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Partial differential equations. — *Homogenization of doubly-nonlinear equations,* by AUGUSTO VISINTIN, communicated by E. Magenes.

ABSTRACT. — This note deals with a doubly-nonlinear parabolic-hyperbolic equation that represents electromagnetic processes in a nonhomogeneous metal surrounded by an insulating environment. Existence of a weak solution is sketched. The constitutive relations are then assumed to exhibit periodic oscillations in space. As the period vanishes, the solution converges in the sense of Nguetseng to that of a corresponding two-scale homogenized problem. The homogenization procedure is then completed by eliminating the dependence on the fine-scale variable. An analogous problem issued from phase transitions is also illustrated.

KEY WORDS: Electromagnetism; Maxwell equations; Stefan problem; homogenization; two-scale convergence.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 35K60, 35Q60, 35R35, 78M40.

INTRODUCTION

In this note we study a doubly-nonlinear model of electromagnetic processes for composite materials, sketch the proof of existence of a weak solution, and deal with its homogenization via so-called two-scale convergence. The analytical structure of this setting is fairly general; this allows us to apply an analogous approach to a doubly-nonlinear model of phase transitions.

Maxwell's equations in a nonhomogeneous and anisotropic metal surrounded by an insulating environment lead to a parabolic-hyperbolic system of partial differential equations in the whole space. In this note we provide a weak formulation for an initial-value problem with nonlinear constitutive relations of the form $\vec{B} = \vec{B}(\vec{H}, x)$ and $\vec{J} = \vec{J}(\vec{E}, \vec{H}, x)$, neglecting hysteresis. We allow for discontinuity of the dependence of $\vec{B}$ on $\vec{H}$, thus accounting for the possible occurrence of *free boundaries.*

In Sect. 1 we study the existence of a solution. Sect. 2 is devoted to reviewing some results of two-scale convergence. In Sect. 3 we then assume that the above constitutive relations exhibit fast periodic oscillations in space, let the space-period vanish, and prove two-scale convergence (in the sense of Nguetseng) to a two-scale homogenized problem. We then complete the homogenization procedure by eliminating the dependence on the fine-scale variable. In Sect. 4 we deal with existence of a weak solution and homogenization of a model of phase transitions, allowing also for a temperature-dependent thermal conductivity. Details and proofs will appear separately in [31].

A large literature has been devoted to homogenization (see e.g. [3, 7, 9, 17, 20, 24, 25]). In [5, 10, 11, 13, 14, 21] homogenization was also applied to the Maxwell equations and/or the Stefan model assuming linearity in the elliptic part. The notion of two-scale convergence was introduced by Nguetseng [22] and then developed by Allaire [1] and others.

Doubly-nonlinear parabolic equations have been dealt with in a number of papers (see e.g. [2, 12, 15] and [29, Chap. III]). It seems that the homogenization of these equations has not yet been addressed. In [28] this author proved existence of a weak solution for a less general model of electromagnetic evolution than that of this paper; that argument was based on a fixed-point technique that apparently does not carry over to the present setting.

1. Parabolic-hyperbolic problem for a nonlinear periodic material

Let Ω be a (possibly unbounded) three-dimensional domain, fix a constant $T > 0$, and set $A_T := A \times]0, T[$ for any $A \subset \mathbb{R}^3$, and $Y := [0, 1[^3$. Let two given functions

$$\varphi : \mathbb{R}^3 \times \Omega \times \mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}, \qquad \vec{\alpha} : \mathbb{R}^3 \times \mathbb{R}^3 \times \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$$

be such that

$$(1.1) \quad \begin{aligned} &\vec{v} \mapsto \varphi(\vec{v}, x, y) \text{ is convex and lower semicontinuous for any } x \text{ and a.a. } y, \\ &x \mapsto \varphi(\vec{v}, x, y) \text{ is piecewise continuous for any } \vec{v} \text{ and a.a. } y, \\ &y \mapsto \varphi(\vec{v}, x, y) \text{ is measurable and } Y\text{-periodic for any } (\vec{v}, x), \\ &\{\vec{v} \in \mathbb{R}^3 : \varphi(\vec{v}, x, y) < +\infty\} \text{ has nonempty interior for any } x \text{ and a.a. } y, \end{aligned}$$

$$(1.2) \quad \begin{aligned} &\vec{v} \mapsto \vec{\alpha}(\vec{v}, \vec{w}, x, y) \text{ is continuous and maximal monotone for any } (\vec{w}, x) \text{ and a.a. } y, \\ &\vec{w} \mapsto \vec{\alpha}(\vec{v}, \vec{w}, x, y) \text{ is uniformly continuous with respect to } \vec{v}, \text{ for any } x \text{ and a.a. } y, \\ &x \mapsto \vec{\alpha}(\vec{v}, \vec{w}, x, y) \text{ is piecewise continuous for any } (\vec{v}, \vec{w}) \text{ and a.a. } y, \\ &y \mapsto \vec{\alpha}(\vec{v}, \vec{w}, x, y) \text{ is measurable and } Y\text{-periodic for any } (\vec{v}, \vec{w}, x). \end{aligned}$$

For any $\varepsilon > 0$ we assume that the following vector fields are also given:

$$(1.3) \qquad \vec{E}^0_\varepsilon \in L^2(\mathbb{R}^3 \setminus \Omega)^3,$$

$$(1.4) \qquad \vec{B}^0_\varepsilon \in L^2(\mathbb{R}^3)^3 \quad \text{such that} \quad \nabla \cdot \vec{B}^0_\varepsilon = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3) \quad (\nabla \cdot := \text{div}),$$

$$(1.5) \qquad \vec{g} \in L^2(\mathbb{R}^3_T)^3 \quad \text{such that} \quad \nabla \cdot \vec{g}(\cdot, t) = 0 \ \text{ in } \mathcal{D}'(\mathbb{R}^3) \text{ for a.a. } t \in]0, T[.$$

We now introduce the weak formulation of an initial-value problem for a system of nonlinear partial differential equations in the whole $\mathbb{R}^3_T$. Here we set $\nabla \times := \text{curl}$, $\chi_\Omega := 1$ in Ω and $\chi_\Omega := 0$ outside Ω.

Problem 1_ε. *Find $\vec{B}_\varepsilon, \vec{H}_\varepsilon, \vec{E}_\varepsilon, \vec{J}_\varepsilon \in L^2(\mathbb{R}^3_T)^3$ such that*

$$(1.6) \qquad \nabla \times \vec{H}_\varepsilon = \vec{J}_\varepsilon + (1 - \chi_\Omega) \frac{\partial \vec{E}_\varepsilon}{\partial t} + \vec{g} \quad \text{in } \mathcal{D}'(\mathbb{R}^3_T)^3,$$

$$(1.7) \qquad \nabla \times \vec{E}_\varepsilon = -\frac{\partial \vec{B}_\varepsilon}{\partial t} \quad \text{in } \mathcal{D}'(\mathbb{R}^3_T)^3,$$

$$(1.8) \qquad \vec{B}_\varepsilon \in \partial\varphi(\vec{H}_\varepsilon, x, x/\varepsilon) \quad \text{a.e. in } \Omega_T, \qquad \vec{B}_\varepsilon = \vec{H}_\varepsilon \quad \text{a.e. in } \mathbb{R}^3_T \setminus \Omega_T,$$

$$(1.9) \qquad \vec{J}_\varepsilon = \vec{\alpha}(\vec{E}_\varepsilon, \vec{H}_\varepsilon, x, x/\varepsilon) \quad \text{a.e. in } \Omega_T, \qquad \vec{J}_\varepsilon = \vec{0} \quad \text{a.e. in } \mathbb{R}^3_T \setminus \Omega_T,$$

$$(1.10) \qquad (1 - \chi_\Omega)\vec{E}_\varepsilon(\cdot, 0) = (1 - \chi_\Omega)\vec{E}^0_\varepsilon, \qquad \vec{B}_\varepsilon(\cdot, 0) = \vec{B}^0_\varepsilon \quad \text{in } \mathcal{D}'(\mathbb{R}^3)^3.$$

Interpretation. This problem is nonlinear parabolic in Ω_T, and linear hyperbolic in $\mathbb{R}^3_T \setminus \Omega_T$. The equations (1.6) and (1.7) correspond to the Ampère–Maxwell and Faraday laws with normalized units. Conduction currents are here confined to Ω; by the so-called *eddy-current approximation*, displacement currents are then neglected in Ω, whereas they are accounted for outside Ω. This model represents electromagnetic evolution in a metal surrounded by an insulator like air. By the properties of electromagnetic radiation, it would not be natural to confine Maxwell equations to Ω and then to prescribe conditions on its boundary [4, 6].

The nonlinear relation (1.8) may represent the constitutive behaviour of a large class of nonhomogeneous magnetic materials. The equation (1.9) generalizes the classical Ohm law $\vec{J}_\varepsilon = \sigma(\vec{E}_\varepsilon + \vec{E}_a)$, σ being the electric conductivity and $\vec{E}_a$ a prescribed applied electromotive force. The nonlinear equation (1.9) may also account for the dependence of the electric conductivity on the magnetic field, as in the classical Hall effect [18; Sect. 21].

Let us set

$$L^2_{\mathrm{rot}}(\mathbb{R}^3)^3 := \{\vec{v} \in L^2(\mathbb{R}^3)^3 : \nabla \times \vec{v} \in L^2(\mathbb{R}^3)^3\},$$
$$L^2_{\mathrm{div}}(\mathbb{R}^3)^3 := \{\vec{v} \in L^2(\mathbb{R}^3)^3 : \nabla \cdot \vec{v} \in L^2(\mathbb{R}^3)\}.$$

It is known that these are Hilbert spaces equipped with the respective graph norms.

THEOREM 1.1 ([31]). *Let the hypotheses* (1.1)–(1.5) *be fulfilled, and assume that*

(1.11) $\exists c > 0, \exists h \in L^1(\Omega) : \varphi(\vec{v}, x, y) \geq c|\vec{v}|^2 + h(x)$ *for any* $(\vec{v}, x)$ *and a.a.* y,

(1.12) $\exists C > 0, \exists \tilde{h} \in L^2(\Omega) : \forall \vec{v} \in \mathbb{R}^3,$

$$\text{if } \vec{w} \in \partial\varphi(\vec{v}, x, y) \text{ then } |\vec{w}| \leq C|\vec{v}| + \tilde{h}(x) \quad \text{for a.a. } (x, y),$$

(1.13) $\vec{v} \mapsto \varphi(\vec{v}, x, y)$ *is strictly convex, for any* x *and a.a.* y,

(1.14) $\exists L > 0, \exists \lambda \in L^2(\mathbb{R}^3) :$

$$|\vec{\alpha}(\vec{v}, \vec{z}, x, y)| \leq L(|\vec{v}| + |\vec{z}|) + \lambda(x) \quad \text{for any } (\vec{v}, x), \text{ for a.a. } y.$$

Then for any $\varepsilon > 0$ *there exists a solution of Problem* 1_ε *such that*

$$\vec{B}_\varepsilon \in L^\infty(0, T; L^2(\mathbb{R}^3)^3) \cap H^1(0, T; (L^2_{\mathrm{rot}}(\mathbb{R}^3)^3)'), \tag{1.15}$$
$$\vec{H}_\varepsilon \in L^\infty(0, T; L^2(\mathbb{R}^3)^3) \cap L^2(0, T; L^2_{\mathrm{rot}}(\Omega)^3), \tag{1.16}$$
$$\vec{E}_\varepsilon \in L^2(\mathbb{R}^3_T)^3 \cap L^\infty(0, T; L^2(\mathbb{R}^3 \setminus \Omega)^3), \tag{1.17}$$
$$\vec{J}_\varepsilon \in L^2(\mathbb{R}^3_T)^3. \tag{1.18}$$

Moreover for any ε *a solution can be selected so that these fields are uniformly bounded with respect to* ε *in these spaces.*

OUTLINE OF PROOF. The argument is based on approximation via an implicit scheme of time-discretization, with time-step T/m ($m \in \mathbb{N}$). At each step the problem is reduced to the minimization of a convex coercive functional, and thus has a solution, that we label by the index m. An energy estimate is then derived by multiplying the time-discretized equations $(1.6)_m$ and $(1.7)_m$ respectively by $\vec{E}_{\varepsilon m}$ and $-\vec{H}_{\varepsilon m}$, and then adding the two equalities. This yields the uniform boundedness of the approximating fields $\vec{B}_{\varepsilon m}$, $\vec{H}_{\varepsilon m}$,

$\vec{E}_{\varepsilon m}$, $\vec{J}_{\varepsilon m}$ in the function spaces (1.15)–(1.18). As $m \to \infty$ along a suitable sequence, these fields then weak* converge to some fields $\vec{B}_\varepsilon$, $\vec{H}_\varepsilon$, $\vec{E}_\varepsilon$, $\vec{J}_\varepsilon$, respectively. Taking the limit in the time-discretized equations, (1.6) and (1.7) are then obtained.

The nonlinear relation (1.8) can be proved via *compensated compactness* [19, 26]. By the hypothesis (1.13) the technique of *compactness by strict convexity* of [27] then yields $\vec{H}_{\varepsilon m} \to \vec{H}_\varepsilon$ strongly in $L^2(\Omega_T)^3$, for any ε. This allows one to derive (1.9) via monotonicity and semicontinuity, without the need of applying any further compactness property. □

Free boundaries. The above hypotheses do not exclude the occurrence of discontinuities in the $\vec{B}_\varepsilon$ vs. $\vec{H}_\varepsilon$ constitutive relation. This may correspond to the occurrence of unknown moving interfaces, or *free boundaries*, across which the fields $\vec{B}_\varepsilon$ and $\vec{E}_\varepsilon$ fulfil discontinuity relations of Rankine–Hugoniot type.

2. TWO-SCALE CONVERGENCE

With a view to the study of the limit as $\varepsilon \to 0$ and to the formulation of a homogenized problem, in this section we review some elements of two-scale convergence mainly along the lines of [1, 22].

We denote by $\mathcal{Y}$ the set $Y := [0, 1[^3$ equipped with the topology of the 3-dimensional torus, and identify any function on $\mathcal{Y}$ with its Y-periodic extension to $\mathbb{R}^3$. We denote by ε a parameter that we assume to tend to zero along a prescribed sequence. For any bounded sequence $\{\vec{u}_\varepsilon\}$ in $L^2(\Omega)^3$ and any $\vec{u} \in L^2(\Omega \times \mathcal{Y})^3$, we say that $\vec{u}_\varepsilon$ *weakly two-scale converges* to $\vec{u}$ in $L^2(\Omega \times \mathcal{Y})^3$, and write $\vec{u}_\varepsilon \underset{2}{\rightharpoonup} \vec{u}$, whenever

$$\int_\Omega \vec{u}_\varepsilon(x) \cdot \vec{v}(x, x/\varepsilon)\, dx \to \iint_{\Omega \times \mathcal{Y}} \vec{u}(x, y) \cdot \vec{v}(x, y)\, dx\, dy \qquad \forall \vec{v} \in \mathcal{D}(\Omega \times \mathcal{Y})^3. \tag{2.1}$$

We similarly define weak* two-scale convergence in $L^\infty(\Omega \times \mathcal{Y})^3$, which we denote by $\vec{u}_\varepsilon \overset{*}{\underset{2}{\rightharpoonup}} \vec{u}$. We also say that $\{\vec{u}_\varepsilon\}$ *strongly two-scale converges* to $\vec{u}$ in $L^2(\Omega \times \mathcal{Y})^3$, and write $\vec{u}_\varepsilon \underset{2}{\to} \vec{u}$, if (2.1) holds and $\|\vec{u}_\varepsilon\|_{L^2(\Omega)^3} \to \|\vec{u}\|_{L^2(\Omega \times \mathcal{Y})^3}$. We denote the standard *one-scale* weak (strong, resp.) convergence by $\rightharpoonup$ ($\to$, resp.). The theorem of the next section is based on Propositions 2.1–2.5 below.

PROPOSITION 2.1 ([1, 22]). *For any bounded sequence $\{\vec{u}_\varepsilon\}$ in $L^2(\Omega)^3$, there exists $\vec{u} \in L^2(\Omega \times \mathcal{Y})^3$ such that, possibly after extracting a subsequence, $\vec{u}_\varepsilon \underset{2}{\rightharpoonup} \vec{u}$ in $L^2(\Omega \times \mathcal{Y})^3$.*

Throughout the whole paper for any function $\vec{v} \in L^1_{\mathrm{loc}}(\mathbb{R}^3 \times \mathcal{Y})^3$ we set

$$\hat{\vec{v}}(x) := \int_{\mathcal{Y}} \vec{v}(x, y)\, dy \qquad \text{for a.a. } x \in \mathbb{R}^3, \tag{2.2}$$

and extend to $\mathbb{R}^3$ with vanishing value any function defined on Ω. For any function of $(x, y) \in \mathbb{R}^3 \times \mathcal{Y}$, we shall denote by ∇_x (∇_y, resp.) the gradient with respect to the first (second, resp.) vector argument.

PROPOSITION 2.2 ([30]). *Let* $\{\vec{u}_\varepsilon\}$ *be a bounded sequence in* $L^2_{\rm rot}(\mathbb{R}^3)^3$ *such that* $\vec{u}_\varepsilon \underset{2}{\rightharpoonup} \vec{u}$ *in* $L^2(\mathbb{R}^3 \times \mathcal{Y})^3$. *Then* $\nabla_y \times \vec{u} = \vec{0}$ *in* $\mathcal{D}'(\mathbb{R}^3 \times \mathcal{Y})^3$. *Moreover there exists* $\vec{u}_1 \in L^2(\mathbb{R}^3; H^1(\mathcal{Y})^3)$ *such that*

$$\hat{\vec{u}}_1 = \vec{0} \quad \text{in } \mathbb{R}^3, \qquad \nabla_y \cdot \vec{u}_1 = 0 \quad \text{a.e. in } \mathbb{R}^3 \times \mathcal{Y}, \tag{2.3}$$

and, possibly after extracting a subsequence,

$$\nabla \times \vec{u}_\varepsilon \underset{2}{\rightharpoonup} \nabla \times \hat{\vec{u}} + \nabla_y \times \vec{u}_1 \quad \text{in } L^2(\mathbb{R}^3 \times \mathcal{Y})^3. \tag{2.4}$$

The same statement holds if in (2.4), $\nabla \times \hat{\vec{u}}$ *is replaced by* $\nabla \times \vec{u}$, *with a different field* $\vec{u}_1$.

Any bounded sequence $\{\vec{u}_\varepsilon\}$ in $L^2_{\rm div}(\mathbb{R}^3)^3$ satisfies an analogous statement with curl and divergence exchanged [30]. In particular, as was already pointed out in [1], in this case $\vec{u}_\varepsilon \underset{2}{\rightharpoonup} \vec{u}$ in $L^2(\mathbb{R}^3 \times \mathcal{Y})^3$ implies $\nabla_y \cdot \vec{u} = 0$ in $\mathcal{D}'(\mathbb{R}^3 \times \mathcal{Y})$.

Now we state a simple extension of the classical *div-curl lemma* of Murat and Tartar [19] to two-scale convergence. We also include time-dependence, for it is in this form that the result is applied to our problem.

PROPOSITION 2.3 ([32]). *Let* $\{\vec{u}_\varepsilon\}$ *be a bounded sequence in* $L^2(0, T; L^2_{\rm rot}(\mathbb{R}^3)^3)$, *and* $\{\vec{w}_\varepsilon\}$ *be a bounded sequence in* $L^2(0, T;\ L^2_{\rm div}(\mathbb{R}^3)^3)$. *If*

$$\exists r, s\ (r > 0) \text{ such that either } \{\vec{u}_\varepsilon\} \text{ or } \{\vec{w}_\varepsilon\} \text{ is bounded in } H^r(0, T; H^{-s}(\mathbb{R}^3)^3), \tag{2.5}$$

$$\vec{u}_\varepsilon \underset{2}{\rightharpoonup} \vec{u}, \quad \vec{w}_\varepsilon \underset{2}{\rightharpoonup} \vec{w} \quad \text{in } L^2(\mathbb{R}^3_T \times \mathcal{Y})^3, \tag{2.6}$$

then

$$\iint_{\mathbb{R}^3_T} \vec{u}_\varepsilon \cdot \vec{w}_\varepsilon \theta\, dx\, dt \to \iiint_{\mathbb{R}^3_T \times \mathcal{Y}} \vec{u} \cdot \vec{w}\, \theta\, dx\, dy\, dt \qquad \forall \theta \in \mathcal{D}([0, T]; \mathcal{D}(\mathbb{R}^3)). \tag{2.7}$$

We also extend to two-scale convergence a property of *compactness by strict convexity* [27; 29, Chap. X].

PROPOSITION 2.4 ([32]). *Let* $p \in]1, +\infty[$, *and let* φ *fulfil* (1.1) *and be such that*

$$\exists c > 0, \exists \vec{w} \in L^{p'}(\Omega), \exists h \in L^1(\Omega): \quad \varphi(\vec{v}, x, y) \ge \vec{w}(x) \cdot \vec{v} + c|\vec{v}|^p + h(x) \quad \forall (\vec{v}, x), \text{ for a.a. } y, \tag{2.8}$$

$$\vec{v} \mapsto \varphi(\vec{v}, x, y) \text{ is strictly convex, for any } x \text{ and a.a. } y. \tag{2.9}$$

For any sequence $\{\vec{u}_\varepsilon\}$ *in* $L^p(\Omega_T)^3$, *if* $\vec{u}_\varepsilon \underset{2}{\rightharpoonup} \vec{u}$ *in* $L^p_{\rm loc}(\Omega_T \times \mathcal{Y})^3$ *and*

$$\iint_{\Omega_T} \varphi_\varepsilon(\vec{u}_\varepsilon(x, t), x, x/\varepsilon)\theta(x, t)\, dx\, dt \to \iiint_{\Omega_T \times \mathcal{Y}} \varphi(\vec{u}(x, y, t), x, y)\theta(x, t)\, dx\, dy\, dt \tag{2.10}$$

for any $\theta \in \mathcal{D}(\Omega_T)$, *then*

$$\vec{u}_\varepsilon \underset{2}{\to} \vec{u} \quad \text{in } L^p_{\rm loc}(\Omega_T; L^p(\mathcal{Y})^3). \tag{2.11}$$

The next result is intended for application to functions that also depend on the coarse-scale variable x. However in constitutive relations usually x plays the role of a parameter, and thus the extension to that setting presents no additional difficulty.

PROPOSITION 2.5 ([32]). *Let $\varphi : \mathbb{R}^3 \times \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ be a (possibly nonconvex) normal integrand. Assume that φ fulfils* (1.11)*, that the same holds for the convex conjugate function φ^*, and set, for all $\vec{\xi}, \vec{\eta} \in \mathbb{R}^3$,*

$$\varphi_0(\vec{\xi}) := \inf\left\{ \int_{\mathcal{Y}} \varphi(\vec{\xi}+\vec{\zeta}, y)\, dy : \vec{\zeta} \in L^2(\mathcal{Y})^3,\ \hat{\vec{\zeta}} = \vec{0},\ \nabla\times\vec{\zeta} = \vec{0} \text{ in } \mathcal{D}'(\mathcal{Y})^3 \right\}, \tag{2.12}$$

$$\psi_0(\vec{\eta}) := \inf\left\{ \int_{\mathcal{Y}} \varphi^*(\vec{\eta}+\vec{\theta}, y)\, dy : \vec{\theta} \in L^2(\mathcal{Y})^3,\ \hat{\vec{\theta}} = \vec{0},\ \nabla\cdot\vec{\theta} = 0 \text{ in } \mathcal{D}'(\mathcal{Y}) \right\}. \tag{2.13}$$

If $\vec{u}, \vec{w} \in L^2(\mathcal{Y})^3$ are such that $\vec{w}(y) \in \partial\varphi(\vec{u}(y), y)$ for a.a. $y \in \mathcal{Y}$ and

$$\nabla\times\vec{u} = \vec{0} \quad \text{in } \mathcal{D}'(\mathcal{Y})^3, \qquad \nabla\cdot\vec{w} = 0 \quad \text{in } \mathcal{D}'(\mathcal{Y}), \tag{2.14}$$

then

$$\hat{\vec{w}} \in \partial\varphi_0(\hat{\vec{u}}), \qquad (\varphi_0)^*(\hat{\vec{w}}) = \psi_0(\hat{\vec{w}}), \tag{2.15}$$

$$\int_{\mathcal{Y}} [\varphi(\vec{u}(y), y) + \varphi^*(\vec{w}(y), y)]\, dy = \varphi_0(\hat{\vec{u}}) + (\varphi_0)^*(\hat{\vec{w}}), \tag{2.16}$$

$$\hat{\vec{u}} \in \partial\psi_0(\hat{\vec{w}}), \qquad (\psi_0)^*(\hat{\vec{u}}) = \varphi_0(\hat{\vec{u}}). \tag{2.17}$$

3. HOMOGENIZATION

Now we let ε tend to zero. First we deal with convergence to a solution of a two-scale problem; we then complete the homogenization procedure by deriving a single-scale problem. Let us set

$$\begin{aligned} W &:= \{\vec{v} \in L^2(\mathcal{Y})^3 : \nabla_y \times \vec{v} = \vec{0} \ \text{ in } \mathcal{D}'(\mathcal{Y})^3\}, \\ V &:= \{\vec{w} \in L^2(\mathcal{Y})^3 : \nabla_y \cdot \vec{w} = 0 \ \text{ in } \mathcal{D}'(\mathcal{Y})\}; \end{aligned} \tag{3.1}$$

these are Banach spaces. Let us assume that φ, $\vec{\alpha}$, $\vec{B}^0_\varepsilon$, $\vec{E}^0_\varepsilon$, $\vec{g}$ fulfil the hypotheses (1.1)–(1.5), and that

$$\vec{B}^0_\varepsilon \underset{2}{\rightharpoonup} \vec{B}^0, \qquad \vec{E}^0_\varepsilon \underset{2}{\rightharpoonup} \vec{E}^0 \qquad \text{in } L^2(\mathbb{R}^3\times\mathcal{Y})^3. \tag{3.2}$$

PROBLEM 2 (two-scale formulation). *Find $\vec{B}, \vec{H}, \vec{E}, \vec{H}_1, \vec{E}_1, \vec{J} \in L^2(\mathbb{R}^3\times\mathcal{Y})^3$ such that*

$$\begin{aligned} &\vec{B} \in L^2(\mathbb{R}^3_T; V) \cap H^1(0,T; L^2(\mathbb{R}^3; W')), \\ &\vec{H}, \vec{E} \in L^2(\mathbb{R}^3_T; W), \qquad \nabla\times\hat{\vec{H}}, \nabla\times\hat{\vec{E}} \in L^2(\mathbb{R}^3_T)^3, \\ &\vec{H}_1, \vec{E}_1 \in L^2(\mathbb{R}^3_T; V), \qquad \vec{J} \in L^2(\mathbb{R}^3_T\times\mathcal{Y})^3, \end{aligned} \tag{3.3}$$

$$\hat{\vec{H}}_1 = \hat{\vec{E}}_1 = \vec{0} \quad \text{a.e. in } \mathbb{R}^3_T, \tag{3.4}$$

$$\nabla \times \hat{\vec{H}} + \nabla_y \times \vec{H}_1 = \vec{J} + \vec{g} + (1 - \chi_\Omega)\frac{\partial \vec{E}}{\partial t} \quad \text{in } \mathcal{D}'(\mathbb{R}^3_T \times \mathcal{Y})^3, \tag{3.5}$$

$$\nabla \times \hat{\vec{E}} + \nabla_y \times \vec{E}_1 = -\frac{\partial \vec{B}}{\partial t} \quad \text{in } \mathcal{D}'(\mathbb{R}^3_T \times \mathcal{Y})^3, \tag{3.6}$$

$$\vec{B} \in \partial\varphi(\vec{H}, x, y) \quad \text{a.e. in } \Omega_T \times \mathcal{Y}, \quad \vec{B} = \vec{H} \quad \text{a.e. in } (\mathbb{R}^3_T \setminus \Omega_T) \times \mathcal{Y}, \tag{3.7}$$

$$\vec{J} = \vec{\alpha}(\vec{E}, \vec{H}, x, y) \quad \text{a.e. in } \Omega_T \times \mathcal{Y}, \quad \vec{J} = \vec{0} \quad \text{a.e. in } (\mathbb{R}^3_T \setminus \Omega_T) \times \mathcal{Y}, \tag{3.8}$$

$$\vec{E}(\cdot, 0) = \vec{E}^0 \quad \text{in } \mathcal{D}'(\Omega \times \mathcal{Y})^3, \quad \vec{B}(\cdot, 0) = \vec{B}^0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3 \times \mathcal{Y})^3. \tag{3.9}$$

THEOREM 3.1 ([31]). *Assume that the hypotheses* (1.1)–(1.5), (1.11)–(1.14), (3.2) *are fulfilled. For any $\varepsilon > 0$, let $(\vec{B}_\varepsilon, \vec{H}_\varepsilon, \vec{E}_\varepsilon, \vec{J}_\varepsilon)$ be a solution of Problem 1_ε, and assume that this family is uniformly bounded with respect to ε in the spaces* (1.15)–(1.18) *(such a family exists by Theorem* 1.1*). Then there exist $\vec{B}, \vec{H}, \vec{E}, \vec{J}$ such that, as $\varepsilon \to 0$ along a suitable sequence,*

$$\vec{B}_\varepsilon \overset{*}{\underset{2}{\rightharpoonup}} \vec{B}, \quad \vec{H}_\varepsilon \overset{*}{\underset{2}{\rightharpoonup}} \vec{H} \quad \text{in } L^\infty(0, T; L^2(\mathbb{R}^3 \times \mathcal{Y})^3), \tag{3.10}$$

$$\vec{E}_\varepsilon \overset{*}{\underset{2}{\rightharpoonup}} \vec{E} \quad \text{in } L^2(\mathbb{R}^3_T \times \mathcal{Y})^3 \cap L^\infty(0, T; L^2((\mathbb{R}^3 \setminus \Omega) \times \mathcal{Y})^3), \tag{3.11}$$

$$\vec{J}_\varepsilon \underset{2}{\rightharpoonup} \vec{J} \quad \text{in } L^2(\mathbb{R}^3_T \times \mathcal{Y})^3. \tag{3.12}$$

Moreover, there exist $\vec{H}_1, \vec{E}_1$ such that $(\vec{B}, \vec{H}, \vec{E}, \vec{J}, \vec{H}_1, \vec{E}_1)$ is a solution of Problem 2.

OUTLINE OF PROOF. By Proposition 2.1 the energy estimate yields (3.10)–(3.12) as ε tends to zero along a suitable subsequence. By Proposition 2.2 the equations (3.5) and (3.6) then follow. (3.7) can be derived from (1.8) via two-scale compensated compactness (cf. Proposition 2.3). By Proposition 2.4 one can then show that $\vec{H}_\varepsilon \underset{2}{\to} \vec{H}$ strongly in $L^2(\mathbb{R}^3_T \times \mathcal{Y})^3$. Finally, (3.8) follows by passing to the two-scale limit in (1.9). □

Elimination of the y-dependence. Let us now assume that $\vec{\alpha}$ is cyclically monotone with respect to the first argument; that is, denoting by ∂ the subdifferential operator with respect to the first argument,

$$\vec{\alpha}(\vec{J}, \vec{E}, x, y) = \partial\varrho(\vec{J}, \vec{E}, x, y) \quad \forall \vec{J}, \vec{E} \in \mathbb{R}^3, \ \forall x \in \mathbb{R}^3, \ \forall y \in \mathcal{Y}, \tag{3.13}$$

for some function $\varrho : \mathbb{R}^3 \times \mathbb{R}^3 \times \Omega \times \mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}$ such that

$$\begin{aligned} &\vec{v} \mapsto \varrho(\vec{v}, \vec{w}, x, y) \text{ is convex and lower semicontinuous for any } \vec{w}, x \text{ and a.a. } y,\\ &\vec{w} \mapsto \varrho(\vec{v}, \vec{w}, x, y) \text{ is continuous for any } \vec{v}, x \text{ and a.a. } y,\\ &x \mapsto \varrho(\vec{v}, \vec{w}, x, y) \text{ is piecewise continuous for any } \vec{v}, \vec{w} \text{ and a.a. } y,\\ &y \mapsto \varrho(\vec{v}, \vec{w}, x, y) \text{ is measurable and } Y\text{-periodic for any } (\vec{v}, \vec{w}, x),\\ &\{\vec{v} \in \mathbb{R}^3 : \varrho(\vec{v}, \vec{w}, x, y) < +\infty\} \text{ has nonempty interior for any } x \text{ and a.a. } y. \end{aligned} \tag{3.14}$$

Let us then set (cf. (2.12)), for all $\vec{H}, \vec{E} \in \mathbb{R}^3$ and a.e. $x \in \Omega$,

$$
\begin{aligned}
\varphi_0(\vec{H}, x) &:= \inf_{\vec{\eta} \in W,\, \hat{\vec{\eta}} = \vec{0}} \int_{\mathcal{Y}} \varphi(\vec{H} + \vec{\eta}(y), x, y)\, dy, \\
\varrho_0(\vec{E}, \vec{H}, x) &:= \inf_{\vec{\eta} \in W,\, \hat{\vec{\eta}} = \vec{0}} \int_{\mathcal{Y}} \varrho(\vec{E} + \vec{\eta}(y), \vec{H}, x, y)\, dy.
\end{aligned}
\tag{3.15}
$$

We are now able to introduce a single-scale homogenized problem. We anticipate the relation with the solution of Problem 2 by appending the hat ($\hat{}$) to these purely coarse-scale-dependent fields (cf. (2.2)).

PROBLEM 3 (single-scale formulation). *Find $\hat{\vec{B}}, \hat{\vec{H}}, \hat{\vec{E}}, \hat{\vec{J}} \in L^2(\mathbb{R}^3_T)^3$ such that*

$$\nabla \times \hat{\vec{H}}, \nabla \times \hat{\vec{E}} \in L^2(\mathbb{R}^3_T)^3, \tag{3.16}$$

$$\nabla \times \hat{\vec{H}} = \hat{\vec{J}} + (1 - \chi_\Omega) \frac{\partial \hat{\vec{E}}}{\partial t} + \vec{g} \quad \textit{in } \mathcal{D}'(\mathbb{R}^3_T)^3, \tag{3.17}$$

$$\nabla \times \hat{\vec{E}} = -\frac{\partial \hat{\vec{B}}}{\partial t} \quad \textit{in } \mathcal{D}'(\mathbb{R}^3_T)^3, \tag{3.18}$$

$$\hat{\vec{B}} \in \partial\varphi_0(\hat{\vec{H}}, x) \quad \textit{a.e. in } \Omega_T, \qquad \hat{\vec{B}} = \hat{\vec{H}} \quad \textit{a.e. in } \mathbb{R}^3_T \setminus \Omega_T, \tag{3.19}$$

$$\hat{\vec{J}} \in \partial\varrho_0(\hat{\vec{E}}, \hat{\vec{H}}, x) \quad \textit{a.e. in } \Omega_T, \qquad \hat{\vec{J}} = \vec{0} \quad \textit{a.e. in } \mathbb{R}^3_T \setminus \Omega_T, \tag{3.20}$$

$$\hat{\vec{E}}(\cdot, 0) = \vec{E}^0 \quad \textit{in } \mathcal{D}'(\Omega)^3, \qquad \hat{\vec{B}}(\cdot, 0) = \vec{B}^0 \quad \textit{in } \mathcal{D}'(\mathbb{R}^3)^3. \tag{3.21}$$

THEOREM 3.2 ([31]). *Assume the hypotheses of Theorem* 3.1 *are satisfied, and let $(\vec{B}, \vec{H}, \vec{E}, \vec{H}_1, \vec{E}_1, \vec{J})$ be a solution of Problem* 2 *(that exists by Theorem* 3.1*). The averaged fields $\hat{\vec{B}}, \hat{\vec{H}}, \hat{\vec{E}}, \hat{\vec{J}}$ (cf.* (2.2)*) then solve Problem* 3.

OUTLINE OF PROOF. It is clear that the fine-scale equations (3.5) and (3.6) imply the corresponding fine-scale equations (3.17) and (3.18). By Proposition 2.5 the relations (3.7) and (3.8) yield (3.19) and (3.20). □

4. HOMOGENIZATION OF A MODEL OF PHASE TRANSITIONS

The above approach can also be applied to other models. Here we outline the homogenization of a doubly-nonlinear Stefan-type model of phase transitions for a nonhomogeneous material, in which the heat flux depends nonlinearly on the temperature gradient and on the temperature itself.

Let $\varphi : \mathbb{R} \times \Omega \times \mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}$ and $\vec{\alpha} : \mathbb{R}^3 \times \mathbb{R} \times \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ have properties analogous to (1.1) and (1.2). For any $\varepsilon > 0$ let us consider the following initial- and boundary-value problem:

$$\text{(4.1)} \qquad \begin{cases} \dfrac{\partial w_\varepsilon}{\partial t} + \nabla \cdot \vec{q}_\varepsilon = f & \text{in } \mathcal{D}'(\Omega_T), \\ w_\varepsilon \in \partial\varphi(u_\varepsilon, x, x/\varepsilon) & \text{a.e. in } \Omega_T, \\ \vec{q}_\varepsilon = -\vec{\alpha}(\nabla u, u_\varepsilon, x, x/\varepsilon) & \text{a.e. in } \Omega_T, \\ w_\varepsilon(\cdot, 0) = w^0 & \text{a.e. in } \Omega, \\ u_\varepsilon = \tilde{u} & \text{on } \partial\Omega \times]0, T[, \end{cases}$$

for given $f, w^0, \tilde{u}$. If we interpret w_ε as the density of internal energy, u_ε as the temperature, $\vec{q}_\varepsilon$ as the heat flux, and f as the density of a distributed heat source or sink, this is a model of nonlinear heat diffusion. It is well known that a multivalued $\partial\varphi$ accounts for phase transitions (cf. e.g. [16, 23, 29]). At variance with the model of electromagnetism of Sect. 1, here it is more natural to prescribe conditions on the boundary of Ω, rather than deal with heat propagation in the whole $\mathbb{R}^3$.

We consider the nonlinear Fourier conduction law $(4.1)_3$, although for many phemomena a linear dependence of $\vec{q}_\varepsilon$ on ∇u_ε is physically acceptable. From the point of view of applications it is more relevant that this relation allows for a dependence of the heat flux on the temperature.

Existence of a solution for the system (4.1) and convergence to a homogenized problem can be proved as for Theorems 1.1, 3.1, 3.2. Under assumptions analogous to (1.1), (1.11)–(1.14) and natural hypotheses on the data $f, w^0, \tilde{u}$, for any $\varepsilon > 0$ there exists a weak solution of (4.1) such that

$$\text{(4.2)} \qquad \begin{aligned} & w_\varepsilon \in L^\infty(0, T; L^2(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)), \\ & u_\varepsilon \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\ & \vec{q}_\varepsilon \in L^2(\Omega_T)^3; \end{aligned}$$

moreover these functions are uniformly bounded with respect to ε in the respective spaces.

For any $\varepsilon > 0$ here also the approximation may be performed by implicit time-discretization. The proof of convergence is even simpler than that of Theorem 1.1, for here one can estimate the approximate solution $u_{\varepsilon m}$ in $L^2(0, T; H^1(\Omega))$, so that one can then take advantage of the compactness of the inclusion $H^1(\Omega) \subset L^2_{\text{loc}}(\Omega)$. One can thus pass to the limit as $m \to \infty$ in the $w_{\varepsilon m}$ vs. $u_{\varepsilon m}$ relation without using compensated compactness. The strong convergence of $u_{\varepsilon m}$ in $L^2(\Omega_T)$ can then be proved via the strict convexity of the functional φ, and this allows one to pass to the limit in the nonlinear Fourier law.

Homogenized problem. We then let the space-period ε tend to zero, getting a two-scale formulation similar to Problem 3. More specifically, there exists a set $(w, u, u_1, \vec{q}, \vec{q}_1)$ of functions of the coarse-scale variable $x \in \Omega$, of the fine-scale variable $y \in \mathcal{Y}$ and of the time variable t, such that, using the notation (2.2),

$$
(4.3) \qquad \begin{cases}
\dfrac{\partial w}{\partial t} + \nabla \cdot \hat{\vec{q}} + \nabla_y \cdot \vec{q}_1 = f & \text{in } \mathcal{D}'(\Omega_T \times \mathcal{Y}), \\
w \in \partial\varphi(u, x, y) & \text{a.e. in } \Omega_T \times \mathcal{Y}, \\
\vec{q} = -\vec{\alpha}(\nabla u + \nabla_y u_1, u, x, y) & \text{a.e. in } \Omega_T \times \mathcal{Y}, \\
\nabla_y \times \vec{q}_1 = \vec{0} & \text{a.e. in } \Omega_T \times \mathcal{Y}, \\
\hat{u}_1 = 0, \quad \hat{\vec{q}}_1 = \vec{0} & \text{a.e. in } \Omega_T, \\
w(\cdot, \cdot, 0) = w^0 & \text{a.e. in } \Omega \times \mathcal{Y}, \\
u = \tilde{u} & \text{on } \partial\Omega \times]0, T[.
\end{cases}
$$

There is a relevant difference between this model and Problem 3: because of the $H^1(\Omega)$-regularity, the two-scale limit u does not depend on y, thus $\hat{u} = u$. This excludes occurrence of a fine-scale structure for the temperature field.

Let us now assume that $\vec{\alpha}$ is cyclically monotone in the first argument, as in (3.13). Defining ϱ_0 as in (3.15) and setting $\bar{\varphi}(\xi, x) := \int_{\mathcal{Y}} \varphi(\xi, x, y)\, dy$ for any $(\xi, x) \in \mathbb{R}^3 \times \Omega$, we then get the following coarse-scale problem:

$$
(4.4) \qquad \begin{cases}
\dfrac{\partial \hat{w}}{\partial t} + \nabla \cdot \hat{\vec{q}} = f & \text{in } \mathcal{D}'(\Omega_T), \\
\hat{w} \in \partial\bar{\varphi}(u, x) & \text{a.e. in } \Omega_T, \\
\hat{\vec{q}} = -\partial\varrho_0(\nabla u, u, x) & \text{a.e. in } \Omega_T, \\
\hat{w}(\cdot, 0) = \hat{w}^0 & \text{a.e. in } \Omega, \\
u = \tilde{u} & \text{on } \partial\Omega \times]0, T[,
\end{cases}
$$

in which the fine-scale variable y does not occur at all. (The form of $(4.4)_2$ is different from that of $(3.19)_1$ because $\hat{u} = u$.) This completes the homogenization procedure.

REMARK. This setting generalizes the classical (multi-dimensional) Stefan problem. If $\partial\varrho_0$ is linear we retrieve the results of [14] in a more general setting, in particular with a temperature- and space-dependent conductivity. If we start from the Fourier law for a composite material, $\vec{q}_\varepsilon = -k(u_\varepsilon, x, x/\varepsilon)\nabla u_\varepsilon$ (k being a Carathéodory function), and define a 3×3 tensor $\hat{k}(u, x)$ via a suitable elliptic cell problem, the homogenized relation $(4.4)_3$ reads

$$
(4.5) \qquad \hat{\vec{q}} = -\hat{k}(u, x)\nabla u \quad \text{a.e. in } \Omega_T.
$$

The above considerations may be extended in several directions; for instance, inserting a relaxation dynamics in the phase evolution equation $(4.1)_2$ and/or in the nonlinear Fourier conduction law $(4.1)_3$ does not raise difficulties. An analogous remark applies to the above electromagnetic problem.

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