Rend. Lincei Mat. Appl. 17 (2006), 243–255

Partial differential equations. — *Weak convergence, local bifurcations and uniqueness theorems,* by E. NORMAN DANCER.

ABSTRACT. — We show how weak convergence techniques can be used to improve classical theorems on local bifurcation and on uniqueness when a parameter is large.

KEY WORDS: Nonlinear; elliptic; partial differential equations.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 35J65.

The purpose of this paper is to show how weak convergence ideas can be used to considerably improve a number of known bifurcation results and uniqueness theorems. We improve the theorems by weakening the assumptions. We improve considerably some ideas of Katriel [19] which were for some particular problems in the one-dimensional case. The main idea is to show that certain terms converge weakly and use this to pass to the limit. The main lemma proves weak convergence in a non-obvious case.

In particular, we weaken the assumptions in classical results that the solutions of

$$\Delta u = \lambda f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

bifurcating from zero (or from infinity) at a simple eigenvalue form a single arc. (In the case of bifurcation from zero, we assume $f(0) = 0$, $f'(0) = 1$.)

In addition, we improve results on when the same equation has a unique positive solution for large λ.

We also improve some results of mine on multiple solutions of Ambrosetti–Prodi type problems for jumping nonlinearities.

It is clear that weak convergence ideas should have other applications.

1. WEAK CONVERGENCE

We start with the key lemma.

LEMMA 1. *Assume that Ω has finite measure, $v \in C^1(\overline{\Omega})$ where $T = \{x \in \Omega : \nabla v(x) = 0\}$ has measure zero, that $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz and that $y^{-1} f(y) \to 0$ as $|y| \to \infty$. In addition assume that $w_n \to 0$ in $C^1(\overline{\Omega})$ and $\alpha_n \to \infty$ as $n \to \infty$. Then $f'(\alpha_n(v + w_n)) \rightharpoonup 0$ weakly in $L^p(\Omega)$ for $1 < p < \infty$.*

REMARK. There is a problem in the definition of $f'(\alpha_n(v + w_n))$ in that it is not defined a.e. on the set where $\nabla(v + w_n) = 0$ and f' is not differentiable at $\alpha_n(v + w_n)$. However, by our assumptions on T, this set has small measure if n is large and hence it only makes a small contribution to the integral for n large. Thus it will not affect the conclusion. (Alternatively, we could define $f'(\alpha_n(v+w_n))$ to be zero on this set.) As another alternative in our applications, $\nabla(v+w_n)$ only vanishes on a set of measure zero and hence $f'(\alpha_n(v+w_n))$ is defined a.e.

PROOF. STEP 1.

$$\int_\Omega f'(\alpha_n(v+w_n))\frac{\partial}{\partial x_i}(v+w_n)\phi \to 0 \quad \text{as } n\to\infty \text{ if } \phi\in C_0^\infty(\Omega).$$

This follows since by integration by parts the integral becomes

$$-\alpha_n^{-1}\int f(\alpha_n(v+w_n))\frac{\partial\phi}{\partial x_i}. \tag{1}$$

Here we have used the fact that the weak ith partial derivative of $f(\alpha_n(v+w_n))$ is $\alpha_n f'(\alpha_n(v+w_n)\frac{\partial}{\partial x_i}(v+w_n)$ (cf. [16]). Since $|f(y)|\le\epsilon|y|+M_\epsilon$ on $\mathbb{R}$ if $\epsilon>0$, we see that

$$|\alpha_n^{-1}f(\alpha_n(v+w_n))|\le\epsilon(|v|+|w_n|)+\alpha_n^{-1}M_\epsilon$$

and hence this term tends to zero uniformly on $\overline{\Omega}$. Thus our claim follows.

STEP 2.

$$\int_\Omega f'(\alpha_n(v+w_n))\frac{\partial v}{\partial x_i}\phi\to 0 \quad \text{as } n\to\infty \text{ if } \phi\in C_0^\infty(\Omega).$$

This follows trivially from Step 1 since f' is bounded on $\mathbb{R}$ and $v+w_n-v\to 0$ in $C^1(\Omega)$ as $n\to\infty$. Since $f'(\alpha_n(v+w_n))\frac{\partial v}{\partial x_i}$ is bounded, we see by density the result is still true if $\phi\in L^2(\Omega)$.

STEP 3. Note that f' is bounded and thus to prove weak convergence, it suffices to prove that

$$\int_\Omega f'(\alpha_n(v+w_n)\phi\to 0 \quad \text{as } n\to\infty$$

for ϕ in a dense subset of $L^q(\Omega)$. Here $q^{-1}+p^{-1}=1$. Now if M is a closed subset of measure zero of Ω, the set of $L^q(\Omega)$ functions vanishing in a neighbourhood of $\partial\Omega\cup M$ is dense in $L^q(\Omega)$. This follows easily from the dominated convergence theorem, since if $M_n=\{x\in\Omega : d(x,M)\ge 1/n, d(x,\partial\Omega)\ge 1/n\}$ the characteristic function of M_n, $\chi_{(M_n)}$, satisfies $\chi_{(M_n)}\to 1$ a.e. on Ω as $n\to\infty$ (because M has measure zero). (We replace ϕ by $\phi\chi_{(M_n)}$.) We set $M=\{x\in\Omega : \nabla v(x)=0\}$. This has measure zero.

If $x\in\Omega\backslash M$, there is a neighbourhood N_x of x such that some partial derivative $\partial v/\partial x_i$ is non-zero on $\overline{N}_x$ and thus $|\partial v/\partial x_i|$ has a positive lower bound on $\overline{N}_x$. If $\phi\in L^q(\Omega)$ vanishes on a neighbourhood of $M\cup\partial\Omega$, then a finite number of N_{x_i}'s cover the support of ϕ (by compactness). Hence we can write

$$\phi=\sum_{j=1}^k\frac{\partial v}{\partial x_{i(j)}}\left(\left(\frac{\partial v}{\partial x_{i(j)}}\right)^{-1}\phi\chi_{(N_j)}\right).$$

Since $(\frac{\partial v}{\partial x_{i(j)}})^{-1}\phi\chi_{(N_{x_j})} \in L^q(\Omega)$, the result now follows from the last remark in Step 2.

COROLLARY. *Assume the assumptions of Lemma* 1 *hold for both* w_n *and* $\widetilde{w}_n$ *and* $w_n - \widetilde{w}_n \neq 0$ *a.e. on* Ω. *Then* $\int_0^1 f'(\alpha_n(v + tw_n + (1-t)\widetilde{w}_n))\,dt \rightharpoonup 0$ *weakly in* $L^p(\Omega)$ *as* $n \to \infty$.

REMARK. f' is measurable and bounded and the integral is defined for almost all x. It would suffice to assume that the set $\{x \in \Omega : w_n(x) = \widetilde{w}_n(x),\ f$ is not differentiable at $\alpha_n(v + w_n)(x)\}$ has measure zero.

PROOF. If $\phi \in C_0^\infty(\Omega)$,

$$\left(\int_0^1 f'(\alpha_n(v + tw_n + (1-t)\widetilde{w}_n))\,dt, \phi\right) = \int_0^1 (f'(\alpha_n(v + tw_n + (1-t)\widetilde{w}_n)), \phi)\,dt.$$

Now the integrand on the right hand side is bounded uniformly in t and tends to zero for each t as $n \to \infty$ by Lemma 1. Thus the dominated convergence theorem implies the right hand side tends to zero and hence so does the left hand side. As before, this suffices to prove the result.

REMARKS. 1. It would suffice that w_n converges to zero in $W^{1,1}_{\mathrm{loc}}(\Omega)$ as $n \to \infty$ if f is C^1.

2. There is an obvious variant where $\alpha_n \to 0$ as $n \to \infty$ and we assume that f is Lipschitz and $f(0) = f'(0) = 0$. The proof is essentially the same.

3. If $v = c \neq 0$ on a set S of positive measure, then $\nabla v = 0$ a.e. on S (by [26]) and thus our assumption fails. Moreover, it is easy to see that the conclusion fails in this case unless $f'(y) \to 0$ as $y \to \infty$. We suspect that the result fails for some f of class C^1 whenever T has positive measure.

4. Our methods also apply to maps $f(x, u(x))$ on Ω provided f is C^1, $|f'_y(x, y)| \leq K$ on $\Omega \times \mathbb{R}$ and $y^{-1}(|f_x(x, y)| + |f(x, y)|) \to 0$ as $|y| \to \infty$ uniformly in x. (It is clear that the condition that f is C^1 can be weakened and that our result is not best possible because it is a consequence of Lemma 1 that if $a \in L^\infty(\Omega)$, then $a(x)f'(\alpha(v + w_n)) \rightharpoonup 0$ weakly in $L^q(\Omega)$ if the assumptions of Lemma 1 hold.

5. Our argument in fact implies that $f'(\alpha_n(v + w_n)) \rightharpoonup 0$ in $L^q(\Omega)$ if and only if $f'(\alpha_n(v + w_n)) \rightharpoonup 0$ in $L^q(T)$.

6. We discuss briefly when $m(T) = 0$ in the important case where v is a non-trivial solution of a linear elliptic equation

$$\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + b_j\frac{\partial u}{\partial x_j} + r(x)u = 0 \quad \text{in } \Omega.$$

We discuss other cases in later sections. Provided the coefficients are regular enough to ensure that $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ except for possibly a set Z of measure zero, then T having positive measure implies that all the second (generalized) partial derivatives of u are zero a.e. on T (cf. [26]) and hence $r(x)u = 0$ a.e. on T provided a_{ij} are locally Lipschitz a.e. on T. On the other hand, under reasonable assumptions (cf. [2], [17] and [8]), v can only vanish on a set of measure zero. Thus, we see that under quite weak assumptions r must vanish on a subset of T of positive measure. Provided r is real analytic in Ω, except possibly for a singular set Z_1 of measure zero which does not disconnect Ω, it follows that $r \equiv 0$. In this case, one can frequently deduce by maximum principles that v is constant (for example for Neumann or Dirichlet or Robin boundary conditions). Thus, we see that in many cases where T has positive measure, v is constant. Note on the other hand that it is easy to construct examples where $m(T) > 0$ and the coefficients are all C^∞ functions.

7. It is not difficult to choose f Lipschitz such that $|f'(y)| = 1$ a.e. and $f(y) \to C$ as $y \to \infty$. In this case, it is easy to see that $f'(\alpha_n(v + w_n))$ does not converge strongly to 0 in any $L^p(\Omega)$ as $n \to \infty$. We can obtain examples with f C^1 by rounding off the corners of f carefully. On the other hand if $f' \in L^1$ (for example if f is eventually monotone), it is not difficult to use the coarea formula much as is Schaaf and Schmitt [24] (but rather more locally) to show that strong convergence holds in $L^1(\Omega)$ and thus in $L^p(\Omega)$ for $1 < p < \infty$. (Note that the contribution near the critical point is small and $f' \in L^\infty$.)

8. Under appropriate hypotheses, we could prove similar results for $f'(\frac{\partial}{\partial x_i}(\alpha_n(v + w_n)))$.

LEMMA 2. *Assume the hypotheses of Lemma* 1 *hold,* $a \in L^\infty(\Omega)$ *and* $-\Delta - a(x)I$ *is invertible on* $\dot{W}^{1,2}(\Omega)$. *Then the operator* $-\Delta - (a(x) + f'(\alpha_n(v + w_n)))I$ *is invertible on* $\dot{W}^{1,2}(\Omega)$ *for large n and the two operators have the same number of negative eigenvalues.*

PROOF. If

$$-\Delta\psi_n = (a(x) + f'(\alpha_n(v + w_n)))\psi_n \tag{2}$$

where $\psi_n \in \dot{W}^{1,2}(\Omega)$ and $\|\psi_n\|_2 = 1$ for all n, then standard estimates ensure that ψ_n is bounded in $\dot{W}^{1,2}(\Omega)$ (since f' is uniformly bounded) and thus a subsequence converges weakly in $\dot{W}^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$ to ψ where $\|\psi\|_2 = 1$. Now, if $\phi \in C_0^\infty(\Omega)$, then

$$(f'(\alpha_n(v + w_n))\psi_n, \phi) = (f'(\alpha_n(v + w_n)), \psi_n\phi) \to 0 \quad \text{as } n \to \infty$$

since $\psi_n\phi \to \psi\phi$ strongly in $L^2(\Omega)$ as $n \to \infty$ and $f'(\alpha_n(v + w_n)) \rightharpoonup 0$ weakly in $L^2(\Omega)$ as $n \to \infty$ by Lemma 1. Thus, we can pass to the weak limit in (2) and deduce that $-\Delta\psi = a(x)\psi$, $\|\psi\|_2 = 1$, which contradicts our assumptions. Since we can use the same argument to prove that $-\Delta - (a + tf'(\alpha_n(v + w_n)))I$ is invertible for large n uniformly in t for $t \in [0, 1]$, we deduce that for large n, $-\Delta - a$ and $-\Delta - (a + f'(\alpha_n(v + w_n)))I$ have the same number of negative eigenvalues counting multiplicity, as required.

REMARKS. 1. If M is a closed subspace of $\dot{W}^{1,2}(\Omega)$ such that $\psi_n \in M$ for a sequence of large n, then our proof shows that there exists $\psi \in M \cap \dot{W}^{1,2}(\Omega)$, $\psi \neq 0$, such that $-\Delta\psi = a(x)\psi$, that is, $-\Delta\psi - aI$ has an eigenfunction in M corresponding to the eigenvalue zero.

2. It is possible to allow perturbations which are quite singular near $\partial\Omega$. We do this in §3 and the idea can be readily generalized. That weak convergence can be used is also implicit in work of Daners [16].

2. Application to bifurcation theorems

In this section, we show that our ideas can be used to weaken the assumptions in some classical bifurcation results. We only look at the simplest cases.

We first consider bifurcation from zero. Assume that $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, $f(0) = 0$, $f'(0)$ exists and $f'(0) = 1$. Let $g(y) = f(y) - y$. We consider the small non-zero solutions of

$$(3) \qquad \begin{aligned} -\Delta u &= \lambda f(u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

with λ near λ_i where λ_i is a simple eigenvalue of $-\Delta$ on Ω for Dirichlet boundary conditions. Here Ω is a bounded domain with smooth boundary. Let $\mathcal{D}$ denote the non-trivial solutions of (3) considered as a subset of $W^{2,p}(\Omega) \times \mathbb{R}$ where $p > \frac{1}{2}N$, and let h_i be a normalized eigenfunction of $-\Delta$ corresponding to λ_i. Let N be a closed complement to span h_i in $W^{2,p}(\Omega)$.

THEOREM 1. *Under the above assumptions, there is a neighbourhood* $\hat{T}$ *of* $(0, \lambda_i)$ *in* $\mathcal{D} \cup \{(0, \lambda_i)\}$, $\epsilon > 0$, *and continuous functions* $v : (-\epsilon, \epsilon) \to N$ *and* $\lambda : (-\epsilon, \epsilon) \to \mathbb{R}$ *such that* $v(0) = 0$, $\lambda(0) = \lambda_i$ *and* $\hat{T} = \{(\alpha(h + v), \lambda(\alpha)) : |\alpha| < \epsilon\}$.

REMARK. Thus the solutions form a continuous arc. This is well known [6] if f is C^1 near zero and $f'(0) \neq 0$. (Note that if $f'(0) \neq 0$, we can rescale so that $f'(0) = 1$.)

PROOF. By a standard degree argument (cf. [22]), for each small non-zero α, there is a $\lambda(\alpha)$ near λ_i and $v(\alpha) \in N$ small such that $(\alpha(h_i + v(\alpha)), \lambda(\alpha)) \in \mathcal{D}$ and in fact these form continua. Moreover, any solution in $\mathcal{D}$ near $(0, \lambda_i)$ has this form. It suffices to prove the uniqueness of $(v(\alpha), \lambda(\alpha))$ because a simple compactness argument then implies their continuity. If not, we have two solutions $(\widetilde{v}_1, \widetilde{\lambda}_1)$, $(\widetilde{v}_2, \widetilde{\lambda}_2)$ of (3) such that $\widetilde{v}_1 - \widetilde{v}_2 \in N$, $\widetilde{v}_i$ are small and $\widetilde{\lambda}_i$ are near λ_0. Now

$$(4) \qquad \begin{aligned} -\Delta(\widetilde{v}_1 - \widetilde{v}_2) &= \widetilde{\lambda}_1(f(\widetilde{v}_1) - f(\widetilde{v}_2)) + (\widetilde{\lambda}_1 - \widetilde{\lambda}_2) f(\widetilde{v}_2) \\ &= \widetilde{\lambda}_1(\widetilde{v}_1 - \widetilde{v}_2) \int_0^1 f'(t\widetilde{v}_1 + (1-t)\widetilde{v}_2)\, dt + (\widetilde{\lambda}_1 - \widetilde{\lambda}_2)(\widetilde{v}_2 + g(\widetilde{v}_2)). \end{aligned}$$

There is a technical point on the existence of the integral which we return to at the end of the proof.

We will deduce from this that

$$(5) \qquad |\alpha|^{-1} \|\widetilde{v}_1 - \widetilde{v}_2\|_{2,p} = o(|\widetilde{\lambda}_1 - \widetilde{\lambda}_2|).$$

We do this in two steps. If we recall that, for all the small solutions, $u = \alpha(h + v)$ where v is small and $v \in N$, we see that $\widetilde{z} \equiv \int_0^1 f'(t\widetilde{v}_1 + (1-t)\widetilde{v}_2)\, dt$ must converge weakly to one in $L^p(\Omega)$ as $\alpha \to 0$ by the corollary to Lemma 1 (using Remark 2 after Lemma 1). It follows easily from this by a simple compactness argument (and the ideas in the proof of Lemma 2) that there exists $k > 0$ such that $\|-\Delta w - \widetilde{\lambda}_1 \widetilde{z} w\|_p \geq k \|w\|_{2,p}$ whenever α is small, $w \in N$ and $w = 0$ on $\partial\Omega$. Now $\widetilde{v}_2 + g(\widetilde{v}_2) = \alpha(h_i + v(\alpha)) + g(\alpha(h_i + v(\alpha)))$. Since

$g'(0) = 0$ and $Ph_i = 0$ and $v(\alpha)$ is small, we easily see that $P(\widetilde{v}_2 + g(\widetilde{v}_2))$ is $o(\alpha)$, where P is the projection with kernel spanned by h_i and range N. Hence the claim (5) follows easily from (4). To prove an estimate the other way, we take the scalar product of (4) with $\alpha^{-1}h_i$. We find

$$\text{(6)} \qquad \widetilde{\lambda}_1 \int_0^1 \langle (f'(t\widetilde{v}_1 + (1-t)\widetilde{v}_2) - \lambda_i)h_i, \alpha^{-1}(\widetilde{v}_1 - \widetilde{v}_2)\rangle\, dt$$
$$= (\widetilde{\lambda}_1 - \widetilde{\lambda}_2)\langle \alpha^{-1}(\widetilde{v}_1 + g(\widetilde{v}_1)), h_i\rangle.$$

Now on the left hand side f' is bounded near the origin so we can easily bound the left hand side by $K|\alpha^{-1}|\,\|\widetilde{v}_1 - \widetilde{v}_2\|_2 \leq K|\alpha|^{-1}\|\widetilde{v}_1 - \widetilde{v}_1\|_{2,p}$. On the other hand, we have $\alpha^{-1}\langle \widetilde{v}_1 + g(\widetilde{v}_1), h_i\rangle = \langle h_i, h_i\rangle + o(1)$ since $g'(0) = 0$ and $\widetilde{v}_1 = \alpha h_i + o(\alpha)$. Hence (6) implies that $|\widetilde{\lambda}_1 - \widetilde{\lambda}_2| \leq K|\alpha|^{-1}\|\widetilde{v}_1 - \widetilde{v}_2\|_{2,p}$. This contradicts (5) unless $\widetilde{\lambda}_1 = \widetilde{\lambda}_2$ and $\widetilde{v}_1 = \widetilde{v}_2$. This completes the proof except for the technical point.

It is easy to see from (4) that

$$\text{(7)} \qquad -\Delta(\widetilde{v}_1 - \widetilde{v}_2) = r(x)(\widetilde{v}_1 - \widetilde{v}_2) + \mu s(x)$$

where $r(x) = (\widetilde{v}_1(x) - \widetilde{v}_2(x))^{-1}(f(\widetilde{v}_1(x)) - f(\widetilde{v}_2(x)) \in L^\infty(\Omega)$ and $s(x) = f(\widetilde{v}_1)$. We prove that if $\widetilde{v}_1 \neq \widetilde{v}_2$, then $\{x \in \Omega : \widetilde{v}_1(x) = \widetilde{v}_2(x)\}$ has measure zero. There are two cases. If $\mu = 0$, then $\widetilde{v}_1 - \widetilde{v}_2$ solves a linear equation and the result follows from [2]. If $\mu \neq 0$, $\mu s(x) \neq 0$ a.e. on Ω. (We prove this claim below). If $v_1 - v_2$ vanishes on a set $\widetilde{T}$ of positive measure, then $\Delta(\widetilde{v}_1 - \widetilde{v}_2) = 0$ a.e. on $\widetilde{T}$. Thus by (7), $\mu s(x) = 0$ a.e. on $\widetilde{T}$, which contradicts our claim. Thus it suffices to prove that $s(x)$ only vanishes on a set of measure zero. Since $f'(0) = 1$ and $\widetilde{v}_1$ is small it is easy to see that s and $\widetilde{v}_1$ have the same zeros. Now $-\Delta\widetilde{v}_1 = k(x)\widetilde{v}_1$ where $k \in L^\infty(\Omega)$ (since $\widetilde{v}_1$ solves (1)). Hence [2] again implies that $\widetilde{v}_1$ only vanishes on a set of measure zero, which proves our claim.

REMARK. Clearly, the arguments are valid for much more general differential operators (using Remark 6 after Lemma 1) and some other boundary conditions. We do not need self-adjointness. By some of the remarks after Lemma 1, we could allow more general nonlinearities. We can also prove the reduction to a finite-dimensional problem if the kernel is multidimensional.

We also have the corresponding result for bifurcation from infinity. We assume f is globally Lipschitz, $y^{-1}f(y) - f'(\infty) \to 0$ as $|y| \to \infty$ and $f'(\infty)$ is non-zero. As before, we can assume without loss of generality that $f'(\infty) = 1$. As before, we assume λ_i is a simple eigenvalue of $-\Delta$. The following theorem generalizes a result in [9].

THEOREM 2. *There exist $K, \epsilon > 0$ and continuous functions $v : \{s \in \mathbb{R} : |s| > K\} \to N$ and $\lambda : \{s \in \mathbb{R} : |s| > K\} \to \mathbb{R}$ with $v(t) \to 0$ as $|t| \to \infty$ and $\lambda(t) \to \lambda_i$ as $|t| \to \infty$ such that the solutions of* (3) *with $\|u\|_p$ large and λ near λ_i are $\{(\alpha(h + v(\alpha)), \lambda(\alpha)) : |\alpha| > K\}$.*

PROOF. As in [23] or [27], one can easily use the asymptotic linearity to show that any large solution with λ near λ_i is of the above form. Moreover, a simple degree argument on N shows that for each large α, v and λ exist. As before, the proof reduces to proving the uniqueness of v and λ. The proof of this is very similar to the proof of Theorem 1 (using Lemma 1 again, or more particularly the corollary to Lemma 1).

REMARK. If f is C^1, one can go further and check that v and λ are C^1 and one can relate the Morse index of the solutions to the sign of $\lambda'(\alpha)$ (as in [6] and [7]).

3. UNIQUENESS THEOREMS WHEN A PARAMETER IS LARGE

We prove several uniqueness theorem when a parameter is large for

$$(8)\qquad \begin{aligned} -\Delta u &= \lambda f(u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Here Ω is a smooth bounded domain in $\mathbb{R}^n$.

THEOREM 3. *Assume that f is globally Lipschitz on $[0,\infty)$, $f(0) \geq 0$, $|yf'(y)| \leq K$ for y large and $f(y) \to M$ as $y \to \infty$. Then there exist $\alpha, \beta > 0$ such that (8) has a unique positive solution u with $\|u\|_\infty \geq \alpha$ if $\lambda > \beta$.*

PROOF. STEP 1. There exists $\alpha > 0$ such that if $\lambda_i \to \infty$ and u_i are positive solutions of (1) with $\|u_i\|_\infty \geq \alpha$ for all i, then $\lambda_i^{-1} u_i \to M(-\Delta)^{-1}1$ in $C^1(\Omega)$ as $i \to \infty$. Let $u_0 = M(-\Delta)^{-1}(1)$.

Suppose that u_i are positive solutions for $\lambda = \lambda_i$ where $\lambda_i \to \infty$ and $\|u_i\|_\infty \to \infty$ as $i \to \infty$. We prove that $\lambda_i^{-1} u_i \to u_0$ in $C^1(\Omega)$ as $i \to \infty$. We write $u_i = \|u_i\|_\infty \widetilde{u}_i$. Then

$$-\Delta \widetilde{u}_i = \lambda_i \|u_i\|_\infty^{-1} f(\|u_i\|_\infty \widetilde{u}_i).$$

Since f is bounded and $\|\widetilde{u}_i\|_\infty = 1$, standard L^∞ estimates (as in [16]) imply that $\lambda_i \|u_i\|_\infty^{-1}$ has a positive lower bound. Either $\lambda_i \|u_i\|_\infty^{-1} \to \gamma > 0$ as $i \to \infty$ or $\lambda_i \|u_i\|_\infty^{-1} \to \infty$ as $i \to \infty$, after taking subsequences.

In the former case, $\{\|\widetilde{u}_i\|_\infty\}$ is bounded in $W^{2,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ if $0 < \alpha < 1$ and $n < p < \infty$ and thus we can choose a subsequence converging strongly in $C^1(\Omega)$ and weakly in $W^{2,p}(\Omega)$ to $\overline{v}$ where $\overline{v} \geq 0$, $\overline{v} = 0$ on $\partial\Omega$ and $\|\overline{v}\|_\infty = 1$. If $\overline{v}(x) > 0$, then $\widetilde{u}_i(x)$ has a positive lower bound for large i and hence $\|u_i\|_\infty \widetilde{u}_i(x) \to \infty$ (since $\|u_i\|_\infty \to \infty$ as $i \to \infty$). Hence $f(\|u_i\|_\infty \widetilde{u}_i(x)) \to f(\infty)$ and thus in the limit we find that $-\Delta\overline{v}(x) = \gamma f(\infty) > 0$ on $\{x \in \Omega : \overline{v}(x) > 0\}$. On the other hand, $\Delta\overline{v}(x) = 0$ a.e. on $\{x \in \Omega : \overline{v}(x) = 0\}$ (cf. [26]). Hence $\overline{v}$ is superharmonic and since it is non-trivial, $\overline{v}(x) > 0$ on Ω. Thus $-\Delta\overline{v}(x) = \gamma f(\infty)$ on Ω and hence $\overline{v} \equiv \gamma u_0$. Thus our claim follows in this case.

If $\lambda_i \|u_i\|_\infty^{-1} \to \infty$ as $i \to \infty$, we show that we get a contradiction. We use a simple and standard blowing up argument near the maximum of $\widetilde{u}_i$ (as in [10]). We rescale the x variables by a factor $\lambda_i^{-1/2} \|u_i\|_\infty^{-1/2}$ and in the new variables we obtain a solution $\hat{u}_i$ of $-\Delta v = f(v)$ on a large domain Ω_i such that $\hat{u}_i$ has its maximum at 0, $\|\hat{u}_i\|_\infty = 1$ and $\hat{u}_i$ is zero on $\partial\Omega_i$. By local estimates, a subsequence of $\hat{u}_i$ converges uniformly on compact sets to a bounded function $\hat{u}$ on either all of $\mathbb{R}^n$ or a half space T such that $\hat{u}(0) = 1$ and $\hat{u} = 0$ on ∂T in the second case. Much as before, we find that $-\Delta\hat{u} = f(\infty)$ where $\hat{u} > 0$ and $\Delta\hat{u} =$

0 a.e. on $\hat{u} = 0$. Thus $\hat{u}$ is superharmonic and hence $\hat{u} > 0$ everywhere. (Remember that $\hat{u}$ is non-trivial.) Thus $-\Delta\hat{u} = f(\infty)$ on $\mathbb{R}^n$ or T (depending on the case). In either case, this contradicts known results. For example, in the full space cases, this contradicts Proposition 3 in [10] while we can use Theorem 2 in [12] (or [1]) to prove that if $\hat{u}$ existed in the half space case, there would have to be a full space solution of the same equation in dimension $N - 1$ (where $\Omega \subseteq \mathbb{R}^N$) and we have a contradiction as before. This proves our claim.

Hence we can find K such that, if $\lambda_i \geq K$ and $\|u_i\|_\infty \geq K$, then $\lambda_i^{-1} u_i$ is C^1 close to u_0. In particular, this implies that, if $\|u_i\| \geq K$ and λ_i is large, $\|u_i\|_\infty$ is large and hence $\lambda_i^{-1} u_i \to u_0$ in $C^1(\overline{\Omega})$. This completes the proof of this step.

STEP 2. We write $u = \lambda v$. Then our equation becomes $-\Delta v = f(\lambda v)$ in Ω with Dirichlet boundary conditions. Now $v_i \to (-\Delta)^{-1} M$ in $C^1(\overline{\Omega})$ as $i \to \infty$, where $v_i = \lambda_i^{-1} u_i$.

Now if $\widetilde{K}$ is a compact subset of Ω which is also a smooth manifold with boundary, then $\lambda_i f'(\lambda_i v_i) \rightharpoonup 0$ in $L^2(\widetilde{K})$ as $i \to \infty$. This follows by a slight modification of the proof of Lemma 1 once we note that $|\lambda_i f'(\lambda_i v_i)| \leq v_i^{-1} |\lambda_i v_i f'(\lambda_i v_i)| \leq K v_i^{-1} \leq K_2$ on $\widetilde{K}$ for i large. If w_i is another solution for $\lambda = \lambda_i$, then the proof of the corollary to Lemma 1 shows that $\int_0^1 \lambda_i f'(tv_i + (1-t)w_i)\, dt \rightharpoonup 0$ in $L^2(\widetilde{K})$ as $i \to \infty$. Note that ∇u_0 cannot vanish on a set of positive measure, and that $v_i - w_i$ satisfies $-\Delta z = k(x) z$ where $k \in L^\infty(\Omega)$ and thus $v_i - w_i$ only vanishes on a set of measure zero. (The claim for u_0 follows because if ∇u_0 vanishes on a set of positive measure T, then $\partial^2 u_0 / \partial x_i \partial x_j$ must vanish a.e. on T and thus $\Delta u_0 = 0$ a.e. on T. This is impossible since $-\Delta u_0 = M$ on Ω.) Now let h_i be $w_i - v_i$ normalized to have L^2 norm 1. Then by a simple calculation

$$-\Delta h_i = h_i \int_0^{\prime} \lambda_i f'(tv_i + (1-t)w_i)\, dt.$$

If we can bound h_i in $\dot{W}^{1,2}(\Omega)$, then we can choose a subsequence converging weakly to h in $\dot{W}^{1,2}(\Omega)$ where $\|h\|_2 = 1$. As in the proof of Lemma 2, we can then pass to the limit in the weak form of the equation for h_i when using a test function $\phi \in C_0^\infty(\widetilde{K})$ (and thus the support of ϕ is contained in $\widetilde{K}$ where $\widetilde{K} \subset\subset \Omega$) to prove h is weakly harmonic. Since $h \in \dot{W}^{1,2}(\Omega)$, we see that $h = 0$ a.e., which contradicts the fact that $\|h\|_2 = 1$.

Thus it remains to bound h_i in $\dot{W}^{1,2}(\Omega)$ where

$$-\Delta h_i = a_i(x) h_i \quad \text{and} \quad |a_i(x)| \leq K(u_0(x))^{-1}.$$

Note that since $\partial u_0 / \partial n > 0$ on $\partial\Omega$, $u_0(x) \geq K_1 d(x, \partial\Omega)$ if x is near $\partial\Omega$. On the other hand, by Hardy's inequality, $\int_\Omega d(x, \partial\Omega)^{-2} h_i^2 \leq K_2 \|\nabla h_i\|_2^2$. Now

$$\int_\Omega |\nabla h_i|^2 = \int_\Omega a_i(x) h_i^2 \leq K_3 \int_{\Omega_\delta} d(x, \partial\Omega)^{-1} h_i^2 + K_4 \int_\Omega h_i^2$$

where $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}$ and we have used the fact that a_i is bounded in the interior of Ω, and also our bound for a_i near $\partial\Omega$. Here K_3 does not depend on δ but K_4 might. Since $\|h_i\|_2 = 1$, we see that if we choose δ small,

$$\int_\Omega |\nabla h_i|^2 \leq \frac{1}{2K_2} \int_{\Omega_\delta} d(x, \partial\Omega)^{-2} |h_i|^2 + K_5 \leq \frac{1}{2} \int_\Omega |\nabla h_i|^2 + K_5$$

by Hardy's inequality. Hence we have the required bound for h_i in $\dot{W}^{1,2}(\Omega)$.

REMARKS. Once again we can clearly allow more general differentiable operators (so that the inverse is positive), some other boundary conditions and multiply f by $g(x)$ where g is non-negative and bounded. If f is non-negative and does not decay too rapidly at zero, one can usually combine our ideas with those in [10] to understand all the positive solutions for λ large. When f has positive limit at ∞ and is space independent, our result is more general than those in [4], [10], [18], [24], [27]. Our ideas could also be used to improve results in [14].

Our second theorem concerns the case where f is globally Lipschitz, $0 < \alpha < 1$, $y^{-\alpha} f(y) \to C > 0$ as $y \to \infty$ and $y^{1-\alpha}|f'(y)| \leq K$ for large (and hence all) non-negative y. Note that it is easy to see that the equation

$$-\Delta u = Cu^{\alpha} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

has a unique positive solution $\widetilde{u}_0$ (cf. [10]).

THEOREM 4. *Assume that the above conditions on f hold and $f(0) \geq 0$. Then there exist $\alpha, \beta > 0$ such that* (8) *has a unique positive solution with $\|u\|_\infty \geq \alpha$ if $\lambda \geq \beta$.*

PROOF. STEP 1. By rescaling, we can assume that $C = 1$. We can easily combine the ideas in the proof of Step 1 of the proof of Theorem 3 with pp. 433–434 in [10] to prove that there is an $\alpha > 0$ such that if u_i are positive solutions of (8) with $\|u_i\|_\infty \geq \alpha$ for all i and $\lambda_i \to \infty$ as $i \to \infty$, then $\|u_i\|_\infty \to \infty$ as $i \to \infty$ and $\lambda_i^{-1/(1-\alpha)} u_i \to \widetilde{u}_0$ in $C^1(\Omega)$ as $i \to \infty$. (We mostly work with the rescaled function $\lambda^{-1/(1-\alpha)} u$.)

STEP 2. Assume that u_i and v_i are distinct positive solutions of (8) for $\lambda = \lambda_i$ such that $\lambda_i \to \infty$ as $i \to \infty$ and $\lambda_i^{-1/(1-\alpha)} u_i \to \widetilde{u}_0$, $\lambda_i^{-1/(1-\alpha)} v_i \to \widetilde{u}_0$ in $C^1(\Omega)$ as $i \to \infty$. Since a large positive constant (depending upon λ) is a supersolution, there must be a maximal solution. Thus, we may assume that $u_i \leq v_i$. Let $\widetilde{u}_i = \lambda_i^{-1/(1-\alpha)} u_i$ and define $\widetilde{v}_i$ analogously. Then $-\Delta \widetilde{u}_i = \lambda_i^{-\alpha/(1-\alpha)} f(\lambda_i^{1/(1-\alpha)} \widetilde{u}_i)$ with an analogous result for $\widetilde{v}_i$. Thus

$$-\Delta(\widetilde{v}_i - \widetilde{u}_i) = \lambda_i(\widetilde{v}_i - \widetilde{u}_i) \int_0^{\prime} f'(\lambda_i^{1/(1-\alpha)}(t\widetilde{u}_i + (1-t)\widetilde{v}_i))\, dt.$$

Now if $\lambda_i^{1/(1-\alpha)} d(x_i, \partial\Omega)$ is large our estimate for f' shows that λ_i times the modulus of the integrand is bounded by $\lambda_i K d(x, \partial\Omega)^{\alpha-1} \lambda_i^{-1}$ which is bounded if x is not close to $\partial\Omega$ and is bounded by $K_1 d(x, \partial\Omega)^{\alpha-1}$ otherwise. (If $\lambda_i^{1/(1-\alpha)} d(x, \partial\Omega)$ is bounded, λ_i times the integrand is bounded by $\lambda_i K_2$, which is bounded by $K_3 d(x, \partial\Omega)^{\alpha-1}$.) Moreover, $\frac{d}{dy}(\lambda^{-\alpha/(1-\alpha)} f(\lambda^{1/(1-\alpha)} y)) = \lambda f'(\lambda^{1/(1-\alpha)} y)$ and $\lambda^{-\alpha/(1-\alpha)} f(\lambda^{1/(1-\alpha)} y) \to y^{\alpha}$ as $\lambda \to \infty$. Thus we are back very much in the situation of Theorem 3 and we can complete the proof as there. There is one difference. The limiting operator in the equation for $\widetilde{v}_i - \widetilde{u}_i$ is

$$-\Delta h - \alpha \widetilde{u}_0^{\alpha-1} h = 0 \quad \text{on } \Omega$$

with Dirichlet boundary conditions. This potential is weakly singular on the boundary but this causes no problem. As in [10], we prove that there is no non-trivial solution of this equation by comparison with $-\Delta h - \widetilde{u}_0^{\alpha-1}h = 0$ in Ω with Dirichlet boundary conditions which has zero as the principal eigenvalue with positive eigenfunction $\widetilde{u}_0$. The proof that $\nabla\widetilde{u}_0$ only vanishes on a set of measure zero is similar to the corresponding argument in the proof of Theorem 3.

REMARKS. 1. We could produce similar results for the uniqueness of the large positive solution of (8) for small λ if $1 < p < (n+2)/(n-2)$, $f(y) \sim y^p$ as $y \to \infty$, $f(0) \geq 0$, f is locally Lipschitz, $|f'(y)| \leq Ky^{p-1}$ for large y, and the positive solution of $-\Delta u = u^p$ in Ω, $u = 0$ on $\partial\Omega$, is unique and non-degenerate. We have an analogous theorem for small solutions for λ large if $f(y) \sim y^p$ as $y \to 0^+$, f is locally Lipschitz and $|f'(y)| \leq Ky^{p-1}$ for small positive y.

2. To prove the theorem for more general differential operators, it is convenient to know that $\nabla\widetilde{u}_0 \neq 0$ a.e. One can frequently use Morrey's Theorem 5.8.5 of [21] to prove $\widetilde{u}_0$ is real analytic in Ω and hence deduce this. We can probably use different rescalings to cover many cases when the growth is not asymptotically a power (as in [18]).

4. MULTIPLE SOLUTIONS OF AMBROSETTI–PRODI PROBLEMS

In this section, we consider the problem

$$(9) \qquad \begin{aligned} -\Delta u &= g(u) - (th_1 + v) \quad &&\text{in } \Omega, \\ u &= 0 &&\text{on } \partial\Omega, \end{aligned}$$

where v is orthogonal to h_1, h_1 is the positive eigenfunction corresponding to the principal eigenvalue λ_1 of $-\Delta$ for Dirichlet boundary conditions, t is large positive, $g : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz and C^1 and $y^{-1}g(y) \to \mu$ (resp. ν) as $y \to \infty$ (resp. $-\infty$). We also assume $\mu, \nu > \lambda_1$, $\mu \neq \nu$, the problem $-\Delta u = \mu u^+ + \nu u^-$ in Ω, $u = 0$ on $\partial\Omega$, has only the trivial solution and neither μ nor ν are eigenvalues of $-\Delta$ (though this could be weakened). We finally assume that the interval (μ, ν) contains an eigenvalue of $-\Delta$ (where without loss of generality we may assume $\mu < \nu$). This is exactly the situation of [11] except that we have considerably weakened the condition on g at $\pm\infty$. Let $s(y) = g(y) - \mu y^+ - \nu y^-$. Note that s satisfies the conditions of Lemma 1.

Let N be the subspace spanned by the eigenspaces corresponding to eigenvalues in (μ, ν). We write elements of $\dot{W}^{1,2}(\Omega)$ as $n + m$ where $n \in N$, $m \in N^\perp$ and let P be the corresponding projection onto $N^\perp$. It is easily proved (as in [11]) that the equation $P(W(m+n)) = 0$ can be uniquely solved for m as a function of n, that is, $m = S(n)$. Here $W(u) = -\Delta u - \mu u^+ - \nu u^-$. Let $F(n) = (I - P)W(n + S(n))$. This is easily seen to be a gradient map (cf. [11]). We prove the following result, which generalizes a result in [11] by weakening a condition on g.

THEOREM 5. *If the critical groups of F (with $\mathbb{Z}$ coefficients) at zero are neither all trivial or are not $\delta^{k0}\mathbb{Z}$ (where δ^{k0} is the Kronecker delta), then* (9) *has at least three distinct solutions for all large positive t.*

REMARK. There are 3-solution results in [20] under weaker assumptions on g. However, they do not apply for all the range of μ and ν where Theorem 5 applies. The ideas in [13] are sometimes useful for calculating the critical groups. As in [11] there is an analogous theorem for t large negative.

PROOF. We can follow the proof in [11] exactly if we prove that the problem

$$P(W(m+n) - t^{-1}s(t(m+n)) + t^{-1}v) = 0 \tag{10}$$

can be uniquely solved for m as a function of n if t is large and n is close to a zero of $(I-P)W(n+S(n)) - h_1 = 0$. (The zeros are a compact set C not containing zero.) The existence follows by a simple degree argument. Moreover, as in §1 of [11], easy estimates show that for t large any solution m of (10) is uniformly close to $S(n)$. If t is large and n is near C, and if m_1, m_2 are solutions of (10) with $m_1 \neq m_2$, then m_1 and m_2 are uniformly close to $S(n)$. Remember as in [11] that W is strictly differentiable at $n_0 + S(n_0)$ in the sense of [3] if $n_0 \in C$. Thus subtracting the equations for m_1 and m_2, we find that

$$(I-P)\left[W'(n_0+S(n_0))(m_1-m_2) + (m_1-m_2)\int_0^1 s'(t(n+rm_1+(1-r)m_2))\,dr\right]$$

where t is large and n is close to $n_0 \in C$. Since $\nabla(n_0 + S(n_0))$ only vanishes on a set of measure zero if $n_0 \in C$ (as we prove below), the integral converges weakly to zero in $L^q(\Omega)$ as $t \to \infty$ for all q with $1 < q < \infty$ (cf. the corollary to Lemma 1). Thus we can pass to the weak limit as $t \to \infty$ much as in §1 to find

$$(I-P)W'(n_0+S(n_0))h = 0$$

where h is the limit of $m_1 - m_2$ normalized, $h \neq 0$ and $h \in N^{\perp}$. By the argument in [11], this is impossible. Hence we have uniqueness. That m depends continuously on n follows by a limit and compactness argument. We can then prove that m depends differentiably on n by first using a similar argument to above to show that $(I-P)(W'(n+S(n)) + s'(t(n+S(n))))$ is invertible on $N^{\perp}$ if n is near $n_0 \in C$ (cf. [11]).

It remains to prove that the gradient of $n_0 + S(n_0)$ can only vanish on a set of measure zero if $n_0 \in C$. In this case, $n + S(n_0)$ is a solution of $-\Delta u = \mu u^+ + \nu u^- - h_1$. First note that such a u cannot vanish on a set of positive measure since if it vanished on such a set $\widetilde{T}$, then $\Delta u = 0$ a.e. on $\widetilde{T}$, which is impossible since $h_1 > 0$ on Ω. Thus if ∇u vanished on a set of positive measure, then it must vanish on a set of positive measure of $u > 0$ (or $u < 0$). We only consider the former case. The other is similar. Then $-\Delta u = \mu u - h_1$ on $u > 0$ and hence by Theorem 5.8.5 in Morrey [21], u is real analytic there. Since u cannot be constant on a component of $u > 0$, it follows that $\partial u/\partial x_i$ can only vanish on a set of measure zero in each component of $u > 0$ and hence on a set of measure zero of $u > 0$. (There are only countably many components of $u > 0$. There are many variants of this last argument.)

REMARK. With much more care it is possible to delete the assumption that g is C^1. Clearly, we could replace $-\Delta$ by rather more general self-adjoint operators, and some other boundary conditions.

The research was partially supported by the Australian Research Council.

REFERENCES

[1] H. BERESTYCKI - L. CAFFARELLI - L. NIRENBERG, *Further qualitative properties of elliptic equations in unbounded domains*. Ann. Scuola Norm. Sup. Pisa 25 (1997), 69–94.

[2] L. CAFFARELLI - A. FRIEDMAN, *Partial regularity of the zero-set of solutions of linear and superlinear elliptic equations*. J. Differential Equations 60 (1985), 420–433.

[3] H. CARTAN, *Calcul Différentiel*. Hermann, Paris, 1967.

[4] A. CASTRO - R. SHIVAJI, *Uniqueness of positive solutions for a class of elliptic boundary value problems*. Proc. Roy. Soc. Edinburgh Sect. A 98 (1984), 267–269.

[5] K. C. CHANG, *Infinite Dimensional Morse Theory and Multiple Solution Problems*. Birkhäuser, Boston, 1993.

[6] M. CRANDALL - P. H. RABINOWITZ, *Bifurcation from simple eigenvalues*. J. Funct. Anal. 8 (1971), 321–340.

[7] M. CRANDALL - P. H. RABINOWITZ, *Bifurcation, perturbation of simple eigenvalues and linearized stability*. Arch. Rat. Mech. Anal. 2 (1973), 161–180.

[8] E. N. DANCER, *Corrigendum: On the Dirichlet problem for weakly nonlinear elliptic differential equations*. Proc. Roy. Soc. Edinburgh Sect. A 89 (1981), 15.

[9] E. N. DANCER, *A note on bifurcation from infinity*. Quart. J. Math. Oxford 25 (1974), 81–84.

[10] E. N. DANCER, *On the number of positive solutions for a nonlinear elliptic equation when a parameter is large*. Proc. London Math. Soc. 53 (1986), 429–452.

[11] E. N. DANCER, *Multiple solutions of asymptotically homogenous problems*. Ann. Mat. Pura Appl. 152 (1988), 63–78.

[12] E. N. DANCER, *A note on the method of moving planes*. Bull. Austral. Math. Soc. 46 (1992), 425–434.

[13] E. N. DANCER, *Some results for jumping nonlinearities*. Topol. Methods Nonlinear Anal. 19 (2002), 221–235.

[14] E. N. DANCER - J. P. SHI, *Uniqueness of positive solutions to sublinear semipositive problems*. Preprint.

[15] D. DANERS, *Dirichlet problems on varying domains*. J. Differential Equations 188 (2003), 591–624.

[16] D. GILBARG - N. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*. Springer, Berlin, 1983.

[17] R. HARDT - L. SIMON, *Nodal sets for solutions for elliptic equations*. J. Differential Geom. 30 (1989), 505–522.

[18] D. D. HOE - R. C. SMITH, *On uniqueness for a class of nonlinear boundary value problems*. Preprint.

[19] G. KATRIEL, *Uniqueness of periodic solutions for asymptotically linear Duffing equations with strong forcing*. Topol. Methods Nonlinear Anal. 12 (1998), 263–274.

[20] A. MARINO - A. MICHELETTI - A. PISTOIA, *A nonsymmetric asymptotically linear problem*. Topol. Methods Nonlinear Anal. 4 (1994), 289–339.

[21] C. MORREY, *Multiple Integrals in the Calculus of Variations*. Springer, Berlin, 1966.

[22] P. H. RABINOWITZ, *Some global results for nonlinear eigenvalue problems*. J. Funct. Anal. 7 (1971), 487–513.

[23] P. H. RABINOWITZ, *Some aspects of nonlinear eigenvalue problems*. Rocky Mountain J. Math. 3 (1973), 161–202.

[24] R. SCHAAF - K. SCHMITT, *Asymptotic behaviour of positive solution branches of elliptic problems with linear part at resonance*. Z. Angew. Math. Phys. 43 (1992), 645–676.

[25] V. SCHUCHMAN, *About uniqueness for nonlinear boundary value problems*. Math. Ann. 267 (1984), 537–542.

[26] G. STAMPACCHIA, *Équations elliptiques du second ordre à coefficients discontinus*. Presses Univ. de Montréal, 1976.

[27] J. TOLAND, *Asymptotic linearity and nonlinear eigenvalue problems*. Quart. J. Math. (Oxford) 24 (1973), 241–250.

[28] M. WIEGNER, *A uniqueness theorem for some nonlinear boundary value problems with a large parameter*. Math. Ann. 270 (1985), 401–402.

Received 11 October 2005,
and in revised form 18 October 2005.

School of Mathematics and Statistics
The University of Sydney
SYDNEY, NSW 2006, Australia
normd@maths.usyd.edu.au