Rend. Lincei Mat. Appl. 17 (2006), 69–85

Mathematical analysis. — *Nodal solutions of nonlinear elliptic Dirichlet problems on radial domains*, by THOMAS BARTSCH and MARCO DEGIOVANNI, presented by A. Ambrosetti.

ABSTRACT. — Let $\Omega \subset \mathbb{R}^N$ be a ball or an annulus and $f : \mathbb{R} \to \mathbb{R}$ absolutely continuous, superlinear, subcritical, and such that $f(0) = 0$. We prove that the least energy nodal solution of $-\Delta u = f(u)$, $u \in H_0^1(\Omega)$, is not radial. We also prove that Fučik eigenfunctions, i.e. solutions $u \in H_0^1(\Omega)$ of $-\Delta u = \lambda u^+ - \mu u^-$, with eigenvalue (λ, μ) on the first nontrivial curve of the Fučik spectrum, are not radial. A related result holds for asymmetric weighted eigenvalue problems. An essential ingredient is a quadratic form generalizing the Hessian of the energy functional $J \in C^1(H_0^1(\Omega))$ at a solution. We give new estimates on the Morse index of this form at a radial solution. These estimates are of independent interest.

KEY WORDS: Nodal solutions; Dirichlet problems; Fučik spectrum; weighted asymmetric eigenvalue problems.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 35J65, 58E05.

1. INTRODUCTION

In this paper we investigate two types of nonlinear elliptic equations on a bounded domain $\Omega \subset \mathbb{R}^N$, with homogeneous Dirichlet boundary conditions. One is the equation

$$-\Delta u = f(u) \quad \text{in } \Omega, \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $f(0) = 0$ and which grows superlinearly and subcritically as $|u| \to \infty$. The second type are the nonlinear eigenvalue equations

$$-\Delta u = \lambda(au^+ - bu^-) \quad \text{in } \Omega \tag{1.2}$$

with given $a, b > 0$, and

$$-\Delta u = \lambda u^+ - \mu u^- \quad \text{in } \Omega. \tag{1.3}$$

Here $u^\pm = \max\{\pm u, 0\}$. Equation (1.2) is an asymmetric (if $a \neq b$) weighted eigenvalue problem, (1.3) the Fučik eigenvalue problem. We are interested in "least energy nodal" solutions of (1.1) and in the first nodal eigenfunctions of (1.2), (1.3). In particular, we prove a symmetry breaking effect in case Ω is radially symmetric.

In recent years there has been an increasing interest in the existence and properties of nodal solutions of semilinear elliptic boundary value problems beginning with [3, 6, 10] in the mid 1990s. Let

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(x,u)\,dx, \qquad F(x,s) = \int_0^s f(x,t)\,dt,$$

be the energy functional associated to (1.1). Then critical points of $J : H_0^1(\Omega) \to \mathbb{R}$ are solutions of (1.1), and nontrivial solutions lie on the Nehari manifold

$$\mathcal{N} = \{u \in H_0^1(\Omega) : u \neq 0,\ J'(u)u = 0\}.$$

Similarly, nodal solutions lie on the nodal Nehari set

$$\mathcal{S} = \{u \in H_0^1(\Omega) : u^\pm \neq 0,\ J'(u)u^\pm = 0\} = \{u \in H_0^1(\Omega) : u^\pm \in \mathcal{N}\}.$$

We are interested in local minimizers of $J|_\mathcal{S}$.

If $f(s)/|s|$ is strictly increasing on $\mathbb{R}^-$ and on $\mathbb{R}^+$, then $\mathcal{N}$ is in fact a topological manifold. Under additional regularity and growth conditions on f, it is a differentiable manifold, and critical points of the constrained functional $J|_\mathcal{N}$ are critical points of J. Since the maps $H_0^1(\Omega) \to H_0^1(\Omega)$, $u \mapsto u^\pm$, are not locally Lipschitz continous, the set $\mathcal{S}$ is not a differentiable manifold, independent of smoothness and growth of f. Under mild conditions on f not involving differentiability, we show that a local minimizer of $J|_\mathcal{S}$ is a critical point of J with "Morse index" at least 2. Since J is only of class C^1 in our setting, we need a generalized version of Morse index. Let us point out that it would also be interesting to apply variational methods, not only minimization, directly to $J|_\mathcal{S}$. However, due to the lack of Lipschitz continuity of $u \mapsto u^\pm$, it seems difficult to apply the critical point theory for continuous functionals on metric spaces (see [9, 14, 15]) to $J|_\mathcal{S}$. In fact, it is not at all clear that a critical point of $J|_\mathcal{S}$ in the generalized sense is a critical point of J.

If Ω is radially symmetric, we prove that a local minimizer of $J|_\mathcal{S}$ cannot be radial. The main idea is to relate the generalized Morse index of a radial critical point u of J to the number of nodal domains of u. Here we improve a recent result of Aftalion and Pacella [1].

For the nonlinear eigenvalue problem (1.2) the solution corresponding to minimizers of $J|_\mathcal{S}$ is the second eigenfunction. This changes sign, and we prove that on a radially symmetric domain, it cannot be radial. Similarly, (1.3) has a curve $\mathcal{C}$ of Fučik eigenvalues corresponding to the second eigenvalue. This has been investigated in [12]. Again we prove that an eigenfunction associated to $(\lambda, \mu) \in \mathcal{C}$ is never radial. The techniques developed to deal with (1.1) with nondifferentiable f are useful when treating (1.2) and (1.3).

The paper is organized as follows. In Section 2 we state our results on (1.1). These will be proved in Sections 5 and 6. In Section 3 we formulate the results on (1.2) and (1.3), which will be proved in Section 7. Section 4 contains results on critical groups and Morse indices for C^1-functionals which are essential for our proofs. We believe they are of independent interest.

2. SUPERLINEAR DIRICHLET PROBLEMS

Let Ω be a bounded, open subset of $\mathbb{R}^N$, $N \geq 2$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Assume that

(f_1) $f(x, 0) = 0$ for a.e. $x \in \Omega$;

(f_2) there exist a function a and constants $b \in \mathbb{R}$, $p > 2$ such that

$$|f(x, s)| \leq a(x)|s| + b|s|^{p-1} \quad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R},$$

with $a \in L^{N/2}(\Omega)$ and $p \leq 2^*$ for $N \geq 3$, $a \in L^r(\Omega)$ for some $r > 1$ and p unrestricted for $N = 2$;

(f_3) for a.e. $x \in \Omega$, the function $s \mapsto f(x, s)/|s|$ is strictly increasing on $\mathbb{R}^-$ and on $\mathbb{R}^+$.

Let $J : H_0^1(\Omega) \to \mathbb{R}$ be defined as

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(x,u)\,dx,$$

where $F(x,s) = \int_0^s f(x,t)\,dt$. Then J is a functional of class C^1 whose critical points are the weak solutions $u \in H_0^1(\Omega)$ of the problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Since we are interested in nodal solutions we consider the constraint

$$\mathcal{S} = \{u \in H_0^1(\Omega) : u^+ \neq 0,\ u^- \neq 0,\ J'(u)u^+ = J'(u)u^- = 0\},$$

where $u^\pm = \max\{\pm u, 0\}$. We will investigate local minima of $J|_\mathcal{S}$. For given $u \in H_0^1(\Omega)$ we define the extended valued, upper semicontinuous functional $Q_u : H_0^1(\Omega) \to \overline{\mathbb{R}}$ by

$$Q_u(v) = \limsup_{\substack{z\to u\\ t\to 0\\ w\to v}} \frac{J(z+tw) + J(z-tw) - 2J(z)}{t^2}.$$

Finally, we denote by $m(J;u)$ the supremum of the dimensions of the linear subspaces V of $H_0^1(\Omega)$ such that

$$Q_u(v) < 0 \quad \text{for every } v \in V \setminus \{0\}.$$

THEOREM 2.1. *Let $u \in \mathcal{S}$ be a local minimum for $J|_\mathcal{S}$. Then u is a free critical point of J with $m(J;u) \le 2$.*

Consider now the particular case in which

(f_4) $f : \mathbb{R} \to \mathbb{R}$ is independent of $x \in \Omega$ and absolutely continuous.

From regularity theory we know that any weak solution $u \in H_0^1(\Omega)$ of (2.1) belongs in fact to $C^1(\Omega)$.

For a continuous function $u : \Omega \to \mathbb{R}$, let $\mathrm{nod}(u) \in \mathbb{N} \cup \{\infty\}$ be the number of nodal domains, i.e. the number of connected components of $\Omega \setminus u^{-1}(0)$.

THEOREM 2.2. *Suppose (f_1)–(f_4) hold and Ω is a ball or an annulus. Then*

$$m(J;u) \ge (\mathrm{nod}(u) - 1)\cdot(N+1)$$

holds for any radial solution $u \in H_0^1(\Omega)$ of (2.1).

Theorem 2.1 will be proved in Section 5 and Theorem 2.2 in Section 6. They immediately imply the next

COROLLARY 2.3. *Suppose (f_1)–(f_4) hold and Ω is a ball or an annulus. Then a local minimum of $J|_\mathcal{S}$ is not radial.*

In [5] it is proved that a least energy nodal solution, i.e. a global minimizer u of $J|_{\mathcal{S}}$, is *foliated Schwarz symmetric*. This means that there exists $P \in \mathbb{R}^N$ with $|P| = 1$ such that $u(x) = v(|x|, \langle x, P\rangle)$ depends only on the euclidean norm $|x|$ of x and on the projection $\langle x, P\rangle$. Moreover, $v(r, s)$ is decreasing in s. The question whether or not least energy nodal solutions are radial has been settled in [1] in the case where f is of class C^1 and f' satisfies appropriate growth conditions so that J is of class C^2. These conditions are essential for the argument of [1] since the authors differentiate the equation and need the Hessian of J at a solution. In the differentiable case $m(J; u)$ is the Morse index of J at u. In this case it has been proved in [1] that $m(J; u) \geq N + 1$ if u is a radial and sign-changing solution of (2.1). Theorem 2.1 is an improvement of this estimate and is new even in the differentiable case. Under the hypotheses of [1] on f' it can however be obtained using the methods of [1]. The generalization to the nondifferentiable case is particularly worthwhile for certain applications to competing species problems in mathematical ecology where one has to deal with jumping nonlinearities. We refer the reader to the papers [7, 11] for the relevance of nodal solutions of (2.1) to mathematical ecology.

3. Asymmetric weighted eigenvalue problems and Fučik eigenfunctions

In this section we first investigate the symmetry of solutions of

$$\begin{cases} -\Delta u = \lambda(au^+ - bu^-) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Here $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a radial bounded domain, i.e. a ball or an annulus, and $a, b > 0$ are fixed constants. Let $\lambda_1 = \lambda_1(\Omega) > 0$ be the first Dirichlet eigenvalue of $-\Delta$ in Ω, and let $e_1 > 0$ be an associated eigenfunction. Clearly, e_1 solves (3.1) for $\lambda = \lambda_1/a$ and $-e_1$ solves (3.1) for $\lambda = \lambda_1/b$. It is well known that there exists a first nontrivial eigenvalue λ_2 with a sign-changing eigenfunction; see [2] for a proof in a much more general situation. In [5] it is proved that every eigenfunction corresponding to λ_2 is foliated Schwarz symmetric. The question whether or not these eigenfunctions are radial has been left open.

THEOREM 3.1. *A nontrivial solution of* (3.1) *for* $\lambda = \lambda_2$ *is not radial.*

Theorem 3.1 will be proved in Section 7. As a corollary, we obtain an analogous result for a solution of

$$\begin{cases} -\Delta u = \lambda u^+ - \mu u^- & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

for $(\lambda, \mu) \in \mathcal{C} \subset \mathbb{R}^+ \times \mathbb{R}^+$ lying in the *first nontrivial curve* of the Fučik spectrum. Recall that the Fučik spectrum consists of all pairs (λ, μ) such that (3.2) has a nontrivial solution. Defining $\eta : (\lambda_1, \infty) \to \mathbb{R}$ by

$$\eta(\lambda) = \inf\{\mu > \lambda_1 : (\lambda, \mu) \text{ is a nontrivial Fučik eigenvalue}\},$$

we have $\lambda_1 < \eta(\lambda) < \infty$ for every λ. By definition, the curve

$$\mathcal{C} := \{(\lambda, \eta(\lambda)) : \lambda \in (\lambda_1, \infty)\}$$

consists of Fučik eigenvalues. Moreover, η is continuous and strictly decreasing with $\eta(\lambda_2) = \lambda_2$, hence $(\lambda_2, \lambda_2) \in \mathcal{C}$. The set $\mathcal{C}$ is symmetric with respect to the diagonal. We refer the reader to [12] for proofs of these statements.

THEOREM 3.2. *A nontrivial solution of* (3.2) *for* $(\lambda, \mu) \in \mathcal{C}$ *is not radial.*

PROOF. By the above discussion $\lambda_2 = 1$ is the first nontrivial eigenvalue of (3.1) if $(a, b) \in \mathcal{C}$. Thus Theorem 3.2 follows immediately from Theorem 3.1. □

The nonlinear eigenvalue problems of this section have in general nondifferentiable right hand sides. They are easy to deal with in the differentiable case $a = b$ or $\lambda = \mu$. The corresponding variational integral is not of class C^2 and Morse index arguments are not straightforward.

4. CRITICAL GROUPS

In the following, H^* will denote Alexander–Spanier cohomology with coefficients in a given ring $\mathcal{R}$.

DEFINITION 4.1. Let X be a topological space, $\Phi : X \to \mathbb{R}$ a continuous function and $u \in X$. We define the *critical groups* of Φ at u by

$$\underline{C}_m(\Phi; u) = H^m(\{v \in X : \Phi(v) < \Phi(u)\} \cup \{u\}, \{v \in X : \Phi(v) < \Phi(u)\}),$$
$$C_m(\Phi; u) = H^m(\{v \in X : \Phi(v) \leq \Phi(u)\}, \{v \in X : \Phi(v) \leq \Phi(u)\} \setminus \{u\}).$$

Because of the excision property, we may replace X by any neighborhood U of u in X. If X is a Banach manifold of class C^1, Φ is a function of class C^1 and u is not a critical point of Φ, then it turns out that $\underline{C}_m(\Phi; u)$ and $C_m(\Phi; u)$ vanish for every m (see e.g. [8, Proposition 3.4 and p. 1064]). Moreover, if u is an isolated critical point of Φ, then we have $\underline{C}_m(\Phi; u) \cong C_m(\Phi; u)$ for every m (see e.g. [8, Proposition 3.7]).

THEOREM 4.2. *Let E be a Banach space which splits into a direct sum $E = V \oplus W$ with* $\dim V = m < \infty$ *and W closed in E. Let $u \in E$, $r > 0$ and let*

$$\Phi : (\mathrm{B}_r(u) \cap V) + (\mathrm{B}_r(u) \cap W) \to \mathbb{R}$$

be a continuous function. Assume that, for every $w \in \mathrm{B}_r(u) \cap W$, the function $v \mapsto \Phi(v + w)$ is strictly concave on $\mathrm{B}_r(u) \cap V$. Then $\underline{C}_k(\Phi; u)$ and $C_k(\Phi; u)$ are both trivial for every $k \leq m - 1$.

PROOF. The assertion concerning $C_k(\Phi; u)$ is proved in [16, Theorem 6.1]. The assertion concerning $\underline{C}_k(\Phi; u)$ can be proved in the same way. □

COROLLARY 4.3. *Let E be a Banach space, U an open subset of E, $\Phi : U \to \mathbb{R}$ a continuous function and $u \in U$. Define $Q_u : E \to \overline{\mathbb{R}}$ by*

$$Q_u(v) = \limsup_{\substack{z \to u \\ t \to 0 \\ w \to v}} \frac{\Phi(z + tw) + \Phi(z - tw) - 2\Phi(z)}{t^2}$$

and assume that there exists a linear subspace V of E, with finite dimension m, such that

$$Q_u(v) < 0 \quad \text{for every } v \in V \setminus \{0\}.$$

Then $\underline{C}_k(\Phi; u)$ and $C_k(\Phi; u)$ are both trivial for every $k \leq m - 1$.

PROOF. Let W be a closed subspace of E such that $E = V \oplus W$. We claim that there exists $r > 0$ such that, for every $w \in \mathrm{B}_r(u) \cap W$, the function $v \mapsto \Phi(v + w)$ is strictly concave on $\mathrm{B}_r(u) \cap V$. Since Φ is continuous, this is equivalent to showing that there exists $r > 0$ such that, for every $u_0, u_1 \in (\mathrm{B}_r(u) \cap V) + (\mathrm{B}_r(u) \cap W)$ with $u_0 \neq u_1$ and $u_1 - u_0 \in V$, we have

$$\Phi\left(\frac{1}{2}\, u_0 + \frac{1}{2}\, u_1\right) > \frac{1}{2}\, \Phi(u_0) + \frac{1}{2}\, \Phi(u_1).$$

By contradiction, let (u_n) be a sequence convergent to u in E and (z_n) be a sequence in $V \setminus \{0\}$ convergent to 0 such that

$$2\Phi(u_n) \leq \Phi(u_n + z_n) + \Phi(u_n - z_n). \tag{4.1}$$

Let $z_n = t_n v_n$ with $t_n > 0$ convergent to 0 and v_n of unit norm. Since V is finite-dimensional, up to a subsequence (v_n) is convergent to some $v \in V$ with $v \neq 0$. It follows that

$$\begin{aligned}\limsup_{n\to\infty} &\frac{\Phi(u_n + z_n) + \Phi(u_n - z_n) - 2\Phi(u_n)}{\|z_n\|^2} \\ &= \limsup_{n\to\infty} \frac{\Phi(u_n + t_n v_n) + \Phi(u_n - t_n v_n) - 2\Phi(u_n)}{t_n^2} \leq Q_u(v) < 0,\end{aligned}$$

which contradicts (4.1). Therefore there exists $r > 0$ with the required property.

By Theorem 4.2, the assertion follows. □

REMARK 4.4. If Φ is of class C^2 in a neighborhood of u, we clearly have $Q_u(v) = \Phi''(u)[v, v]$. If Φ is of class C^1 in a neighborhood of u, we have

$$Q_u(v) \leq \limsup_{\substack{z\to u \\ (\tau,\vartheta)\to(0,0) \\ w\to v}} \frac{\Phi'(z + \tau w)w - \Phi'(z + \vartheta w)w}{\tau - \vartheta}. \tag{4.2}$$

To see this, it is enough to observe that we can assume, in the definition of Q_u, that $t > 0$ and then apply the mean value theorem to the function

$$\tau \mapsto \Phi(z + \tau w) - \Phi(z + (\tau - t)w)$$

on the interval $[0, t]$.

In the last part of this section, we provide an estimate of the right hand side of (4.2) in a specific case.

Let Ω be a bounded, open subset of $\mathbb{R}^N$, $N \geq 2$, and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Assume that there exist functions a_1, a_2 and constants $b \in \mathbb{R}$, $p > 2$ such that

$$|g(x,s)| \leq a_1(x) + b|s|^{p-1}, \tag{4.3}$$

$$(g(x,s) - g(x,t))(s-t) \geq -(a_2(x) + b|s|^{p-2} + b|t|^{p-2})(s-t)^2, \tag{4.4}$$

for a.e. $x \in \Omega$ and every $s, t \in \mathbb{R}$, where $a_1 \in L^{2N/(N+2)}(\Omega)$, $a_2 \in L^{N/2}(\Omega)$ and $p \leq 2^*$ for $N \geq 3$, $a_1, a_2 \in L^r(\Omega)$ for some $r > 1$ and p unrestricted for $N = 2$.

For every $(x, s) \in \Omega \times \mathbb{R}$, set

$$\underline{D}_s g(x,s) = \liminf_{(t,\tau)\to(s,s)} \frac{g(x,t) - g(x,\tau)}{t - \tau}. \tag{4.5}$$

The continuity of $s \mapsto g(x, s)$ implies that

$$\underline{D}_s g(x,s) = \liminf_{\substack{(t,\tau)\to(0,0)\\ t,\tau\in\mathbb{Q}}} \frac{g(x,s+t) - g(x,s+\tau)}{t-\tau} \quad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}.$$

Therefore, for every measurable function $u : \Omega \to \mathbb{R}$, the function $\underline{D}_s g(x, u)$ is measurable and satisfies, by (4.4), the inequality

$$\underline{D}_s g(x, u(x)) \geq -a_2(x) - 2b|u(x)|^{p-2} \quad \text{for a.e. } x \in \Omega. \tag{4.6}$$

Consider the C^1-functional $\Psi : H_0^1(\Omega) \to \mathbb{R}$ defined by $\Psi(u) = -\int_\Omega G(x,u)\,dx$, where $G(x,s) = \int_0^s g(x,t)\,dt$.

PROPOSITION 4.5. *If* (4.3) *and* (4.4) *hold then for every* $u, v \in H_0^1(\Omega)$, *we have*

$$\limsup_{\substack{z\to u\\ (\tau,\vartheta)\to(0,0)\\ w\to v}} \frac{\Psi'(z+\tau w)w - \Psi'(z+\vartheta w)w}{\tau - \vartheta} \leq -\int_\Omega \underline{D}_s g(x,u)v^2\,dx < \infty$$

(we agree that $+\infty \cdot 0 = 0$*).*

PROOF. The latter inequality follows from (4.6). To prove the former, consider two sequences (u_n), (v_n) in $H_0^1(\Omega)$ converging to u, v, respectively, and two sequences (τ_n), (ϑ_n) in $\mathbb{R}$, with $\tau_n \neq \vartheta_n$, converging to 0. Up to a subsequence, we may also assume that (u_n) is convergent to u and (v_n) is convergent to v a.e. in Ω.

By (4.4), for a.e. $x \in \Omega$ and every $s, \tau, \vartheta, r \in \mathbb{R}$ with $\tau \neq \vartheta$, we have

$$\frac{(g(x,s+\tau r) - g(x,s+\vartheta r))r}{\tau - \vartheta} \geq -a(x)r^2 - b|s+\tau r|^{p-2}r^2 - b|s+\vartheta r|^{p-2}r^2,$$

whence

$$\frac{(g(x,u_n+\tau_n v_n) - g(x,u_n+\vartheta_n v_n))v_n}{\tau_n - \vartheta_n} \geq -av_n^2 - b|u_n+\tau_n v_n|^{p-2}v_n^2 - b|u_n+\vartheta_n v_n|^{p-2}v_n^2.$$

Moreover, for a.e. $x \in \Omega$ and every $r, s \in \mathbb{R}$, we have

$$\liminf_{\substack{\sigma \to s \\ (\tau,\vartheta)\to(0,0) \\ \varrho \to r}} \frac{(g(x,\sigma+\tau\varrho)-g(x,\sigma+\vartheta\varrho))\varrho}{\tau-\vartheta} = \underline{D}_s g(x,s) r^2$$

with the convention $+\infty \cdot 0 = 0$.

By Fatou's lemma the assertion follows. □

5. Proof of Theorem 2.1

We consider the Nehari manifold

$$\mathcal{N} := \{u \in H_0^1(\Omega) \setminus \{0\} : J'(u)u = 0\}.$$

This is a topological manifold as a consequence of (f_1)–(f_3) but not a differentiable manifold in general. Regularity conditions on f which imply that $\mathcal{N}$ is a C^1-submanifold of $H_0^1(\Omega)$ can be found in [4]. Condition (f_3) implies that, for every $u \in H_0^1(\Omega) \setminus \{0\}$, the map

$$g_u : \mathbb{R}^+ \to \mathbb{R}, \qquad \lambda \mapsto \frac{1}{\lambda} J'(\lambda u)u = \int_\Omega |\nabla u|^2 \, dx - \int_\Omega \frac{f(x,\lambda u)}{\lambda} u \, dx,$$

is strictly decreasing. Since $(u,\lambda) \mapsto g_u(\lambda)$ is continuous, the set

$$\mathcal{O}_0 := \{u \in H_0^1(\Omega) \setminus \{0\} : g_u(\lambda_2) < 0 < g_u(\lambda_1) \text{ for some } 0 < \lambda_1 < \lambda_2\}$$

is open. Clearly, we have $\mathcal{O}_0 = \{\lambda u : u \in \mathcal{N}, \lambda > 0\}$. For $u \in \mathcal{O}_0$, there exists a unique $\lambda_u > 0$ such that $g_u(\lambda_u) = 0$, that is, $\lambda_u u \in \mathcal{N}$. Using the fact that g_u is strictly decreasing, it is easy to check that the map $\mathcal{O}_0 \to \mathbb{R}^+$, $u \mapsto \lambda_u$, is continuous. Setting

$$\mathcal{O} := \{u \in H_0^1(\Omega) : u^+, u^- \in \mathcal{O}_0\},$$

we see that $\mathcal{O} \subset H_0^1(\Omega)$ is open and obtain

LEMMA 5.1. *The maps*

$$h_0 : \mathcal{N} \times \mathbb{R}^+ \to \mathcal{O}_0, \qquad (u,\lambda) \mapsto \lambda u,$$

and

$$h : \mathcal{S} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathcal{O}, \qquad (u,s,t) \mapsto su^+ - tu^-,$$

are homeomorphisms onto open subsets $\mathcal{O}_0$, $\mathcal{O}$ of $H_0^1(\Omega)$.

THEOREM 5.2. *For every $u \in \mathcal{S}$ and every k, we have*

$$\underline{C}_k(J;u) \cong \underline{C}_{k-2}(J|_{\mathcal{S}};u) \quad \text{and} \quad C_k(J;u) \cong C_{k-2}(J|_{\mathcal{S}};u).$$

PROOF. Let $u \in \mathcal{S}$ and set

$$C_- = \{sv^+ - tv^- : v \in \mathcal{S},\ s > 0,\ 0 < t \leq 1\},$$
$$C_+ = \{sv^+ - tv^- : v \in \mathcal{S},\ s > 0,\ t \geq 1\}.$$

As a consequence of Lemma 5.1, and observing that $J(su^+ - tu^-) < J(u)$ whenever $(s, t) \neq (1, 1)$, we have

$$\underline{C}_k(J; u) \cong H^k(\{v \in C_- \cup C_+ : J(v) < J(u)\} \cup \{u\}, \{v \in C_- \cup C_+ : J(v) < J(u)\}).$$

Since we are working with Alexander–Spanier cohomology in a metric space, we have the Mayer–Vietoris exact sequence for (relatively) closed sets

$$\begin{aligned}
&H^{k-1}(\{v \in C_- : J(v) < J(u)\} \cup \{u\}, \{v \in C_- : J(v) < J(u)\})\\
&\quad\oplus H^{k-1}(\{v \in C_+ : J(v) < J(u)\} \cup \{u\}, \{v \in C_+ : J(v) < J(u)\})\\
&\to H^{k-1}(\{v \in C_- \cap C_+ : J(v) < J(u)\} \cup \{u\}, \{v \in C_- \cap C_+ : J(v) < J(u)\})\\
&\to H^{k}(\{v \in C_- \cup C_+ : J(v) < J(u)\} \cup \{u\}, \{v \in C_- \cup C_+ : J(v) < J(u)\})\\
&\to H^{k}(\{v \in C_- : J(v) < J(u)\} \cup \{u\}, \{v \in C_- : J(v) < J(u)\})\\
&\quad\oplus H^{k}(\{v \in C_+ : J(v) < J(u)\} \cup \{u\}, \{v \in C_+ : J(v) < J(u)\}).
\end{aligned}$$

It is easy to see that $\{v \in C_\pm : J(v) < J(u)\}$ is a weak deformation retract of $\{v \in C_\pm : J(v) < J(u)\} \cup \{u\}$. It follows that

$$\begin{aligned}
&H^{k-1}(\{v \in C_- \cap C_+ : J(v) < J(u)\} \cup \{u\}, \{v \in C_- \cap C_+ : J(v) < J(u)\})\\
&\quad\cong H^{k}(\{v \in C_- \cup C_+ : J(v) < J(u)\} \cup \{u\}, \{v \in C_- \cup C_+ : J(v) < J(u)\}).
\end{aligned}$$

Setting

$$D_- = \{sv^+ - v^- : v \in \mathcal{S},\ 0 < s \leq 1\},$$
$$D_+ = \{sv^+ - v^- : v \in \mathcal{S},\ s \geq 1\},$$

we obtain in a similar way

$$\begin{aligned}
&\underline{C}_{k-2}(J|_{\mathcal{S}}; u)\\
&\cong H^{k-2}(\{v \in D_- \cap D_+ : J(v) < J(u)\} \cup \{u\}, \{v \in D_- \cap D_+ : J(v) < J(u)\})\\
&\cong H^{k-1}(\{v \in C_- \cap C_+ : J(v) < J(u)\} \cup \{u\}, \{v \in C_- \cap C_+ : J(v) < J(u)\}),
\end{aligned}$$

hence the assertion concerning the critical groups $\underline{C}_*$ follows.

The assertion concerning the critical groups C_* can be proved in a similar way. □

COROLLARY 5.3. *Let $u \in \mathcal{S}$ be a local minimum of $J|_{\mathcal{S}}$. Then*

$$\underline{C}_2(J; u) \cong \mathcal{R}, \qquad \underline{C}_k(J; u) \cong \{0\} \quad \textit{for every } k \neq 2.$$

PROOF. By the dimension axiom, we have

$$\underline{C}_0(J|_{\mathcal{S}}; u) \cong \mathcal{R}, \quad \underline{C}_k(J|_{\mathcal{S}}; u) \cong \{0\} \quad \text{for } k \neq 0.$$

Then the assertion follows from Theorem 5.2. □

PROOF OF THEOREM 2.1. From Corollary 5.3 it follows that $\underline{C}_2(J; u)$ is not trivial. Therefore u is a free critical point of J. This fact can also be proved arguing as in [5, Proposition 3.1].

Assume now, for a contradiction, that $m(J; u) \geq 3$ and consider a finite-dimensional subspace V of $H_0^1(\Omega)$ with $\dim V \geq 3$ such that Q_u is strictly negative on $V \setminus \{0\}$. From Corollary 4.3 we deduce that $\underline{C}_2(J; u)$ is trivial, and a contradiction follows. □

6. PROOF OF THEOREM 2.2

From assumption (f_2) and using Young's inequality, it readily follows that f satisfies (4.3). Moreover, assumption (f_3) implies that

$$\frac{f(x,s) - f(x,t)}{s-t} > \frac{f(x,s)}{s}$$

for a.e. $x \in \Omega$ and every $s, t \in \mathbb{R}$ with either $0 < s < t$ or $t < s < 0$.

Combining this fact with assumption (f_2), it is easy to show that f satisfies also (4.4). Finally, if we define $\underline{D}_s f$ as in (4.5), we have

$$\underline{D}_s f(x,s) \geq -a(x) - b|s|^{p-2} \quad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}. \tag{6.1}$$

PROPOSITION 6.1. *For every $u, v \in H_0^1(\Omega)$, we have*

$$Q_u(v) \leq \int_\Omega |\nabla v|^2 \, dx - \int_\Omega \underline{D}_s f(x,u) v^2 \, dx < \infty$$

(again, we agree that $+\infty \cdot 0 = 0$).

PROOF. This follows from Remark 4.4 and Proposition 4.5. □

Now we consider the particular case in which Ω is a ball or an annulus centered at the origin of $\mathbb{R}^N$, $N \geq 2$, and $f : \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function. Since f is independent of x, we may assume that a in hypothesis (f_2) is a constant. In particular, the estimate (6.1) takes the form

$$\underline{D}_s f(s) \geq -a - b|s|^{p-2} \quad \text{for every } s \in \mathbb{R}. \tag{6.2}$$

LEMMA 6.2. *Let $u \in H_0^1(\Omega)$ be a radial, sign-changing solution of* (2.1) *and let*

$$\Omega_1 = \{x \in \Omega : x_1 < 0\}.$$

Then there exists $v \in H_0^1(\Omega_1)$ such that

$$\int_{\Omega_1} |\nabla v|^2 \, dx - \int_{\Omega_1} \underline{D}_s f(u) v^2 \, dx < 0.$$

PROOF. From regularity theory we know that $u \in C^{1,\alpha}(\overline{\Omega}) \cap W^{2,q}(\Omega)$ for every $\alpha < 1$ and every $q < \infty$.

By contradiction, assume that

$$\int_{\Omega_1} |\nabla v|^2\,dx - \int_{\Omega_1} \underline{D}_s f(u)v^2\,dx \geq 0 \quad \text{for every } v \in H_0^1(\Omega_1). \tag{6.3}$$

Since

$$\underline{D}_s f(u)vw = \frac{1}{4}\,\underline{D}_s f(u)(v+w)^2 - \frac{1}{4}\,\underline{D}_s f(u)(v-w)^2,$$

by (6.2) and (6.3) it follows that

$$\begin{cases} \underline{D}_s f(u)vw \in L^1(\Omega_1) \quad \text{for every } v, w \in H_0^1(\Omega_1), \\ \text{and there exists a constant } C > 0 \text{ such that} \\ \left| \int_{\Omega_1} \underline{D}_s f(u)vw\,dx \right| \leq C\,\|v\|\,\|w\| \quad \text{for every } v, w \in H_0^1(\Omega_1). \end{cases} \tag{6.4}$$

Since u is radial and sign-changing, there exists an open set $\omega \subset\subset \Omega$ such that $D_{x_1}u = 0$ on $\partial\omega$, $D_1 := \Omega_1 \cap \omega \neq \emptyset$, and either $D_{x_1}u > 0$ or $D_{x_1}u < 0$ on ω. Consequently, we have $D_{x_1}u \in H_0^1(D_1)$.

From (6.4) it follows that $\underline{D}_s f(u)D_{x_1}u \in L^1_{\mathrm{loc}}(D_1)$. Combining (2.1) with [17, Corollary 8], we deduce that

$$\int_{D_1} \nabla(D_{x_1}u)\cdot\nabla v\,dx = -\int_{D_1} f(u)D_{x_1}v\,dx = \int_{D_1} \underline{D}_s f(u)D_{x_1}u\,v\,dx$$

for every $v \in C_c^\infty(D_1)$, hence, by (6.4), for every $v \in H_0^1(D_1)$. In particular, we have

$$\int_{D_1} |\nabla(D_{x_1}u)|^2\,dx = \int_{D_1} \underline{D}_s f(u)(D_{x_1}u)^2\,dx.$$

If we consider the function

$$z = \begin{cases} D_{x_1}u & \text{on } D_1, \\ 0 & \text{on } \Omega_1 \setminus \omega, \end{cases}$$

we see that $z \in H_0^1(\Omega_1)$ and

$$\int_{\Omega_1} |\nabla z|^2\,dx - \int_{\Omega_1} \underline{D}_s f(u)z^2\,dx = 0.$$

Combining this fact with (6.3) and (6.4), we deduce that

$$\int_{\Omega_1} \nabla z\cdot\nabla v\,dx - \int_{\Omega_1} \underline{D}_s f(u)zv\,dx = 0 \quad \text{for every } v \in H_0^1(\Omega_1).$$

If $z \geq 0$, (6.2) implies that

$$\int_{\Omega_1} \nabla z\cdot\nabla v\,dx + \int_{\Omega_1} (a + b|u|^{p-2})zv\,dx \geq 0 \quad \text{for every } v \in H_0^1(\Omega_1) \text{ with } v \geq 0,$$

hence z is a nonnegative supersolution of

$$-\Delta w + (a + b|u|^{p-2})w = 0. \tag{6.5}$$

Since $z = 0$ at any point of $\Omega_1 \cap \partial\omega$, but not in a neighborhood, from the weak Harnack inequality (see e.g. [13, Theorem 8.18]) a contradiction follows. If $z \leq 0$, we proceed analogously. □

PROOF OF THEOREM 2.2. If u does not change sign there is nothing to prove. Thus we may assume that $n := \mathrm{nod}(u) \geq 2$. Let $\Omega = A(r, R) := \mathrm{int}\{x \in \mathbb{R}^N : r \leq |x| \leq R\}$ with $0 \leq r < R$, and let $r = r_0 < r_1 < \cdots < r_{n-1} < r_n = R$ be such that $A_i = A(r_{i-1}, r_i)$, $i = 1, \ldots, n$, are the nodal domains of u. We consider the domains $B_i = A(r_{i-1}, r_{i+1})$ and $B_{ij} = \{x \in B_i : x_j < 0\}$, $i = 1, \ldots, n-1$, $j = 1, \ldots, N$. Then $u \in H_0^1(B_i)$ is a radial, sign-changing solution of $-\Delta u = f(u)$. By Lemma 6.2 there exists $v_{ij} \in H_0^1(B_{ij})$ with

$$\int_{B_{ij}} |\nabla v_{ij}|^2 \, dx - \int_{B_{ij}} \underline{D}_s f(u) v_{ij}^2 \, dx < 0.$$

If we set

$$\alpha_k = \min\{\underline{D}_s f(u), k\},$$

we have $\alpha_k \in L^\infty(\Omega)$ by (6.2). By the monotone convergence theorem, there exists $k \in \mathbb{N}$ such that

$$\int_{B_{ij}} |\nabla v_{ij}|^2 \, dx - \int_{B_{ij}} \alpha_k v_{ij}^2 \, dx < 0, \qquad i = 1, \ldots, n-1, \; j = 1, \ldots, N.$$

Therefore the first eigenvalue of $-\Delta - \alpha_k$ on B_{ij} with homogeneous Dirichlet condition is strictly negative. Let ψ_{ij} be an associated eigenfunction. If we extend ψ_{ij} to all of B_i as an odd function with respect to x_j, we get a sign-changing eigenfunction $\varphi_{ij} \in H_0^1(B_i)$ of $-\Delta - \alpha_k$ on B_i with homogeneous Dirichlet condition and strictly negative eigenvalue. Let also $\varphi_{i0} \in H_0^1(B_i)$ be a positive eigenfunction of $-\Delta - \alpha_k$ on B_i with homogeneous Dirichlet condition; of course the associated eigenvalue is also negative. If we extend each φ_{ij} to all Ω with value 0 outside B_i, we find that the quadratic form

$$w \mapsto \int_\Omega |\nabla w|^2 \, dx - \int_\Omega \alpha_k w^2 \, dx$$

is negative definite on

$$V := \mathrm{span}\{\varphi_{ij} : i = 1, \ldots, n-1, \; j = 0, \ldots, N\} \subset H_0^1(\Omega).$$

By construction, the φ_{ij}'s are linearly independent, hence $\dim(V) = (n-1) \cdot (N+1)$. Since

$$Q_u(w) \leq \int_\Omega |\nabla w|^2 \, dx - \int_\Omega \underline{D}_s f(u) w^2 \, dx \leq \int_\Omega |\nabla w|^2 \, dx - \int_\Omega \alpha_k w^2 \, dx,$$

the assertion follows. □

7. Proof of Theorem 3.1

We consider the functionals $A, B : E = H_0^1(\Omega) \to \mathbb{R}$ defined by

$$A(u) = \int_\Omega |\nabla u|^2\, dx \quad \text{and} \quad B(u) := \int_\Omega (a|u^+|^2 + b|u^-|^2)\, dx.$$

Then $A \in C^\infty(E)$ and $B \in C^1(E)$ with

$$B'(u)v = 2\int_\Omega (au^+ - bu^-)v\, dx.$$

In particular, B' is Lipschitz continuous. Clearly, every $c \in \mathbb{R} \setminus \{0\}$ is a regular value of B, so $\mathcal{M} := B^{-1}(1)$ is a C^1-submanifold of codimension 1. A solution $u \neq 0$ of (3.1) yields a critical point $(1/\sqrt{B(u)})u \in \mathcal{M}$ of the constrained functional $A|_\mathcal{M}$ with $\lambda = A(u)/B(u)$, the Lagrange multiplier. Conversely, any critical point $u \in \mathcal{M}$ of $A|_\mathcal{M}$ solves (3.1) with $\lambda = A(u)$. Also, nontrivial solutions of (3.1) correspond to critical points of $A/B : E \setminus \{0\} \to \mathbb{R}$. For any $u \in \mathcal{M}$, we consider the functional

$$Q_u : T_u\mathcal{M} = \{v \in E : B'(u)v = 0\} \to \mathbb{R},$$

$$Q_u(v) := \limsup_{\substack{z\to 0\\ t\to 0\\ w\to v}} \frac{1}{t^2}\left(\frac{A}{B}(u+z+tw) + \frac{A}{B}(u+z-tw) - 2\frac{A}{B}(u+z)\right),$$

where $z, w \in T_u\mathcal{M}$, $t \in \mathbb{R}$. Analogously to Section 2, we denote by $m(A/B; u)$ the supremum of the dimensions of the linear subspaces $V \subset T_u\mathcal{M}$ such that Q_u is strictly negative on $V \setminus \{0\}$. We shall prove:

PROPOSITION 7.1. *If* $\underline{C}_k(A|_\mathcal{M}; u) \neq 0$ *then* $m(A/B; u) \leq k$.

PROPOSITION 7.2. *If* $u \in \mathcal{M}$ *solves* (3.1) *with* $\lambda = \lambda_2$ *then* $\underline{C}_1(A|_\mathcal{M}; u) \neq 0$.

These two results do not require Ω to be radial.

PROPOSITION 7.3. *If* Ω *is a ball or an annulus, and if* u *is a radial solution of* (3.1), *then* $m(A/B; u) \geq (\mathrm{nod}(u) - 1)\cdot(N+1) - 1$.

Clearly, Theorem 3.1 follows from these propositions.

PROOF OF PROPOSITION 7.1. For $\varepsilon > 0$ small the map

$$h_\varepsilon : U_\varepsilon := \{v \in T_u\mathcal{M} : \|v\| < \varepsilon\} \to \mathcal{M}, \qquad h_\varepsilon(v) := \frac{u+v}{\sqrt{B(u+v)}},$$

defines a C^1-diffeomorphism of U_ε onto the open neighborhood $V_\varepsilon := h_\varepsilon(U_\varepsilon)$ of u in $\mathcal{M}$. It follows that

$$\underline{C}_k(A|_\mathcal{M}; u) \cong \underline{C}_k(A \circ h_\varepsilon; 0).$$

Since $A \circ h_\varepsilon(v) = \frac{A}{B}(u+v)$ we deduce that $\underline{C}_k(A\circ h_\varepsilon; 0) \cong \underline{C}_k(\frac{A}{B} \circ \tau_u|_{T_u\mathcal{M}}, 0)$ where $\tau_u(v) := u + v$. Proposition 7.1 follows now from Corollary 4.3. □

PROOF OF PROPOSITION 7.2. This follows from the fact that a solution $u \in \mathcal{M}$ of (3.1) with $\lambda = \lambda_2$ is a critical point of $A|_{\mathcal{M}}$ of mountain pass type. For the proof set

$$\Gamma := \{\gamma \in C([0,1], \mathcal{M}) : \gamma(0) \geq 0,\ \gamma(1) \leq 0\}.$$

It has been proved in [2] that

$$A(u) = \lambda_2 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} A(\gamma(t)).$$

Now, the path

$$\gamma(t) := \frac{\cos(\pi t/2)}{\sqrt{B(u^+)}} u^+ - \frac{\sin(\pi t/2)}{\sqrt{B(u^-)}} u^-, \qquad 0 \leq t \leq 1,$$

satisfies $\gamma(0) \geq 0$, $\gamma(1) \leq 0$, and

$$B(\gamma(t)) = \frac{\cos^2(\pi t/2)}{B(u^+)} B(u^+) + \frac{\sin^2(\pi t/2)}{B(u^-)} B(u^-) = 1,$$

hence $\gamma(t) \in \mathcal{M}$ and $\gamma \in \Gamma$. Moreover, since u solves (3.1), it follows that $A(u^\pm/\sqrt{B(u^\pm)}) = \lambda_2$ and therefore $A(\gamma(t)) = \lambda_2$ for all $t \in [0,1]$. It is also easily seen that there exists one $\bar{t} \in]0,1[$ with $\gamma(\bar{t}) = u$ and that $\gamma(t)$ is not a critical point of $A|_{\mathcal{M}}$ for any $t \neq \bar{t}$. Since B is C^1 with Lipschitz gradient and A is smooth, we may deform $\gamma =: \gamma_0$ on $\mathcal{M}$, using the negative gradient flow for $A|_{\mathcal{M}}$, to a path γ_1 such that $A(\gamma_1(t)) < \lambda_2$ for all $t \in [0,1]$ with $t \neq \bar{t}$. After the deformation one still has $\gamma_1(0) \geq 0$, $\gamma_1(1) \leq 0$. This follows from the fact that $\nabla(A|_{\mathcal{M}})(u) = u - K(u)$ and K is order preserving as a consequence of the maximum principle. Thus we have proved that $\gamma_1 \in \Gamma$.

Now, $\gamma_1(0)$ and $\gamma_1(1)$ cannot be connected by a path in $A^{<\lambda_2} := \{v \in \mathcal{M} : A(v) < \lambda_2\}$. Since this is an open subset of a manifold, we infer that $\gamma_1(0)$ and $\gamma_1(1)$ lie in different components of $A^{<\lambda_2}$. On the other hand, $\gamma_1(0)$ and $\gamma_1(1)$ are connected in $A^{<\lambda_2} \cup \{u\}$ by γ_1. According to [18, Theorem 6.4.5], $H^0(A^{<\lambda_2})$ contains a nontrivial element that does not come from $H^0(A^{<\lambda_2} \cup \{u\})$. By the exact sequence

$$H^0(A^{<\lambda_2} \cup \{u\}) \to H^0(A^{<\lambda_2}) \to H^1(A^{<\lambda_2} \cup \{u\}, A^{<\lambda_2})$$

we conclude that $\underline{C}_1(A|_{\mathcal{M}}; u) = H^1(A^{<\lambda_2} \cup \{u\}, A^{<\lambda_2})$ is not trivial. □

In order to prove Proposition 7.3, we compare Q_u with the quadratic form

$$P_u : E \to \mathbb{R}, \qquad P_u(v) := A(v) - A(u) \cdot \int_\Omega (a\chi_+ + b\chi_- + \min\{a,b\}\chi_0) v^2 \, dx;$$

here $\chi_\pm$ is the characteristic function of the set $\{x \in \Omega : \pm u(x) > 0\}$ and χ_0 is the characteristic function of the set $\{x \in \Omega : u(x) = 0\}$.

LEMMA 7.4. *For every $u \in \mathcal{M}$ and $v \in T_u\mathcal{M}$, we have $Q_u(v) \leq 2P_u(v)$.*

PROOF. It is clear that, for every $u, v \in H_0^1(\Omega)$ with $u \neq 0$, we have

$$\left(\frac{A}{B}\right)'(u)v = \frac{1}{B(u)}\,A'(u)v - \frac{A(u)}{B^2(u)}\,B'(u)v.$$

It follows from Remark 4.4 that, for every $u \in \mathcal{M}$ and $v \in T_u\mathcal{M}$, we have

$$Q_u(v) \leq 2A(v) - A(u) \liminf_{\substack{z\to u\\ (\tau,\vartheta)\to(0,0)\\ w\to v}} \frac{B'(z+\tau w)w - B'(z+\vartheta w)w}{\tau - \vartheta}. \tag{7.1}$$

We also used the fact that $B'(u)v = 0$ for $u \in \mathcal{M}$ and $v \in T_u\mathcal{M}$. On the other hand, if we set $g(s) = as^+ - bs^-$, it is easily seen that (4.3) and (4.4) are satisfied and

$$\underline{D}_s g(s) = \begin{cases} b & \text{if } s < 0,\\ \min\{a,b\} & \text{if } s = 0,\\ a & \text{if } s > 0. \end{cases}$$

From Proposition 4.5 we deduce that

$$\liminf_{\substack{z\to u\\ (\tau,\vartheta)\to(0,0)\\ w\to v}} \frac{B'(z+\tau w)w - B'(z+\vartheta w)w}{\tau - \vartheta} \geq 2\int_\Omega (a\chi_+ + b\chi_- + \min\{a,b\}\chi_0)v^2\,dx. \tag{7.2}$$

Combining (7.1) and (7.2) the assertion follows. □

PROOF OF PROPOSITION 7.3. We argue as in the proof of Theorem 2.2. Let $\Omega = A(r,R) := \operatorname{int}\{x \in \mathbb{R}^N : r \leq |x| \leq R\}$, and let $r = r_0 < r_1 < \cdots < r_{n-1} < r_n = R$ be such that $A_i = A(r_{i-1}, r_i)$, $i = 1,\dots,n = \operatorname{nod}(u)$, are the nodal domains of u. As before we set $B_i = A(r_{i-1}, r_{i+1})$ and $B_{ij} = \{x \in B_i : x_j < 0\}$, $i = 1,\dots,n-1$, $j = 1,\dots,N$. And we define $v_j := D_{x_j}u \in C^{1,\alpha}(\overline{\Omega})$. Since u is radial, it has constant sign near the outer boundary $\partial_o(B_i) = \{x : |x| = r_{i+1}\}$. If $u < 0$ near $\partial_o(B_i)$ we set $\Omega_{ij} := \{x \in B_{ij} : v_j(x) > 0\}$, so that $v_j \in H_0^1(\Omega_{ij})$. Differentiating (3.1) we see that v_j is a weak solution of

$$\begin{cases} -\Delta v = \mu(a\chi_+ + b\chi_-)v & \text{in } D,\\ \quad v = 0 & \text{on } \partial D, \end{cases} \tag{7.3}$$

on $D = \Omega_{ij}$ with $\mu = \lambda = A(u)$. Let $\mu_k(D)$, $k \in \mathbb{N}$, denote the eigenvalues of (7.3) counted with multiplicities.

We have just proved that

$$\mu_1(B_{ij}) < \mu_1(\Omega_{ij}) = \lambda, \qquad i = 1,\dots,n-1,\ j = 1,\dots,N.$$

Let $\psi_{ij} > 0$ be a positive eigenfunction of (7.3) on $D = B_{ij}$. Extend ψ_{ij} to $\varphi_{ij} : B_i \to \mathbb{R}$ so that φ_{ij} is odd in x_j. Then φ_{ij} is a sign-changing eigenfunction of (7.3) on $D = B_i$ with corresponding eigenvalue $\mu = \mu_1(B_{ij}) < \lambda$. Let φ_{i0} be a positive eigenfunction of (7.3) on $D = B_i$ and set

$$V = \text{span}\{\varphi_{ij} : i = 1, \dots, n-1, \ j = 0, \dots, N\} \subset H_0^1(\Omega).$$

Then $\dim(V) = (n-1)\cdot(N+1)$ because the functions φ_{ij} are linearly independent by construction. It follows that the quadratic form

$$E \to \mathbb{R}, \qquad v \mapsto \int_\Omega |\nabla v|^2 dx - \lambda \int_\Omega (a\chi_+ + b\chi_-) v^2\, dx,$$

is negative definite on V. This implies that P_u is negative definite on $V \cap T_u\mathcal{M}$, hence Q_u is negative definite on $V \cap T_u\mathcal{M}$ by Lemma 7.4, and we obtain

$$\begin{aligned} m(A/B; u) &\geq \dim(V \cap T_u\mathcal{M}) \geq (n-1)\cdot(N+1) - 1 \\ &= (\text{nod}(u) - 1)\cdot(N+1) - 1. \qquad \square \end{aligned}$$

ACKNOWLEDGEMENTS. The research of the authors was partially supported by the MIUR project “Variational and topological methods in the study of nonlinear phenomena” (PRIN 2003) and by Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (INdAM).

T. B. thanks the members of the Dipartimento di Matematica e Fisica of the Università Cattolica del Sacro Cuore in Brescia for the invitation and hospitality.

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Received 9 June 2005,
and in revised form 31 August 2005.

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