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Group theory. — *A curious identity and the volume of the root spherical simplex*, by CORRADO DE CONCINI and CLAUDIO PROCESI with an appendix by JOHN R. STEMBRIDGE, communicated on 10 March 2006.

Dedicated to Guido Zappa in honor of his 90th birthday

ABSTRACT. — We show a curious identity on root systems which gives the evaluation of the volume of the spherical simplices cut by the cone generated by simple roots. In the appendix John Stembridge gives a conceptual proof of our identity.

KEY WORDS: Root systems; Weyl groups.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 20F55.

1. INTRODUCTION

In this note we shall consider a finite root system R spanning a Euclidean space E of dimension ℓ (for all the facts about root systems which we are going to use in this note we refer to [1]). ℓ is called the *rank* of R. We choose once and for all a set R^+ of positive roots and in R^+ the set $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ of simple roots. We also denote by W the Weyl group of R, i.e. the finite group generated by the reflections with respect to the hyperplanes orthogonal to the roots in R. Given such a root $\alpha \in R$, we denote by $s_\alpha \in W$ the reflection with respect to the hyperplane orthogonal to α. Set $s_i = s_{\alpha_i}$ for each $i = 1, \dots, \ell$ and call $S = \{s_1, \dots, s_\ell\}$ the set of *simple reflections*. It is known that S generates W and that the pair (W, S) is a Coxeter group.

We know that the ring of regular functions on E, invariant under the action of W, is a polynomial ring generated by homogeneous elements of degrees $d_1 \le \cdots \le d_\ell$. The d_i's are called the degrees. We shall also consider the sequence of exponents $d_1 - 1, \dots, d_\ell - 1$. Recall that $\prod_i d_i = |W|$.

In E we have the *affine arrangement* of the hyperplanes orthogonal to the roots and their translates under the weight lattice Λ, a locally finite configuration invariant under the affine Weyl group $\hat{W}$. The latter group is the semidirect product of W and of the lattice Q spanned by the roots, thought of as translation operators.

$\hat{W}$ is itself a Coxeter group. If E is irreducible, its Coxeter generators are given by the reflections $\{s_0, s_1, \dots, s_\ell\}$, where for $i \ge 1$ the s_i's are the simple generators of W and

$$s_0(v) = s_\theta(v) + \theta,$$

θ being the highest root. It is known that, for each $0 \le i \le \ell$, the subgroup W_i of $\hat{W}$ generated by the reflections $(s_0, \dots, \check{s}_i, \dots, s_\ell)$ is finite, and it is the Weyl group of a root system $R^{(i)}$ which will be discussed presently. Hence we can consider the degrees $d_1^{(i)} \leq \cdots \leq d_\ell^{(i)}$. Our main result is the identity (Theorem 1.2)

$$\sum_{i=0}^{\ell} \frac{(d_1^{(i)} - 1) \cdots (d_\ell^{(i)} - 1)}{d_1^{(i)} \cdots d_\ell^{(i)}} = 1.$$

The proof is a case by case computation using the classification of irreducible root systems. It is quite desirable to give a more conceptual deduction of our identity.

In the last section we show, following a suggestion of Vinberg, that our identity implies the following geometric identity. Take the unit sphere $S(E)$ in E and consider the spherical simplex $S(E) = C(\Delta) \cap S(E)$, $C(\Delta)$ being the cone of positive linear combinations of the simple roots. Then

$$\frac{\mathrm{Vol}\, S(\Delta)}{\mathrm{Vol}\, S(E)} = \frac{(d_1 - 1) \cdots (d_\ell - 1)}{d_1 \cdots d_\ell}.$$

We have discovered this identity while trying to understand the following fact. Consider the complex space $V = E \otimes_{\mathbb{R}} \mathbb{C}$, and take the algebraic torus $T = V/Q$. For any root $\alpha \in R$ the linear form $\check{\alpha}$ defined by

$$\check{\alpha}(v) = 2\frac{(\alpha, v)}{(\alpha, \alpha)}$$

takes integer values on Q, hence we get the character $e^{2\pi\sqrt{-1}\check{\alpha}}$ of T.

Denote its kernel by D_α. In our work on toric arrangements (see [2], and also [4]–[7]) we have shown that the Euler characteristic of the open set $\mathcal{A} := T - \bigcup_{\alpha \in R^+} D_\alpha$ equals $(-1)^\ell |W|$. The only proof we know of this fact is via a combinatorial topological construction of Salvetti [8], [3]. The above identity has been the result of an attempt to compute this Euler characteristic directly.

1.1. *The main identity*

We are interested in the numbers

$$\nu(R) = \prod_{i=1}^{\ell} \frac{d_i - 1}{d_i}.$$

The following list gives $\nu(R)$ in the case of irreducible root systems:

$$\nu_{A_n} = \frac{1}{n+1}, \quad n \geq 1,$$

$$\nu_{B_n} = \nu_{C_n} = \frac{1}{4^n}\binom{2n}{n}, \quad n \geq 2,$$

$$\nu_{D_n} = \frac{n-1}{4^{n-1}n}\binom{2(n-1)}{n-1}, \quad n \geq 4,$$

$$\nu_{G_2} = \frac{5}{12}, \quad \nu_{F_4} = \frac{385}{1152},$$

$$\nu_{E_6} = \frac{77}{324}, \quad \nu_{E_7} = \frac{2431}{9216}, \quad \nu_{E_8} = \frac{30808063}{99532800}.$$

Notice that if R is reducible, i.e. $R = R_1 \cup R_2$ with $R_1 \perp R_2$, then clearly

$$\nu(R) = \nu(R_1)\nu(R_2).$$

We normalize the scalar product so that the short roots have length $\sqrt{2}$ and we denote by D the Dynkin diagram of R.

From now on we assume that R is irreducible and we denote by $\hat{D}$ the extended Dynkin diagram. Set $\alpha_0 = -\theta$, with θ the highest root. We have a bijection between the set $\hat{\Delta} = \{\alpha_0, \dots, \alpha_\ell\}$ and the nodes of $\hat{D}$. For every $i = 0, \dots, \ell$ the diagram D_i obtained from $\hat{D}$ by removing the node corresponding to the root α_i (and all the edges having that node as a vertex) is of finite type. So we can consider the corresponding root system $R^{(i)}$ consisting of all roots in R which are integral linear combinations of the roots $\alpha_0, \dots, \check{\alpha}_i, \dots, \alpha_\ell$ and the corresponding number $\nu(R^{(i)})$.

In the proof of our result we shall need the following well known

LEMMA 1.1. *The following identities hold:*

$$4^n = \sum_{h=0}^{n} \binom{2h}{h}\binom{2(n-h)}{n-h}, \qquad n \geq 0. \tag{1}$$

Furthermore, when $n \geq 2$ we have

$$4^{n-1} = \frac{1}{2}\binom{2n}{n} + \sum_{h=2}^{n} \frac{h-1}{h}\binom{2(h-1)}{h-1}\binom{2(n-h)}{n-h}, \tag{2}$$

$$4^{n-2} = \frac{n-1}{4n}\binom{2(n-1)}{n-1} + \sum_{h=2}^{n-2} \frac{(h-1)(n-h-1)}{h(n-h)}\binom{2(h-1)}{h-1}\binom{2(n-1-h)}{n-1-h}. \tag{3}$$

PROOF. The first identity follows immediately from the power series expansion

$$\frac{1}{\sqrt{1-4t}} = \sum_{n\geq 0} \binom{2n}{n} t^n. \tag{4}$$

To see this notice that setting

$$f(t) = \frac{1}{\sqrt{1-4t}}$$

we have

$$\frac{df(t)}{dt} = 2f(t)^3$$

from which we deduce that

$$(1-4t)\frac{df(t)}{dt} = 2f(t).$$

Writing $f(t) = \sum_{n\geq 0} a_n t^n$ we deduce that

$$\sum_{n\geq 0} n a_n t^{n-1} - 4\sum_{n\geq 0} n a_n t^n = 2\sum_{n\geq 0} a_n t^n.$$

Equating coefficients, we get $a_0 = 1$ and for $h \geq 0$,

$$a_{h+1} = \frac{2(2h+1)}{h+1} a_h.$$

On the other hand, if we set $b_h := \binom{2h}{h}$, we get $b_0 = 1$ and

$$b_{h+1} = \frac{2(h+1)!}{(h+1)!(h+1)!} = \frac{2(h+1)(2h+1)}{(h+1)^2} b_h = \frac{2(2h+1)}{h+1} b_h$$

so $a_n = b_n$ and everything follows.

To see the second identity, notice that using (4) and integrating, we get

$$\frac{1}{2} - \sum_{h \geq 1} \frac{1}{h} \binom{2(h-1)}{h-1} t^h = \frac{1}{2}\sqrt{1-4t}. \tag{5}$$

Again using (4) we deduce

$$\begin{aligned} \frac{1}{2} + \sum_{h \geq 2} \frac{h-1}{h} \binom{2(h-1)}{h-1} t^h &= \sum_{h \geq 1} \binom{2(h-1)}{h-1} t^h + \frac{1}{2} - \sum \frac{1}{h} \binom{2(h-1)}{h-1} t^h \\ &= \frac{1-2t}{2\sqrt{1-4t}}. \end{aligned} \tag{6}$$

This together with (4) implies that

$$\frac{1}{2}\binom{2n}{n} + \sum_{h=2}^{n} \frac{h-1}{h} \binom{2(h-1)}{h-1} \binom{2(n-h)}{n-h}$$

is the coefficient of t^n in the power series expansion of

$$\frac{1-2t}{2\sqrt{1-4t}} \frac{1}{\sqrt{1-4t}} = \frac{1}{2} + \frac{t}{1-4t}.$$

Since $n \geq 2$ the claim follows. To see the last identity, let us remark that its left hand side is the coefficient of t^n in the power series expansion of the function

$$\left(\frac{1-2t}{2\sqrt{1-4t}}\right)^2 = \frac{1+4t^2-4t}{4-16t}.$$

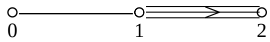
From this everything follows. $\square$

THEOREM 1.2. $\sum_{i=0}^{\ell} \nu(R^{(i)}) = 1$.

PROOF. The proof is by a case by case computation.

Let us deal first with the exceptional cases. In order to make the computation transparent it is more convenient to multiply our sum by $|W|$, the order of the Weyl group.

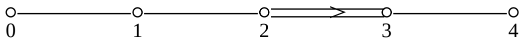
Case G_2. In this case $|W| = 12$. By looking at the extended Dynkin diagram



we get

$$|W| \sum_{i=0}^{2} \nu(R^{(i)}) = 12(\nu(G_2) + \nu(A_2) + \nu(A_1 \times A_1)) = 5 + 3 + 4 = 12.$$

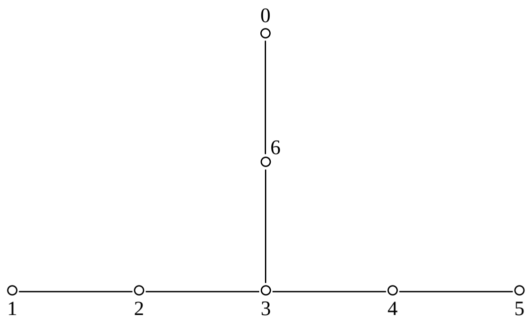
Case F_4. The order of the Weyl group is 1152. By looking at the extended Dynkin diagram



we get

$$\begin{aligned} |W| \sum_{i=0}^{4} \nu(R^{(i)}) &= |W|(\nu(F_4) + \nu(A_1 \times C_3) + \nu(A_2 \times A_2) + \nu(A_3 \times A_1) + \nu(B_4)) \\ &= 385 + 180 + 128 + 144 + 315 = 1152. \end{aligned}$$

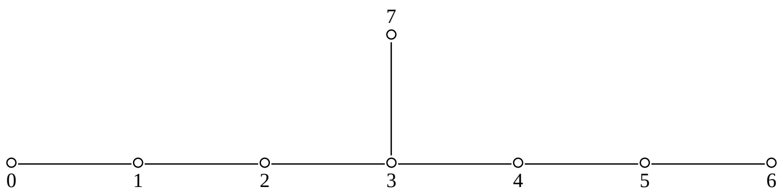
Case E_6. In this case $|W| = 51840$. By looking at the extended Dynkin diagram



we get

$$\begin{aligned} |W| \sum_{i=0}^{6} \nu(R^{(i)}) &= |W|(3\nu(E_6) + \nu(A_2 \times A_2 \times A_2) + 3\nu(A_1 \times A_5)) \\ &= 36960 + 1920 + 12960 = 51840. \end{aligned}$$

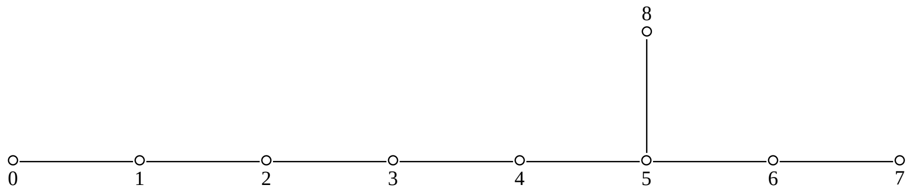
Case E_7. In this case $|W| = 2903040$. By looking at the extended Dynkin diagram



we get

$$|W|\sum_{i=0}^{7}\nu(R^{(i)}) = |W|(2\nu(E_7)+2\nu(A_1\times D_6)+2\nu(A_2\times A_5) \\ +\nu(A_3\times A_3\times A_1)+\nu(A_7)) \\ = 1531530+595350+322560+90720+362880 = 2903040.$$

Case E_8. In this case $|W| = 696729600$. By looking at the extended Dynkin diagram



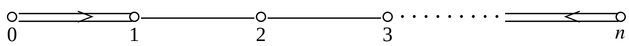
we get

$$|W|\sum_{i=0}^{8}\nu(R^{(i)}) = |W|(\nu(E_8)+\nu(A_1\times E_7)+\nu(A_2\times E_6)+\nu(A_3\times D_5)+\nu(A_4\times A_4) \\ +\nu(A_5\times A_2\times A_1)+\nu(A_1\times A_7)+\nu(D_8)+\nu(A_8)) \\ = 215656441+91891800+55193600+38102400+27869184 \\ +19353600+43545600+127702575+77414400 = 696729600.$$

Case A_n. In this case each $R^{(i)}$ is of type A_n. It follows that

$$\sum_{i=0}^{n}\nu(R^{(i)}) = (n+1)\nu_{A_n} = (n+1)\frac{1}{n+1} = 1.$$

Case C_n. The extended Dynkin diagram is

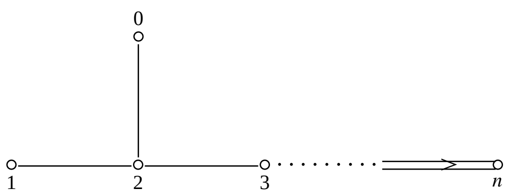


Denoting by C_0 the trivial root system and setting $C_1 = A_1$, we get

$$\sum_{i=0}^{n}\nu(R^{(i)}) = \sum_{h=0}^{n}\nu(C_h\times C_{n-h}) = \sum_{h=0}^{n}\frac{1}{4^{h+n-h}}\binom{2h}{h}\binom{2(n-h)}{n-h} = 1$$

by Lemma 1.1(1).

Case B_n. The extended Dynkin diagram is

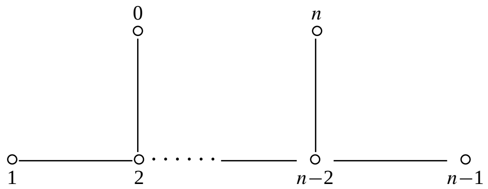


Denoting by B_0 the trivial root system, and setting $C_1 = A_1$, $D_2 = A_1 \times A_1$ and $D_3 = A_3$, we get

$$\begin{aligned}\sum_{i=0}^{n} \nu(R^{(i)}) &= 2\nu(B_n) + \sum_{h=2}^{n} \nu(D_h \times B_{n-h}) \\ &= \frac{2}{4^n}\binom{2n}{n} + \sum_{h=2}^{n} \frac{h-1}{h}\binom{2(h-1)}{h-1}\binom{2(n-h)}{n-h} \\ &= \frac{1}{4^{n-1}}\left(\frac{1}{2}\binom{2n}{n} + \sum_{h=2}^{n} \frac{h-1}{h}\binom{2(h-1)}{h-1}\binom{2(n-h)}{n-h}\right) = 1\end{aligned}$$

by Lemma 1.1(2).

Case D_n. The extended Dynkin diagram is



Setting $D_2 = A_1 \times A_1$ and $D_3 = A_3$, we get

$$\begin{aligned}\sum_{i=0}^{n} \nu(R^{(i)}) &= 4\nu(D_n) + \sum_{h=2}^{n-2} \nu(D_h \times D_{n-h}) \\ &= \frac{1}{4^{n-2}}\left(\frac{n-1}{4n}\binom{2(n-1)}{n-1}\right. \\ &\qquad \left. + \sum_{h=2}^{n-2} \frac{(h-1)(n-h-1)}{h(n-h)}\binom{2(h-1)}{h-1}\binom{2(n-1-h)}{n-1-h}\right),\end{aligned}$$

which equals 1 by Lemma 1.1(3). $\square$

1.2. *The volume of $S(\Delta)$*

Recall that we have introduced the spherical simplex as the intersection of the unit sphere $S(E)$ in E with the cone $C(\Delta)$ of nonnegative linear combinations of the simple roots $\{\alpha_1, \ldots, \alpha_\ell\}$ for the root system R. Our purpose is to show

THEOREM 1.3.
$$\frac{\operatorname{Vol} S(\Delta)}{\operatorname{Vol} S(E)} = \nu(R) = \frac{(d_1 - 1)\cdots(d_\ell - 1)}{d_1 \cdots d_\ell}.$$

PROOF. For simplicity we normalize in such a way that $\operatorname{Vol} S(E) = 1$. We then set $\operatorname{Vol} S(\Delta) = V(R)$. If R is reducible, i.e. $R = R_1 \cup R_2$ with $R_1 \perp R_2$, we have

$$V(R) = V(R_1)V(R_2).$$

Since we also have

$$\nu(R) = \nu(R_1)\nu(R_2),$$

an easy induction implies that we are reduced to showing our claim under the assumption that R is irreducible.

So assume R is irreducible and set $\alpha_0 = -\theta$, with θ the highest root. Write $\alpha_0 = \sum_{j=1}^{\ell} n_j\alpha_j$ with n_j a negative integer for all $j = 1, \dots, \ell$.

As in the previous section, for every $i = 0, \dots, \ell$ let $R^{(i)}$ be the root system consisting of all roots in R which are integral linear combinations of the roots $\alpha_0, \dots, \check{\alpha}_i, \dots, \alpha_\ell$ so that in particular $|R^{(i)}| \leq |R|$. Recall that the Dynkin diagram of $R^{(i)}$ is the subdiagram of $\hat{D}$ obtained by removing the node corresponding to α_i. The roots $\Delta^{(i)} = \{\alpha_0, \dots, \check{\alpha}_i, \dots, \alpha_\ell\}$ are simple roots for $R^{(i)}$.

We claim that E is the union of the cones $C(\Delta^{(i)})$ whose interiors are disjoint. To see this take $u \in E$ and write $u = \sum_{h=1}^{\ell} b_h\alpha_h$. If all b_h are nonnegative then $u \in C(\Delta) = C(\Delta^{(0)})$, otherwise $b_h < 0$ for at least one index $1 \leq h \leq \ell$. Take an index i for which b_i/n_i is maximum. Notice that necessarily $b_i/n_i > 0$. We can clearly write

$$u = \frac{b_i}{n_i}\alpha_0 + \sum_{h=1,\, h\neq i}^{\ell} \left(b_h - \frac{n_h b_i}{n_i}\right)\alpha_h$$

and all coefficients are nonnegative.

Now observe that if, for any $i = 0, \dots, \ell$, we write α_i as a linear combination of $\alpha_0, \dots, \check{\alpha}_i, \dots, \alpha_\ell$ then all coefficients are negative. We leave to the reader the easy verification that this implies that the interiors of the cones $C(\Delta^{(i)})$ are mutually disjoint.

We deduce that

$$\sum_{h=0}^{\ell} V(R^{(h)}) = 1. \tag{7}$$

Now set $\Gamma = \{i \mid R^{(i)} = R\}$. Then Γ is not empty since $0 \in \Gamma$. We can rewrite (7) as

$$|\Gamma| V(R) + \sum_{h \notin \Gamma} V(R^{(h)}) = 1.$$

Similarly by Theorem 1.2 we get

$$|\Gamma| \nu(R) + \sum_{h \notin \Gamma} \nu(R^{(h)}) = 1.$$

Since, by the definition of Γ, for $h \notin \Gamma$ we have $|R^{(h)}| < |R|$, by induction (the case of A_1 in which we have two roots is trivial) we can assume $V(R^{(h)}) = \nu(R^{(h)})$. We get

$$V(R) = \frac{1}{|\Gamma|}\Big(1 - \sum_{h\notin\Gamma} V(R^{(h)})\Big) = \frac{1}{|\Gamma|}\Big(1 - \sum_{h\notin\Gamma} \nu(R^{(h)})\Big) = \nu(R),$$

proving our claim. □

APPENDIX
by John R. Stembridge

In this appendix, we provide an explanation for the "curious identity" (Theorem 1.2) without any case-by-case considerations. The proof is based on two elegant formulas, one due to L. Solomon, the other due to R. Steinberg. Both of these results deserve to be better known.

If W is a finite group generated by reflections in a real Euclidean space E, consider the class function on W defined by

$$\delta_W(q,t)(w) := \frac{\det(1-qw)}{\det(1-tw)} \quad (w \in W),$$

where the determinants are evaluated as endomorphisms of E, and q, t are indeterminates. This may be viewed as a bi-graded character for $S(E) \otimes \Lambda(E)$, the tensor product of the symmetric and exterior algebras of E.

In his 1963 paper on invariants of finite reflection groups [9], Solomon explicitly determined the structure of the W-invariants of $S(E) \otimes \Lambda(E)$. At the level of characters, his structure theorem implies

$$\langle 1_W, \delta_W(q,t)\rangle_W = \prod_{i=1}^{\ell} \frac{1-qt^{d_i-1}}{1-t^{d_i}}, \tag{1}$$

where $d_1, \ldots, d_\ell$ are the degrees ($\ell = \dim E$), 1_W denotes the trivial character of W, and $\langle f, g\rangle_W := |W|^{-1}\sum_{w\in W} f(w)g(w)$ is the usual pairing of real-valued class functions f and g.

Henceforth, assume that W is a Weyl group with an irreducible root system $R \subset E$ of rank ℓ and simple reflections $S = \{s_1, \ldots, s_\ell\}$. Note that by setting $q = 1$ and letting $t \to 1$ in (1), we obtain the quantity $\nu(R)$.

We let $s_0 \in W$ denote the reflection corresponding to the highest root and set $S_0 = S \cup \{s_0\}$. One may interpret S_0 as the W-image of the simple reflections of the associated affine Weyl group $\widehat{W}$.

Following Steinberg (see Section 3 of [10]), the action of $\widehat{W}$ on E descends to a W-action on the ℓ-torus E/Q (where Q denotes the root lattice), and the decomposition of E into simplicial alcoves by the arrangement of affine hyperplanes associated to R induces a simplicial decomposition of E/Q with a compatible W-action. Moreover, the W-stabilizers of the faces of E/Q are (up to conjugacy) generated by the various proper subsets of S_0.

Given $w \in W$, Steinberg computes the Euler characteristic of the w-fixed subcomplex of E/Q in two different ways (see Theorem 3.12 of [10]), thereby obtaining the identity

$$\det(1-w) = \sum_{J \subset S_0} (-1)^{|S|-|J|} 1_{W_J}^W(w), \tag{2}$$

where W_J denotes the reflection subgroup generated by J, and $1_{W_J}^W$ denotes the permutation character of the action of W on W/W_J. It is important to note that J ranges over *proper* subsets of S_0.

Steinberg actually proves a more general identity that involves twisting by an involution; the above instance corresponds to the trivial involution. One may also recognize (2) as a companion to the more familiar identity

$$\det(w) = \sum_{J \subseteq S} (-1)^{|J|} 1_{W_J}^W(w).$$

Now consider the evaluation of

$$\lim_{t \to 1} \langle \delta_W(1,0), \delta_W(1,t) \rangle_W.$$

First, notice that $\delta_W(1,t) \to 1_W$ as $t \to 1$, so we obtain

$$\lim_{t \to 1} \langle \delta_W(1,0), \delta_W(1,t) \rangle_W = \langle \delta_W(1,0), 1_W \rangle_W = 1 \tag{3}$$

by setting $(q,t) = (1,0)$ in (1).

Second, notice that $\delta_W(1,0)(w) = \det(1-w)$, so (2) implies

$$\begin{aligned} \langle \delta_W(1,0), \delta_W(1,t) \rangle_W &= \sum_{J \subset S_0} (-1)^{|S|-|J|} \langle 1_{W_J}^W, \delta_W(1,t) \rangle_W \\ &= \sum_{J \subset S_0} (-1)^{|S|-|J|} \langle 1_{W_J}, \delta_{W_J}(1,t) \rangle_{W_J}, \end{aligned} \tag{4}$$

by Frobenius reciprocity. We can evaluate each of these terms by applying Solomon's formula to the reflection group W_J. But we need to be careful, because the action of W_J on E will have linear invariants if the rank of W_J is less than $\ell = |S|$. In such cases, this means that some of the degrees of W_J will equal 1, which introduces factors of $(1-q)/(1-t)$ in (1). Since we have set $q = 1$, these factors vanish.

Thus (4) should be restricted to ℓ-subsets of S_0, and we obtain

$$\langle \delta_W(1,0), \delta_W(1,t) \rangle_W = \sum_{j=0}^{\ell} \prod_{i=1}^{\ell} \frac{1 - t^{d_i^{(j)}-1}}{1 - t^{d_i^{(j)}}},$$

where $d_1^{(j)}, \dots, d_l^{(j)}$ are the degrees of W_J for $J = S_0 - \{s_j\}$. Comparing this with (3) in the limit $t \to 1$, we obtain the "curious identity"

$$\sum_{j=0}^{\ell} \prod_{i=1}^{\ell} \frac{d_i^{(j)} - 1}{d_i^{(j)}} = 1.$$

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REFERENCES

[1] N. BOURBAKI, *Groupes at algèbres de Lie, Chapitres 4,5,6*. Act. Sci. Indust. 1337, Hermann, Paris, 1968.

[2] C. DE CONCINI - C. PROCESI, *On the geometry of toric arrangements.* Transform. Groups 10 (2005), 387–422.

[3] C. DE CONCINI - M. SALVETTI, *Cohomology of Artin groups: Addendum: "The homotopy type of Artin groups"*. Math. Res. Lett. 3 (1996), 293–297.

[4] A. HENDERSON, *The symmetric group representation on cohomology of the regular elements of a maximal torus of the special linear group*. Preprint, http://www.maths.usyd.edu.au/u/anthonyh/.

[5] G. J. LEHRER, *The cohomology of the regular semisimple variety*. J. Algebra 199 (1998), 666–689.

[6] G. I. LEHRER, *A toral configuration space and regular semisimple conjugacy classes*. Math. Proc. Cambridge Philos. Soc. 118 (1995), 105–113.

[7] E. LOOIJENGA, *Cohomology of $\mathcal{M}_3$ and $\mathcal{M}_3^1$*. In: Mapping Class Groups and Moduli Spaces of Riemann Surfaces (Göttingen, 1991/Seattle, WA, 1991), Contemp. Math. 150, Amer. Math. Soc., 1993, 205–228.

[8] M. SALVETTI, *The homotopy type of Artin groups*. Math. Res. Lett. 1 (1994), 565–577.

[9] L. SOLOMON, *Invariants of finite reflection groups*. Nagoya Math. J. 22 (1963) 57–64.

[10] R. STEINBERG, *Endomorphisms of linear algebraic groups*. Mem. Amer. Math. Soc. 80 (1968).

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