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Partial differential equations. — *A note on nonlinear elliptic problems with singular potentials*, by MARINO BADIALE and SERGIO ROLANDO, presented by A. Ambrosetti.

ABSTRACT. — We deal with the semi-linear elliptic problem

$$-\Delta u + V(|x|)u = f(u), \quad u \in D^{1,2}(\mathbb{R}^N;\mathbb{R}),$$

where the potential $V > 0$ is measurable, singular at the origin and may also have a continuous set of singularities. The nonlinearity is continuous and has a super-linear power-like behaviour; both sub-critical and super-critical cases are considered. We prove the existence of positive radial solutions. If f is odd, we show that the problem has infinitely many radial solutions. Nonexistence results for particular potentials and nonlinearities are also given.

KEY WORDS: Semilinear elliptic equation; singular potential; radial solution.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 35J60.

1. INTRODUCTION

In this paper we consider the semi-linear elliptic problem

$$\begin{cases} -\Delta u + V(|x|)u = f(u), \\ u \in D^{1,2}(\mathbb{R}^N;\mathbb{R}), \quad N \geq 3, \end{cases} \tag{1}$$

where the potential $V : [0,+\infty) \to (0,+\infty]$ is a measurable function, the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is continuous and they satisfy the following assumptions:

($\mathbf{V}_\alpha$) *there exist* $A, \alpha > 0$ *such that* $V(s) \geq As^{-\alpha}$ *for almost every* $s > 0$*;*
($\mathbf{f}_p$) *there exist* $M > 0$ *and* $p > 2$ *such that* $|f(s)| \leq M|s|^{p-1}$ *for all* $s \in \mathbb{R}$.

Further assumptions on V and f, as well as restrictions on the exponents α and p, will be required in the following (see Section 3). As concerns the integrability properties of the potential, we shall assume that

$(\mathbf{V})_1$ $V \in L^1(a,b)$ *for some open bounded interval* (a,b) *with* $b > a > 0$.

Let us point out that assumption ($\mathbf{V}_\alpha$) implies that V is singular at the origin. Other singularities are allowed by $(\mathbf{V})_1$.

Our existence results for problem (1) are stated in Section 3 (Theorems 8 and 11) and proved in Section 5. They give a generalization of the results of [6] and partially extend the ones of [7] (see Remarks 10 and 12).

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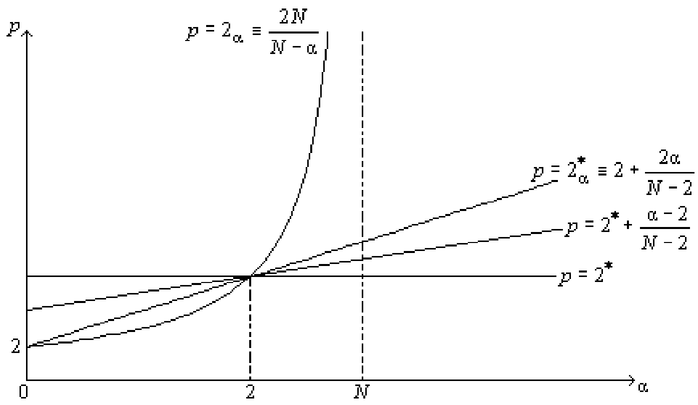
The simplest case in which the assumptions of our results are satisfied (see Example 5) is given by the problem

$$\begin{cases} -\Delta u + \dfrac{A}{|x|^\alpha} u = |u|^{p-2} u, \\ u \in D^{1,2}(\mathbb{R}^N; \mathbb{R}), \quad N \geq 3, \end{cases} \tag{2}$$

where $A > 0$. In the critical case $p = 2^* := 2N/(N-2)$ and $\alpha = 2$, the problem of positive solutions to (2) is studied in [16] (see Remark 4), where Terracini also proves that there are no positive solutions either in $L^p(\mathbb{R}^N)$ for $p \neq 2^*$ and $\alpha = 2$, or in $L^2(\mathbb{R}^N; |x|^{-\alpha}\, dx)$ for $p = 2^*$ and $\alpha \neq 2$. The same problem is handled in [6] where it is shown that no positive solution exists in $L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; |x|^{-\alpha}\, dx)$ if $\alpha \in (0, 2)$ and $p > 2^*$, or $\alpha \in (2, +\infty)$ and $p < 2^*$. On the other hand, the authors prove the existence of a positive radial solution provided that $\alpha \in (0, 2)$ and $p \in (2^* + (\alpha - 2)/(N-2), 2^*)$, or $\alpha \in (2, +\infty)$ and $p \in (2^*, 2^* + (\alpha - 2)/(N-2))$.



Here we generalize the above mentioned nonexistence and existence results. Letting $2_\alpha := 2N/(N-\alpha)$, in Theorem 3 of Section 2 we show that problem (2) has no solution in $L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; |x|^{-\alpha}\, dx)$ whenever $\alpha \in (0, 2)$ and $p \notin (2_\alpha, 2^*)$, or $\alpha = 2$ and $p \neq 2^*$, or $\alpha \in (2, N)$ and $p \notin (2^*, 2_\alpha)$, or $\alpha \geq N$ and $p \leq 2^*$. This will follow from a Pohožaev type identity related to problem (2), which we prove for a general nonlinearity $f \in C^0(\mathbb{R}; \mathbb{R})$ (Lemma 1). As concerns existence, from our results it readily follows that problem (2) admits radial solutions if $\alpha \in (0, 2)$ and $p \in (2^*_\alpha, 2^*)$, or $\alpha \in (2, +\infty)$ and $p \in (2^*, 2^*_\alpha)$, where $2^*_\alpha := 2 + 2\alpha/(N-2)$. Note that $|2^* - 2^*_\alpha| = 2|(\alpha - 2)/(N-2)|$ for every $\alpha > 0$ and $N \geq 3$.

The study developed here constitutes a part of the PhD thesis [13] of the second author. We wish to thank the referee for his helpful remarks.

NOTATIONS

- We denote by $2^* := 2N/(N-2)$ the critical exponent for the Sobolev embedding in dimension $N \geq 3$. Moreover we set $2_\alpha := 2N/(N-\alpha)$ for $\alpha \in (0, N)$ and $2^*_\alpha := 2 + 2\alpha/(N-2)$ for $\alpha \in (0, +\infty)$.

- The open ball $B_\rho(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < \rho\}$ will be simply denoted by B_ρ whenever $x_0 = 0$. The closure of B_ρ is $\bar{B}_\rho$.
- $O(N)$ is the orthogonal group of $\mathbb{R}^N$.
- By $\to$ and $\rightharpoonup$ we respectively denote *strong* and *weak* convergence in a Banach space X, whose dual space is denoted by X'.
- C will stand for any positive constant, which may change from line to line.

2. Pohožaev identity and nonexistence result

Let $A, \alpha > 0$ and $p > 2$. By means of integral identities, we prove the nonexistence result for problem (2) announced in the introduction.

LEMMA 1. *Let $f \in C^0(\mathbb{R}; \mathbb{R})$ and let $u \in C^2(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ be a classical solution to the equation*

$$-\Delta u + \frac{A}{|x|^\alpha} u = f(u) \quad \text{in } \mathbb{R}^N \setminus \{0\},\ N \geq 3. \tag{3}$$

Set $F(s) := \int_0^s f(t)\, dt$ for all $s \in \mathbb{R}$. If

$$\int_{\mathbb{R}^N} \left(|\nabla u|^2 + \frac{u^2}{|x|^\alpha} + |F(u)| \right) dx < +\infty \tag{4}$$

then

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2\, dx + \frac{N-\alpha}{2} \int_{\mathbb{R}^N} \frac{Au^2}{|x|^\alpha}\, dx = N \int_{\mathbb{R}^N} F(u)\, dx. \tag{5}$$

PROOF. The proof relies on a standard argument [10], adapted to the case under discussion. The starting point are the following identities, which hold true on $\mathbb{R}^N \setminus \{0\}$:

$$\begin{aligned}
(x \cdot \nabla u)\Delta u &= \operatorname{div}\left[(x \cdot \nabla u)\nabla u - \frac{1}{2}|\nabla u|^2 x \right] + \frac{N-2}{2}|\nabla u|^2, \\
(x \cdot \nabla u)\frac{Au}{|x|^\alpha} &= \operatorname{div}\left[\frac{A}{2}\frac{u^2}{|x|^\alpha} x \right] - \frac{N-\alpha}{2}\frac{Au^2}{|x|^\alpha}, \\
(x \cdot \nabla u) f(u) &= \operatorname{div}[F(u)x] - NF(u).
\end{aligned}$$

Then, for $R_2 > R_1 > 0$, upon multiplying equation (3) by $x \cdot \nabla u$ and applying the Divergence Theorem on the open annulus $\Omega := \Omega_{R_1,R_2} := B_{R_2} \setminus \bar{B}_{R_1}$, we get

$$\begin{aligned}
-\int_{\partial\Omega} (x \cdot \nabla u)(\nabla u \cdot \nu)\, d\sigma + \frac{1}{2}\int_{\partial\Omega} \left(|\nabla u|^2 + \frac{Au^2}{|x|^\alpha} \right) x \cdot \nu\, d\sigma - \int_{\partial\Omega} F(u) x \cdot \nu\, d\sigma \\
= \frac{N-2}{2}\int_\Omega |\nabla u|^2\, dx + \frac{N-\alpha}{2}\int_\Omega \frac{Au^2}{|x|^\alpha}\, dx - N\int_\Omega F(u)\, dx
\end{aligned} \tag{6}$$

where $\nu(x)$ is the outward normal of $\partial\Omega$ at x and $d\sigma$ is the $(N-1)$-dimensional measure of $\partial\Omega$. Note that $\partial\Omega = \partial B_{R_1} \cup \partial B_{R_2}$, $\nu(x) = -x/R_1$ on ∂B_{R_1} and $\nu(x) = x/R_2$ on ∂B_{R_2}. Hence

$$\left| \int_{\partial B_{R_i}} (x \cdot \nabla u)(\nabla u \cdot \nu)\, d\sigma \right| = \frac{1}{R_i} \int_{\partial B_{R_i}} (x \cdot \nabla u)^2\, d\sigma \le R_i \int_{\partial B_{R_i}} |\nabla u|^2\, d\sigma, \tag{7}$$

$$\left| \int_{\partial B_{R_i}} \left(|\nabla u|^2 + \frac{Au^2}{|x|^\alpha} \right) x \cdot \nu\, d\sigma \right| = R_i \int_{\partial B_{R_i}} \left(|\nabla u|^2 + \frac{Au^2}{|x|^\alpha} \right) d\sigma, \tag{8}$$

$$\left| \int_{\partial B_{R_i}} F(u) x \cdot \nu\, d\sigma \right| \le R_i \int_{\partial B_{R_i}} |F(u)|\, d\sigma, \tag{9}$$

for $i = 1, 2$. Now, arguing by contradiction, it is easy to prove that there exists a sequence $R_{1,n} \to 0$, $R_{1,n} > 0$, such that

$$R_{1,n} \int_{\partial B_{R_{1,n}}} \left(|\nabla u|^2 + \frac{u^2}{|x|^\alpha} + |F(u)| \right) d\sigma \to 0.$$

By (7)–(9), this implies

$$-\int_{\partial B_{R_{1,n}}} (x \cdot \nabla u)(\nabla u \cdot \nu)\, d\sigma + \frac{1}{2} \int_{\partial B_{R_{1,n}}} \left(|\nabla u|^2 + \frac{Au^2}{|x|^\alpha} \right) x \cdot \nu\, d\sigma - \int_{\partial B_{R_{1,n}}} F(u) x \cdot \nu\, d\sigma \to 0$$

so that, evaluating (6) for $R_1 = R_{1,n}$ and passing to the limit as $n \to \infty$, we get

$$\begin{aligned} -\int_{\partial B_{R_2}} (x \cdot \nabla u)(\nabla u \cdot \nu)\, d\sigma + \frac{1}{2} \int_{\partial B_{R_2}} \left(|\nabla u|^2 + \frac{Au^2}{|x|^\alpha} \right) x \cdot \nu\, d\sigma - \int_{\partial B_{R_2}} F(u) x \cdot \nu\, d\sigma \\ = \frac{N-2}{2} \int_{B_{R_2}} |\nabla u|^2\, dx + \frac{N-\alpha}{2} \int_{B_{R_2}} \frac{Au^2}{|x|^\alpha}\, dx - N \int_{B_{R_2}} F(u)\, dx. \end{aligned} \tag{10}$$

Arguing again by contradiction, one infers the existence of a sequence $R_{2,n} \to +\infty$ such that

$$R_{2,n} \int_{\partial B_{R_{2,n}}} \left(|\nabla u|^2 + \frac{u^2}{|x|^\alpha} + |F(u)| \right) d\sigma \to 0.$$

Hence, upon recalling (7)–(9), (10) yields the conclusion. □

LEMMA 2. *Let $u \in C^2(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ be a classical solution to the equation*

$$-\Delta u + \frac{A}{|x|^\alpha} u = |u|^{p-2} u \quad \text{in } \mathbb{R}^N \setminus \{0\},\ N \ge 3. \tag{11}$$

If $u \in D^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; |x|^{-\alpha}\, dx) \cap L^p(\mathbb{R}^N)$ then

$$\int_{\mathbb{R}^N} \left(|\nabla u|^2 + \frac{Au^2}{|x|^\alpha} \right) dx = \int_{\mathbb{R}^N} |u|^p\, dx. \tag{12}$$

PROOF. Multiplying (11) by u, using the identity $u\,\Delta u = \operatorname{div}[u\,\nabla u] - |\nabla u|^2$ in $\mathbb{R}^N \setminus \{0\}$ and applying the Divergence Theorem on the open annulus $\Omega := B_{R_2} \setminus \bar{B}_{R_1}$, we obtain

$$\frac{1}{R_1}\int_{\partial B_{R_1}} u(\nabla u \cdot x)\,d\sigma - \frac{1}{R_2}\int_{\partial B_{R_2}} u(\nabla u \cdot x)\,d\sigma + \int_\Omega \left(|\nabla u|^2 + \frac{Au^2}{|x|^\alpha}\right)dx = \int_\Omega |u|^p\,dx.$$

Now, since $\frac{2^*}{2^*-1} = \frac{2N}{N+2} < 2$ and $\frac{N-1}{N} = \frac{1}{2^*} + \frac{1}{2}$, by the Hölder inequality we get

$$\begin{aligned}
\left|\frac{1}{R_i}\int_{\partial B_{R_i}} u(\nabla u \cdot x)\,d\sigma\right| &\le \int_{\partial B_{R_i}} |u||\nabla u|\,d\sigma \\
&\le \left(\int_{\partial B_{R_i}} |u|^{2^*}\,d\sigma\right)^{1/2^*}\left(\int_{\partial B_{R_i}} |\nabla u|^{2N/(N+2)}\,d\sigma\right)^{(N+2)/2N} \\
&\le C\left(\int_{\partial B_{R_i}} |u|^{2^*}\,d\sigma\right)^{1/2^*}\left(\int_{\partial B_{R_i}} |\nabla u|^2\,d\sigma\right)^{1/2} R_i^{(N-1)/N} \\
&= C\left(R_i\int_{\partial B_{R_i}} |u|^{2^*}\,d\sigma\right)^{1/2^*}\left(R_i\int_{\partial B_{R_i}} |\nabla u|^2\,d\sigma\right)^{1/2}.
\end{aligned}$$

Then, as in the proof of Lemma 1, one can take $R_{1,n} \to 0^+$ and $R_{2,n} \to +\infty$ such that

$$R_{i,n}\int_{\partial B_{R_{i,n}}} (|u|^{2^*} + |\nabla u|^2)\,d\sigma \to 0$$

and the conclusion follows. □

THEOREM 3. *If* $\alpha \in (0, 2)$ *and* $p \notin (2_\alpha, 2^*)$, *or* $\alpha = 2$ *and* $p \neq 2^*$, *or* $\alpha \in (2, N)$ *and* $p \notin (2^*, 2_\alpha)$, *or* $\alpha \ge N$ *and* $p \le 2^*$, *then equation* (11) *has no nontrivial classical solution* $u \in C^2(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ *such that* $u \in D^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; |x|^{-\alpha}\,dx) \cap L^p(\mathbb{R}^N)$.

PROOF. Assuming that the assertion of the theorem is false, we can apply both Lemma 1 and Lemma 2. Thus plugging (12) into (5) one gets

$$\left(\frac{N-2}{2} - \frac{N}{p}\right)\int_{\mathbb{R}^N} |\nabla u|^2\,dx = \left(\frac{N}{p} - \frac{N-\alpha}{2}\right)\int_{\mathbb{R}^N} \frac{Au^2}{|x|^\alpha}\,dx,$$

which is not possible for $u \neq 0$ if

$$\left(\frac{N-2}{2} - \frac{N}{p}\right)\left(\frac{N}{p} - \frac{N-\alpha}{2}\right) < 0.$$

Since this inequality is actually equivalent to the assumptions of the theorem, we have a contradiction. □

REMARK 4. In the case $\alpha = 2$ and $p = 2^*$, equation (11) admits solutions. This has been proved in [16], where the positive radial solutions are completely classified; in particular, all the radial positive solutions in $D^{1,2}(\mathbb{R}^N)$ are explicitly found. Related results can be found in [5] and [8].

3. EXISTENCE RESULTS

In order to state our existence results we need some notation. For $N \geq 3$ and for any given measurable function $V : [0, +\infty) \to (0, +\infty]$ satisfying assumption $(\mathbf{V})_1$ we define the weighted Sobolev space

$$X := X(\mathbb{R}^N; V) := \left\{ u \in D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(|x|)u^2\,dx < +\infty \right\}, \tag{13}$$

which will be studied in Section 4. Note that $(\mathbf{V})_1$ ensures $X \neq \{0\}$.

For any given $f \in C^0(\mathbb{R}; \mathbb{R})$ set $F(s) := \int_0^s f(t)\,dt$.

Our existence results rely on assumptions $(\mathbf{V}_\alpha)$, $(\mathbf{f}_p)$ and $(\mathbf{V})_1$ suitably combined with some of the following conditions:

- $(\mathbf{V})_2$ *there exist* $B, \beta, \mu_0 > 0$ *such that* $V(\mu s) \leq \mu^{-\beta} BV(s)$ *for almost every* $\mu > \mu_0$ *and* $s > 0$*;*
- $(\mathbf{f})_1$ *there exists* $\vartheta > 2$ *such that* $\vartheta F(s) \leq f(s)s$ *for all* $s \in \mathbb{R}$*;*
- $(\mathbf{f})_2$ $F(s_*) > 0$ *for some* $s_* \in (0, +\infty)$*;*
- $(\mathbf{f})_3$ $F(s) > 0$ *for all* $s \in (0, +\infty)$*;*
- $(\mathbf{f})_4$ f *is odd;*
- $(\mathbf{F}_p)$ *there exists* $m > 0$ *such that* $F(s) \geq m|s|^p$ *for all* $s \in \mathbb{R}$.

Note that $V(s) \not\equiv +\infty$ and $(\mathbf{V})_2$ ensure $V \in L^\infty(c, +\infty)$ for some $c > 0$.

EXAMPLE 5. If $V : [0, +\infty) \to (0, +\infty]$ satisfies $As^{-\alpha} \leq V(s) \leq Cs^{-\alpha}$ for some $C \geq A > 0$ and for almost every $s > 0$, then assumptions $(\mathbf{V}_\alpha)$ and $(\mathbf{V})_2$ hold with $\beta = \alpha$, $B \geq C/A$ and $\mu_0 > 0$ arbitrary.

EXAMPLE 6. For any given $A > 0$ and $\alpha \geq \beta > 0$, the function

$$V(s) = \begin{cases} +\infty & \text{for } s = 0, \\ As^{-\alpha} & \text{for } s \in (0, 1], \\ As^{-\beta} & \text{for } s \in [1, +\infty), \end{cases}$$

satisfies $(\mathbf{V}_\alpha)$ and $(\mathbf{V})_2$ with $B = \mu_0 = 1$.

EXAMPLE 7. For any given $s_0 > 0$, $A > 0$ and $\alpha \geq \beta > 0$, assumptions $(\mathbf{V}_\alpha)$ and $(\mathbf{V})_2$ hold for the function

$$V(s) = \begin{cases} +\infty & \text{for } s \in [0, s_0], \\ B(s - s_0)^{-\beta} & \text{for } s \in (s_0, +\infty), \end{cases}$$

provided that $B = B(s_0, A, \alpha, \beta) > 0$ is large enough.

THEOREM 8. *Let* $f \in C^0(\mathbb{R}; \mathbb{R})$ *satisfy* $(\mathbf{f})_1$ *and let* $V : [0, +\infty) \to (0, +\infty]$ *be a measurable function satisfying* $(\mathbf{V})_1$*. Assume that* $(\mathbf{V}_\alpha)$ *and* $(\mathbf{f}_p)$ *hold with* $\alpha \in (0, 2)$ *and* $p \in (2^*_\alpha, 2^*)$*, or* $\alpha \in (2, +\infty)$ *and* $p \in (2^*, 2^*_\alpha)$*. Assume furthermore that either* V *satisfies* $(\mathbf{V})_2$ *and* f *satisfies* $(\mathbf{f})_2$*, or* f *satisfies* $(\mathbf{f})_3$*. Then problem* (1) *has a nontrivial nonnegative* *radial solution $u \in X(\mathbb{R}^N; V)$, by which we mean*

$$\int_{\mathbb{R}^N} (\nabla u \cdot \nabla h + V(|x|)uh)\, dx = \int_{\mathbb{R}^N} f(u)h\, dx \quad \text{for all } h \in X(\mathbb{R}^N; V). \tag{14}$$

REMARK 9. If we replace assumption $(\mathbf{f})_2$ with the requirement of the existence of $s_* < 0$ such that $F(s_*) > 0$, similar arguments ensure the existence of a nonpositive solution.

REMARK 10. Theorem 8 generalizes the existence results of [6] in two directions: first, as discussed in the introduction, it enlarges the range of p's for which problem (2) admits nonnegative radial solutions; second, it covers more general classes of potentials and nonlinearities.

THEOREM 11. *Let $f \in C^0(\mathbb{R}; \mathbb{R})$ satisfy $(\mathbf{f})_1$ and $(\mathbf{f})_4$, and let $V : [0, +\infty) \to (0, +\infty]$ be a measurable function satisfying $(\mathbf{V})_1$. Assume that $(\mathbf{V}_\alpha)$, $(\mathbf{f}_p)$ and $(\mathbf{F}_p)$ hold with $\alpha \in (0, 2)$ and $p \in (2^*_\alpha, 2^*)$, or $\alpha \in (2, +\infty)$ and $p \in (2^*, 2^*_\alpha)$. Then there exist in $X(\mathbb{R}^N; V)$ infinitely many radial solutions to problem* (1) *in the sense of* (14).

REMARK 12. If $f(u) = |u|^{p-2}u$, $V \in L^\infty$ and $\lim_{s\to\infty} s^\alpha V(s) = 1$, problem (1) admits infinitely many radial solutions for $0 < \alpha < 2$ and a slightly larger range of exponents than in Theorem 11, namely $p \in (p_\alpha, 2^*)$ with $p_\alpha := 2 + 2\alpha/(N - 1 - \alpha/2) < 2^*_\alpha$. This has been proved in [7], where the authors are also able to prescribe nodal properties to the solutions. Multiplicity results for problem (1) with continuous potentials which do not vanish at infinity are contained in [2] and [3].

Theorems 8 and 11 will be proved in Section 5 by means of Mountain Pass theorems, for which the reader is referred to the celebrated paper [1] by Ambrosetti–Rabinowitz, or to some more recent books such as [9], [12], [15].

4. A WEIGHTED SOBOLEV SPACE

Let $N \geq 3$ and assume that $V : [0, +\infty) \to (0, +\infty]$ is a measurable function satisfying $(\mathbf{V}_\alpha)$ and $(\mathbf{V})_1$. The aim of this section is to study the weighted Sobolev space X introduced in (13).

It is well known that the Sobolev space

$$D^{1,2} := D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$$

equipped with the norm $\|u\|_{D^{1,2}} := \|\nabla u\|_{L^2}$ is a Hilbert space, which can be regarded as the completion of $C_c^\infty(\mathbb{R}^N)$. Let us just recall that weak $D^{1,2}$-convergence implies pointwise convergence on $\mathbb{R}^N$ (up to a subsequence and almost everywhere).

Since the convergence in the weighted Lebesgue space $L^2(\mathbb{R}^N; V(|x|)\, dx)$ implies pointwise convergence (up to a subsequence and almost everywhere), the space $X = D^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; V(|x|)\, dx)$ is a Hilbert space with respect to the norm $\|u\|^2 := (u \mid u)$ induced by the scalar product

$$(u \mid v) := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(|x|)uv)\, dx \quad \text{for all } u, v \in X(\mathbb{R}^N; V).$$

Note that $C_c^\infty(B_b \setminus \bar{B}_a) \subset X$ thanks to assumption $(\mathbf{V})_1$.

By the continuous embedding $X \hookrightarrow D^{1,2}$, weak X-convergence implies pointwise convergence on $\mathbb{R}^N$ (up to a subsequence and almost everywhere). As a consequence, the subspace

$$X_{\mathrm{r}} := X_{\mathrm{r}}(\mathbb{R}^N; V) := \{u \in X(\mathbb{R}^N; V) : u(x) = u(gx) \text{ for all } g \in O(N)\}$$

(which is nonempty by assumption $(\mathbf{V})_1$) is closed in X, and thus it is a Hilbert space itself.

PROPOSITION 13. *The embedding $X_{\mathrm{r}}(\mathbb{R}^N; V) \hookrightarrow L^{2^*_\alpha}(\mathbb{R}^N)$ is continuous.*

PROOF. By a well known radial lemma [4], there exists $C_N > 0$ such that

$$(15) \quad \forall u \in D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N) \quad |u(x)| \leq C_N \|\nabla u\|_{L^2(\mathbb{R}^N)} |x|^{-(N-2)/2} \quad \text{for almost every } x \in \mathbb{R}^N$$

(we notice that in [4] the authors prove this estimate for $|x| \geq 1$, but their proof actually works for $x \neq 0$). Hence for any $u \in X_{\mathrm{r}}$ we have

$$\begin{aligned} |u(x)|^{2^*_\alpha} &= |u(x)|^2 |u(x)|^{2\alpha/(N-2)} \leq |u(x)|^2 C \|\nabla u\|^{2\alpha/(N-2)}_{L^2(\mathbb{R}^N)} |x|^{-\alpha} \\ &\leq A^{-1} C \|\nabla u\|^{2\alpha/(N-2)}_{L^2(\mathbb{R}^N)} V(|x|) |u(x)|^2 \end{aligned}$$

for almost every $x \in \mathbb{R}^N$, which implies

$$\int_{\mathbb{R}^N} |u|^{2^*_\alpha}\, dx \leq C (\|\nabla u\|^2_{L^2(\mathbb{R}^N)})^{\alpha/(N-2)} \int_{\mathbb{R}^N} V(|x|) u^2\, dx \leq C \|u\|^{2\alpha/(N-2)} \|u\|^2$$

so that $\|u\|_{L^{2^*_\alpha}(\mathbb{R}^N)} \leq C\|u\|$. □

By the Hölder inequality, Proposition 13 together with Sobolev embedding yields the following continuous embeddings:

$$(16) \qquad X_{\mathrm{r}}(\mathbb{R}^N; V) \hookrightarrow L^p(\mathbb{R}^N) \quad \text{for } 0 < \alpha < 2 \text{ and } p \in [2^*_\alpha, 2^*]$$

and

$$(17) \qquad X_{\mathrm{r}}(\mathbb{R}^N; V) \hookrightarrow L^p(\mathbb{R}^N) \quad \text{for } \alpha > 2 \text{ and } p \in [2^*, 2^*_\alpha].$$

PROPOSITION 14. *The embeddings* (16)–(17) *are compact for $p \neq 2^*, 2^*_\alpha$.*

PROOF. Let $\{u_n\} \subset X_{\mathrm{r}}$ be a bounded sequence and let $P(s) := |s|^p$ and $Q(s) := |s|^{2^*} + |s|^{2^*_\alpha}$. Then $u_n \rightharpoonup u$ in X_{r} (up to a subsequence) and $\lim_{|s|\to\infty} P(s)/Q(s) = \lim_{s\to 0} P(s)/Q(s) = 0$. Thus we can apply a well known compactness lemma [14] (see also [4]), by which $u_n \to u$ in $L^p(\mathbb{R}^N)$ provided that the following conditions hold:

(i) $\lim_{|x|\to\infty} |u_n(x) - u(x)| = 0$ uniformly with respect to n;
(ii) $|u_n - u|^p \to 0$ almost everywhere on $\mathbb{R}^N$;
(iii) $\sup_n \int_{\mathbb{R}^N} Q(u_n - u)\, dx < +\infty$.

Actually, (i) is ensured by (15) together with the boundedness of $\{\|\nabla(u_n - u)\|_{L^2}\}$, whereas (ii) is satisfied, up to a subsequence, by pointwise convergence of weakly convergent sequences in X_{r}. Finally, since

$$\int_{\mathbb{R}^N} Q(u_n - u)\, dx = \|u_n - u\|^{2^*_\alpha}_{L^{2^*_\alpha}} + \|u_n - u\|^{2^*}_{L^{2^*}},$$

condition (iii) follows from the continuity of the embeddings (16)–(17) together with the boundedness of $\{\|u_n - u\|\}$. □

The following extendibility propositions show that if $u \in X_{\mathrm{r}}$ then $|u|^{p-1}$ defines an element of the dual space X' provided that conditions (16)–(17) on p and α are satisfied.

PROPOSITION 15. *If $u \in X_{\mathrm{r}}(\mathbb{R}^N; V)$ then there exists $C = C(N, \alpha, A) > 0$ such that*

$$\int_{\mathbb{R}^N} |u|^{2_\alpha^*-1}|v|\,dx \le C\|u\|^{2_\alpha^*-1}\|v\| \quad \text{for all } v \in X(\mathbb{R}^N; V).$$

PROOF. One has

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^{2_\alpha^*-1}|v|\,dx &= \int_{\mathbb{R}^N} |x|^{\alpha/2}|u|^{2_\alpha^*-1}\frac{|v|}{|x|^{\alpha/2}}\,dx \\ &\le \left(\int_{\mathbb{R}^N} |x|^\alpha |u|^{2(2_\alpha^*-1)}\,dx\right)^{1/2}\left(\int_{\mathbb{R}^N}\frac{v^2}{|x|^\alpha}\,dx\right)^{1/2} \\ &\le \left(\int_{\mathbb{R}^N} |x|^\alpha |u|^{2(2_\alpha^*-1)}\,dx\right)^{1/2}\left(A^{-1}\int_{\mathbb{R}^N} V(|x|)v^2\,dx\right)^{1/2} \\ &\le C\left(\int_{\mathbb{R}^N} |x|^\alpha |u|^{2(2_\alpha^*-1)}\,dx\right)^{1/2}\|v\|.\end{aligned}$$

Now we write $2(2_\alpha^* - 1) = 2 + 4\alpha/(N-2)$ and use the estimate (15) to obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^\alpha |u|^{2(2_\alpha^*-1)}\,dx &= \int_{\mathbb{R}^N} \frac{u^2}{|x|^\alpha}|x|^{2\alpha}|u|^{4\alpha/(N-2)}\,dx \\ &\le C\|\nabla u\|_{L^2(\mathbb{R}^N)}^{4\alpha/(N-2)}\int_{\mathbb{R}^N}\frac{u^2}{|x|^\alpha}|x|^{2\alpha}|x|^{-\frac{N-2}{2}\frac{4\alpha}{N-2}}\,dx \\ &= C\|\nabla u\|_{L^2(\mathbb{R}^N)}^{4\alpha/(N-2)}\int_{\mathbb{R}^N}\frac{u^2}{|x|^\alpha}\,dx \\ &\le A^{-1}C\|\nabla u\|_{L^2(\mathbb{R}^N)}^{4\alpha/(N-2)}\int_{\mathbb{R}^N} V(|x|)u^2\,dx \\ &\le C\|u\|^{4\alpha/(N-2)}\|u\|^2 = C\|u\|^{2(2_\alpha^*-1)},\end{aligned}$$

which completes the proof. □

PROPOSITION 16. *If $u \in X_{\mathrm{r}}(\mathbb{R}^N; V)$ then there exists $C = C(N) > 0$ such that*

$$\int_{\mathbb{R}^N} |u|^{2^*-1}|v|\,dx \le C\|u\|^{2^*-1}\|v\| \quad \text{for all } v \in X(\mathbb{R}^N; V).$$

PROOF. Use the Hölder inequality and Sobolev embedding. □

PROPOSITION 17. *If $u \in X_{\mathrm{r}}(\mathbb{R}^N; V)$ then there exists $C = C(N, \alpha, A) > 0$ such that for every $p \in [2_\alpha^*, 2^*]$ or $p \in [2^*, 2_\alpha^*]$, according as $\alpha \in (0, 2)$ or $\alpha \in (2, +\infty)$, one has*

$$\int_{\mathbb{R}^N} |u|^{p-1}|v|\,dx \le C\|u\|^{p-1}\|v\| \quad \text{for all } v \in X(\mathbb{R}^N; V).$$

PROOF. This follows, by interpolation, from Propositions 15 and 16. □

We conclude the study of X_{r} by proving the following technical lemma.

LEMMA 18. *Assume that V satisfies $(\mathbf{V})_2$ and let $F \in C^0(\mathbb{R};\mathbb{R})$ satisfy condition $(\mathbf{f})_2$. Then there exists $u \in X_{\mathrm{r}}(\mathbb{R}^N;V)$ such that $\int_{\mathbb{R}^N} F(u)\,dx > 0$.*

PROOF. First, from assumption $(\mathbf{V})_2$ we deduce that $V \in L^\infty(c,+\infty)$ for some $c>0$, so that $C_{\mathrm{c}}^\infty(\mathbb{R}^N \setminus \bar{B}_c) \subset X$. In order to prove the lemma, for any $R \geq 3$ we let $\phi_R \in C_{\mathrm{c}}^\infty(c,+\infty)$ be such that $0 \leq \phi_R \leq 1$, $\phi_R(t)=0$ for $t \leq c+1$ or $t \geq c+1+R$ and $\phi_R(t)=1$ for $c+2 \leq t \leq c+R$. Then, for all $x \in \mathbb{R}^N$ define $u_R(x) := \phi_R(|x|)$, so that $u_R \in X_{\mathrm{r}}$ satisfies $\operatorname{supp} u_R \subset B_{c+1+R} \setminus \bar{B}_{c+1}$, $0 \leq u_R \leq 1$ in $\mathbb{R}^N$ and $u_R = 1$ in $\bar{B}_{c+R} \setminus B_{c+2}$. Using condition $(\mathbf{f})_2$, it is now easy to see that $\int_{\mathbb{R}^N} F(s_* u_R)\,dx \to +\infty$ as $R \to +\infty$. $\square$

5. PROOF OF THEOREMS 8 AND 11

In this section we assume that $f \in C^0(\mathbb{R};\mathbb{R})$ satisfies $(\mathbf{f})_1$ and $V:[0,+\infty) \to (0,+\infty]$ is a measurable function satisfying $(\mathbf{V})_1$. We also assume that $(\mathbf{V}_\alpha)$ and $(\mathbf{f}_p)$ hold with $\alpha \in (0,2)$ and $p \in (2_\alpha^*, 2^*)$, or $\alpha \in (2,+\infty)$ and $p \in (2^*, 2_\alpha^*)$. Set $F(s) := \int_0^s f(t)\,dt$ and let $N \geq 3$. Our aim is to give the proofs of Theorems 8 and 11, which will be achieved through some preliminary lemmata.

We are going to look for critical points of the functional $I : X_{\mathrm{r}}(\mathbb{R}^N;V) \to \mathbb{R}$ defined by

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(|x|)u^2)\,dx - \int_{\mathbb{R}^N} F(u)\,dx \quad \text{for all } u \in X_{\mathrm{r}}(\mathbb{R}^N;V).$$

By assumption $(\mathbf{f}_p)$ and the continuous embeddings (16)–(17), one easily sees that I is of class C^1 on X_{r} with Fréchet derivative $I'(u)$ at any $u \in X_{\mathrm{r}}$ given by

$$I'(u)h = (u \mid h) - \int_{\mathbb{R}^N} f(u)h\,dx \quad \text{for all } h \in X_{\mathrm{r}}(\mathbb{R}^N;V).$$

The next lemma shows that any critical point of I gives rise to a solution to problem (1). Notice that the lemma is a version of the Principle of Symmetric Criticality (see [11]).

LEMMA 19. *Every critical point of $I : X_{\mathrm{r}}(\mathbb{R}^N;V) \to \mathbb{R}$ satisfies* (14).

PROOF. Let $u \in X_{\mathrm{r}}$. For $h \in X$ we define

$$T(u)h := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla h + V(|x|)uh)\,dx - \int_{\mathbb{R}^N} f(u)h\,dx.$$

The linear functional $T(u)$ is well defined and continuous on X, i.e., $T(u) \in X'$, since

$$\left|\int_{\mathbb{R}^N} f(u)h\,dx\right| \leq \int_{\mathbb{R}^N} |f(u)||h|\,dx \leq M\int_{\mathbb{R}^N} |u|^{p-1}|h|\,dx \leq C\|u\|^{p-1}\|h\|$$

for all $h \in X$ by Proposition 17. Hence there exists a unique $\tilde{u} \in X$ such that $T(u)h = (\tilde{u} \mid h)$ for all $h \in X$. By obvious changes of variable, it is easy to see that for every $h \in X$

and $g \in O(N)$ one has $(\tilde{u} \mid h(g\,\cdot)) = (\tilde{u}(g^{-1}\,\cdot) \mid h)$ and $T(u)h(g\,\cdot) = T(u)h$, so that $(\tilde{u}(g^{-1}\,\cdot) \mid h) = (\tilde{u} \mid h)$. This means $\tilde{u}(g^{-1}\,\cdot) = \tilde{u}$ for all $g \in O(N)$, i.e., $\tilde{u} \in X_{\mathrm{r}}$. Now assume $I'(u) = 0$ in X_{r}'. Then $(\tilde{u} \mid h) = T(u)h = I'(u)h = 0$ for all $h \in X_{\mathrm{r}}$ implies $\tilde{u} = 0$. This means $T(u)h = 0$ for all $h \in X$, that is, (14) holds. □

LEMMA 20. *The functional $I : X_{\mathrm{r}}(\mathbb{R}^N; V) \to \mathbb{R}$ satisfies the Palais–Smale condition.*

PROOF. Let $\{u_n\} \subset X_{\mathrm{r}}$ be a sequence such that $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ in X_{r}'. We have to show that $\{u_n\}$ contains an X_{r}-converging subsequence. A standard argument employing assumption $(\mathbf{f})_1$ shows that $\{u_n\}$ is bounded in X_{r}. Then up to a subsequence $u_n \rightharpoonup u$ in X_{r} and, by Proposition 14, $u_n \to u$ in L^p for some $u \in X_{\mathrm{r}}$. It is a standard exercise to conclude that $u_n \to u$ in X_{r}. □

PROOF OF THEOREM 8. As we are interested in nonnegative solutions, it is not restrictive to assume $f(s) = 0$ for all $s \le 0$. Indeed, $(\mathbf{f}_p)$ implies $f(0) = 0$ and the hypotheses of the theorem are still satisfied upon replacing $f(s)$ with $f(s_+)$. We want to apply the well known Mountain Pass Theorem. To this end we observe that, by the continuous embeddings (16)–(17), assumption $(\mathbf{f}_p)$ yields

$$\left| \int_{\mathbb{R}^N} F(u)\,dx \right| \le C \int_{\mathbb{R}^N} |u|^p\,dx \le C\|u\|^p \quad \text{for all } u \in X_{\mathrm{r}}$$

so that

$$I(u) \ge \frac{1}{2}\|u\|^2 - C\|u\|^p \quad \text{for all } u \in X_{\mathrm{r}}. \tag{18}$$

Since $p > 2$, this proves that I has a mountain-pass geometry near the origin, namely there exist $\delta, \rho > 0$ such that for all $u \in X_{\mathrm{r}}$ with $\|u\| = \rho$ one has $I(u) \ge \delta$. On the other hand, we are going to show that, under assumptions $(\mathbf{V})_2$ and $(\mathbf{f})_2$ or under assumption $(\mathbf{f})_3$, there exists $\bar{u} \in X_{\mathrm{r}}$ such that $\|\bar{u}\| > \rho$ and $I(\bar{u}) < 0$.

First assume that $(\mathbf{V})_2$ and $(\mathbf{f})_2$ hold. Take $u \in X_{\mathrm{r}}$ such that $\int_{\mathbb{R}^N} F(u)\,dx > 0$ (see Lemma 18) and set $u_n(x) := u(\mu_n^{-1}x)$ where $\{\mu_n\} \subset (\mu_0, +\infty)$ is any diverging sequence such that the inequality in $(\mathbf{V})_2$ holds for every $\mu = \mu_n$ and almost every $s > 0$. Then $u_n \in X_{\mathrm{r}}$ is such that

$$\begin{aligned}\|u_n\|^2 &= \mu_n^{N-2} \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \mu_n^N \int_{\mathbb{R}^N} V(\mu_n|x|)u^2\,dx \\ &\ge \mu_n^{N-2} \int_{\mathbb{R}^N} |\nabla u|^2\,dx + A\mu_n^{N-\alpha} \int_{\mathbb{R}^N} \frac{u^2}{|x|^\alpha}\,dx \to +\infty\end{aligned}$$

and

$$\begin{aligned}I(u_n) &= \frac{1}{2}\mu_n^{N-2} \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{1}{2}\mu_n^N \int_{\mathbb{R}^N} V(\mu_n|x|)u^2\,dx - \mu_n^N \int_{\mathbb{R}^N} F(u)\,dx \\ &\le \frac{1}{2}\mu_n^{N-2} \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{B}{2}\mu_n^{N-\beta} \int_{\mathbb{R}^N} V(|x|)u^2\,dx - \mu_n^N \int_{\mathbb{R}^N} F(u)\,dx \to -\infty\end{aligned}$$

as $n \to \infty$.

We now assume hypothesis $(\mathbf{f})_3$. Together with $(\mathbf{f})_1$, it implies $F(s) \geq F(1)s^{\vartheta}$ for every $s \geq 1$. Hence for any $\lambda > 1$ and $u \in X_{\mathrm{r}}$ nonnegative one has

$$\int_{\mathbb{R}^N} F(\lambda u)\,dx = \int_{\lambda u \geq 1} F(\lambda u)\,dx + \int_{0 \leq \lambda u < 1} F(\lambda u)\,dx \geq F(1)\lambda^{\vartheta} \int_{\lambda u \geq 1} u^{\vartheta}\,dx$$
$$\geq F(1)\lambda^{\vartheta} \int_{u \geq 1} u^{\vartheta}\,dx.$$

Since $\vartheta > 2$, this gives

$$I(\lambda u) \leq \frac{1}{2}\lambda^2 \|u\|^2 - F(1)\lambda^{\vartheta} \int_{u \geq 1} u^{\vartheta}\,dx \to -\infty \quad \text{as } \lambda \to +\infty.$$

As a conclusion, I exhibits a mountain-pass geometry. Hence Lemma 20 and the Mountain Pass Theorem provide the existence of a nontrivial critical point for I. By Lemma 19, it solves equation (1) in the sense of (14). Finally, we observe that, since $f(s) = 0$ for all $s \leq 0$, any $u \in X$ satisfying (14) has to be nonnegative. Indeed, one has $f(u)u_- = 0$ almost everywhere on $\mathbb{R}^N$, so that

$$0 = \int_{\mathbb{R}^N} \nabla u \cdot \nabla u_-\,dx + \int_{\mathbb{R}^N} V(|x|)uu_-\,dx - \int_{\mathbb{R}^N} f(u)u_-\,dx = -\|u_-\|^2$$

implies $u = u_+$. □

PROOF OF THEOREM 11. Since assumption $(\mathbf{f})_4$ implies $I(u) = I(-u)$ for all $u \in X_{\mathrm{r}}$, we can apply the Symmetric Mountain Pass Theorem (see for example Theorem 6.5 in [15]). To this end, taking into account (18) and Lemma 20, we need only show that I satisfies the following geometrical condition: for any finite-dimensional subspace $Y \neq \{0\}$ of X_{r} there exists $R > 0$ such that for all $u \in Y$ with $\|u\| \geq R$ one has $I(u) \leq 0$. In fact it is sufficient to prove that any sequence $\{u_n\} \subset Y$ with $\|u_n\| \to +\infty$ admits a subsequence such that $I(u_n) \leq 0$. This is ensured by assumption $(\mathbf{F}_p)$. Indeed, since all norms are equivalent on Y, one has

$$\int_{\mathbb{R}^N} F(u_n)\,dx \geq m \int_{\mathbb{R}^N} |u_n|^p\,dx \geq C\|u_n\|^p$$

so that, since $p > 2$, we get

$$I(u_n) = \frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^N} F(u_n)\,dx \leq \frac{1}{2}\|u_n\|^2 - C\|u_n\|^p \to -\infty$$

and hence the claim follows. Then the Symmetric Mountain Pass Theorem yields the existence of an unbounded sequence of critical values for I, to which there corresponds a sequence of nontrivial critical points for I. Lemma 19 completes the proof. □

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