(\frac{a}{2}+2i)_{i-1}}. \tag{8.8}$$

Combining (8.3), (8.7), and (8.8) we obtain

$$\begin{aligned}\omega_r(a,b,k) &= \lim_{n\to\infty}\frac{\mathrm{M}_r(S_{n,a,b,k})}{\mathrm{M}_r(S_{n,a,b,0})}\\ &= \frac{(\frac{a}{4}+\frac{3k}{2}+\frac{1}{2})_{b/2}}{(\frac{a}{4}+\frac{1}{2})_{b/2}}\prod_{i=1}^{b/2}\frac{1}{[(i)_k]^2}\\ &\quad\times\frac{(\frac{a}{2}+3i-1)_{k-i}(\frac{a}{2}+3i-1)_{k-i+1}}{(\frac{a}{2}+3i-1)_{-i}(\frac{a}{2}+3i-1)_{-i+1}}\left[\frac{(\frac{a}{2}+3k+2i)_{i-1}}{(\frac{a}{2}+2i)_{i-1}}\right]^2.\end{aligned} \tag{8.9}$$

To finish proving this case, we need to determine the asymptotics of the right-hand side above as $k \to \infty$.

As a and b are fixed, we have

$$\frac{(\frac{a}{4}+\frac{3k}{2}+\frac{1}{2})_{b/2}}{(\frac{a}{4}+\frac{1}{2})_{b/2}} \sim \frac{(\frac{3}{2})^{b/2}}{(\frac{a}{4}+\frac{1}{2})_{b/2}} k^{b/2}, \quad k \to \infty. \tag{8.10}$$

Expressing the factor in the product in (8.9) in terms of Gamma functions via formula (2.3), and using that for any fixed a and b we have

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim x^{a-b}, \quad x \to \infty \tag{8.11}$$

(see, e.g., [46, p. 119, (5.02)]), we are led to

$$\begin{aligned}&\frac{1}{[(i)_k]^2}\frac{(\frac{a}{2}+3i-1)_{k-i}(\frac{a}{2}+3i-1)_{k-i+1}}{(\frac{a}{2}+3i-1)_{-i}(\frac{a}{2}+3i-1)_{-i+1}}\left[\frac{(\frac{a}{2}+3k+2i)_{i-1}}{(\frac{a}{2}+2i)_{i-1}}\right]^2\\ &\quad\sim 3^{2(i-1)}\left(\frac{a}{2}+2i-1\right)\frac{[\Gamma(i)]^2}{[\Gamma(\frac{a}{2}+3i-1)]^2}k^{a+4i-3}, \quad k \to \infty.\end{aligned} \tag{8.12}$$

Using (8.10) and (8.12) in equation (8.9), we arrive at

$$\omega_r(a,b,k) \sim 3^{\frac{b^2}{4}} \prod_{i=1}^{b/2} \frac{\Gamma(i)^2}{\Gamma\left(\frac{a}{2}+3i-1\right)^2} k^{ab/2+b^2/2}, \quad k \to \infty. \tag{8.13}$$

Clearly,

$$\prod_{i=1}^{b/2} \Gamma(i) = G(\frac{b}{2}+1).$$

Furthermore, it is a straightforward exercise to show that

$$\prod_{i=1}^{b/2} \Gamma\left(\frac{a}{2}+3i-1\right) = \left\{\frac{\Gamma(\frac{a}{6}+\frac{b}{2}+\frac{1}{3})}{\Gamma(\frac{a}{6}+\frac{b}{2}+\frac{2}{3})}\frac{\Gamma(\frac{a}{6}+\frac{2}{3})}{\Gamma(\frac{a}{6}+\frac{1}{3})}\frac{G(\frac{a}{2}+\frac{3b}{2}+1)}{G(\frac{a}{2}+1)}\right\}^{1/3}. \tag{8.14}$$

Using this, (8.13) can, after some manipulation, be rewritten as (8.2).

Case 2: b odd. In the same fashion as we obtained (8.6) for even b, we get for odd b that

$$\frac{\mathrm{M}_r(S_{n,a,b,k})}{\mathrm{M}_r(S_{n,a,b,0})} = \frac{F_1(\frac{a}{2}+1,k,\frac{n}{2}-k-1,\frac{b+1}{2})}{F_1(\frac{a}{2}+1,0,\frac{n}{2}-1,\frac{b+1}{2})}\frac{F_2(\frac{a}{2}+1,k,\frac{n}{2}-k,\frac{b-1}{2})}{F_2(\frac{a}{2}+1,0,\frac{n}{2},\frac{b-1}{2})}, \tag{8.15}$$

where F_1 and F_2 are given by formulas [43, (2.6) and (2.7)] (these are more involved expressions than the ones for P_1 and P_2, and to keep the focus we do not list them here). Just as it was the case with equations (8.7) and (8.8), it is straightforward to see that

$$\lim_{n\to\infty} \frac{F_1(\frac{a}{2}+1, k, \frac{n}{2}-k-1, \frac{b+1}{2})}{F_1(\frac{a}{2}+1, 0, \frac{n}{2}-1, \frac{b+1}{2})}$$
$$= \frac{1}{2^k(\frac{a}{4}+\frac{b}{2}+\frac{k}{2}+\frac{1}{2})_k}$$
$$\times \prod_{i=1}^{\lfloor (b+1)/6 \rfloor} \frac{(\frac{a}{2}+3k+6i-2)_{3(b+1)/2-9i+1}}{(\frac{a}{2}+6i-2)_{3(b+1)/2-9i+1}}$$
$$\times \prod_{i=1}^{\lfloor (b-1)/6 \rfloor} \frac{\frac{a}{2}+6i-1}{\frac{a}{2}+6i+3k-1} \prod_{i=1}^{(b-1)/2} \frac{(\frac{a}{2}+3i-1)_{k-i+1}}{(\frac{a}{2}+3i-1)_{-i+1}}$$
$$\times \prod_{i=1}^{k} \frac{\Gamma(\frac{b}{2}+i+\frac{3}{2})}{\Gamma(i+\frac{3}{2})} \frac{\Gamma(\frac{a}{2}+\frac{3b}{2}+3i-1)}{\Gamma(\frac{a}{2}+\frac{3b}{2}+3i-\frac{5}{2})} \frac{1}{(i)_{(b+3)/2}(i+\frac{3}{2})_{(b-3)/2}} \tag{8.16}$$

and

$$\lim_{n\to\infty} \frac{F_2(\frac{a}{2}+1, k, \frac{n}{2}-k, \frac{b-1}{2})}{F_2(\frac{a}{2}+1, 0, \frac{n}{2}, \frac{b-1}{2})}$$
$$= \frac{\prod_{i=1}^{\lfloor (k+1)/3 \rfloor} (\frac{a}{2}+3i-1)_{3k-9i+4}}{\prod_{i=1}^{\lfloor k/3 \rfloor} \frac{a}{2}+3k-6i+1}$$
$$\times \prod_{i=1}^{k} \frac{\Gamma(\frac{b}{2}+i+\frac{3}{2})}{\Gamma(i+\frac{3}{2})} \frac{\Gamma(\frac{a}{2}+\frac{3b}{2}+3i-1)}{\Gamma(\frac{a}{2}+b+k+2i-1)}$$
$$\times \frac{1}{(i)_{(b+3)/2}(i+\frac{3}{2})_{(b-3)/2}}. \tag{8.17}$$

Looking back at (8.15), we see that we need to determine the $k \to \infty$ asymptotics of the expressions (8.16) and (8.17).

One new feature we have now is that these expressions contain products whose upper limit involves k. Because of this, in addition to (8.11) we also need to use the asymptotics of the Barnes G-function given by (2.19).

The details of the calculations depend on the residue of a modulo 3. If a is a multiple of 3, since by assumptions a is even, it is in fact a multiple of 6. Writing then $6a$ for the size of the core, and $2b+1$ for the size of the satellite (as the latter is assumed odd in the current case), we obtain after some straightforward if lengthy manipulations that the $k \to \infty$ asymptotics of the expression on the right in (8.16) (in which a is replaced by $6a$ and b by $2b+1$) is

$$\frac{2^{41/36}\pi e^{1/12}}{3^{23/24}A\Gamma\left(\frac{2}{3}\right)^2} \times \frac{\prod_{i=0}^{b}\Gamma(i+\frac{1}{2})\prod_{i=1}^{a+b}\Gamma(3i-1)\prod_{i=1}^{\lfloor b/3\rfloor}(3a+6i-1)}{\prod_{i=0}^{a+b}\Gamma(3i+\frac{1}{2})\prod_{i=1}^{b}\Gamma(3a+2i)\prod_{i=1}^{\lfloor (b+1)/3\rfloor}(3a+6i-2)_{3b-9i+4}} \times 3^{b^2/2}k^{3ab+b^2+3a/2+b+1/4}, \tag{8.18}$$

while the $k\to\infty$ asymptotics of the expression on the right in (8.17) (with a replaced by $6a$ and b by $2b+1$) is

$$\frac{2^{41/36}\pi e^{1/12}}{3^{11/24}A\Gamma\left(\frac{2}{3}\right)^2}\frac{\prod_{i=1}^{a}\Gamma(3i-1)\prod_{i=0}^{b}\Gamma(i+\frac{1}{2})}{\prod_{i=0}^{a+b}\Gamma(3i+\frac{1}{2})}3^{b^2/2+b}k^{3ab+b^2+3a/2+b+1/4}. \tag{8.19}$$

Then by (8.15)–(8.19) we obtain

$$\omega_r(6a,2b+1,k) \sim \sqrt{3}\left[\frac{2^{41/36}\pi e^{1/12}}{3^{23/24}A\Gamma(\frac{2}{3})^2}\right]^2\left[\frac{\prod_{i=0}^{b}\Gamma(i+\frac{1}{2})}{\prod_{i=0}^{a+b}\Gamma(3i+\frac{1}{2})}\right]^2\frac{\prod_{i=1}^{a}\Gamma(3i-1)\prod_{i=1}^{a+b}\Gamma(3i-1)}{\prod_{i=1}^{b}\Gamma(3a+2i)} \times \frac{\prod_{i=1}^{\lfloor b/3\rfloor}3a+6i-1}{\prod_{i=1}^{\lfloor (b+1)/3\rfloor}(3a+6i-2)_{3b-9i+4}}3^{b(b+1)}k^{6ab+2b^2+3a+2b+1/2}, \quad k\to\infty. \tag{8.20}$$

After some manipulation, using the recurrence (2.17) and the value of $G(1/2)$ given by (2.18), one sees that (8.20) can be written in terms of the Barnes G-function as

$$\omega_r(6a,2b+1,k) \sim 3^{(2b+1)^2/4}\frac{G\left(\frac{2b+1}{2}+1\right)^2}{\left\{\frac{\Gamma(a+\frac{2b+1}{2}+\frac{1}{3})}{\Gamma(a+\frac{2b+1}{2}+\frac{2}{3})}\frac{\Gamma(a+\frac{2}{3})}{\Gamma(a+\frac{1}{3})}\frac{G(3a+\frac{3(2b+1)}{2}+1)}{G(3a+1)}\right\}^{2/3}} \times k^{(2b+1)(6a+2b+1)/2}, \quad k\to\infty. \tag{8.21}$$

The remaining cases, when the size of the core is of the form $6a+2$ or $6a+4$ for some integer a, are handled similarly. Together they prove that for all even core sizes

a and odd satellite sizes b we have

$$\omega_r(a,b,k) \sim 3^{b^2/4} \frac{G\left(\frac{b}{2}+1\right)^2}{\left\{\frac{\Gamma(\frac{a}{6}+\frac{b}{2}+\frac{1}{3})}{\Gamma(\frac{a}{6}+\frac{b}{2}+\frac{2}{3})}\frac{\Gamma(\frac{a}{6}+\frac{2}{3})}{\Gamma(\frac{a}{6}+\frac{1}{3})}\frac{G(\frac{a}{2}+\frac{3b}{2}+1)}{G(\frac{a}{2}+1)}\right\}^{2/3}} k^{b(a+b)/2}, \quad k\to\infty. \tag{8.22}$$

This completes the proof. ∎

Proof of Theorem 2.4. Taking the limit as $n\to\infty$ in the statement of Conjecture 2.1, it follows by (2.4) and (2.5) that

$$\frac{\omega(S_{n,a,b,k})}{\omega_r(S_{n,a,b,k})^3} = \left[\prod_{i=1}^{k}\frac{(a+6i-4)(a+3b+6i-2)}{(a+6i-2)(a+3b+6i-4)}\right]^2. \tag{8.23}$$

One readily gets, using (8.11), that

$$\prod_{i=1}^{k}\frac{(a+6i-4)(a+3b+6i-2)}{(a+6i-2)(a+3b+6i-4)} \to \frac{\Gamma(\frac{a}{6}+\frac{2}{3})\Gamma(\frac{a}{6}+\frac{b}{2}+\frac{1}{3})}{\Gamma(\frac{a}{6}+\frac{1}{3})\Gamma(\frac{a}{6}+\frac{b}{2}+\frac{2}{3})}, \quad k\to\infty. \tag{8.24}$$

Thus, the constant approached by the right-hand side of (8.23) as $k\to\infty$ precisely cancels the factors involving the Gamma function at the denominator in the cube of the right-hand side of (8.2). Using (8.24) and the expression (8.2) for $\omega_r(S_{n,a,b,k})$, equation (8.23) yields then formula (2.20). ∎

9. Proof of Theorems 2.5 and 2.6

Proof of Theorem 2.5. Consider the region $S_{n,0,B,k}$ (illustrated on the left in Figure 13 when $n=6$, $B=4$ and $k=1$). It is what is called in [24] a *triad hexagon* – a region obtained from a lattice hexagon by removing three bowties in a triad formation (i.e., the nodes of the bowties form a lattice equilateral triangle, housing at its corners the inner lobes of the bowties; for $S_{n,0,B,k}$ these inner lobes are empty).

Recall the bowtie squeezing operation from [24]. It transforms a triad hexagon into another as follows. Choose one of the bowties $\mathcal{B}$, denote its node by v, and decrease the size of its outer lobe by d units, while increasing the size of its inner lobe by d units. Translate the other two bowties d units away from v. Finally, from the point of view of the node v, push out the three sides of the hexagon closest to the inner lobe of $\mathcal{B}$ d lattice spacings, and push in the remaining three sides d lattice spacings. Then we say that the obtained triad hexagon was obtained from the original one by squeezing in bowtie $\mathcal{B}$ d units.

A conceptual product formula for the ratio of the number of tilings of two triad hexagons related by a sequence of bowtie squeezing operations in provided in [24, Theorem 1]. It has the form of the product of ratios of hyperfactorials whose arguments have a simple geometric meaning, namely the number of lattice spacings between a bowtie node and a hexagon side.

Starting with the region $S_{n,0,B,k}$ viewed as a triad hexagon, squeeze in the top bowtie $a' = B - a$ units, then the left bowtie $b' = B - b$, and finally the right bowtie $c' = B - c$ units. It is straightforward to check that the resulting triad hexagon is precisely the region $T_{n,k,B,a,b,c}$ in the statement of Theorem 2.5 (for $n = 6$, $B = 4$ and $k = 1$, this is shown on the right in Figure 13).

Therefore, [24, Theorem 1] provides a formula for $M(T_{n,k,B,a,b,c})/\,\mathrm{M}(S_{n,0,B,k})$ as a product of ratios of hyperfactorials evaluated at certain integers representing distances from the bowtie nodes to the sides of the outer hexagons. All these distances can readily be read off from Figure 13. After simplification we obtain formula (2.21). ■

In the proof of Theorem 2.6 we will use the following result.

Lemma 9.1. *For integers $\alpha_1, \dots, \alpha_s$, $\beta_1, \dots, \beta_t$ and non-negative integers $\gamma_1, \dots, \gamma_s$, $\delta_1, \dots, \delta_t$ with $\sum_{i=1}^{s} \gamma_i = \sum_{j=1}^{t} \delta_j$, as $n \to \infty$ we have*

$$\frac{\langle n + \alpha_1 + \gamma_1 \rangle}{\langle n + \alpha_1 \rangle} \cdots \frac{\langle n + \alpha_s + \gamma_s \rangle}{\langle n + \alpha_s \rangle} \frac{\langle n + \beta_1 \rangle}{\langle n + \beta_1 + \delta_1 \rangle} \cdots \frac{\langle n + \beta_t \rangle}{\langle n + \beta_t + \delta_t \rangle} \sim n^{\sum_{i=1}^{s}[\gamma_i \alpha_i + \gamma_i(\gamma_i+1)/2] - \sum_{j=1}^{t}[\delta_j \beta_j + \delta_j(\delta_j+1)/2]}. \tag{9.1}$$

Proof. Write

$$\frac{\langle n + \alpha_i + \gamma_i \rangle}{\langle n + \alpha_i \rangle} = \Gamma(n + \alpha_i + 1) \dots \Gamma(n + \alpha_i + \gamma_i)$$

and

$$\frac{\langle n + \beta_j \rangle}{\langle n + \beta_j + \delta_j \rangle} = 1/(\Gamma(n + \beta_j + 1) \dots \Gamma(n + \beta_j + \delta_j)).$$

Since

$$\sum_{i=1}^{s} \gamma_i = \sum_{j=1}^{t} \delta_j,$$

we end up with as many factors at the numerator as at the denominator. Match them in pairs and apply equation (8.11) to each pair to obtain (9.1). ■

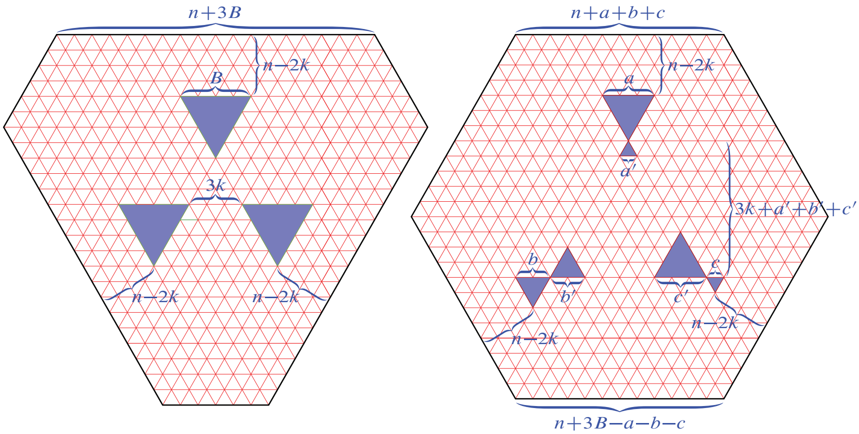

Figure 13. Distances in the region $S_{n,0,B,k}$ (left), and in the region $T_{n,k,B,a,b,c}$ (right; $a' = B - a$, $b' = B - b$, $c' = B - c$).

Proposition 9.2. *Writing $a' = B - a$, $b' = B - b$, $c' = B - c$, we have*

$$\lim_{n\to\infty} \frac{\dfrac{\mathrm{M}(T_{n,k,B,a,b,c})}{\mathrm{M}(T_{n,0,B,a,b,c})}}{\dfrac{\mathrm{M}(S_{n,0,B,k})}{\mathrm{M}(S_{n,0,B,0})}} = \frac{\langle B+b'+c'\rangle\langle b'+c'\rangle\langle B+a'+c'\rangle\langle a'+c'\rangle\langle B+a'+b'\rangle\langle a'+b'\rangle}{\langle B\rangle^3\langle a'+b'+c'\rangle^4}$$
$$\times \frac{\langle 3k+a'+b'+c'\rangle^4}{\langle 3k\rangle^4}\frac{\langle 3k+B\rangle}{\langle 3k+B+b'+c'\rangle}\frac{\langle 3k\rangle}{\langle 3k+b'+c'\rangle}$$
$$\times \frac{\langle 3k+B\rangle}{\langle 3k+B+a'+c'\rangle}\frac{\langle 3k\rangle}{\langle 3k+a'+c'\rangle}\frac{\langle 3k+B\rangle}{\langle 3k+B+a'+b'\rangle}\frac{\langle 3k\rangle}{\langle 3k+a'+b'\rangle}. \tag{9.2}$$

Proof. Write the quantity inside the limit on the left-hand side of (9.2) as

$$\frac{\dfrac{\mathrm{M}(T_{n,k,B,a,b,c})}{\mathrm{M}(S_{n,0,B,k})}}{\dfrac{\mathrm{M}(T_{n,0,B,a,b,c})}{\mathrm{M}(S_{n,0,B,0})}}.$$

Then the numerator of the resulting fraction is given by equation (2.21), and the denominator by its $k = 0$ specialization.

In the formula on the right-hand side of (2.21), the second fraction and the last three depend on n, while the others don't. We claim that the asymptotics of the part depending on n is independent of k. Then, when taking the limit as n goes to infinity in $\frac{\mathrm{M}(T_{n,k,B,a,b,c})}{\mathrm{M}(S_{n,0,B,k})} / \frac{\mathrm{M}(T_{n,0,B,a,b,c})}{\mathrm{M}(S_{n,0,B,0})}$ (which is just the left-hand side of (9.2)), the contributions coming from the factors on the right-hand side of equation (2.21) that involve n cancel out. The limit is then just the ratio between the part of the right-hand side of (2.21) which does not depend on n and its $k = 0$ specialization. This visibly leads to the expression in equation (9.2).

Therefore, to finish the proof it suffices to prove our claim. One readily sees that the product of the four fractions involving n on the right-hand side of (2.21) is an expression of the kind treated in Lemma 9.1, and thus its asymptotics is given by equation (9.1). A straightforward calculation checks that the resulting asymptotics is $n^{3B(a'+b'+c')-(a'+b'+c')^2}$, which is independent of k. ∎

When $a + b + c = a' + b' + c'$, the following result supplies the exact value of the correlation $\bar{\omega}(X_1, X_2, X_3; k)$ (see its defining equation (2.22)) of the three bowties at the center of the region $T_{n,k,B,a,b,c}$ in the special case $k = 0$.

Proposition 9.3. *If $a + b + c = a' + b' + c'$, we have*

$$\bar{\omega}(X_1, X_2, X_3; 0) = \frac{3^{\frac{1}{2}(a+b+c)^2}}{(2\pi)^{a+b+c}} \times \frac{\langle a+b+c\rangle^6 \langle a\rangle\langle b\rangle\langle c\rangle\langle a'\rangle\langle b'\rangle\langle c'\rangle}{\langle 2a+b+c\rangle\langle a+2b+c\rangle\langle a+b+2c\rangle\langle a'+b'\rangle\langle a'+c'\rangle\langle b'+c'\rangle}. \tag{9.3}$$

Proof. It is not hard to see that the hexagonal subregion of sides a', b', c', a', b', c' (clockwise from top) determined by the innermost sides of the inner lobes of the bowties in the region $T_{n,0,B,a,b,c}$ must be internally tiled. Therefore, it follows that $\mathrm{M}(T_{n,0,B,a,b,c})$ is equal to the number of tilings of this inner hexagon, which, by a rewriting of the formula in footnote 2, is

$$\langle a'\rangle\langle b'\rangle\langle c'\rangle\langle a'+b'+c'\rangle/(\langle a'+b'\rangle\langle a'+c'\rangle\langle b'+c'\rangle),$$

times the number of tilings of the region obtained from $T_{n,0,B,a,b,c}$ by removing this inner hexagon. However, the latter is a special case of the shamrock regions whose tilings were enumerated in [23]. By [23, Theorem 2.1] (with $m = a' + b' + c'$) we obtain

$$\begin{aligned}
&\frac{\mathrm{M}(T_{n,0,B,a,b,c})}{\mathrm{M}(H_{n+a+b+c})}\\
&\quad= \frac{\langle a+b+c\rangle^5\langle a'+b'+c'\rangle\langle a\rangle\langle b\rangle\langle c\rangle\langle a'\rangle\langle b'\rangle\langle c'\rangle}{\langle 2a+b+c\rangle\langle a+2b+c\rangle\langle a+b+2c\rangle\langle a'+b'\rangle\langle a'+c'\rangle\langle b'+c'\rangle}\\
&\qquad\times\frac{\langle n+a+b+c\rangle^3}{\langle n\rangle^3}\frac{\langle n+a\rangle}{\langle n+a+b\rangle}\frac{\langle n+b\rangle}{\langle n+b+c\rangle}\frac{\langle n+c\rangle}{\langle n+a+c\rangle}\\
&\qquad\times\frac{\langle n+a+2b+2c\rangle}{\langle n+a+2b+c\rangle}\frac{\langle n+2a+b+2c\rangle}{\langle n+a+b+2c\rangle}\frac{\langle n+2a+2b+c\rangle}{\langle n+2a+b+c\rangle}\\
&\qquad\times\frac{\langle \frac{n}{2}\rangle^6}{\langle \frac{n}{2}+a+b+c\rangle^6}\frac{\langle \frac{3n}{2}+2a+2b+2c\rangle^2}{\langle \frac{3n}{2}+a+b+c\rangle^2}\frac{\langle 3n+2a+2b+2c\rangle}{\langle 3n+3a+3b+3c\rangle}. \qquad (9.4)
\end{aligned}$$

The asymptotics of $\langle n+\alpha\rangle$ readily follows from the defining relation (2.20) of the Glaisher–Kinkelin constant A to be

$$\langle n+\alpha\rangle \sim \frac{e^{\frac{1}{12}}}{A}(2\pi)^{\frac{n+\alpha}{2}}e^{-\frac{3}{4}n^2-\alpha n}n^{\frac{(n+\alpha)^2}{2}-\frac{1}{12}}, \quad n\to\infty.$$

Using this for all factors involving n in equation (9.4) yields, after simplifications, equation (9.3). ■

Proof of Theorem 2.6. Multiply (9.2) by the denominator of its left-hand side, and divide both the numerator and denominator inside the limit of the resulting left-hand side by $\mathrm{M}(H_{n+a+b+c})$. By the definition of $\bar{\omega}$ (see (2.22)) and ω (see (2.12)), this gives

$$\begin{aligned}
&\frac{\bar{\omega}(X_1,X_2,X_3;k)}{\bar{\omega}(X_1,X_2,X_3;0)}\\
&\quad=\omega(0,B,k)\frac{\langle B+b'+c'\rangle\langle b'+c'\rangle\langle B+a'+c'\rangle\langle a'+c'\rangle\langle B+a'+b'\rangle\langle a'+b'\rangle}{\langle B\rangle^3\langle a'+b'+c'\rangle^4}\\
&\qquad\times\frac{\langle 3k+a'+b'+c'\rangle^4}{\langle 3k\rangle^4}\frac{\langle 3k+B\rangle}{\langle 3k+B+b'+c'\rangle}\frac{\langle 3k\rangle}{\langle 3k+b'+c'\rangle}\\
&\qquad\times\frac{\langle 3k+B\rangle}{\langle 3k+B+a'+c'\rangle}\frac{\langle 3k\rangle}{\langle 3k+a'+c'\rangle}\frac{\langle 3k+B\rangle}{\langle 3k+B+a'+b'\rangle}\frac{\langle 3k\rangle}{\langle 3k+a'+b'\rangle}. \qquad (9.5)
\end{aligned}$$

The asymptotics of the part involving k on the right-hand side of (9.5) is readily seen, using Lemma 9.1, to be $(3k)^{2[a'b'+a'c'+b'c'-B(a'+b'+c')]}$. Using then Proposition 9.3 and Theorem 2.3, equation (9.5) implies (2.23). ■

10. Concluding remarks

In this paper we presented an "experiment" designed to give the exact value of the correlation of a core and three satellite triangular holes. It relies on the first author's two decade old observation that if the satellites are enclosed symmetrically by a hexagon, the number of lozenge tilings of the resulting region is round, and on its almost decade-old generalization that brings in the presence of the core. We also presented asymptotic consequences of our exact formulas, which include the verification of the electrostatic conjecture ([9, Conjecture 1]) for the system of gaps consisting of the core and satellites (it was the special case of this when the core is empty that was the original motivation for this work). In fact, combining our results with those in [24], we obtain a verification of [9, Conjecture 1] for arbitrary triples of bowtie gaps arranged in a triad, a satisfying generalization of our motivating case. Other consequences we present include a strengthening of [9, Conjecture 1] (by specifying exactly the multiplicative constant), an unexpected exact way to calibrate the hexagonal lattice against the square lattice so that the monomer–monomer correlations decay at precisely the same rate, and a heuristic derivation of the special value $G(1/2)$ of the Barnes G-function.

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