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Partial differential equations. — *Periodic solutions of Birkhoff–Lewis type for the nonlinear wave equation*, by LUCA BIASCO and LAURA DI GREGORIO, presented by A. Ambrosetti.

ABSTRACT. — We prove existence and multiplicity of small amplitude periodic solutions with large period for the wave equation with small "mass". Such solutions bifurcate from resonant finite-dimensional invariant tori of the fourth order Birkhoff normal form of the associated hamiltonian system. The number of geometrically distinct solutions and their minimal periods tend to infinity when the "mass" tends to zero.

KEY WORDS: Nonlinear wave equation; infinite-dimensional hamiltonian systems; periodic solutions; Birkhoff normal form.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 34C25, 35L05, 37K50.

INTRODUCTION AND MAIN RESULT

We outline in this note some recent results on time periodic solutions with long period for the nonlinear wave equation, obtained in [5], to which we refer for complete proofs.

Let us consider the nonlinear wave equation on the interval $[0, \pi]$ with Dirichlet boundary conditions

$$\begin{cases} u_{tt} - u_{xx} + \mu u + f(u) = 0, \\ u(t, 0) = u(t, \pi) = 0, \end{cases} \tag{1}$$

where $\mu > 0$ and f is a real analytic, odd function of the form $f(u) = \sum_{k\geq 3} f_k u^k$, $f_3 \neq 0$.

Equation (1) can be studied as an infinite-dimensional hamiltonian system. If we set $v = u_t$ the Hamiltonian is

$$H(u, v) = \int_0^\pi \left(\frac{v^2}{2} + \frac{u_x^2}{2} + \mu \frac{u^2}{2} + g(u) \right) dx,$$

where $g(u) = \int_0^u f(s)\, ds$. Introduce coordinates $q = (q_1, q_2, \ldots)$, $p = (p_1, p_2, \ldots)$ through the relations

$$u(x) = \sum_{i\geq 1} \frac{q_i}{\sqrt{\omega_i}} \chi_i(x), \qquad v(x) = \sum_{i\geq 1} \sqrt{\omega_i}\, p_i\, \chi_i(x),$$

where $\chi_i(x) := \sqrt{2/\pi} \sin ix$ and $\omega_i := \sqrt{i^2 + \mu}$; then the Hamiltonian takes the form

$$H = \Lambda + G(q) = \frac{1}{2} \sum_{i\geq 1} \omega_i (q_i^2 + p_i^2) + G(q), \tag{2}$$

where

$$G(q) := \int_0^\pi g\Big(\sum_{i\geq 1} \frac{q_i}{\sqrt{\omega_i}}\,\chi_i(x)\Big)\,dx = \frac{f_3}{4}\sum_{i,j,k,l} G_{ijkl}\, q_i q_j q_k q_l + \text{h.o.t.}$$

and $G_{ijkl} := (\omega_i\omega_j\omega_k\omega_l)^{-1/2}\int_0^\pi \chi_i\chi_j\chi_k\chi_l\,dx$. The origin is an elliptic equilibrium point for the Hamiltonian H.

A classical method to find orbits with long period, close to an elliptic equilibrium point, was given by Birkhoff and Lewis for finite-dimensional systems in [6] (see also [8]). Their procedure consists in putting the system in fourth order Birkhoff normal form: the truncated Hamiltonian obtained by neglecting the five or higher order terms is integrable. If the so called “twist” condition on the action-to-frequency map holds, there exist infinitely many resonant tori on which the motion of the truncated Hamiltonian is periodic. By the Implicit Function Theorem and topological arguments, Birkhoff and Lewis showed the existence of a sequence of resonant tori accumulating at the origin with the property that at least two periodic orbits bifurcate from each of them.

In [5] we adapt the Birkhoff–Lewis procedure to the nonlinear wave equation (1). This approach was recently carried out in [1] for the beam equation and the NLS (see Remark 2 below for comparison).

For hamiltonian PDEs, one meets two difficulties that do not appear in the finite-dimensional case: the generalization of the Birkhoff normal form and a small divisors problem.

Concerning the first difficulty we consider only a Birkhoff “seminormal form”. We fix a *finite* subset of indices $\mathcal{I} := \{i_1, \ldots, i_N\} \subset \mathbb{N}$, $1 \leq i_1 < \cdots < i_N$, and we put the Hamiltonian H in (2) in the seminormal form

$$H = \Lambda + \bar{G} + \hat{G} + K$$

where $\bar{G} + \hat{G}$ is the fourth order term with $\bar{G}$ depending only on $p_i^2 + q_i^2$, $i \in \mathbb{N}^+$, $\hat{G}$ depending only on p_i, q_i, $i \notin \mathcal{I}$, and K is a sixth or higher order term. The truncated Hamiltonian $\Lambda + \bar{G} + \hat{G}$ possesses the $2N$-dimensional invariant manifold $\{p_i = q_i = 0,$ $i \notin \mathcal{I}\}$, which is foliated by N-dimensional invariant tori. Since $f_3 \neq 0$, $\bar{G}$ satisfies the “twist” property and, therefore, the linear frequencies of such tori form an open subset of $\mathbb{R}^N$. We focus on completely resonant tori which are foliated by periodic orbits with frequency $\tilde{\omega} = (\tilde{\omega}_{i_1}, \ldots, \tilde{\omega}_{i_N})$. Such lower dimensional tori are highly degenerate. Hence, in order to show the persistence of periodic orbits for the whole Hamiltonian H, we have to impose some nondegeneracy conditions to avoid resonances between the torus frequencies and the normal ones. This is exactly the point in which the small divisors problem appears (see p. 30).

The estimate on the small divisors is the crucial step. To overcome this problem we impose the strong condition (16) on the small divisors, avoiding KAM analysis (see Remark 1). For this reason we consider only periods T which are integer multiples of π (compare with the classical variational approach of Rabinowitz, Brezis, Nirenberg etc.) and we need

$$\mu^2 T \ll 1. \tag{3}$$

We consider the "mass" $\mu > 0$ as a *small* parameter. For $\mu \to 0$ the frequencies $\omega_i \to i$, and the Birkhoff seminormal form *degenerates*, in the sense that its domain of definition shrinks to zero while the remainder term K blows up, see Proposition 1 (note that for $\mu = 0$ equation (1) becomes completely resonant).

Next we perform a Lyapunov–Schmidt reduction as in [3], [1]. We solve the range equation by the Contraction Mapping Theorem using the above estimate on the small divisors and controlling the blow-up of the remainder term K for μ going to zero (see p. 32). The bifurcation equation is solved by variational arguments.

THEOREM 1. *Let $N \geq 2$ and fix $\mathcal{I} := \{i_1, \ldots, i_N\} \subset \mathbb{N}^+$. Then there exists a constant $0 < c = c(\mathcal{I}) < 1$ such that, if $0 < \mu \leq c$, there exist at least c/μ geometrically distinct smooth periodic solutions $u(t,x)$ of* (1) *with minimal period $T^{\min} \in \pi\,\mathbb{Q}$,*

$$\frac{c}{\mu} \leq T^{\min} \leq \frac{c}{\mu^2},$$

satisfying

$$\sup_{t\in\mathbb{R},\, x\in[0,\pi]} \left| u(t,x) - \mu \sum_{i\in\mathcal{I}} a_i \cos(\tilde{\omega}_i t + \varphi_i) \sin ix \right| \leq c^{-1}\mu^2, \tag{4}$$

where $a_i \geq c$, $\varphi_i \in \mathbb{R}$ and $\tilde{\omega}_i \in \mathbb{R}$ with

$$|\tilde{\omega}_i - \omega_i| \leq c^{-1}\mu^2 \tag{5}$$

for any $i \in \mathcal{I}$.

The solutions u are actually analytic in the spatial variable and C^∞ in time. One could obtain estimates similar to (4) for the derivatives of u of every order.

Theorem 1 is the first existence result on periodic solutions with *large minimal period* for the autonomous nonlinear wave equation (for different-type existence results of periodic solutions with large minimal period in the forced case see [10]).

Other authors, e.g. Kuksin, Wayne, Craig, Bourgain, Bambusi, constructed families of periodic solutions of hamiltonian PDEs being inspired by the Lyapunov Center Theorem. These periodic solutions are the continuation of one linear mode, the amplitudes of the other modes being much smaller (unlike the resonant case $\mu = 0$ considered in [2], [4], [7]). On the other hand, the periodic solutions of Theorem 1 involve $N \geq 2$ modes, oscillating with the same order of magnitude. These orbits are a *strictly nonlinear phenomenon*.

SCHEME OF THE PROOF

We introduce complex coordinates

$$\mathrm{z}_i = \frac{1}{\sqrt{2}}(q_i + \mathrm{i}p_i), \qquad \bar{\mathrm{z}}_i = \frac{1}{\sqrt{2}}(q_i - \mathrm{i}p_i), \tag{6}$$

where $\mathrm{i} = \sqrt{-1}$, living in the complex Hilbert algebra

$$\ell^{a,s} := \Big\{ \mathrm{z} = (\mathrm{z}_1, \mathrm{z}_2, \ldots) : \mathrm{z}_i \in \mathbb{C},\ i \ge 1,\ \|\mathrm{z}\|^2_{a,s} = \sum_{i\ge 1} |\mathrm{z}_i|^2 i^{2s} e^{2ai} < \infty \Big\}$$

(here $a > 0$ and $s > 1/2$). The symplectic structure is $-\mathrm{i}\sum_{i\ge1} d\mathrm{z}_i \wedge d\bar{\mathrm{z}}_i = \sum_{i\ge1} dp_i \wedge dq_i$, and the Hamiltonian in (2) reads

$$H = \Lambda + G = \sum_{i\ge1} \omega_i |\mathrm{z}_i|^2 + G(\mathrm{z}, \bar{\mathrm{z}}). \tag{7}$$

Hamilton's equations are $\dot{\mathrm{z}} = \mathrm{i}\partial_{\bar{\mathrm{z}}} H$, $\dot{\bar{\mathrm{z}}} = -\mathrm{i}\partial_{\mathrm{z}} H$. The Hamiltonian H is real analytic in the sense that H is a function of z and $\bar{\mathrm{z}}$, real analytic in the real and imaginary part of z. Denote by $A(\ell^{a,s}, \ell^{a,s+1})$ the class of all real analytic maps from some neighborhood of the origin in $\ell^{a,s}$ into $\ell^{a,s+1}$. Since $\omega_i \sim i$, the hamiltonian vector field X_G belongs to $A(\ell^{a,s}, \ell^{a,s+1})$, that is, the nonlinearity is *smoothing of order* 1.

Given a finite multi-index $\mathcal{I}$, we will denote by $\hat{\mathrm{z}}$ the infinite vector obtained by deleting from $\mathrm{z} = (\mathrm{z}_1, \mathrm{z}_2, \ldots)$ its $\mathcal{I}$-components, namely $\hat{\mathrm{z}} := (\ldots, \mathrm{z}_{i_1-1}, \mathrm{z}_{i_1+1}, \ldots, \mathrm{z}_{i_j-1}, \mathrm{z}_{i_j+1}, \ldots, \mathrm{z}_{i_N-1}, \mathrm{z}_{i_N+1}, \ldots) = (\mathrm{z}_i)_{i\in\mathcal{I}^c}$, where $\mathcal{I}^c := \mathbb{N}^+ \setminus \mathcal{I}$.

Following [9] we put the Hamiltonian H in Birkhoff seminormal form. We explicitly investigate its dependence on μ for μ small.

PROPOSITION 1 (Birkhoff seminormal form). *Let* $0 < \mu < 1$, $\mathcal{I} \subset \mathbb{N}^+$. *There exists a real analytic, close to the identity, symplectic change of coordinates* $\mathrm{z}_* \mapsto \mathrm{z}$ *defined in* $B_r \subset \ell^{a,s}$ *into* $B_{2r} \subset \ell^{a,s}$ *with* $r := \mathrm{const}\sqrt{\mu}$, *satisfying* $\|\mathrm{z} - \mathrm{z}_*\|_{a,s+1} = O(\|\mathrm{z}_*\|^3_{a,s}/\mu)$, *transforming the Hamiltonian* $H = \Lambda + G$ *in* (7) *into seminormal form up to order six. That is,* $H_* = \Lambda + \bar{G} + \hat{G} + K$, *where*

$$X_{\bar{G}}, X_{\hat{G}}, X_K \in A(\ell^{a,s}, \ell^{a,s+1}), \tag{8}$$

$\bar{G} = \frac{1}{2}\sum_{i \text{ or } j \in \mathcal{I}} \bar{G}_{ij} |\mathrm{z}_{*i}|^2 |\mathrm{z}_{*j}|^2$, $|\hat{G}| = O(\|\hat{\mathrm{z}}_*\|^4_{a,s})$ *and* $|K| = O(\|\mathrm{z}_*\|^6_{a,s}/\mu)$.

Note that $\Lambda + \bar{G}$ is integrable with integrals $|\mathrm{z}_{*i}|^2$, $i = 1, 2, \ldots$. Moreover, although the fourth order term $\hat{G}$ is not integrable, it only depends on $\hat{\mathrm{z}}_* = (\mathrm{z}_{*i})_{i\in\mathcal{I}^c}$, that is, it is independent of the $\mathcal{I}$-modes. We also remark that this normal form *degenerates* as μ goes to zero.

Since we are looking for small amplitude solutions it is convenient to introduce the small parameter $\eta := \tau^{-1/2}$, where

$$\tau := \frac{T}{2\pi},$$

and to perform the rescaling

$$\mathrm{z}_* =: \eta z, \quad \bar{\mathrm{z}}_* =: \eta \bar{z}, \quad H_* \mapsto \eta^{-2} H_* =: \mathcal{H},$$

after which the Hamiltonian reads

$$\mathcal{H}(z, \bar{z}; \eta) = \Lambda + \eta^2(\bar{G} + \hat{G}) + \eta^4 \widetilde{K}(z, \bar{z}; \eta), \qquad \|z\|_{a,s}, \|\bar{z}\|_{a,s} \le \mathrm{const}\frac{\sqrt{\mu}}{\eta},$$

where $\widetilde{K}(z, \bar{z}; \eta) := \eta^{-2} K(\eta z, \eta \bar{z})$, $|\widetilde{K}| = O(\|z\|^6_{a,s}/\mu)$.

We now introduce action-angle variables $(I, \phi) \in \mathbb{R}_+^N \times \mathbb{T}^N$ on the $\mathcal{I}$-modes by the following symplectic change of coordinates: $z_i := \sqrt{I_i}(\cos\phi_i - \mathrm{i}\sin\phi_i)$, $\bar{z}_i := \sqrt{I_i}(\cos\phi_i + \mathrm{i}\sin\phi_i)$, for $i \in \mathcal{I}$. The action $I := (I_i)_{i\in\mathcal{I}}$, $I_i := z_i\bar{z}_i$, is defined for $|I| \le \text{const}\,\mu/\eta^2$. We note that $\sum_{i\in\mathcal{I}} dI_i \wedge d\phi_i = -\mathrm{i}\sum_{i\in\mathcal{I}} dz_i \wedge d\bar{z}_i = \sum_{i\in\mathcal{I}} dp_i \wedge dq_i$ and the *phase space* is

$$\mathcal{P}_{a,s} := \mathbb{R}_+^N \times \mathbb{T}^N \times \ell^{a,s} \ni (I, \phi, \hat{z}). \tag{9}$$

In these variables the Hamiltonian becomes

$$\begin{aligned}\widetilde{\mathcal{H}}(I, \phi, \hat{z}, \bar{\hat{z}}; \eta) = \omega \cdot I + \Omega \cdot \hat{z}\bar{\hat{z}} + \eta^2\left[\frac{1}{2}(AI, I) + (BI, \hat{z}\bar{\hat{z}}) + \hat{G}(\hat{z}, \bar{\hat{z}})\right] \\ + \eta^4 \widetilde{K}(I, \phi, \hat{z}, \bar{\hat{z}}; \eta),\end{aligned} \tag{10}$$

where

$$\begin{aligned}\omega &:= (\omega_{i_1}, \ldots, \omega_{i_N}),\\ \Omega &:= (\ldots, \omega_{i_1-1}, \omega_{i_1+1}, \ldots, \omega_{i_j-1}, \omega_{i_j+1}, \ldots, \omega_{i_N-1}, \omega_{i_N+1}, \ldots),\end{aligned}$$

$\Omega \cdot \hat{z}\bar{\hat{z}}$ is short for $\sum_{i\in\mathcal{I}^c} \omega_i \hat{z}_i \bar{\hat{z}}_i$ and the matrices $A \in \mathrm{Mat}(N \times N)$, $B \in \mathrm{Mat}(\infty \times N)$ can be explicitly evaluated. In particular one can verify (see [9]) that, since $f_3 \neq 0$, the "twist" property $\det A \neq 0$ holds.

Resonant tori

We will find periodic solutions of the Hamiltonian $\widetilde{\mathcal{H}}$ in (10) close to the ones of the integrable Hamiltonian

$$\omega \cdot I + \Omega \cdot \hat{z}\bar{\hat{z}} + \eta^2\left[\frac{1}{2}(AI, I) + (BI, \hat{z}\bar{\hat{z}})\right], \tag{11}$$

in which $\hat{G}$ and $\widetilde{K}$ have been neglected. The manifold $\{\hat{z} = 0\}$ is invariant for the Hamiltonian in (11) and it is completely filled up by the N-dimensional invariant tori

$$\mathcal{T}(I_0) := \{I = I_0,\ \phi \in \mathbb{T}^N,\ \hat{z} = 0\},$$

on which the motion is linear with frequencies

$$\tilde{\omega} := \tilde{\omega}(I_0, \tau) = \omega + \eta^2 A I_0. \tag{12}$$

Such tori are linearly stable with *shifted elliptic frequencies*

$$\tilde{\Omega}_i := \tilde{\Omega}_i(I_0, \tau) = \Omega_i + \eta^2 (BI_0)_i = \omega_i + \eta^2 (BI_0)_i, \quad i \in \mathcal{I}^c. \tag{13}$$

On $\mathcal{T}(I_0)$ the flow $t \mapsto (I_0, \phi_0 + \tilde{\omega}t, 0)$ is T-periodic, $T > 0$, if and only if

$$\tilde{\omega}(I_0, \tau)\tau =: k \in \mathbb{Z}^N. \tag{14}$$

Hence, if (14) holds, the torus $\mathcal{T}(I_0)$ is completely resonant and supports the infinitely many T-periodic orbits of the family

$$\mathcal{F} := \{I(t) = I_0,\ \phi(t) = \phi_0 + \tilde{\omega} t,\ \ \hat{z}(t) = 0\}. \tag{15}$$

The family $\mathcal{F}$ will not persist in its entirety for the Hamiltonian $\widetilde{\mathcal{H}}$. However, if the period T is "sufficiently nonresonant" with the shifted elliptic frequencies, we can prove the persistence of at least N geometrically distinct T-periodic solutions of $\widetilde{\mathcal{H}}$ close to $\mathcal{F}$. More precisely, the required nonresonance condition is

$$|\ell - \tilde{\Omega}_i(I_0, \tau)\tau| \geq \frac{\text{const}}{i}, \qquad \forall \ell \in \mathbb{Z},\ \forall i \in \mathcal{I}^c, \tag{16}$$

in which the "small divisor" $\ell - \tilde{\Omega}_i(I_0, \tau)\tau$ appears.

We now consider the periodicity condition (14). Since A is invertible ("twist" condition) we can choose I_0 and k as functions of τ so that (14) is always satisfied:

$$I_0 := I_0(\tau) := A^{-1}(\kappa - \{\omega\tau\}), \tag{17}$$

$$k := k(\tau) := [\omega\tau] + \kappa, \tag{18}$$

where[1] $[(x_1, \dots, x_N)] := ([x_1], \dots, [x_N])$, $\{(x_1, \dots, x_N)\} := (\{x_1\}, \dots, \{x_N\})$ and $\kappa \in \mathbb{Z}^N$ is a constant vector added to have

$$(I_0)_i \geq \text{const} > 0, \qquad \forall i \in \mathcal{I}. \tag{19}$$

By (17) the small divisor in (16) takes the form

$$\ell - (\tau\Omega + BA^{-1}(\kappa - \{\omega\tau\}))_i, \qquad \ell \in \mathbb{Z},\ i \in \mathcal{I}^c. \tag{20}$$

The small divisors estimate (16)

By (20), we have to prove that

$$|\ell - \tau\sqrt{i^2 + \mu} - (BA^{-1}(\kappa - \{\omega\tau\}))_i| \geq \frac{\text{const}}{i}, \qquad \forall \ell \in \mathbb{Z},\ i \in \mathcal{I}^c. \tag{21}$$

In order to show (21), we perform the expansion $\tau\omega_i = \tau\sqrt{i^2 + \mu} = i\tau + \mu\tau/2i + O(\mu^2\tau)$ requiring that $\mu^2\tau$ is small (recall (3)). Using the explicit formulas for A and B, one can prove

$$\ell - \tau\sqrt{i^2 + \mu} - \Big(BA^{-1}(\kappa - \{\omega\tau\})\Big)_i = \ell - \tau i - \frac{\mu\tau}{2i} + \frac{4}{(4N-1)i}\left(\frac{N\mu\tau}{2} + m\right) + O\left(\frac{\mu}{i} + \frac{\mu^2\tau}{i}\right), \qquad \forall \ell \in \mathbb{Z},\ i \in \mathcal{I}^c,$$

[1] Here the functions $[\cdot] : \mathbb{R} \to \mathbb{Z}$ and $\{\cdot\} : \mathbb{R} \to [0, 1)$ denote the integer part and the fractional part respectively.

for a suitable $m \in \mathbb{Z}$. If we take μ and $\mu^2\tau$ small enough and choose the constant in (21) equal, e.g., to $1/6(4N-1)$, estimate (21) follows from

$$|\mu\tau + 2(i(4N-1)(\ell - \tau i) + 4m)| \geq 1/2, \quad \forall \ell \in \mathbb{Z},\ i \in \mathcal{I}^c. \tag{22}$$

To satisfy (22) we take $\tau \in \mathbb{N}$ so that $i(4N-1)(\ell - \tau i) + 4m$ is an integer; in addition, assuming that τ satisfies

$$|\mu\tau + 2j| \geq 1/2, \quad \forall j \in \mathbb{Z},$$

we have proved the following

PROPOSITION 2. *There exist* $O(\mu^{-2})$ *rescaled periods* $\tau \in \mathbb{N}$ *satisfying* $\mu^2\tau \ll 1$ *such that the small divisors estimate* (16) *holds.*

The functional setting

We look for periodic orbits of the Hamiltonian $\widetilde{\mathcal{H}}$ near the family $\mathcal{F}$ defined in (15), namely we seek solutions of the form $(I(t), \phi(t), \hat{z}(t)) = (I_0, \tilde{\omega}t + \phi_0, 0) + \zeta(t)$, where I_0 was defined in (17), $\phi_0 \in \mathbb{T}^N$ is a parameter to be determined and $\zeta(t) = (J(t), \psi(t), w(t))$ is a T-periodic curve taking values in the covering space $\mathbb{R}^N \times \mathbb{R}^N \times \ell^{a,s}$ (which for simplicity we will still denote by $\mathcal{P}_{a,s}$). For $\zeta = (J, \psi, w) \in \mathbb{R}^N \times \mathbb{R}^N \times \ell^{a,s}$ we define the norm

$$\|\zeta\|_{\mathcal{P}_{a,s}} = \|(J, \psi, w)\|_{\mathcal{P}_{a,s}} := |J| + |\psi| + \|w\|_{a,s}.$$

We look for H^1-solutions $\zeta(t)$ in the Banach space

$$\overline{H}^1_{T,a,s} := \left\{ \zeta \in H^1_{T,a,s} : \int_0^T \psi(t)\, dt = 0 \right\}$$

where $H^1_{T,a,s} := \{\zeta \in H^1(\mathbb{R}, \mathcal{P}_{a,s}) : \zeta(t+T) = \zeta(t)\}$ and $H^1(\mathbb{R}, \mathcal{P}_{a,s})$ is the Sobolev space of the L^2-functions $\zeta : \mathbb{R} \to \mathcal{P}_{a,s}$ with weak derivative in L^2. The space $H^1_{T,a,s}$ is endowed with the norm $\|\zeta\|_{H^1_{T,a,s}} := \|\zeta\|_{T,a,s} + T\|\partial_t\zeta\|_{T,a,s}$, where

$$\|\zeta\|^2_{T,a,s} := \frac{1}{T}\int_0^T (|J(t)|^2 + |\psi(t)|^2 + \|w(t)\|^2_{a,s})\, dt.$$

With this norm, $H^1_{T,a,s}$ is a Banach algebra with constant independent of T (see [1]). We have to solve the functional equation

$$L\zeta = N(\zeta; \phi_0) \tag{23}$$

where L is the linear operator

$$L\zeta = L(J, \psi, w) := (\dot{\psi} - \eta^2 AJ,\ \dot{J},\ \dot{w}_i - \mathrm{i}\tilde{\Omega}_i w_i) \tag{24}$$

and N is the Nemytskiĭ operator coming from the nonlinearity. By (8), N is smoothing of order 1, namely

$$N(\cdot; \phi_0) \in \mathcal{C}^\infty(H^1_{T,a,s}, H^1_{T,a,s+1}). \tag{25}$$

The Lyapunov–Schmidt reduction

By the “twist” condition and the nonresonance property (16), it results that the kernel $\mathcal{K}$ and the range $\mathcal{R}$ of L are $\{\psi \equiv \text{const}\}$ and $\{\int_0^T \tilde{\psi} = 0\}$ respectively. Define the projections $\Pi_{\mathcal{K}}\tilde{\zeta} := (0, \langle\tilde{\psi}\rangle, 0)$, $\Pi_{\mathcal{R}}\tilde{\zeta} := (\tilde{J}, \tilde{\psi} - \langle\tilde{\psi}\rangle, \tilde{w})$, where $\langle\tilde{\psi}\rangle := \int_0^T \tilde{\psi}$; then the equation $L\zeta = N(\zeta;\phi_0)$ decomposes into the kernel equation

$$0 = \Pi_{\mathcal{K}} N(\zeta;\phi_0)$$

and the range equation

$$L\zeta = \Pi_{\mathcal{R}} N(\zeta;\phi_0)$$

which is equivalent to $\zeta = \Phi\,\zeta$, with $\Phi\,\zeta := L^{-1}\Pi_{\mathcal{R}}N(\zeta;\phi_0)$. By (16), L^{-1} “looses one derivative”, which is compensated by the smoothing property (25). Using the estimates on the blow-up of the remainder term K for μ going to zero, we prove that Φ is a contraction on a suitable closed ball of $\overline{H}^1_{T,a,s}$. Hence, for any fixed ϕ_0, we can solve the range equation finding a solution $\zeta(t) = \zeta_{\phi_0}(t)$ by the Contraction Mapping Theorem. Inserting $\zeta = \zeta_{\phi_0}$ in the kernel equation, it remains to solve the finite-dimensional bifurcation equation $0 = \Pi_{\mathcal{K}}N(\zeta_{\phi_0};\phi_0)$, determining $\phi_0 \in \mathbb{T}^N$ by standard variational arguments (see [3], [1]).

Finally, the very precise estimates (4), (5) allow us to prove the lower bounds on the minimal period, $T^{\min} \geq c/\mu$. In Proposition 2 the total number of periods satisfying (16) is estimated from below by const/μ^2. However, not all the T-periodic solutions corresponding to different T’s are necessarily distinct. Using (4), (5) we prove that the total number of geometrically distinct solutions found in Theorem 1 is estimated from below by const/μ. The estimate on the amplitudes $a_i \geq c > 0$ follows by (19). Regularity in time follows by a bootstrap-type argument.

REMARK 1. In (16) we imposed a strong condition on the small divisors in order to use the standard Contraction Mapping Theorem to solve the range equation. For that reason we can consider only a finite number of periods. In order to obtain a positive measure set of periods, one should solve the small divisors problem by a Nash–Moser Implicit Function Theorem. Thereafter, one should prove that the bifurcation equation $0 = \Pi_{\mathcal{K}}N(\zeta_{\phi_0};\phi_0)$ has a solution for ϕ_0 belonging to a suitable Cantor set (see [4]). A way to proceed is to develop the reduced action functional in powers of the perturbation parameter and to prove that the first nontrivial term has a nondegenerate critical point. In the present case, when the perturbation parameter η goes to zero, the period $T = 2\pi\eta^{-2}$ goes to infinity and all the low order terms in the development of the reduced action functional vanish.

REMARK 2. In [1], instead of (16), the weaker “diophantine-type” condition $|\ell - \tilde{\Omega}_i\tau| \geq \text{const}\, i^{-\sigma}$ is imposed on the small divisors; for all $\sigma > 1$, almost every rescaled period τ is admitted. Hence, if the nonlinearity is smoothing of order $d > 1$, taking $1 < \sigma < d$, the Contraction Mapping Theorem can still be used to have existence. In particular, $d = 2$ for the beam equation and $d > 1$ for the NLS. Unfortunately, we have exactly $d = 1$ for the wave equation.

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