Rend. Lincei Mat. Appl. 17 (2006), 35–67

Partial differential equations. — *Parabolic problems with dynamical boundary conditions: eigenvalue expansions and blow up*, by CATHERINE BANDLE, JOACHIM VON BELOW and WOLFGANG REICHEL, presented by A. Ambrosetti.

ABSTRACT. — An existence theory for local solutions of a parabolic problem under a dynamical boundary condition $\sigma u_t + u_n = 0$ is developed and a spectral representation formula is derived. It relies on the spectral theory of an associated elliptic problem with the eigenvalue parameter both in the equation and the boundary condition. The well-posedness of the parabolic problem holds in some natural space only if the number of negative eigenvalues is finite. This depends on the parameter σ in the boundary condition. If $\sigma \geq 0$ the parabolic problem is always well-posed. For $\sigma < 0$ it is well-posed only if the space dimension is 1 and ill-posed in space dimension ≥ 2. By means of the theory of compact operators the spectrum is analyzed and some qualitative properties of the eigenfunctions are derived. An interesting phenomenon is the "parameter-resonance", where for a specific parameter-value σ_0 two eigenvalues of the elliptic problem cross. Examples are given for which the eigenvalues and eigenfunctions can be computed explicitly. The last part of the paper deals with blow up of solutions of the parabolic problem with nonlinear positive sources.

KEY WORDS: Compact operator; comparison principles for elliptic and parabolic equations; variational characterization of eigenvalues; existence theory for local solutions; series representation; blow up phenomena.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 35K20, 47A75, 35K60, 35C10.

1. INTRODUCTION

Let $D \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary ∂D, and let n be its outer normal. Let $q(x)$ and $\mathcal{F}(x,t,u)$ be positive functions and σ be an arbitrary real number. In this paper we shall discuss parabolic problems of the form

$$u_t - \Delta u + q(x)u = \mathcal{F}(x,t,u) \quad \text{in } D \times (0,T), \tag{1.1}$$

$$\sigma u_t + u_n = 0 \quad \text{on } \partial D \times (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x) \quad \text{in } D. \tag{1.3}$$

We will study the existence of local and global weak solutions by means of a Hilbert space approach and derive a representation formula for the solutions.

The most general result concerning existence of local solutions for positive σ was established by Escher [10] in the framework of semigroups (cf. also the various references to previous work cited therein). The question of blow up and of the blow up time for the nonlinear problem was addressed in [7] where it was shown, again for positive σ, that for large initial conditions blow up occurs for the same type of nonlinearities as for Dirichlet and Neumann boundary conditions.

Many authors have studied different variants of (1.1)–(1.3). Vitillaro [20] considered nonlinear boundary conditions where u_t is replaced by $|u_t|^{m-1}u_t$. Fila and Quittner [11] treated the problem (1.1)–(1.3) with a nonlinear term on the right-hand side of (1.2) of the form $\mu|u|^{q-1}u$. They were mainly interested under what conditions solutions exist globally for all times or when they blow up in finite time. In all papers it is assumed that σ is positive. Problems with nonlinear Neumann boundary conditions ($\sigma = 0$) with (1.2) replaced by $u_n = g(u)$ have been the object of many papers (cf. e.g. [16]).

The case of $\sigma < 0$ is less studied [4], [19]. It turns out that it is much more delicate and gives rise to unexpected phenomena. In fact the parabolic problem is well-posed in some natural Hilbert space $L^2((0,T), H^1(D))$ if σ is nonnegative. For negative σ, however, well-posedness holds in the space $C([0,T], H^1(D))$ only in space dimension one. In higher space dimensions the parabolic problem is ill-posed in this space.

For the expansion of the solutions of the heat equation with dynamical boundary conditions into a Fourier series we are led to the following eigenvalue problem:

$$-\Delta\varphi + q(x)\varphi = \lambda\varphi \quad \text{in } D, \qquad \varphi_n = \lambda\sigma\varphi \quad \text{on } \partial D. \tag{1.4}$$

The corresponding Rayleigh quotient reads

$$R[v] = \frac{\int_D |\nabla v|^2\,dx + \int_D qv^2\,dx}{\int_D v^2\,dx + \sigma \oint_{\partial D} v^2\,ds},$$

which is not positive definite for negative σ.

The spectral theory for such problems has been treated by Ercolano and Schechter [9] for formally self-adjoint elliptic operators of second and higher order under lower boundedness assumptions. For positive space-dependent σ and for second order elliptic operators it has been shown by François [12] that the spectrum consists of countably many eigenvalues bounded from below and tending to $+\infty$. In the present paper the corresponding result is derived for negative constant σ, more precisely, the spectrum has an additional negative set which is infinite in dimensions $N \geq 2$ and finite for $N = 1$. The eigenfunctions are complete in $H^1(D)$ except in the *resonance case* $|D| + \sigma|\partial D| = 0$ where it has to be supplemented with an additional element.

The main existence results for the linear heat equation with dynamical boundary conditions are presented at the beginning of Section 2. We follow a Hilbert space approach as in [17] which has the advantage of providing an expansion formula for the solutions. The main part of Section 3 deals with the eigenvalue problem (1.4). The spectrum will be described completely by means of the theory of compact linear operators. As an illustration we compute the explicit form of the eigenfunctions on intervals $(0, L) \subset \mathbb{R}$ and balls $B_R(0) \subset \mathbb{R}^N$.

In Section 4 we study the phenomenon of parameter resonance in detail. As σ passes through a critical value $\sigma_0 = -|D|/|\partial D| < 0$ the first positive eigenvalue $\lambda_1(\sigma)$ varies continuously into the first negative eigenvalue $\lambda_{-1}(\sigma)$. At the resonance value σ_0 itself, this gives rise to an incomplete system of eigenfunctions, which needs to be augmented by an extra element.

In Section 5 we study the long-time behavior of the solutions of (1.1)–(1.3) with nonlinear positive sources. The main result concerns the case where $\sigma \geq 0$ and the initial data are also positive. It turns out that all solutions blow up in finite time if the corresponding reaction equation $\dot{z} = f(z)$ has no global solutions. Loosely speaking, the mechanism resembles the one with Neumann rather than Dirichlet boundary conditions. This completes the results on existence of blow up initiated in [7]. The case of negative σ seems to be much more involved. The difficulty comes mainly from the fact that the standard comparison principles do not hold, and the method of upper and lower solutions is not applicable. For instance in contrast to the case of positive σ, a solution with positive initial data can become negative in finite time. Currently no results on blow up or global existence seem to be available and these questions remain open.

After completion of the manuscript it was brought to our attention that J. L. Vazquez and E. Vitillaro [19] have studied (1.1)–(1.3) with $\mathcal{F} = 0$, $q = 0$ and arbitrary σ. Their results on well- and ill-posedness of the parabolic problem (1.1)–(1.3) and on the elliptic eigenvalue problem (1.4) overlap with ours. Their results are stated for C^∞-domains. Their approach is different, e.g., they show that $\langle u, v\rangle + L_1(u)L_1(v)u$ with $L_1(u) = (\int_D u\,dx + \sigma \oint u\,ds)/|D| + \sigma|\partial D|$ is an equivalent norm in $H^1(D)$ except in the resonance case $|D| + \sigma|\partial D| = 0$.

2. Main existence results

2.1. *Introduction*

Consider the linear problem

$$u_t - \Delta u + q(x)u = f(x,t) \quad \text{in } D \times \mathbb{R}^+, \tag{2.1}$$

$$\sigma u_t + u_n = 0 \quad \text{on } \partial D \times \mathbb{R}^+, \tag{2.2}$$

$$u(x,0) = u_0(x) \quad \text{in } D, \tag{2.3}$$

with $q \in L^\infty(D)$. Let us introduce some notation. For $u, v \in H^1(D)$ we set

$$\langle u, v\rangle = \int_D (\nabla u \nabla v + q(x)uv)\,dx,$$

$$(u, v) = \int_D uv\,dx, \quad (u, v)_0 = \oint_{\partial D} uv\,ds.$$

If $q \geq 0$ and $\int_D q\,dx > 0$ then the form $\langle\cdot,\cdot\rangle$ induces a norm which is equivalent to the standard norm of $H^1(D)$. Assume that $u_0 \in H^1(D)$ and $f \in L^2((0,T), L^2(D))$. A function $u \in C([0,T], H^1(D))$ is called a *weak solution* of (2.1)-(2.3) if

$$-\int_0^T ((u, \phi_t) + \sigma(u, \phi_t)_0)\,dt + \int_0^T \langle u, \phi\rangle\,dt = \int_0^T (f, \phi)\,dt + (u_0, \phi) + \sigma(u_0, \phi)_0$$

for all $\phi \in C^1([0,T], H^1(D))$ with $\phi(\cdot, T) \equiv 0$.

We will use the space $L^2(D) \times L^2(\partial D)$ consisting of the pairs (f, g) of functions $f \in L^2(D)$ and $g \in L^2(\partial D)$. Writing $u \in L^2(D) \times L^2(\partial D)$ means that $u : D \to \mathbb{R}$ is in $L^2(D)$ and $u : \partial D \to \mathbb{R}$ is in $L^2(\partial D)$, but in general there is no coupling between u in D and u on ∂D. Recall however that for domains D with Lipschitz boundary every function $u \in H^1(D)$ has a trace in $H^{1/2}(\partial D)$ (cf. Alt [1]). Hence $H^1(D) \hookrightarrow L^2(D) \times L^2(\partial D)$ compactly.

2.2. *Results for the eigenvalue problem*

If we are looking for solutions of the homogeneous heat equation (2.1) with $f \equiv 0$ satisfying the boundary conditions (2.2), of the form $u(x,t) = e^{-\lambda t}\varphi(x)$, then $\varphi(x)$ is a solution of the eigenvalue problem

$$-\Delta\varphi + q(x)\varphi = \lambda\varphi \quad \text{in } D, \qquad \varphi_n = \sigma\lambda\varphi \quad \text{on } \partial D. \tag{2.4}$$

We first collect some results on the eigenvalue problem (2.4) and refer to a later Section 3 for the proofs. Define

$$a(u,v) := (u,v) + \sigma(u,v)_0, \qquad u, v \in H^1(D).$$

The eigenvalue problem (2.4) can be expressed in the weak form as

$$\langle\varphi, z\rangle = \lambda a(\varphi, z) \qquad \forall z \in H^1(D).$$

We shall use the following notation:

$$\mathbb{N} = \{1, 2 \dots\}, \qquad \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \qquad \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}.$$

Let φ_i and λ_i, $i \in I$, denote all the eigenfunctions and eigenvalues of (2.4). We shall show that the index set I is countably infinite. A negative (resp. positive) index will stand for a negative (resp. positive) eigenvalue. If zero is an eigenvalue then it will be denoted by λ_0. Our results on the eigenvalue problem are as follows.

THEOREM 1. *Let $q \geq 0$ with $\int_D q\,dx > 0$. Then there exists a complete set $\{\psi_i\}_{i\in I} \subset H^1(D)$ of eigenfunctions of* (2.4) *with $\langle\psi_i, \psi_j\rangle = \delta_{ij}$. For every $u \in H^1(D)$ we have $u = \sum_{i\in I}\langle u, \psi_i\rangle\psi_i$ in $H^1(D)$.*

(i) *If $\sigma \geq 0$ then the eigenvalues are positive and $I = \mathbb{N}$. For every $u \in L^2(D) \times L^2(\partial D)$ we have*
$$u = \sum_{i=1}^{\infty} a(u, \varphi_i)\varphi_i$$
in $L^2(D) \times L^2(\partial D)$ with $\varphi_i = \sqrt{\lambda_i}\psi_i$.

(ii) *If $N \geq 2$ and $\sigma < 0$ then there are countably many positive and negative eigenvalues, i.e. $I = \mathbb{Z} \setminus \{0\}$.*

(iii) *If $N = 1$ and $\sigma < 0$ then $I = \{-k, \dots, -1\} \cup \mathbb{N}$ with $k \geq 2$.*

REMARK. For $\sigma < 0$ and $u \in H^1(D)$ the expansion $u = \sum_{i\in I}\langle u, \psi_i\rangle\psi_i$ can be rewritten as

$$u = \sum_{i\in I} \operatorname{sign}(\lambda_i) a(u, \varphi_i)\varphi_i \tag{2.5}$$

with $\varphi_i = \sqrt{|\lambda_i|}\psi_i$. Unlike the case $\sigma \geq 0$ we do not know if this expansion is true for $u \in L^2(D) \times L^2(\partial D)$ even if only finitely many negative eigenvalues are present.

The case $q \equiv 0$ is more involved. We shall distinguish between two cases: $|D| + \sigma|\partial D| \neq 0$ and $|D| + \sigma|\partial D| = 0$.

THEOREM 2. *Assume* $q \equiv 0$ *and* $|D| + \sigma|\partial D| \neq 0$. *Then there exists a complete set* $\{\psi_i\}_{i\in I} \subset H^1(D)$ *of eigenfunctions of* (2.4) *with* $\langle \psi_i, \psi_j\rangle = \delta_{ij}$ *for* $i, j \neq 0$. *For every* $u \in H^1(D)$ *we have* $u = \sum_{i\in I\setminus\{0\}} \langle u, \psi_i\rangle \psi_i + P(u)$ *in* $H^1(D)$ *where*

$$P(u) := \frac{a(u,1)}{a(1,1)} = \frac{\int_D u\,dx + \sigma \oint_{\partial D} u\,ds}{|D| + \sigma|\partial D|}$$

is a projection into the eigenspace corresponding to λ_0.

(i) *If* $\sigma \geq 0$ *then the eigenvalues are nonnegative and* $I = \mathbb{N}_0$. *For every* $u \in L^2(D) \times L^2(\partial D)$ *we have*

$$u = \sum_{i=1}^{\infty} a(u, \varphi_i)\varphi_i + P(u)$$

in $L^2(D) \times L^2(\partial D)$ *with* $\varphi_i = \sqrt{\lambda_i}\psi_i$.

(ii) *If* $N \geq 2$ *and* $\sigma < 0$ *then* $I = \mathbb{Z}$.

(iii) *Let* $N = 1$, $D = (0, L)$ *and* $\sigma < 0$. *If* $|\sigma| < L/2$ *then* $I = \{-2, -1\} \cup \mathbb{N}_0$ *and if* $|\sigma| > L/2$ *then* $I = \{-1\} \cup \mathbb{N}_0$.

In order to describe the situation in the case $|D| + \sigma|\partial D| = 0$ if $N \geq 2$ and correspondingly $|D| + 2\sigma = 0$ if $N = 1$ we consider an arbitrary solution w of the boundary value problem

$$-\Delta w = 1 \quad \text{in } D, \qquad w_n = \sigma \quad \text{on } \partial D. \tag{2.6}$$

Note that all eigenfunctions, including the constant ones, belong to the space

$$\mathcal{V} = \left\{ v \in H^1(D) : a(v,1) = \int_D v\,dx + \sigma \oint_{\partial D} v\,ds = 0 \right\}.$$

In addition, all eigenfunctions except the constants lie in the subspace

$$\mathcal{V}_w = \left\{ v \in \mathcal{V} : a(v,w) = \int_D vw\,dx + \sigma \oint_{\partial D} vw\,ds = 0 \right\},$$

where w is an arbitrary but fixed solution of (2.6). Hence every element $u \in H^1(D)$ can be split into

$$u = u_w + P(u) + Q(u)w$$

where

$$u_w \in \mathcal{V}_w, \qquad P(u) = \frac{a(u,w)}{a(1,w)} - \frac{a(w,w)a(u,1)}{a(1,w)^2}, \qquad Q(u) = \frac{a(u,1)}{a(w,1)}.$$

THEOREM 3. *Assume* $q \equiv 0$ *and* $|D| + \sigma|\partial D| = 0$. *Then there exists an orthonormal system* $\{\psi_i\}_{i\in I} \subset H^1(D)$ *of eigenfunctions of* (2.4) *with* $\langle \psi_i, \psi_j\rangle = \delta_{ij}$ *for* $i, j \neq 0$. *By adding the functions* 1 *and* w *the expansion* $u = \sum_{i\in I\setminus\{0\}} \langle u, \psi_i\rangle\psi_i + P(u) + Q(u)w$ *holds in* $H^1(D)$ *for every* $u \in H^1(D)$.

(i) *If* $N \geq 2$ *then* $I = \mathbb{Z}$.

(ii) *Let* $N = 1$ *and* $D = (0, L)$. *Then* $I = \{-1\} \cup \mathbb{N}_0$.

2.3. *Existence results for the linear parabolic problem*

The weak solution of the parabolic problem (2.1)–(2.3) can now be constructed by means of the complete system of eigenfunctions introduced in the previous section. Let ψ_i and w have the same meaning as in Theorems 1–3.

The following simple result will be helpful for the convergence proof of the formal solution of the parabolic problem.

LEMMA 4. *There exists a constant $C > 0$ such that for every $f \in L^2(D)$ one has*

$$\sum_{i \in I} (f, \psi_i)^2 \leq C \|f\|^2_{L^2(D)}.$$

PROOF. For $k \in \mathbb{N}$ let $I_k = I \setminus \{0\} \cap \{-k, \dots, k\}$ and let $Z_k = \text{span}[\psi_i : i \in I_k]$. The function $z_k := \sum_{i \in I_k} (f, \psi_i)\psi_i$ satisfies $\langle z_k, \phi \rangle = (f, \phi)$ for all $\phi \in Z_k$. Hence

$$\langle z_k, z_k \rangle = \sum_{i \in I_k} (f, \psi_i)^2 = (f, z_k) \leq \|f\|_{L^2(D)} \|z_k\|_{L^2(D)} \leq C \|f\|_{L^2(D)} \|z_k\|_{H^1(D)}.$$

Since the constants do not belong to Z_k the bilinear form $\langle \cdot, \cdot \rangle$ produces a norm on Z_k which is equivalent to the $H^1(D)$-norm. Therefore $\langle z_k, z_k \rangle = \sum_{i \in I_k} (f, \psi_i)^2 \leq C \|f\|^2_{L^2(D)}$. Letting $k \to \infty$ we get $\sum_{i \in I \setminus \{0\}} (f, \psi_i)^2 \leq C \|f\|^2_{L^2(D)}$ and the same holds if $i = 0$ is included. □

THEOREM 5. *Suppose there are at most finitely many negative eigenvalues. Let $f \in H^1((0, T), L^2(D))$ and $u_0 \in H^1(D)$. Then problem* (2.1)–(2.3) *is well-posed and has a unique solution $u \in C([0, T], H^1(D))$ in the sense of Section* 2.1. *The solution has the following form:*

(i) *If $q \geq 0$ and $\int_D q \, dx > 0$ then*

$$u(x, t) = \sum_{i \in I} \langle u_0, \psi_i \rangle e^{-\lambda_i t} \psi_i(x) + \sum_{i \in I} f_i(t) \lambda_i e^{-\lambda_i t} \psi_i(x),$$

where $f_i(t) = \int_0^t (f(\cdot, \tau), \psi_i) e^{\lambda_i \tau} \, d\tau$ for $i \in I$.

(ii) *If $q \equiv 0$ and $|D| + \sigma |\partial D| \neq 0$ then*

$$u(x, t) = \sum_{i \in I \setminus \{0\}} \langle u_0, \psi_i \rangle e^{-\lambda_i t} \psi_i(x) + \frac{a(u_0, 1)}{a(1, 1)} + \sum_{i \in I \setminus \{0\}} f_i(t) \lambda_i e^{-\lambda_i t} \psi_i(x) + \frac{f_0(t)}{a(1, 1)},$$

where $f_i(t) = \int_0^t (f(\cdot, \tau), \psi_i) e^{\lambda_i \tau} \, d\tau$ for $i \in I \setminus \{0\}$ and $f_0(t) = \int_0^t (f(\cdot, \tau), 1) \, d\tau$.

(iii) *If $q \equiv 0$ and $|D| + \sigma |\partial D| = 0$ then*

$$\begin{aligned} u(x, t) = & \sum_{i \in I \setminus \{0\}} \langle u_0, \psi_i \rangle e^{-\lambda_i t} \psi_i(x) \\ & + \frac{a(u_0, w)}{a(1, w)} - \frac{a(w, w) a(u_0, 1)}{a(1, w)^2} + \frac{a(u_0, 1)}{a(w, 1)} (w(x) - t) \\ & + \sum_{i \in I \setminus \{0\}} f_i(t) \lambda_i e^{-\lambda_i t} \psi_i(x) + f_0(t) + \tilde{f}(t)(w(x) - t) \end{aligned}$$

where

$$f_i(t) = \int_0^t (f(\cdot,\tau),\psi_i)e^{\lambda_i\tau}\,d\tau \text{ for } i \in I\setminus\{0\},$$

$$f_0(t) = \int_0^t \left(\frac{(f(\cdot,\tau),w)}{a(1,w)} - \frac{a(w,w)(f(\cdot,\tau),1)}{a(1,w)^2} + \frac{\tau(f(\cdot,\tau),1)}{a(w,1)}\right)d\tau,$$

$$\tilde f(t) = \int_0^t \frac{(f(\cdot,\tau),1)}{a(w,1)}\,d\tau.$$

REMARK. For $\sigma \geq 0$ the above theorem can be improved as follows. Suppose $f \in L^2((0,T),L^2(D))$ and $u_0 \in L^2(D)\times L^2(\partial D)$. Then the solution has the properties $u \in L^2((0,T),H^1(D))$, $u_t \in L^2((0,T),L^2(D))$, trace $u_t \in L^2((0,T),L^2(\partial D))$ and moreover $\|u-u_0\|_{L^2(D)}$, $\|$ trace $u-u_0\|_{L^2(\partial D)} \to 0$ as $t\to 0$. Furthermore, u is a solution in the weak sense, i.e., for all $\tau \in (0,T)$ and for all $\phi \in L^2((0,T),H^1(D))$,

$$\int_0^\tau ((u_t,\phi)+\sigma(u_t,\phi)_0)\,dt + \int_0^\tau \langle u,\phi\rangle = \int_0^\tau (f,\phi)\,dt.$$

PROOF OF THEOREM 5. As an illustration we prove (iii). The proofs of statements (i) and (ii) are almost the same. In view of Theorem 3 we look for a solution of the form

$$u(x,t) = \sum_{j\in I\setminus\{0\}} \alpha_j(t)\psi_i(x) + \alpha_0(t) + \tilde\alpha(t)(w(x)-t).$$

First we replace the infinite sum $\sum_{j\in I\setminus\{0\}}$ by a finite sum $\sum_{j\in I_k}$, $I_k = I\setminus\{0\}\cap\{-k,\dots,k\}$, and show that the coefficients $\alpha_j(t)$ have the form given in the theorem. We insert the finite sum expression u^k into the weak form of (2.1)–(2.3), where u_0 is replaced by the projection of u_0^k into $Z_k = \operatorname{span}[\psi_i : i\in I_k]\oplus\operatorname{span}[1,w]$. For finite sums u^k we can use the concept of classical solution of (2.1)–(2.3). Testing with a function $\phi\in H^1(D)$ this means

$$a(u_t^k,\phi) + \langle u^k,\phi\rangle = (f,\phi) + (g,\phi)_0.$$

Replacing ϕ successively with ψ_i, 1 and w and keeping in mind that

$$\lambda_i a(\psi_i,\psi_j) = \delta_{ij}, \quad a(\psi_i,1) = a(1,1) = a(\psi_i,w) = 0,$$
$$\langle\psi_i,\psi_j\rangle = \delta_{ij}, \quad \langle\psi_i,w\rangle = 0,$$

we obtain the following set of equations:

$$\frac{\dot\alpha_i}{\lambda_i} + \alpha_i = (f,\psi_i) \quad \text{if } i\in I\setminus\{0\},$$
$$\dot{\tilde\alpha}(t)a(w,1) = (f,1),$$
$$\dot\alpha_0(t)a(w,1) + \dot{\tilde\alpha}(t)a(w-t,w) = (f,w).$$

The expressions for the coefficients $\alpha_i,\alpha_0,\tilde\alpha$ follow by straightforward integration if we impose the initial condition $u^k(0) = u_0^k$. Now we can build the full series defining $u(x,t)$. We will show next that $u\in C([0,T],H^1(D))$. This then establishes that u is a weak solution in the sense of Section 2.1. Note that $\langle\cdot,\cdot\rangle$ introduces an equivalent norm on $H^1(D)$ only in the case $\int_D q\,dx > 0$. For $q \equiv 0$ it is an equivalent norm only on the subspaces $\mathcal{V}$, $\mathcal{V}_w$. But since these subspaces have codimension 1 or 2, it is enough to control $\langle u,u\rangle$. Let

$$u_a(x,t) = \sum_{i\in I\setminus\{0\}} \langle u_0,\psi_i\rangle e^{-\lambda_i t}\psi_i(x), \qquad u_b(x,t) = \sum_{i\in I\setminus\{0\}} f_i(t)\lambda_i e^{-\lambda_i t}\psi_i(x),$$

where $f_i(t) = \int_0^t (f,\psi_i)e^{\lambda_i s}\,ds$. Then for $t\in[0,T]$ one finds

$$\langle u_a,u_a\rangle \le \sum_{i\in I,\, i<0} \langle u_0,\psi_i\rangle^2 e^{-2\lambda_i t} + \sum_{i\in I,\, i>0} \langle u_0,\psi_i\rangle^2 \le C\|u_0\|^2_{H^1(D)},$$

where $C = e^{2|\lambda_{\min}|T}$ and $\lambda_{\min}$ is the smallest (negative) eigenvalue. Lebesgue's dominated convergence theorem implies that $u_a(\cdot,t)$ is continuous as a function from $[0,T]$ to $H^1(D)$. Next we need to show the same for u_b. Note first that

$$\langle u_b,u_b\rangle = \sum_{i\in I\setminus\{0\}} f_i(t)^2\lambda_i^2 e^{-2\lambda_i t}. \tag{2.7}$$

For $i\neq 0$ one has

$$f_i(t) = \frac{1}{\lambda_i}(f(\cdot,t),\psi_i)e^{\lambda_i t} - \frac{1}{\lambda_i}(f(\cdot,0),\psi_i) - \frac{1}{\lambda_i}\int_0^t (f_t(\cdot,s),\psi_i)e^{\lambda_i s}\,ds$$

and hence

$$f_i(t)^2 \le \frac{2}{\lambda_i^2}(f(\cdot,t),\psi_i)^2 e^{2\lambda_i t} + \frac{2}{\lambda_i^2}(f(\cdot,0),\psi_i)^2 + \int_0^t (f_t(\cdot,s),\psi_i)^2\,ds\,\frac{e^{2\lambda_i t}-1}{\lambda_i^3}.$$

Finally, this leads to

$$\sum_{i\in I\setminus\{0\}} f_i(t)^2\lambda_i^2 e^{-2\lambda_i t} \le C\sum_{i\in I\setminus\{0\}}\left((f(\cdot,t),\psi_i)^2 + (f(\cdot,0),\psi_i)^2 + \int_0^t (f_t(\cdot,s),\psi_i)^2\,ds\right).$$

Applying Lemma 4 we obtain

$$\sum_{i\in I\setminus\{0\}} f_i(t)^2\lambda_i^2 e^{-2\lambda_i t} \le C\int_0^T (\|f(\cdot,s)\|^2_{L^2(D)} + \|f_t(\cdot,s)\|^2_{L^2(D)})\,ds,$$

where the constants C depend on $|\lambda_{\min}|$ and T. This shows that the series on the right-hand side of (2.7) converges uniformly in t. As before, Lebesgue's dominated convergence theorem implies that $u_b(\cdot,t)$ is continuous as a function from $[0,T]$ to $H^1(D)$. This finishes the proof of the theorem. □

3. Spectral theory

In this section we shall prove Theorems 1–3 on the structure of the spectrum of (2.4).

If $\sigma \geq 0$ and $q \equiv 0$ it is known from [6] that the eigenvalue problem (2.4) has countably many positive eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ such that $\lambda_k \to \infty$ as $k \to \infty$. We shall extend this result to the cases of Theorems 1–3. Frequently in this section we use the following well known result (cf. Alt [1]):

Lemma 6. *If $\mathcal{V}$ is a closed subspace of $H^1(D)$ not containing the constants, then $(\int_D |\nabla u|^2\,dx)^{1/2}$ is an equivalent norm on $\mathcal{V}$. In particular, there exist constants $C_1, C_2 > 0$ such that for all $v \in \mathcal{V}$,*

$$\int_D v^2\,dx \leq C_1 \int_D |\nabla v|^2\,dx, \qquad \oint_{\partial D} v^2\,ds \leq C_2 \int_D |\nabla v|^2\,dx. \tag{3.1}$$

3.1. *The case $q(x) \geq 0$, $\int_D q\,dx > 0$*

In this case the form $\langle\cdot,\cdot\rangle$ generates the norm $(\int_D(|\nabla v|^2 + q(x)v^2)\,dx)^{1/2}$, which is equivalent to the standard norm of $H^1(D)$. To see this note first that by Lemma 6,

$$\left\| v - \frac{\int_D vq\,dx}{\int_D q\,dx} \right\|^2_{H^1(D)} \leq C \int_D |\nabla v|^2\,dx \tag{3.2}$$

since the space $\{v \in H^1(D) : \int_D vq\,dx = 0\}$ does not contain the constants. It follows from (3.2) that $\|v\|^2_{H^1(D)} \leq C \int_D(|\nabla v|^2 + q(x)v^2)\,dx$. Now we can describe the eigenvalues of (2.4) as eigenvalues of a compact operator as follows.

Lemma 7. *For $h \in H^1(D)$ there exists a unique $v \in H^1(D)$ such that*

$$-\Delta v + q(x)v = h \quad \text{in } D, \qquad v_n = \sigma h \quad \text{on } \partial D. \tag{3.3}$$

The operator

$$K : H^1(D) \to H^1(D), \qquad h \mapsto v,$$

is compact, invertible and self-adjoint with respect to the inner product $\langle\cdot,\cdot\rangle$. Hence it has countably many eigenvalues $\{\mu_k\}_{k\in I}$ and the eigenfunctions form a complete system in $H^1(D)$. The eigenvalues of (2.4) *are the reciprocals $\lambda_i = \mu_i^{-1}$.*

Proof. For $h \in H^1(D)$ the functional $L_h : H^1(D) \to \mathbb{R}$ given by $L_h(\phi) = \int_D h\phi\,dx + \sigma \oint_{\partial D} h\phi\,ds$ is continuous and hence by the Riesz representation theorem there exists a unique $v \in H^1(D)$ such that $\langle v, \phi\rangle = L_h(\phi)$ for all $\phi \in H^1(D)$. Thus v is the weak solution of (3.3) and the operator K is well defined. Continuity and compactness of K are standard and invertibility and symmetry are immediate. □

Remark. The following is a more general version of Lemma 7. Let $\mathcal{W} = \{(f, g) \in L^2(D) \times L^2(\partial D)\}$ be equipped with the norm $\|(f, g)\| = (\|f\|^2_{L^2(D)} + \|g\|^2_{L^2(\partial D)})^{1/2}$. Then for every $(f, g) \in \mathcal{W}$ there exists a unique $v \in H^1(D)$ such that $-\Delta v + q(x)v = f$

in D, $v_n = \sigma g$ on ∂D. The corresponding solution operator $T : (f, g) \mapsto v$ from $\mathcal{W}$ to $H^1(D)$ is compact.

As a consequence of Lemma 7 eigenvalues of (2.4) can be described variationally as critical values of the functional

$$J(v) := a(v, v) = \int_D v^2\,dx + \sigma \oint_{\partial D} v^2\,ds$$

in the set $\{v \in H^1(D) : \int_D (|\nabla v|^2 + q(x)v^2)\,dx = 1\}$. Obviously, if $\sigma \geq 0$ there are only positive eigenvalues. This explains part (i) of Theorem 1. For the remaining parts of Theorem 1 note the following (see e.g. De Figueiredo [8]):

LEMMA 8. *Assume $\sigma < 0$. Suppose that*

$$\mu_1 = \sup\{J(v) : \langle v, v\rangle = 1\} > 0, \qquad \mu_{-1} = \inf\{J(v) : \langle v, v\rangle = 1\} < 0.$$

Then $\lambda_1 = \mu_1^{-1}$ and $\lambda_{-1} = \mu_{-1}^{-1}$ are the first positive and first negative eigenvalues of (2.4)*. Moreover, the following holds:*

(a) *Let $k \in \mathbb{N}$. Suppose $0 < \lambda_1 \leq \ldots \leq \lambda_k$ are the (not necessarily different) first k positive eigenvalues with eigenfunctions $\psi_1, \ldots, \psi_k$. Suppose that*

$$\mu_{k+1} = \sup\{J(v) : \langle v, v\rangle = 1,\ a(\psi_j, v) = 0,\ j = 1, \ldots, k\} > 0.$$

Then $\lambda_{k+1} = \mu_{k+1}^{-1}$ is the next positive eigenvalue.

(b) *Let $k \in \mathbb{N}$. Suppose $\lambda_{-k} \leq \cdots \leq \lambda_{-1} < 0$ are the (not necessarily different) first k negative eigenvalues with eigenfunctions $\psi_{-k}, \ldots, \psi_{-1}$. Suppose that*

$$\mu_{-k-1} = \inf\{J(v) : \langle v, v\rangle = 1,\ a(\psi_j, v) = 0,\ j = -k, \ldots, -1\} < 0.$$

Then $\lambda_{-k-1} = \mu_{-k-1}^{-1}$ is the next negative eigenvalue.

It is easy to see that the critical values μ_j, μ_{-j} are attained provided they are positive, negative, resp.

THEOREM 9. *Let $\sigma < 0$.*

(a) *Then* (2.4) *has an unbounded sequence of positive eigenvalues.*
(b) *If $N \geq 2$ then* (2.4) *has an unbounded sequence of negative eigenvalues.*
(c) *If $N = 1$ then* (2.4) *has only finitely many but at least two negative eigenvalues, of multiplicity one.*

PROOF. (a) For any function $v \in H_0^1(D)$ we have $J(v) > 0$ since the boundary integral vanishes. Thus we see that $\mu_1 > 0$ is attained. Now it suffices to show that for any $k \in \mathbb{N}$ there exists a trial function v such that $a(\psi_j, v) = 0$ for $j = 1, \ldots, k$ and $J(v) > 0$. Such a choice is always possible in any $(k+1)$-dimensional subspace of $H_0^1(D)$.

(b) We need to show that $\mu_{-k-1} < 0$ for all $k \in \mathbb{N}_0$. By rotating and shifting D let us suppose that $\inf_D x_1 = 0$, $\sup_D x_1 = d$ with $d = \operatorname{diam} D$ and $x_2 \geq 0$ in D. We divide D into $k+1$ domains as follows:

$$D_i = \left\{x \in D : x_1 \in \left(\frac{(i-1)d}{k+1}, \frac{id}{k+1}\right)\right\}, \qquad i = 1, \ldots, k+1.$$

Clearly $\overline{D} = \overline{D}_1 \cup \cdots \cup \overline{D}_{k+1}$. Next we fix $\alpha > 0$ and define $k+1$ functions

$$v_i(x) = \begin{cases} x_2^\alpha \sin(\pi(k+1)x_1/d) & \text{if } x \in D_i, \\ 0 & \text{else,} \end{cases}$$

for $i = 1, \ldots, k+1$. Let $v = \sum_{i=1}^{k+1} c_i v_i$ and determine the vector $c = (c_1, \ldots, c_{k+1})$ from the condition $a(\psi_j, v) = 0$ for $j = -k, \ldots, -1$. These k conditions are represented by the linear system

$$\sum_{i=1}^{k+1} c_i a(\psi_j, v_i) = 0 \quad \text{for } j = -k, \ldots, -1,$$

where in case $k = 0$ there are no conditions on the value c. Since the linear system consists of k equations in $k+1$ unknowns $(c_1, \ldots, c_{k+1})$ we have at least a one-dimensional space of nontrivial solutions $0 \neq c = (c_1, \ldots, c_{k+1}) \in \mathbb{R}^{k+1}$. With $e_2 = (0, 1, 0, \ldots, 0) \in \mathbb{R}^N$ we compute

$$\begin{aligned} \int_{D_i} v^2\,dx &= c_i^2 \int_{D_i} x_2^{2\alpha} \sin^2(\pi(k+1)x_1/d)\,dx \\ &= \frac{c_i^2}{2\alpha+1} \int_{D_i} \nabla \cdot (x_2^{2\alpha+1} \sin^2(\pi(k+1)x_1/d) e_2)\,dx \\ &= \frac{c_i^2}{2\alpha+1} \oint_{\partial D_i} x_2^{2\alpha+1} \sin^2(\pi(k+1)x_1/d)\, e_2 \cdot n\,ds \\ &\le \frac{\operatorname{diam} D}{2\alpha+1} \oint_{\partial D_i} v^2\,ds. \end{aligned}$$

By superposition we obtain

$$\int_D v^2\,dx \le \frac{\operatorname{diam} D}{2\alpha+1} \oint_{\partial D} v^2\,ds.$$

If $\alpha > 0$ is so large that $\operatorname{diam} D/(2\alpha+1) < -\sigma$ then $J(v) = \int_D v^2\,dx + \sigma \oint_{\partial D} v^2\,ds < 0$. The remaining degree of freedom is the multiple of c which is chosen such that $\int_D (|\nabla v|^2 + q(x)v^2)\,dx = 1$. This shows that $\mu_{-k-1} < 0$.

(c) Let $D = (0, L)$ and $q_\infty = \|q\|_\infty$. Suppose v satisfies for $\lambda < 0$ the eigenvalue equation

$$-v'' + q(x)v = \lambda v \quad \text{in } (0, L), \qquad -v'(0) = \sigma\lambda v(0), \qquad v'(L) = \sigma\lambda v(L).$$

Since $q \in L^\infty(D)$ the solution v is understood as a $C^1(\overline{D})$-solution in the sense of Carathéodory. Consider also the solution u of the initial-value problem

$$-u'' + q_\infty u = \lambda u \quad \text{in } (0, L), \qquad -u'(0) = \sigma\lambda u(0),$$

which is unique up to multiples. Note that $v(0)/v'(0) = u(0)/u'(0)$ and assume $v(0) = u(0) = 1$. For negative λ the solution u is given by

$$u(x) = \cosh(\sqrt{-\lambda + q_\infty}\, x) - \frac{\sigma\lambda}{\sqrt{-\lambda + q_\infty}} \sinh(\sqrt{-\lambda + q_\infty}\, x).$$

Notice that u and v are convex as long as they are positive and concave as long as they are negative. For λ sufficiently small a computation shows that $u'(x)$ has no zero in $[0, L]$. In this case $u(x)$ is decreasing and vanishes exactly once at $\xi_u \in (0, L)$. By Sturm's Comparison Theorem (cf. [21], especially the observation that it also holds for C^1-solutions in the sense of Carathéodory), v has a zero $\xi_v \leq \xi_u$. From the identity

$$(v'u - u'v)' = (q - q_\infty)uv \leq 0 \quad \text{in } (\xi_u, L)$$

we deduce that

$$v'(L)u(L) - u'(L)v(L) + \underbrace{u'(\xi_u)v(\xi_u)}_{\geq 0} \leq 0.$$

Since $v'(L) = \sigma\lambda v(L)$ this implies

$$\sigma\lambda - \frac{u'(L)}{u(L)} \leq 0.$$

Since $u'(L)/u(L) \approx \sqrt{-\lambda}$ as $\lambda \to -\infty$ we conclude that all eigenvalues are bounded from below by some number Λ_0. Since the negative eigenvalues cannot have a finite accumulation point we have proved that there are at most finitely many.

It remains to show that there are at least two negative eigenvalues. For λ_{-1} it suffices to construct a function v such that $J(v) = \int_0^L v^2\,dx + \sigma(v^2(0) + v^2(L)) < 0$. This is achieved by $v(x) = x^\alpha$ with sufficiently large α. For λ_{-2} one needs a function v such that $J(v) < 0$ and $a(v, \psi_{-1}) = \int_0^L v\psi_{-1}\,dx + \sigma(v(0)\psi_{-1}(0) + v(L)\psi_{-1}(L)) = 0$. This can be obtained by taking

$$v(x) = \begin{cases} a_1|x - L/2|^\alpha & \text{on } [0, L/2], \\ a_2|x - L/2|^\alpha & \text{on } [L/2, L]. \end{cases}$$

For sufficiently large α the functional is negative independent of the choice of a_1, a_2. By choosing a_1, a_2 appropriately one can achieve $a(v, \psi_{-1}) = 0$. This finishes the proof of (c). □

3.1.1. *Qualitative properties of eigenfunctions.* Next we will show that the eigenvalues λ_{-1} and λ_1 are simple. We need the following version of the maximum principle. Note that for smooth domains and smooth solutions it is a simple consequence of Hopf's maximum principle.

LEMMA 10. *Suppose $D \subset \mathbb{R}^N$ is a bounded Lipschitz domain. Let $v \in H^1(D)$ be a weak solution of*

$$-\Delta v = a(x)v \quad \text{in } D, \qquad v_n = b(x)v \quad \text{on } \partial D$$

with $a \in L^\infty(D)$ and $b \in L^\infty(\partial D)$. If $v \geq 0$ in D then either $v \equiv 0$ or there exists $\delta > 0$ such that $v \geq \delta$ in D. In the second case the trace of v satisfies $v \geq \delta$ on ∂D in the L^2-sense.

PROOF. By interior regularity, $v \in C^{1,\alpha}_{\text{loc}}(D)$. Suppose $v \not\equiv 0$. Then v cannot have an interior zero in D by the strong maximum principle and hence $v > 0$ in D. It remains to show that there exists $\delta > 0$ such that $v \geq \delta$ in D. By Lemma 30 of the Appendix, there exists a smooth function $t(y) \geq \delta > 0$ in a neighborhood U of ∂D and a constant $K > 0$ such that

$$|\nabla t(y)| \geq \frac{1}{2K} \quad \text{for all } y \in U,$$
$$\nabla t(x) \cdot n(x) \leq \frac{-1}{K} \quad \text{for almost all } x \in \partial D.$$

Let $0 \leq h \in C^\infty(U)$ be a test function with $h = 0$ on $\partial U \cap D$ extended by 0 in $D \setminus U$. Then

$$\begin{aligned} \int_\omega \nabla v \nabla h \, dx &= \oint_{\partial D} b(x) v h \, ds + \int_U a(x) v h \, dx \\ &\geq \oint_{\partial D} -b^-(x) v h \, ds + \int_U -a^-(x) v h \, dx \end{aligned} \tag{3.4}$$

where we use the convention $a(x) = a^+(x) - a^-(x)$ and likewise for b. Next we construct a weak subsolution. Let $z(y) = e^{\alpha t(y)}$. For sufficiently large $\alpha > 0$ we find

$$-\Delta z(y) = (-\alpha \Delta t(y) - \alpha^2 |\nabla t(y)|^2) z \leq -a^-(y) z \quad \text{in } U,$$
$$z_n(x) = \alpha \nabla t(x) \cdot n(x) z(x) \leq -b^-(x) z(x) \quad \text{a.e. on } \partial D.$$

Hence, testing with $0 \leq h \in C^\infty(U)$ with $h = 0$ on $\partial U \cap D$ one obtains

$$\int_U \nabla z \nabla h \, dx \leq \oint_{\partial D} -b^-(x) z h \, ds + \int_U -a^-(x) w h \, dx. \tag{3.5}$$

By the assumption $v > 0$ in D one can choose $\tau > 0$ such that $\tau z < v$ on $\partial U \cap D$. Hence the test function $h = (v - \tau z)^- \geq 0$ vanishes on $\partial U \cap D$ and can be inserted in the difference of (3.4) and (3.5). This yields

$$\int_U -|\nabla (v - \tau z)^-|^2 \, dx \geq \oint_{\partial D} b^-(x)[(v - \tau z)^-]^2 \, ds + \int_U a^-(x)[(v - \tau z)^-]^2 \, dx.$$

This shows that $v \geq \tau z$ in U. Since $z \geq \delta > 0$ in U this proves the result. □

THEOREM 11. *Assume $q \geq 0$.*

(i) *If $\int_D q \, dx > 0$ then the eigenvalues λ_{-1} and λ_1 are simple and their eigenfunctions are of constant sign.*

(ii) *If $\lambda \in \mathbb{R}$ is an eigenvalue such that one eigenfunction is of constant sign, then $\lambda = \lambda_1$, $\lambda = 0$ or $\lambda = \lambda_{-1}$.*

PROOF. Let ψ be an eigenfunction associated to μ and φ be a nonnegative eigenfunction associated with $\lambda \neq \mu$. By Lemma 10 we find that $\varphi \geq \delta > 0$ in D. Moreover, standard regularity arguments based on bootstrapping (see e.g. Struwe [18, Appendix B]) imply that φ, ψ are in $L^\infty(D)$. Hence ψ^2/φ is in $H^1(D)$ and can be used as a test function for the φ-equation. This implies

$$\begin{aligned} \int_D \nabla \varphi \frac{2\psi\varphi\nabla\psi - \psi^2\nabla\varphi}{\varphi^2} \, dx &= \int_D (\lambda - q)\psi^2 \, dx + \oint_{\partial D} \sigma\lambda\psi^2 \, dx \\ &= (\lambda - \mu) a(\psi, \psi) + \int_D |\nabla \psi|^2 \, dx. \end{aligned}$$

Hence

$$0 \le \int_D \left| \frac{\psi}{\varphi} \nabla\varphi - \nabla\psi \right|^2 dx = (\mu - \lambda) a(\psi, \psi). \tag{3.6}$$

(i) Let ψ be an eigenfunction associated to λ_{-1}. The variational principle of Lemma 8 implies that $\varphi = |\psi|$ is also an eigenfunction to λ_{-1}. By (3.6) we have

$$\left| \frac{\psi}{\varphi} \nabla\varphi - \nabla\psi \right|^2 = 0 \quad \text{a.e. in } D,$$

which proves that $\varphi = c\psi$ in D. Hence λ_{-1} is simple and by Lemma 10 the associated eigenfunction is bounded away from 0. The same argument works for λ_1.

(ii) If we choose $\mu = \lambda_1$ then it follows from $a(\psi, \psi) > 0$ and (3.6) that $\lambda \le \lambda_1$. If we take $\mu = \lambda_{-1}$ then $a(\psi, \psi) < 0$ and (10) imply $\lambda \ge \lambda_{-1}$. Since there are no eigenvalues in $(\lambda_{-1}, 0)$ and $(0, \lambda_1)$ it follows that $\lambda = \lambda_{-1}$, $\lambda = 0$ or $\lambda = \lambda_1$. □

3.2. *The case $q(x) \equiv 0$*

Again we want to apply the theory of compact self-adjoint operators in order to describe the eigenvalues of (2.4). This requires the Hilbert space $\mathcal{V} = \{v \in H^1(D) : a(v, 1) = 0\}$ with $a(u, v) = \int_D uv\,dx + \sigma \oint_{\partial D} uv\,ds$. In the case $|D| + \sigma|\partial D| \neq 0$ the space $\mathcal{V}$ does not contain the constants and hence $(\int_D |\nabla v|^2\,dx)^{1/2}$ is an equivalent norm on $\mathcal{V}$. All solutions of (2.4) except the constants belong to $\mathcal{V}$.

However, if $|D| + \sigma|\partial D| = 0$ then the constants do belong to $\mathcal{V}$. We must therefore change the setting and define a proper subspace of $\mathcal{V}$ by $\mathcal{V}_w = \{v \in \mathcal{V} : a(v, w) = 0\}$ where w is a solution of the problem $-\Delta w = 1$ in D, $w_n = \sigma$ on ∂D. The constants do not belong to $\mathcal{V}_w$ and $(\int_D |\nabla v|^2\,dx)^{1/2}$ is an equivalent norm on $\mathcal{V}_w$. The choice of w may seem arbitrary. In Section 4 we show why no other choice for w is possible.

LEMMA 12 (Existence and uniqueness).

(i) *Let $|D| + \sigma|\partial D| \neq 0$. For any $h \in \mathcal{V}$ there exists a unique $v \in \mathcal{V}$ such that*

$$-\Delta v = h \quad \text{in } D, \qquad v_n = \sigma h \quad \text{on } \partial D. \tag{3.7}$$

The operator

$$K : \mathcal{V} \to \mathcal{V}, \qquad h \mapsto v,$$

is compact, invertible and self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle$. Hence it has countably many eigenvalues $\{\mu_k\}_{k \in I}$ and the eigenfunctions form a complete system in $\mathcal{V}$. The eigenvalues of (2.4) *except $\lambda_0 = 0$ are the reciprocals $\lambda_i = \mu_i^{-1}$.*

(ii) *The same holds in the case $|D| + \sigma|\partial D| = 0$ if $\mathcal{V}$ is replaced by $\mathcal{V}_w$.*

PROOF. We give the proof in the "resonance" case $|D| + \sigma|\partial D| = 0$. In the "nonresonance" case $|D| + \sigma|\partial D| \neq 0$ the same proof works by formally setting $w = 0$ in all of the following. For given $h \in \mathcal{V}_w$ the functional $L_h : \mathcal{V}_w \to \mathbb{R}$ given by

$$L_h(\phi) = \int_D h\phi\,dx + \sigma \oint_{\partial D} h\phi\,ds$$

is continuous and hence by the Riesz representation theorem there exists a unique $v \in \mathcal{V}_w$ such that $\langle v, \phi \rangle = L_h(\phi)$ for all $\phi \in \mathcal{V}_w$. We want to deduce that v is a weak solution of (3.7). Since $H^1(D) = \mathcal{V}_w \oplus \text{span}[1, w]$ this follows once we show that

$$\langle v, \phi \rangle = L_h(\phi) \quad \forall \phi \in \text{span}[1, w]. \tag{3.8}$$

The right-hand side $L_h(\phi) = a(\phi, h)$ in (3.8) vanishes for $\phi \in \{1, w\}$ by the assumption $h \in \mathcal{V}_w$. For $\phi = 1$ also the left-hand side of (3.8) vanishes. It remains to compute $\langle v, w \rangle$. Since w weakly solves the equation $-\Delta w = 1$ in D and $w_n = \sigma$ on ∂D we find $\int_D \nabla v \nabla w \, dx = \int_D v \, dx + \sigma \oint_{\partial D} v \, ds = 0$ by definition of $\mathcal{V}_w$. Hence the operator K is well defined. Continuity and compactness of K are again standard, and so are invertibility and symmetry. □

REMARK. There is a more general version of Lemma 12. If $|D| + \sigma|\partial D| \neq 0$ then let $\mathcal{W} = \{(f, g) \in L^2(D) \times L^2(\partial D) : \int_D f \, dx + \sigma \int_D g \, ds = 0\}$ with the norm $\|(f, g)\| = (\|f\|^2_{L^2(D)} + \|g\|^2_{L^2(\partial D)})^{1/2}$. For every $(f, g) \in \mathcal{W}$ there exists a unique $v \in \mathcal{V}$ such that $-\Delta v = f$ in D, $v_n = \sigma g$ on ∂D. The corresponding solution operator $T : (f, g) \mapsto v$ from $\mathcal{W}$ to $\mathcal{V}$ is compact. If $|D| + \sigma|\partial D| = 0$ then the same result holds if $\mathcal{W}$, $\mathcal{V}$ are replaced by $\mathcal{W}_w$, $\mathcal{V}_w$, where $\mathcal{W}_w = \{(f, g) \in \mathcal{W} : \int_D f w \, dx + \sigma \oint_{\partial D} g w \, ds = 0\}$.

Since $\lambda_0 = 0$ is an eigenvalue with the constants as eigenfunctions, the variational description of the eigenvalues of (2.4) differs slightly from the one given in the case where $q(x) \geq 0, \not\equiv 0$. The eigenvalues except 0 are critical values of

$$J(v) := a(v, v) = \int_D v^2 \, dx + \sigma \oint_{\partial D} v^2 \, ds$$

in the set $\{v \in \mathcal{V} : \int_D |\nabla v|^2 \, dx = 1\}$ or $\{v \in \mathcal{V}_w : \int_D |\nabla v|^2 \, dx = 1\}$. Obviously, if $\sigma \geq 0$ there are only positive critical values. This explains part (i) of Theorem 2. For the remaining parts of Theorem 2 and Theorem 3 we have the following result (see e.g. De Figueiredo [8]):

LEMMA 13. *Assume $\sigma < 0$ and $|D| + \sigma|\partial D| \neq 0$. Suppose that*

$$\mu_1 = \sup\{J(v) : v \in \mathcal{V}, \ \langle v, v \rangle = 1\} > 0, \qquad \mu_{-1} = \inf\{J(v) : v \in \mathcal{V}, \ \langle v, v \rangle = 1\} < 0.$$

Then $\lambda_1 = \mu_1^{-1}$ and $\lambda_{-1} = \mu_{-1}^{-1}$ are the first positive and first negative eigenvalues of (2.4). *Moreover, the following holds:*

(a) *Let $k \in \mathbb{N}$. Suppose $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_k$ are the (not necessarily different) first $k + 1$ nonnegative eigenvalues with eigenfunctions $\psi_0, \ldots, \psi_k$. Suppose that*

$$\mu_{k+1} = \sup\{J(v) : v \in \mathcal{V}, \ \langle v, v \rangle = 1, \ a(\psi_j, v) = 0, \ j = 1, \ldots, k\} > 0.$$

Then $\lambda_{k+1} = \mu_{k+1}^{-1}$ is the next positive eigenvalue.

(b) *Let $k \in \mathbb{N}$. Suppose $\lambda_{-k} \leq \cdots \leq \lambda_{-1} < \lambda_0 = 0$ are the (not necessarily different) first $k + 1$ nonpositive eigenvalues with eigenfunctions $\psi_{-k}, \ldots, \psi_0$. Suppose that*

$$\mu_{-k-1} = \inf\{J(v) : v \in \mathcal{V}, \ \langle v, v \rangle = 1, \ a(\psi_j, v) = 0, \ j = -k, \ldots, -1\} < 0.$$

Then $\lambda_{-k-1} = \mu_{-k-1}^{-1}$ is the next negative eigenvalue.

The same holds in the case $|D| + \sigma|\partial D| = 0$ if $\mathcal{V}$ is replaced by $\mathcal{V}_w$.

Since $\langle v, v\rangle^{1/2}$ is an equivalent norm on $\mathcal{V}$, $\mathcal{V}_w$ any sequence of extremal functions is bounded in the full H^1-norm. Provided $\mu_j > 0$, $\mu_{-j} < 0$ it is easy to see that these values are attained.

THEOREM 14. *Let* $\sigma < 0$.

(a) *Then* (2.4) *has an unbounded sequence of positive eigenvalues.*
(b) *If* $N \geq 2$ *then* (2.4) *has an unbounded sequence of negative eigenvalues.*
(c) *If* $N = 1$ *and* $D = (0, L)$ *then* (2.4) *has exactly one negative eigenvalue if* $|\sigma| \geq L/2$ *and exactly two negative eigenvalues if* $|\sigma| < L/2$.

PROOF. The proof is almost identical with the proof of Theorem 9. The one-dimensional case is computed explicitly in the following subsection. □

REMARK. Note that in contrast to Theorem 11 we do not claim for $\sigma < 0$ that the eigenfunctions associated with λ_{-1}, λ_1 have constant sign. In fact the properties of λ_{-1}, λ_1 depend on the value of σ and change near the critical value $-|D|/|\partial D|$ (see Corollary 22 below):

$$\sigma > -|D|/|\partial D| \;\Rightarrow\; \lambda_{-1} \text{ simple, } \psi_{-1} \text{ of constant sign, } \psi_1 \text{ sign-changing,}$$
$$\sigma < -|D|/|\partial D| \;\Rightarrow\; \lambda_1 \text{ simple, } \psi_1 \text{ of constant sign, } \psi_{-1} \text{ sign-changing,}$$
$$\sigma = -|D|/|\partial D| \;\Rightarrow\; \psi_{-1}, \psi_1 \text{ both sign-changing.}$$

3.3. *Examples*

The interval with constant potential q. Consider the one-dimensional case with constant coefficient $q \in \mathbb{R}$,

$$\varphi'' + (\lambda - q)\varphi = 0 \quad \text{in } (0, L)$$

together with the boundary conditions

$$-\varphi'(0) = \sigma\lambda\varphi(0) \quad \text{and} \quad \varphi'(L) = \sigma\lambda\varphi(L).$$

Solutions are of the following form:

$$\varphi(x) = \begin{cases} A\cos(\sqrt{\lambda - q}\,x) + B\sin(\sqrt{\lambda - q}\,x) & \text{if } \lambda > q, \\ A\cosh(\sqrt{q - \lambda}\,x) + B\sinh(\sqrt{q - \lambda}\,x) & \text{if } \lambda < q, \\ Ax + B & \text{if } \lambda = q. \end{cases}$$

From the boundary conditions we deduce that eigenvalues are determined by the equations

$$\tan(\sqrt{\lambda - q}\,L) = \frac{2\sigma\lambda\sqrt{\lambda - q}}{\sigma^2\lambda^2 + q - \lambda} \quad \text{if } \lambda > q, \tag{3.9}$$

$$\tanh(\sqrt{q - \lambda}\,L) = \frac{2\sigma\lambda\sqrt{q - \lambda}}{\sigma^2\lambda^2 + q - \lambda} \quad \text{if } \lambda < q. \tag{3.10}$$

In the case $\lambda = q$ we obtain the zero eigenvalue if $q = 0$ and the eigenvalue $\lambda = 2/(\sigma L)$ if $q = 2/(\sigma L)$.

Due to the asymptotes of the tan-function and its periodicity there are infinitely many positive solutions of (3.9) regardless of the sign of σ and q.

Next we look at the number of negative eigenvalues in the case $q \geq 0$ (the case $q < 0$ is omitted due to the complexity of different cases). Clearly such eigenvalues can only exist for $\sigma < 0$ and they are governed by (3.10).

LEMMA 15. *Assume $\sigma < 0$.*

(a) *If $q > 0$ then there exist exactly two negative eigenvalues $\lambda_{-2} < \lambda_{-1} < 0$. The corresponding eigenfunction φ_{-1} has constant sign whereas φ_{-2} is sign-changing and antisymmetric with respect to its zero at $x = L/2$.*

(b) *If $q = 0$ and $|\sigma| \geq L/2$ then there exists exactly one negative eigenvalue λ_{-1} with sign-changing eigenfunction φ_{-1}, which is antisymmetric with respect to its zero at $L/2$.*

(c) *If $q = 0$ and $|\sigma| < L/2$ then there exist exactly two negative eigenvalues $\lambda_{-2} < \lambda_{-1} < 0$. The corresponding eigenfunction φ_{-1} has constant sign whereas φ_{-2} is sign-changing. Moreover, φ_{-2} is antisymmetric with respect to its zero at $L/2$.*

PROOF. Using the half-angle formula $\tanh(2a) = 2\tanh a/(1 + \tanh^2 a)$ let us rewrite (3.10) as

$$\frac{\tanh(\sqrt{q-\lambda}L/2)}{1+\tanh^2(\sqrt{q-\lambda}L/2)} = \frac{\sigma\lambda/\sqrt{q-\lambda}}{1+\sigma^2\lambda^2/(q-\lambda)}. \tag{3.11}$$

Let λ^* be the negative root of $\sigma^2\lambda^2 + \lambda - q = 0$. Then (3.11) is equivalent to

$$\tanh(\sqrt{q-\lambda}L/2) = \frac{\sigma\lambda}{\sqrt{q-\lambda}} \quad \text{if } \lambda \in (\lambda^*, 0), \tag{3.12}$$

$$\tanh(\sqrt{q-\lambda}L/2) = \frac{\sqrt{q-\lambda}}{\sigma\lambda} \quad \text{if } \lambda \in (-\infty, \lambda^*). \tag{3.13}$$

(a) There is exactly one solution $\lambda_{-2} \in (-\infty, \lambda^*)$ of (3.13) and at least one solution $\lambda_{-1} \in (\lambda^*, 0)$ of (3.12). To see the uniqueness of the eigenvalue in $(\lambda^*, 0)$ note that any such eigenfunction is given by (assuming $A > 0$)

$$\begin{aligned}
\varphi(x) &= A\cosh(\sqrt{q-\lambda}x)\left(1 - \frac{\sigma\lambda}{\sqrt{q-\lambda}}\tanh(\sqrt{q-\lambda}x)\right) \\
&\geq A\cosh(\sqrt{q-\lambda}x)\left(1 - \frac{\sigma\lambda}{\sqrt{q-\lambda}}\tanh(\sqrt{q-\lambda}L)\right) \\
&= A\cosh(\sqrt{q-\lambda}x)\left(1 - \frac{2\sigma^2\lambda^2}{\sigma^2\lambda^2+q-\lambda}\right)
\end{aligned}$$

and it is easy to see that the last expression is positive if $\lambda \in (\lambda^*, 0)$ and negative if $\lambda < \lambda^*$. Hence Theorem 11 applies and shows that for $\lambda \in (\lambda^*, 0)$ there is exactly one eigenvalue.

(b) & (c) As before there is exactly one solution of (3.13) with a corresponding sign-changing eigenfunction. It remains to discuss (3.12) which takes the form

$$\tanh(\sqrt{-\lambda}L/2) = -\sigma\sqrt{-\lambda} \quad \text{for } \lambda \in (\lambda^*, 0),$$

i.e. the intersection of a concave curve with a linear curve of the variable $\sqrt{-\lambda}$. Since $\lambda = 0$ is an intersection point the existence of a further intersection point depends on the derivatives at 0, i.e., for $L/2 < |\sigma|$ the two curves intersect exactly once in $(\lambda^*, 0)$ whereas for $L/2 \geq |\sigma|$ they do not intersect. The eigenfunction corresponding to an intersection point in $(\lambda^*, 0)$ has constant sign. $\square$

The N-dimensional ball with constant potential q. For $N > 1$ let $D = B_R(0) \subset \mathbb{R}^N$ be the ball of radius R and let (r, θ), $r \in [0, R]$, $\theta \in \mathbb{S}^{N-1}$, be its polar coordinates. Consider for constant $q \in \mathbb{R}$ the problem

$$\Delta\varphi + (\lambda - q)\varphi = 0 \quad \text{in } B_R(0), \qquad \partial_r\varphi(x) = \sigma\lambda\varphi(x) \quad \text{on } \partial B_R(0). \tag{3.14}$$

The separation of variables $\varphi(x) = w(r)\alpha(\theta)$ yields

$$w'' + \frac{N-1}{r}w' + \left(\lambda - q - \frac{\nu}{r^2}\right)w = 0, \qquad \Delta_\theta\alpha + \nu\alpha = 0,$$

where Δ_θ is the Laplace–Beltrami operator on $\mathbb{S}^{N-1}$ with eigenfunction α and eigenvalue ν. Hence α must be a spherical harmonic and $\nu = \nu_k = k(k + N - 2)$, $k = 0, 1, 2, \ldots$. The equation for w then becomes

$$w'' + \frac{N-1}{r}w' + \left(\lambda - q - \frac{\nu_k}{r^2}\right)w = 0 \quad \text{in } (0, R), \tag{3.15}$$

$$w'(R) = \sigma\lambda w(R). \tag{3.16}$$

By the usual transformation $z(r) = r^{(N-2)/2}w(r)$ one finds

$$z'' + \frac{z'}{r} + \left(\lambda - q - \frac{(k + (N-2)/2)^2}{r^2}\right)z = 0 \quad \text{in } (0, R), \tag{3.17}$$

$$z'(R) = \left(\sigma\lambda + \frac{N-2}{2R}\right)z(R). \tag{3.18}$$

Solutions of (3.17) are of the form

$$z(r) = \begin{cases} J_{k+(N-2)/2}(\sqrt{\lambda - q}\,r) & \text{if } \lambda > q, \\ I_{k+(N-2)/2}(\sqrt{q - \lambda}\,r) & \text{if } \lambda < q, \\ r^{k+(N-2)/2} & \text{if } \lambda = q, \end{cases} \tag{3.19}$$

where J_ν is the regular Bessel function of index ν and I_ν is the regular modified Bessel function of index ν. The eigenvalues are determined from (3.18) by the equations

$$\frac{J'_{k+(N-2)/2}(\sqrt{\lambda - q}\,R)}{J_{k+(N-2)/2}(\sqrt{\lambda - q}\,R)} = \frac{\sigma\lambda + \frac{N-2}{2R}}{\sqrt{\lambda - q}} \quad \text{if } \lambda > q, \tag{3.20}$$

$$\frac{I'_{k+(N-2)/2}(\sqrt{q - \lambda}\,R)}{I_{k+(N-2)/2}(\sqrt{q - \lambda}\,R)} = \frac{\sigma\lambda + \frac{N-2}{2R}}{\sqrt{q - \lambda}} \quad \text{if } \lambda < q. \tag{3.21}$$

In the case $\lambda = q$ we obtain the eigenvalue $\lambda = k/(\sigma R)$ if $q = k/(\sigma R)$ for some $k \in \mathbb{N}_0$.

Recall that J_ν, J_ν' have infinitely many zeroes tending to ∞ and that the zeroes of J_ν and J_ν' separate each other. This implies that for every fixed $k \in \mathbb{N}_0$ there are infinitely many positive solutions of (3.20) regardless of the sign of σ and q.

As before we look at the number of negative eigenvalues in the case $q \geq 0$ (the case $q < 0$ is omitted). We know from Theorems 1–3 that such eigenvalues only exist for $\sigma < 0$ and they are governed by (3.21).

THEOREM 16. *Assume $\sigma < 0$. Every eigenfunction of* (3.14) *corresponding to a negative eigenvalue has the form $\varphi(x) = z(r)r^{(2-N)/2}\alpha(\theta)$ where α is a spherical harmonic and $z(r)$ is a modified Bessel function as in* (3.19) *and has constant sign. For every $k \in \mathbb{N}$ there exists exactly one negative eigenvalue with a spherical harmonic of index $k(k + N - 2)$. The situation for $k = 0$ is different:*

(a) *Let $q > 0$ or $q = 0$ and $|\sigma| < R/N$. Then there exists exactly one negative eigenvalue with a radially symmetric eigenfunction.*
(b) *Let $q = 0$ and $|\sigma| \geq R/N$. Then there is no negative eigenvalue corresponding to a radially symmetric eigenfunction.*

REMARK. The eigenfunctions corresponding to negative eigenvalues are either of constant sign or such that every nodal domain intersects the boundary. This follows from the representation $\varphi(x) = z(r)r^{(2-N)/2}\alpha_k(\theta)$ since $z(r)$ is of constant sign and the sign changes are due to the spherical harmonic α_k.

First we collect some properties of the modified Bessel functions. There is the integral representation

$$I_\nu(s) = \frac{(s/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu + 1/2)} \int_{-1}^{1} (1 - t^2)^{\nu - 1/2} \cosh(st)\, dt$$

and the series representation

$$I_\nu(s) = \sum_{k=0}^{\infty} \frac{(s/2)^{2k+\nu}}{k!\,\Gamma(\nu + k + 1)},$$

which lead to

$$I_\nu'(s) = I_{\nu+1}(s) + \frac{\nu}{s} I_\nu(s). \tag{3.22}$$

The previous identity implies

$$\frac{I_\nu'(s)}{I_\nu(s)} > \frac{\nu}{s} \quad \text{for all } s > 0. \tag{3.23}$$

Moreover $I_\nu(s) \sim e^s/\sqrt{2\pi s}$ as $s \to \infty$ and $I_\nu \sim (s/2)^\nu/\Gamma(\nu + 1)$ as $s \to 0$. Consequently, we have

$$\frac{I_\nu'(s)}{I_\nu(s)} \to 1 \quad \text{as } s \to \infty, \qquad \frac{I_\nu'(s)}{I_\nu(s)} = \frac{\nu}{s} + \frac{s}{2(\nu + 1)} + O(s^2) \quad \text{as } s \to 0. \tag{3.24}$$

PROOF OF THEOREM 16. *Completeness.* Suppose $\varphi(r,\theta)$ is an eigenfunction of (3.14) corresponding to a negative eigenvalue λ. We need to show that $\varphi(x) = z(r)r^{(2-N)/2}\alpha(\theta)$ with a spherical harmonic α and a modified Bessel function z of the form (3.19). For fixed r expand $\varphi(r,\theta)$ in spherical harmonics $\alpha_k^l(\theta)$ of degree k,

$$\varphi(r,\theta) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k} g_k^l(r)\alpha_k^l(\theta), \tag{3.25}$$

with $g_k^l \in L^2_{r^{N-1}}(0,R)$. Next we expand g_k^l, $l = 1,\dots,d(k)$, into Bessel or modified Bessel functions ψ_i^k which are the eigenfunctions of (3.17), (3.18). The same arguments as in the proof of Theorem 1 show that this system is complete in the space $H^1_{\text{rad}}(R)$, the completion of the differentiable radial functions in $(0,R)$ under the norm

$$\langle v,v\rangle = \int_0^R r^{N-1}\left(v'^2 + \left(q + \frac{k(k+N-2)}{r^2}\right)v^2\right)dr.$$

The corresponding bilinear form $a(v,u)$ is

$$a_r(u,v) = \int_0^R r^{N-1}uv\,dr + \sigma R^{N-1}u(R)v(R).$$

The eigenfunction ψ_i^k solves (3.17), (3.18) with eigenvalue λ_i^k. According to the classical results on the solutions of the Bessel differential equation there is only one solution which is contained in $H^1_{\text{rad}}(R)$, namely the function given in (3.19). Hence

$$g_k^l = \sum_{i=1}^{\infty}\gamma_{ki}^l\psi_i^k, \qquad \varphi(r,\theta) = \sum_{k,l,i\geq 1}\gamma_{ki}^l\psi_i^k(r)\alpha_k^l(\theta).$$

Observe that $\psi_i^k(r)\alpha_k^l(\theta)$ is also an eigenfunction of (3.14) corresponding to λ_i^k. From the above representation it follows immediately that $\varphi = \sum_{l=1}^{d_k}\gamma_{ki}^l\psi_k^i\alpha_i^k$. This proves the completeness.

Uniqueness. Fix $k \in \mathbb{N}_0$ and a corresponding spherical harmonic α_k. Suppose there are two values $\lambda < \tilde\lambda < 0$ solving (3.21). Let $w, \tilde w$ be the corresponding solutions of (3.15)–(3.16) normalized such that $w(R) = \tilde w(R)$. Since $w, \tilde w > 0$ in $(0,R]$ we see that $h = w - \tilde w$ satisfies

$$h'' + \frac{N-1}{r}h' + \left(\tilde\lambda - q - \frac{\nu_k}{r^2}\right)h > 0 \quad \text{in } (0,R], \qquad h'(0) = 0, \qquad h(R) = 0.$$

If $\nu_k = 0$ then a standard application of the maximum principle implies $h < 0$ in $[0,R)$. If $\nu_k > 0$ then we know that $w(0) = \tilde w(0) = 0$ and hence $h(0) = 0$. The assumption that h attains a non-negative maximum somewhere in $(0,R)$ immediately gives a contradiction. Hence also in this case we have $h < 0$, i.e., $w < \tilde w$ in $(0,R)$. Using the orthogonality relation between eigenfunctions and $w(R) = \tilde w(R)$ we find

$$\begin{aligned}0 &= \int_D w(r)\tilde w(r)\alpha_k(\theta)^2\,dx + \sigma\oint_{\partial D} w(R)\tilde w(R)\alpha_k(\theta)^2\,ds\\ &\leq \int_D \tilde w(r)^2\alpha_k(\theta)^2\,dx + \sigma\oint_{\partial D}\tilde w(R)^2\alpha_k(\theta)^2\,ds = J(\tilde w\alpha_k).\end{aligned}$$

This is a contradiction since $J(\tilde w\alpha_k) < 0$.

Existence for $q > 0$. Fix $k \in \mathbb{N}_0$. Let $f(\lambda)$ denote the left-hand side and $g(\lambda)$ the right-hand side of (3.21). By (3.24) we know that $f(\lambda) < g(\lambda)$ for λ sufficiently negative. Moreover, by (3.23) we have

$$f(0) > \frac{k + (N-2)/2}{\sqrt{q}R} \geq \frac{N-2}{2\sqrt{q}R} = g(0).$$

Hence there exists at least one (and by the previous uniqueness proof in fact exactly one) solution of (3.21) in $(-\infty, 0)$.

Existence for $q = 0$. Fix $k \in \mathbb{N}_0$. With the same functions $f(\lambda), g(\lambda)$ as before we deduce from (3.24) that

$$f(\lambda) \sim \frac{k + (N-2)/2}{\sqrt{-\lambda}R} + \frac{\sqrt{-\lambda}R}{2k+N} + O(\lambda^2) \quad \text{as } \lambda \to 0,$$
$$g(\lambda) = -\sigma\sqrt{-\lambda} + \frac{(N-2)/2}{\sqrt{-\lambda}R}.$$

Hence, for every $k \in \mathbb{N}$ we have $f(\lambda) > g(\lambda)$ near $\lambda = 0$, which implies as before existence of a unique solution of (3.21) in $(-\infty, 0)$. For $k = 0$ this also holds under the additional assumption $R/N > -\sigma$.

Nonexistence for $q = 0$, $k = 0$ *and* $|\sigma| \geq R/N$. In this case we claim that $f(\lambda) < g(\lambda)$ for all $\lambda < 0$. This is equivalent to $I_\nu'(t)t < I_\nu(t)(\frac{-\sigma}{R}t^2 + \nu)$ for all $t > 0$ with $\nu = (N-2)/2$. Computing the series we find

$$I_\nu'(t)t = \sum_{j=0}^{\infty} \frac{(2j+\nu)(t/2)^{2j+\nu}}{j!\Gamma(\nu+j+1)},$$
$$I_\nu(t)\left(\frac{-\sigma}{R}t^2 + \nu\right) = \sum_{j=0}^{\infty} \frac{\nu(t/2)^{2j+\nu}}{j!\Gamma(\nu+j+1)} - \sum_{j=0}^{\infty} \frac{4\sigma(t/2)^{2j+2+\nu}}{Rj!\Gamma(\nu+j+1)}$$
$$= \sum_{j=0}^{\infty} \left(\nu - \frac{4\sigma j(\nu+j)}{R}\right) \frac{(t/2)^{2j+\nu}}{j!\Gamma(\nu+j+1)}.$$

Using $\nu = (N-2)/2$ and comparing the coefficients of the two series we find equality for $j = 0$, weak inequality for $j = 1$ and strict inequality for $j \geq 2$ provided $1 \leq -\sigma N/R$. Hence there is no solution of (3.21) in $(-\infty, 0)$. □

COROLLARY 17. *Suppose* $q = \text{const} \geq 0$ *and* $\sigma < 0$. *Then the eigenvalue problem* (3.14) *restricted to radially symmetric functions has infinitely many positive eigenvalues and at most one negative eigenvalue. The parabolic problem* (2.1)–(2.3) *restricted to radially symmetric initial data is well-posed.*

PROPOSITION 18. *Suppose* $0 \leq q \in L^\infty(0, R)$, $\int_0^R r^{n-1}q(r)\,dr > 0$ *and* $\sigma < 0$. *Then the eigenvalue problem* (3.14) *restricted to radially symmetric functions has infinitely many positive eigenvalues and exactly one negative eigenvalue. The parabolic problem* (2.1)–(2.3) *restricted to radially symmetric initial data is well-posed.*

PROOF. The existence of a first negative eigenvalue λ_{-1} follows from Theorem 9. We may assume that the eigenfunction $\phi = \phi_{-1}$ is positive. It satisfies

$$-\Delta\phi + q(r)\phi = \lambda_{-1}\phi \quad \text{in } B_R(0), \qquad \phi'(R) = \sigma\lambda\phi(R). \tag{3.26}$$

It is radially symmetric because it is simple. It remains to show that there is exactly one negative "radial" eigenvalue. Let ψ be an arbitrary radial eigenfunction

$$-\Delta\psi + q(r)\psi = \lambda\psi \quad \text{in } B_R(0), \qquad \psi'(R) = \sigma\lambda\psi(R)$$

with $\lambda < \lambda_{-1}$. The maximum principle implies that ψ cannot have a closed nodal line. Hence we may assume $\psi > 0$ in $\overline{B}_R(0)$ and obtain

$$-\Delta\psi + q(r)\psi < \lambda_{-1}\psi \quad \text{in } B_R(0).$$

By a suitable scaling we may assume that $\phi(R) = \psi(R) = 1$. Since ψ is a subsolution of (3.26) we get $0 < \psi < \phi$ in $B_R(0)$. By the orthogonality condition we find

$$0 = \int_D \phi\psi\, dx + \sigma \oint_{\partial D} \phi\psi\, ds \le \int_D \phi^2\, dx + \sigma \oint_{\partial D} \phi^2\, ds = J(\phi).$$

This is a contradiction since $J(\phi) < 0$. □

4. EIGENVALUES IN THE RESONANCE CASE

As we have seen in Sections 2.2 and 2.3 the resonance case $q \equiv 0$ and $\sigma_0 = -|D|/|\partial D|$ displays special spectral properties, that are discussed in detail in this section.

4.1. *The choice of the space $\mathcal{V}_w$*

Suppose one wants to solve

$$-\Delta v = h \quad \text{in } D, \qquad v_n = \sigma h \quad \text{on } \partial D \tag{4.1}$$

for $h \in H^1(D)$. Then necessarily $h \in \mathcal{V} = \{v \in H^1(D) : a(1, v) = 0\}$, where $a(u, v) = \int_D uv\, dx + \sigma \oint_{\partial D} uv\, ds$. The next lemma explains why in the resonance case $\sigma = \sigma_0$ one has to choose h with the extra condition $a(w, h) = 0$ in order to obtain $v \in \mathcal{V}$. The only possible choice for w is a solution of $-\Delta w = 1$ in D and $w_n = \sigma_0$ on ∂D.

LEMMA 19. *Let $h \in \mathcal{V}$ and $\sigma \in \mathbb{R}$. Then there exists a one-parameter family $\mathcal{S} = \{v_0 + \gamma\}_{\gamma\in\mathbb{R}} \subset H^1(D)$ of solutions of* (4.1).

PROOF. Let $h \in \mathcal{V}$ and define $\bar{h} = |D|^{-1} \int_D h\, dx$. Let $a, b, c \in H^1(D)$ be solutions of

$$(A)\ \begin{cases} -\Delta a = h - \bar{h} & \text{in } D, \\ a_n = 0 & \text{on } \partial D, \end{cases} \qquad (B)\ \begin{cases} -\Delta b = \bar{h} & \text{in } D, \\ b_n = -\bar{h}\dfrac{|D|}{|\partial D|} & \text{on } \partial D, \end{cases}$$

$$(C)\ \begin{cases} -\Delta c = 0 & \text{in } D, \\ c_n = \sigma h + \bar{h}\dfrac{|D|}{|\partial D|} & \text{on } \partial D. \end{cases}$$

Solutions for (A) and (B) exist for every $h \in H^1(D)$ whereas the solution of (C) only exists if additionally $a(1, h) = 0$. Moreover, a, b, c are unique up to additive constants. Finally, $v_0 = a + b + c$ solves (4.1). □

LEMMA 20. *Let $h \in \mathcal{V}$.*

(i) *There exists a unique element in $\mathcal{S} \cap \mathcal{V}$ if and only if $\sigma \neq \sigma_0$.*
(ii) *Let $\sigma = \sigma_0$. Then $\mathcal{S} \subset \mathcal{V}$ if and only if $h \in \mathcal{V}_w = \{h \in \mathcal{V} : a(w, h) = 0\}$. Furthermore, if $h \in \mathcal{V}_w$ then there exists a unique element in $\mathcal{S} \cap \mathcal{V}_w$.*

PROOF. (i) A unique solution $v_0 + \gamma \in \mathcal{V}$ can be selected provided $a(1, \gamma) \neq 0$. This is the case if and only if $\sigma \neq \sigma_0$.

(ii) If w is the solution of $-\Delta w = 1$ in D, $w_n = \sigma_0$ on ∂D then $b = \bar{h} w$. Testing the equation for w with c and rearranging terms one finds

$$\int_D c\, dx + \sigma_0 \oint_{\partial D} c\, ds = \sigma_0 \oint_{\partial D} (h - \bar{h}) w\, ds,$$

and likewise by testing with a one obtains

$$\int_D a\, dx + \sigma_0 \oint_{\partial D} a\, ds = \int_D (h - \bar{h}) w\, dx.$$

Hence, the condition $v \in \mathcal{V}$ reads

$$\int_D (a + b + c)\, dx + \sigma_0 \oint_{\partial D} (a + b + c)\, ds = \int_D hw\, dx + \sigma_0 \oint_{\partial D} hw\, ds = 0,$$

i.e., one needs the additional condition $a(w, h) = 0$. Uniqueness of the solution in the space $\mathcal{V}_w$ holds provided $a(1, w) \neq 0$. This is true since $a(1, w) = \int_D |\nabla w|^2\, dx$. □

4.2. *Behavior of λ_{-1}, λ_1 near $\sigma = \sigma_0$*

THEOREM 21. *There exists $\epsilon > 0$ and an C^1-curve $\sigma \mapsto (\lambda(\sigma), v(\sigma))$ for $\sigma \in (\sigma_0 - \epsilon, \sigma_0 + \epsilon)$ with values in $\mathbb{R} \times H^1(D)$ such that $(\lambda(\sigma), v(\sigma))$ is an eigenpair for the eigenvalue problem* (2.4) *with the properties $\int_D v(\sigma)\, dx = |D|$ and*

$$\lambda(\sigma) = \frac{-|\partial D|}{\int_D |\nabla w|^2\, dx}(\sigma - \sigma_0) + O(\sigma - \sigma_0)^2,$$
$$v(\sigma) = 1 - \frac{|\partial D|(w - \bar{w})}{\int_D |\nabla w|^2\, dx}(\sigma - \sigma_0) + O(\sigma - \sigma_0)^2.$$

Moreover, if (λ, v) is an eigenpair of (2.4) *with $|\sigma - \sigma_0| < \epsilon$, $|\lambda| < \epsilon$ and $v > 0$, $\int_D v\, dx = |D|$ then either (λ, v) lies on the curve or $(\lambda, v) = (0, 1)$.*

REMARK. Note that $|\sigma| < |\sigma_0|$ implies $\lambda(\sigma) < 0$ and $|\sigma| > |\sigma_0|$ implies $\lambda(\sigma) > 0$. Hence $\lambda(\sigma)$ parameterizes λ_{-1} if $|\sigma| < |\sigma_0|$ and λ_1 if $|\sigma| > |\sigma_0|$. It shows how λ_{-1} passes through 0 and becomes λ_1 as σ passes through the critical value σ_0 (see Figure 1). The positivity of the eigenfunction also passes from ψ_{-1} to ψ_1. Note that the min-max principle implies that the eigenvalues are decreasing with respect to σ.

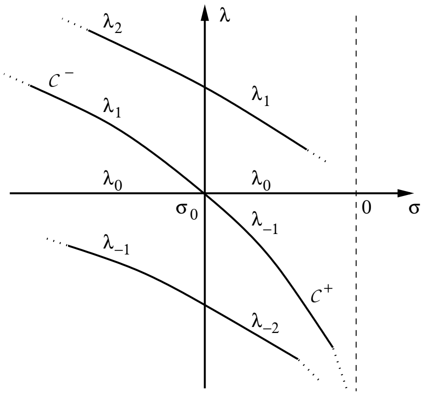

FIG. 1. Eigenvalues as functions of σ.

PROOF. Consider the normalized eigenvalue problem

$$(P) \qquad -\Delta v = \lambda v \quad \text{in } D, \qquad v_n = \sigma\lambda v \quad \text{on } \partial D, \qquad \int_D v\,dx = |D|.$$

In the following we describe the solutions of (P) as the zero-set of a nonlinear function $F(\sigma, \lambda, v)$ where $F : \mathbb{R} \times \mathbb{R} \times H^1(D) \to \mathbb{R} \times H^1(D)$.

Construction of the curve. Define the operator

$$T : \mathbb{R} \times \mathbb{R} \times H^1(D) \to H^1(D), \qquad (\sigma, \lambda, v) \mapsto T(\sigma, \lambda, v) := \xi,$$

where ξ is the unique solution of

$$\text{(4.2)} \qquad \begin{cases} -\Delta\xi = \lambda v & \text{in } D, \\ \xi_n = \sigma\lambda v - \dfrac{\lambda}{|\partial D|}\displaystyle\int_D v\,dx - \dfrac{\lambda\sigma}{|\partial D|}\oint_{\partial D} v\,ds & \text{on } \partial D, \\ \displaystyle\int_D \xi\,dx = |D|. \end{cases}$$

Let

$$F : \mathbb{R}\times\mathbb{R}\times H^1(D) \to \mathbb{R}\times H^1(D), \quad (\sigma, \lambda, v) \mapsto \left(\int_D v\,dx + \sigma\oint_{\partial D} v\,ds,\, T(\sigma, \lambda, v) - v\right).$$

Note that $F(\sigma_0, 0, 1) = (0, 0)$. Moreover, the following relation holds between zeroes of F and solutions of (P):

$$F(\sigma, \lambda, v) = (0, 0) \quad \Rightarrow \quad (\sigma, \lambda, v) \text{ solves } (P),$$

$$\left\{ \begin{array}{l} (\sigma, \lambda, v) \text{ solves } (P) \text{ and} \\ (\lambda, v) \neq (0, 1) \text{ or } \sigma = \sigma_0 \end{array} \right\} \quad \Rightarrow \quad F(\sigma, \lambda, v) = (0, 0).$$

Therefore, solving $F(\sigma, \lambda, v) = (0, 0)$ near $\sigma = \sigma_0$, $\lambda = 0$, $v = 1$ by the implicit function theorem will give all statements of the theorem since

$$\text{(4.3)} \quad (\lambda(\sigma), v(\sigma)) = (0, 1) - \left[\frac{\partial F}{\partial(\lambda, v)}\bigg|_{(\sigma_0,0,1)} \right]^{-1} \frac{\partial F}{\partial\sigma}\bigg|_{(\sigma_0,0,1)} (\sigma - \sigma_0) + O(\sigma - \sigma_0)^2.$$

It remains to show the invertibility of $\frac{\partial F}{\partial(\lambda,v)}|_{(\sigma_0,0,1)}$ and to compute the inverse $(\beta, z) := [\frac{\partial F}{\partial(\lambda,v)}|_{(\sigma_0,0,1)}]^{-1}(\alpha, y)$ for given $(\alpha, y) \in \mathbb{R} \times H^1(D)$. This requires finding the solution (β, z) of

$$\text{(4.4)} \qquad \int_D z\,dx + \sigma_0 \oint_{\partial D} z\,ds = \alpha, \qquad \frac{\partial T}{\partial(\lambda, v)}\bigg|_{(\sigma_0,0,1)} (\beta, z) - z = y.$$

Differentiation of (4.2) with respect to (λ, v) yields $\frac{\partial T}{\partial(\lambda,v)}|_{(\sigma_0,0,1)}(\beta, z) = \zeta$, where ζ solves

$$-\Delta\zeta = \beta \quad \text{in } D, \qquad \zeta_n = \beta\sigma_0 \quad \text{on } \partial D, \qquad \int_D \zeta\,dx = 0,$$

i.e. $\zeta = \beta(w - \bar{w})$. Therefore the solution (β, z) of (4.4) is determined by

$$\int_D z\,dx + \sigma_0 \oint_{\partial D} z = \alpha, \qquad \beta(w - \bar{w}) - z = y.$$

The solution (β, z) can now be computed as

$$\text{(4.5)} \qquad \beta = \frac{\alpha + \int_D y\,dx + \sigma_0 \oint_{\partial D} y\,ds}{\int_D w\,dx + \sigma_0 \oint_{\partial D} w\,ds},$$

$$\text{(4.6)} \qquad z = \frac{\alpha + \int_D y\,dx + \sigma_0 \oint_{\partial D} y\,ds}{\int_D w\,dx + \sigma_0 \oint_{\partial D} w\,ds}(w - \bar{w}) - y.$$

The uniqueness of (β, z) shows the invertibility of $\frac{\partial F}{\partial(\lambda,v)}|_{(\sigma_0,0,1)}$. Notice that the denominator in the above formula is $\int_D |\nabla w|^2\,dx$. Finally, $\frac{\partial F}{\partial\sigma}|_{(\sigma_0,0,1)} = (|\partial D|, 0) \in \mathbb{R} \times H^1(D)$. Inserting $(\alpha, y) = (|\partial D|, 0)$ into (4.5)–(4.6) and (4.3) gives the expansion of $\lambda(\sigma)$ and $v(\sigma)$ as claimed in the theorem. □

COROLLARY 22. *For every $\sigma \in (-\infty, \sigma_0)$ the eigenfunctions corresponding to λ_1 have constant sign. For every $\sigma \in (\sigma_0, 0)$ the eigenfunctions corresponding to λ_{-1} have constant sign.*

PROOF. Let $B_\delta(0, 1) \subset \mathbb{R} \times H^1(D)$ be the open unit ball of radius δ centered at $(0, 1) \in \mathbb{R} \times H^1(D)$. For small $\delta > 0$ we know that

$$\deg(F(\sigma_0, \cdot, \cdot), B_\delta(0, 1), (0, 0)) \neq 0$$

due to the invertibility of $\frac{\partial F}{\partial(\lambda,v)}|_{(\sigma_0,0,1)}$. Therefore, the global continuation theorem (see e.g. [3]) applies and shows the existence of two continua $\mathcal{C}^+ \subset [\sigma_0, \infty) \times \mathbb{R} \times H^1(D)$ and $\mathcal{C}^- \subset (-\infty, \sigma_0] \times \mathbb{R} \times H^1(D)$ of solutions (σ, λ, v) of $F(\sigma, \lambda, v) = (0, 0)$ containing the point $(\sigma_0, 0, 1)$. Locally near $(\sigma, \lambda, v) = (\sigma_0, 0, 1)$ the two continua $\mathcal{C}^+, \mathcal{C}^-$ are described by the curve of Theorem 21. Note that the condition $\int_D v\,dx = |D|$ shows that $v \neq 0$ for every element $(\sigma, \lambda, v) \in \mathcal{C}^+, \mathcal{C}^-$. Thus, the maximum principle of Lemma 10 and a continuity argument show that $v > 0$ for every $(\sigma, \lambda, v) \in \mathcal{C}^+, \mathcal{C}^-$. Similarly, $\lambda > 0$ for every $(\sigma, \lambda, v) \in \mathcal{C}^-$ except for $\sigma = \sigma_0$, and $\lambda < 0$ for every $(\sigma, \lambda, v) \in \mathcal{C}^+$ except for $\sigma = \sigma_0$. Therefore Theorem 11(ii) shows that $\lambda = \lambda_{\mp 1}$ if $(\sigma, \lambda, v) \in \mathcal{C}^\pm$ and $\sigma \neq \sigma_0$ and that $\mathcal{C}^+, \mathcal{C}^-$ can be parameterized as single-valued continuous curves depending on σ, where the λ-part is decreasing in σ. Hence the global continuation theorem implies that both $\mathcal{C}^+$ and $\mathcal{C}^-$ are unbounded continua. Finally, note that the projection of $\mathcal{C}^+$ onto the σ-axis is $[\sigma_0, 0)$ and that $\mathcal{C}^+$ becomes unbounded in the negative λ-direction as $\sigma \to 0$. On the other hand, the projection of $\mathcal{C}^-$ onto the σ-axis is $(-\infty, \sigma_0]$. □

5. Parabolic problems with nonlinear sources

In this section we consider problem (1.1)–(1.3) with positive sources $\mathcal{F}(x, t, u)$. For simplicity we restrict ourselves to the case

$$\mathcal{F}(x, t, u) = m(x, t) f(u),$$

where $m \in L^\infty((0, T), L^\infty(D))$, $m \geq 0$ and $0 \leq f \in C^1$. Throughout this section we shall suppose that there exists a local weak solution u of (1.1)–(1.3) in $(0, T)$ in the sense that $u \in \mathcal{B} := L^2((0, T), H^1(D))$, $u_t \in L^2((0, T), L^2(D))$, trace $u_t \in L^2((0, T), L^2(\partial D))$, $\mathcal{F}(\cdot, \cdot, u(\cdot)) \in L^2((0, T), L^2(D))$ and

$$\int_0^\tau ((u_t, \phi) + \sigma(u_t, \phi)_0)\, dt + \int_0^\tau \langle u, \phi\rangle = \int_0^\tau (f, \phi)\, dt$$

for all $\tau \in (0, T)$ and for all $\phi \in \mathcal{B}$. Moreover we suppose that $u_0 \in L^2(D) \times L^2(\partial D)$ and $\|u - u_0\|_{L^2(D)}$, $\|\text{trace}\, u - u_0\|_{L^2(\partial D)} \to 0$ as $t \to 0$. This is the case if $\sigma \geq 0$.

For the study of blow up behavior we consider nonlinearities f subject to the following additional conditions:

$$f(s) > 0 \quad \text{for } s > 0, \qquad f(s),\ f'(s) \geq 0 \quad \text{for all } s \in \mathbb{R}, \tag{5.1}$$

$$\int_{s_0}^\infty \frac{d\xi}{f(\xi)} < \infty \qquad \text{for some positive } s_0. \tag{5.2}$$

Our main result for $\sigma \geq 0$ is based on a slight modification of an elegant argument by Rial and Rossi [16] who studied the question of blow up for problems with nonlinear Neumann boundary conditions.

THEOREM 23. *Assume $\sigma \geq 0$, $q \equiv 0$, $\int_D u_0\, dx + \sigma \oint_{\partial D} u_0\, ds > 0$ and* (5.1), (5.2). *If $\int_0^t (\int_D m(x, \tau)\, dx)\, d\tau \to \infty$ as $t \to \infty$ then $u(x, t)$ does not exist as a weak solution for all times.*

5.1. *Comparison principles*

We start with some auxiliary result which shows that the qualitative behavior depends heavily on the sign of σ.

LEMMA 24. *Let $\sigma \geq 0$ and let u be a weak solution of* (1.1)–(1.3) *in* $(0, T)$ *with* $f \geq 0$.

(i) *Let h be a weak solution of the homogeneous equation* (1.1)–(1.2) *with zero right-hand side and $h(x, 0) \leq u_0(x)$ a.e. in D. Then $h \leq u$.*

(ii) *If $u_0 \geq 0$ then $u \geq 0$, i.e., all weak solutions of* (1.1)–(1.3) *with nonnegative sources and nonnegative initial data u_0 are nonnegative.*

PROOF. (i) For all $\tau \in (0, T)$ we have

$$\int_0^\tau (a(u_t - h_t, \phi) + \langle u - h, \phi\rangle)\, dt = \int_0^\tau (\mathcal{F}, \phi)\, dt \geq 0 \quad \text{if } 0 \leq \phi \in \mathcal{B}.$$

If we choose $\phi = (u - h)^-$ then we obtain

$$-\int_0^\tau (a((u-h)_t^-, (u-h)^-) - \langle (u-h)^-, (u-h)^-\rangle)\, dt \geq 0,$$

which implies, keeping in mind that by assumption $(u - h)^-(x, 0) = 0$,

$$a((u-h)^-, (u-h)^-)|_\tau \leq 0 \quad \text{for all } \tau \in (0, T).$$

This leads immediately to the conclusion $h \leq u$.

(ii) follows from (i) since $h(x, 0) = u_0(x) \geq 0$ implies $h \geq 0$, which follows as in (i) by testing the equation for h with h^-. □

REMARK. The lemma is not true in general if $\sigma < 0$. For instance let $D = (0, L)$, $\mathcal{F} \equiv 1$, $q \equiv 0$ and $u_0 \geq 0$. Let $h(x, t)$ be the solution of the homogeneous parabolic problem with $\mathcal{F} \equiv 0$ and $h(x, 0) = u_0(x)$. By Theorem 5,

$$\delta(x, t) = u(x, t) - h(x, t) = \sum_{i \in I \setminus \{0\}} \frac{\operatorname{sign}(\lambda_i) L}{\lambda_i} \bar{\varphi}_i (1 - e^{-\lambda_i t}) \varphi_i + \frac{Lt}{L + 2\sigma}$$

provided $L + 2\sigma \neq 0$. Here $\bar{\varphi}_i = L^{-1} \int_0^L \varphi_i(x)\, dx$.

If $0 > \sigma > -L/2$ then $I = \{-2, -1\} \cup \mathbb{N}$, φ_{-1} is positive, and φ_{-2} is sign-changing with $\bar{\varphi}_{-2} = 0$ due to antisymmetry with respect to its zero at $x = L/2$ (cf. Lemma 15). Therefore

$$\delta(x, t) \approx -\frac{L\bar{\varphi}_{-1}}{|\lambda_{-1}|} e^{-\lambda_{-1} t} \varphi_{-1} \to -\infty \quad \text{as } t \to \infty.$$

If $-L/2 > \sigma$ then $I = \{-1\} \cup \mathbb{N}$, φ_{-1} is sign-changing and $\bar{\varphi}_{-1} = 0$ again by antisymmetry so that $\delta(x, t) \approx Lt/(L + 2\sigma) \to -\infty$ as $t \to \infty$. And even in the resonance case $\sigma = -L/2$ the difference $\delta(x, t)$ tends to $-\infty$ as $t \to \infty$. Hence in all three cases $u < h$ for large t.

DEFINITION 25. *A function $\bar{u} \in \mathcal{B}$ with $\bar{u}_t \in L^2((0, T), L^2(D))$ and with* trace $\bar{u}_t \in L^2((0, T), L^2(\partial D))$ *is called an* upper solution *of* (1.1)–(1.3) *if $\bar{u}(x, 0) \geq u_0(x)$ in D and if for all $\phi \in \mathcal{B}$ with $\phi \geq 0$ and all $\tau \in (0, T)$,*

$$\int_0^\tau (a(\bar{u}_t, \phi) + \langle \bar{u}, \phi\rangle)\, dt \geq \int_0^\tau (\mathcal{F}(x, t, \bar{u}), \phi)\, dt.$$

Similarly $\underline{u}(x, t)$ is called a lower solution *of* (1.1)–(1.3) *if the inequality signs are reversed.*

PROPOSITION 26. *Assume $\sigma \geq 0$. Let $\underline{u}$ resp. $\bar{u}$ be a lower resp. upper solution of* (1.1)–(1.3) *with $0 \leq f \in C^1$. Suppose in addition that for all $\tau \in (0, T)$,*

$$\sup_{t \in (0,\tau)} \|\underline{u}(\cdot, \tau)\|_{L^\infty(D)} < \infty, \qquad \sup_{t \in (0,\tau)} \|m(\cdot, \tau)\|_{L^\infty(D)} < \infty.$$

Then $\bar{u}(x, t) \geq \underline{u}(x, t)$.

PROOF. For all $\phi \in \mathcal{B}$ with $\phi \geq 0$ and all $\tau \in (0, T)$ we have

$$\int_0^\tau (a(\underline{u}_t - \bar{u}_t, \phi) + \langle \underline{u} - \bar{u}, \phi \rangle)\, dt \leq \int_0^\tau (\mathcal{F}(x, t, \underline{u}) - \mathcal{F}(x, t, \bar{u}), \phi)\, dt.$$

Let $\phi = (\underline{u} - \bar{u})^+$ and $\Omega^+ = \{(x, t) \in D \times (0, T) : \underline{u}(x, t) \geq \bar{u}(x, t)\}$. By the boundedness assumption on $\underline{u}$ and the C^1-property of f one obtains

$$\begin{aligned}\int_0^\tau (\mathcal{F}(x, t, \underline{u}) - \mathcal{F}(x, t, \bar{u}), \phi)\, d\tau &= \int_{\Omega^+} (\mathcal{F}(x, t, \underline{u}) - \mathcal{F}(x, t, \bar{u}))(\underline{u} - \bar{u})\, dx\, dt \\ &\leq c_\tau \int_{\Omega^+} (\underline{u} - \bar{u})^2\, dx\, dt = c_\tau \int_0^\tau ((\underline{u} - \bar{u})^+, (\underline{u} - \bar{u})^+)\, dt.\end{aligned}$$

Integration gives

$$\begin{aligned}\frac{1}{2} a((\underline{u} - \bar{u})^+, (\underline{u} - \bar{u})^+)|_{t=\tau} - \frac{1}{2} a((\underline{u} - \bar{u})^+, (\underline{u} - \bar{u})^+)|_{t=0}& \\ \leq c_\tau \int_0^\tau ((\underline{u} - \bar{u})^+, (\underline{u} - \bar{u})^+)\, dt.&\end{aligned}$$

Since $(u - \bar{u})^+|_{t=0} = 0$ a Gronwall-type argument shows the assertion $(\underline{u} - \bar{u})^+ \equiv 0$. □

Comparison principles for classical solutions of parabolic problems with dynamical boundary conditions and $\sigma \geq 0$ can be found in [5].

5.2. *Proof of Theorem 23*

Suppose that $u(x, t)$ exists for all times. By the comparison principle of Lemma 24 we have $u \geq h$ where h is the solution of the corresponding homogeneous linear problem (2.1)–(2.3) with $h(x, 0) = u_0(x)$. From Theorem 5 we get the formula

$$h(x, t) = \sum_{i=1}^\infty e^{-\lambda_i t} a(u_0, \varphi_i)\varphi_i(x) + h_0, \qquad h_0 = \frac{a(u_0, 1)}{a(1, 1)}.$$

Since $\lambda_i > 0$ for $i > 0$ we deduce that

$$h(x, t) \to h_0 = \frac{\int_D u_0\, dx + \sigma \oint_{\partial D} u_0\, ds}{|D| + \sigma |\partial D|} > 0 \quad \text{as } t \to \infty.$$

Consequently, if a solution of (1.1)–(1.3) exists for all times then for any positive ϵ there exists t_0 such that $u \geq h_0 - \epsilon$ in $\overline{D} \times (t_0, \infty)$ and $f(u) > c_0$ for $t \geq t_0$. Introducing $1/f(u)$ as a test function in the weak formulation of (1.1)–(1.3) on the time interval (t_0, τ) we get for $\tau \geq t_0$,

$$\int_{t_0}^\tau \left(\int_D \frac{u_t}{f(u)}\, dx + \sigma \oint_{\partial D} \frac{u_t}{f(u)}\, ds - \int_D \frac{|\nabla u|^2 f'(u)}{f^2(u)}\, dx \right) dt = \int_{t_0}^\tau \int_D m(x, t)\, dx\, dt.$$

Put

$$g(s) := \int_{s_0}^s \frac{d\xi}{f(\xi)}.$$

Since f is nondecreasing we have

$$\int_D g(T,x)\,dx + \sigma \oint_{\partial D} g(T,x)\,ds - \int_D g(t_0,x)\,dx - \sigma \oint_{\partial D} g(t_0,x)\,ds \geq \int_{t_0}^T \int_D m(x,t)\,dx\,dt.$$

If we let $T \to \infty$ the right-hand side tends to infinity whereas the left-hand side remains bounded. This is a contradiction and thus $u(x,t)$ cannot be a global solution. □

The argument fails if σ is negative.

EXAMPLE 1. Let $f(s) = e^s$. Then by the previous theorem all solutions blow up in finite time independent of the size of the domain. This is in contrast to the Dirichlet boundary conditions where for small domains there exist stationary solutions $U(x)$. Hence for all $u_0(x) \leq U(x)$ the solutions are global. For large domains, however, no stationary solutions exist and all solutions blow up in finite time [13].

EXAMPLE 2. For $p > 1$ let

$$f(s) = \begin{cases} s^p & \text{if } s > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

Then for all positive nonvanishing initial data the solution explodes in finite time. On the other hand, all negative constants are stationary solutions. If $u_0(x) < 0$, the solutions are negative and exist globally because $u = 0$ is an upper solution. If h_0 is negative and u_0 changes sign it is not clear whether the solutions are global or blow up in finite time. Under Dirichlet conditions, however, global solutions exists.

5.3. *Outer domains*

The situation is different in outer domains. By means of a standard procedure it is possible to construct solutions of D in outer domains with Lipschitz boundary, for uniformly bounded, continuous initial conditions $u_0 \geq 0$. Let $\{D_k\}_{k=1}^\infty$ be a sequence of nested bounded domains $D^c \subset D_1 \subset D_2 \subset \cdots$ with $\bigcup_{k=1}^\infty D_k = \mathbb{R}^N$. Let u_k be the solution of

$$\begin{aligned} u_t - \Delta u &= f(u) && \text{in } D \cap D_k \times \mathbb{R}^+, \\ \sigma u_t + u_n &= 0 && \text{on } \partial D \times \mathbb{R}^+, \\ u &= 0 && \text{on } \partial D_k \times \mathbb{R}^+, \\ u(x,0) &= u_0(x) && \text{in } D. \end{aligned}$$

By comparison $u_k(x,t) \leq u_{k+1}(x,t)$ in $D \cap D_k$. Let $z(t)$ be the solution of $\dot{z} = f(z)$, $z(0) = \|u_0\|_\infty$. It exists for, say, $t < T$. Then $u_k \leq z$ for $t < T$. Standard arguments based on a priori estimates for the heat equation imply that $u_k \to u$ in $C^{2,1}_{\text{loc}}$ as $k \to \infty$, where u is a solution in $D \times (0,T)$.

In what follows, we shall consider for simplicity the function f defined in (5.3). In the case of Dirichlet boundary conditions it is well known from [2] and [14] that no nontrivial positive solutions exist for all times if $p \in (1, 1 + 2/N]$. This implies the following result.

LEMMA 27. *Let D be an outer domain. Assume $\sigma > 0$ and $u_0 \in C(\overline{D})$. For $p \in (1, 1 + 2/N]$ all positive weak solutions blow up in finite time.*

PROOF. The assertion follows immediately from a comparison argument. Indeed, $u(x,t) \geq u^D(x,t)$ where $u^D(x,t)$ is the solution of (1.1)–(1.3) with Dirichlet boundary conditions on a smooth domain $D' \subset D$ and with the same initial condition $u_0(x)$. □

Next we derive that global solutions can exist for small initial data, in contrast to the case of bounded domains. This is essentially due to solutions that vanish at infinity.

LEMMA 28. *Let $\sigma > 0$ and $p > 1 + 2/N$. If the complement D^c of D is star-shaped with respect to the origin and if $\min_{\partial D} |x \cdot n(x))| > \sigma N$ on ∂D, then there exist global solutions for sufficiently small initial data.*

PROOF. The idea is to construct a positive upper solution $U(x,t)$, i.e. a function satisfying

$$U_t - \Delta U \geq f(U) \quad \text{in } D, \qquad \sigma U_t + U_n \geq 0 \quad \text{on } \partial D \times \mathbb{R}^+. \tag{5.4}$$

A standard choice is [2]

$$U(x,t) = A(t+t_0)^{-\gamma} e^{-r^2/(4(t+t_0))} \qquad \text{with } A, \gamma, t_0 > 0.$$

We have

$$U_t = U\left\{-\frac{\gamma}{t+t_0} + \frac{r^2}{4(t+t_0)^2}\right\}, \qquad U_r = -U\left(\frac{r}{2(t+t_0)}\right),$$

$$U_{rr} = -U\left\{\left(\frac{r}{2(t+t_0)}\right)^2 + \frac{1}{2(t+t_0)}\right\},$$

and thus

$$U_t - \Delta U - U^p = U\left(\frac{N-2\gamma}{2(t+t_0)} + \frac{r^2}{2(t+t_0)^2}\right) - U^p, \tag{5.5}$$

$$\sigma U_t + U_n = U\left(-\frac{\sigma\gamma}{t+t_0} + \frac{\sigma r^2}{4(t+t_0)^2} - \frac{x \cdot n(x)}{2(t+t_0)}\right). \tag{5.6}$$

If we choose $\gamma < N/2$, say $\gamma = N/2 - \epsilon$, then (5.5) is positive for all $t \in \mathbb{R}^+$ provided that

$$\epsilon(t+t_0)^{-1} > A^{p-1}(t+t_0)^{-\gamma(p-1)}.$$

This can only be achieved if $p > 1 + 2/N$ and A is sufficiently small. For the expression in (5.6) to be positive for all positive t we must have $-x \cdot n(x) > 0$ and $-\sigma N/2 > x \cdot n(x)/2$. The assertion now follows by comparison. □

6. OPEN PROBLEMS

We finish with a list of open problems.

1. The assumption on the star-shapedness and the size of σ in Lemma 28 seems to be unnatural since the result is true in the limit $\sigma \to \infty$ for general outer domains.
2. It is not clear whether in the case of negative σ the blow up result of Theorem 23 remains true. Can one find conditions on blow up for $\sigma < 0$? Do there exist global (likely sign-changing) solutions? Since f is bounded for negative values, the solution can never tend to $-\infty$ in finite time. If blow up occurs then only in regions where u is positive.

3. Can one find an example in the one-dimensional case of a potential $q \geq 0$ with more than two negative eigenvalues in the case $\sigma < 0$?
4. (See Section 2.2.) Does the expansion (2.5) hold in the sense of $L^2(D) \times L^2(\partial D)$ if $\sigma < 0$?

7. Appendix

Let us recall an approximation property of Lipschitz domains (cf. Nečas [15]). There exists a sequence of C^∞-domains $D_i \supset D$ with the following properties:

(i) There exist $k \in \mathbb{N}$, balls $B_1, \ldots, B_k$, neighborhoods $U_1, \ldots, U_k \subset \mathbb{R}^{N-1}$, C^∞-functions $\phi_l^i : U_l \to \mathbb{R}$ and Lipschitz functions $\phi_l : U_l \to \mathbb{R}$ for $l = 1, \ldots, k$ such that

$$\partial D_i \cap B_l = \operatorname{graph} \phi_l^i, \qquad \partial D \cap B_l = \operatorname{graph} \phi_l$$

and $\|\phi_l^i - \phi_l\|_\infty \to 0$ as $i \to \infty$, $\|\nabla \phi_l^i\|_\infty \leq \|\nabla \phi_l\|_\infty$ for $l = 1, \ldots, k$. Let $\operatorname{Lip}(\partial D) = \max_{l=1,\ldots,k} \|\nabla \phi_l\|_\infty$.

(ii) There exist homeomorphisms $\Lambda_i : \partial D \to \partial D_i$ such that

$$n^i \circ \Lambda_i \to n \qquad \text{a.e. on } \partial D \text{ as } i \to \infty.$$

(iii) There exists a smooth vector field $h : \mathbb{R}^N \to \mathbb{R}^N$ and $K > 0$ such that

$$h(\Lambda_i(x)) \cdot n^i(\Lambda_i(x)) \leq -K \qquad \forall i \in \mathbb{N},\ \forall x \in \partial D.$$

LEMMA 29. *For fixed $i \in \mathbb{N}$ let $\chi^i(x,t)$ denote the solution of the system*

$$\dot{X} = h(X), \qquad X(0) = x \in \partial D^i,$$

at time t, i.e., χ_i is a map $\partial D_i \times \mathbb{R} \to \mathbb{R}^N$. Then the maps

$$D_{(x,t)}\chi^i(x,t) : T_x\partial D^i \times \mathbb{R} \to \mathbb{R}^N, \qquad D_{(x,t)}\chi^i(x,t)^{-1} : \mathbb{R}^N \to T_x\partial D^i \times \mathbb{R}$$

have the expansions

$$D_{(x,t)}\chi^i(x,t) = (\mathrm{Id}|_{T_x\partial D^i}, h(x)) + O(t), \tag{7.1}$$

and

$$[D_{(x,t)}\chi^i(x,t)]^{-1} z = \left(z - \frac{z \cdot n^i(x)}{h(x) \cdot n^i(x)} h(x), \frac{z \cdot n^i(x)}{h(x) \cdot n^i(x)} \right) + O(t)z \tag{7.2}$$

for all $z \in \mathbb{R}^n$ where $\|O(t)\|_{L^\infty(\partial D^i)} \leq Ct$ for all $i \in \mathbb{N}$.

PROOF. Differentiation with respect to $x \in \partial D^i$ will be denoted by D_x (the index i is dropped). If one differentiates $\dot{X} = h(X)$, $X(0) = x$ with respect to $x \in \partial D^i$ and integrates then one finds

$$D_x\chi^i(x,t) = e^{t(D_x h)(\chi^i(x,t))}.$$

Likewise one finds $D_t\chi^i(x,t) = h(\chi^i(x,t))$. Note that $\|D_x h\|_\infty \le \|\mathbf{D}h\|_\infty \,\mathrm{Lip}(\partial D)$, where $\mathbf{D}h(y)$ is the Jacobian of $h(y)$ with respect to $y \in \mathbb{R}^N$. Expanding the expressions of $D_x\chi^i$ and $D_t\chi^i$ in t gives (7.1). The expansion (7.2) follows from (7.1) if one observes that

$$[(\mathrm{Id}|_{T_x\partial D^i}, h(x))]^{-1}z = \left(z - \frac{z\cdot n^i(x)}{h(x)\cdot n^i(x)}h(x), \frac{z\cdot n^i(x)}{h(x)\cdot n^i(x)}\right). \qquad \Box$$

For given $\delta > 0$ let $\Omega^i_\delta = \{y \in D^i : \mathrm{dist}(y, \partial D^i) < \delta\}$ and define the map

$$t^i : \Omega^i_\delta \to \mathbb{R}, \qquad y \mapsto t\text{-component of } \chi^i(y)^{-1}.$$

LEMMA 30. *There exist $\delta > 0$ and $i_0 \in \mathbb{N}$ such that for all $i \ge i_0$,*

$$|\nabla t^i(y)| \ge \frac{1}{2\|h\|_{L^\infty(\Omega^i_\delta)}} \quad \textit{for all } y \in \Omega^i_\delta, \tag{7.3}$$

$$\nabla t^i(x)\cdot n(x) \le \frac{-1}{\|h\|_{L^\infty(\Omega^i_\delta)}} \quad \textit{for almost all } x \in \partial D. \tag{7.4}$$

PROOF. First we compute from (7.2) in Lemma 29 that

$$\nabla t^i(y) = \frac{1}{h(x)\cdot n^i(x)} n^i(x) + O(\mathrm{dist}(y, \partial D^i))$$

where $x = x$-component of $\chi^i(y)^{-1} \in \partial D^i$. If i is sufficiently large and $\mathrm{dist}(y, \partial D^i) \le \delta$ sufficiently small we obtain (7.3). Moreover

$$\nabla t^i(x)\cdot n^i(x) \le \frac{-1}{\|h\|_{L^\infty(\Omega^i_\delta)}} \quad \text{for all } x \in \partial D^i.$$

Since $n^i \circ \Lambda^i \to n$ as $i \to \infty$ for almost all $x \in \partial D$, this implies (7.4). $\Box$

REFERENCES

[1] H. W. ALT, *Lineare Funktionalanalysis*. 3rd edition, Springer, Berlin, 1999.

[2] C. BANDLE - H. A. LEVINE, *Fujita type results for convective-like reaction diffusion equations in exterior domains*. ZAMP 40 (1989), 665–676.

[3] C. BANDLE - W. REICHEL, *Solutions of quasilinear second order elliptic boundary value problems via degree theory.* In: Handbook of Differential Equations: Stationary Partial Differential Equations, Vol. 1, M. Chipot and P. Quittner (eds.), Elsevier, 2004, 1–70.

[4] J. VON BELOW, *Parabolic Network Equations*. Tübingen, 1994, 3rd ed. to appear.

[5] J. VON BELOW - C. DE COSTER, *A qualitative theory for parabolic problems under dynamical boundary conditions*. J. Inequal. Appl. 5 (2000), 467–486.

[6] J. VON BELOW - G. FRANÇOIS, *Spectral asymptotics for the Laplacian under an eigenvalue dependent boundary condition*. Bull. Belgian Math. Soc. 2005/2006, to appear.

[7] J. VON BELOW - G. PINCET MAILLY, *Blow up for reaction diffusion equations under dynamical boundary conditions*. Comm. Partial Differential Equations 28 (2003), 223–247.

[8] D. G. DE FIGUEIREDO, *Positive solutions of semilinear elliptic problems*. In: Differential Equations (Sao Paulo, 1981), Lecture Notes in Math. 957, Springer, Berlin, 1982, 34–87.

[9] J. ERCOLANO - M. SCHECHTER, *Spectral theory for operators generated by elliptic boundary problems with eigenvalue parameter in boundary conditions, I*. Comm. Pure Appl. Math. 18 (1965), 83–105.

[10] J. ESCHER, *Quasilinear parabolic systems with dynamical boundary conditions*. Comm. Partial Differential Equations 18 (1993), 1309–1364.

[11] M. FILA - P. QUITTNER, *Large time behavior of solutions of a semilinear parabolic equation with a nonlinear dynamical boundary condition*. In: Topics in Nonlinear Analysis, Progr. Nonlinear Differential Equations Appl. 35, Birkhäuser, Basel, 1999, 251–272.

[12] G. FRANÇOIS, *Spectral asymptotics stemming from parabolic equations under dynamical boundary conditions*. Preprint LMPA Nr. 225 (2004), submitted.

[13] A. LACEY, *Mathematical analysis of thermal runaway for spatially inhomogeneous reactions*. SIAM J. Appl. Anal. 43 (1983), 1350–1366.

[14] K. MOCHIZUKI - R. SUZUKI, *Critical exponent and critical blow up for quasi-linear parabolic equations*. Israel J. Math. 98 (1997), 141–156.

[15] J. NEČAS, *Les méthodes directes en théorie des équations elliptiques*. Masson, Paris, 1967.

[16] D. F. RIAL - J. D. ROSSI, *Blow-up results and localization of blow-up points in an N-dimensional smooth domain*. Duke Math. J. 88 (1997), 391–405.

[17] R. SHOWALTER, *Degenerate evolution equations*. Indiana Univ. Math. J. 23 (1974), 655–677.

[18] M. STRUWE, *Variational Methods*. 2nd ed., Ergeb. Math. Grenzgeb. 34, Springer, Berlin, 1996.

[19] J. L. VAZQUEZ - E. VITILLARO, *Heat equation with dynamical boundary conditions of reactive type*. Preprint, 2004.

[20] E. VITILLARO, *Global existence and blow-up for Laplace equation with nonlinear dynamical boundary conditions and for abstract evolution equations*. To appear.

[21] W. WALTER, *Gewöhnliche Differentialgleichungen*. 7th ed., Springer, Berlin, 2000.

Received 27 April 2005,
and in revised form 20 June 2005.

C. Bandle
Mathematisches Institut
Universität Basel
Rheinsprung 21
CH-4051 BASEL, Switzerland
catherine.bandle@unibas.ch

J. von Below
LMPA J. Liouville, EA 2597
Université du Littoral Côte d'Opale
50, rue F. Buisson, B.P. 699
F-62228 CALAIS Cedex, France
joachim.von.below@lmpa.univ-littoral.fr

W. Reichel
Mathematisches Institut
Universität Basel
Rheinsprung 21
CH-4051 BASEL, Switzerland
wolfgang.reichel@unibas.ch