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Partial differential equations. — *Approximating the inverse matrix of the G-limit through changes of variables in the plane*, by GIOCONDA MOSCARIELLO, CARLO SBORDONE and FRANÇOIS MURAT, communicated on 10 March 2006.

ABSTRACT. — Let A_j be a sequence of coercive symmetric matrices in $L^\infty(\mathbb{R}^2)^{2\times 2}$ with $\det A_j = 1$ which G-converges to A. We prove that there exists a sequence of K-quasiconformal mappings F_j which converge locally uniformly to a K-quasiconformal mapping F such that $A_j^{-1} \circ F_j^{-1}$ G-converges to $A^{-1} \circ F^{-1}$. The result is specific to the two-dimensional case but a similar result holds in dimension 1.

KEY WORDS: G-convergence; quasiconformal mappings; Beltrami operators; elliptic equations.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 35J25, 30C62.

1. INTRODUCTION

Let K be a fixed real number such that $K \geq 1$. We denote by $\mathcal{M}(K)$ the set of 2×2 symmetric matrices

$$A = A(x), \qquad x \in \mathbb{R}^2,$$

with $L^\infty(\mathbb{R}^2)$ coefficients which satisfy the ellipticity bounds

$$|\xi|^2/K \leq \langle A(x)\xi, \xi\rangle \leq K|\xi|^2 \quad \text{a.e. } x \in \mathbb{R}^2,\ \forall \xi \in \mathbb{R}^2. \tag{1.1}$$

We denote by $\mathcal{M}_1(K)$ the subset of $\mathcal{M}(K)$ whose elements satisfy the condition

$$\det A(x) = 1 \quad \text{a.e. } x \in \mathbb{R}^2. \tag{1.2}$$

It is well known that $\mathcal{M}(K)$ is compact with respect to G-convergence ([M], [S1], [T]) and rather surprising that $\mathcal{M}_1(K)$ enjoys the same property ([FM]).

Our aim is to prove that $\mathcal{M}_1(K)$ enjoys another interesting property: if A_j is a sequence in $\mathcal{M}_1(K)$ and if we consider the inverse matrices A_j^{-1}, for a subsequence we may assume that

$$A_j \xrightarrow{G} A \quad \text{and} \quad A_j^{-1} \xrightarrow{G} B^{-1},$$

and in general B is different from A. Indeed, it is well known that the inverse of the G-limit does not coincide with the G-limit of the inverses. However, the situation is different if we allow suitable changes of variables: we prove in the present paper that for every ball B there exists a sequence F_j of K-quasiconformal mappings in the plane which locally uniformly converges to a K-quasiconformal mapping F, such that defining $\hat{A}_j$ and $\hat{A}$ as A_j and A in the ball B and as the identity outside B, one has

$$\hat{A}_j^{-1} \circ F_j^{-1} \xrightarrow{G} \hat{A}^{-1} \circ F^{-1}.$$

Let us emphasize that our result is restricted to the two-dimensional case; a similar result holds in one dimension, but the result dramatically fails if $d \geq 3$.

2. DEFINITIONS AND PRELIMINARY RESULTS

Let us first recall the definition of the G-convergence (see [S1], [T]) of a sequence of 2×2 symmetric matrices $A_j = A_j(x)$ with $L^\infty(\mathbb{R}^2)$ coefficients which belong to $\mathcal{M}(K)$, i.e. satisfy (1.1) uniformly in j.

We say that a sequence $A_j \in \mathcal{M}(K)$ *G-converges* to A, and write

$$A_j \xrightarrow{G} A,$$

where A also belongs to $\mathcal{M}(K)$, if for every bounded open subset Ω of $\mathbb{R}^2$ and for every $f \in L^2(\Omega)$ one has

$$u_j \rightharpoonup u \quad \text{weakly in } W_0^{1,2}(\Omega),$$

where u_j and u are defined by

$$\begin{cases} -\operatorname{div}(A_j(x)\nabla u_j) = f & \text{in } \mathcal{D}'(\Omega), \\ u_j \in W_0^{1,2}(\Omega), \end{cases}$$

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = f & \text{in } \mathcal{D}'(\Omega), \\ u \in W_0^{1,2}(\Omega). \end{cases}$$

One of the main properties of the G-convergence in $\mathcal{M}(K)$ is its sequential compactness: from every sequence of matrices in $\mathcal{M}(K)$ one can extract a subsequence G-converging to a matrix which also belongs to $\mathcal{M}(K)$ (see [S1], [T]).

Despite the fact that the condition

$$\det A_j(x) = 1 \quad \text{a.e. } x \in \mathbb{R}^2$$

is not necessarily preserved under any familiar weak convergence of the sequence A_j, it is preserved under G-convergence (see e.g. [FM]). So the subset $\mathcal{M}_1(K)$ of $\mathcal{M}(K)$ consisting of matrices with determinant equal to one is G-closed. This result is specific to the two-dimensional case.

Let us now recall the definition of K-quasiconformal mappings (still in the two-dimensional setting; see [AIM]).

We say that F is a *K-quasiconformal mapping* if $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism with F and F^{-1} in $W^{1,2}_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ such that

$$|DF(x)|^2 \leq \left(K + \frac{1}{K}\right) J(x, F) \quad \text{a.e. } x \in \mathbb{R}^2, \tag{2.1}$$

where $|DF(x)|$ stands for the Hilbert–Schmidt norm of the differential matrix $DF(x) \in \mathbb{R}^{2\times 2}$ and $J(x, F)$ for the Jacobian determinant of F, i.e.

$$J(x, F) = \det DF(x).$$

3. Main result

Let A_j be a sequence of matrices such that

$$A_j \in \mathcal{M}_1(K), \qquad A_j \xrightarrow{G} A; \tag{3.1}$$

then, as mentioned in Section 2, $A \in \mathcal{M}_1(K)$. Fix a ball B in $\mathbb{R}^2$ and define the matrices $\hat{A}_j$ and $\hat{A}$ by

$$\hat{A}_j(x) = \begin{cases} A_j(x), & x \in B, \\ I, & x \notin B, \end{cases} \qquad \hat{A}(x) = \begin{cases} A(x), & x \in B, \\ I, & x \notin B. \end{cases}$$

Our main result is as follows.

THEOREM 3.1. *Under assumption* (3.1), *there exists a sequence F_j of K-quasiconformal mappings which satisfies*

$$F_j \to F \qquad \textit{locally uniformly}, \tag{3.2}$$

with F a K-quasiconformal mapping, such that

$$\hat{A}_j^{-1} \circ F_j^{-1} \xrightarrow{G} \hat{A}^{-1} \circ F^{-1}. \tag{3.3}$$

PROOF. Since $\hat{A}_j \in \mathcal{M}_1(K)$, by the so-called measurable Riemann Mapping Theorem ([IM]) there exists a unique K-quasiconformal mapping $F_j : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F_j(0,0) = (0,0)$, $F_j(1,0) = (1,0)$, $F_j(\infty) = \infty$ and

$$\hat{A}_j(x) = J(x, F_j)\,(DF_j)^{-1}(x)\,{}^t(DF_j)^{-1}(x), \tag{3.4}$$

where ${}^tD$ and D^{-1} denote the transpose and the inverse of the matrix D. Note that a matrix A of the form

$$A(x) = J(x, F)\,(DF)^{-1}(x)\,{}^t(DF)^{-1}(x) \qquad \text{a.e. } x \in \mathbb{R}^2$$

(whose determinant is always equal to 1) belongs to $\mathcal{M}_1(K)$ if and only if condition (2.1) holds true: indeed, since the Hilbert–Schmidt norm of the matrix D is given by

$$|D|^2 = \operatorname{tr} D\,{}^tD,$$

the distortion inequality

$$|D|^2 \le \left(K + \frac{1}{K}\right) \det D$$

is equivalent to

$$\operatorname{tr}\left(\frac{D\,{}^tD}{\det D}\right) \le K + \frac{1}{K};$$

if λ and $1/\lambda$ are the eigenvalues of $D\,{}^tD/\det D$ (this matrix is always symmetric, non-negative and with determinant 1), then the last inequality is equivalent to

$$\lambda + \frac{1}{\lambda} \le K + \frac{1}{K},$$

and to $1/K \le \lambda \le K$.

Since the mapping F_j is K-quasiconformal for every j, there exists (see [GIKMS, Lemma 5.2]) a subsequence F_{j_h} and a K-quasiconformal mapping F such that $F(0,0) = (0,0)$, $F(1,0) = (1,0)$, $F(\infty) = \infty$ and

$$F_{j_h} \to F \quad \text{locally uniformly.} \tag{3.5}$$

By a result of S. Spagnolo ([S2, Theorem 2], see also [F]) one has

$$J(x, F_{j_h})\,(DF_{j_h})^{-1}(x)\,{}^t(DF_{j_h})^{-1}(x) \overset{G}{\to} J(x,F)\,(DF)^{-1}(x)\,{}^t(DF)^{-1}(x). \tag{3.6}$$

Since $\hat{A}_j$ G-converges to $\hat{A}$ by the local character of G-convergence, we have

$$\hat{A}(x) = J(x,F)\,(DF)^{-1}(x)\,{}^t(DF^{-1})(x) \quad \text{a.e. } x \in \mathbb{R}^2. \tag{3.7}$$

Let us now show that the whole sequence F_j locally uniformly converges to F. Indeed, let F_{j_k} be any subsequence of F_j. By Lemma 5.2 of [GIKMS] we can extract from F_{j_k} a further subsequence still denoted by F_{j_k} such that

$$F_{j_k} \to \hat{F} \quad \text{locally uniformly,}$$

where $\hat{F}$ is a K-quasiconformal mapping which satisfies $\hat{F}(0,0) = (0,0)$, $\hat{F}(1,0) = (1,0)$, $\hat{F}(\infty) = \infty$. Again by Spagnolo's result we deduce that

$$A(x) = J(x,\hat{F})\,(D\hat{F})^{-1}(x)\,{}^t(D\hat{F}^{-1})(x) \quad \text{a.e. } x \in \mathbb{R}^2. \tag{3.8}$$

Since the matrices of the right-hand sides of (3.7) and (3.8) coincide, there exists (see [LV]) a Möbius transformation H such that

$$\hat{F} = H \circ F. \tag{3.9}$$

In complex notation, a Möbius transformation can be written as

$$H(z) = \frac{az+b}{cz+d}, \quad z \in \mathbb{C},$$

for some a, b, c, d in $\mathbb{C}$. In the same notation we have

$$F(0) = 0, \quad F(1) = 1, \quad F(\infty) = \infty,$$

and

$$\hat{F}(0) = 0, \quad \hat{F}(1) = 1, \quad \hat{F}(\infty) = \infty,$$

which together with (3.9) implies that

$$H(z) = z,$$

and therefore that $F = \hat{F}$. This implies that the whole sequence F_j converges to F. Applying the chain rule in (3.4) allows one to write

$$\hat{A}_j^{-1} \circ F_j^{-1}(y) = J(y, F_j^{-1})\,{}^t(DF_j^{-1})^{-1}(y)\,(DF_j^{-1})^{-1}(y). \tag{3.10}$$

Since the inverse mappings F_j^{-1} are also K-quasiconformal and satisfy

$$F_j^{-1} \to F^{-1} \quad \text{locally uniformly,}$$

applying once more Spagnolo's result, we infer that the sequence of matrices defined by the right-hand side of (3.10) G-converges to

$$J(y, F^{-1})\, {}^t(DF^{-1})^{-1}(y)(DF^{-1})^{-1}(y),$$

which coincides with the matrix $\hat{A}^{-1} \circ F^{-1}$. This proves the desired result. □

4. THE ONE-DIMENSIONAL CASE

In this section we consider the one-dimensional case for which we prove a result similar to Theorem 3.1. Let $a_j \in L^\infty(\mathbb{R})$ be a sequence of measurable functions $a_j : \mathbb{R} \to \mathbb{R}$ satisfying the uniform bounds

$$1/K \le a_j(x) \le K \quad \text{a.e. } x \in \mathbb{R}, \tag{4.1}$$

where $K \ge 1$ is a given constant. Up to a subsequence we may assume that

$$a_j \rightharpoonup a \quad \text{in } \sigma(L^\infty, L^1), \tag{4.2}$$

$$\frac{1}{a_j} \rightharpoonup \frac{1}{b} \quad \text{in } \sigma(L^\infty, L^1), \tag{4.3}$$

for some functions $a, b \in L^\infty(\mathbb{R})$ with

$$1/K \le b(x) \le a(x) \le K \quad \text{a.e. } x \in \mathbb{R}.$$

Actually $b(x)$ may be strictly less than $a(x)$.

Nevertheless, the composition with suitable changes of variable $h_j : \mathbb{R} \to \mathbb{R}$ allows the inverses of a_j to weakly converge to the inverse of a.

THEOREM 4.1. *Let $a_j \in L^\infty(\mathbb{R})$ be a sequence satisfying* (4.1) *and* (4.2). *Then there exists a sequence of increasing homeomorphisms $h_j : \mathbb{R} \to \mathbb{R}$ which are uniformly Lipschitz continuous together with their inverses and which converge locally uniformly on $\mathbb{R}$ to a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that*

$$\frac{1}{a_j(h_j^{-1})} \rightharpoonup \frac{1}{a(h^{-1})} \quad \text{in } \sigma(L^\infty, L^1). \tag{4.4}$$

PROOF. Define

$$h_j(s) = c_j \int_0^s a_j(t)\, dt \quad \forall s \in \mathbb{R}, \tag{4.5}$$

where $c_j > 0$ is a sequence of constants which converges to some $c > 0$. Then

$$h_j(s) \to h(s) = c \int_0^s a(t)\, dt \quad \forall s \in \mathbb{R},$$

and also locally uniformly, because h_j is bounded in $W^{1,\infty}_{\text{loc}}(\mathbb{R})$.

By the chain rule, the inverse h_j^{-1} of h_j is given by

$$h_j^{-1}(\sigma) = \int_0^\sigma \frac{1}{c_j a_j(h_j^{-1}(\tau))}\, d\tau \quad \forall \sigma \in \mathbb{R}. \tag{4.6}$$

Since the sequence h_j^{-1} is bounded in $W^{1,\infty}_{\text{loc}}(\mathbb{R})$ and since the sequence h_j converges to h, one has

$$h_j^{-1}(\sigma) \to h^{-1}(\sigma) \quad \forall \sigma \in \mathbb{R}$$

where h^{-1} is given by

$$h^{-1}(\sigma) = \int_0^\sigma \frac{1}{ca(h^{-1}(\tau))}\, d\tau. \tag{4.7}$$

Since h_j^{-1} is bounded in $W^{1,\infty}_{\text{loc}}(\mathbb{R})$, h_j^{-1} also weak star converges to h^{-1} in $W^{1,\infty}_{\text{loc}}(\mathbb{R})$. The weak star convergence of the derivatives, namely

$$(h_j^{-1})' \rightharpoonup (h^{-1})' \quad \text{in } \sigma(L^\infty, L^1),$$

implies (4.4) in view of (4.6) and (4.7), since c_j converges to c. □

REMARK 4.1. One could think that the definition (4.5) of h_j is the only possible choice in order to have (4.4) when (4.1) and (4.2) hold true, or in other words that h_j is uniquely determined up to the multiplicative constant c_j. We will prove in this remark that in general this is not the case, and that there are many other choices of functions h_j which satisfy (4.4) and the assumption of Theorem 4.1.

Observe indeed that (4.4) is equivalent to

$$\int_0^\sigma \frac{1}{a_j(h_j^{-1}(\tau))}\, d\tau \to \int_0^\sigma \frac{1}{a(h^{-1}(\tau))}\, d\tau \quad \forall \sigma \in \mathbb{R}.$$

By the changes of variable $h_j^{-1}(\tau) = t$, this can be rewritten as

$$\int_0^{h_j^{-1}(\sigma)} \frac{h_j'(t)}{a_j(t)}\, dt \to \int_0^{h^{-1}(\sigma)} \frac{h'(t)}{a(t)}\, dt. \tag{4.8}$$

Since h_j^{-1} tends to h^{-1} locally uniformly, (4.8), and therefore (4.4), is equivalent to

$$\int_0^s \frac{h_j'(t)}{a_j(t)}\, dt \to \int_0^s \frac{h'(t}{a(t)}\, dt \quad \forall s \in \mathbb{R},$$

that is, to

$$\frac{h_j'}{a_j} \rightharpoonup \frac{h'}{a} \quad \text{in } \sigma(L^\infty, L^1). \tag{4.9}$$

To prove that (4.5) is not the only possible choice for h_j, let us now consider the following example. Let a_j and h_j' be given by

$$a_j(x) = \varphi(jx), \quad h_j'(x) = \psi(jx), \quad h_j(0) = 0,$$

where φ and ψ are the periodic functions of period 1 defined on $(0, 1)$ by

$$\varphi(y) = A\chi_{(0,1/3)}(y) + B\chi_{(1/3,2/3)}(y) + C\chi_{(2/3,1)}(y),$$
$$\psi(y) = X\chi_{(0,1/3)}(y) + Y\chi_{(1/3,2/3)}(y) + Z\chi_{(2/3,1)}(y),$$

where $\chi_{(a,b)}$ denotes the characteristic function of the interval (a, b) and where A, B, C, X, Y, Z are strictly positive constants. Then a_j satisfies (4.1) and (4.2) for some $K \geq 1$ and for a given by

$$a(x) = \frac{1}{3}(A + B + C).$$

The functions $h_j : \mathbb{R} \to \mathbb{R}$ are increasing homeomorphisms which are uniformly Lipschitz continuous together with their inverses, and which converge locally uniformly on $\mathbb{R}$ to the homeomorphism $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h'(x) = \frac{1}{3}(X + Y + Z), \quad h(0) = 0.$$

In this example, convergence (4.9), which is equivalent to (4.4), amounts to

$$\frac{\frac{1}{3}(X + Y + Z)}{\frac{1}{3}(A + B + C)} = \frac{1}{3}\left(\frac{X}{A} + \frac{Y}{B} + \frac{Z}{C}\right), \tag{4.10}$$

since the right-hand side of (4.10) is the $\sigma(L^\infty, L^1)$ limit of h_j'/a_j. Formula (4.10) is equivalent to

$$Z = \frac{3C(A + B + C)}{2C - (A + B)}\left[\frac{1}{3}\left(\frac{X}{A} + \frac{Y}{B}\right) - \frac{X + Y}{A + B + C}\right]. \tag{4.11}$$

For every $A, B, X, Y > 0$, if we choose $C > 0$ sufficiently large, then the number Z defined by (4.11) satisfies $Z > 0$, and all the assumptions of Theorem 4.1 are satisfied. But in general we do not have (4.5), since this would imply

$$h_j'(s) = c_j a_j(s) \quad \forall s \in \mathbb{R},$$

that is, $\psi(y) = c\varphi(y)$, an assertion which is false when we choose $X/A \neq Y/B$, a choice which is always possible.

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