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Partial differential equations. — *Free energies and Fichera's quasi-static problem for materials with fading memory*, by GIOVAMBATTISTA AMENDOLA and MAURO FABRIZIO, presented by T. Manacorda.

ABSTRACT. — In this work we are concerned with the Fichera problem related to the quasi-static equation of viscoelasticity. By the use of the space derived from the minimum free energy, connected with a viscoelastic material, we prove a uniqueness and existence theorem in a new large space of data.

KEY WORDS: Free energy; fading memory; viscoelasticity.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 74A15, 74D05, 30E20.

1. INTRODUCTION

The Fichera problem for the integro-differential equation of viscoelasticity is connected with the study of the corresponding quasi-static problem. Fichera [11]–[13] observed that, in contrast to the case of linear elasticity, this problem cannot be resolved without providing the datum of the initial history and so working on the time interval $[0, \infty)$. Other authors [8], [9] studied the same problem on the interval $(-\infty, \infty)$, but they had to assign suitable decay conditions for $t \to \infty$ on the solutions, which is the same as giving the initial conditions.

In this paper, following the Fichera point of view, we prove that it is more convenient to work on new certain spaces more natural than the ones studied up to now. Therefore, we shall consider the topology connected with the minimum free energy, which provides the largest space $\mathcal{H}$ from which to take the initial history. On the other hand, the space of solutions is built by means of the dual space of the set $\mathcal{H}$ defined by the histories. For such a Fichera problem, by means of these new spaces, we are able to prove existence and uniqueness for a wide family of initial data.

2. FADING MEMORY AND THERMODYNAMICS

A viscoelastic material is defined by a constitutive equation which relates the stress tensor T to the deformation gradient F by means of a functional of the type

$$T(x,t) = \hat{T}(F^t(x)),$$

where $F^t(x,s) = F(x, t-s)$ for $s \in \mathbb{R}^+ = [0, \infty)$ is the history of the deformation gradient F, x denotes the position vector and t is the time.

In the linear case we have

$$T(x,t) = G_0(x)E(x,t) + \int_0^\infty G'(x,s)E^t(x,s)\,ds,$$

where $E = (\nabla u + (\nabla u)^T)/2$ is the infinitesimal strain tensor. Moreover, $G_0(x)$ and $-G'(x,s)$ are symmetric tensors, as also is

$$G_\infty(x) := G_0(x) + \int_0^\infty G'(x,s)\,ds.$$

Furthermore, the tensors $G_0(x)$ and $G_\infty(x)$ are positive definite. Of course, the study can be generalized to a non-linear constitutive equation of the type

$$T(x,t) = G_0(x)\Phi(F(x,t)) + \int_0^\infty G'(x,s)\Phi(F^t(x,s))\,ds, \tag{1}$$

where $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ is a non-linear, regular and odd function of F.

The dependence on x will be understood and not explicitly written.

A map $P : [0, d_p) \to \mathrm{Lin}$, piecewise continuous and defined by

$$P(\tau) = \dot{E}(\tau), \qquad \tau \in [0, d_p),$$

is called a *kinetic process* of duration $d_p \in \mathbb{R}^{++}$, a strictly positive real. By Π we denote the set of kinetic processes such that if $P_1, P_2 \in \Pi$ then $P_1 * P_2 \in \Pi$, where

$$P_1 * P_2(\tau) = \begin{cases} P_1(\tau) & \text{if } \tau \in [0, d_{p_1}), \\ P_2(\tau - d_{p_1}) & \text{if } \tau \in [d_{p_1}, d_{p_1} + d_{p_2}); \end{cases}$$

moreover, if $P \in \Pi$, then $P_{[\tau_1,\tau_2)} \in \Pi$ denotes the restriction of P to $[\tau_1, \tau_2) \subset [0, d_p)$. The state of the system will be denoted by σ, while Σ is the state space. The map $\hat{\rho} : \Sigma \times \Pi \to \Sigma$, which associates to any initial state $\sigma_i \in \Sigma$ and any process $P \in \Pi$ the final state $\sigma_f = \hat{\rho}(\sigma_i, P)$, is called the *evolution function*.

In general the stress $T(t)$ is a function of the state $\sigma(t)$ and of the process $P(t)$, that is,

$$T(t) = \hat{T}(\sigma(t), P(t)).$$

For a viscoelastic material the state is given by the history $\sigma(t) = E^t = (E(t), E_r^t)$, where $E_r^t(s) = E^t(s) - E(t)$ for $s > 0$ denotes the history relative to the instantaneous value of E, and the process by

$$P(\tau) = \dot{E}^P(\tau), \qquad \tau \in [0, d_p),$$

while the stress is now a function of the state $\sigma(t)$ only.

In the following we shall consider the work W done on the process P, defined as

$$W(\sigma, P) = \int_0^{d_p} \hat{T}(E^t, \dot{E}^P(\tau)) \cdot \dot{E}^P(\tau)\,d\tau.$$

DEFINITION 1 ([17]). *Two strain histories E_1^t, E_2^t are said to be* equivalent *if, for every process $\dot{E}^P : [0,\tau) \to \mathrm{Sym}$, they satisfy*

$$\hat{T}(E_1^t, \dot{E}^P(\tau)) = \hat{T}(E_2^t, \dot{E}^P(\tau)) \qquad \forall \tau > 0;$$

equivalently, as proved by Gentili [14], *two histories are said to be equivalent if for every process $\dot{E}^P : [0,\tau) \to \mathrm{Sym}$ and for every $\tau > 0$ we have*

$$W(E_1^t, \dot{E}^P) = W(E_2^t, \dot{E}^P).$$

For the linear case we have the following result.

THEOREM 2 ([5], [6]). *In the linear case, two histories* E_1^t, E_2^t *are equivalent if and only if* $E_1(t) = E_2(t)$ *and*

$$\int_0^\infty G'(s+\tau)E_1^t(s)\,ds = \int_0^\infty G'(s+\tau)E_2^t(s)\,ds \qquad \forall\tau \geq 0. \tag{2}$$

According to the definition of the state σ of a viscoelastic material, two couples $(E_1(t), E_{r_1}^t(\tau))$ and $(E_2(t), E_{r_2}^t(\tau))$ equivalent in the Noll sense are represented by the same state $\sigma(t)$ (see [16]). In other words, the state can be expressed by the pair

$$\sigma = (E(t), \tilde{I}^t(\tau))$$

where

$$\tilde{I}^t(\tau) = -\int_0^\infty G'(s+\tau)E^t(s)\,ds,$$

or equivalently by the function

$$I^t(\tau) = -G(\tau)E(t) + \tilde{I}^t(\tau).$$

We also use the representation

$$\check{I}^t(\tau) = -\int_0^\infty G'(s+\tau)[E^t(s) - E(t)]\,ds,$$

which can be chosen to characterize the state as well.

As shown in [9], from the Second Law of Thermodynamics we have the following restriction on the half-range Fourier sine transform of $G'(s)$:

$$G'_s(\omega) = \int_0^\infty G'(s)\sin\omega s\,ds < 0 \qquad \forall\omega \in \mathbb{R}^{++}, \tag{3}$$

or equivalently one on the half-range Fourier cosine transform:

$$\check{G}_c(\omega) = \int_0^\infty \check{G}(s)\cos\omega s\,ds > 0 \qquad \forall\omega \in \mathbb{R}, \tag{4}$$

where $\check{G} = G(s) - G_\infty$. From the Strong Form of the Second Law we also obtain (see [4])

$$G_\infty \geq 0; \tag{5}$$

in particular, for a solid we have $G_\infty > 0$, while for a fluid $G_\infty = 0$.

DEFINITION 3 ([14]). *A process* $\dot{E}^P : [0, d_P) \to \text{Sym}$ $(d_P < \infty)$ *is said to be a* finite work process *if*

$$W(0^\dagger, \dot{E}^P) = \int_0^{d_P} T(E^\tau, \dot{E}^P_{[0,\tau)}) \cdot \dot{E}^P(\tau)\,d\tau < \infty,$$

where $0^\dagger$ *denotes the null history,* $0^\dagger(s) = 0$ *for all* $s \in \mathbb{R}^+$, *and* E^τ *is the ensuing strain in* $(0, d_P]$.

Moreover, since the duration d_P of a process P is usually finite, we can define P on $\mathbb{R}^+$ by putting $P(\tau) = \dot{E}^P(\tau) = 0$ for all $\tau \geq d_P$ and, assuming also $E(\tau) = 0$ for all $\tau > d_P$, the work $W(0^\dagger, \dot{E}^P)$ in the linear case can be written in the form

$$\begin{aligned} W(0^\dagger, \dot{E}^P) &= \frac{1}{2} G_\infty E(d_P) \cdot E(d_P) \\ &\quad + \frac{1}{2} \int_0^\infty \int_0^\infty \check{G}(|\tau - s|) \dot{E}^P(\tau) \cdot \dot{E}^P(s)\, d\tau\, ds \\ &= \frac{1}{2} G_\infty E(d_P) \cdot E(d_P) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \check{G}_c(\omega) \dot{E}_+^P(\omega) \cdot \overline{\dot{E}_+^P(\omega)}\, d\omega, \end{aligned} \tag{6}$$

where $\dot{E}_+^P$ is the Fourier transform of $\dot{E}^P$.

The set of finite work processes is given by

$$\mathcal{H}_G = \left\{ \dot{E}^P;\ \int_0^\infty \int_0^\infty \check{G}(|\tau - s|) \dot{E}^P(\tau) \cdot \dot{E}^P(s)\, d\tau\, ds < \infty \right\}. \tag{7}$$

This set is a Hilbert space if the kernel $\check{G}$ satisfies the condition imposed on the constitutive equation by the Second Law of Thermodynamics (i.e. $\check{G}_c(\omega) > 0$) with the norm given by

$$\begin{aligned} \|\dot{E}^P\|^2 &= \int_0^\infty \int_0^\infty \check{G}(|\tau - s|) \dot{E}^P(\tau) \cdot \dot{E}^P(s)\, d\tau\, ds \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \check{G}_c(\omega) \dot{E}_+^P(\omega) \cdot \overline{\dot{E}_+^P(\omega)}\, d\omega. \end{aligned}$$

The domain of definition of the admissible states is the set of all the strain histories rendering the work well defined when the process belongs to $\mathcal{H}_G$.

The work done on a process P, of duration $d_P < \infty$, on supposing that $P(\tau) = 0$ for $\tau \geq d_P$ and $E(\tau) = 0$ for $\tau > d_P$, assumes the form

$$\begin{aligned} W(I^t, \dot{E}^P) &= \frac{1}{2} G_\infty E(d_P) \cdot E(d_P) \\ &\quad + \frac{1}{2} \int_0^\infty \int_0^\infty \check{G}(|\tau - s|) \dot{E}^P(\tau) \cdot \dot{E}^P(s)\, d\tau\, ds \\ &\quad - \frac{1}{2} G_\infty E(t) \cdot E(t) - \int_0^\infty \check{I}^t(\tau) \cdot \dot{E}^P(\tau)\, d\tau < \infty. \end{aligned} \tag{8}$$

Therefore the set of admissible states $\check{I}^t(\tau)$ is given by the dual $\mathcal{H}'_G$ of $\mathcal{H}_G$, namely

$$\mathcal{H}'_G = \left\{ \check{I}^t(\tau);\ \int_0^\infty \check{I}^t(\tau) \cdot \dot{E}^P(\tau) d\tau < \infty,\ \forall \dot{E}^P \in \mathcal{H}_G \right\}.$$

DEFINITION 4. *A function $\psi : \mathcal{D}_\psi \to \mathbb{R}^+$ is called a* free energy *if*

(a) *the domain $\mathcal{D}_\psi \subset \mathcal{D}$ is such that, for every $\sigma_1 \in \mathcal{D}_\psi$ and $P \in \Pi$, the state $\sigma = \hat{\rho}(\sigma_1, P) \in \mathcal{D}_\psi$,*
(b) *if $\sigma^\dagger = (0, 0^t)$ is the zero state, then $\sigma^\dagger \in \mathcal{D}_\psi$ and $\psi(\sigma^\dagger) = 0$,*

(c) *for any pair* $\sigma_1, \sigma_2 \in \mathcal{D}_\psi$ *and* $P \in \Pi$ *such that* $\hat{\rho}(\sigma_1, P) = \sigma_2$ *we have*

$$\psi(\sigma_2) - \psi(\sigma_1) \leq W(\sigma_1, P). \tag{9}$$

In linear viscoelasticity, many free energies can be considered [2]. The family $\mathcal{F}$ of free energies is a convex set; $\mathcal{F}$ has a minimum and a maximum element, ψ_m and ψ_M. In particular, we recall that there is an intermediate free energy called the Graffi–Volterra free energy, given by

$$\psi_G(E^t) = \frac{1}{2}G_\infty E(t)\cdot E(t) - \frac{1}{2}\int_0^\infty G'(s)[E^t(s)-E(t)]\cdot[E^t(s)-E(t)]\,ds, \tag{10}$$

where $G'(s) < 0,\ G''(s) \geq 0$.

The maximum free energy, considered in [7], has the following expression:

$$\begin{aligned}\psi_M(E^t) &= \frac{1}{2}G_\infty E(t)\cdot E(t)\\ &\quad + \frac{1}{2}\int_0^\infty\int_0^\infty G_{12}(|s_1-s_2|)[E^t(s_1)-E(t)]\cdot[E^t(s_2)-E(t)]\,ds_1\,ds_2,\end{aligned} \tag{11}$$

where $G_{12}(|s_1 - s_2|) = \frac{\partial^2}{\partial s_1 \partial s_2} G(|s_1 - s_2|)$.

Finally, the minimum free energy was found by Breuer and Onat [1] in 1964 in the following form:

$$\begin{aligned}\psi_m(\dot{E}_m) &= \frac{1}{2}G_\infty E(t)\cdot E(t)\\ &\quad + \frac{1}{2}\int_0^\infty\int_0^\infty \breve{G}(|s_1-s_2|)\dot{E}_m(s_1)\cdot\dot{E}_m(s_2)\,ds_1\,ds_2,\end{aligned} \tag{12}$$

where $\dot{E}_m$ is the optimal process which yields the maximum recoverable work, but it is not a functional of the history $E^t(s)$, or of the state $I^t = (E(t), \tilde{I}^t(\tau))$.

Golden [15] was able to give a representation of the minimum free energy in terms of $\breve{I}^t(\tau)$ as

$$\begin{aligned}\psi_m(I^t) &= \frac{1}{2}G_\infty E(t)\cdot E(t)\\ &\quad + \frac{1}{2}\int_0^\infty\int_0^\infty H(\tau,\tau')\breve{I}^t(\tau)\cdot\breve{I}^t(\tau')\,d\tau\,d\tau',\end{aligned} \tag{13}$$

where $H(\tau, \tau')$ is a suitable kernel depending on $\breve{G}$; moreover he proved that $\mathcal{H}'_G$ is the domain on which the minimum free energy is defined.

3. Quasi-static problem in linear viscoelasticity

Because the constitutive equation of linear viscoelasticity can be written in the form

$$T(x,t) = G(x,t)\nabla u(x,0) + \int_0^t G(x,t-\tau)\nabla\dot{u}(x,\tau)\,d\tau - \tilde{I}^0(x,t), \tag{14}$$

the equation of motion becomes

$$\rho\ddot{u}(x,t) = \nabla\cdot\int_0^t G(x,t-\tau)\nabla\dot{u}(x,\tau)\,d\tau + b(x,t), \tag{15}$$

where

$$b(x,t) = f(x,t) + \nabla\cdot[-\tilde{I}^0(x,t) + G(x,t)\nabla u(x,0)]$$

is a given function in $\Omega\times\mathbb{R}^+$. Letting $v=\dot{u}$ we can write the differential equation (15) in the form (see [3])

$$\rho\dot{v}(x,t) = \nabla\cdot\int_0^t G(x,t-\tau)\nabla v(x,\tau)\,d\tau + b(x,t) \quad \text{on } \Omega\times\mathbb{R}^+. \tag{16}$$

This is a differential equation in the unknown function v, to which we must associate initial and boundary conditions. For this purpose, we note that in the general case we have initial conditions expressed by a known function $v(x,0)=v_0(x)$; however, this function can always be supposed to be zero by changing the sources and, therefore, we assume the following data:

$$v(x,0)=0 \quad \forall x\in\bar{\Omega}, \qquad v(x,t)=0 \quad \forall(x,t)\in\partial\Omega\times\mathbb{R}^+. \tag{17}$$

Let w be a smooth vector function on $\bar{\Omega}\times\mathbb{R}^+$, vanishing on the boundary $\partial\Omega$ and such that $w(x,T)=0$ for all $x\in\Omega$. Inner multiply (16) by w and integrate on $\Omega\times[0,T]$ to obtain

$$\begin{aligned}&\int_0^T\int_\Omega \rho\dot{v}(x,t)\cdot w(x,t)\,dx\,dt\\ &\quad+\int_0^T\int_\Omega\int_0^t G(x,t-\tau)\nabla v(x,t)\cdot\nabla w(x,t)\,d\tau\,dx\,dt\\ &\quad-\int_0^T\int_\Omega I^0(x,t)\cdot\nabla w(x,t)\,dx\,dt - \int_0^T\int_\Omega f(x,t)\cdot w(x,t)\,dx\,dt = 0,\end{aligned} \tag{18}$$

which corresponds to the Virtual Work Principle and leads to a new definition of weak solution.

Now, we are in a position to consider the quasi-static problem, which is connected with equations (18) and (17), on the time domain $(0,\infty)$, and is given in weak sense by the system

$$\begin{aligned}&\int_0^\infty\int_\Omega\int_0^t G(x,t-\tau)\nabla v(x,\tau)\cdot\nabla w(x,t)\,d\tau\,dx\,dt\\ &\qquad-\int_0^\infty\int_\Omega [f(x,t)\cdot w(x,t) + I^0(x,t)\cdot\nabla w(x,t)]\,dx\,dt = 0,\end{aligned} \tag{19}$$

$$v(x,0)=0 \quad \forall x\in\bar{\Omega}, \qquad v(x,t)=0 \quad \forall(x,t)\in\partial\Omega\times\mathbb{R}^+. \tag{20}$$

So, in order to obtain a rigorous definition of solution according to the equality (19), we need to fix the function space for the functions $v(x,t)$ and $w(x,t)$. We introduce the spaces

$$\mathcal{H}_G(\mathbb{R}^+; H_0^1(\Omega)) = \Big\{ v \in L^2_{\mathrm{loc}}(\mathbb{R}^+; H_0^1(\Omega)); \\ \int_0^\infty \int_0^\infty \int_\Omega G(x, |\tau - \tau'|) \nabla v(x, \tau') \cdot \nabla v(x, \tau)\, dx\, d\tau'\, d\tau < \infty \Big\},$$

while the states $I^0(x,t)$ will be elements of the space $\mathcal{H}'_G(\mathbb{R}^+; H_0^1(\Omega))$.

The space $\mathcal{H}_G(\mathbb{R}^+; H_0^1(\Omega))$ is a Hilbert space with respect to the inner product

$$(v_1, v_2)_{\mathcal{H}_G} = \int_0^\infty \int_0^\tau \int_\Omega [G_\infty(x) + \check{G}(x, \tau - \tau')] \\ \times [\nabla v_1(x, \tau') \cdot \nabla v_2(x, \tau) + \nabla v_1(x, \tau) \cdot \nabla v_2(x, \tau')]\, dx\, d\tau'\, d\tau.$$

DEFINITION 5. *A function $v \in \mathcal{H}_G(\mathbb{R}^+; H_0^1(\Omega))$ is called a* weak solution *of the quasi-static problem related to equations* (16)–(17) *with data $f \in \mathcal{H}'_G(\mathbb{R}^+; H^{-1}(\Omega))$ and $I^0 \in \mathcal{H}'_G(\mathbb{R}^+; H_0^1(\Omega))$ if it satisfies the identity* (19) *for any $w \in \mathcal{H}_G(\mathbb{R}^+; H_0^1(\Omega))$.*

Now we are in a position to state the following theorem.

THEOREM 6. *Assume that the kernel $\check{G}(x, \cdot) \in L^1(\mathbb{R}^+; L^\infty(\Omega))$ satisfies the thermodynamic condition $\check{G}_c(x, \omega) > 0$ for any $(x, \omega) \in \bar{\Omega} \times \mathbb{R}$. Then there exists a unique weak solution $v \in \mathcal{H}_G(\mathbb{R}^+; H_0^1(\Omega))$ of the problem* (16)–(17) *for any $I^0 \in \mathcal{H}'_G(\mathbb{R}^+; H_0^1(\Omega))$ and $f \in L^2(\mathbb{R}^+; H^{-1}(\Omega))$.*

PROOF. Consider the Fourier transform of the system (16)–(17) under the quasi-static hypothesis

$$\nabla \cdot [G_+(x, \omega) \nabla v_+(x, \omega)] = -f_+(x, \omega) + \nabla \cdot I_+^0(x, \omega), \tag{21}$$

$$v_+(x, \omega)|_{\partial\Omega} = 0. \tag{22}$$

For any fixed $\omega \in \mathbb{R}$, the sesquilinear form

$$a(v_+(x, \omega), w_+(x, \omega)) = \int_\Omega [G_+(x, \omega) \nabla v_+(x, \omega) \cdot \overline{\nabla w_+(x, \omega)}]\, dx \tag{23}$$

is a bounded and coercive form in $H_0^1(\Omega)$. Indeed, it is easy to verify that it is bounded. In order to obtain the coercivity, we have to prove that for any fixed $\omega \in \mathbb{R}$ the inequality

$$|a(v_+(\omega), v_+(\omega))| \geq k(\omega) \|v_+(\omega)\|_{H_0^1}$$

holds for all $v_+ \in H_0^1(\Omega)$, where $k(\omega)$ is a positive constant. By the definition (23), since $G_+(x, \omega) = G_c(x, \omega) - iG_s(x, \omega)$, for any $\omega \in \mathbb{R}$ we have

$$|a(v_+(\omega), v_+(\omega))| \geq \int_\Omega G_c(x, \omega) \nabla v_+(x, \omega) \cdot \overline{\nabla v_+(x, \omega)}\, dx \geq k(\omega) \|v_+(\omega)\|_{H_0^1}$$

where $k(\omega) = \inf\{|G_c(x, \omega)|;\ x \in \Omega\}$.

This proves that for any fixed $\omega \in \mathbb{R}$, the problem (21)–(22) admits a solution $v_+ \in H_0^1(\Omega)$ if the difference

$$F_+(x,\omega) = f_+(x,\omega) - \nabla \cdot I_+^0(x,\omega)$$

is an element of $H^{-1}(\Omega)$.

Now, in order to study the behavior of v_+ when $\omega \to \infty$, we apply the Parseval theorem to (18); after an integration by parts, we get

$$\begin{aligned}(24)\qquad &\int_{-\infty}^{\infty}\int_{\Omega}\{\rho v_+(x,\omega)\cdot\overline{(i\omega w_+(x,\omega)-w_0(x))}\\ &\qquad - G_+(x,\omega)\nabla v_+(x,\omega)\cdot\overline{\nabla w_+(x,\omega)}\}\,dx\,d\omega\\ &= -\int_{-\infty}^{\infty}\int_{\Omega}\{f_+(x,\omega)\cdot\overline{w_+(x,\omega)} + I_+^0(x,\omega)\cdot\overline{\nabla w_+(x,\omega)}\}\,dx\,d\omega.\end{aligned}$$

It is known that for any vector function $f \in H^{-1}(\Omega)$ there exists a tensor function $A \in L^2(\Omega)$ such that for all $v \in H^1(\Omega)$,

$$\int_0^{\infty}\int_{\Omega} f(x,t)\cdot v(x,t)\,dx = \int_0^{\infty}\int_{\Omega} A(x,t)\cdot\nabla v(x,t)\,dx;$$

moreover, we recall that in our case

$$v(x,0) = \frac{1}{\pi}\int_{-\infty}^{\infty} v_+(x,\omega)\,d\omega = 0.$$

Hence, equation (24) can be written for $w(x,t) = v(x,t)$, that is,

$$\begin{aligned}(25)\qquad &\int_{-\infty}^{\infty}\int_{\Omega}\{-\rho v_+(x,\omega)\cdot\overline{i\omega v_+(x,\omega)} + G_+(x,\omega)\nabla v_+(x,\omega)\cdot\overline{\nabla v_+(x,\omega)}\}\,dx\,d\omega\\ &= \int_{-\infty}^{\infty}\int_{\Omega}[A_+(x,\omega) + I_+^0(x,\omega)]\cdot\overline{\nabla v_+(x,\omega)}\,dx\,d\omega.\end{aligned}$$

We note that the first term in the integral on the left hand side of this equation is an odd function of ω and, therefore, its integral over $\mathbb{R}$ vanishes; moreover, the integral of the second term reduces to

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{\Omega} G_+(x,\omega)\nabla v_+(x,\omega)\cdot\overline{\nabla v_+(x,\omega)}\,dx\,d\omega\\ &\qquad = \int_{-\infty}^{\infty}\int_{\Omega} G_c(x,\omega)\nabla v_+(x,\omega)\cdot\overline{\nabla v_+(x,\omega)}\,dx\,d\omega,\end{aligned}$$

since G is symmetric, $G_+ = G_c - iG_s$ and G_s is an odd function.

Application of the Schwarz inequality to the integral on the right-hand side of (25) yields

$$(26)\quad \int_{-\infty}^{\infty}\int_{\Omega} G_+(x,\omega)\nabla v_+(x,\omega)\cdot\overline{\nabla v_+(x,\omega)}\,dx\,d\omega$$
$$\le \left\{\int_{-\infty}^{\infty}\int_{\Omega} G_+^{-1}(x,\omega)[A_+(x,\omega)+I_+^0(x,\omega)]\cdot\overline{[A_+(x,\omega)+I_+^0(x,\omega)]}\,dx\,d\omega\right\}^{1/2}$$
$$\times\left\{\int_{-\infty}^{\infty}\int_{\Omega} G_+(x,\omega)\nabla v_+(x,\omega)\cdot\overline{\nabla v_+(x,\omega)}\,dx\,d\omega\right\}^{1/2},$$

whence it follows that

$$(27)\quad \int_{-\infty}^{\infty}\int_{\Omega} G_c(x,\omega)\nabla v_+(x,\omega)\cdot\overline{\nabla v_+(x,\omega)}\,dx\,d\omega$$
$$\le \int_{-\infty}^{\infty}\int_{\Omega} G_+^{-1}(x,\omega)[A_+(x,\omega)+I_+^0(x,\omega)]\cdot\overline{[A_+(x,\omega)+I_+^0(x,\omega)]}\,dx\,d\omega.$$

Therefore, if $A + I^0 \in \mathcal{H}'_G(\mathbb{R}^+; L^2(\Omega))$ we have $v \in \mathcal{H}_G(\mathbb{R}^+; H_0^1(\Omega))$. From (27) and by the Poincaré theorem, it follows that there exists a constant $C(\Omega)$ such that

$$(28)\quad C(\Omega)\int_{-\infty}^{\infty}\int_{\Omega} G_c(x,\omega)v_+(x,\omega)\cdot\overline{v_+(x,\omega)}\,dx\,d\omega$$
$$\le \int_{-\infty}^{\infty}\int_{\Omega} G_+^{-1}(x,\omega)[A_+(x,\omega)+I_+^0(x,\omega)]\cdot\overline{[A_+(x,\omega)+I_+^0(x,\omega)]}\,dx\,d\omega.$$

Hence, if $f \in \mathcal{H}'_G(\mathbb{R}^+; H^{-1}(\Omega))$ and $I^0 \in \mathcal{H}'_G(\mathbb{R}^+; L^2(\Omega))$, then we see that the function v belongs to $\mathcal{H}_G(\mathbb{R}^+; H^1(\Omega))$ and it is a virtual work solution of the problem (16)–(17) in the sense of Definition 1.1.

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G. Amendola
Dipartimento di Matematica Applicata ”U.Dini”
Facoltà di Ingegneria
Via Diotisalvi, 2
56126 PISA, Italy
amendola@dma.unipi.it

M. Fabrizio
Dipartimento di Matematica
Università degli Studi di Bologna
Piazza di Porta S. Donato, 5
40127 BOLOGNA, Italy
fabrizio@morgana.dm.unibo.it