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Complex variables functions. — *Schröder equation in several variables and composition operators*, by CINZIA BISI and GRAZIANO GENTILI, communicated on 10 March 2006.

ABSTRACT. — Let φ be a holomorphic self-map of the open unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ such that $\varphi(0) = 0$ and the differential $d\varphi_0$ of φ at 0 is non-singular. The study of the Schröder equation in several complex variables

$$\sigma \circ \varphi = d\varphi_0 \circ \sigma$$

is naturally related to the theory of composition operators on Hardy spaces of holomorphic maps on $\mathbb{B}^n$ and to the theory of discrete, complex dynamical systems. An extensive use of the infinite matrix which represents the composition operator associated to the map φ leads to a simpler approach, and provides new proofs, to results on existence of solutions for the Schröder equation.

KEY WORDS: Schröder equation; composition operators.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 32A10, 47B33; Secondary 32A30, 32H15, 30C80.

1. INTRODUCTION

Let φ be a holomorphic self-map of the open unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ such that $\varphi(0) = 0$ and the differential $d\varphi_0$ of φ at 0 is non-singular. A $\mathbb{C}^n$-valued holomorphic map σ defined on $\mathbb{B}^n$ is a solution of the Schröder equation associated to φ in several variables (and will be called a *Schröder map* for φ, [5]) if

$$\sigma \circ \varphi = d\varphi_0 \circ \sigma. \tag{1.1}$$

We will call the intersection of $\mathbb{B}^n$ with any one-dimensional complex subspace of $\mathbb{C}^n$ a *slice* of $\mathbb{B}^n$. The map φ is often assumed to be *non-unitary on any slice*, i.e. such that there are no ζ and η in $\partial\mathbb{B}^n$ with $\varphi(\lambda\zeta) = \lambda\eta$ for all λ in the unit disk Δ. Since φ maps $\mathbb{B}^n$ into itself and $\varphi(0) = 0$, the Schwarz Lemma implies $\|d\varphi_0\| \le 1$, and strict inequality occurs precisely when φ is non-unitary on any slice (see, e.g., [1], [8]. In this case the differential $d\varphi_0$ has no eigenvalue of modulus 1. Since we are interested only in locally univalent solutions of the Schröder equation (1.1), we assume that $d\varphi_0$ is diagonalizable to guarantee that the Schröder map σ of φ is invertible at 0 (see, e.g., [3]).

Let C_φ denote the composition operator $C_\varphi(f) = f \circ \varphi$ on the Hardy space $H^2(\mathbb{B}^n)$ (see, e.g., [4]). Cowen and MacCluer [3] give necessary and sufficient conditions on φ to guarantee the existence of solutions of the Schröder equation associated to φ. Namely, they prove the following:

THEOREM 1.1. *Suppose φ is a holomorphic map of $\mathbb{B}^n$ into $\mathbb{B}^n$ with $\varphi(0) = 0$ and suppose that $d\varphi_0$ is upper triangular in the standard basis and diagonalizable, with* *diagonal entries* $\lambda_1, \dots, \lambda_n$ *such that* $0 < |\lambda_j| < 1$ *for* $j = 1, \dots, n$. *Let* X *be any size, square upper left corner of the matrix* A_φ *representing* C_φ *with respect to the standard (non-normalized) basis of the Hardy space* $H^2(\mathbb{B}^n)$. *If the Schröder equation has a solution* σ *on* $\mathbb{B}^n$, *i.e.* $\sigma \circ \varphi = d\varphi_0 \circ \sigma$ *and* $d\sigma_0 = I$, *then* X *is diagonalizable.*

THEOREM 1.2. *Suppose* φ *is a holomorphic map of* $\mathbb{B}^n$ *into* $\mathbb{B}^n$ *with* $\varphi(0) = 0$ *and suppose that* $d\varphi_0$ *is upper triangular in the standard basis and diagonalizable, with diagonal entries* $\lambda_1, \dots, \lambda_n$ *such that* $0 < |\lambda_j| < 1$ *for* $j = 1, \dots, n$. *Let* $\lambda_j = \lambda_1^{k_1} \cdots \lambda_n^{k_n}$ *be the longest expression (with maximal* $\sum k_i$*) for an eigenvalue of* $d\varphi_0$ *as a product of any number of powers of eigenvalues of* $d\varphi_0$. *Set* $m = k_1 + \cdots + k_n$ *and let* M *be the number of multi-indices* $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ *of total order* $|\alpha| = \alpha_1 + \cdots + \alpha_n$ *less than or equal to* m; *equivalently* M *is the dimension of the vector space* $\mathcal{H}_m$ *spanned by the set* $\{z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} : |\alpha| \le m\}$. *Let* $\mathcal{M}$ *be the upper left* $M \times M$ *corner of the infinite matrix* A_φ *associated to* C_φ *with respect to the standard (non-normalized) basis of the Hardy space* $H^2(\mathbb{B}^n)$. *If* $\mathcal{M}$ *is diagonalizable, then the Schröder equation has a solution* σ *with* $d\sigma_0$ *invertible.*

For what concerns the study of the classical Schröder equation we refer the reader to [5] and [6].

In Sections 3 and 4 of this paper we present new and simpler proofs of the above results. The new proofs are based on techniques borrowed from the theory of complex, discrete dynamical systems and on classical results due to Sternberg [10], and rely upon the fact that the study of composition operators on spaces of holomorphic maps on the open unit ball of $\mathbb{C}^n$ plays a fundamental role in determining the solutions of the complex Schröder equation (see, e.g., [4]) .

We proceed as follows. We choose the standard orthogonal (non-normalized) basis $\{1, z_1, \dots, z_n, z_1^2, z_1 z_2, \dots\}$ for the Hardy space $H^2(\mathbb{B}^n)$, ordered by degree and lexicographically for any degree. We associate to the composition operator C_φ acting on $H^2(\mathbb{B}^n)$ an infinite matrix A_φ which "represents" the action of C_φ on $H^2(\mathbb{B}^n)$ with respect to the above basis. If $\varphi = (\varphi_1, \dots, \varphi_n)$ and if $z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ is the j-th monomial of the basis, then the entries of the j-th column of the matrix A_φ are the coefficients of $\varphi_1^{\alpha_1} \varphi_2^{\alpha_2} \cdots \varphi_n^{\alpha_n}$ with respect to the basis. Notice that, for $j = 2, \dots, n+1$, the entries of the j-th column of A_φ are the coefficients of the series expansion of φ_{j-1} at 0. An extensive use of this infinite matrix representation, together with techniques of discrete dynamical systems, leads to a different, simpler approach to some instrumental results of Sternberg (see Section 2) and to Theorems 1.1 and 1.2 by Cowen and MacCluer.

We believe that the direct techniques presented in this paper will be useful in finding a different approach to the study of the boundary Schröder equation in several complex variables (see [2]).

2. PRELIMINARY RESULTS

If $H^2(\mathbb{B}^n)$ is the Hardy space of L^2 holomorphic maps on $\mathbb{B}^n$, we will denote by $\mathcal{B}$ the standard orthogonal (non-normalized) basis $\{1, z_1, \dots, z_n, z_1^2, z_1 z_2, \dots\}$, where the monomials are ordered by degree and lexicographically for any degree. Let $\varphi = (\varphi_1, \dots, \varphi_n)$ be a holomorphic self-map of the open unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$, such that $\varphi(0) = 0$

and the differential $d\varphi_0$ is non-singular. Let C_φ be the composition operator associated to φ.

DEFINITION 2.1. The infinite matrix A_φ *representing* C_φ with respect to the basis $\mathcal{B} = \{1, z_1, \ldots, z_n, z_1^2, z_1z_2, \ldots\}$ of the Hardy space $H^2(\mathbb{B}^n)$ is the matrix whose coefficients $A^{\beta_1\ldots\beta_n}_{\alpha_1\ldots\alpha_n}$ are defined by

$$(2.1)\qquad C_\varphi(z_1^{\alpha_1}z_2^{\alpha_2}\cdots z_n^{\alpha_n}) = \varphi_1^{\alpha_1}(z_1,\ldots,z_n)\varphi_2^{\alpha_2}(z_1,\ldots,z_n)\cdots\varphi_n^{\alpha_n}(z_1,\ldots,z_n) = \sum_{\beta_1,\ldots,\beta_n\in\mathbb{N}} A^{\beta_1\ldots\beta_n}_{\alpha_1\ldots\alpha_n} z_1^{\beta_1}z_2^{\beta_2}\cdots z_n^{\beta_n}$$

where both the row-indices $(\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ and the column-indices $(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ inherit their well-ordering from the basis $\mathcal{B}$.

REMARK 2.2. For all the $(n+1)$ row-indices $(\beta_1, \ldots, \beta_n)$ with order $\beta_1 + \cdots + \beta_n \leq 1$, the entry $A^{\beta_1\ldots\beta_n}_{\alpha_1\ldots\alpha_n}$ of the infinite matrix A_φ vanishes if the column-index $(\alpha_1, \ldots, \alpha_n)$ has order $\alpha_1 + \cdots + \alpha_n \geq 2$. In fact the series expansion of the map $\varphi_1^{\alpha_1}(z_1,\ldots,z_n)\varphi_2^{\alpha_2}(z_1,\ldots,z_n)\cdots\varphi_n^{\alpha_n}(z_1,\ldots,z_n)$ has neither linear nor constant terms if $\alpha_1 + \cdots + \alpha_n \geq 2$. This property can be rephrased by saying that all the entries in the first $n+1$ rows of the infinite matrix A_φ vanish if they stay "on the right side" of the $(n+1)$-th column.

LEMMA 2.3. *If $d\varphi_0$ is non-singular and upper triangular, then the infinite matrix A_φ associated to C_φ is lower triangular.*

PROOF. Since $d\varphi_0$ is non-singular, the monomial with the "smallest" index appearing in (2.1) is exactly $z_1^{\alpha_1}z_2^{\alpha_2}\cdots z_n^{\alpha_n}$. Therefore, $(\beta_1, \ldots, \beta_n) < (\alpha_1, \ldots, \alpha_n)$ implies that $A^{\beta_1\ldots\beta_n}_{\alpha_1\ldots\alpha_n} = 0$. □

Let us recall that, given $j \in \{1, \ldots, n\}$, a *resonance* for the eigenvalue λ_j of $d\varphi_0$ is a relation of the form

$$\lambda_j = \lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_n^{k_n}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $d\varphi_0$, $k_i \geq 0$ for all $i = 1, \ldots, n$ and $\sum_{i=1}^n k_i \geq 2$. We set

$$R(\lambda_j) = \{(k_1, \ldots, k_n) \in \mathbb{N}^n : \lambda_j = \lambda_1^{k_1}\cdots\lambda_n^{k_n} \text{ is a resonance for } \lambda_j\}.$$

Let P^∞ denote the set of n-tuples of formal power series without constant terms, in n variables, and let F^∞ denote the group of those elements of P^∞ whose matrix of linear terms is non-singular. The following lemma is due to Sternberg [10].

LEMMA 2.4. *Let T be an element of F^∞ whose matrix S of linear terms is diagonalizable. Assume there are no resonances among the eigenvalues of S. Then T is equivalent to S by an inner automorphism of F^∞.*

PROOF. A proof of Lemma 2.4 can be found in [10]. We present here a different proof, based on the use of composition operators. As pointed out in the introduction, for the purposes of this paper we can assume that the matrix S is diagonal. We wish to find an $R = (r_1, \dots, r_n) \in F^\infty$ such that $RTR^{-1} = S = (\lambda_1 z_1, \dots, \lambda_n z_n)$. We can assume that the matrix of linear terms of R is the identity matrix and rewrite the desired equation as $RT = SR$.

In the language of composition operators the equation $RT = SR$ becomes

$$C_T \circ C_R = C_R \circ C_S,$$

and in terms of the associated infinite matrices A_T, A_R and A_S,

$$\begin{pmatrix} 1 & 0 & \cdot & 0 & 0 & 0 & \cdot \\ 0 & \lambda_1 & \cdot & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \lambda_n & 0 & 0 & \cdot \\ 0 & t_2^{n+2} & \cdot & t_{n+1}^{n+2} & \lambda_1^2 & 0 & \cdot \\ 0 & t_2^{n+3} & \cdot & t_{n+1}^{n+3} & t_{n+2}^{n+3} & \lambda_1\lambda_2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdot & 0 & 0 & 0 & \cdot \\ 0 & 1 & \cdot & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 1 & 0 & 0 & \cdot \\ 0 & r_2^{n+2} & \cdot & r_{n+1}^{n+2} & 1 & 0 & \cdot \\ 0 & r_2^{n+3} & \cdot & r_{n+1}^{n+3} & r_{n+2}^{n+3} & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdot & 0 & 0 & 0 & \cdot \\ 0 & 1 & \cdot & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 1 & 0 & 0 & \cdot \\ 0 & r_2^{n+2} & \cdot & r_{n+1}^{n+2} & 1 & 0 & \cdot \\ 0 & r_2^{n+3} & \cdot & r_{n+1}^{n+3} & r_{n+2}^{n+3} & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdot & 0 & 0 & 0 & \cdot \\ 0 & \lambda_1 & \cdot & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \lambda_n & 0 & 0 & \cdot \\ 0 & 0 & \cdot & 0 & \lambda_1^2 & 0 & \cdot \\ 0 & 0 & \cdot & 0 & 0 & \lambda_1\lambda_2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Given the above equality among infinite matrices, to determine R it is enough to find, for all $j = 2, \dots, n+1$, the j-th column of the infinite matrix A_R. By equating the j-th column of the matrix on the left hand side with the j-th column of the matrix on the right hand side ($j = 2, \dots, n+1$) we obtain an infinite, linear system of equations

$$\text{(2.2)} \qquad \begin{cases} t_j^{n+2} + \lambda_1^2 r_j^{n+2} & = \lambda_{j-1} r_j^{n+2}, \\ t_j^{n+3} + t_{n+2}^{n+3} r_j^{n+2} + \lambda_1\lambda_2 r_j^{n+3} & = \lambda_{j-1} r_j^{n+3}, \\ t_j^{n+4} + t_{n+2}^{n+4} r_j^{n+2} + t_{n+3}^{n+4} r_j^{n+3} + \lambda_1\lambda_3 r_j^{n+4} & = \lambda_{j-1} r_j^{n+4}, \\ \dots \end{cases}$$

i.e. an infinite, lower triangular, linear system in the unknown variables r_j^k ($2 \leq j \leq n+1$; $k \geq n+2$). Since no one of the $\lambda_1, \dots, \lambda_n$ vanishes, and since they have no resonance relations, all the linear systems of type (2.2) can be solved inductively. □

In the presence of resonances for the eigenvalues of the matrix S of linear terms of $T \in F^\infty$, the "linearization procedure" does not work any more. Nevertheless Sternberg [10] proved the following

LEMMA 2.5. *Let T be an element of F^∞ whose matrix S of linear terms is diagonalizable and has eigenvalues $\lambda_1, \dots, \lambda_n$. Suppose the set $R(\lambda_j)$ is not empty for some $j \in \{1, \dots, n\}$. Then there exists a transformation R in F^∞ such that RTR^{-1} has the form $N = (N_1, \dots, N_n)$ where*

$$N_j = \lambda_j z_j + \sum_{(k_1,\dots,k_n)\in R(\lambda_j)} A^j_{k_1\dots k_n} z_1^{k_1} \cdots z_n^{k_n} \tag{2.3}$$

for $j = 1, \dots, n$.

PROOF. Again, for the original proof we refer the reader to [10]. A different, direct proof will be presented here. In the language of composition operators, the equation $RT = NR$ becomes

$$C_T \circ C_R = C_R \circ C_N,$$

and in terms of the infinite matrices associated to composition operators,

$$\begin{pmatrix} 1 & 0 & \cdot & 0 & 0 & 0 & \cdot \\ 0 & \lambda_1 & \cdot & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \lambda_n & 0 & 0 & \cdot \\ 0 & t_2^{n+2} & \cdot & t_{n+1}^{n+2} & \lambda_1^2 & 0 & \cdot \\ 0 & t_2^{n+3} & \cdot & t_{n+1}^{n+3} & t_{n+2}^{n+3} & \lambda_1\lambda_2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdot & 0 & 0 & 0 & \cdot \\ 0 & 1 & \cdot & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 1 & 0 & 0 & \cdot \\ 0 & r_2^{n+2} & \cdot & r_{n+1}^{n+2} & 1 & 0 & \cdot \\ 0 & r_2^{n+3} & \cdot & r_{n+1}^{n+3} & r_{n+2}^{n+3} & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdot & 0 & 0 & 0 & \cdot \\ 0 & 1 & \cdot & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 1 & 0 & 0 & \cdot \\ 0 & r_2^{n+2} & \cdot & r_{n+1}^{n+2} & 1 & 0 & \cdot \\ 0 & r_2^{n+3} & \cdot & r_{n+1}^{n+3} & r_{n+2}^{n+3} & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdot & 0 & 0 & 0 & \cdot \\ 0 & \lambda_1 & \cdot & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \lambda_n & 0 & 0 & \cdot \\ 0 & A^1_{20\cdot 0} & \cdot & A^n_{20\cdot 0} & \lambda_1^2 & 0 & \cdot \\ 0 & A^1_{11\cdot 0} & \cdot & A^n_{11\cdot 0} & A^{n+1}_{11\cdot 0} & \lambda_1\lambda_2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

where $A^j_{k_1\dots k_n} = 0$ if $(k_1, \dots, k_n) \notin R(\lambda_j)$, for all $j \in \{1, \dots, n\}$. As in Lemma 2.4, the above equality among infinite matrices produces n triangular, infinite, linear systems of equations that can be solved inductively to construct R. It turns out that the non-vanishing coefficients $A^j_{k_1\dots k_n}$, corresponding to the indices of resonances, are not uniquely determined. □

3. SOLVING THE SCHRÖDER EQUATION

Lemmas 2.4 and 2.5 were proved for formal power series. The results that they state actually hold in the environment of convergent complex power series (see, e.g., [7] and [9]) and therefore they can be applied to the case of germs of holomorphic maps. In fact the following local result holds.

THEOREM 3.1. *Let $\varphi : U \to \mathbb{C}^n$ be a holomorphic map, where U is a neighborhood of $0 \in \mathbb{C}^n$ and $\varphi(0) = 0$. Suppose that $d\varphi_0$ is diagonalizable and that its eigenvalues satisfy $0 < |\lambda_n| \leq \cdots \leq |\lambda_1| < 1$. Then there exist a germ of biholomorphism σ and a holomorphic map g associated to $\lambda_1, \ldots, \lambda_n$ such that, in a neighborhood of 0,*

$$\sigma \circ \varphi = g \circ \sigma \tag{3.1}$$

with $\sigma(0) = 0$ and $d\sigma_0 = I$. In particular, if $R(\lambda_j)$ is empty for all $j \in \{1, \ldots, n\}$, then $g = d\varphi_0$, i.e. φ can be linearized.

In the absence of resonances, the germ of biholomorphism provided by Theorem 3.1 can be extended to a solution of the Schröder equation on the open unit ball of $\mathbb{C}^n$, to obtain the following

THEOREM 3.2. *Let $\varphi : \mathbb{B}^n \to \mathbb{B}^n$ be a holomorphic map such that $\varphi(0) = 0$, φ is non-unitary on any slice of $\mathbb{B}^n$ and $d\varphi_0$ is non-singular and diagonalizable. Then, in the absence of resonances, there exists a holomorphic map $\tilde{\sigma}$ defined on $\mathbb{B}^n$ which solves the Schröder equation*

$$\tilde{\sigma} \circ \varphi = d\varphi_0 \circ \tilde{\sigma}$$

and is such that $\tilde{\sigma}(0) = 0$ and $d\tilde{\sigma}_0 = I$.

PROOF. As pointed out in the introduction, we assume that $d\varphi_0$ is diagonalizable to guarantee that the Schröder map $\tilde{\sigma}$ is invertible at 0; examples of non-locally invertible Schröder maps with $d\varphi_0$ not diagonalizable are given in [3]. We are then able to use Theorem 3.1 to find a local solution σ defined in a neighborhood U of 0. Since in our hypotheses the basin of attraction of 0 is equal to $\mathbb{B}^n$ (see, e.g., [1, Proposition 2.2.33]), for every $z \in \mathbb{B}^n$ there is $m(z) \in \mathbb{N}$ such that, for all $m \in \mathbb{N}$ with $m \geq m(z)$, we have $\varphi^m(z) \in U$. We can therefore define

$$\tilde{\sigma}(z) = d\varphi_0^{-m(z)} \circ \sigma \circ \varphi^{m(z)}(z). \tag{3.2}$$

Let us check that $\tilde{\sigma}$ is well defined and holomorphic on the whole of $\mathbb{B}^n$. In fact, if $p, q \in \mathbb{N}$ are such that $p > q \geq m(z)$, then

$$\begin{aligned} d\varphi_0^{-p} \circ \sigma \circ \varphi^p(z) &= d\varphi_0^{-p} \circ \sigma \circ \varphi^{p-q}(\varphi^q(z)) = d\varphi_0^{-p} \circ d\varphi_0^{p-q} \circ \sigma(\varphi^q(z)) \\ &= d\varphi_0^{-q} \circ \sigma \circ \varphi^q(z). \end{aligned}$$

Since U is open and since $\varphi^{m(z)}(z) \in U$, the expression (3.2) defines $\tilde{\sigma}$ locally, in a neighborhood of $z \in \mathbb{B}^n$; hence $\tilde{\sigma}$ is holomorphic on $\mathbb{B}^n$.

We are left to prove that $\tilde{\sigma}$ satisfies the Schröder equation for φ on $\mathbb{B}^n$. We have

$$\begin{aligned} \tilde{\sigma} \circ \varphi(z) &= d\varphi_0^{-m(z)} \circ \sigma \circ \varphi^{m(z)}(\varphi(z)) = d\varphi_0^{-m(z)} \circ \sigma \circ \varphi(\varphi^{m(z)}(z)) \\ &= d\varphi_0^{-m(z)} \circ d\varphi_0 \circ \sigma(\varphi^{m(z)}(z)) = d\varphi_0 \circ d\varphi_0^{-m(z)} \circ \sigma \circ \varphi^{m(z)}(z) \\ &= d\varphi_0 \circ \tilde{\sigma}(z). \qquad \square \end{aligned}$$

4. THE SCHRÖDER MAP IN THE PRESENCE OF RESONANCES

We will now consider the case in which there are resonances for the eigenvalues of the differential $d\varphi_0$ of the holomorphic map φ and prove Theorem 1.2. We will expand in details a nice example to explore, with our new approach, the connection between Theorem 1.2 and Lemma 2.5.

EXAMPLE 4.1. Suppose $n = 2$ and let

$$\varphi(z_1, z_2) = (c_1 z_1, c_1^3 z_2 + c_2 z_1^2)$$

with $c_1 \neq 0, 1$ and $c_2 \neq 0$. Hence, if c_1, c_2 are sufficiently small then $\varphi(\mathbb{B}^2) \subseteq \mathbb{B}^2$, the map φ is injective on $\mathbb{B}^2$ and

$$d\varphi_0 = \begin{pmatrix} c_1 & 0 \\ 0 & c_1^3 \end{pmatrix}.$$

Therefore $d\varphi_0$ is diagonal and its eigenvalues, $\lambda_1 = c_1$ and $\lambda_2 = c_1^3$, have a unique resonance of order 3. Then, with reference to the notations of Theorem 1.2, we have $\dim \mathcal{H}_3 = 10 = \dim \mathcal{M}$, and hence $\mathcal{M}$ is the upper left 10×10 corner of the matrix A_φ representing C_φ with respect to the basis $\mathcal{B}$ of the Hardy space $H^2(\mathbb{B}^2)$. A simple computation shows that

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_2 & c_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_1^6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1 c_2 & 0 & c_1^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c_1^3 c_2 & 0 & c_1^5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1^7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1^9 \end{pmatrix}. \tag{4.1}$$

The matrix $\mathcal{M}$ has distinct eigenvalues (indeed $1 \neq c_1 \neq c_1^n$, $\forall n \in \mathbb{N}$). Therefore $\mathcal{M}$ is diagonalizable and, by Theorem 1.2, there exists a solution σ of the Schröder equation. We want to directly construct a Schröder map, i.e. a map σ such that

$$\sigma \circ \varphi = d\varphi_0 \circ \sigma. \tag{4.2}$$

In terms of the associated composition operators equation (4.2) becomes

$$C_\varphi \circ C_\sigma = C_\sigma \circ C_{d\varphi_0}. \tag{4.3}$$

The infinite matrix A_φ associated to C_φ with respect to the basis $\mathcal{B}$ contains $\mathcal{M}$ as upper left 10×10 corner and is such that all its entries a_{ij} with $1 \leq i \leq 10$ and $j > 10$, or $1 \leq j \leq 10$ and $i > 10$, vanish. A direct computation shows that the infinite matrix $A_{d\varphi_0}$

representing $C_{d\varphi_0}$ in $H^2(\mathbb{B}^2)$ is diagonal (since $d\varphi_0$ is diagonal) and its upper left 10×10 corner is

$$\mathcal{Z} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_1^6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_1^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1^5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1^7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1^9 \end{pmatrix}.$$

In view of the structure of the infinite matrices A_φ and $A_{d\varphi_0}$, to find a solution of equation (4.2) it is enough to determine a 10×10 block, $\mathcal{Y}$, which will play the role of the upper left corner of the matrix representing C_σ, that is (see (4.3)), such that

$$\mathcal{Y}^{-1}\mathcal{M}\mathcal{Y} = \mathcal{Z}. \tag{4.4}$$

The matrix $\mathcal{M}$ decomposes into three blocks along the diagonal:

$$\mathcal{M}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 \\ 0 & 0 & c_1^3 & 0 \\ 0 & 0 & c_2 & c_1^2 \end{pmatrix}, \quad \mathcal{M}_2 = \begin{pmatrix} c_1^4 & 0 & 0 & 0 \\ 0 & c_1^6 & 0 & 0 \\ c_1c_2 & 0 & c_1^3 & 0 \\ 0 & 2c_1^3c_2 & 0 & c_1^5 \end{pmatrix}, \quad \mathcal{M}_3 = \begin{pmatrix} c_1^7 & 0 \\ 0 & c_1^9 \end{pmatrix}.$$

As a consequence, equation (4.4) decomposes in turn into three simpler matrix equations of the same type. After assuming $d\sigma_0 = I$, a direct computation leads to

$$\mathcal{Y} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{c_2}{c_1^3 - c_1^2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{c_2}{c_1^3 - c_1^2} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2c_2}{c_1^3 - c_1^2} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.5}$$

Since, by definition, $C_\sigma(z_1) = (\sigma)_1$ and $C_\sigma(z_2) = (\sigma)_2$, we obtain

$$\sigma(z_1, z_2) = \left(z_1, z_2 + \frac{c_2}{c_1^3 - c_1^2} z_1^2\right).$$

Example 4.1 points out clearly the main feature of the resonant Schröder equation: even in the presence of resonances, precisely when the matrix $\mathcal{M}$ associated to φ is diagonalizable, the normal form of φ can be linear. This last fact happens when, for all $j = 1, \dots, n$, the coefficients $A^j_{k_1 \dots k_n}$ can be chosen to vanish for all $(k_1, \dots, k_n) \in R(\lambda_j)$ (see (2.3), Lemma 2.5).

We are now ready to state and prove the following

LEMMA 4.2. *Suppose $\varphi : \mathbb{B}^n \to \mathbb{B}^n$ and M are as described in Theorem* 1.2. *Let $\mathcal{M}$ be the upper left $M \times M$ corner of the infinite matrix A_φ representing the composition operator C_φ with respect to the basis $\mathcal{B}$ of the Hardy space $H^2(\mathbb{B}^n)$. If $\mathcal{M}$ is diagonalizable, then the coefficients $A^j_{k_1 \dots k_n}$ in the normal form* (2.3) *of the representation of φ can be chosen to be zero, for all $j = 1, \dots, n$ and all $(k_1, \dots, k_n) \in \mathbb{N}^n$. Therefore there exists a germ of biholomorphism σ such that, in a neighborhood of* 0,

$$\sigma \circ \varphi = d\varphi_0 \circ \sigma$$

with $\sigma(0) = 0$ and $d\sigma_0 = I$.

PROOF. If $\mathcal{M}$ is diagonalizable then there exists an infinite matrix A_σ whose upper left $M \times M$ corner, $\mathcal{Y}$, is such that $\mathcal{Y}^{-1}\mathcal{M}\mathcal{Y} = \mathcal{Z}$ is diagonal. Now, since

$$\sum k_i \leq m$$

by Lemma 2.5, all possibly non-zero coefficients $A^j_{k_1 \dots k_n}$ in the infinite matrix $A_\sigma^{-1} A_\varphi A_\sigma$ fall in the upper left $M \times M$ corner $\mathcal{Z}$. The fact that the eigenvalues of $d\varphi_0$ appear along the diagonal of $\mathcal{Z}$ forces all the coefficients $A^j_{k_1 \dots k_n}$ to be zero. As a consequence, by means of the infinite matrix A_σ, we can construct the power series of the desired germ σ according to Lemma 2.5 which, as already pointed out, holds in the environment of convergent, complex power series. □

The application of the same "globalizing techniques" used in the proof of Theorem 3.2 completes our new approach to the proof of Theorem 1.2.

Note that, with our approach to the problem, the proof of Theorem 1.1 is straightforward.

We believe that our direct techniques will be useful to find a different approach to the study of the boundary Schröder equation in several complex variables (see [2]).

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