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Apéry-like numbers for non-commutative harmonic oscillators and automorphic integrals

Kazufumi Kimoto and Masato Wakayama

Abstract. The purpose of the present paper is to study the number theoretic properties of the special values of the spectral zeta functions of the non-commutative harmonic oscillator (NcHO), especially in relation to modular forms and elliptic curves from the viewpoint of Fuchsian differential equations, and deepen the understanding of the spectrum of the NcHO. We study first the general expression of special values of the spectral zeta function $\zeta_Q(s)$ of the NcHO at $s = n$ $(n = 2, 3, \dots)$ and then the generating and meta-generating functions for *Apéry-like numbers* defined through the analysis of special values $\zeta_Q(n)$. Actually, we show that the generating function w_{2n} of such Apéry-like numbers appearing (as the "first anomaly") in $\zeta_Q(2n)$ for $n = 2$ gives an example of *automorphic integral with rational period functions* in the sense of Knopp, but still a better explanation remains to be clarified explicitly for $n > 2$. This is a generalization of our earlier result on showing that w_2 is interpreted as a $\Gamma(2)$-modular form of weight 1. Moreover, certain congruence relations over primes for "normalized" Apéry-like numbers are also proven. In order to describe w_{2n} in a similar manner as w_2, we introduce a *differential Eisenstein series* by using analytic continuation of a classical generalized Eisenstein series due to Berndt. The differential Eisenstein series is actually a typical example of the automorphic integral of negative weight. We then have an explicit expression of w_4 in terms of the differential Eisenstein series. We discuss also shortly the Hecke operators acting on such automorphic integrals and relating Eichler's cohomology group.

Contents

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1. Introduction

Let Q be a parity preserving matrix valued ordinary differential operator defined by

$$Q = Q_{\alpha,\beta} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\Big(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2\Big) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\Big(x\frac{d}{dx} + \frac{1}{2}\Big).$$

The system defined by Q is called the *non-commutative harmonic oscillator* (NcHO), which was introduced in [35–38] (see also [33,34] for references therein and for recent progress). Throughout the paper, we always assume that $\alpha, \beta > 0$ and $\alpha\beta > 1$. Under this assumption, the operator Q becomes a positive self-adjoint unbounded operator on $L^2(\mathbb{R}; \mathbb{C}^2)$, the space of $\mathbb{C}^2$-valued square-integrable functions on $\mathbb{R}$, and Q has only a discrete spectrum with uniformly bounded multiplicity:

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots (\nearrow \infty).$$

It was proved recently that the lowest eigenstate is multiplicity free [14] and also the multiplicity of general eigenstate is less than or equal to 2 [46] (see [47] for the proof).

The aim of the present paper is to advance a number theoretic study of the spectrum of the NcHO through observing special values of the spectral zeta function $\zeta_Q(s)$ ([16, 17]) defined by

$$\zeta_Q(s) := \sum_{n=1}^{\infty} \lambda_n^{-s} \quad (\Re(s) > 1),$$

and further to deepen the understanding of the spectrum. It is noted that, when $\alpha = \beta$, $Q = Q_{\alpha,\alpha}$ is unitarily equivalent to the couple of quantum harmonic oscillators, whence the eigenvalues are easily calculated as $\{\sqrt{\alpha^2 - 1}(n + \frac{1}{2}) \mid n \in \mathbb{Z}_{\ge 0}\}$ having multiplicity 2. Actually, when $\alpha = \beta$, behind Q, there exists a structure corresponding to the tensor product of the 2-dimensional trivial representation and the oscillator representation (see, e.g., [15]) of the Lie algebra $\mathfrak{sl}_2$. Namely, in this case, $\zeta_Q(s)$ is essentially given by the Riemann zeta function $\zeta(s)$ as $\zeta_Q(s) = 2(2^s - 1)\sqrt{\alpha^2 - 1}\zeta(s)$. In other words, $\zeta_Q(s)$ is a $\frac{\alpha}{\beta}$-analogue of $\zeta(s)$. The clarification of the spectrum in the general $\alpha \neq \beta$ case is, however, considered to be highly non-trivial. Indeed, while the spectrum is described theoretically by using certain continued fractions [36–38] and also by Heun's ordinary differential equations (those have four regular singular points) [40] in a certain complex domain [31, 47], almost no satisfactory information on each eigenvalue is available in reality when $\alpha \neq \beta$ (see [33] and references therein).

It is nevertheless worth mentioning that, in recent years, special attention has been paid to studying the spectrum of self-adjoint operators with non-commutative coefficients, like the Jaynes–Cummings model, the quantum Rabi model and its generalized version, etc., not only in mathematics but also in theoretical/experimental

physics (see, e.g., [2, 8, 11, 50] and references therein). The NcHO has been expected similarly to provide one of these Hamiltonians describing such quantum interacting systems, i.e., a Hamiltonian describing such an interaction between photons and atoms. Although it does not seem to be expected, it has been shown in [47] that (the "Heun picture" of) the quantum Rabi model can be obtained by the second order element of the universal enveloping algebra $U(\mathfrak{sl}_2)$ naturally arising from the NcHO through the oscillator representation. It is, in fact, caught by taking particular parameters and considering general *confluence procedure,* i.e., *confluence of two singular points* in Heun's ordinary differential equation obtained in the action of the non-unitary principal series representation of $\mathfrak{sl}_2$.

Therefore, in place of hunting each eigenvalue of Q, it is significant to study the spectral zeta function $\zeta_Q(s)$ of the NcHOs as a sort of generating function of the eigenvalues. From the physical point of view, $\zeta_Q(s)$ is also regarded as the Mellin transform of the partition function of the system defined by the NcHO. This paper discusses the number theoretic properties of the special values of $\zeta_Q(s)$ at integer points. We notice that special values are considered as moments of the partition functions. We have actually studied congruence properties of the *Apéry-like numbers* in [20] that have arisen naturally from the special values $\zeta_Q(2)$ at $s = 2$ by the same idea guided in the studies for the Apéry numbers for $\zeta(2)$ in [5] (and references therein). This study of congruence properties led us further to show that the generating function w_2 of the Apéry-like numbers for $\zeta_Q(2)$ is interpreted as a $\Gamma(2)$-modular form of weight 1 [21] in the same way as in a pioneering study by Beukers [4,6] for the Apéry numbers. In other words, the recurrence equation of these Apéry-like numbers defined in [20] provide one of the particular examples listed in Zagier [51, (#19)].[1] Moreover, it is known in [23] that the Apéry-like numbers corresponding to $\zeta_Q(2)$ are described by a finite convolution of the Hurwitz zeta function and certain variation of multiple L-values. Also, recently, certain nice congruence relations among these Apéry-like numbers that are quite resembled to the Rodriguez Villegas type congruence [30] and conjectured in [20] are proved in [29]. Further interesting congruence that involves Bernoulli numbers has been obtained in [28] (see also [45]). The congruence in [28] can be considered as a one step deep congruence of the one proved in [29] corrected by the remainder term.

It is hard in general to obtain the precise information of the higher special values of $\zeta_Q(n)$ $(n > 2)$ as the same level of $\zeta_Q(2)$. Thus, in this paper we introduce the Apéry-like numbers $J_k(n)$ $(k = 0, 1, 2, \ldots)$ for each n defined through the *first anomaly* of $\zeta_Q(n)$ $(n > 2)$. These Apéry-like numbers share the properties of the one

[1] Although the terminology "*Apéry-like*" is the identical one, the usage/definition of the name in the current paper is different from the one in the title of [51].

for $\zeta_Q(2)$, e.g., satisfy a similar recurrence relation as in the case of $\zeta_Q(2)$ and hence the ordinary differential equation satisfied by the generating function follows from the recurrence relation. Remarkably, each of the homogeneous part of those differential equations is identified to be a (n dependent) power of the homogeneous part of the one corresponding to $\zeta_Q(2)$. Further, we observe that the meta-generating functions of Apéry-like numbers are described explicitly by the modular Mahler measures studied by Rodriguez Villegas in [39]. Through this relation, we may expect to discuss an interesting aspect of a discrete dynamical system behind the NcHO defined by some group via (weighted) Cayley graphs (see [9]; also, e.g., [27]) in the future. Moreover, we show that the generating function w_{2n} of Apéry-like numbers corresponding to the first anomaly in $\zeta_Q(2n)$ when $n = 2$ is given by an automorphic integral with a rational period function in the sense of Knopp [24]. This is obviously a generalization of our earlier result [21] showing that w_2 is interpreted as a $\Gamma(2)$-modular form of weight 1. However, it is still unclear whether there is a similar explicit (geometric and algebraic) interpretation in general for $\zeta_Q(n)$ ($n > 2$). Further, the study of the special values of the spectral zeta function for the quantum Rabi model [44] and comparison to the ones for NcHO is a quite interesting future problem as NcHO is a "covering" of the model.

The organization of the paper is as follows. In §2 we calculate (Theorem 2.6) the special values of the spectral zeta function for the NcHO. These explicit formulas are found already in [22] (see [18]) by multiple integrals like (a generalization of) the original Apéry cases for $\zeta(2)$ and $\zeta(3)$ using Legendre functions [4,7]. The basic idea is on the same line as [16] but some essentially new techniques are explored.

In §3 we derive the recursion formula for the Apéry-like numbers associated to the first anomalies of special values of $\zeta_Q(s)$ and the differential equations satisfied by the generating functions of such Apéry-like numbers. Although our study is very much influenced by the classical (algebro-geometric) work on Apéry numbers in [4,6,7] and its subsequent developments, since the family of generating functions for Apéry-like numbers arising via the NcHO possesses a remarkable hierarchical structure, there is a decisive difference between these two. We then define the *normalized* Apéry-like numbers which are shown to be rational numbers, and present numerical data of these numbers. In the end of this section, we give a certain conjecture (Conjecture 3.6) for the congruence among those normalized Apéry-like numbers which are the generalization of the results in [20] based on numerical experiments. We can only show in this paper a weaker/partial result in Theorem 3.10, which may be considered as a version of the classical Kummer congruence for the special values at negative odd integer points of $\zeta(s)$. We remark that, however, it is quite difficult to expect an exact generalization of the congruence relation (i.e., of the same shape which is relevant to the hypergeometric series) shown by employing p-adic analysis in [29] (and [28]) for $\zeta(2)$.

We study in §4 also meta-generating functions for Apéry-like numbers in relation to the study on modular Mahler measures in [39]. In §5, we first recall briefly the modular form interpretation of the generating function for the Apéry-like numbers for $\zeta_Q(2)$ from [21] and discuss the corresponding generating function w_{2n} for the Apéry-like numbers for (the first anomaly in) $\zeta_Q(2n)$. We may also study the Apéry numbers associated with $\zeta_Q(2n+1)$, but the structure behind this is different from the one in [7] that is relating with $K3$ surfaces. Actually, although the homogeneous part of the differential equation satisfied by the Apéry-like numbers arisen from odd special values are the same as the even case, even the value $\zeta_Q(3)$ can not be interpreted as a picture of $K3$ spaces. We recall then in §5 a notion of *automorphic integrals with rational period functions* in the sense of Knopp [24] (that is a slightly generalized notion of the automorphic integrals [10]). Then we study w_{2n} from the viewpoint of Fuchsian differential equations. Indeed, we show that w_{2n} can be expressed by the linear space spanned by higher derivatives of automorphic integrals and w_2. In other words, we observe that w_{2k} is obtained by some linear combination of the multiple integral of the (same) modular forms. For instance, the explicit expression of w_6 by such a linear span of integrals is given in §5.5. In order to describe w_{2n} in a similar manner as w_2, it is necessary to introduce a *differential Eisenstein series* by using analytic continuation of a classical generalized Eisenstein series due to Berndt [3] in §6. These differential Eisenstein series provide typical examples of the automorphic integral of negative weight and we have an explicit expression of w_4 in terms of the differential Eisenstein series. We notice that the differential Eisenstein series is periodic, whence has a Fourier expansion at the infinity. Further, we discuss shortly the Hecke operators acting on such automorphic integrals and compute the associated L-function of the differential Eisenstein series (which has an Euler product). In the final § 7, we discuss briefly the Eichler cohomology groups relevant to the periodic automorphic integrals. A part of ideas of the paper has been discussed in our proceedings paper [22], but there is a certain misleading terminology [22] so that we will fix that in this paper.[2]

[2]The general definition of "residual modular forms" in [22] is too demanding. Although the example given in [22] satisfies such strong condition in the definition, if the level N is large, i.e., the number of inequivalent cusps is increasing, the definition of residual modular forms allows only the zero form. In this paper, we find actually that the notion of the automorphic integrals in the sense of [24] is sufficient for our study.

2. Special values of the spectral zeta function

From the sequence of the eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots (\to \infty)$ of Q, we define the spectral zeta function of Q by the Dirichlet series

$$\zeta_Q(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}.$$

This series is absolutely convergent and defines a holomorphic function in s in the region $\Re s > 1$. We call this function $\zeta_Q(s)$ the *spectral zeta function* for the non-commutative harmonic oscillator Q, see [17]. The zeta function $\zeta_Q(s)$ is analytically continued to the whole complex plane $\mathbb{C}$ as a single-valued meromorphic function which is holomorphic except for the simple pole at $s = 1$. It is notable that $\zeta_Q(s)$ has 'trivial zeros' at $s = 0, -2, -4, \ldots$ from the presence of $\Gamma(s)^{-1}$ at the analytic continuation to the whole complex plane [17]. When the two parameters α and β are equal, then $\zeta_Q(s)$ essentially gives the Riemann zeta function $\zeta(s)$ (see Remark 2.2).

We are interested in the special values of $\zeta_Q(s)$, that is, the values $\zeta_Q(s)$ at $s = 2, 3, 4, \ldots$. In [16] the first two special values are calculated as

$$\begin{aligned}
\zeta_Q(2) &= 2\Big(\frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\Big)^2 \\
&\quad \times \Bigg(\zeta\Big(2, \frac{1}{2}\Big) \\
&\qquad + \Big(\frac{\alpha-\beta}{\alpha+\beta}\Big)^2 \int_{[0,1]^2} \frac{4du_1du_2}{\sqrt{(1-u_1^2u_2^2)^2 + (1-u_1^4)(1-u_2^4)/(\alpha\beta-1)}}\Bigg), \\
\zeta_Q(3) &= 2\Big(\frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\Big)^3 \\
&\quad \times \Bigg(\zeta\Big(3, \frac{1}{2}\Big) \\
&\qquad + 3\Big(\frac{\alpha-\beta}{\alpha+\beta}\Big)^2 \int_{[0,1]^3} \frac{8du_1du_2du_3}{\sqrt{(1-u_1^2u_2^2u_3^2)^2 + (1-u_1^4)(1-u_2^4u_3^4)/(\alpha\beta-1)}}\Bigg),
\end{aligned}$$

where $\zeta(s, x) = \sum_{n=0}^{\infty}(n+x)^{-s}$ is the Hurwitz zeta function. These values are also given by the contour integral expressions using a solution of a certain Fuchsian differential equation. Later, in [32] Ochiai gave an expression of $\zeta_Q(2)$ using the complete elliptic integral or the hypergeometric function, and the present authors [20] gave a similar formula for $\zeta_Q(3)$.

In this section, we present an explicit calculation of the special values of the spectral zeta function $\zeta_Q(k)$ of the non-commutative harmonic oscillator Q for all positive integers $k > 1$, and express them in terms of integrals of certain algebraic functions (see Theorem 2.6 for the formula).

2.1. Preliminaries for calculating special values

Following to the method in [16], we first explain how to calculate the special values of $\zeta_Q(s)$.

Put

$$\varepsilon := \frac{1}{\sqrt{\alpha\beta}}, \quad \kappa := \varepsilon(1-\varepsilon^2)^{-1/2} = \frac{1}{\sqrt{\alpha\beta - 1}}$$

and

$$A := \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Notice that $0 < \varepsilon < 1$ and $\kappa > 0$. Since it is difficult to find the heat kernel of the NcHO

$$Q = \frac{1}{2}(-\partial_x^2 + x^2)A + \Big(x\partial_x + \frac{1}{2}\Big)J,$$

we look at a slightly modified one

$$Q' = A^{-1/2}QA^{-1/2} = \frac{1}{2}(-\partial_x^2 + x^2) + \varepsilon J\Big(x\partial_x + \frac{1}{2}\Big),$$

whose heat kernel is explicitly obtained as we see below.

The heat kernel of the usual quantum harmonic oscillator is known as the Mehler kernel and is given by

$$p(t,x,y) = \pi^{-1/2}e^{-t/2}(1-e^{-2t})^{-1/2}\exp\Big(-\frac{x^2-y^2}{2} - \frac{(e^{-t}x-y)^2}{1-e^{-2t}}\Big).$$

Namely, $p(t,x,y)$ satisfies

$$-\partial_t p(t,x,y) = \frac{1}{2}(-\partial_x^2 + x^2)p(t,x,y), \quad p(t,x,y) \to \delta(x-y) \quad (t \downarrow 0).$$

Put

$$q(t,x,y) = (1-\varepsilon^2)^{1/4}p((1-\varepsilon^2)^{1/2}t, (1-\varepsilon^2)^{1/4}x, (1-\varepsilon^2)^{1/4}y).$$

Then

$$-\partial_t q(t,x,y) = \frac{1}{2}(-\partial_x^2 + (1-\varepsilon^2)x^2)q(t,x,y), \quad q(t,x,y) \to \delta(x-y) \quad (t \downarrow 0).$$

Define

$$K'(t,x,y) = q(t,x,y)\exp\Big(\frac{\varepsilon(x^2-y^2)}{2}J\Big).$$

We see that

$$-\partial_t K'(t,x,y) = \frac{1}{2}(-\partial_x^2 + x^2)K'(t,x,y) + \varepsilon J\Big(x\partial_x + \frac{1}{2}\Big)K'(t,x,y),$$
$$K'(t,x,y) \to \delta(x-y)I \quad (t \downarrow 0),$$

which implies that $K'(t,x,y)$ is the heat kernel of Q' (see [16] for detail[3]). Hence, the integral kernel $Q^{-1}(x,y)$ of Q^{-1} is

$$\begin{aligned} Q^{-1}(x,y) &= \int_0^\infty A^{-1/2}K'(t,x,y)A^{-1/2}\,dt \\ &= \int_0^1 A^{1/2}K(u,(1-\varepsilon^2)^{1/4}x,(1-\varepsilon^2)^{1/4}y)A^{-1/2}\,du \quad (u=e^{-\frac{t}{2}}) \end{aligned}$$

since $Q^{-1} = A^{-1/2}Q'^{-1}A^{-1/2}$, where we put

$$\begin{aligned} K(u,x,y) &:= 2\pi^{-1/2}(1-\varepsilon^2)^{-1/4}(1-u^4)^{-1/2}E(u,x,y)B(x,y), \\ E(u,x,y) &:= \exp\left(-\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} \frac{1+u^4}{2(1-u^4)} & \frac{-u^2}{1-u^4} \\ \frac{-u^2}{1-u^4} & \frac{1+u^4}{2(1-u^4)} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right), \\ B(x,y) &:= A^{-1}\exp\frac{\kappa(x^2-y^2)}{2}J. \end{aligned}$$

Furthermore, we introduce the following functions

$$\begin{aligned} \mathbf{B}(x_1,\dots,x_k) &:= \mathrm{tr}(B(x_1,x_2)B(x_2,x_3)\dots B(x_k,x_1)), \\ \mathbf{E}(u_1,\dots,u_k;x_1,\dots,x_k) &:= E(u_1,x_1,x_2)E(u_2,x_2,x_3)\dots E(u_k,x_k,x_1), \\ \mathbf{F}(u_1,\dots,u_k) &:= \int_{\mathbb{R}^k}\mathbf{E}(u_1,\dots,u_k;x_1,\dots,x_k)\mathbf{B}(x_1,\dots,x_k)dx_1\dots dx_k, \end{aligned}$$

where the symbol tr represents the matrix trace. Hence, for a positive integer k, we have

$$\begin{aligned} \zeta_Q(k) &= \mathrm{Tr}\,Q^{-k} \\ &= \int_{[0,1]^k}\left(\int_{\mathbb{R}^k}\mathrm{tr}(K(u_1,x_1,x_2)K(u_2,x_2,x_3)\cdots K(u_k,x_k,x_1))d\mathbf{x}\right)d\mathbf{u} \\ &= \left(\frac{2}{\sqrt{\pi(1-\varepsilon^2)}}\right)^k\int_{[0,1]^k}\mathbf{F}(u_1,\dots,u_k)\frac{d\mathbf{u}}{\sqrt{\prod_{j=1}^k(1-u_j^4)}}, \end{aligned} \tag{2.1}$$

[3]There is a typo in [16, (2.11b)]. The right equation should be

$$\partial_t p_\gamma(t,x,y) = -\frac{1}{2}[\partial_x^2 + (1-\gamma^2)x^2]p_\gamma(t,x,y),$$

in which the coefficient of x^2 is replaced from $(1-\gamma^2)^{1/2}$ to $1-\gamma^2$. The result itself is, however, correct.

where $d\mathbf{x} = dx_1 \dots dx_k$ and $d\mathbf{u} = du_1 \dots du_k$ for short, and the symbol Tr denotes the operator trace. This is our basis to calculate the special values. Thus, we have only to calculate $\mathbf{F}(u_1, \dots, u_k)$ to get the special values of the spectral zeta function $\zeta_Q(s)$.

2.2. Special values $\zeta_Q(k)$

The following lemma is crucial.

Lemma 2.1. *For any positive integer k, it holds that*

$$\mathbf{B}(x_1, x_2, \dots, x_k) = 2\Big(\frac{\alpha+\beta}{2\alpha\beta}\Big)^k \Big\{1 + \sum_{0<2j\le k} \Big(\frac{\alpha-\beta}{\alpha+\beta}\Big)^{2j} \sum_{1\le i_1<i_2<\dots<i_{2j}\le k} \cos\Big(\kappa \sum_{r=1}^{2j} (-1)^r x_{i_r}^2\Big)\Big\}.$$

Proof. For convenience, let us put $i = \sqrt{-1}$, $a_1 = \alpha^{-1}$, $a_2 = \beta^{-1}$ and $t_j = \kappa x_j^2/2$ $(j = 1, 2, \dots, k)$. The function $\mathbf{B}(x_1, x_2, \dots, x_k)$ is then calculated as follows:

$$\begin{aligned}
&\mathbf{B}(x_1, x_2, \dots, x_k) \\
&= \sum_{s_1,s_2,\dots,s_k\in\{1,2\}} a_{s_1}a_{s_2}\dots a_{s_k} \prod_{m=1}^{k} \cos\Big(t_m - t_{m+1} + \frac{s_{m+1}-s_m}{2}\pi\Big) \\
&= \sum_{s_1,s_2,\dots,s_k\in\{1,2\}} a_{s_1}a_{s_2}\dots a_{s_k} \\
&\qquad \times \prod_{m=1}^{k} \frac{i^{s_{m+1}-s_m} e^{i(t_m-t_{m+1})} + i^{-(s_{m+1}-s_m)} e^{-i(t_m-t_{m+1})}}{2} \\
&= \frac{1}{2^k} \sum_{s_1,s_2,\dots,s_k\in\{1,2\}} \sum_{l_1,l_2,\dots,l_k\in\{1,-1\}} \prod_{m=1}^{k} a_{s_m} i^{l_m(s_{m+1}-s_m)} e^{i l_m (t_m - t_{m+1})} \\
&= \frac{1}{2^k} \sum_{l_1,l_2,\dots,l_k\in\{1,-1\}} \sum_{s_1,s_2,\dots,s_k\in\{1,2\}} \prod_{m=1}^{k} a_{s_m} i^{s_m(l_{m-1}-l_m)} e^{i t_m (l_m - l_{m-1})} \\
&= \frac{1}{2^k} \sum_{l_1,l_2,\dots,l_k\in\{1,-1\}} \prod_{m=1}^{k} (a_1 i^{l_{m-1}-l_m} + a_2 i^{2(l_{m-1}-l_m)}) e^{i(l_m-l_{m-1})t_m},
\end{aligned}$$

where we set $s_0 = s_k$, $s_{k+1} = s_1$, $l_0 = l_k$, $l_{k+1} = l_1$, $t_0 = t_k$ and $t_{k+1} = t_1$. Here we notice that

(i) $i^{l_{m-1}-l_m} = -(-1)^{\delta_{l_m,l_{m-1}}}$,

(ii) $\#\{m \in \{1, 2, \dots, k\} \mid l_{m-1} \neq l_m\}$ is even (remark that $l_0 = l_k$),

(iii) if there exist $i_1, \dots, i_{2j} \in \{1, 2, \dots, k\}$ such that $i_1 < \cdots < i_{2j}$, $l_{i_r - 1} \neq l_{i_r}$ for each $r = 1, 2, \dots, 2j$ and $l_{m-1} = l_m$ for $m \in \{1, 2, \dots, k\} \setminus \{i_1, \dots, i_{2j}\}$, then $\sum_{m=1}^{k} (l_m - l_{m-1}) t_m = -2 l_{i_1} \sum_{r=1}^{2j} (-1)^r t_{i_r}$.

Thus, it follows that

$$\begin{aligned}
&\mathbf{B}(x_1, x_2, \dots, x_k) \\
&= \frac{1}{2^k} \sum_{l \in \{1,-1\}} \Big((\alpha^{-1} + \beta^{-1})^k \\
&\qquad + \sum_{0 < 2j \le k} (\beta^{-1} - \alpha^{-1})^{2j} (\beta^{-1} + \alpha^{-1})^{k-2j} \\
&\qquad\qquad \times \sum_{1 \le i_1 < \cdots < i_{2j} \le k} \cos\Big(2l \sum_{r=1}^{2j} (-1)^r t_{i_r} \Big) \Big) \\
&= 2 \Big(\frac{\alpha + \beta}{2\alpha\beta} \Big)^k \Big(1 + \sum_{0 < 2j \le k} \Big(\frac{\alpha - \beta}{\alpha + \beta} \Big)^{2j} \sum_{1 \le i_1 < \cdots < i_{2j} \le k} \cos\Big(\sum_{r=1}^{2j} (-1)^r \kappa x_{i_r}^2 \Big) \Big).
\end{aligned}$$

This is the desired conclusion. ■

For $\mathbf{u} = (u_1, u_2, \dots, u_k)$, we define the k by k matrix $\Delta_k(\mathbf{u})$ by

$$\begin{aligned}
\Delta_k(\mathbf{u}) &:= \begin{pmatrix}
\frac{1-u_k^4 u_1^4}{(1-u_k^4)(1-u_1^4)} & \frac{-u_1^2}{1-u_1^4} & 0 & 0 & \dots & \frac{-u_k^2}{1-u_k^4} \\
\frac{-u_1^2}{1-u_1^4} & \frac{1-u_1^4 u_2^4}{(1-u_1^4)(1-u_2^4)} & \frac{-u_2^2}{1-u_2^4} & 0 & \dots & 0 \\
0 & \frac{-u_2^2}{1-u_2^4} & \frac{1-u_2^4 u_3^4}{(1-u_2^4)(1-u_3^4)} & \frac{-u_3^2}{1-u_3^4} & \dots & 0 \\
0 & 0 & \frac{-u_3^2}{1-u_3^4} & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \frac{-u_{k-1}^2}{1-u_{k-1}^4} \\
\frac{-u_k^2}{1-u_k^4} & 0 & 0 & \dots & \frac{-u_{k-1}^2}{1-u_{k-1}^4} & \frac{1-u_{k-1}^4 u_k^4}{(1-u_{k-1}^4)(1-u_k^4)}
\end{pmatrix} \\
&= \sum_{i=1}^{k} \Big\{ (E_{ii}^{(k)} + E_{i+1,i+1}^{(k)}) \Big(\frac{1}{1-u_i^4} - \frac{1}{2} \Big) + (E_{i,i+1}^{(k)} + E_{i+1,i}^{(k)}) \frac{-u_i^2}{1-u_i^4} \Big\}.
\end{aligned}$$

It then follows that

$$\mathbf{E}(u_1, \dots, u_k; x_1, \dots, x_k) = \exp(-\mathbf{x} \Delta_k(\mathbf{u}) \mathbf{x}') \tag{2.2}$$

and

$$\det \Delta_k(\mathbf{u}) = \frac{(1 - u_1^2 \dots u_k^2)^2}{(1 - u_1^4) \dots (1 - u_k^4)} \tag{2.3}$$

(see [16, Theorem A.2]). Here $\mathbf{x} = (x_1, x_2, \dots, x_k)$, $E_{ij}^{(k)}$ denotes the matrix unit of size k. We also assume that the indices of $E_{ij}^{(k)}$ are understood modulo k, i.e.,

$E_{0,j}^{(k)} = E_{k,j}^{(k)}$, $E_{k+1,j}^{(k)} = E_{1,j}^{(k)}$, etc. The prime $'$ indicates the matrix transpose. Notice that $\Delta_k(\mathbf{u})$ is real symmetric and positive definite for any $\mathbf{u} \in (0,1)^k$. For $\{i_1, i_2, \ldots, i_{2j}\} \subset [k] = \{1, 2, \ldots, k\}$, we also put

$$\Xi_k(i_1, \ldots, i_{2j}) := \sqrt{-1} \sum_{r=1}^{2j} (-1)^r E_{i_r, i_r}^{(k)}.$$

Since

$$\sum_{r=1}^{2j} (-1)^r x_{i_r}^2 = \sqrt{-1}\, \mathbf{x}\, \Xi_k(i_1, \ldots, i_{2j}) \mathbf{x}'$$

and

$$\cos\Big(\kappa \sum_{r=1}^{2j} (-1)^r x_{i_r}^2\Big) = \frac{1}{2}\Big\{\exp\Big(\sqrt{-1}\kappa \sum_{r=1}^{2j} (-1)^r x_{i_r}^2\Big) + \exp\Big(-\sqrt{-1}\kappa \sum_{r=1}^{2j} (-1)^r x_{i_r}^2\Big)\Big\},$$

we have

$$\begin{aligned} &\mathbf{E}(u_1, \ldots, u_k; x_1, \ldots, x_k) \cos\Big(\kappa \sum_{r=1}^{2j} (-1)^r x_{i_r}^2\Big) \\ &\quad = \frac{1}{2} \exp(-\mathbf{x}(\Delta_k(\mathbf{u}) + \kappa\, \Xi_k(i_1, \ldots, i_{2j}))\mathbf{x}') \\ &\qquad + \frac{1}{2} \exp(-\mathbf{x}(\Delta_k(\mathbf{u}) - \kappa\, \Xi_k(i_1, \ldots, i_{2j}))\mathbf{x}'). \end{aligned} \tag{2.4}$$

As in [16, Lemma A.1], one proves the following result.

Lemma 2.2. *The determinant*

$$\det(\Delta_k(\mathbf{u}) + \kappa\, \Xi_k(i_1, \ldots, i_{2j})) \tag{2.5}$$

is even in κ. In particular, this determinant is real-valued for each $\mathbf{u} \in (0,1)^k$ and $\kappa > 0$.

Let $\mathcal{C}_m$ denote the cyclic subgroup of the symmetric group $\mathfrak{S}_m$ of degree m generated by the cyclic permutation $(1, 2, \ldots, m) \in \mathfrak{S}_m$. By Lemma 2.2, it follows that

$$\det(\Delta_k(\mathbf{u}) + \kappa\, \Xi_k(i_1, \ldots, i_{2j})) = \det(\Delta_k(\mathbf{u}) + \kappa\, \Xi_k(j_{\sigma(1)}, \ldots, j_{\sigma(2j)}))$$

for any $\sigma \in \mathcal{C}_{2k}$ since $\Xi_k(j_{\sigma(1)}, \ldots, j_{\sigma(2j)}) = \mathrm{sgn}(\sigma)\, \Xi_k(i_1, \ldots, i_{2j})$.

Let $\mathrm{Sym}_k^{\times}$ be the set of k by k *complex* symmetric matrices such that all principal minors are invertible, and $\mathrm{Sym}_k^{+}(\mathbb{R})$ be the set of k by k *positive real* symmetric matrices. Notice that $\Delta_k(\mathbf{u}) \in \mathrm{Sym}_k^{+}(\mathbb{R})$ for any $\mathbf{u} \in (0,1)^k$. We need the following two lemmas for later use in the evaluation of $\mathbf{F}(u_1, \ldots, u_k)$.

Lemma 2.3 (LDU decomposition). *Let k be a positive integer. For any $A \in \mathrm{Sym}_k^{\times}$, there exists a lower unitriangular matrix L and a diagonal matrix D such that $A = LDL'$. Moreover, D is given by*

$$D = \mathrm{diag}(d_1, d_2/d_1, d_3/d_2, \ldots, d_k/d_{k-1}),$$

where d_j denotes the j-th principal minor determinant of A.

Proof. Let us prove the lemma by induction on k. The assertion is clear if $k = 1$. Suppose that the assertion is true for $k-1$. Take $A \in \mathrm{Sym}_k^{\times}$ and write

$$A = \begin{pmatrix} A_0 & \mathbf{a} \\ \mathbf{a}' & \alpha \end{pmatrix}$$

with $A_0 \in \mathrm{Sym}_{k-1}^{\times}$, $\mathbf{a} \in \mathbb{C}^{k-1}$ and $\alpha \in \mathbb{C}$. By the induction hypothesis, there exist lower unitriangular matrix L_0 and diagonal matrix D_0 of size $k-1$ such that $A_0 = L_0 D_0 L_0'$. Put

$$L = \begin{pmatrix} L_0 & \mathbf{o} \\ \mathbf{v}' & 1 \end{pmatrix}, \quad D = \begin{pmatrix} D_0 & \mathbf{o} \\ \mathbf{o}' & d \end{pmatrix},$$

where $\mathbf{v} = (L_0 D_0)^{-1}\mathbf{a}$ and $d = \alpha - \mathbf{a}' A_0^{-1}\mathbf{a}$ (notice that $(L_0 D_0)^{-1}$ and A_0^{-1} exist by the induction hypothesis) and $\mathbf{o} \in \mathbb{C}^{k-1}$ represents the zero vector. Then it is straightforward to check that $A = LDL'$. This proves the first assertion of the lemma. The second assertion is obvious by the construction of D above. ∎

Lemma 2.4. *Let $T \in \mathrm{Sym}_k^{+}(\mathbb{R})$ and D be a real diagonal matrix of size k. Denote by d_m the principal m-minor determinant of $T + \sqrt{-1}D$. Then it follows that*

$$\Re(d_{m+1}\overline{d_m}) > 0 \quad \textit{for } m = 1, 2, \ldots, k-1.$$

Proof. Clearly, it is enough to prove the positivity of $\Re(d_{m+1}\overline{d_m})$ with $m = k-1$. Write T and D as

$$T = \begin{pmatrix} A & \mathbf{a} \\ \mathbf{a}' & \alpha \end{pmatrix}, \quad D = \begin{pmatrix} U & \mathbf{o} \\ \mathbf{o}' & u \end{pmatrix}$$

with $A \in \mathrm{Sym}_m^{+}(\mathbb{R})$, $\mathbf{a} \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$, $u \in \mathbb{R}$ and a real diagonal matrix U of size m. Here $\mathbf{o} \in \mathbb{R}^m$ is the zero vector. Since T is positive, we must have $0 < \mathbf{a}' A^{-1}\mathbf{a} < \alpha$. Put $B = \sqrt{A} \in \mathrm{Sym}_m^{+}(\mathbb{R})$, $X = B^{-1}UB^{-1} \in \mathrm{Sym}_m(\mathbb{R})$ and $\mathbf{b} = B^{-1}\mathbf{a}$. Then we have

$$\begin{aligned} d_{m+1}\overline{d_m} &= \det \begin{pmatrix} A + \sqrt{-1}U & \mathbf{a} \\ \mathbf{a}' & \alpha + \sqrt{-1}u \end{pmatrix} \begin{pmatrix} A - \sqrt{-1}U & \mathbf{o} \\ \mathbf{o} & 1 \end{pmatrix} \\ &= \det \begin{pmatrix} (A + \sqrt{-1}U)(A - \sqrt{-1}U) & \mathbf{a} \\ \mathbf{a}'(A - \sqrt{-1}U) & \alpha + \sqrt{-1}u \end{pmatrix} \end{aligned}$$

$$= |\det(A + \sqrt{-1}U)|^2(\alpha + \sqrt{-1}u - \mathbf{a}'(A + \sqrt{-1}U)^{-1}\mathbf{a})$$
$$= \det B^4 \det(I + X^2)(\alpha + \sqrt{-1}u - \mathbf{b}'(I + \sqrt{-1}X)^{-1}\mathbf{b}).$$

Since

$$(I + \sqrt{-1}X)^{-1} = (I + X^2)^{-1} - \sqrt{-1}X(I + X^2)^{-1},$$

it follows that

$$\Re(\mathbf{b}'(I + \sqrt{-1}X)^{-1}\mathbf{b}) = \mathbf{b}'(I + X^2)^{-1}\mathbf{b} \le \mathbf{b}'\mathbf{b} = \mathbf{a}'A^{-1}\mathbf{a} < \alpha$$

or

$$\Re(\alpha + \sqrt{-1}u - \mathbf{b}'(I + \sqrt{-1}X)^{-1}\mathbf{b}) > 0.$$

Thus, we have $\Re(d_{m+1}\overline{d_m}) > 0$ as desired. ■

We recall the well-known fact.

Lemma 2.5 (Gaussian integral). *For any $a, b \in \mathbb{C}$ with $\Re a > 0$, it follows that*

$$\int_{\mathbb{R}} \exp(-a(x-b)^2)dx = \sqrt{\frac{\pi}{a}}.$$

Here $\sqrt{a}$ is chosen as $\Re\sqrt{a} > 0$.

By Lemma 2.3, $A \in \mathrm{Sym}_k^\times$ is decomposed as $A = LDL'$ with a certain lower unitriangular matrix L and a diagonal matrix $D = \mathrm{diag}(d_1, d_2/d_1, \ldots, d_k/d_{k-1})$, where d_j is the j-th principal minor determinant of A. If all entries of D have positive real parts, then it follows from Lemma 2.5 that

$$\int_{\mathbb{R}^k} \exp(-\mathbf{x}A\mathbf{x}')d\mathbf{x} = \frac{\pi^{k/2}}{\sqrt{\det A}}. \tag{2.6}$$

Now, the matrix $\Delta_k(\mathbf{u}) + \kappa\Xi_k(\mathbf{i})$ belongs to $\mathrm{Sym}_k^\times$ for any $\mathbf{u} \in (0,1)^k$. Denote by $d_k = d_k(k, \mathbf{u}, \kappa, \mathbf{i})$ the k-th principal minor determinant of $\Delta_k(\mathbf{u}) + \kappa\Xi_k(\mathbf{i})$, and put $d_0 = 1$. It then follows from Lemma 2.4 that $\Re(d_j/d_{j-1}) > 0$ for $j = 1, 2, \ldots, k$. Consequently, in view of (2.4), (2.5), (2.6), and Lemma 2.1, we can calculate $\mathbf{F}(u_1, \ldots, u_k)$ as

$$\begin{aligned}
&\mathbf{F}(u_1, \ldots, u_k) \\
&\quad = 2\sqrt{\pi^k}\Big(\frac{\alpha+\beta}{2\alpha\beta}\Big)^k \Big\{\frac{1}{\sqrt{\det \Delta_k(\mathbf{u})}} \\
&\qquad\qquad + \sum_{0<2j\le k}\Big(\frac{\alpha-\beta}{\alpha+\beta}\Big)^{2j} \\
&\qquad\qquad \times \sum_{1\le i_1<i_2<\cdots<i_{2j}\le k} \frac{1}{\sqrt{\det(\Delta_k(\mathbf{u}) + \kappa\Xi_k(i_1, \ldots, i_{2j}))}}\Big\}.
\end{aligned}$$

We also notice that

$$\int_{[0,1]^k} \frac{2^k d\mathbf{u}}{1-u_1^2u_2^2\cdots u_k^2} = \zeta(k,1/2)$$

for $k \geq 2$. From these equations together with (2.1) and (2.3), we now obtain the

Theorem 2.6. *For each positive integer $k \geq 2$, it follows that*

$$\zeta_Q(k) = 2\Big(\frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\Big)^k\Big(\zeta(k,1/2) + \sum_{0<2j\leq k}\Big(\frac{\alpha-\beta}{\alpha+\beta}\Big)^{2j} R_{k,j}(\kappa)\Big). \tag{2.7}$$

Here $R_{k,j}(\kappa)$ is given by a sum of integrals

$$R_{k,j}(\kappa) = \sum_{1\leq i_1<i_2<\cdots<i_{2j}\leq k} \int_{[0,1]^k} \frac{2^k du_1\ldots du_k}{\sqrt{\mathcal{W}_k(\mathbf{u};\kappa;i_1,\ldots,i_{2j})}},$$

where the function $\mathcal{W}_k(\mathbf{u};\kappa;i_1,\ldots,i_{2j})$ is given by

$$\mathcal{W}_k(\mathbf{u};\kappa;i_1,\ldots,i_{2j}) = \det(\Delta_k(\mathbf{u}) + \kappa\,\Xi_k(i_1,\ldots,i_{2j}))\prod_{r=1}^{k}(1-u_r^4).$$

Remark 2.1. The algebraic variety $\mathcal{W}_k(\mathbf{u};\kappa;i_1,\ldots,i_{2j}) = 0$ defined by the denominator of the integral $R_{k,j}(\kappa)$ above is worth studying, e.g., from the viewpoint in [5–7].

Remark 2.2. If $\alpha = \beta$, then we have $\zeta_Q(k) = 2(\alpha^2-1)^{-k/2}\zeta(k,1/2)$, which is a special case of the fact that $\zeta_Q(s) = 2(\alpha^2-1)^{-s/2}\zeta(s,1/2)$ for $\alpha = \beta$. In fact, when α and β are equal, we can show that $Q \cong \sqrt{\alpha^2-1}(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2)I$ (see [33]).

We give an expansion of the determinants $\mathcal{W}_k(\mathbf{u};\kappa;i_1,\ldots,i_{2j})$ appearing in (2.7). For $\mathbf{j} = \{j_1, j_2, \ldots, j_r\} \subset [k]$ with $r > 0$ and $j_1 < j_2 < \cdots < j_r$, define

$$C_k(\mathbf{u};\mathbf{j}) = \prod_{i=1}^{r}\Big(1 - \prod_{j_i\leq s<j_{i+1}} u_s^4\Big).$$

We also define $C_k(\mathbf{u};\emptyset) = (1-u_1^2u_2^2\ldots u_k^2)^2$. Here we regard that $j_{r+1} = k + j_1$ and $u_{i+k} = u_i$. For instance, if $k = 9$ and $\mathbf{j} = \{3,6,8\}$, then

$$C_9(\mathbf{u};\mathbf{j}) = (1-u_3^4u_4^4u_5^4)(1-u_6^4u_7^4)(1-u_8^4u_9^4u_1^4u_2^4).$$

Lemma 2.7. *For a given subset $\mathbf{i} = \{i_1, i_2, \ldots, i_{2j}\} \subset [k]$ with $i_1 < i_2 < \cdots < i_{2j}$, it follows that*

$$\mathcal{W}_k(\mathbf{u};\kappa;\mathbf{i}) = \sum_{d\geq 0}(-\kappa^2)^d\,\mathcal{W}_{k,d}(\mathbf{u};\mathbf{i}) \tag{2.8}$$

with

$$\mathcal{W}_{k,d}(\mathbf{u};\mathbf{i}) := \sum_{\substack{S\subset[2j]\\ \#S=2d}} (-1)^{\|S\|} C_k(\mathbf{u};\mathbf{i}(S)). \tag{2.9}$$

Here $\|S\| := \sum_{s\in S} s$ *is the sum of the elements in* S *and* $\mathbf{i}(S) := \{i_{s_1},\dots,i_{s_l}\}$ *if* $S = \{s_1,\dots,s_l\}$ *with* $s_1 < \cdots < s_l$.

Proof. Let $\mathbf{d}_i$ be the i-th column vector of $\Delta_k(\mathbf{u})$. We also denote by $\{\mathbf{e}_i\}_{i=1}^k$ the standard basis of $\mathbb{C}^k$. By the multilinearity of a determinant, we readily get

$$\begin{aligned}&\det(\Delta_k(\mathbf{u}) + \kappa\Xi_k(\mathbf{i}))\\ &\quad = \det\Delta_k(\mathbf{u}) + \sum_{r=1}^{2j}(\sqrt{-1}\kappa)^r \sum_{1\le s_1<\cdots<s_r\le 2j} (-1)^{s_1+\cdots+s_r}\det(\mathbf{d}_1,\dots,\mathbf{e}_{i_{s_1}},\dots,\\ &\qquad\qquad \mathbf{e}_{i_{s_r}},\dots,\mathbf{d}_k).\end{aligned}$$

The determinant $\det(\mathbf{d}_1,\dots,\mathbf{e}_{i_{s_1}},\dots,\mathbf{e}_{i_{s_r}},\dots,\mathbf{d}_k)$ is a product of r tridiagonal determinants

$$D_p = \begin{vmatrix} d_{a(p)+1,a(p)+1} & d_{a(p)+1,a(p)+2} & \cdots & d_{a(p)+1,a(p+1)-1}\\ d_{a(p)+2,a(p)+1} & d_{a(p)+2,a(p)+2} & \cdots & d_{a(p)+2,a(p+1)-1}\\ \vdots & \vdots & \ddots & \vdots\\ d_{a(p+1)-1,a(p)+1} & d_{a(p+1)-1,a(p)+2} & \cdots & d_{a(p+1)-1,a(p+1)-1}\end{vmatrix},$$

where $a(p) = i_{s_p}$, d_{ij} is the (i,j)-entry of $\Delta_k(\mathbf{u})$, and the indices are understood modulo k. If $a(p+1) = a(p)+1$, then we understand that $D_p = 1$. It is easy to see that

$$D_p = \frac{1-u_{a(p)}^4 u_{a(p)+1}^4\cdots u_{a(p+1)-1}^4}{(1-u_{a(p)}^4)(1-u_{a(p)+1}^4)\cdots(1-u_{a(p+1)-1}^4)}.$$

Hence, we have

$$\mathcal{W}_k(\mathbf{u};\kappa;\mathbf{i}) = \sum_{S\subset[2j]}(-1)^{\|S\|}(\sqrt{-1}\kappa)^{\#S}C_k(\mathbf{u};\mathbf{i}(S)).$$

Since $\mathcal{W}_k(\mathbf{u};\kappa;\mathbf{i})$ is real-valued by Lemma 2.2, we have the conclusion by taking the real parts. ∎

2.3. Examples

2.3.1. $\mathcal{W}_{k,d}(\mathbf{u};\mathbf{i})$ and $R_{k,j}(\kappa)$. We give several examples of $\mathcal{W}_{k,d}(\mathbf{u};\mathbf{i})$. For convenience, we prepare some notation for abbreviation. Let us put

$$\mathbf{V}_k(\mathbf{u}) := (1-u_1^2\dots u_k^2)^2, \quad \mathbf{U}_{\mathbf{t}}(\mathbf{u}) := \prod_{i=1}^m\Big(1-\prod_{j=1}^{t_i} u_{j+\sum_{k<i}t_k}^4\Big)$$

for a positive integer k and a sequence $\mathbf{t} = (t_1, \dots, t_m) \in \mathcal{T}_m(k)$, where

$$\mathcal{T}_m(k) := \{(t_1, \dots, t_m) \in [k]^m \mid t_1 + \cdots + t_m = k\}.$$

For instance, if $\mathbf{t} = (2, 3, 2, 1) \in \mathcal{T}_4(8)$, then

$$\mathbf{U}_{\mathbf{t}}(u_1, \dots, u_8) = (1 - u_1^4 u_2^4)(1 - u_3^4 u_4^4 u_5^4)(1 - u_6^4 u_7^4)(1 - u_8^4).$$

Notice that $\mathcal{W}_{k,0}(\mathbf{u}; \mathbf{i}) = \mathbf{V}_k(\mathbf{u})$ for any $\mathbf{i}$.

Example 2.1. For $\mathbf{i} = \{i, j\} \subset [k]$ with $i < j$, we have

$$\begin{aligned}\mathcal{W}_{k,1}(\mathbf{u}; \mathbf{i}) &= (-1)^{1+2} C_k(\mathbf{u}; \mathbf{i}(1, 2)) \\ &= -\mathbf{U}_{(r,k-r)}(u_i, u_{i+1}, \dots, u_k, u_1, \dots, u_{i-1}),\end{aligned}$$

where $r = j - i$. This fact immediately implies that $R_{k,1}(\kappa)$ in (2.7) is given by

$$\begin{aligned}R_{k,1}(\kappa) &= \frac{k}{2} \sum_{r=1}^{k-1} \int_{[0,1]^k} \frac{2^k d\mathbf{u}}{\sqrt{\mathbf{V}_k(\mathbf{u}) + \kappa^2 \mathbf{U}_{(r,k-r)}(\mathbf{u})}} \\ &= \sum_{0<2r\le k} \frac{k}{1 + \delta_{2r,k}} \int_{[0,1]^k} \frac{2^k d\mathbf{u}}{\sqrt{\mathbf{V}_k(\mathbf{u}) + \kappa^2 \mathbf{U}_{(k-r,r)}(\mathbf{u})}}. \qquad (2.10)\end{aligned}$$

Example 2.2. For $\mathbf{i} \subset [k]$ with $\#\mathbf{i} = 2j$, it follows in general that

$$\mathcal{W}_{k,j}(\mathbf{u}; \mathbf{i}) = (-1)^j C_k(\mathbf{u}; \mathbf{i})$$

since $\|[2j]\| = j(2j - 1) \equiv j \pmod 2$.

Example 2.3. For $\mathbf{i} = \{i_1, i_2, i_3, i_4\} \subset [k]$ with $i_1 < i_2 < i_3 < i_4$, we have

$$\begin{aligned}\mathcal{W}_{k,1}(\mathbf{u}; \mathbf{i}) &= (-1)^{1+2} C_k(\mathbf{u}; i_1, i_2) + (-1)^{1+3} C_k(\mathbf{u}; i_1, i_3) + (-1)^{1+4} C_k(\mathbf{u}; i_1, i_4) \\ &\quad + (-1)^{2+3} C_k(\mathbf{u}; i_2, i_3) + (-1)^{2+4} C_k(\mathbf{u}; i_2, i_4) \\ &\quad + (-1)^{3+4} C_k(\mathbf{u}; i_3, i_4) \\ &= -(1 - u_{i_1}^4 \dots u_{i_2-1}^4)(1 - u_{i_3}^4 \dots u_{i_4-1}^4) \\ &\quad \times (1 - u_{i_2}^4 \dots u_{i_3-1}^4)(1 - u_{i_4}^4 \dots u_{i_1-1}^4) \\ &\quad - (1 - u_{i_1}^4 \dots u_{i_2-1}^4 u_{i_3}^4 \dots u_{i_4-1}^4)(1 - u_{i_2}^4 \dots u_{i_3-1}^4 u_{i_4}^4 \dots u_{i_1-1}^4).\end{aligned}$$

By Example 2.2, we also see that

$$\begin{aligned}\mathcal{W}_{k,2}(\mathbf{u}; \mathbf{i}) &= C_k(\mathbf{u}; \mathbf{i}) \\ &= (1 - u_{i_1}^4 \dots u_{i_2-1}^4)(1 - u_{i_2}^4 \dots u_{i_3-1}^4) \\ &\quad \times (1 - u_{i_3}^4 \dots u_{i_4-1}^4)(1 - u_{i_4}^4 \dots u_{i_1-1}^4).\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\det(\Delta_k(\mathbf{u}) + \kappa\,\Xi_k(i_1,i_2,i_3,i_4))\prod_{i=1}^{k}(1-u_i^4)\\
&\quad= \mathbf{V}_k(\mathbf{u}) + (\kappa^2+\kappa^4)(1-u_{i_1}^4\dots u_{i_2-1}^4)(1-u_{i_3}^4\dots u_{i_4-1}^4)\\
&\qquad\qquad\times(1-u_{i_2}^4\dots u_{i_3-1}^4)(1-u_{i_4}^4\dots u_{i_1-1}^4)\\
&\qquad+\kappa^2(1-u_{i_1}^4\dots u_{i_2-1}^4u_{i_3}^4\dots u_{i_4-1}^4)(1-u_{i_2}^4\dots u_{i_3-1}^4u_{i_4}^4\dots u_{i_1-1}^4).
\end{aligned}$$

If we take $(t_1,t_2,t_3,t_4)\in\mathcal{T}_4(k)$ such that $i_{p+1}\equiv i_p+t_p \pmod k$ $(p=1,2,3,4;$ $i_5=i_1)$, then it follows that

$$\begin{aligned}
&\int_{[0,1]^k}\frac{d\mathbf{u}}{\sqrt{\det(\Delta_k(\mathbf{u})+\kappa\,\Xi_k(i_1,i_2,i_3,i_4))\prod_{i=1}^{k}(1-u_i^4)}}\\
&\quad=\int_{[0,1]^k}\frac{d\mathbf{u}}{\sqrt{\mathbf{V}_k(\mathbf{u})+\kappa^2\mathbf{U}_{(t_1+t_3,t_2+t_4)}(\mathbf{u})+(\kappa^2+\kappa^4)\mathbf{U}_{(t_1,t_2,t_3,t_4)}(\mathbf{u})}}
\end{aligned}$$

by changing the variables of the integration as

$$(u_{i_1},\dots,u_{i_2-1},u_{i_3},\dots,u_{i_4-1},u_{i_2},\dots,u_{i_3-1},u_{i_4},\dots,u_{i_1-1})\mapsto(u_1,\dots,u_k).$$

The cyclic group $\mathcal{C}_4$ of order 4 naturally acts on $\mathcal{T}_4(k)$ by

$$\sigma.(t_1,t_2,t_3,t_4):=(t_{\sigma(1)},t_{\sigma(2)},t_{\sigma(3)},t_{\sigma(4)})\quad(\sigma\in\mathcal{C}_4).$$

Notice that the integral above is $\mathcal{C}_4$-invariant. For a given $\mathbf{t}=(t_1,t_2,t_3,t_4)\in\mathcal{T}_4(k)$, the number of subsets $\mathbf{i}=\{i_1,i_2,i_3,i_4\}$ in $[k]$ satisfying the condition $i_{p+1}\equiv i_p+t_p$ $\pmod k$ is equal to $k/\#C_4(\mathbf{t})$, where $\mathcal{C}_4(\mathbf{t})$ denotes the stabilizer of $\mathbf{t}$ in $\mathcal{C}_4$. Consequently,

$$R_{k,2}(\kappa)=\sum_{\mathbf{t}\in\mathcal{T}_4(k)/\mathcal{C}_4}\frac{k}{\#\mathcal{C}_4(\mathbf{t})}\int_{[0,1]^k}\frac{2^k d\mathbf{u}}{\mathfrak{S}}=\frac{k}{4}\sum_{\mathbf{t}\in\mathcal{T}_4(k)}\int_{[0,1]^k}\frac{2^k d\mathbf{u}}{\mathfrak{S}},\tag{2.11}$$

where

$$\mathfrak{S}:=\sqrt{\mathbf{V}_k(\mathbf{u})+\kappa^2\mathbf{U}_{(t_1+t_3,t_2+t_4)}(\mathbf{u})+(\kappa^2+\kappa^4)\mathbf{U}_{(t_1,t_2,t_3,t_4)}(\mathbf{u})}$$

and $\mathbf{t}=(t_1,t_2,t_3,t_4)$. Similarly, the result in Example 2.1 can be also rewritten as

$$\begin{aligned}
R_{k,1}(\kappa)&=\frac{k}{2}\sum_{\mathbf{t}\in\mathcal{T}_2(k)}\int_{[0,1]^k}\frac{2^k d\mathbf{u}}{\sqrt{\mathbf{V}_k(\mathbf{u})+\kappa^2\mathbf{U}_{(t_1,t_2)}(\mathbf{u})}}\\
&=\sum_{\mathbf{t}\in\mathcal{T}_2(k)/\mathcal{C}_2}\frac{k}{\#\mathcal{C}_2(\mathbf{t})}\int_{[0,1]^k}\frac{2^k d\mathbf{u}}{\sqrt{\mathbf{V}_k(\mathbf{u})+\kappa^2\mathbf{U}_{(t_1,t_2)}(\mathbf{u})}}.
\end{aligned}$$

2.3.2. Several special values. Using Theorem 2.6 and the formulas (2.10) and (2.11) for $R_{k,1}(\kappa)$ and $R_{k,2}(\kappa)$ given in the previous examples, we show several examples of the special values of $\zeta_Q(s)$.

Example 2.4. The values $\zeta_Q(2)$ and $\zeta_Q(3)$ are given by

$$\zeta_Q(2) = 2\Big(\frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\Big)^2\Big(\zeta\Big(2,\frac{1}{2}\Big) + \Big(\frac{\alpha-\beta}{\alpha+\beta}\Big)^2 R_{2,1}(\kappa)\Big),$$
$$\zeta_Q(3) = 2\Big(\frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\Big)^3\Big(\zeta\Big(3,\frac{1}{2}\Big) + \Big(\frac{\alpha-\beta}{\alpha+\beta}\Big)^2 R_{3,1}(\kappa)\Big)$$

with

$$\begin{aligned} R_{2,1}(\kappa) &= \int_{[0,1]^2} \frac{4du_1du_2}{\sqrt{\mathbf{V}_2(\mathbf{u}) + \kappa^2\mathbf{U}_{(1,1)}(\mathbf{u})}} \\ &= \int_{[0,1]^2} \frac{4du_1du_2}{\sqrt{(1-u_1^2u_2^2)^2 + \kappa^2(1-u_1^4)(1-u_2^4)}}, \\ R_{3,1}(\kappa) &= 3\int_{[0,1]^3} \frac{8du_1du_2du_3}{\sqrt{\mathbf{V}_3(\mathbf{u}) + \kappa^2\mathbf{U}_{(2,1)}(\mathbf{u})}} \\ &= 3\int_{[0,1]^3} \frac{8du_1du_2du_3}{\sqrt{(1-u_1^2u_2^2u_3^2)^2 + \kappa^2(1-u_1^4)(1-u_2^4u_3^4)}}. \end{aligned}$$

This recovers the result obtained in [16].

Example 2.5. The values $\zeta_Q(4)$ and $\zeta_Q(5)$ are given by

$$\zeta_Q(4) = 2\Big(\frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\Big)^4\Big(\zeta\Big(4,\frac{1}{2}\Big) + \Big(\frac{\alpha-\beta}{\alpha+\beta}\Big)^2 R_{4,1}(\kappa) + \Big(\frac{\alpha-\beta}{\alpha+\beta}\Big)^4 R_{4,2}(\kappa)\Big),$$
$$\zeta_Q(5) = 2\Big(\frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\Big)^5\Big(\zeta\Big(5,\frac{1}{2}\Big) + \Big(\frac{\alpha-\beta}{\alpha+\beta}\Big)^2 R_{5,1}(\kappa) + \Big(\frac{\alpha-\beta}{\alpha+\beta}\Big)^4 R_{5,2}(\kappa)\Big)$$

with

$$\begin{aligned} R_{4,1}(\kappa) &= 4\int_{[0,1]^4} \frac{16d\mathbf{u}}{\sqrt{\mathbf{V}_4(\mathbf{u}) + \kappa^2\mathbf{U}_{(3,1)}(\mathbf{u})}} + 2\int_{[0,1]^4} \frac{16d\mathbf{u}}{\sqrt{\mathbf{V}_4(\mathbf{u}) + \kappa^2\mathbf{U}_{(2,2)}(\mathbf{u})}}, \\ R_{5,1}(\kappa) &= 5\int_{[0,1]^5} \frac{32d\mathbf{u}}{\sqrt{\mathbf{V}_5(\mathbf{u}) + \kappa^2\mathbf{U}_{(4,1)}(\mathbf{u})}} + 5\int_{[0,1]^5} \frac{32d\mathbf{u}}{\sqrt{\mathbf{V}_5(\mathbf{u}) + \kappa^2\mathbf{U}_{(3,2)}(\mathbf{u})}} \end{aligned}$$

and

$$R_{4,2}(\kappa) = \int_{[0,1]^4} \frac{16d\mathbf{u}}{\sqrt{\mathbf{V}_4(\mathbf{u}) + \kappa^2\mathbf{U}_{(2,2)}(\mathbf{u}) + (\kappa^2+\kappa^4)\mathbf{U}_{(1,1,1,1)}(\mathbf{u})}},$$

$$R_{5,2}(\kappa) = 5\int_{[0,1]^5} \frac{32d\mathbf{u}}{\sqrt{\mathbf{V}_5(\mathbf{u}) + \kappa^2\mathbf{U}_{(3,2)}(\mathbf{u}) + (\kappa^2+\kappa^4)\mathbf{U}_{(2,1,1,1)}(\mathbf{u})}}.$$

2.3.3. Apéry-like numbers for $\zeta_Q(2)$ and the elliptic integral. We define the numbers $J_2(m)$ $(m \geq 0)$ by the expansion

$$R_{2,1}(\kappa) = \sum_{m=0}^{\infty} \binom{-1/2}{m} J_2(m)\kappa^{2m}.$$

Then they satisfy the three-term recurrence relation [16]

$$4m^2 J_2(m) - (8m^2 - 8m + 3)J_2(m-1) + 4(m-1)^2 J_2(m-2) = 0 \quad (m \geq 2). \tag{2.12}$$

This implies that the generating function $w_2(z) = \sum_{m=0}^{\infty} J_2(m)z^m$ satisfies

$$\left\{z(1-z)^2\frac{d^2}{dz^2} + (1-3z)(1-z)\frac{d}{dz} + z - \frac{3}{4}\right\}w_2(z) = 0. \tag{2.13}$$

This differential equation is the Picard–Fuchs equation for the universal family of elliptic curves equipped with rational 4-torsion. In fact, each elliptic curve in the family is birationally equivalent to one of the curves

$$(1-u^2v^2)^2 + x^2(1-u^4)(1-v^4) = 0$$

in the (u, v)-plane, which appears in the denominator of the integrand of $R_{2,1}(x)$.

The equation (2.13) can be reduced to the Gaussian hypergeometric differential equation by a suitable change of variable and solved as follows [32]:

$$w_2(z) = \frac{3\zeta(2)}{1-z}\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{z}{z-1}\right),$$

from which we obtain

$$R_{2,1}(\kappa) = 3\zeta(2)\,{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; -\kappa^2\right)^2.$$

Thus, we have the following formulas for $\zeta_Q(2)$, see [16, 32]:

$$\begin{aligned}\zeta_Q(2) &= \left(\frac{\pi(\alpha+\beta)}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\right)^2\left(1 + \frac{1}{2\pi\sqrt{-1}}\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2\int_{|z|=r}\frac{u(z)}{z(1+\kappa^2 z)^{1/2}}dz\right)\\ &= \left(\frac{\pi(\alpha+\beta)}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\right)^2\left(1 + \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; -\kappa^2\right)^2\right),\end{aligned}$$

where $u(z) = w_2(z)/3\zeta(2)$ is a normalized (unique) holomorphic solution of (2.13) in $|z| < 1$ and $\kappa^2 < r < 1$. We also have similar formulas for $\zeta_Q(3)$, see [16, 20].

3. Apéry-like numbers

In what follows, we restrict our attention on $R_{k,1}(\kappa)$ appearing in the special value formula for $\zeta_Q(s)$. We may sometimes refer to $R_{k,1}(\kappa)$ as the *first anomaly* in $\zeta_Q(k)$ for short. In this section, we define Apéry-like numbers $J_k(n)$, and study their recurrence equation and the differential equation satisfied by the generating function of $J_k(n)$. We finally discuss congruence properties for the *normalized* Apéry-like numbers $\widetilde{J}_k(n)$ (§3.4).

3.1. Apéry-like numbers associated to the first anomaly

We expand the first anomaly $R_{k,1}(\kappa)$ as follows:

$$\begin{aligned} R_{k,1}(\kappa) &= \frac{k}{2}\sum_{r=1}^{k-1}\int_{[0,1]^k}\frac{2^k\,du_1\cdots du_k}{\sqrt{(1-u_1^2\cdots u_k^2)^2+\kappa^2(1-u_1^4\cdots u_r^4)(1-u_{r+1}^4\cdots u_k^4)}} \\ &= \frac{k}{2}\sum_{r=1}^{k-1}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}J_{r,k-r}(n)\kappa^{2n} \\ &= \frac{k}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}J_k(n)\kappa^{2n}, \end{aligned}$$

where we put

$$\begin{aligned} J_k(n) &= \sum_{r=1}^{k-1}J_{r,k-r}(n), \\ J_{r,k-r}(n) &= 2^k\int_{[0,1]^k}\frac{(1-u_1^4\cdots u_r^4)^n(1-u_{r+1}^4\cdots u_k^4)^n}{(1-u_1^2\cdots u_k^2)^{2n+1}}du_1\cdots du_k. \end{aligned}$$

If we change the variables of the integral by

$$\begin{aligned} u_j &= e^{-\frac{1}{2}x_j} \quad (j=1,2,\ldots,r), \\ u_{r+j} &= e^{-\frac{1}{2}y_j} \quad (j=1,2,\ldots,k-r), \end{aligned}$$

then the corresponding domain of integration is

$$0 \le x_1 \le x_2 \le \cdots \le x_r, \quad 0 \le y_1 \le y_2 \le \cdots \le y_{k-r},$$

so that we have

$$\begin{aligned}
J_{r,k-r}(n) &= \int\limits_{\substack{0\le x_1\le x_2\le\cdots\le x_r\\ 0\le y_1\le y_2\le\cdots\le y_{k-r}}} \frac{(1-e^{-2x_r})^n(1-e^{-2y_{k-r}})^n}{(1-e^{-x_r-y_{k-r}})^{2n+1}} \\
&\qquad\qquad \times e^{-\frac{1}{2}(x_r+y_{k-r})}dx_1dx_2\cdots dx_r dy_1dy_2\cdots dy_{k-r} \\
&= \int\limits_0^\infty\int\limits_0^\infty \frac{x^{r-1}}{(r-1)!}\frac{y^{k-r-1}}{(k-r-1)!}\frac{e^{-\frac{1}{2}(x+y)}}{(1-e^{-x-y})^{2n+1}}(1-e^{-2x})^n \\
&\qquad\qquad \times (1-e^{-2y})^n dxdy \\
&= \int\limits_0^\infty \frac{e^{-\frac{1}{2}u}}{(1-e^{-u})^{2n+1}}du \int\limits_0^u \frac{t^{r-1}}{(r-1)!}\frac{(u-t)^{k-r-1}}{(k-r-1)!}(1-e^{-2t})^n \\
&\qquad\qquad \times (1-e^{-2(u-t)})^n dt.
\end{aligned}$$

By the binomial theorem, we have

$$\begin{aligned}
J_k(n) = \sum_{r=1}^{k-1} J_{r,k-r}(n) &= \int\limits_0^\infty \frac{u^{k-2}}{(k-2)!}\frac{e^{-\frac{1}{2}u}}{(1-e^{-u})^{2n+1}}du \\
&\qquad \times \int\limits_0^u (1-e^{-2t})^n(1-e^{-2(u-t)})^n dt \\
&= \frac{1}{2^{2n+1}}\int\limits_0^\infty \frac{u^{k-2}}{(k-2)!}\frac{e^{nu}}{(\sinh\frac{u}{2})^{2n+1}}du \\
&\qquad \times \int\limits_0^u (1-e^{-2t})^n(1-e^{-2(u-t)})^n dt. \qquad (3.1)
\end{aligned}$$

We call the numbers $J_k(n)$ the *Apéry-like numbers* associated to the first anomaly $R_{k,1}(\kappa)$ of $\zeta_Q(k)$, or *k-th Apéry-like numbers* for short.[4] By the equation (3.1) above, one has

$$J_k(n) = \frac{1}{2^{2n+1}}\int\limits_0^\infty \frac{u^{k-2}}{(k-2)!}B_n(u)du, \qquad (3.2)$$

$$B_n(u) = \frac{e^{nu}}{(\sinh\frac{u}{2})^{2n+1}}\int\limits_0^u (1-e^{-2t})^n(1-e^{-2(u-t)})^n dt \qquad (3.3)$$

for $k = 2, 3, 4, \ldots$ and $n = 0, 1, 2, \ldots$. We notice that the function $B_n(u)$ is continuous at $u = 0$ and is of exponential decay as $u \to +\infty$ (see [16, Proposition 4.10]). It is

[4][Differences of conventions] $J_2(n)$ in this article is equal to J_n in [16] (and $J_2(n)$ in [20]). $J_3(n)$ in this article is equal to $2J_n^1$ in [16] (and $2J_3(n)$ in [20]), since our $J_3(n)$ is defined to be the sum $J_{1,3-1}(n) + J_{2,3-2}(n)$, each summand in which is equal to J_n^1 in [16].

convenient to introduce the numbers $J_0(n)$ and $J_1(n)$ by

$$J_0(n) = 0, \quad J_1(n) = \frac{2^n n!}{(2n+1)!!} = \frac{(1)_n (1)_n}{(\frac{3}{2})_n (1)_n} \quad (n = 0, 1, 2, \ldots),$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol.

Example 3.1. We see that

$$\begin{aligned}
B_0(u) &= \frac{1}{\sinh \frac{u}{2}} \int_0^u dt = \frac{u}{\sinh \frac{u}{2}}, \\
B_1(u) &= \frac{e^u}{(\sinh \frac{u}{2})^3} \int_0^u (1 - e^{-2t})(1 - e^{-2(u-t)}) dt \\
&= 4\frac{u}{\sinh \frac{u}{2}} + 2\frac{u - \sinh u}{(\sinh \frac{u}{2})^3} \\
&= 4\frac{u}{\sinh \frac{u}{2}} + 2\frac{u}{(\sinh \frac{u}{2})^3} - 4\frac{\cosh \frac{u}{2}}{(\sinh \frac{u}{2})^2}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
J_k(0) &= \frac{1}{2 \cdot (k-2)!} \int_0^\infty \frac{u^{k-1}}{\sinh \frac{u}{2}} du, \\
J_k(1) &= \frac{1}{2 \cdot (k-2)!} \int_0^\infty \frac{u^{k-1}}{\sinh \frac{u}{2}} du + \frac{1}{4 \cdot (k-2)!} \int_0^\infty \frac{u^{k-2}(u - \sinh u)}{(\sinh \frac{u}{2})^3} du \\
&= \frac{1}{2 \cdot (k-2)!} \int_0^\infty \frac{u^{k-1}}{\sinh \frac{u}{2}} du + \frac{1}{4 \cdot (k-2)!} \int_0^\infty \frac{u^{k-1}}{(\sinh \frac{u}{2})^3} du \\
&\quad - \frac{1}{2 \cdot (k-2)!} \int_0^\infty \frac{\cosh \frac{u}{2}}{(\sinh \frac{u}{2})^2} u^{k-2} du.
\end{aligned}$$

Using the formulas

$$\zeta\Big(s, \frac{1}{2}\Big) = \frac{1}{2\Gamma(s)} \int_0^\infty \frac{u^{s-1}}{\sinh \frac{u}{2}} du = \frac{1}{4\Gamma(s+1)} \int_0^\infty \frac{\cosh \frac{u}{2}}{(\sinh \frac{u}{2})^2} u^s du \quad (\Re(s) > 1),$$

$$\int_0^\infty \frac{u^{s-1}}{(\sinh \frac{u}{2})^3} du = (s-1) \int_0^\infty \frac{\cosh \frac{u}{2}}{(\sinh \frac{u}{2})^2} u^{s-2} du - \frac{1}{2} \int_0^\infty \frac{u^{s-1}}{\sinh \frac{u}{2}} du \quad (\Re(s) > 3),$$

we get

$$J_k(0) = (k-1)\zeta\Big(k, \frac{1}{2}\Big),$$

$$J_k(1) = (k-3)\zeta\Big(k-2, \frac{1}{2}\Big) + \frac{3(k-1)}{4}\zeta\Big(k, \frac{1}{2}\Big)\Big(= J_{k-2}(0) + \frac{3}{4}J_k(0)\Big)$$

for $k \geq 4$. It is worth noting that these formulas are also valid for $k = 2$ and $k = 3$;

$$J_2(0) = \zeta\Big(2, \frac{1}{2}\Big), \quad J_2(1) = \frac{3}{4}\zeta\Big(2, \frac{1}{2}\Big);$$

$$J_3(0) = 2\zeta\Big(3, \frac{1}{2}\Big), \quad J_3(1) = 1 + \frac{3}{2}\zeta\Big(3, \frac{1}{2}\Big).$$

Here we use the fact that

$$\zeta\Big(0, \frac{1}{2}\Big) = 0, \quad \lim_{s\to 1}(s-1)\zeta\Big(s, \frac{1}{2}\Big) = 1.$$

We have now the following series expansion of $J_k(n)$.

Lemma 3.1 (Series expression). *We have*

$$J_k(n) = \sum_{r=1}^{k-1}\sum_{m=0}^{\infty}\binom{2n+m}{m}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{1}{(\frac{1}{2}+m+2j)^r}$$
$$\times \sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{1}{(\frac{1}{2}+m+2j)^{k-r}}.$$

Proof. It is elementary to see that

$$\int_0^\infty\int_0^\infty \frac{x^{r-1}}{(r-1)!}\frac{y^{k-r-1}}{(k-r-1)!}\frac{e^{-\frac{1}{2}(x+y)}(1-e^{-2x})^n(1-e^{-2y})^n}{(1-e^{-x-y})^{2n+1}}dxdy$$
$$= \sum_{m=0}^{\infty}(-1)^m\binom{-2n-1}{m}\int_0^\infty \frac{x^{r-1}}{(r-1)!}e^{-\frac{1}{2}x-mx}(1-e^{-2x})^n dx$$
$$\times \int_0^\infty \frac{y^{k-r-1}}{(k-r-1)!}e^{-\frac{1}{2}y-my}(1-e^{-2y})^n dy.$$

Since

$$\begin{aligned}
&\int_0^\infty \frac{z^{a-1}}{(a-1)!} e^{-\frac{1}{2}z-mz}(1-e^{-2z})^n dz \\
&= \frac{1}{(a-1)!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\int_0^\infty z^{a-1}e^{-(\frac{1}{2}+m+2j)z}dz \\
&= \sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{1}{(\frac{1}{2}+m+2j)^a},
\end{aligned}$$

the desired series expansion follows immediately. ■

3.2. Recurrence relations among Apéry-like numbers

Third Apéry-like numbers $J_3(n)$ satisfy the following inhomogeneous recurrence formula, which is obtained in [16]:

$$4n^2J_3(n) - (8n^2-8n+3)J_3(n-1) + 4(n-1)^2J_3(n-2) = 4J_1(n-1) \quad (3.4)$$

for each $n \geq 2$. One should remark that the homogeneous part of this recurrence formula is the same as the one for $J_2(n)$ given in (2.12).

We here show that the Apéry-like numbers $J_k(n)$ for $k \geq 4$ also satisfy a similar three-term recurrence formula. Put

$$T_{l,p}(n) = \frac{1}{2^{2n+1}}\int_0^\infty \frac{u^l}{l!}\Big(\tanh\frac{u}{2}\Big)^p B_n(u)du$$

for $l, p, n = 0, 1, 2, \ldots$. Notice that $J_k(n) = T_{k-2,0}(n)$ for $k \geq 2$. We also note that

$$T_{l,p}(0) = \frac{1}{2\cdot l!}\int_0^\infty \Big(\tanh\frac{u}{2}\Big)^p \frac{u^{l+1}}{\sinh\frac{u}{2}}du.$$

We need [16, formulas (4.36) and (4.37)]:

$$2\tanh\frac{u}{2}B_n'(u) = 8nB_{n-1}(u) - (2n+1)B_n(u), \quad (3.5)$$

$$\begin{aligned}
n\Big(\tanh\frac{u}{2}\Big)^2 B_n(u) &= 2(2n-1)B_{n-1}(u) + 2(2n-1)\Big(\tanh\frac{u}{2}\Big)^2 B_{n-1}(u) \\
&\quad - 16(n-1)B_{n-2}(u). \quad (3.6)
\end{aligned}$$

It follows from (3.5) that

$$(p+1)T_{l,p+2}(n) - 2T_{l-1,p+1}(n) = 2nT_{l,p}(n-1) - (2n-p)T_{l,p}(n).$$

Moreover, it follows from (3.6) that

$$2nT_{l,p+2}(n) = (2n-1)T_{l,p}(n-1) + (2n-1)T_{l,p+2}(n-1) - 2(n-1)T_{l,p}(n-2).$$

Combining these, we get

$$\begin{aligned}
&2n(2n-p)T_{l,p}(n)\\
&-(8n^2-4(p+2)n+3+2p)T_{l,p}(n-1)+2(n-1)(2n-2-p)T_{l,p}(n-2)\\
&\quad = 4nT_{l-1,p+1}(n) - 2(2n-1)T_{l-1,p+1}(n).
\end{aligned}$$

We see that

$$\begin{aligned}
&n\tanh\frac{u}{2}B_n(u) - 2(2n-1)\tanh\frac{u}{2}B_{n-1}(u)\\
&= -2\cdot\frac{8(n-1)B_{n-2}(u)-(2n-1)B_{n-1}(u)}{\tanh\frac{u}{2}}\\
&= -4B'_{n-1}(u)
\end{aligned}$$

by (3.6) and (3.5). This implies that

$$\begin{aligned}
&\frac{4n}{2^{2n+1}}\int_0^\infty \frac{u^l}{l!}\Big(\tanh\frac{u}{2}\Big)^{p+1}B_n(u)du - \frac{2(2n-1)}{2^{2n-1}}\int_0^\infty \frac{u^l}{l!}\Big(\tanh\frac{u}{2}\Big)^{p+1}B_{n-1}(u)du\\
&\quad = -\frac{4}{2^{2n-1}}\int_0^\infty \frac{u^l}{l!}\Big(\tanh\frac{u}{2}\Big)^{p}B'_{n-1}(u)du,
\end{aligned}$$

or

$$\begin{aligned}
&4nT_{l,p+1}(n) - 2(2n-1)T_{l,p+1}(n-1)\\
&\quad = 4T_{l-1,p}(n-1) + 2pT_{l,p-1}(n-1) + 2pT_{l,p+1}(n-1).
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
&2n(2n-p)T_{l,p}(n) - (8n^2-4(p+2)n+3+2p)T_{l,p}(n-1)\\
&+2(n-1)(2n-2-p)T_{l,p}(n-2)\\
&\quad = 4T_{l-2,p}(n-1) + 2pT_{l-1,p-1}(n-1) + 2pT_{l-1,p+1}(n-1). \qquad (3.7)
\end{aligned}$$

In particular, if we put $p=0$ and $l=k-2$ in the equation above and join (3.4), we obtain the following recurrence equation for $J_k(n)$ ($k\geq 2$), which was announced in [22].

Theorem 3.2. *For $k\geq 2$ and $n\geq 2$,*

$$4n^2J_k(n) - (8n^2-8n+3)J_k(n-1) + 4(n-1)^2J_k(n-2) = 4J_{k-2}(n-1). \quad (3.8)$$

Remark 3.1. The generalized Apéry-like numbers defined in [19] (which was named as $J_k(n)$ in [19]) is identical to $J_{1,k-1}(n)$ in this paper. It is quite interesting that those generalized Apéry-like numbers, i.e., $J_{1,k-1}(n)$, satisfy the following recurrence relation similarly to (3.8) (i.e., having the same homogeneous part of the Apéry-like numbers)

$$\begin{aligned}4n^2 J_{1,k-1}(n) - (8n^2 - 8n + 3)J_{1,k-1}(n-1) + 4(n-1)^2 J_{1,k-1}(n-2)\\ = 4J_{1,k-3}(n)\end{aligned} \tag{3.9}$$

for $k \geq 4$ and $n \geq 2$. From this observation, although $J_{1,k-1}(n)$ does not describe the special values $\zeta_Q(n)$, various $J_{r,k-r}(n)$ $(r = 1, 2, \ldots, k-1)$ are having similar nature as the Apéry-like numbers possess. This may suggest that there are certain unexpectically significant number theoretic properties behind NcHO that should be clarified.

3.3. Differential equations for the generating functions

For $k \geq 0$, we define

$$w_k(z) = \sum_{n=0}^{\infty} J_k(n) z^n, \tag{3.10}$$

$$g_k(x) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} J_k(n) x^n. \tag{3.11}$$

We call $w_k(z)$ the k-th generating function of the Apéry-like numbers. It is immediate to see that $w_0(z) = 0$, $g_0(x) = 0$ and

$$\begin{aligned} w_1(z) &= {}_2F_1(1, 1; \frac{3}{2}; z) = \frac{1}{\sqrt{1-z}} \frac{\arcsin\sqrt{z}}{\sqrt{z}}, \\ g_1(x) &= {}_2F_1(\frac{1}{2}, 1; \frac{3}{2}; -x) = \frac{\arctan\sqrt{z}}{\sqrt{z}}. \end{aligned}$$

For later use, we notice two differential equations for $w_1(z)$:

$$\left\{ z(1-z)\frac{d^2}{dz^2} + \frac{3}{2}(1-2z)\frac{d}{dz} - 1 \right\} w_1(z) = 0, \tag{3.12}$$

$$\left\{ 2z(1-z)\frac{d}{dz} + 1 - 2z \right\} w_1(z) = 1. \tag{3.13}$$

Let us translate the formula (3.8) into the differential equations for the generating functions $w_k(z)$. We have

$$\begin{aligned}\Big\{z(1-z)^2\frac{d^2}{dz^2}+(1-z)(1-3z)\frac{d}{dz}+z-\frac{3}{4}\Big\}\sum_{n=0}^{\infty}J_k(n)z^n\\ =J_k(1)-\frac{3}{4}J_k(0)+\frac{1}{4}\sum_{n=2}^{\infty}(4n^2J_k(n)\\ -(8n^2-8n+3)J_k(n-1)+4(n-1)^2J_k(n-2))z^{n-1}.\end{aligned}$$

Using (3.8) and $J_k(1)=J_{k-2}(0)+\frac{3}{4}J_k(0)$, we obtain

Theorem 3.3. *One has*

$$\Big\{z(1-z)^2\frac{d^2}{dz^2}+(1-z)(1-3z)\frac{d}{dz}+z-\frac{3}{4}\Big\}w_k(z)=w_{k-2}(z) \tag{3.14}$$

for $k\geq 2$.

Remark 3.2. We have

$$\left\{z(1-z)^2\frac{d^2}{dz^2}+(1-z)(1-3z)\frac{d}{dz}+z-\frac{3}{4}\right\}^k w_{2k}(z)=0$$

and

$$\begin{aligned}&\Big\{z(1-z)\frac{d^2}{dz^2}+\frac{3}{2}(1-2z)\frac{d}{dz}-1\Big\}\\ &\quad\times\Big\{z(1-z)^2\frac{d^2}{dz^2}+(1-z)(1-3z)\frac{d}{dz}+z-\frac{3}{4}\Big\}^k w_{2k+1}(z)=0\end{aligned}$$

for each $k\geq 0$. Namely, $w_k(z)$ is a power series solution of a linear differential equation, which is holomorphic at $z=0$.

To find an explicit formula for $J_k(n)$, it is useful to introduce the function

$$v_k(t)=(1-z)w_k(z),\quad t=\frac{z}{z-1}\quad\Big(\Longleftrightarrow\ w_k(z)=(1-t)v_k(t),\quad z=\frac{t}{t-1}\Big). \tag{3.15}$$

Note that

$$\begin{aligned}v_2(t)&=J_2(0)\cdot{}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;t\Big)=J_2(0)\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}^2t^n,\\ v_1(t)&=\frac{1}{1-t}{}_2F_1\Big(1,1;\frac{3}{2};\frac{t}{t-1}\Big)=\sum_{n=0}^{\infty}\frac{t^n}{2n+1}.\end{aligned}$$

The formula (3.14) is translated as

$$\left\{t(1-t)\frac{d^2}{dt^2}+(1-2t)\frac{d}{dt}-\frac{1}{4}\right\}v_k(t)=-v_{k-2}(t). \tag{3.16}$$

Let us look at the (hypergeometric differential) operator

$$D_t=t(1-t)\frac{d^2}{dt^2}+(1-2t)\frac{d}{dt}-\frac{1}{4}. \tag{D}$$

It is straightforward to check that

$$p_n(t)=-\frac{1}{(n+\frac{1}{2})^2}\binom{-\frac{1}{2}}{n}^{-2}\sum_{k=0}^{n}\binom{-\frac{1}{2}}{k}^2 t^k,$$

satisfies the equation $D_t\ p_n(t)=t^n$ by using the fact

$$D_t\,t^n=-\frac{(2n+1)^2}{4}t^n+n^2t^{n-1}$$

(see [20, Section 4]). Thus, if we put

$$\xi_l(t)=\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}^2 A_{l,n}t^n \quad (l\geq 0),$$

then

$$D_t\left\{-\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}^2 A_{l,n}\,p_n(t)\right\}=-\xi_l(t).$$

On the other hand, we see that

$$\begin{aligned}-\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}^2 A_{l,n}\,p_n(t)&=\sum_{n=0}^{\infty}A_{l,n}\frac{1}{(n+\frac{1}{2})^2}\sum_{k=0}^{n}\binom{-\frac{1}{2}}{k}^2 t^k\\&=\sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}^2\left\{\sum_{n=k}^{\infty}\frac{A_{l,n}}{(n+\frac{1}{2})^2}\right\}t^k.\end{aligned}$$

Hence, if we *assume* that the numbers $A_{l,k}$ satisfy the condition

$$A_{l+2,k}=\sum_{n=k}^{\infty}\frac{A_{l,n}}{(n+\frac{1}{2})^2}, \tag{3.17}$$

then the functions $\xi_l(t)$ satisfy the relation

$$D_t\,\xi_{l+2}(t)=-\xi_l(t) \quad (l\geq 0).$$

Notice that we have

$$A_{l+2m,n} = \sum_{n \le s_1 \le s_2 \le \cdots \le s_m} \frac{A_{l,s_m}}{(s_1+\frac{1}{2})^2(s_2+\frac{1}{2})^2\cdots(s_m+\frac{1}{2})^2} \tag{3.18}$$

under the assumption (3.17).

Now, we determine the numbers $A_{l,n}$ so that they satisfy (3.17). If we set

$$A_{l,n} = \begin{cases} \dfrac{1}{2}\dfrac{1}{n+\frac{1}{2}}\dbinom{-\frac{1}{2}}{n}^{-2} & l = 1, \\ J_2(0) & l = 2, \end{cases}$$

and extend by the relation (3.18), then the relation (3.17) is surely satisfied. We remark that the series (3.18) indeed converges since $A_{1,n}$ and $A_{2,n}$ are bounded so that the positive series $A_{l+2m,n}$ is dominated by a constant multiple of the series (*multiple zeta-star value*)

$$\zeta_m^\star(2,2,\ldots,2) = \sum_{0<k_1\le k_2\le\cdots\le k_m} (k_1k_2\ldots k_m)^{-2}.$$

Notice that

$$\xi_1(t) = \frac{1}{1-t}{}_2F_1\Big(1,1;\frac{3}{2};\frac{t}{t-1}\Big) = v_1(t),$$
$$\xi_2(t) = J_2(0)\cdot{}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;t\Big) = v_2(t).$$

From the discussion above we have the following (see [22] for the proof).

Proposition 3.4. *There exist constants C_j $(j = 1,2,\ldots)$ such that $v_l(t)$ is given by*

$$v_l(t) = \xi_l(t) + \sum_{0<j<l/2} C_{l-2j}v_{2j}(t). \tag{3.19}$$

Moreover, the coefficients $C_{l-2}, C_{l-4},\ldots$ are determined inductively.

From this proposition, we observe

$$w_l(z) = \frac{1}{1-z}\xi_l\Big(\frac{z}{z-1}\Big) + \sum_{0<j<l/2} C_{l-2j}w_{2j}(z), \tag{3.20}$$

and in particular

$$J_l(n) = \sum_{k=0}^{n}(-1)^k\binom{-\frac{1}{2}}{k}^2\binom{n}{k}A_{l,k} + \sum_{0<j<l/2} C_{l-2j}J_{2j}(n). \tag{3.21}$$

By this equation, we can determine C_{l-2j} by putting $n = 0$ inductively and obtain explicit formulas of $J_l(n)$ for each l. We give first a few examples.

Example 3.2. For $l = 2, 3, 4$, we have

$$\begin{aligned}
J_2(n) &= \zeta\Big(2, \frac{1}{2}\Big) \sum_{k=0}^{n} (-1)^k \binom{-\frac{1}{2}}{k}^2 \binom{n}{k}, \\
J_3(n) &= -\frac{1}{2} \sum_{k=0}^{n} (-1)^k \binom{-\frac{1}{2}}{k}^2 \binom{n}{k} \sum_{0 \le j < k} \frac{1}{(j+\frac{1}{2})^3} \binom{-\frac{1}{2}}{j}^{-2} \\
&\quad + 2\zeta\Big(3, \frac{1}{2}\Big) \sum_{k=0}^{n} (-1)^k \binom{-\frac{1}{2}}{k}^2 \binom{n}{k}, \\
J_4(n) &= -\zeta\Big(2, \frac{1}{2}\Big) \sum_{k=0}^{n} (-1)^k \binom{-\frac{1}{2}}{k}^2 \binom{n}{k} \sum_{0 \le j < k} \frac{1}{(j+\frac{1}{2})^2} \\
&\quad + 3\zeta\Big(4, \frac{1}{2}\Big) \sum_{k=0}^{n} (-1)^k \binom{-\frac{1}{2}}{k}^2 \binom{n}{k}.
\end{aligned}$$

3.4. Numerical data of normalized Apéry-like sequences

In this section, certain numerical data of the Apéry-like sequences is presented.

The *normalized Apéry-like numbers* $\widetilde{J}_k(n)$ are defined inductively by the conditions

$$\begin{aligned}
J_{2s}(n) &= \sum_{j=0}^{s-1} J_{2s-2j}(0) \widetilde{J}_{2j+2}(n), \\
J_{2s+1}(n) &= \sum_{j=0}^{s-1} J_{2s+1-2j}(0) \widetilde{J}_{2j+2}(n) + \widetilde{J}_{2s+1}(n).
\end{aligned}$$

It is equivalent to define $\widetilde{J}_k(n)$ by the recurrence relation

$$\widetilde{J}_k(n) = J_r(0)^{-1}\Big(J_k(n) - \sum_{0<2j<k} J_{k-2j}(0) \widetilde{J}_{2j+2}(n)\Big), \quad r = \begin{cases} 1 & k \equiv 1 \pmod 2, \\ 2 & k \equiv 0 \pmod 2. \end{cases}$$

The numbers $\widetilde{J}_k(n)$ satisfy the relation

$$4n^2 \widetilde{J}_k(n) - (8n^2 - 8n + 3)\widetilde{J}_k(n-1) + 4(n-1)^2 \widetilde{J}_k(n-2) = 4\widetilde{J}_{k-2}(n-1). \tag{3.22}$$

Notice that this is identical to the one for $J_k(n)$. It is elementary to check that

$$\widetilde{J}_1(n) = J_1(n) = \frac{2^n n!}{(2n+1)!!}, \quad \widetilde{J}_2(n) = \frac{J_2(n)}{J_2(0)} = \sum_{j=0}^{n} (-1)^j \binom{-\frac{1}{2}}{j}^2 \binom{n}{j}$$

and

$$\widetilde{J}_k(0) = \begin{cases} 1 & k = 1, 2, \\ 0 & \text{otherwise}, \end{cases} \qquad \widetilde{J}_k(1) = \begin{cases} \dfrac{2}{3} & k = 1, \\ \dfrac{3}{4} & k = 2, \\ 1 & k = 3, 4, \\ 0 & \text{otherwise}. \end{cases}$$

These are all *rational numbers*. Hence, by the recurrence relation (3.22) for $\widetilde{J}_k(n)$, all the normalized Apéry-like numbers $\widetilde{J}_k(n)$ are rational.

Let us put

$$Z_s^{\text{even}}(k) := (-1)^s \sum_{k > j_1 > \cdots > j_s \geq 0} \frac{1}{(j_1 + \frac{1}{2})^2 \dots (j_s + \frac{1}{2})^2},$$

$$Z_s^{\text{odd}}(k) := \frac{(-1)^s}{2} \sum_{k > j_1 > \cdots > j_s \geq 0} \frac{1}{(j_1 + \frac{1}{2})^2 \dots (j_{s-1} + \frac{1}{2})^2 (j_s + \frac{1}{2})^3} \binom{-\frac{1}{2}}{j_s}^{-2},$$

for $s = 1, 2, 3, \dots$. We also set $Z_0^{\text{even}}(k) = 1$ for convenience. Then we have the

Theorem 3.5. *For $s = 1, 2, 3, \dots$, we have*

$$\widetilde{J}_{2s+2}(n) = \sum_{k=0}^{n} (-1)^k \binom{-\frac{1}{2}}{k}^2 \binom{n}{k} Z_s^{\text{even}}(k),$$

$$\widetilde{J}_{2s+1}(n) = \sum_{k=0}^{n} (-1)^k \binom{-\frac{1}{2}}{k}^2 \binom{n}{k} Z_s^{\text{odd}}(k).$$

Proof. Define the numbers $\widetilde{A}_{l,n}$ by the relation (3.17) satisfied by $A_{l,n}$ together with the normalized initial condition

$$\widetilde{A}_{1,n} = A_{1,n} = \frac{1}{2} \frac{1}{n + \frac{1}{2}} \binom{-\frac{1}{2}}{n}^{-2}, \quad \widetilde{A}_{2,n} = 1.$$

We immediately have

$$Y_s^{\text{even}}(n) := \widetilde{A}_{2s+2,n} = \sum_{n \leq j_1 \leq \cdots \leq j_s} \frac{1}{(j_1 + \frac{1}{2})^2 \dots (j_s + \frac{1}{2})^2},$$

$$Y_s^{\text{odd}}(n) := \widetilde{A}_{2s+1,n} = \frac{1}{2} \sum_{n \leq j_1 \leq \cdots \leq j_s} \frac{1}{(j_1 + \frac{1}{2})^2 \dots (j_s + \frac{1}{2})^3} \binom{-\frac{1}{2}}{j_s}^{-2}.$$

By the same discussion as in the previous section, we see that there exist certain numbers C_j $(j = 1, 2, 3, \dots)$ such that

$$\widetilde{J}_l(n) = \sum_{k=0}^{n} (-1)^k \binom{-\frac{1}{2}}{k}^2 \binom{n}{k} \widetilde{A}_{l,k} + \sum_{0<j<l/2} C_{l-2j} \widetilde{J}_{2j}(n). \tag{3.23}$$

Put $n = 0$ in (3.23), we have $0 = \widetilde{A}_{l,0} + C_{l-2}$ if $l \geq 3$ since $\widetilde{J}_2(0) = 1$ and $\widetilde{J}_k(0) = 0$ if $k > 2$. Thus, we see that $\widetilde{J}_l(n)$ are of the form

$$\widetilde{J}_l(n) = \sum_{k=0}^{n} (-1)^k \binom{-\frac{1}{2}}{k}^2 \binom{n}{k} \widetilde{B}_{l,k}$$

with

$$\widetilde{B}_{l,k} = \widetilde{A}_{l,k} - \sum_{0<j<l/2} \widetilde{A}_{l-2j+2,0} \widetilde{B}_{2j,k}, \quad \widetilde{B}_{2,k} = \widetilde{A}_{2,k}, \quad \widetilde{B}_{3,k} = \widetilde{A}_{3,k}.$$

Therefore, it is enough to show that $Z_s^{\mathrm{even}}(k)$'s and $Z_s^{\mathrm{odd}}(k)$'s satisfy the relations

$$Z_s^{\mathrm{even}}(k) = Y_s^{\mathrm{even}}(k) - \sum_{j=0}^{s-1} Y_{s-j}^{\mathrm{even}}(0) Z_j^{\mathrm{even}}(k), \tag{3.24}$$

$$Z_s^{\mathrm{odd}}(k) = Y_s^{\mathrm{odd}}(k) - \sum_{j=0}^{s-1} Y_{s-j}^{\mathrm{odd}}(0) Z_j^{\mathrm{even}}(k). \tag{3.25}$$

Assume that $s \geq 2$, since these are directly proved when $s = 1$. We only prove (3.25) by induction on k (the proof of (3.24) is parallel). If $k = 0$, then the both sides of (3.25) is zero. Suppose that (3.25) is true for k. Notice that

$$Z_s^{\mathrm{even}}(k+1) = -\frac{1}{(k+\frac{1}{2})^2} Z_{s-1}^{\mathrm{even}}(k) + Z_s^{\mathrm{even}}(k),$$

$$Y_s^{\mathrm{even}}(k+1) = -\frac{1}{(k+\frac{1}{2})^2} Y_{s-1}^{\mathrm{even}}(k) + Y_s^{\mathrm{even}}(k),$$

$$Y_s^{\mathrm{odd}}(k+1) = -\frac{1}{(k+\frac{1}{2})^2} Y_{s-1}^{\mathrm{odd}}(k) + Y_s^{\mathrm{odd}}(k).$$

Using these relations together with the induction assumption, it is straightforward to verify that the both sides of (3.25) for $k + 1$ coincide. ∎

Remark 3.3. Note that

$$Z_s^{\mathrm{even}}(n) = Z_s^{\mathrm{odd}}(n) = 0 \quad (0 \leq n < s), \quad Z_s^{\mathrm{even}}(s) = Z_s^{\mathrm{odd}}(s) = \frac{1}{(s!)^2} \binom{-\frac{1}{2}}{s}^{-2},$$

and hence

$$\widetilde{J}_{2s+2}(n) = \widetilde{J}_{2s+1}(n) = 0 \quad (0 \le n < s), \quad \widetilde{J}_{2s+2}(s) = \widetilde{J}_{2s+1}(s) = \frac{1}{(s!)^2}.$$

We now provide several numerical data of $\widetilde{J}_k(n)$:

$$\widetilde{J}_1(n)\colon 1,\ \frac{2}{3},\ \frac{8}{15},\ \frac{16}{35},\ \frac{128}{315},\ \frac{256}{693},\ \frac{1024}{3003},\ \frac{2048}{6435},\ \frac{32768}{109395};$$

$$\widetilde{J}_1(n)\colon 1,\ \frac{3}{4},\ \frac{41}{64},\ \frac{147}{256},\ \frac{8649}{16384},\ \frac{32307}{65536},\ \frac{487889}{1048576},\ \frac{1856307}{4194304},\ \frac{454689481}{1073741824};$$

$$\widetilde{J}_3(n)\colon 0,\ 1,\ \frac{65}{48},\ \frac{13247}{8640},\ \frac{704707}{430080},\ \frac{660278641}{387072000},\ \frac{357852111131}{204374016000},\ \frac{309349386395887}{173581664256000};$$

$$\widetilde{J}_4(n)\colon 0,\ 1,\ \frac{11}{8},\ \frac{907}{576},\ \frac{1739}{1024},\ \frac{6567221}{3686400},\ \frac{54281321}{29491200},\ \frac{7260544493}{3853516800},\ \frac{709180003579}{369937612800};$$

$$\widetilde{J}_5(n)\colon 0,\ 0,\ \frac{1}{4},\ \frac{109}{216},\ \frac{101717}{138240},\ \frac{4557449}{4838400},\ \frac{15689290781}{13934592000},\ \frac{131932666373}{102187008000},$$
$$\frac{144010453389429161}{99983038611456000};$$

$$\widetilde{J}_6(n)\colon 0,\ 0,\ \frac{1}{4},\ \frac{73}{144},\ \frac{3419}{4608},\ \frac{29273}{30720},\ \frac{151587391}{132710400},\ \frac{232347221}{176947200},\ \frac{2444144299823}{1664719257600};$$

$$\widetilde{J}_7(n)\colon 0,\ 0,\ 0,\ \frac{1}{36},\ \frac{515}{6912},\ \frac{76667}{576000},\ \frac{115560397}{580608000},\ \frac{1051251017}{3901685760},\ \frac{18813135818903}{54935735500800};$$

$$\widetilde{J}_8(n)\colon 0,\ 0,\ 0,\ \frac{1}{36},\ \frac{43}{576},\ \frac{15389}{115200},\ \frac{1659311}{8294400},\ \frac{251914357}{928972800},\ \frac{10258433947}{29727129600}.$$

3.5. Congruence of normalized Apéry-like numbers

The congruence properties of $\widetilde{J}_k(2)$ (and $\widetilde{J}_k(3)$) obtained in [20] (see also [29]) are considered to be one of the consequences of the modular property that the generating function w_2 possesses (i.e., w_2 is an automorphic form for $\Gamma(2)(\cong \Gamma_0(4))$). As we will show in §5, there is a "weak modularity" for w_{2n} (i.e., w_{2n} is an automorphic integral for $\mathfrak{G}(2)$: see §5.1). Therefore, we may expect similar congruence properties among $\widetilde{J}_k(n)$ $(n \ge 4)$. In fact, we provide below a certain reasonable conjecture on congruence relations among $\widetilde{J}_k(n)$. The aim of this section is to show some weak and restricted version of the conjecture.

Based on a numerical experiment, we conjecture that the following congruence relations among the normalized Apéry-like numbers should hold.

Conjecture 3.6. *For positive integers m, n, s and a prime number p such that $mp \geq s$ and $mp^{n-1} \geq s$, we have*

$$\frac{p^{(2s+1)n}\,\widetilde{J}_{2s+1}(mp^n)}{p^{2s+1}\,\widetilde{J}_{2s+1}(mp)} \equiv \frac{p^{(2s+1)(n-1)}\,\widetilde{J}_{2s+1}(mp^{n-1})}{p^{2s+1}\,\widetilde{J}_{2s+1}(mp)} (\not\equiv 0) \pmod{p^n},$$

$$\frac{p^{2sn}\,\widetilde{J}_{2s+2}(mp^n)}{p^{2s}\,\widetilde{J}_{2s+2}(mp)} \equiv \frac{p^{2s(n-1)}\,\widetilde{J}_{2s+2}(mp^{n-1})}{p^{2s}\,\widetilde{J}_{2s+2}(mp)} (\not\equiv 0) \pmod{p^n}.$$

Remark 3.4. When $m < \frac{p}{2}$, the denominator of $p^{2s}\,\widetilde{J}_{2s+2}(mp)$ is indivisible by p, that is, $p^{2s}\,\widetilde{J}_{2s+2}(mp) \equiv p^t a \pmod{p^n}$ for some $t \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{Z}\backslash p\mathbb{Z}$. In this case, the second one in the conjecture above is equivalent to

$$p^{2sn-t}\,\widetilde{J}_{2s+2}(mp^n) \equiv p^{2s(n-1)-t}\,\widetilde{J}_{2s+2}(mp^{n-1}) \pmod{p^n}.$$

Here we prove slightly weaker results (Theorem 3.10). *In what follows in Section 3.6, p always denotes an odd prime.* We recall the following basic congruences on binomial coefficients (see [20, (6.7), (6.12), and (6.13)]).

Lemma 3.7. *For any positive integers m, n, j, the following congruence relations hold:*

$$\binom{-\frac{1}{2}}{pj}^2 \binom{mp^n}{pj} \equiv \binom{-\frac{1}{2}}{j}^2 \binom{mp^{n-1}}{j} \pmod{p^n},$$

$$p \nmid j \implies \binom{mp^n}{j} \equiv 0 \pmod{p^n}.$$

We also need the following elementary facts.

Lemma 3.8. *Let $\mathrm{ord}_p\, x$ be the exponent of p in $x \in \mathbb{Q}$, i.e., $x = \prod_p p^{\mathrm{ord}_p x}$ for $x \in \mathbb{Q}$. If $1 \leq 2j + 1 < p^{n+1}$, then*

$$\mathrm{ord}_p \binom{-\frac{1}{2}}{j} \leq n - \mathrm{ord}_p(2j+1).$$

Proof. Put $r = \mathrm{ord}_p(2j+1)$. Then there is some odd integer m such that $2j + 1 = mp^r < p^{n+1}$. In general, we see that

$$\mathrm{ord}_p \binom{-\frac{1}{2}}{j} = \mathrm{ord}_p \binom{2j}{j} = \sum_{l \geq 1} \left(\left\lfloor \frac{2j}{p^l} \right\rfloor - 2\left\lfloor \frac{2j}{p^l} \right\rfloor \right) = \#\left\{ l \geq 1 \;\middle|\; \left\{ \frac{j}{p^l} \right\} \geq \frac{1}{2} \right\},$$

where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of $x \in \mathbb{R}$. Notice that $\{x+1\} = \{x\}$. It follows then

$$1 \leq l \leq r \implies \left\{ \frac{j}{p^l} \right\} = \left\{ \frac{mp^r - 1}{2p^l} \right\} = \left\{ \frac{m}{2} p^{r-l} - \frac{1}{2p^l} \right\} = \left\{ \frac{1}{2} - \frac{1}{2p^l} \right\} < \frac{1}{2},$$

and

$$l \geq n+1 \implies 0 \leq \frac{j}{p^l} < \frac{p^{n+1-l}}{2} \leq \frac{1}{2} \implies \left\{\frac{j}{p^l}\right\} < \frac{1}{2}.$$

Thus, we have

$$\mathrm{ord}_p \binom{-\frac{1}{2}}{j} = \#\left\{l \geq 1 \;\middle|\; \left\{\frac{j}{p^l}\right\} \geq \frac{1}{2},\ r < l \leq n\right\} \leq n - r$$

as desired. ■

Lemma 3.9. *For* $k = 0, 1, \ldots, \frac{p^n-1}{2}$,

$$p^{2sn} Z_s^{\mathrm{even}}(kp) \equiv p^{2s(n-1)} Z_s^{\mathrm{even}}(k) \pmod{p^n} \tag{3.26}$$

holds.

Notice that the denominator of $p^{2sn} Z_s^{\mathrm{even}}(k)$ is not divisible by p if $2k-1 < p^{n+1}$.

Proof. First we notice that

$$\{2j+1 \mid 0 \leq j < kp\} \cap p\mathbb{Z} = \{p(2j+1) \mid 0 \leq j < k\}.$$

In the sum

$$p^{2sn} Z_s^{\mathrm{even}}(kp) = (-1)^s \sum_{kp > j_1 > \cdots > j_s \geq 0} \frac{p^{2sn}}{(j_1+\frac{1}{2})^2 \ldots (j_s+\frac{1}{2})^2},$$

the summand is $\equiv 0 \pmod{p^n}$ if any of $2j_1+1, \ldots, 2j_s+1$ is indivisible by p. Hence, we have

$$\begin{aligned}
&(-1)^s p^{2sn} Z_s^{\mathrm{even}}(kp) \\
&\quad = \sum_{k > j_1 > \cdots > j_s \geq 0} \frac{p^{2sn}}{(j_1+\frac{1}{2})^2 \ldots (j_s+\frac{1}{2})^2} \\
&\quad \equiv \sum_{k > j_1 > \cdots > j_s \geq 0} \frac{p^{2sn}}{(p(j_1+\frac{1}{2}))^2 \ldots (p(j_s+\frac{1}{2}))^2} \pmod{p^n} \\
&\quad = \sum_{k > j_1 > \cdots > j_s \geq 0} \frac{p^{2s(n-1)}}{(j_1+\frac{1}{2})^2 \ldots (j_s+\frac{1}{2})^2} \\
&\quad = (-1)^s p^{2s(n-1)} Z_s^{\mathrm{even}}(k),
\end{aligned}$$

which implies (3.26). ■

Theorem 3.10. *If* $1 \le m < \frac{p}{2}$, *then*

$$p^{2sn}\widetilde{J}_{2s+2}(mp^n) \equiv p^{2s(n-1)}\widetilde{J}_{2s+2}(mp^{n-1}) \pmod{p^n} \tag{3.27}$$

holds.

Proof. Using the lemma above, we have

$$\begin{aligned}
p^{2sn}\widetilde{J}_{2s+2}(mp^n) &= \sum_{k=0}^{mp^n}(-1)^k\binom{-\frac{1}{2}}{k}^2\binom{mp^n}{k}p^{2sn}Z_s^{\mathrm{even}}(k)\\
&\equiv \sum_{k=0}^{mp^{n-1}}(-1)^{kp}\binom{-\frac{1}{2}}{kp}^2\binom{mp^n}{kp}p^{2sn}Z_s^{\mathrm{even}}(kp) \pmod{p^n}\\
&\equiv \sum_{k=0}^{mp^{n-1}}(-1)^{k}\binom{-\frac{1}{2}}{k}^2\binom{mp^{n-1}}{k}p^{2s(n-1)}Z_s^{\mathrm{even}}(k) \pmod{p^n}\\
&= p^{2s(n-1)}\widetilde{J}_{2s+2}(mp^{n-1})
\end{aligned}$$

as desired. ■

Remark 3.5 (Odd case). We expect that the congruence formula in Theorem 3.10 also holds for the odd case. Explicitly, we conjecture that

$$p^{(2s+1)n}\widetilde{J}_{2s+1}(mp^n) \equiv p^{(2s+1)(n-1)}\widetilde{J}_{2s+1}(mp^{n-1}) \pmod{p^n} \tag{3.28}$$

holds for $1 \le m < \frac{p}{2}$. This is reduced to the congruence

$$p^{(2s+1)n}Z_s^{\mathrm{odd}}(kp) \equiv p^{(2s+1)(n-1)}Z_s^{\mathrm{odd}}(k) \pmod{p^n} \tag{3.29}$$

as in the even case. To prove this, we need the following fact, which we have not managed to prove. Let $j' = \frac{p(2j+1)-1}{2}$, i.e., $2j'+1 = p(2j+1)$. Then

$$\frac{p^{3(n-1)}}{(j+\frac{1}{2})^3}\left(\binom{-\frac{1}{2}}{j'}^{-2} - \binom{-\frac{1}{2}}{j}^{-2}\right) \equiv 0 \pmod{p^n} \tag{3.30}$$

when $1 \le 2j+1 < p^n$. We note that by an elementary discussion, this congruence is reduced to

$$\mathrm{ord}_p\left(\frac{1}{2j+1}\left(1 - \frac{\binom{-\frac{1}{2}}{j'}^2}{\binom{-\frac{1}{2}}{j}^2}\right)\right) \ge 1, \tag{3.31}$$

where $\mathrm{ord}_p\, x$ for $x \in \mathbb{Q}$ is the exponent of p in x, i.e., $x = \prod_p p^{\mathrm{ord}_p\, x}$. If $2j+1 = mp^r$ $(p \nmid m)$ and $s = \mathrm{ord}_p \binom{2j}{j}$, then (3.31) is equivalent to

$$\binom{2j'}{j'} \equiv (-1)^{\frac{p-1}{2}} \binom{2j}{j} \pmod{p^{s+r+1}}. \tag{3.32}$$

We note that the *modulo* p^{r+1} version of the congruence (3.32) can be proved easily as we sketch in the following. By a repeated use of the binomial theorem (see, e.g., [6]), we get

$$(1-X)^{mp^{r+1}-1} \equiv (1-X^p)^{mp^r-1} \sum_{j=0}^{p-1} X^j \pmod{p^{r+1}}.$$

Comparing the coefficients of X^{ap+b} $(0 \le b < p)$ in the both sides, we have

$$\binom{mp^{r+1}-1}{ap+b} \equiv (-1)^b \binom{mp^r-1}{a} \pmod{p^{r+1}}$$

in general. In particular, when $a = j$ and $b = \frac{p-1}{2}$, we obtain

$$\binom{2j'}{j'} \equiv (-1)^{\frac{p-1}{2}} \binom{2j}{j} \pmod{p^{r+1}}.$$

3.6. A remark on Euler's constant for the NcHO

We know that the spectral zeta function $\zeta_Q(s)$ can be meromorphically continued to the whole complex plane with unique pole at $s = 1$, see [17]. Actually, it has a simple pole at $s = 1$ with residue $\frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}}$. By this fact, it would be reasonable to define the Euler(-Mascheroni) constant γ_Q for the NcHO by

$$\gamma_Q := \lim_{s\to 1} \left\{ \zeta_Q(s) - \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}} \frac{1}{s-1} \right\}.$$

Since we can not expect neither an Euler product nor functional equation for $\zeta_Q(s)$, the analysis and results developed, e.g., in [12] for the Dedekind and Selberg zeta function is seemingly difficult. Nevertheless, we expect that γ_Q may possess certain arithmetic significance like Kronecker's limit formula [43], since it can be regarded as a regularized value of "$\zeta_Q(1)$". Exploring this problem would be desirable to obtain new information of the spectrum.

4. Apéry-like numbers and Mahler measures

In this section, we observe a certain mysterious relation between our Apéry-like numbers and the modular Mahler measures discussed in [39] through a generating function of the generating functions $v_k(t)$ of the Apéry-like numbers.

4.1. Meta-generating functions

We study a generating function of $v_k(t)$ (sometimes we refer to it as the meta-generating (grandmother) function of Apéry-like numbers) as

$$V^e(t,\lambda) := \sum_{k=0}^{\infty} v_{2k+2}(t)(-1)^k\lambda^{2k},$$
$$V^o(t,\lambda) := \sum_{k=0}^{\infty} v_{2k+1}(t)(-1)^k\lambda^{2k}.$$

For the time being, we will concentrate on the even meta-generating function $V^e(t,\lambda)$. Since $v_k(0) = w_k(0) = J_k(0) = (k-1)\zeta(k,1/2)$, we have

$$V^e(0,\lambda) = \sum_{k=0}^{\infty}(2k+1)\zeta\Big(2k+2,\frac{1}{2}\Big)(-1)^k\lambda^{2k} = \frac{\pi^2}{2\cosh^2\pi\lambda}.$$

Lemma 4.1. *For a sufficiently small $|\lambda|$, the function $V^e(t,\lambda)$ (resp. $V^o(t,\lambda)$) in the variable t is holomorphic around $t=0$.*

Proof. Recall the integral expression (3.1) of $J_k(n)$

$$J_k(n) = \frac{1}{2^{2n+1}}\int_0^{\infty}\frac{u^{k-2}}{(k-2)!}\frac{e^{nu}}{(\sinh\frac{u}{2})^{2n+1}}du\int_0^{u}(1-e^{-2t})^n(1-e^{-2(u-t)})^n dt.$$

Since

$$(1-e^{-2t})^n(1-e^{-2(u-t)})^n \le (1-e^{-u})^{2n}$$

one observes that

$$0 < J_k(n) = \frac{1}{(k-2)!}\int_0^{\infty}u^{k-2}e^{-\frac{u}{2}}\frac{u}{1-e^{-u}}du.$$

Since $\frac{u}{1-e^{-u}} < \frac{1}{1-e^{-1}} < 2$ for $0 < u < 1$ and $\frac{u}{1-e^{-u}} < 2u$ for $u > 1$, for any $\varepsilon > 0$, one sees that there exists a constant C_ε such that

$$\int_0^\infty u^{k-2}e^{-\frac{u}{2}}\frac{u}{1-e^{-u}}du < 2\int_0^1 u^{k-2}e^{-\frac{u}{2}}du + 2\int_1^\infty u^{k-1}e^{-\frac{u}{2}}du$$
$$< C_\varepsilon \int_0^\infty u^{k-2}e^{-\frac{u}{2+\varepsilon}}du = C_\varepsilon(2+\varepsilon)^{k-1}\Gamma(k-1).$$

It follows that $0 < J_k(n) < C_\varepsilon(2+\varepsilon)^{k-1}$ (independent of n). Since

$$w_k(z) = \sum_{n=0}^\infty J_k(n)z^n,$$

one has

$$|w_k(z)| \le C_\varepsilon(2+\varepsilon)^{k-1}(1-|z|)^{-1} \quad (|z| < 1).$$

Recall that $v_k(t) = (1-z)w_k(z)$ with $z = \frac{t}{t-1}$. Hence, one obtains

$$|v_k(t)| \le C_\varepsilon(2+\varepsilon)^{k-1}(|t-1|-|t|)^{-1} \quad \left(\Re(t) < \frac{1}{2}\right).$$

This immediately shows that if $|\lambda| < 1/\sqrt{2+\varepsilon}$

$$|V^e(t,\lambda)| \le \sum_{k=0}^\infty |v_{2k+2}(t)(-1)^k\lambda^{2k}| < C_\varepsilon(|t-1|-|t|)^{-1}(1-(2+\varepsilon)|\lambda|^2)^{-1}.$$

This shows the assertion for $V^e(t,\lambda)$. For the odd parity function $V^o(t,\lambda)$, the proof is the same. ■

It follows from the equation (3.16) that

$$D_t V^e(t,\lambda) = \lambda^2 V^e(t,\lambda), \tag{4.1}$$

where D_t is the differential operator defined in (D). Namely, the function $V^e(t,\lambda)$ is an eigenfunction of D_t with eigenvalue λ^2. Note that $v_2(t) = V^e(t,0)$ is a modular form for $\Gamma_0(4)$.

Remark 4.1. Similarly to the even case, we have

$$\begin{aligned} D_t V^o(t,\lambda) &= \sum_{k=0}^\infty (-1)^k D_t v_{2k+1}(t)\lambda^{2k} \\ &= \sum_{k=0}^\infty (-1)^{k-1} v_{2k-1}(t)\lambda^{2k} \\ &= -v_{-1}(t) + \lambda^2 V^o(t,\lambda), \end{aligned}$$

where

$$v_{-1}(t) = -D_t v_1(t) = -\frac{1}{4}\sum_{n=0}^{\infty}\Big\{4(n+1)^2\frac{1}{2n+3} - (2n+1)^2\frac{1}{2n+1}\Big\}t^n$$
$$= -\frac{1}{4}\sum_{n=0}^{\infty}\frac{t^n}{2n+3} = -\frac{1}{4}\frac{v_1(t)-1}{t}.$$

Namely, one has

$$(D_t - \lambda^2)V^o(t,\lambda) = \frac{1}{4t}\Big\{\frac{1}{1-t}{}_2F_1\Big(1,1;\frac{3}{2};\frac{t}{t-1}\Big) - 1\Big\}.$$

Rewrite (4.1) as

$$\Big(t(1-t)\frac{d^2}{dt^2} + (1-2t)\frac{d}{dt} - \frac{1}{4} - \lambda^2\Big)V^e(t,\lambda) = 0.$$

Since $V^e(t,\lambda)$ is holomorphic around $t = 0$ and $v_k(0) = (k-1)\zeta(k,1/2)$, one has

$$V^e(t,\lambda) = \frac{\pi^2}{2\cosh^2\pi\lambda}{}_2F_1\Big(\frac{1}{2}+i\lambda,\frac{1}{2}-i\lambda;1;t\Big). \tag{4.2}$$

From this, it is immediate to see that

$$v_{2k+2}(t) = (2k)!\frac{d^{2k}}{d\lambda^{2k}}\Big\{\frac{\pi^2}{2\cosh^2\pi\lambda}{}_2F_1\Big(\frac{1}{2}+i\lambda,\frac{1}{2}-i\lambda;1;t\Big)\Big\}\Big|_{\lambda=0}. \tag{4.3}$$

In relation with the modular form interpretation of the generating function of Apéry-like numbers for $\zeta_Q(2)$ developed in [21] (see §5) and [51], we naturally come to the following problems.

Problem 4.1. Determine whether there are any pair of $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$ such that the function $\frac{d^{2k}}{d\lambda^{2k}}\frac{\pi^2}{2\cosh^2\pi\lambda}{}_2F_1(\frac{1}{2}+i\lambda,\frac{1}{2}-i\lambda;1;t(\tau))$ in τ can be a modular form for some congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ and for some modular function $t = t(\tau)$ for Γ.

Problem 4.2. For some $t = t(\tau)$, is there any λ such that ${}_2F_1(\frac{1}{2}+i\lambda,\frac{1}{2}-i\lambda;1;t(\tau))$ is a modular form for some congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$? Moreover, how many such λ's; are these either finite or countably infinite, etc. locating on a certain line or algebraic curve? Notice that, if $\lambda \in \mathbb{Z}\backslash\{0\}$ then ${}_2F_1(\frac{1}{2}+i\lambda,\frac{1}{2}-i\lambda;1;t)$ is a polynomial so that the function in question is trivially a modular function.

Problem 4.3. Can we treat directly $V^e(t,\lambda)$ in the context of modular forms?

4.2. Integral expression of $V^e(t,\lambda)$

Concerning the question as is stated in Problem 4.3, we first provide the integral expression of $V^e(t,\lambda)$.

Proposition 4.2.

$$V^e(t,\lambda) = \frac{1}{2}\int_0^\infty\int_0^\infty \frac{2e^{-\frac{u+v}{2}}\cos(\lambda(u+v))(1-e^{-u-v})}{(1-e^{-u-v})^2 - t(e^{-v}-e^{-u})^2}\,du\,dv.$$

Proof. Recalling (3.2), (3.3), (3.10), and (3.15), we have

$$\begin{aligned}
V^e(t,\lambda) &= (1-z)\sum_{k=0}^\infty (i\lambda)^{2k}\sum_{n=0}^\infty \frac{z^n}{2^{2n+1}}\int_0^\infty \frac{u^{2k}}{(2k)!}\left\{\frac{e^{nu}}{(\sinh\frac{u}{2})^{2n+1}}\right.\\
&\qquad\qquad \left.\times\int_0^u (1-e^{-2s})^n(1-e^{-2(u-s)})^n\,ds\right\}du\\
&= (1-z)\int_0^\infty \cos(\lambda u)\left(\int_0^u\left\{\sum_{n=0}^\infty \frac{z^n e^{nu}}{(2\sinh\frac{u}{2})^{2n+1}}\right.\right.\\
&\qquad\qquad \left.\left.\times(1-e^{-2s})^n(1-e^{-2(u-s)})^n\right\}ds\right)du\\
&= (1-z)\int_0^\infty \cos(\lambda u)\left(\int_0^u \frac{1}{2\sinh\frac{u}{2}}\left\{1-\frac{ze^u}{4\sinh^2\frac{u}{2}}\right.\right.\\
&\qquad\qquad \left.\left.\times(1-e^{-2s})(1-e^{-2(u-s)})\right\}^{-1}ds\right)du\\
&= \frac{1}{2}\int_0^\infty\left(\int_0^u \frac{2e^{-\frac{u}{2}}\cos(\lambda u)(1-e^{-u})}{(1-e^{-u})^2 - t(e^{-s}-e^{s-u})^2}ds\right)du\\
&= \frac{1}{2}\int_0^\infty\int_0^\infty \frac{2e^{-\frac{u+v}{2}}\cos(\lambda(u+v))(1-e^{-u-v})}{(1-e^{-u-v})^2 - t(e^{-v}-e^{-u})^2}\,du\,dv
\end{aligned}$$

as desired. ∎

When $\lambda = \frac{1}{i}(\frac{1}{2}-\frac{1}{l})$ for some integer $l \geq 2$, assuming $t = T^2$, we have

$$V^e\left(T^2, \frac{1}{i}\left(\frac{1}{2}-\frac{1}{l}\right)\right) = \frac{l^2}{2}\int_0^1\int_0^1 \frac{1+(xy)^{l-2}}{1-(xy)^l - T(x^l-y^l)}dx\,dy. \tag{4.4}$$

On the other hand, it follows from (4.2) that

$$V^e\Big(T^2, \frac{1}{i}\Big(\frac{1}{2}-\frac{1}{l}\Big)\Big) = \frac{\pi^2}{2\sin^2\frac{\pi}{l}}\,{}_2F_1\Big(\frac{1}{l}, 1-\frac{1}{l}; 1; T^2\Big). \tag{4.5}$$

It follows then

$$\pi^2\,{}_2F_1\Big(\frac{1}{l}, 1-\frac{1}{l}; 1; T^2\Big) = l^2\sin^2\frac{\pi}{l}\int_0^1\int_0^1 \frac{1+(xy)^{l-2}}{1-(xy)^l - T(x^l-y^l)}\,dx\,dy. \tag{4.6}$$

Related to this function/integral, we now recall the result in [39], which is discussing the relation between Mahler measures and the special value of L-functions for elliptic curves from the modular form point of view. The (logarithmic) Mahler measure $m(P)$ of a Laurent polynomial $P \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ is defined as the following integral over the torus:

$$m(P) = \int_0^1 \cdots \int_0^1 \log|P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})|\, d\theta_1 \dots d\theta_n.$$

It is known that, for instance, there is a remarkable identity such as $m(1+x+y) = L'(\chi, -1)$, where $L(\chi, s)$ is the Dirichlet series associated to the quadratic character χ of conductor 3. Among others, the study in [39] shows the following result, which asserts very explicitly the relation between Mahler measures and the special value of L-functions for elliptic curves.

Proposition 4.3. *For $l = 2, 3, 4$ and 6, put*

$$u_l(\lambda) = \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2} \frac{1}{1-\lambda P_l(x,y)}\frac{dx}{x}\frac{dy}{y}, \tag{4.7}$$

where $\mathbb{T}^2 = \big\{(z,w) \in \mathbb{C}^2 \,\big|\, |z| = |w| = 1\big\}$ and

$$P_2(x,y) = x + \frac{1}{x} + y + \frac{1}{y}, \quad P_3(x,y) = \frac{x^2}{y} + \frac{y^2}{x} + \frac{1}{xy},$$
$$P_4(x,y) = xy^2 + \frac{x}{y^2} + \frac{1}{x}, \quad P_6(x,y) = \frac{x^2}{y} - \frac{y}{x} - \frac{1}{xy}.$$

Then one finds

$$u_l(\lambda) = {}_2F_1\Big(\frac{1}{l}, 1-\frac{1}{l}; 1; C_l\lambda^l\Big) \quad (l = 2, 3, 4, 6),$$
$$C_2 = 2^4, \quad C_3 = 3^3, \quad C_4 = 2^6, \quad C_6 = 2^4 3^3.$$

The function $u_l(\lambda)$ is related to the Mahler measure of the polynomial $P_l(x,y)-1/\lambda$ (which defines an elliptic curve) as

$$m(P_l(x,y)-1/\lambda) = \Re\left\{-\log\lambda - \int_0^\lambda (u_l(t)-1)\frac{dt}{t}\right\}.$$

It is worth noting that, via the hypergeometric representation (4.5), this proposition shows that the following relation between Mahler measures associated with curves and our meta-generation functions of Apéry-like numbers holds.

Corollary 4.4. *The following holds.*

$$V^e\left(C_l\lambda^l, \frac{1}{i}\left(\frac{1}{2}-\frac{1}{l}\right)\right) = \frac{\pi^2}{2\sin^2(\frac{\pi}{l})}u_l(\lambda) \quad (l=2,3,4,6). \tag{4.8}$$

Remark 4.2. Since there is an intimate relation between the Mahler measure for elements in group ring of a finite group and the characteristic polynomial of the adjacency matrix of a weighted Cayley graph and characters of the group [9], it is natural to expect the existence of a certain dynamical system behind the NcHO.

5. Automorphic integrals associated with Apéry-like numbers

The function $w_2(t)$ becomes a modular form of weight 1 with respect to the congruent subgroup $\Gamma(2)$ if we take t as a suitable $\Gamma(2)$-modular function. This is a reflection of the fact that the differential equation for $w_2(t)$ is the Picard–Fuchs equation for an associated family of elliptic curves. In this section, we recall this story for $w_2(t)$ and other generating functions $w_k(t)$ of Apéry-like numbers from [22] and study the Fourier expansions of certain integrals of modular forms, which appear naturally in the story.

5.1. Automorphic integrals

We summarize notations used in what follows, and we briefly recall the notion of automorphic integrals due to Knopp [24]. This is a slightly extended notion of automorphic integrals studied in [10].

Let Γ be a Fuchsian group and m be an integer. Let $\tau \in \mathfrak{h}$, $\mathfrak{h}$ being the complex upper half plane, and $q := e^{2\pi i\tau}$. Denote by $F(\mathfrak{h})$ the linear space of all $\mathbb{C}$-valued functions on the complex upper half plane. The group Γ acts on $\mathfrak{h}$ by $\gamma\tau := \frac{a\tau+b}{c\tau+d}$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $\tau \in \mathfrak{h}$. The space $F(\mathfrak{h})$ becomes a (right) Γ-module by the map $F(\mathfrak{h}) \times \Gamma \ni (f,\gamma) \mapsto f|_m\gamma \in F(\mathfrak{h})$ defined by

$$(f|_m\gamma)(\tau) = j(\gamma,\tau)^{-m} f(\gamma\tau). \tag{5.1}$$

Here $j(\gamma,\tau) := c\tau + d$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\tau \in \mathfrak{h}$. We denote by $\mathcal{H}(\mathfrak{h})$, $\mathcal{M}(\mathfrak{h})$, $\mathbb{C}(\tau)$ the subspaces of $F(\mathfrak{h})$ consisting of holomorphic functions on $\mathfrak{h}$, meromorphic functions on $\mathfrak{h}$ and rational functions on $\mathfrak{h}$ respectively. We also set $\mathbb{C}[\tau]_k$ to be the space of polynomial functions on $\mathfrak{h}$ which is of at most degree k. Notice that these spaces are Γ-submodules of $F(\mathfrak{h})$ under the action $f|_{-k}\gamma$.

The standard generators of the modular group $\mathrm{SL}_2(\mathbb{Z})$ are denoted by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Define the subgroup $\mathfrak{G}(2)$ of $\mathrm{SL}_2(\mathbb{Z})$ by $\mathfrak{G}(2) := \langle T^2, S \rangle$. Notice that $\mathfrak{G}(2)$ contains $\Gamma(2)$, the principal congruence subgroup of level 2;

$$\mathfrak{G}(2) \supset \Gamma(2) := \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv I \bmod 2\} = \langle T^2, ST^{-2}S^{-1} \rangle.$$

If $f(\tau)$ is an automorphic form of even integral weight $m+2$ for Γ, then an $(m+1)$-fold iterated integral $F(\tau)$ of $f(\tau)$ is called an *automorphic integral* of $f(\tau)$. By the Bol formula

$$\frac{d^{m+1}}{d\tau^{m+1}}(j(\gamma,\tau)^m F(\gamma\tau)) = j(\gamma,\tau)^{-m-2} F^{(m+1)}(\gamma\tau) \quad (\gamma \in \mathrm{SL}_2(\mathbb{R})), \tag{5.2}$$

we see that $(F|_{-m}\gamma)(\tau) - F(\tau)$ is a *polynomial* in τ of degree at most $m+1$.

In [24], Knopp introduced an extended notion of the automorphic integrals; a meromorphic function F on the upper half plane $\mathfrak{h}$ is called an *automorphic integral* of weight $2k$ for Γ with rational *period functions* $\{R_F(\gamma)(\tau) \in \mathbb{C}(\tau) \mid \gamma \in \Gamma\}$ if

$$(F|_{2k}\gamma)(\tau) = F(\tau) + R_F(\gamma)(\tau)$$

for each $\gamma \in \Gamma$ and F is meromorphic at each cusp of Γ.

Example 5.1. The Eisenstein series $E_2(\tau)$ of weight 2 satisfies

$$E_2(\tau+1) = E_2(\tau), \quad \tau^{-2}E_2\left(-\frac{1}{\tau}\right) - E_2(\tau) = \frac{12}{2\pi i\tau}.$$

Hence, $E_2(\tau)$ is an automorphic integral of weight 2 with for $\mathrm{SL}_2(\mathbb{Z})$.

Notice that an automorphic integral obtained by an $(m+1)$-fold iterated integral of the automorphic form of weight $m+2$ is an automorphic integral of weight $-m$ with *polynomial* period functions. To emphasize the polynomiality of the period functions, in what follows, we call an automorphic integral with polynomial period functions an automorphic integral.

5.2. Modular form interpretation of $w_2(t)$

We first recall the result on the modularity of $w_2(t)$ in [21] briefly. We recall the following standard functions; the elliptic theta functions

$$\theta_2(\tau) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2/2}, \quad \theta_3(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2/2}, \quad \theta_4(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2},$$

and normalized Eisenstein series

$$E_k(\tau) = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \quad (k = 2, 4, 6, \dots),$$

where $\sigma_s(n) = \sum_{d|n} d^s$. Put

$$t = t(\tau) = -\frac{\theta_2(\tau)^4}{\theta_4(\tau)^4} = \frac{\lambda(\tau)^2}{\lambda(\tau)^2 - 1} = \frac{\eta(\tau)^8 \eta(4\tau)^{16}}{\eta(2\tau)^{24}}, \tag{5.3}$$

which is a $\Gamma(2)$-modular function such that $t(i\infty) = 0$. Here $\lambda(\tau)$ is the Legendre modular function and $\eta(\tau)$ is the Dedekind eta function:

$$\lambda(\tau) = 16\eta(\tau)^8 \eta(2\tau)^{-24} \eta(4\tau)^{16}.$$

We see that

$$1 - t = \frac{\theta_3(\tau)^4}{\theta_4(\tau)^4}, \quad \frac{t}{t-1} = \frac{\theta_2(\tau)^4}{\theta_3(\tau)^4}, \quad \frac{q}{t}\frac{dt}{dq} = \frac{1}{2}\theta_3(\tau)^4.$$

By the formula ([48, Section 22.3]),

$${}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; \frac{\theta_2(\tau)^4}{\theta_3(\tau)^4}\Big) = \theta_3(\tau)^2,$$

it follows from (3.14) for $k = 2$ that

$$w_2(t) = \frac{J_2(0)}{1-t} {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; \frac{t}{t-1}\Big) = J_2(0)\frac{\theta_4(\tau)^4}{\theta_3(\tau)^2} = J_2(0)\frac{\eta(2\tau)^{22}}{\eta(\tau)^{12}\eta(4\tau)^8},$$

which is a $\Gamma(2)$-modular form of weight 1.

5.3. Toward modular interpretation of $w_k(t)$

The fact mentioned above on $w_2(t)$ naturally leads us to a question what the nature of $w_k(t)$ is in general. In order to answer this question for the special case $w_4(t)$, we recall the following general fact (Lemma 5.1), which is a slight modification of [49, Lemma 1] and is proved in the same manner. Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ commensurable with the modular group.

Lemma 5.1. *Let $A(\tau)$ be a modular form of weight k and $t(\tau)$ be a non-constant modular function on Γ such that $t(i\infty) = 0$. Let*

$$L := \vartheta^{k+1} + r_k(t)\vartheta^k + \cdots + r_0(t) \quad \left(\vartheta = t\frac{d}{dt}\right)$$

be the differential operator with rational coefficients $r_j(t)$. Assume that A, as a function in t, satisfies the differential equation $LA(t) = 0$. Let $g(t) = g(t(\tau))$ be another modular form. Then a solution of the inhomogeneous differential equation $LB(t) = g(t)$ is given by the iterated integration

$$B(t) = A(t)\underbrace{\int^q \cdots \int^q}_{k+1}\left(\frac{q\,dt/dq}{t}\right)^{k+1}\frac{g(t)}{A(t)}\frac{dq}{q}\cdots\frac{dq}{q}.$$

From Theorem 3.3, it follows that

$$\left(t(1-t)^2\frac{d^2}{dt^2} + (1-t)(1-3t)\frac{d}{dt} + t - \frac{3}{4}\right)^k w_{2k+2}(t) = w_2(t) \tag{5.4}$$

for $k \geq 1$, which can be also written in terms of the Euler operator ϑ as

$$L_k w_{2k+2}(t) = \frac{t^k}{(1-t)^{2k}} w_2(t) \quad (k \geq 1), \tag{5.5a}$$

$$L_k = \vartheta^{2k} + r_{2k-1}(t)\vartheta^{2k-1} + \cdots + r_0(t) \quad (r_0(t), \ldots, r_{2k-1}(t) \in \mathbb{C}(t)). \tag{5.5b}$$

Let us consider the function

$$\begin{aligned} W_k(t) &:= w_2(t)\underbrace{\int_0^q \cdots \int_0^q}_{2k}\left(\frac{q\,dt/dq}{t}\right)^{2k}\frac{t^k}{(1-t)^{2k}}\frac{dq}{q}\cdots\frac{dq}{q} \\ &= \left(-\frac{1}{4}\right)^k J_2(0)\frac{\theta_4(\tau)^4}{\theta_3(\tau)^2}\underbrace{\int_0^q \cdots \int_0^q}_{2k}\left(\theta_2(\tau)^4\theta_4(\tau)^4\right)^k\frac{dq}{q}\cdots\frac{dq}{q}. \end{aligned}$$

Let us look at the case where $k = 1$. If we apply Lemma 5.1 to (5.5), then we see that the integral $W_1(t)$ is a solution to (5.5), and hence $w_4(t) - W_1(t)$ is a solution of the homogeneous equation $L_1 f = 0$ of degree 2 which is holomorphic at $t = 0$. This implies that $w_4(t) - W_1(t)$ is a constant multiple of $w_2(t)$. Thus, we have $w_4(t) = Cw_2(t) + W_1(t)$ for a constant C, which is determined to be $\pi^2 (= J_4(0)/J_2(0))$ by looking at the constant terms. Namely, we get

$$w_4(t) = \pi^2 w_2(t) + W_1(t). \tag{5.6}$$

5.4. Automorphic integrals approach to $W_k(t)$

In what follows, we consider $W_k(t)$ for $k \in \mathbb{N}$ in general. For convenience, let us put

$$f(\tau) = \theta_2(\tau)^4\theta_4(\tau)^4 = \frac{1}{15}(E_4(\tau/2) - 17E_4(\tau) + 16E_4(2\tau)), \tag{5.7}$$

$$\mathbf{E}_k(\tau) = \underbrace{\int_0^q \cdots \int_0^q}_{2k} f(\tau)^k \frac{dq}{q} \cdots \frac{dq}{q}, \tag{5.8}$$

$$\mathbf{G}_k(\tau) = \underbrace{\int_0^q \cdots \int_0^q}_{4k-1} f(\tau)^k \frac{dq}{q} \cdots \frac{dq}{q} = \underbrace{\int_0^q \cdots \int_0^q}_{2k-1} \mathbf{E}_k(\tau) \frac{dq}{q} \cdots \frac{dq}{q}. \tag{5.9}$$

Notice that

$$W_k(t) = \left(-\frac{1}{4}\right)^k w_2(t)\mathbf{E}_k(\tau) = \frac{2\pi i}{(16\pi^2)^k} w_2(t) \frac{d^{2k-1}}{d\tau^{2k-1}} \mathbf{G}_k(\tau).$$

Clearly, $\mathbf{G}_k(\tau)$ is a periodic function with period 2 and $\mathbf{G}_k(i\infty) = 0$. Since $f(\tau)^k$ is a modular form of weight $4k$ with respect to $\Gamma(2)$ (or $\mathfrak{G}(2)$), the function $\mathbf{G}_k(\tau)$ is an automorphic integral for $f(\tau)^k$ by definition. Hence, by (5.6), we have the

Theorem 5.2. *The fourth generating function $w_4(t)$ of Apéry-like numbers is a linear combination of $w_2(t)$ and*

$$W_1(t) = \frac{2\pi i}{16\pi^2} w_2(t)\mathbf{G}_1'(\tau),$$

which is a product of $w_2(t)$ and the derivative $\mathbf{G}_1'(\tau)$ of an automorphic integral for $\mathfrak{G}(2)$ of weight -2, as

$$w_4(t) = \pi^2 w_2(t) + W_1(t). \tag{5.10}$$

Note that the Fourier expansion of $\mathbf{G}_1(\tau)$ is given by

$$\begin{aligned}\mathbf{G}_1(\tau) &= \int_0^q \int_0^q \int_0^q \frac{1}{15}(E_4(\tau/2) - 17E_4(\tau) + 16E_4(2\tau)) \frac{dq}{q}\frac{dq}{q}\frac{dq}{q} \\ &= 16\Big(8\sum_{n\geq 1}\sigma_{-3}(n)q^{\frac{n}{2}} - 17\sum_{n\geq 1}\sigma_{-3}(n)q^n + 2\sum_{n\geq 1}\sigma_{-3}(n)q^{2n}\Big).\end{aligned} \tag{5.11}$$

In the next section we will give a formula for $\mathbf{G}_1(\tau)$ and $w_4(t)$, in which they are expressed in terms of *differential Eisenstein series* ((6.8) and Theorem 6.6).

We calculate the period function of $\mathbf{G}_k(\tau)$, especially to describe $w_4(t)$ concretely via $\mathbf{G}_1(\tau)$. The L-function corresponding to $f(\tau)^k$ is

$$\Lambda_k(s) = \int_0^\infty t^s f(it)^k \frac{dt}{t}, \tag{5.12}$$

which satisfies the functional equation $\Lambda_k(4k-s) = \Lambda_k(s)$. By the inversion formula of Mellin's transform, one notices that

$$f(iy)^k = \frac{1}{2\pi i} \int_{\Re s=\alpha} y^{-s}\Lambda_k(s)ds \quad (y > 0, \alpha \gg 0).$$

Put

$$\Xi_k(s) = \frac{\Lambda_k(s+2k)}{\prod_{j=1-2k}^{2k-1}(s-j)}, \quad \rho_{k,j} = \operatorname*{Res}_{s=j} \Xi_k(s) \quad (j = 1-2k, \ldots, 2k-1).$$

The functional equation for $\Lambda_k(s)$ implies the oddness $\Xi_k(-s) = -\Xi_k(s)$, from which we see that $\rho_{k,-j} = \rho_{k,j}$. Define $R_S^k(\tau)$ by

$$R_S^k(\tau) = -(2\pi)^{4k-1} \sum_{j=1-2k}^{2k-1} \rho_{k,j} \left(\frac{\tau}{i}\right)^{2k-1-j}.$$

Notice that $R_S^k(\tau)$ is a polynomial in τ of degree $4k-2$. We have the

Lemma 5.3 ([22, Theorem 4]). *One has*

$$\mathbf{G}_k(\tau+2) = \mathbf{G}_k(\tau), \quad \tau^{4k-2}\mathbf{G}_k\left(-\frac{1}{\tau}\right) - \mathbf{G}_k(\tau) = R_S^k(\tau).$$

Let us consider the particular case where $k = 1$. Explicitly, we have

$$\Lambda_1(s) = 16\pi^{-s}\Gamma(s)\zeta(s)\zeta(s-3)(1-2^{-s})(1-2^{4-s}),$$
$$\rho_{1,-1} = \rho_{1,1} = \frac{7\zeta(3)}{\pi^3}, \quad \rho_{1,0} = -\frac{1}{2}, \quad R_S^1(\tau) = 56\zeta(3)(\tau^2-1) + \frac{4\pi^3}{i}\tau.$$

Lemma 5.3 then reads

Lemma 5.4. *The function*

$$\widetilde{\mathbf{G}}_1(\tau) := \mathbf{G}_1(\tau) - 56\zeta(3) \tag{5.13}$$

satisfies

$$\widetilde{\mathbf{G}}_1(\tau+2) = \widetilde{\mathbf{G}}_1(\tau), \quad \tau^2\widetilde{\mathbf{G}}_1\left(-\frac{1}{\tau}\right) - \widetilde{\mathbf{G}}_1(\tau) = \frac{4\pi^3}{i}\tau. \tag{5.14}$$

5.5. Experimental calculation to determine the coefficients

In this section, we observe that the generating function $w_{2k}(t)$ of the Apéry numbers for $\zeta_Q(2k)$ is expressed by a certain linear combination of the multiple integral of the (same) modular forms. Namely, we try to determine the coefficients $c'_{k,j}$ in the equation

$$w_{2k+2}(t) = \frac{J_{2k+2}(0)}{J_2(0)} w_2(t) + \sum_{j=1}^{k-1} c'_{k,j} W_j(t) + W_k(t) \tag{5.15}$$

$$= w_2(t)\Big\{\frac{J_{2k+2}(0)}{J_2(0)} + \sum_{j=1}^{k}\Big(-\frac{1}{4}\Big)^j c'_{k,j}\mathbf{E}_j(\tau)\Big\} \quad (c'_{k,k} = 1). \tag{5.16}$$

(Recall that $W_j(t) = (-\frac{1}{4})^j w_2(t)\mathbf{E}_j(\tau)$.) Let $\alpha_l^{(m)}$, $\beta_l^{(m)}$ and $\gamma_l^{(m)}$ be the l-th Fourier coefficients of $w_m(t)$, $\mathbf{E}_m(\tau)$ and t^m respectively, that is,

$$w_m(t) = \sum_{l=0}^{\infty} \alpha_l^{(m)} q^{\frac{l}{2}}, \quad \mathbf{E}_m(\tau) = \sum_{l=0}^{\infty} \beta_l^{(m)} q^{\frac{l}{2}}, \quad t^m = \sum_{l=0}^{\infty} \gamma_l^{(m)} q^{\frac{l}{2}}.$$

Trivially we have $\gamma_l^{(0)} = \delta_{l,0}$. We also note that $\gamma_l^{(m)} = \beta_l^{(m)} = 0$ if $l < m$ since t and $f(\tau)$ vanish at $i\infty$. Thus, we have

$$w_m(t) = \sum_{n=0}^{\infty} J_m(n) t^n = \sum_{n=0}^{\infty} J_m(n) \sum_{l=0}^{\infty} \gamma_l^{(n)} q^{\frac{l}{2}} = \sum_{l=0}^{\infty}\Big(\sum_{n=0}^{l} J_m(n)\gamma_l^{(n)}\Big) q^{\frac{l}{2}},$$

or

$$\alpha_l^{(m)} = \sum_{n=0}^{l} J_m(n)\gamma_l^{(n)}.$$

Recall that $W_j(t) = (-\frac{1}{4})^j w_2(t)\mathbf{E}_j(\tau)$. Now, (5.15) reads

$$\sum_{l=0}^{\infty} \alpha_l^{(2k+2)} q^{\frac{l}{2}} = \frac{J_{2k+2}(0)}{J_2(0)} \sum_{l=0}^{\infty} \alpha_l^{(2)} q^{\frac{l}{2}} + \sum_{j=1}^{k}\Big(-\frac{1}{4}\Big)^j c'_{k,j} \sum_{l,m\geq 0} \alpha_l^{(2)} \beta_m^{(j)} q^{\frac{l+m}{2}}$$

Comparing the coefficients of $q^{l/2}$ of the both sides, we have

$$\sum_{j=1}^{k}\Big(-\frac{1}{4}\Big)^j \Big\{\sum_{m=0}^{l} \alpha_{l-m}^{(2)} \beta_m^{(j)}\Big\} c'_{k,j} = \alpha_l^{(2k+2)} - \frac{J_{2k+2}(0)}{J_2(0)} \alpha_l^{(2)}$$

for $l = 1, \ldots, k$.

Example 5.2 ($l = 1$). We have

$$-\frac{1}{4}J_2(0)\gamma_0^{(0)}\beta_1^{(1)}c'_{k,1} = \frac{J_{2k+2}(1)J_2(0) - J_{2k+2}(0)J_2(1)}{J_2(0)}\gamma_1^{(1)} = J_{2k}(0)\gamma_1^{(1)}$$

since

$$\begin{aligned} &J_{2k+2}(1)J_2(0) - J_{2k+2}(0)J_2(1) \\ &\quad = \Big(J_{2k}(0) + \frac{3}{4}J_{2k+2}(0)\Big)J_2(0) - \frac{3}{4}J_2(0)J_{2k+2}(0) = J_{2k}(0)J_2(0). \end{aligned}$$

We see that

$$\gamma_1^{(1)} = -16, \quad \beta_1^{(1)} = 64$$

Thus, we have

$$c'_{k,1} = \frac{J_{2k}(0)}{J_2(0)}.$$

For instance, we have

$$\begin{aligned} w_6(t) &= w_2(t)\Big\{\frac{J_6(0)}{J_2(0)} - \frac{1}{4}\frac{J_4(0)}{J_2(0)}\mathbf{E}_1(\tau) + \frac{1}{16}\mathbf{E}_2(\tau)\Big\} \\ &= w_2(t)\Big\{\frac{J_6(0)}{J_2(0)} + \frac{J_4(0)}{J_2(0)}\frac{2\pi i}{16\pi^2}\frac{d}{d\tau}\mathbf{G}_1(\tau) + \frac{2\pi i}{256\pi^4}\frac{d^3}{d\tau^3}\mathbf{G}_2(\tau)\Big\}. \end{aligned}$$

Example 5.3 ($l = 2$). We see that

$$\gamma_2^{(1)} = -128, \quad \gamma_2^{(2)} = 256, \quad \beta_2^{(1)} = -128, \quad \beta_2^{(2)} = 256.$$

For $k \geq 2$, we have

$$\begin{aligned} &-\frac{1}{4}\Big\{J_2(0)\beta_2^{(1)} + J_2(0)\gamma_1^{(0)}\beta_1^{(1)} + J_2(1)\gamma_1^{(1)}\beta_1^{(1)}\Big\}c'_{k,1} + \frac{1}{16}J_2(0)\beta_2^{(2)}c'_{k,2} \\ &\quad = \frac{J_{2k+2}(1)J_2(0) - J_{2k+2}(0)J_2(1)}{J_2(0)}\gamma_2^{(1)} + \frac{J_{2k+2}(2)J_2(0) - J_{2k+2}(0)J_2(2)}{J_2(0)}\gamma_2^{(2)}, \end{aligned}$$

which is reduced to

$$\begin{aligned} c'_{k,2} &= -8 - 14c'_{k,1} + 16\frac{J_{2k+2}(2)J_2(0) - J_{2k+2}(0)J_2(2)}{J_2(0)^2} \\ &= -8 + 8\frac{J_{2k}(0)}{J_2(0)} + 4\frac{J_{2k-2}(0)}{J_2(0)}. \end{aligned}$$

Example 5.4. We have

$$c'_{k,1} = \frac{J_{2k}(0)}{J_2(0)}, \quad c'_{k,2} = \frac{4J_{2k-2}(0)}{J_2(0)}, \quad c'_{k,3} = \frac{117J_{2k-2}(0) + 162J_{2k-4}(0)}{8J_2(0)},$$

$$c'_{k,4} = \frac{-695J_{2k-2}(0) + 2794J_{2k-4}(0) + 1024J_{2k-6}(0)}{9J_2(0)}.$$

A systematic study of the generating functions w_{2n} for higher special values is desirable.

6. Differential Eisenstein series

We have shown in Theorem 5.2 that the function $w_4(t)$ is a linear combination of $w_2(t)$ and the derivative $\mathbf{G}_1'$ of an automorphic integral. To understand the integrals $\mathbf{G}_j(\tau)$ more concretely, we introduce a family of functions called differential Eisenstein series which play a role analogous to the ordinary Eisenstein series.

6.1. Periodic automorphic integrals

Let Γ be a congruence subgroup of level N and m be an integer. We take a Γ-submodule $\mathfrak{X}$ of $F(\mathfrak{h})$. We focus our attention on automorphic integrals of special types defined as follows.

Definition 6.1 (periodic automorphic integrals). Let χ be a (multiplicative) character of Γ such that $\chi(T^N) = 1$. A holomorphic function $f \in \mathcal{H}(\mathfrak{h})$ is called a *periodic automorphic integral* for Γ of weight m with character χ and *period functions* $\{R_{f,\chi}(\gamma)\}_{\gamma\in\Gamma} \subset \mathfrak{X}$ if

$$f(\tau + N) = f(\tau), \tag{6.1}$$

$$(f|_m\gamma)(\tau) - \chi(\gamma)f(\tau) = R_{f,\chi}(\gamma)(\tau) \quad \text{for all } \gamma \in \Gamma, \tag{6.2}$$

and for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ there exists $\{a_n\}_{n\in\mathbb{Z}}$ such that

$$(f|_m\gamma)(\tau) - \sum_{n\in\mathbb{Z}} a_n q^{\frac{n}{N}} \in \mathfrak{X}, \quad a_n = 0 \quad (n \ll 0). \tag{6.3}$$

We denote by $M_m^\chi(\Gamma, \mathfrak{X})$ the set consisting of such periodic automorphic integral. When χ is the trivial character, we omit the symbol χ and simply write $M_m(\Gamma, \mathfrak{X})$. We call f an *Eichler cusp forms* if it is a periodic automorphic integral such that the Fourier expansion part of $f|_m\gamma$ in (6.3) has no constant term for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. The space of Eichler cusp forms is denoted by $C_m^\chi(\Gamma, \mathfrak{X})$.

When $m > 0$, $M_m(\Gamma) := M_m(\Gamma, \{0\})$ and $C_m(\Gamma) := C_m(\Gamma, \{0\})$ are nothing but the spaces of classical modular forms and cusp forms of weight m respectively. Indeed, $f \in M_m(\Gamma)$ is holomorphic at every cusp of Γ in this case.

Remark 6.1. If $1 \in \mathfrak{X}$, that is, $\mathfrak{X}$ contains constant functions, then any constant shift $f(\tau) + c$ $(c \in \mathbb{C})$ of $f \in M_m(\Gamma, \mathfrak{X})$ also belongs to $M_m(\Gamma, \mathfrak{X})$. In this case, it is natural to study the quotient space $M_m(\Gamma, \mathfrak{X})/(\text{constants})$.

Example 6.1. We give a non-trivial example of Γ-submodule $\mathfrak{X}$ of $F(\mathfrak{h})$ as follows. Let $V_{2k,m}$ be a subspace of $F(\mathfrak{h})$ generated by τ^j $(j = 0, 1, \ldots, 2k)$ and $(\tau - \alpha)^{-j}$ $(\alpha \in \mathbb{C}\setminus\{\gamma\infty \mid \gamma \in \Gamma(2)\},\ j = 1, 2, \ldots, m)$. Notice that $0 \notin \{\gamma\infty \mid \gamma \in \Gamma(2)\}$. Then

the space $V_{2k,m}$ is a $\Gamma(2)$-stable subspace of $\mathbb{C}(\tau)$ under the action $(f|_{-2k}\gamma)(\tau) = j(\gamma,\tau)^{2k} f(\gamma\tau)$ ($\gamma \in \Gamma(2)$). In fact, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, if we put $f(\tau) = (\tau-\alpha)^{-j}$ ($1 \le j \le m$), we observe

$$\begin{aligned}(f|_{-2k}\gamma)(\tau) &= (c\tau+d)^{2k}\Big(\frac{a\tau+b}{c\tau+d} - \alpha\Big)^{-j} \\ &= \frac{(c\tau+d)^{2k+j}}{((a-c\alpha)\tau + (b-d\alpha))^j} \\ &= \text{a polynomial of degree } 2k + \frac{\text{a polynomial of degree } j-1}{(\tau-\gamma\alpha)^j}.\end{aligned}$$

This clearly shows that $(f|_{-2k}\gamma)(\tau) \in V_{2k,m}$.

The period functions $\{R_{f,\chi}(\gamma)\}$ for $f \in M_m(\Gamma,\mathfrak{X})$ obey the relation

$$R_{f,\chi}(T^N) = 0, \tag{6.4}$$

$$R_{f,\chi}(\gamma_1\gamma_2) = \chi(\gamma_1)R_{f,\chi}(\gamma_2) + R_{f,\chi}(\gamma_1)|_m\gamma_2 \quad (\gamma_1,\gamma_2 \in \Gamma). \tag{6.5}$$

The latter identity is readily checked as follows.

$$\begin{aligned}R_{f,\chi}(\gamma_1\gamma_2) &= f|_m\gamma_1\gamma_2 - \chi(\gamma_1\gamma_2)f \\ &= \chi(\gamma_1)\big(f|_m\gamma_2 - \chi(\gamma_2)f\big) + \big(f|_m\gamma_1 - \chi(\gamma_1)f\big)|_m\gamma_2 \\ &= \chi(\gamma_1)R_{f,\chi}(\gamma_2) + R_{f,\chi}(\gamma_1)|_m\gamma_2. \end{aligned}\tag{6.6}$$

Hence, by (6.5), the condition (6.2) can be replaced by the one only for generators of Γ.

For convenience, we give the definitions of the space of negative weight holomorphic automorphic integrals (with characters) in terms of the generators for the specific groups $\mathfrak{G}(2)$ and $\Gamma(2)$.

Definition 6.2 (periodic automorphic integrals for $\mathfrak{G}(2)$ and $\Gamma(2)$). We set

$$M_m(\mathfrak{G}(2),\mathfrak{X}) := \left\{ f \in \mathcal{H}(\mathfrak{h}) \;\middle|\; \begin{array}{l} f(\tau+2) = f(\tau), \\ \tau^{-m} f\Big(-\dfrac{1}{\tau}\Big) - f(\tau) \in \mathfrak{X}, \\ f \text{ is holomorphic at } i\infty \end{array} \right\},$$

$$M_m(\Gamma(2),\mathfrak{X}) := \left\{ f \in \mathcal{H}(\mathfrak{h}) \;\middle|\; \begin{array}{l} f(\tau+2) = f(\tau), \\ (2\tau+1)^{-m} f\Big(\dfrac{\tau}{2\tau+1}\Big) - f(\tau) \in \mathfrak{X}, \\ f \text{ satisfies (6.3)} \end{array} \right\}.$$

Remark 6.2. If $f \in \mathcal{H}(\mathfrak{h})$ is holomorphic at $i\infty$ and satisfies the conditions

$$f(\tau+2) = f(\tau), \quad \tau^{-m} f\Big(-\frac{1}{\tau}\Big) - f(\tau) = R_{f,\chi}(S)(\tau) \in \mathfrak{X},$$

then we see that

$$(f|_m T)(\tau) = \sum_{n\geq 0}(-1)^n a_n q^{\frac{n}{2}}$$

when the Fourier expansion of f is given by $f(\tau) = \sum_{n\geq 0} a_n q^{\frac{n}{2}}$. Namely, f satisfies the condition (6.3) in the definition of periodic automorphic integrals for $\mathfrak{G}(2)$.

Example 6.2. By Lemma 5.3, we have $\mathbf{G}_k(\tau) \in M_{2-4k}(\mathfrak{G}(2), \mathbb{C}[\tau])$ for each positive integer k.

Remark 6.3. When $f(\tau) \in M_m(\mathfrak{G}(2), \mathfrak{X})$ with period functions $\{R_f(\gamma)\}$, by virtue of (6.5), we have $f(\tau) \in M_m(\Gamma(2), \mathfrak{X})$. Indeed, we have

$$R_f(ST^{-2}S^{-1}) = R_f(S)|_m T^{-2}S^{-1} + R_f(T^{-2})|_m S^{-1} + R_f(S^{-1}) \in \mathfrak{X}.$$

6.2. Differential Eisenstein series

We always assume that $-\pi \leq \arg z < \pi$ for $z \in \mathbb{C}$ to determine the branch of complex powers. Define

$$\begin{aligned} G(s,x,\tau) &:= \sum_{m,n\in\mathbb{Z}}{}' (m\tau + n + x)^{-s}, \\ G(s,\tau) &:= G(s,0,\tau), \\ G^{(N;a,b)}(s,\tau) &:= \sum_{\substack{m,n\in\mathbb{Z} \\ m\equiv a \pmod N \\ n\equiv b \pmod N}}{}' (m\tau + n)^{-s} \quad (a,b \in \{0,1,\ldots,N-1\}) \end{aligned}$$

for $s \in \mathbb{C}$ such that $\Re(s) > 2$. Here $\sum'_{m,n\in\mathbb{Z}}$ means the sum over all pairs (m,n) of integers such that the summand is defined. We sometimes refer to these series as *generalized Eisenstein series* (e.g., [3]). Remark that

$$G^{(N;a,b)}(s,\tau) = N^{-s} G\Big(s, \frac{a\tau + b}{N}, \tau\Big),$$

in particular that $G^{(N;0,0)}(s,\tau) = N^{-s}G(s,\tau)$.

It is known that $G(s,x,\tau)$ is analytically continued to the whole s-plane, and $G(s,x,\tau)$ can be written in the form

$$G(s,x,\tau) = \sum_{n>-x} \frac{1}{(n+x)^s} + \frac{1}{\Gamma(s)} A(s,x,\tau),$$

when $x \in \mathbb{R}$, where $A(s,x,\tau)$ is holomorphic in s and τ. In particular, we see that

$$G(-2k,\tau) = G^{(2;1,1)}(-2k,\tau) = 0$$

for any positive integer k (see [26, Theorem 1]; see also [3]). We now introduce the notion of *differential Eisenstein series*.

Definition 6.3 (differential Eisenstein series[5]). For $m \in \mathbb{Z}$, define

$$\mathrm{dG}_m(\tau) := \frac{\partial}{\partial s} G(s,\tau)\Big|_{s=m},$$
$$\mathrm{dG}_m^{(N;a,b)}(\tau) := \frac{\partial}{\partial s} G^{(N;a,b)}(s,\tau)\Big|_{s=m} \qquad (a,b \in \{0,1,\dots,N-1\}).$$

It is immediate to see that $\mathrm{dG}_m(\tau+1) = \mathrm{dG}_m(\tau)$ and $\mathrm{dG}_m^{(N;a,b)}(\tau+N) = \mathrm{dG}_m^{(N;a,b)}(\tau)$. In the case where $N=2$, it is convenient to introduce an abbreviation $\mathrm{dG}_m^{a,b}(\tau)$ for $\mathrm{dG}_m^{(2;a,b)}(\tau)$, which will appear frequently below.

For later use, we recall the definitions and several results on the double zeta functions and double Bernoulli numbers [1]. Let $\underline{\omega} = (\omega_1, \omega_2)$ be a pair of complex parameters. *Barnes' double zeta function* is defined by

$$\zeta_2(s, z \mid \underline{\omega}) := \sum_{m,n\geq 0} (m\omega_1 + n\omega_2 + z)^{-s} \quad (\Re s > 2),$$

and the *double Bernoulli polynomials* $B_{2,k}(z \mid \underline{\omega})$ are defined by the generating function

$$\frac{t^2 e^{zt}}{(e^{\omega_1 t}-1)(e^{\omega_2 t}-1)} = \sum_{k=0}^{\infty} B_{2,k}(z \mid \underline{\omega}) \frac{t^k}{k!}.$$

It is well known that the Barnes double zeta function is extended meromorphically to the whole complex plane and the special values at the non-positive integer points are given by (see, e.g., [1])

Lemma 6.1. *For each $m \in \mathbb{N}$, one has*

$$\zeta_2(1-m, z \mid \underline{\omega}) = \frac{B_{2,m+1}(z \mid \underline{\omega})}{m(m+1)}.$$

Example 6.3. We have

$$\zeta_2\Big(-2k, \frac{\tau-1}{2} \,\Big|\, (-1,\tau)\Big) = \frac{B_{2,2k+2}(\frac{\tau-1}{2} \mid (-1,\tau))}{(2k+1)(2k+2)} \in \frac{1}{\tau}\mathbb{C}[\tau],$$
$$\zeta_2(-2k, \tau \mid (-1,\tau)) = \frac{B_{2,2k+2}(\tau \mid (-1,\tau))}{(2k+1)(2k+2)} \in \frac{1}{\tau}\mathbb{C}[\tau].$$

[5] We have used the notation $dE_m(\tau)$ in [22] in place of $\mathrm{dG}_m(\tau)$. In this paper, however, we use the notation $\mathrm{dE}_m(\tau)$ for representing the *normalized* differential Eisenstein series in §6.6, which follows the standard use of Eisenstein series in the classical theory of modular forms.

6.3. dG_{-2k} is an automorphic integral

We notice the following elementary fact.

Lemma 6.2. *If $\tau \in \mathfrak{h}$ and $(a, b) \in \mathbb{R}^2 - \{(0,0)\}$, then*

$$\arg\Big(-\frac{1}{\tau}\Big) + \arg(a\tau + b) \geq \pi \iff a > 0,\ b \leq 0.$$

Lemma 6.3. *For each $k \in \mathbb{N}$, one has*

$$\mathrm{dG}_{-2k}\Big(-\frac{1}{\tau}\Big) = \Big(-\frac{1}{\tau}\Big)^{2k}\{\mathrm{dG}_{-2k}(\tau) - 4k\pi i\zeta_2(-2k, \tau \mid (-1, \tau))\}.$$

Proof. It follows from Lemma 6.2 that

$$\begin{aligned}
G\Big(s, -\frac{1}{\tau}\Big) &= \sum_{m,n\in\mathbb{Z}}{}' \Big(-m\frac{1}{\tau} + n\Big)^{-s} \\
&= \sum_{m,n\in\mathbb{Z}}{}' \Big(\Big(-\frac{1}{\tau}\Big)(m\tau + n)\Big)^{-s} \\
&= \Big(-\frac{1}{\tau}\Big)^{-s}\Big\{\sum_{m,n\in\mathbb{Z}}{}' (m\tau + n)^{-s} + (e^{2\pi i s} - 1)\sum_{\substack{m>0\\ n\leq 0}}(m\tau + n)^{-s}\Big\} \\
&= \Big(-\frac{1}{\tau}\Big)^{-s}\{G(s, \tau) + (e^{2\pi i s} - 1)\zeta_2(s, \tau \mid (-1, \tau))\}.
\end{aligned}$$

This yields

$$\begin{aligned}
&\frac{\partial}{\partial s}G\Big(s, -\frac{1}{\tau}\Big)\Big|_{s=-2k} \\
&\quad = \frac{\partial}{\partial s}\Big(-\frac{1}{\tau}\Big)^{-s}\Big|_{s=-2k}\{G(-2k, \tau) + (e^{-4k\pi} - 1)\zeta_2(-2k, \tau \mid (-1, \tau))\} \\
&\qquad + \Big(-\frac{1}{\tau}\Big)^{2k}\frac{\partial}{\partial s}\{G(s, \tau) + (e^{2\pi i s} - 1)\zeta_2(s, \tau \mid (-1, \tau))\}|_{s=-2k} \\
&\quad = \Big(-\frac{1}{\tau}\Big)^{2k}\Big\{\frac{\partial}{\partial s}G(s, \tau)\Big|_{s=-2k} - 4k\pi i\zeta_2(-2k, \tau \mid (-1, \tau))\Big\}.
\end{aligned}$$

Thus, we have

$$\mathrm{dG}_{-2k}\Big(-\frac{1}{\tau}\Big) = \Big(-\frac{1}{\tau}\Big)^{2k}\{\mathrm{dG}_{-2k}(\tau) - 4k\pi i\zeta_2(-2k, \tau \mid (-1, \tau))\}. \qquad \blacksquare$$

By a similar calculation, we also have the following result.

Lemma 6.4. *For each $k \in \mathbb{N}$, one has*

$$\mathrm{dG}^{1,1}_{-2k}\Big(-\frac{1}{\tau}\Big) = \tau^{-2k}(\mathrm{dG}^{1,1}_{-2k}(\tau) - 4k\pi i\zeta_2(-2k, \tau - 1 \mid (-2, 2\tau))).$$

By the lemmas above, we obtain the

Corollary 6.5. *One has*

$$\mathrm{dG}_{-2k}(\tau) \in M_{-2k}(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}(\tau)), \quad \mathrm{dG}^{0,0}_{-2k}(\tau), \mathrm{dG}^{1,1}_{-2k}(\tau) \in M_{-2k}(\mathfrak{G}(2), \mathbb{C}(\tau))$$

for each $k \in \mathbb{N}$.

Remark 6.4. We observe that $\mathrm{dG}^{0,0}_{-2k}(\tau), \mathrm{dG}^{1,1}_{-2}(\tau) \in M_{-2}(\Gamma(2), V_{2,1})$, $V_{2,1}$ being the space defined in Example 6.1.

Remark 6.5. A recent calculation due to Shibukawa [41] on the same analysis of the lemmas above shows that $\mathrm{dG}_{-2k}(\tau) \in M_{-2k}(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}(\mathfrak{h}))$ but $\notin M_{-2k}(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}(\tau))$ for $k > 0$.

Remark 6.6. Although we have given the proof of lemmas above directly, we may extend these relations to the general case by a similar analysis in [3].

Remark 6.7. The function $\mathrm{dG}^{1,1}_m(\tau)$ can be written as

$$\mathrm{dG}^{1,1}_m(\tau) = (1 + 2^{-m})\mathrm{dG}_m(\tau) - 2^{-m}\mathrm{dG}_m(\tau/2) - \mathrm{dG}_m(2\tau).$$

6.4. An expression of $w_4(t)$ in terms of differential Eisenstein series

By Lemmas 6.3, 6.4 and 6.1, we have

$$\mathrm{dG}^{1,1}_{-2}\Big(-\frac{1}{\tau}\Big) = \tau^{-2}\Big(\mathrm{dG}^{1,1}_{-2}(\tau) - \frac{\pi i}{3} B_{2,4}(\tau - 1 \mid (-2, 2\tau))\Big),$$
$$\mathrm{dG}_{-2}\Big(-\frac{1}{\tau}\Big) = \tau^{-2}\Big(\mathrm{dG}_{-2}(\tau) - \frac{\pi i}{3} B_{2,4}(\tau \mid (-1, \tau))\Big).$$

A straightforward calculation using these formulas shows that

$$7B_{2,4}(\tau \mid (-1, \tau)) + 2B_{2,4}(\tau - 1 \mid (-2, 2\tau)) = -\frac{3}{2}\tau.$$

Therefore, if we put

$$\phi_1(\tau) := -8\pi^2\{7\mathrm{dG}_{-2}(\tau) + 2\mathrm{dG}^{1,1}_{-2}(\tau)\}, \tag{6.7}$$

then we have

$$\phi_1(\tau + 2) = \phi_1(\tau), \quad \tau^2\phi_1\Big(-\frac{1}{\tau}\Big) - \phi_1(\tau) = 4\pi^3\frac{\tau}{i}.$$

These relations are exactly the same as the ones (5.14) for $\widetilde{\mathbf{G}}_1(\tau) = \mathbf{G}_1(\tau) - 56\zeta(3)$. Therefore, the difference $\mathbf{G}_1(\tau) - 56\zeta(3) - \phi_1(\tau)$ is a classical holomorphic modular forms of weight -2 for $\Gamma(2)$. Since $M_{-2}(\Gamma(2)) = \{0\}$, we have

$$\mathbf{G}_1(\tau) = \phi_1(\tau) + 56\zeta(3). \tag{6.8}$$

Putting this expression into (5.10), we obtain the following

Theorem 6.6. *The generating function $w_4(t)$ of Apéry-like numbers $J_4(n)$ is given by*

$$w_4(t) = \frac{\pi^4}{2}\frac{\theta_4(\tau)^4}{\theta_3(\tau)^2}\Big[1 + \frac{1}{\pi i}\frac{d}{d\tau}\{7\mathrm{dG}_{-2}(\tau) + 2\mathrm{dG}_{-2}^{1,1}(\tau)\}\Big],$$

where $t = t(\tau) = -\theta_2(\tau)^4\theta_4(\tau)^{-4}$.

6.5. Fourier expansion of $\mathrm{dG}_{-2k}(\tau)$

We now compute the Fourier expansion of the differential Eisenstein series $\mathrm{dG}_{-2k}(\tau)$ using the result in [41]. Similarly to the classical Eisenstein series, we will find that the Fourier expansion of $\mathrm{dG}_{-2k}(\tau)$ is given by the Lambert series. In particular, we notice that $\mathrm{dG}_{-2k}(\tau)$ is not a cusp form.

We first recall the result for the bilateral zeta function in [41]. For (ω_1, ω_2), the bilateral zeta function $\xi_2(s, z \mid \omega_1, \omega_2)$ is defined by

$$\xi_2(s, z \mid \omega_1, \omega_2) := \zeta_2(s, z + \omega_1 \mid \omega_1, \omega_2) + \zeta_2(s, z \mid -\omega_1, \omega_2). \tag{6.9}$$

We take ω_1 as $0 < \arg(\omega_1) \leq \pi$ in the subsequent discussion. Then the following Fourier expansion of $\xi_2(s, z \mid -1, \omega)$ is known ([41, Theorem 4.7 and Corollary 4.8]).

Proposition 6.7. *Suppose $z, \omega \in \mathfrak{h}$. Then we have*

$$\xi_2(s, z \mid -1, \omega) = \frac{e^{-(\pi/2)is}(2\pi)^s}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{n^{s-1}e^{2\pi inz}}{1 - e^{2\pi in\omega}}. \tag{6.10}$$

Moreover, one notices that the bilateral zeta function $\xi_2(s, z \mid \omega_1, \omega)$ is an entire function in $s \in \mathbb{C}$ and for $m \in \mathbb{N}$

$$\xi_2(1 - m, z \mid \omega_1, \omega) = 0. \tag{6.11}$$

Using this proposition, we prove the following

Theorem 6.8. *The Fourier expansion of the differential Eisenstein series $\mathrm{dG}_{-2k}(\tau)$ $(k \in \mathbb{N})$ is expressed by the Lambert series as*

$$\begin{aligned}\mathrm{dG}_{-2k}(\tau) &= \frac{(-1)^k(2k)!}{(2\pi)^{2k}}\Big\{2\sum_{n=1}^{\infty}\frac{n^{-2k-1}}{1 - e^{2\pi in\tau}} - \zeta(2k+1)\Big\} \\ &= \frac{(-1)^k(2k)!}{(2\pi)^{2k}}\Big\{\zeta(2k+1) + 2\sum_{n=1}^{\infty}\sigma_{-2k-1}(n)e^{2\pi in\tau}\Big\}.\end{aligned}$$

In particular, the constant term is given by the multiple of $\zeta(2k+1)$ *as*

$$\mathrm{dG}_{-2k}(i\infty) = \frac{(-1)^k(2k)!}{(2\pi)^{2k}}\zeta(2k+1) \neq 0.$$

Proof. We observe that

$$\begin{aligned} G(s,\tau) &= \sum_{m_0,m_1\in\mathbb{Z}}{}' (m_0+m_1\tau)^{-s} \\ &= \sum_{m_0\in\mathbb{Z},\, m_1\in\mathbb{Z}_{\geq 0}}{}' (m_0+m_1\tau)^{-s} + \sum_{m_0\in\mathbb{Z},\, m_1\in\mathbb{Z}_{\geq 0}}{}' (m_0-m_1\tau)^{-s} - \sum_{m_0\neq 0} m_0^{-s} \\ &= (1+e^{\pi i s})\Big\{ \sum_{m_0\in\mathbb{Z},\, m_1\in\mathbb{Z}_{\geq 0}}{}' (m_0+m_1\tau)^{-s} - \zeta(s)\Big\}. \end{aligned}$$

By (6.11), since

$$0 = \xi_2(1-2k, z \mid -1, \tau) = \sum_{m_0\in\mathbb{Z},\, m_1\in\mathbb{Z}_{\geq 0}}{}' (z+m_0+m_1\tau)^{-s}|_{s=-2k} + z^{2k},$$

for $k \in \mathbb{N}$, we have

$$\sum_{m_0\in\mathbb{Z},\, m_1\in\mathbb{Z}_{\geq 0}}{}' (m_0+m_1\tau)^{-s}|_{s=-2k} = 0.$$

It follows that

$$\frac{\partial}{\partial s}G(s,\tau)\Big|_{s=-2k} = 2\frac{\partial}{\partial s}\Big\{ \sum_{m_0\in\mathbb{Z},\, m_1\in\mathbb{Z}_{\geq 0}}{}' (m_0+m_1\tau)^{-s}\Big\}\Big|_{s=-2k} - 2\zeta'(-2k). \quad (6.12)$$

On the other hand, we observe

$$\begin{aligned} \xi_2(s, z \mid -1, -\tau) &= \zeta_2(s, z-1 \mid -1, -\tau) + \zeta_2(s, z \mid 1, -\tau) \\ &= \sum_{m_0,m_1=0}^{\infty} (z-1-m_0+m_1\tau)^{-s} + \sum_{m_0,m_1=0}^{\infty} (z+m_0+m_1\tau)^{-s} \\ &= \sum_{m_0\in\mathbb{Z}} \sum_{m_1=0}^{\infty} (z+m_0+m_1\tau)^{-s} \\ &= \sum_{m_0\in\mathbb{Z},\, m_1\in\mathbb{Z}_{\geq 0}}{}' (z+m_0+m_1\tau)^{-s} + z^{-s}. \end{aligned}$$

By the Fourier expansion (6.10), using the fact

$$\frac{d}{ds}\frac{1}{\Gamma(s)}\Big|_{s=-2k} = (2k)!,$$

we have

$$\frac{\partial}{\partial s}\xi_2(s,z\mid -1,\tau)\Big|_{s=-2k}=\frac{(-1)^k(2k)!}{(2\pi)^{2k}}\sum_{n=1}^{\infty}\frac{n^{-2k-1}e^{2\pi inz}}{1-e^{2\pi in\tau}}. \tag{6.13}$$

Therefore, we have

$$\begin{aligned}&\frac{\partial}{\partial s}\Big\{\sum_{m_0\in\mathbb{Z},\,m_1\in\mathbb{Z}_{\geq 0}}{}'(m_0+m_1\tau)^{-s}\Big\}\Big|_{s=-2k}\\&=\Big\{\frac{\partial}{\partial s}\xi_2(s,z\mid -1,\tau)\Big|_{s=-2k}+(\log z)z^{2k}\Big\}\Big|_{z=0}\\&=\frac{(-1)^k(2k)!}{(2\pi)^{2k}}\sum_{n=1}^{\infty}\frac{n^{-2k-1}}{1-e^{2\pi in\tau}}.\end{aligned}$$

It follows from (6.12) that

$$\mathrm{dG}_{-2k}(\tau)=\frac{\partial}{\partial s}G(s,\tau)\Big|_{s=-2k}=\frac{2(-1)^k(2k)!}{(2\pi)^{2k}}\sum_{n=1}^{\infty}\frac{n^{-2k-1}}{1-e^{2\pi in\tau}}-2\zeta'(-2k). \tag{6.14}$$

Using the functional equation

$$\frac{2^{s-1}\pi^s}{\Gamma(s)}\zeta(1-s)=\cos\frac{\pi s}{2}\zeta(s)$$

and

$$\frac{d}{ds}\frac{1}{\Gamma(s)}\Big|_{s=-2k}=(2k)!,$$

we have

$$\zeta'(-2k)=\frac{(-1)^k(2k)!}{2(2\pi)^{2k}}\zeta(2k+1).$$

Hence, we complete the proof of the theorem. ■

Remark 6.8. We note that the function $\phi_1(\tau)$ is expressible only by dG_{-2} as

$$\phi_1(\tau)=8\pi^2(8\mathrm{dG}_{-2}(\tau/2)-17\mathrm{dG}_{-2}(\tau)+2\mathrm{dG}_{-2}(2\tau)).$$

By Theorem 6.8, we have

$$\mathrm{dG}_{-2}(\tau)=-\frac{1}{2\pi^2}\Big\{\zeta(3)+2\sum_{n=1}^{\infty}\sigma_{-3}(n)q^n\Big\}.$$

Comparing with (5.11), we obtain (6.8) again.

Remark 6.9. Like Ramanujan did, we may evaluate values of the Lambert series at $\tau = i$ if $k \in \mathbb{N}$ is odd as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k+1}(1-e^{-2\pi n})} = \frac{ki(2\pi)^{2k+1}}{2(-1)^k(2k+2)!} B_{2,2k+2}(i \mid (-1,i)) + \frac{1}{2}\zeta(2k+1). \tag{6.15}$$

In fact, by Lemma 6.3 together with Example 6.3, we have

$$\mathrm{dG}_{-2k}(i) = 2k\pi i \frac{B_{2,2k+2}(i \mid (-1,i))}{(2k+1)(2k+2)},$$

whenever k is odd. Hence, the formula follows immediately from Theorem 6.8.

6.6. Hecke operators acting on automorphic forms of negative weight

We give a short remark on the Hecke operators acting on the negative weight automorphic forms.

Let $n \in \mathbb{N}$ and set $\mathbb{M}_n := \{g \in \mathrm{Mat}_2(\mathbb{Z}) \mid \det g = n\}$. Since the group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{M}_n$ on the left, one may decompose $\mathbb{M}_n$ into orbits. We now consider the automorphic forms of weight $-k$ ($k \in \mathbb{N}$). For $f \in M_{-k}(\mathrm{SL}_2(\mathbb{Z}), \mathfrak{X})$, we set

$$(T(n)f)(\tau) = n^{-k/2-1} \sum_{\mu \in \mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{M}_n} (f|_{-k}\mu)(\tau). \tag{6.16}$$

Here we notice that the sum $\sum_{\mu \in \mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{M}_n} f|_{-k}\mu$ depends on the choice of a system of representatives $\{\mu\}$ for the orbits $\mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{M}_n$. Actually, if we take another representatives $\{\gamma\mu\}$ ($\gamma \in \mathrm{SL}_2(\mathbb{Z})$) we observe that

$$\begin{aligned}\sum_{\mu \in \mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{M}_n} f|_{-k}(\gamma\mu) &= \sum_{\mu \in \mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{M}_n} f|_{-k}\gamma|_{-k}\mu \\ &= \sum_{\mu \in \mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{M}_n} (f + R_f^{-k}(\gamma))|_{-k}\mu \in \sum_{\mu \in \mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{M}_n} f|_{-k}\mu + \mathfrak{X}.\end{aligned}$$

This fact shows that $T(n)f$ is determined modulo the space $\mathfrak{X}$ for another choice $\{\gamma\mu\}$ of the representatives for $\mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{M}_n$. This observation, however, proves also that

Lemma 6.9. *Let $f \in M_{-k}(\mathrm{SL}_2(\mathbb{Z}), \mathfrak{X})$. Then for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have*

$$T(n)f|_{-k}\gamma \equiv T(n)f \mod \mathfrak{X} \tag{6.17}$$

for any choice of a system of representatives $\{\mu\}$ for $\mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{M}_n$.

This lemma shows that for $n \in \mathbb{N}$ the operator $T(n)$ defines a well-defined linear endomorphism of $M_{-k}(\mathrm{SL}_2(\mathbb{Z}), \mathfrak{X})$. We call $T(n)$ the *Hecke operator* of index n (acting on the automorphic integrals of negative weight). Similarly to the classical case, we have the following

Proposition 6.10. *The Hecke operator $T(n)$ $(n = 1, 2, \ldots)$ on $M_{-k}(\mathrm{SL}_2(\mathbb{Z}), \mathfrak{X})$ possesses the following properties.*

(i) *The operator $T(n)$ has the following expression.*

$$(T(n)f)(\tau) := n^{-k-1} \sum_{\substack{a \geq 1 \\ ad = n \\ 0 \leq b < d}} d^k f\Big(\frac{a\tau + b}{d}\Big) = \frac{1}{n} \sum_{\substack{a \geq 1 \\ ad = n \\ 0 \leq b < d}} a^{-k} f\Big(\frac{a\tau + b}{d}\Big). \tag{6.18}$$

(ii) *Let $f(\tau) = \sum_{l=0}^{\infty} \lambda(l) q^l$ $(q = e^{2\pi i \tau})$. Then*

$$(T(n)f)(\tau) = \sum_{l=0}^{\infty} \Big(\sum_{d \mid (n,l)} d^{-k-1} \lambda\Big(\frac{nl}{d^2}\Big)\Big) q^l. \tag{6.19}$$

In particular, the space of cusp forms $C_{-k}(\mathrm{SL}_2(\mathbb{Z}), \mathfrak{X})$ is stable under $T(n)$.

(iii) *Let $m, n \in \mathbb{N}$. Then*

$$T(n)T(m) = \sum_{d \mid (n,m)} d^{-k-1} T(nm/d^2) = T(m)T(n). \tag{6.20}$$

In particular, $T(n)T(m) = T(nm)$ whenever $(n, m) = 1$.

Proof. The proof can be done in the same way as the classical case. Actually, since every matrix $\mu \in \mathbb{M}_n$ can be made upper triangular by multiplying it on the left by $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have a system of representatives for $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{M}_n$ as $\pm\left(\begin{smallmatrix} a & b+dr \\ 0 & d \end{smallmatrix}\right)$ $(ad = n)$ with $a > 0$ and $0 \leq b < d$. With this choice of representatives, by the definition (6.16), we have the expression (i). Using the elementary relation

$$\sum_{b=0}^{d-1} f\Big(\frac{a\tau + b}{d}\Big) = d \sum_{m=0}^{\infty} \lambda(md) q^{md},$$

we have the formula of (ii) from (i). We notice that the constant term in q equals $\sigma_{-k-1}(d)\lambda(0)$, whence $C_{-k}(\mathrm{SL}_2(\mathbb{Z}), \mathfrak{X})$ is stable under $T(n)$. The last assertion (iii) can be deduced from the formula (ii) by computation. This completes the proof. ∎

We now show that the differential Eisenstein series $\mathrm{dG}_{-2k} \in M_{-2k}(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}(\tau))$ is a joint eigenfunction of $T(n)$ for all $n \in \mathbb{N}$.

Lemma 6.11. *We have*

$$(T(n)\mathrm{dG}_{-2k})(\tau) = \sigma_{-2k-1}(n)\mathrm{dG}_{-2k}(\tau)$$

for each $n, k \in \mathbb{N}$.

Proof. Consider the function

$$F_n(s,\tau) = n^{s-1}\sum_{\substack{a\geq 1\\ ad=n\\ 0\leq b<d}} d^{-s}G\Big(s, \frac{a\tau+b}{d}\Big) = \frac{1}{n}\sum_{\substack{a\geq 1\\ ad=n\\ 0\leq b<d}} a^{s}G\Big(s, \frac{a\tau+b}{d}\Big).$$

Then we observe

$$\begin{aligned}
&\frac{\partial}{\partial s}F_n(s,\tau)\Big|_{s=-2k}\\
&\quad= \frac{1}{n}\sum_{\substack{a\geq 1\\ ad=n\\ 0\leq b<d}}\Big\{\frac{\partial}{\partial s}a^{s}\Big|_{s=-2k}G\Big(-2k, \frac{a\tau+b}{d}\Big) + a^{-2k}\frac{\partial}{\partial s}G\Big(s, \frac{a\tau+b}{d}\Big)\Big|_{s=-2k}\Big\}\\
&\quad= \frac{1}{n}\sum_{\substack{a\geq 1\\ ad=n\\ 0\leq b<d}} a^{-2k}\mathrm{dG}_{-2k}\Big(\frac{a\tau+b}{d}\Big) = (T(n)\mathrm{dG}_{-2k})(\tau).
\end{aligned}$$

On the other hand, we have

$$F_n(s,\tau) = n^{s-1}\sum_{\substack{a\geq 1\\ ad=n\\ 0\leq b<d}}\ \sum_{m,l\in\mathbb{Z}}{}' (dm + (a\tau+b)l)^{-s} = \sigma_{s-1}(n)G(s,\tau),$$

and hence

$$\begin{aligned}
\frac{\partial}{\partial s}F_n(s,\tau)\Big|_{s=-2k} &= \frac{\partial}{\partial s}\sigma_{s-1}(n)\Big|_{s=-2k}G(-2k,\tau) + \sigma_{-2k-1}(n)\frac{\partial}{\partial s}G(s,\tau)\Big|_{s=-2k}\\
&= \sigma_{-2k-1}(n)\mathrm{dG}_{-2k}(\tau).
\end{aligned}$$

Thus, we have the lemma. ∎

This lemma implies again that dG_{-2k} can have the Fourier series expansion as

$$\mathrm{dG}_{-2k}(\tau) = \mathrm{dG}_{-2k}(i\infty) + C_k\sum_{n=1}^{\infty}\sigma_{-2k-1}(n)q^n \quad (q = e^{2\pi i\tau}),$$

for some constant C_k. From Theorem 6.8 one finds that $C_k = 2\frac{(-1)^k(2k)!}{(2\pi)^{2k}}$. We define the *normalized* differential Eisenstein series dE_{-2k} of weight $-2k$ as

$$\begin{aligned}\mathrm{dE}_{-2k}(\tau) &:= \frac{2}{C_k\zeta(2k+1)}\mathrm{dG}_{-2k}(\tau)\\ &= 1 + \frac{2}{\zeta(2k+1)}\sum_{n=1}^{\infty}\sigma_{-2k-1}(n)q^n.\end{aligned}$$

Then the associated L-function of dE_{-2k} is given by

$$L(\mathrm{dE}_{-2k}, s) = \sum_{n=1}^{\infty}\frac{\sigma_{-2k-1}(n)}{n^s} = \zeta(s)\zeta(s+2k+1).$$

Namely, dE_{-2k} is a Hecke form (see [13]). We observe in particular that $L(\mathrm{dE}_{-2k}, s)$ has a unique pole at $s = 1$, while there is no pole at $s = -2k$. Notice that since $L(E_{2k}, s) = \zeta(s)\zeta(s-2k+1)$, E_{2k} being the classical Eisenstein series of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$, $L(E_{2k}, s)$ has a unique pole at $s = 2k$ but not at $s = 1$ for $k > 1$.

Further, the completed L-function

$$\Xi_{-2k}(s) = \xi(s)\xi(s+2k+1), \quad \xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

satisfies the functional equation

$$\Xi_{-2k}(-2k-s) = \Xi_{-2k}(s).$$

Remark 6.10. Note that the function $\Xi_{-2k}(s)$ is meromorphic but not entire. It would be interesting to study a Hecke-Weil type theorem about the correspondence between negative weight automorphic integrals and their L-functions (Euler products).

7. Periodic Eichler cohomology for automorphic integrals

We construct a cochain complex from the period functions of negative weight periodic automorphic integrals. Let us fix an integer m. Denote by Γ a congruence subgroup of level N, and χ a multiplicative character of Γ such that $\chi(T^N) = 1$. Suppose that $\mathfrak{X}$ is a Γ-submodule of the space $F(\mathfrak{h})$ of functions on $\mathfrak{h}$ via the action $f\big|_m\gamma$ ($\gamma \in \Gamma$).

7.1. First cohomology

Let $C^1(\Gamma, \mathfrak{X})$ be the space of all maps from Γ to $\mathfrak{X}$. We call $R_\chi \in C^1(\Gamma, \mathfrak{X})$ a (*twisted*) 1-*cocycle* with weight χ if it satisfies

$$R_\chi(\gamma_1\gamma_2) = \chi(\gamma_1)R_\chi(\gamma_2) + R_\chi(\gamma_1)\big|_m\gamma_2.$$

Notice that $R_\chi(I) = 0$ if R_χ is a 1-cocycle. We denote by $Z^1_{[m]}(\Gamma, \mathfrak{X})$ the set of all (twisted) 1-cocycles (Here and after, to avoid complications, we do not specify the character χ in notation). Obviously $Z^1_{[m]}(\Gamma, \mathfrak{X})$ is a subspace of $C^1(\Gamma, \mathfrak{X})$.

Define the element $\delta f \in C^1(\Gamma, \mathfrak{X})$ for $f \in \mathfrak{X}$ by

$$(\delta f)(\gamma) = f\big|_m \gamma - \chi(\gamma) f \quad (\gamma \in \Gamma).$$

By a similar calculation as in (6.6) one has the

Lemma 7.1. *$\delta f \in Z^1_{[m]}(\Gamma, \mathfrak{X})$ for each $f \in \mathfrak{X}$.*

Define the subgroup $B^1_{[m]}(\Gamma, \mathfrak{X})$ of $Z^1_{[m]}(\Gamma, \mathfrak{X})$ by

$$B^1_{[m]}(\Gamma, \mathfrak{X}) = \operatorname{im} \delta = \{\delta f \mid f \in \mathfrak{X}\}.$$

We call an element of $B^1_{[m]}(\Gamma, \mathfrak{X})$ by a (*twisted*) 1-*coboundary*. The quotient group defined by

$$H^1_{[m]}(\Gamma, \mathfrak{X}) := Z^1_{[m]}(\Gamma, \mathfrak{X}) / B^1_{[m]}(\Gamma, \mathfrak{X})$$

is called the first *Eichler cohomology group* of weight m for the Γ-module $\mathfrak{X}$.

Periodic cohomology. Assume that Γ is a congruence subgroup of level N. For $f \in M_m(\Gamma, \mathfrak{X})$, put

$$R^m_f(\gamma) := f\big|_m \gamma - \chi(\gamma) f \quad (\gamma \in \Gamma).$$

It is easy to check that R^m_f gives an element in $Z^1_{[m]}(\Gamma, \mathfrak{X})$ (see (6.5)). We notice that $R^m_f(T^N) = f(\tau + N) - f(\tau) = 0$ by definition.

Definition 7.1. Define

$$\begin{aligned}
&\widetilde{Z}^1_{[m]}(\Gamma, \mathfrak{X}) := \{R \in Z^1_{[m]}(\Gamma, \mathfrak{X}) \mid R(T^N) = 0\},\\
&\widetilde{B}^1_{[m]}(\Gamma, \mathfrak{X}) := \{\delta f \in B^1_{[m]}(\Gamma, \mathfrak{X}) \mid f \in \mathfrak{X},\ f(\tau + N) = f(\tau)\} (\subset \widetilde{Z}^1_{[m]}(\Gamma, \mathfrak{X})),\\
&\widetilde{H}^1_{[m]}(\Gamma, \mathfrak{X}) := \widetilde{Z}^1_{[m]}(\Gamma, \mathfrak{X}) / \widetilde{B}^1_{[m]}(\Gamma, \mathfrak{X}).
\end{aligned}$$

We call $\widetilde{H}^1_{[m]}(\Gamma, \mathfrak{X})$ the first *periodic* Eichler cohomology group (of weight m).

Let $f \in M_m(\Gamma, \mathfrak{X})$. We see that $R^m_f \in \widetilde{Z}^1_{[m]}(\Gamma, \mathfrak{X})$. If $R^m_f \in \widetilde{B}^1_{[m]}(\Gamma, \mathfrak{X})$, then there exists some $g \in \mathfrak{X}$ such that $R^m_f = \delta g$ and $g(\tau + N) = g(\tau)$. It follows that $(f - g)\big|_m \gamma = f - g$, which implies that $f - g \in M_m(\Gamma)$ and hence $g \in M_m(\Gamma, \mathfrak{X}) + M_m(\Gamma) = M_m(\Gamma, \mathfrak{X})$. Thus, we have an injection

$$M^*_m(\Gamma, \mathfrak{X}) \hookrightarrow \widetilde{H}^1_{[m]}(\Gamma, \mathfrak{X}),$$

where we put

$$M^*_m(\Gamma, \mathfrak{X}) := M_m(\Gamma, \mathfrak{X}) / (\mathfrak{X} \cap M_m(\Gamma, \mathfrak{X}) + M_m(\Gamma)). \tag{7.1}$$

If $m < 0$ and $\mathfrak{X} \subset \mathbb{C}(\tau)$, then we have

$$M_m^*(\Gamma, \mathfrak{X}) = \begin{cases} M_m(\Gamma, \mathfrak{X})/(\text{constants}) & 1 \in \mathfrak{X}, \\ M_m(\Gamma, \mathfrak{X}) & 1 \notin \mathfrak{X}. \end{cases}$$

In particular, we have the inequality

$$\dim M_m^*(\Gamma, \mathfrak{X}) \leq \dim \widetilde{H}^1_{[m]}(\Gamma, \mathfrak{X}). \tag{7.2}$$

We also have

$$\dim \widetilde{H}^1_{[m]}(\Gamma, \mathfrak{X}) \leq \dim H^1_{[m]}(\Gamma, \mathfrak{X}) - 1 \tag{7.3}$$

when $\mathfrak{X} = \mathbb{C}(\tau)$ or $\mathfrak{X} = \mathbb{C}[\tau]_{-m}$ (see [22, Lemma 17]).

Notice that $1 = j(\gamma, \tau)^{-2}\{1 - (1 - j(\gamma, \tau)^2)\} \in M_{-2}(\mathfrak{G}(2), \mathbb{C}[\tau]_2)$. By (7.2) and (7.3), we have

$$1 \leq \dim_{\mathbb{C}} M_{-2}^*(\mathfrak{G}(2), \mathbb{C}[\tau]_2) \leq \dim_{\mathbb{C}} H^1_{[-2]}(\Gamma(2), \mathbb{C}[\tau]_2) - 1$$

since $\widetilde{\mathbf{G}}_1 \in M_{-2}(\mathfrak{G}(2), \mathbb{C}[\tau]_2)$. It is known in [10] that

$$H^1_{[-2k]}(\Gamma(2), \mathbb{C}[\tau]_{2k}) \cong M_{2k+2}(\Gamma(2)) \oplus C_{2k+2}(\Gamma(2)),$$

$C_{2k+2}(\Gamma(2))$ being the space of cusp forms of weight $2k + 2$ for $\Gamma(2)$. Since one has both $\dim_{\mathbb{C}} M_4(\Gamma(2)) = 2$ and $\dim_{\mathbb{C}} C_4(\Gamma(2)) = 0$ (see, e.g., [42]), one concludes that $\dim_{\mathbb{C}} M_{-2}^*(\Gamma(2), \mathbb{C}[\tau]_2) = 1$. Thus, we have the following result.

Corollary 7.2. $M_{-2}(\mathfrak{G}(2), \mathbb{C}[\tau]_2) = M_{-2}(\Gamma(2), \mathbb{C}[\tau]_2) = \mathbb{C} \cdot \widetilde{\mathbf{G}}_1 \oplus \mathbb{C} \cdot 1$.

This shows that the $M_{-2}(\mathfrak{G}(2), \mathbb{C}[\tau]_2)$ is essentially given by w_4, i.e., the special value $\zeta_Q(4)$.

The following lemma is obvious.

Lemma 7.3. *Assume that a congruence subgroup Γ of level N contains S. If $f \in M_{-k}(\Gamma, \mathfrak{X})$ we have*

$$R_f^{-k}(T^N)(\tau) = 0, \quad R_f^{-k}(S)(S\tau) = -\tau^{-k} R_f^{-k}(S)(\tau).$$

In particular, $R_f^{-k}(\gamma) \in \widetilde{Z}^1_{[m]}(\Gamma, \mathfrak{X})$. From the cocycle condition, one knows that $R \in \widetilde{Z}^1_{[m]}(\Gamma, \mathfrak{X})$ is determined by the double coset of $\Gamma_\infty = \langle T^N \rangle$:

$$R(T^N \gamma)(\tau) = R(\gamma)(\tau), \quad R(\gamma T^N)(\tau) = R(\gamma)(T^N \tau).$$

7.2. Cochain complex

Let us put

$$C^n = C^n(\Gamma, \mathfrak{X}) := \mathrm{Map}(\Gamma^n, \mathfrak{X}),$$

for $n = 1, 2, 3, \ldots$ and $C^0 = C^0(\Gamma, \mathfrak{X}) := \mathfrak{X}$. For an n-tuple $\underline{\gamma} = (\gamma_1, \ldots, \gamma_n) \in \Gamma^n$, we define

$$\mathcal{T}_j \underline{\gamma} := (\gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1}, \ldots, \gamma_n), \quad \mathcal{C}_j \underline{\gamma} := (\gamma_1, \ldots, \underbrace{\gamma_{j+1}\gamma_j}_{j\text{-th}}, \ldots, \gamma_n)$$

$(j = 1, 2, \ldots, n)$ for convenience. Define the linear operator $\delta^n \colon C^n \to C^{n+1}$ by

$$(\delta^n f)(\underline{\gamma}) := f(\mathcal{T}_1 \underline{\gamma})|_m \gamma_1 + (-1)^{n+1} \chi(\gamma_{n+1}) f(\mathcal{T}_{n+1} \underline{\gamma}) + \sum_{j=1}^{n} (-1)^j f(\mathcal{C}_j \underline{\gamma}) \quad (7.4)$$

for $f \in C^n$ and $\underline{\gamma} = (\gamma_1, \ldots, \gamma_{n+1}) \in \Gamma^{n+1}$.

Although we have given the proof of the following fact in [22], we give here a shorter one.

Lemma 7.4. $\delta^{n+1} \circ \delta^n = 0$.

Proof. Take arbitrary $f \in C^n$. Let $\underline{\gamma} = (\gamma_1, \ldots, \gamma_{n+2}) \in \Gamma^{n+2}$. One has

$$\begin{aligned}(\delta^n f)(\mathcal{T}_k \underline{\gamma}) &= f(\mathcal{T}_1 \mathcal{T}_k \underline{\gamma})\big|_m \mathcal{T}_k \underline{\gamma}_1 + (-1)^{n+1} \chi(\mathcal{T}_k \underline{\gamma}_{n+1}) f(\mathcal{T}_{n+1} \mathcal{T}_k \underline{\gamma}) \\ &\quad + \sum_{j=1}^{n} (-1)^j f(\mathcal{C}_j \mathcal{T}_k \underline{\gamma})\end{aligned}$$

and

$$\begin{aligned}(\delta^n f)(\mathcal{C}_j \underline{\gamma}) &= f(\mathcal{T}_1 \mathcal{C}_j \underline{\gamma})\big|_m \mathcal{C}_j \underline{\gamma}_1 + (-1)^{n+1} \chi(\mathcal{C}_j \underline{\gamma}_{n+1}) f(\mathcal{T}_{n+1} \mathcal{C}_j \underline{\gamma}) \\ &\quad + \sum_{l=1}^{n} (-1)^l f(\mathcal{C}_l \mathcal{C}_j \underline{\gamma})\end{aligned}$$

for $1 \le k \le n+2$ and $1 \le j \le n+1$, where $\mathcal{T}_k \underline{\gamma}_r$, $\mathcal{C}_j \underline{\gamma}_r$ are the r-th entry of $\mathcal{T}_k \underline{\gamma}$, $\mathcal{C}_j \underline{\gamma}$. We have

$$\mathcal{T}_k \underline{\gamma}_1 = \begin{cases} \gamma_2 & k = 1, \\ \gamma_1 & k > 1, \end{cases} \qquad \mathcal{T}_k \underline{\gamma}_{n+1} = \begin{cases} \gamma_{n+1} & k = n+2, \\ \gamma_{n+2} & k < n+2, \end{cases}$$

and

$$\mathcal{C}_j \underline{\gamma}_1 = \begin{cases} \gamma_2\gamma_1 & j = 1, \\ \gamma_1 & j > 1, \end{cases} \quad \mathcal{C}_j \underline{\gamma}_{n+1} = \begin{cases} \gamma_{n+2}\gamma_{n+1} & j = n+1, \\ \gamma_{n+2} & j < n+1. \end{cases}$$

Using these, we have

$$\begin{aligned}((\delta^{n+1}\circ\delta^n)f)(\underline{\gamma}) &= (\delta^n f)(\mathcal{T}_1\underline{\gamma})\big|_m\gamma_1 + (-1)^n\chi(\gamma_{n+2})(\delta^n f)(\mathcal{T}_{n+2}\underline{\gamma})\\ &\quad+\sum_{j=1}^{n+1}(-1)^j(\delta^n f)(\mathcal{C}_j\underline{\gamma})\\ &= \sum_{j=1}^{n+1}(-1)^j\sum_{l=1}^{n}(-1)^l f(\mathcal{C}_l\mathcal{C}_j\underline{\gamma}),\end{aligned}\tag{7.5}$$

which surely vanishes. ■

Thus, we can now define cocycles and coboundaries

$$Z^n_{[m]}(\Gamma,\mathfrak{X}) := \ker\delta^n,\quad B^n_{[m]}(\Gamma,\mathfrak{X}) := \operatorname{im}\delta^{n-1}$$

in $C^n(\Gamma,\mathfrak{X})$ and the cohomology group

$$H^n_{[m]}(\Gamma,\mathfrak{X}) := Z^n_{[m]}(\Gamma,\mathfrak{X})/B^n_{[m]}(\Gamma,\mathfrak{X})$$

for each $n = 0, 1, 2, \ldots$.

The following is a special case of the result by Gunning [10].

Proposition 7.5. *$H^n_{[m]}(\Gamma,\mathbb{C}[\tau]_{-m}) = 0$ if $n > 1$ and $m < 0$.*

Periodic cohomology. We define the groups $\widetilde{Z}^n_{[m]}(\Gamma,\mathfrak{X})$, $\widetilde{B}^n_{[m]}(\Gamma,\mathfrak{X})$ and $\widetilde{H}^n_{[m]}(\Gamma,\mathfrak{X})$ as follows:

$$\begin{aligned}\widetilde{C}^n(\Gamma,\mathfrak{X}) &:= \{f\in C^n(\Gamma,\mathfrak{X}) \mid f(T^{k_1N},\ldots,T^{k_nN}) = 0,\ k_1,\ldots,k_n\in\mathbb{Z}\},\\ \widetilde{Z}^n_{[m]}(\Gamma,\mathfrak{X}) &:= Z^n_{[m]}(\Gamma,\mathfrak{X})\cap\widetilde{C}^n(\Gamma,\mathfrak{X}),\\ \widetilde{B}^n_{[m]}(\Gamma,\mathfrak{X}) &:= B^n_{[m]}(\Gamma,\mathfrak{X})\cap\widetilde{C}^n(\Gamma,\mathfrak{X}),\\ \widetilde{H}^n_{[m]}(\Gamma,\mathfrak{X}) &:= \widetilde{Z}^n_{[m]}(\Gamma,\mathfrak{X})/\widetilde{B}^n_{[m]}(\Gamma,\mathfrak{X}).\end{aligned}$$

Proposition 7.6. *If $H^n_{[m]}(\Gamma,\mathfrak{X}) = 0$, then $\widetilde{H}^n_{[m]}(\Gamma,\mathfrak{X}) = 0$. In particular,*

$$\widetilde{H}^n_{[m]}(\Gamma,\mathbb{C}[\tau]_{-m}) = 0 \quad \text{if } n > 1 \text{ and } m < 0.$$

Proof. This is obvious because $Z^n_{[m]}(\Gamma,\mathfrak{X}) = B^n_{[m]}(\Gamma,\mathfrak{X})$ readily implies

$$\widetilde{Z}^n_{[m]}(\Gamma,\mathfrak{X}) = \widetilde{B}^n_{[m]}(\Gamma,\mathfrak{X})$$

by definition. ■

Problem 7.1. When $H^n_{[m]}(\Gamma,\mathfrak{X})$ vanishes? How about the periodic case $\widetilde{H}^n_{[m]}(\Gamma,\mathfrak{X})$?

Zero-dimensional cohomology. The group $H^0_{[m]}(\Gamma, \mathfrak{X})$ is easily described. Indeed, since $B^0_{[m]}(\Gamma, \mathfrak{X}) = \operatorname{im} \delta^{-1} = 0$, we have

$$H^0_{[m]}(\Gamma, \mathfrak{X}) = Z^0_{[m]}(\Gamma, \mathfrak{X}) = \{f \in \mathfrak{X} \mid f(\gamma\tau) = \chi(\gamma) j(\gamma, \tau)^m f(\tau) \text{ for all } \gamma \in \Gamma\}. \tag{7.6}$$

If $m < 0$, $\mathfrak{X} = \mathbb{C}[\tau]_{-m}$, χ is trivial and Γ is a congruent subgroup of level N, then we have

$$H^0_{[m]}(\Gamma, \mathbb{C}[\tau]_{-m}) \subset \Big\{ f \in \mathbb{C}[\tau]_{-m} \,\Big|\, f(\tau + N) = f(\tau), \\ f\Big(\frac{1}{N\tau + 1}\Big) = (N\tau + 1)^m f(\tau) \Big\} = 0,$$

from which it also follows that $\widetilde{H}^0_{[m]}(\Gamma, \mathbb{C}[\tau]_{-m}) = 0$.

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Kazufumi Kimoto
Department of Mathematical Sciences, University of the Ryukyus, 1 Senbaru, Nishihara, Okinawa 903-0213, Japan; kimoto@math.u-ryukyu.ac.jp

Masato Wakayama
NTT Institute for Fundamental Mathematics, 3-9-1 Midoricho, Musashiro, Tokyo 180-8585; and Institute of Mathematics for Industry, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan; wakayama@imi.kyushu-u.ac.jp