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Partial differential equations. — *Periodic solutions of nonlinear wave equations for asymptotically full measure sets of frequencies*, by PIETRO BALDI and MASSIMILIANO BERTI, communicated on 10 March 2006.

ABSTRACT. — We prove existence and multiplicity of small amplitude periodic solutions of completely resonant nonlinear wave equations with Dirichlet boundary conditions for asymptotically full measure sets of frequencies, extending the results of [7] to new types of nonlinearities.

KEY WORDS: Nonlinear wave equation; infinite-dimensional Hamiltonian systems; periodic solutions; Lyapunov–Schmidt reduction; small divisors problem.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 35L05, 35B10, 37K50.

1. INTRODUCTION

The aim of this note is to prove existence and multiplicity of small amplitude periodic solutions of the completely resonant wave equation

$$\begin{cases} \Box u + f(x,u) = 0, \\ u(t,0) = u(t,\pi) = 0, \end{cases} \tag{1}$$

where $\Box := \partial_{tt} - \partial_{xx}$ is the d'Alembertian operator and

$$f(x,u) = a_2 u^2 + a_3(x) u^3 + O(u^4) \quad \text{or} \quad f(x,u) = a_4 u^4 + O(u^5), \tag{2}$$

for a Cantor-like set of frequencies ω of asymptotically full measure at $\omega = 1$.

Equation (1) is said to be completely resonant because any solution $v = \sum_{j \geq 1} a_j \cos(jt + \vartheta_j) \sin(jx)$ of the linearized equation at $u = 0$,

$$\begin{cases} u_{tt} - u_{xx} = 0, \\ u(t,0) = u(t,\pi) = 0, \end{cases} \tag{3}$$

is 2π-periodic in time.

Existence and multiplicity of periodic solutions of (1) were proved for a zero measure, uncountable Cantor set of frequencies in [4] for $f(u) = u^3 + O(u^5)$ and in [5]–[6] for any nonlinearity $f(u) = a_p u^p + O(u^{p+1})$, $p \geq 2$.

Existence of periodic solutions for a Cantor-like set of frequencies of asymptotically full measure has recently been proved in [7] where, due to the well known "small divisor difficulty", the "0th order bifurcation equation" is required to have nondegenerate periodic solutions. This property was verified in [7] for nonlinearities like $f = a_2 u^2 + O(u^4)$,

$f = a_3(x)u^3 + O(u^4)$. See also [11] for $f = u^3 + O(u^5)$ (and [9] in the case of periodic boundary conditions).

In this note we shall prove that, for quadratic, cubic and quartic nonlinearities $f(x, u)$ as in (2), the corresponding 0th order bifurcation equation has nondegenerate periodic solutions (Propositions 1 and 2), implying, by the results of [7], Theorem 1 and Corollary 1 below.

We remark that our proof is purely analytic (it does not use numerical calculations) being based on the analysis of the variational equation and exploiting properties of the Jacobi elliptic functions.

1.1. *Main results*

Normalizing the period to 2π, we look for solutions of

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + f(x, u) = 0, \\ u(t, 0) = u(t, \pi) = 0, \end{cases}$$

in the Hilbert algebra (for $s > 1/2$, $\sigma > 0$)

$$X_{\sigma,s} := \Big\{ u(t, x) = \sum_{l \geq 0} \cos(lt) u_l(x) \ \Big| \ u_l \in H_0^1((0, \pi), \mathbb{R}) \ \forall l \in \mathbb{N} \text{ and } \|u\|^2_{\sigma,s} := \sum_{l \geq 0} \exp(2\sigma l)(l^{2s} + 1)\|u_l\|^2_{H^1} < +\infty \Big\}.$$

It is natural to look for solutions which are even in time because equation (1) is reversible.

We look as well for solutions of (1) in the subalgebras

$$X_{\sigma,s,n} := \{u \in X_{\sigma,s} \mid u \text{ is } 2\pi/n\text{-periodic}\} \subset X_{\sigma,s}, \qquad n \in \mathbb{N}$$

(they are particular 2π-periodic solutions).

The space of solutions of the linear equation (3) that belong to $H_0^1(\mathbb{T} \times (0, \pi), \mathbb{R})$ and are even in time is

$$\begin{aligned} V &:= \Big\{ v(t, x) = \sum_{l \geq 1} \cos(lt) u_l \sin(lx) \ \Big| \ u_l \in \mathbb{R}, \ \sum_{l \geq 1} l^2 |u_l|^2 < +\infty \Big\} \\ &= \Big\{ v(t, x) = \eta(t + x) - \eta(t - x) \ \Big| \ \eta \in H^1(\mathbb{T}, \mathbb{R}) \text{ odd} \Big\}. \end{aligned}$$

THEOREM 1. *Let either*

$$f(x, u) = a_2 u^2 + a_3(x) u^3 + \sum_{k \geq 4} a_k(x) u^k \tag{4}$$

where $(a_2, \langle a_3 \rangle) \neq (0, 0)$, $\langle a_3 \rangle := \pi^{-1} \int_0^\pi a_3(x)\, dx$, *or*

$$f(x, u) = a_4 u^4 + \sum_{k \geq 5} a_k(x) u^k \tag{5}$$

where $a_4 \neq 0$, $a_5(\pi - x) = -a_5(x)$, $a_6(\pi - x) = a_6(x)$, $a_7(\pi - x) = -a_7(x)$. *Assume moreover* $a_k(x) \in H^1((0,\pi),\mathbb{R})$ *with* $\sum_k \|a_k\|_{H^1}\rho^k < +\infty$ *for some* $\rho > 0$. *Then there exists* $n_0 \in \mathbb{N}$ *such that for every* $n \geq n_0$ *there are* $\delta_0, \bar\sigma > 0$ *and a* C^∞*-curve* $[0,\delta_0) \ni \delta \mapsto u_\delta \in X_{\bar\sigma/2,s,n}$ *with the following properties:*

(i) $\|u_\delta - \delta\bar v_n\|_{\bar\sigma/2,s,n} = O(\delta^2)$ *for some* $\bar v_n \in V \cap X_{\bar\sigma,s,n} \setminus \{0\}$ *with minimal period* $2\pi/n$;
(ii) *there exists a Cantor set* $\mathcal{C}_n \subset [0,\delta_0)$ *of asymptotically full measure at* $\delta = 0$, *i.e. satisfying*

$$\lim_{\varepsilon\to 0^+} \frac{\text{meas}(\mathcal{C}_n \cap (0,\varepsilon))}{\varepsilon} = 1, \tag{6}$$

such that, for each $\delta \in \mathcal{C}_n$, $u_\delta(\omega(\delta)t, x)$ *is a* $2\pi/(\omega(\delta)n)$*-periodic, classical solution of* (1) *with*

$$\omega(\delta) = \begin{cases} \sqrt{1 - 2s^*\delta^2} & \text{if } f \text{ is as in (4),} \\ \sqrt{1 - 2\delta^6} & \text{if } f \text{ is as in (5),} \end{cases}$$

and

$$s^* = \begin{cases} -1 & \text{if } \langle a_3\rangle \geq \pi^2 a_2^2/12, \\ \pm 1 & \text{if } 0 < \langle a_3\rangle < \pi^2 a_2^2/12, \\ 1 & \text{if } \langle a_3\rangle \leq 0. \end{cases}$$

By (6) also each Cantor-like set of frequencies $\mathcal{W}_n := \{\omega(\delta) \mid \delta \in \mathcal{C}_n\}$ has asymptotically full measure at $\omega = 1$.

Note how the interaction between the second and third order terms a_2u^2, $a_3(x)u^3$ changes the bifurcation diagram, i.e. existence of periodic solutions for frequencies ω less than or/and greater than $\omega = 1$.

COROLLARY 1 (Multiplicity). *There exists a Cantor-like set* $\mathcal{W}$ *of asymptotically full measure at* $\omega = 1$ *such that for each* $\omega \in \mathcal{C}$, *equation* (1) *has geometrically distinct periodic solutions*

$$u_{n_0}, \ldots, u_n, \ldots, u_{N_\omega}, \qquad N_\omega \in \mathbb{N},$$

with the same period $2\pi/\omega$. *Their number increases indefinitely as* ω *tends to 1:*

$$\lim_{\omega\to 1} N_\omega = +\infty.$$

PROOF. The proof is as in [7] and we repeat it for completeness. If δ belongs to the asymptotically full measure set (by (6))

$$D_n := \mathcal{C}_{n_0} \cap \ldots \cap \mathcal{C}_n, \qquad n \geq n_0,$$

then there exist $n - n_0 + 1$ geometrically distinct periodic solutions of (1) with the same period $2\pi/\omega(\delta)$ (each u_n has minimal period $2\pi/(n\omega(\delta))$).

There exists a decreasing sequence of positive $\varepsilon_n \to 0$ such that

$$\text{meas}(D_n^c \cap (0,\varepsilon_n)) \leq \varepsilon_n 2^{-n}.$$

Define the set $\mathcal{C} \equiv D_n$ on each $[\varepsilon_{n+1}, \varepsilon_n)$. Then $\mathcal{C}$ has asymptotically full measure at $\delta = 0$ and for each $\delta \in \mathcal{C}$ there exist $N(\delta) := \max\{n \in \mathbb{N} : \delta < \varepsilon_n\}$ geometrically distinct periodic solutions of (1) with the same period $2\pi/\omega(\delta)$, and $N(\delta) \to +\infty$ as $\delta \to 0$. □

REMARK 1. Corollary 1 is an analogue for equation (1) of the well known multiplicity results of Weinstein–Moser [13]–[12] and Fadell–Rabinowitz [10] which hold in finite dimensions. The solutions form a sequence of functions with increasing norms and decreasing minimal periods. Multiplicity of solutions was also obtained in [6] (with the "optimal" number $N_\omega \approx C/\sqrt{|\omega - 1|}$) but only for a zero measure set of frequencies.

The main point in proving Theorem 1 is to show the existence of nondegenerate solutions of the 0th order bifurcation equation for f as in (2). In these cases the 0th order bifurcation equation involves higher order terms of the nonlinearity, and, for n large, can be reduced to an integro-differential equation (which physically describes an averaged effect of the nonlinearity with Dirichlet boundary conditions).

CASE $f(x,u) = a_4 u^4 + O(u^5)$. Performing the rescaling $u \to \delta u$, $\delta > 0$, we look for $2\pi/n$-periodic solutions in $X_{\sigma,s,n}$ of

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + \delta^3 g(\delta, x, u) = 0, \\ u(t,0) = u(t,\pi) = 0, \end{cases} \tag{7}$$

where

$$g(\delta, x, u) := \frac{f(x, \delta u)}{\delta^4} = a_4 u^4 + \delta a_5(x) u^5 + \delta^2 a_6(x) u^6 + \dots .$$

To find solutions of (7) we implement the Lyapunov–Schmidt reduction according to the orthogonal decomposition $X_{\sigma,s,n} = (V_n \cap X_{\sigma,s,n}) \oplus (W \cap X_{\sigma,s,n})$ where

$$V_n := \{v(t,x) = \eta(nt + nx) - \eta(nt - nx) \mid \eta \in H^1(\mathbb{T}, \mathbb{R}) \text{ odd}\},$$
$$W := \left\{ w = \sum_{l \geq 0} \cos(lt) w_l(x) \in X_{0,s} \;\middle|\; \int_0^\pi w_l(x) \sin(lx)\, dx = 0,\ \forall l \geq 0 \right\}.$$

Looking for solutions $u = v + w$ with $v \in V_n \cap X_{\sigma,s,n}$, $w \in W \cap X_{\sigma,s,n}$, and imposing the frequency-amplitude relation

$$\frac{\omega^2 - 1}{2} = -\delta^6$$

we are led to solve the bifurcation equation and the range equation

$$\begin{cases} \Delta v = \delta^{-3} \Pi_{V_n} g(\delta, x, v + w), \\ L_\omega w = \delta^3 \Pi_{W_n} g(\delta, x, v + w), \end{cases}$$

where $\Delta v := v_{xx} + v_{tt}$, $L_\omega := -\omega^2 \partial_{tt} + \partial_{xx}$ and $\Pi_{V_n} : X_{\sigma,s,n} \to V_n \cap X_{\sigma,s,n}$, $\Pi_{W_n} : X_{\sigma,s,n} \to W \cap X_{\sigma,s,n}$ denote the projectors.

With the further rescaling $w \mapsto \delta^3 w$ and since $v^4 \in W_n$ (Lemma 3.4 of [5]), $a_5(x)v^5$, $a_6(x)v^6$, $a_7(x)v^7 \in W_n$ because $a_5(\pi - x) = -a_5(x)$, $a_6(\pi - x) = a_6(x)$, $a_7(\pi - x) = -a_7(x)$ (Lemma 7.1 of [7]), the system is equivalent to

$$\begin{cases} \Delta v = \Pi_{V_n}(4a_4 v^3 w + \delta r(\delta, x, v, w)), \\ L_\omega w = a_4 v^4 + \delta \Pi_{W_n} \tilde{r}(\delta, x, v, w), \end{cases} \tag{8}$$

where $r(\delta, x, v, w) = a_8(x)v^8 + 5a_5(x)v^4 w + O(\delta)$ and $\tilde{r}(\delta, x, v, w) = a_5(x)v^5 + O(\delta)$.

For $\delta = 0$ system (8) reduces to $w = -a_4 \Box^{-1} v^4$ and to the 0th order bifurcation equation

$$\Delta v + 4a_4^2 \Pi_{V_n}(v^3 \Box^{-1} v^4) = 0, \tag{9}$$

which is the Euler–Lagrange equation of the functional $\Phi_0 : V_n \to \mathbb{R}$,

$$\Phi_0(v) = \frac{\|v\|_{H_1}^2}{2} - \frac{a_4^2}{2} \int_\Omega v^4 \Box^{-1} v^4 \tag{10}$$

where $\Omega := \mathbb{T} \times (0, \pi)$.

PROPOSITION 1. *Let $a_4 \neq 0$. There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the 0th order bifurcation equation* (9) *has a solution $\bar{v}_n \in V_n$ which is nondegenerate in V_n (i.e.* Ker $D^2\Phi_0 = \{0\}$*), with minimal period $2\pi/n$.*

CASE $f(x, u) = a_2 u^2 + a_3(x)u^3 + O(u^4)$. Performing the rescaling $u \mapsto \delta u$ we look for $2\pi/n$-periodic solutions of

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + \delta g(\delta, x, u) = 0, \\ u(t, 0) = u(t, \pi) = 0, \end{cases}$$

where

$$g(\delta, x, u) := \frac{f(x, \delta u)}{\delta^2} = a_2 u^2 + \delta a_3(x)u^3 + \delta^2 a_4(x)u^4 + \dots.$$

With the frequency-amplitude relation

$$\frac{\omega^2 - 1}{2} = -s^* \delta^2$$

where $s^* = \pm 1$, we have to solve

$$\begin{cases} -\Delta v = -s^* \delta^{-1} \Pi_{V_n} g(\delta, x, v + w), \\ L_\omega w = \delta \Pi_{W_n} g(\delta, x, v + w). \end{cases}$$

With the further rescaling $w \mapsto \delta w$ and since $v^2 \in W_n$, the system is equivalent to

$$\begin{cases} -\Delta v = s^* \Pi_{V_n}(-2a_2 v w - a_2 \delta w^2 - a_3(x)(v + \delta w)^3 - \delta r(\delta, x, v + \delta w)), \\ L_\omega w = a_2 v^2 + \delta \Pi_{W_n}(2a_2 v w + \delta a_2 w^2 + a_3(x)(v + \delta w)^3 + \delta r(\delta, x, v + \delta w)), \end{cases}$$

where $r(\delta, x, u) := \delta^{-4}[f(x, \delta u) - a_2 \delta^2 u^2 - \delta^3 a_3(x)u^3] = a_4(x)u^4 + \dots.$

For $\delta = 0$ the system reduces to $w = -a_2 \Box^{-1} v^2$ and the 0th order bifurcation equation

$$-s^* \Delta v = 2a_2^2 \Pi_{V_n}(v \Box^{-1} v^2) - \Pi_{V_n}(a_3(x) v^3), \tag{11}$$

which is the Euler–Lagrange equation of $\Phi_0 : V_n \to \mathbb{R}$,

$$\Phi_0(v) := s^* \frac{\|v\|_{H^1}^2}{2} - \frac{a_2^2}{2} \int_\Omega v^2 \Box^{-1} v^2 + \frac{1}{4} \int_\Omega a_3(x) v^4. \tag{12}$$

PROPOSITION 2. *Let* $(a_2, \langle a_3 \rangle) \neq (0, 0)$. *There exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$ *the* 0*th order bifurcation equation* (11) *has a solution* $\bar{v}_n \in V_n$ *which is nondegenerate in* V_n, *with minimal period* $2\pi/n$.

2. CASE $f(x, u) = a_4 u^4 + O(u^5)$

We have to prove the existence of *nondegenerate* critical points of the functional

$$\Phi_n : V \to \mathbb{R}, \qquad \Phi_n(v) := \Phi_0(\mathcal{H}_n v),$$

where Φ_0 is defined in (10). Let $\mathcal{H}_n : V \to V$ be the linear isomorphism defined, for $v(t, x) = \eta(t + x) - \eta(t - x) \in V$, by

$$(\mathcal{H}_n v)(t, x) := \eta(n(t + x)) - \eta(n(t - x))$$

so that $V_n \equiv \mathcal{H}_n V$.

LEMMA 1 (see [6]). *Φ_n has the following development: for $v(t, x) = \eta(t + x) - \eta(t - x) \in V$,*

$$\Phi_n(\beta n^{1/3} v) = 4\pi \beta^2 n^{8/3} \left[\Psi(\eta) + \alpha \frac{\mathcal{R}(\eta)}{n^2} \right], \tag{13}$$

where $\beta := (3/(\pi^2 a_4^2))^{1/6}$, $\alpha := 3/(8\pi^3)$,

$$\Psi(\eta) := \frac{1}{2} \int_{\mathbb{T}} \eta'^2(t)\, dt - \frac{\pi}{4} (\langle \eta^4 \rangle + 3 \langle \eta^2 \rangle^2)^2, \tag{14}$$

$\langle \, \rangle$ *denotes the average on* $\mathbb{T}$, *and*

$$\mathcal{R}(\eta) := - \int_\Omega v^4 \Box^{-1} v^4 \, dt\, dx + \frac{2\pi^4}{3} 4 (\langle \eta^4 \rangle + 3 \langle \eta^2 \rangle^2)^2. \tag{15}$$

PROOF. First, the quadratic term is

$$\frac{1}{2} \|\mathcal{H}_n v\|_{H^1}^2 = \frac{n^2}{2} \|v\|_{H^1}^2 = n^2 2\pi \int_{\mathbb{T}} \eta'^2(t)\, dt.$$

By Lemma 4.8 in [6] the nonquadratic term can be developed as

$$\int_\Omega (\mathcal{H}_n v)^4 \Box^{-1} (\mathcal{H}_n v)^4 = \frac{\pi^4}{6} \langle m \rangle^2 - \frac{\mathcal{R}(\eta)}{n^2}$$

where $m : \mathbb{T}^2 \to \mathbb{R}$ is defined by $m(s_1, s_2) := (\eta(s_1) - \eta(s_2))^4$, its average is $\langle m\rangle := (2\pi)^{-2} \int_{\mathbb{T}^2} m(s_1, s_2)\, ds_1\, ds_2$ and

$$\mathcal{R}(\eta) := -\int_{\Omega} v^4 \Box^{-1} v^4 + \frac{\pi^4}{6} \langle m\rangle^2$$

is homogeneous of degree 8. Since η is odd we find $\langle m\rangle = 2(\langle\eta^4\rangle + 3\langle\eta^2\rangle^2)$, where $\langle\ \rangle$ denotes the average on $\mathbb{T}$. Collecting these equalities we find that

$$\Phi_n(\eta) = 2\pi n^2 \int_{\mathbb{T}} \eta'^2(t)\, dt - \frac{\pi^4}{3} a_4^2 (\langle\eta^4\rangle + 3\langle\eta^2\rangle^2)^2 + \frac{a_4^2}{2n^2} \mathcal{R}(\eta).$$

Via the rescaling $\eta \mapsto \beta n^{1/3} \eta$ we get expressions (14) and (15). $\Box$

By (13), in order to find for n large enough a nondegenerate critical point of Φ_n, it is sufficient to find nondegenerate critical points of $\Psi(\eta)$ defined on

$$E := \{\eta \in H^1(\mathbb{T}) \mid \eta \text{ odd}\},$$

namely nondegenerate solutions in E of

$$\ddot{\eta} + A(\eta)(3\langle\eta^2\rangle\eta + \eta^3) = 0, \qquad A(\eta) := \langle\eta^4\rangle + 3\langle\eta^2\rangle^2. \tag{16}$$

PROPOSITION 3. *There exists an odd, analytic, 2π-periodic solution $g(t)$ of* (16) *which is nondegenerate in E. It is given by $g(t) = V \operatorname{sn}(\Omega t, m)$ where* sn *is the Jacobi elliptic sine and $V, \Omega > 0$ and $m \in (-1, 0)$ are suitable constants (therefore $g(t)$ has minimal period 2π).*

We will construct the solution g of (16) by means of the Jacobi elliptic sine in Lemma 6. The existence of a solution g also follows directly by applying to $\Psi : E \to \mathbb{R}$ the Mountain-Pass Theorem [2]. Furthermore this solution is an analytic function by arguing as in Lemma 2.1 of [7].

2.1. *Nondegeneracy of g*

We now want to prove that g is nondegenerate. The linearized equation of (16) at g is

$$\ddot{h} + 3A(g)(\langle g^2\rangle h + g^2 h) + 6A(g) g \langle gh\rangle + A'(g)[h](3\langle g^2\rangle g + g^3) = 0,$$

which we write as

$$\ddot{h} + 3A(g)(\langle g^2\rangle + g^2) h = -\langle gh\rangle I_1 - \langle g^3 h\rangle I_2 \tag{17}$$

where

$$\begin{cases} I_1 := 6(9\langle g^2\rangle^2 + \langle g^4\rangle) g + 12\langle g^2\rangle g^3, \\ I_2 := 12 g\langle g^2\rangle + 4g^3. \end{cases} \tag{18}$$

For $f \in E$, let $H := L(f)$ be the unique solution belonging to E of the nonhomogeneous linear system

$$\ddot{H} + 3A(g)(\langle g^2 \rangle + g^2)H = f; \tag{19}$$

an integral representation of the Green operator L is given in Lemma 4 below. Thus (17) becomes

$$h = -\langle gh \rangle L(I_1) - \langle g^3 h \rangle L(I_2). \tag{20}$$

Multiplying (20) by g and taking averages we get

$$\langle gh \rangle [1 + \langle gL(I_1) \rangle] = -\langle g^3 h \rangle \langle gL(I_2) \rangle, \tag{21}$$

while multiplying (20) by g^3 and taking averages yields

$$\langle g^3 h \rangle [1 + \langle g^3 L(I_2) \rangle] = -\langle gh \rangle \langle g^3 L(I_1) \rangle. \tag{22}$$

Since g solves (16) we can deduce the following identities.

LEMMA 2. *We have*

$$2A(g)\langle g^3 L(g) \rangle = \langle g^2 \rangle, \qquad 2A(g)\langle g^3 L(g^3) \rangle = \langle g^4 \rangle.$$

PROOF. The first equality is obtained from the identity for $L(g)$,

$$\frac{d^2}{dt^2}(L(g)) + 3A(g)(\langle g^2 \rangle + g^2)L(g) = g,$$

by multiplying by g, taking averages, integrating by parts,

$$\langle \ddot{g} L(g) \rangle + 3A(g)[\langle g^2 \rangle \langle L(g) g \rangle + \langle g^3 L(g) \rangle] = \langle g^2 \rangle,$$

and using the fact that g solves (16).

Analogously, the second equality is obtained from the identity for $L(g^3)$,

$$\frac{d^2}{dt^2}(L(g^3)) + 3A(g)(\langle g^2 \rangle + g^2)L(g^3) = g^3,$$

by multiplying by g, taking averages, integrating by parts, and using the fact that g solves (16). □

Since L is a symmetric operator we can compute the following averages using (18) and Lemma 2:

$$\begin{cases} \langle gL(I_1) \rangle = 6(\langle g^4 \rangle + 9\langle g^2 \rangle^2)\langle gL(g) \rangle + 6A(g)^{-1}\langle g^2 \rangle^2, \\ \langle gL(I_2) \rangle = 12\langle g^2 \rangle \langle gL(g) \rangle + 2A(g)^{-1}\langle g^2 \rangle, \\ \langle g^3 L(I_1) \rangle = 9\langle g^2 \rangle, \\ \langle g^3 L(I_2) \rangle = 2. \end{cases} \tag{23}$$

Thanks to the identities (23), equations (21), (22) simplify to

$$\begin{cases} \langle gh\rangle[A(g)+6\langle g^2\rangle^2]B(g) = -2\langle g^2\rangle B(g)\langle g^3 h\rangle, \\ \langle g^3 h\rangle = -3\langle g^2\rangle\langle gh\rangle, \end{cases} \tag{24}$$

where

$$B(g) := 1 + 6A(g)\langle gL(g)\rangle. \tag{25}$$

Solving (24) we get $B(g)\langle gh\rangle = 0$. We will prove in Lemma 5 that $B(g) \neq 0$, so $\langle gh\rangle = 0$. Hence by (24) also $\langle g^3 h\rangle = 0$ and therefore, by (20), $h = 0$. This concludes the proof of the nondegeneracy of the solution g of (16).

It remains to prove that $B(g) \neq 0$. The key is to express the function $L(g)$ by means of the variation of constants formula.

We first look for a fundamental set of solutions of the homogeneous equation

$$\ddot{h} + 3A(g)(\langle g^2\rangle + g^2)h = 0. \tag{HOM}$$

LEMMA 3. *There exist two linearly independent solutions of* (HOM), $\bar{u} := \dot{g}(t)/\dot{g}(0)$ *and* $\bar{v}$, *such that*

$$\begin{cases} \bar{u} \text{ is even}, \ 2\pi\text{-periodic}, \\ \bar{u}(0) = 1, \quad \dot{\bar{u}}(0) = 0, \end{cases} \qquad \begin{cases} \bar{v} \text{ is odd}, \ \text{not periodic}, \\ \bar{v}(0) = 0, \quad \dot{\bar{v}}(0) = 1, \end{cases}$$

and

$$\bar{v}(t+2\pi) - \bar{v}(t) = \rho\bar{u}(t) \quad \textit{for some } \rho > 0. \tag{26}$$

PROOF. Since (16) is autonomous, $\dot{g}(t)$ is a solution of the linearized equation (HOM); it is even and 2π-periodic.

We can construct another solution of (HOM) in the following way. The superquadratic Hamiltonian system (with constant coefficients)

$$\ddot{y} + 3A(g)\langle g^2\rangle y + A(g)y^3 = 0 \tag{27}$$

has a one-parameter family of odd, $T(E)$-periodic solutions $y(E,t)$, close to g, parametrized by the energy E. Let $\bar{E}$ denote the energy level of g, i.e. $g = y(\bar{E},t)$ and $T(\bar{E}) = 2\pi$. Then $l(t) := (\partial_E y(E,t))_{|E=\bar{E}}$ is an odd solution of (HOM). Differentiating the identity $y(E, t+T(E)) = y(E,t)$ with respect to E we obtain, at $E = \bar{E}$, $l(t+2\pi) - l(t) = -(\partial_E T(E))_{|E=\bar{E}}\dot{g}(t)$ and, normalizing $\bar{v}(t) := l(t)/\dot{l}(0)$, we get (26) with $\rho := -(\partial_E T(E))_{|E=\bar{E}}\dot{g}(0)/\dot{l}(0)$.

Since $y(E,0) = 0$ for all E, the energy identity gives $E = \frac{1}{2}(\dot{y}(E,0))^2$. Differentiating with respect to E at $E = \bar{E}$ yields $1 = \dot{g}(0)\dot{l}(0)$, so

$$\rho = -(\partial_E T(E))_{|E=\bar{E}}(\dot{g}(0))^2. \tag{28}$$

We have $\rho > 0$ because $(\partial_E T(E))_{|E=\bar{E}} < 0$ by the superquadraticity of the potential of (27). This can also be checked by a computation (see Remark after Lemma 6). □

Now we write an integral formula for the Green operator L.

LEMMA 4. *For every $f \in E$ there exists a unique solution $H = L(f)$ of* (19) *which can be written as*

$$L(f) = \left(\int_0^t f(s)\bar{u}(s)\, ds + \frac{1}{\rho} \int_0^{2\pi} f\bar{v} \right) \bar{v}(t) - \left(\int_0^t f(s)\bar{v}(s)\, ds \right) \bar{u}(t) \in E.$$

PROOF. The nonhomogeneous equation (19) has the particular solution

$$\bar{H}(t) = \left(\int_0^t f(s)\bar{u}(s)\, ds \right) \bar{v}(t) - \left(\int_0^t f(s)\bar{v}(s)\, ds \right) \bar{u}(t)$$

as can be verified by observing that the Wronskian $\bar{u}(t)\dot{\bar{v}}(t) - \dot{\bar{u}}(t)\bar{v}(t) \equiv 1$ for all t. Notice that $\bar{H}$ is odd.

Any solution $H(t)$ of (19) can be written as $H(t) = \bar{H}(t) + a\bar{u} + b\bar{v}$, $a, b \in \mathbb{R}$. Since $\bar{H}$ is odd, $\bar{u}$ is even and $\bar{v}$ is odd, requiring H to be odd implies $a = 0$. Imposing now the 2π-periodicity yields

$$\begin{aligned} 0 &= \left(\int_0^{t+2\pi} f\bar{u} \right) \bar{v}(t+2\pi) - \left(\int_0^{t+2\pi} f\bar{v} \right) \bar{u}(t+2\pi) - \left(\int_0^t f\bar{u} \right) \bar{v}(t) \\ &\quad + \left(\int_0^t f\bar{v} \right) \bar{u}(t) + b(\bar{v}(t+2\pi) - \bar{v}(t)) \\ &= \left(b + \int_0^t f\bar{u} \right) (\bar{v}(t+2\pi) - \bar{v}(t)) - \bar{u}(t) \left(\int_t^{t+2\pi} f\bar{v} \right), \end{aligned}$$

because $\bar{u}$ and $f\bar{u}$ are 2π-periodic and $\langle f\bar{u} \rangle = 0$. By (26) we have

$$\rho \left(b + \int_0^t f\bar{u} \right) - \int_t^{t+2\pi} f\bar{v} = 0.$$

This expression is constant in time, because, by differentiating in t,

$$\rho f(t)\bar{u}(t) - f(t)(\bar{v}(t+2\pi) - \bar{v}(t)) = 0$$

again by (26). Hence evaluating at $t = 0$ yields $b = \rho^{-1} \int_0^{2\pi} f\bar{v}$. So there exists a unique solution $H = L(f)$ of (19) belonging to E, and the integral representation of L follows. □

LEMMA 5. *We have*

$$\langle gL(g) \rangle = \frac{\rho}{4\pi A(g)} + \frac{1}{2\pi\rho} \left(\int_0^{2\pi} g\bar{v} \right)^2 > 0$$

because $A(g) > 0$ and $\rho > 0$.

PROOF. Using the formula of Lemma 4 and integrating by parts we can compute

$$\begin{aligned}\langle gL(g)\rangle &= \frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^t g\bar u\right)\bar v(t)g(t)\,dt + \frac{1}{2\pi\rho}\left(\int_0^{2\pi} g\bar v\right)^2\\ &\quad -\frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^t g\bar v\right)\bar u(t)g(t)\,dt\\ &= 2\frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^t g\bar u\right)\bar v(t)g(t)\,dt\ + \frac{1}{2\pi\rho}\left(\int_0^{2\pi} g\bar v\right)^2\end{aligned}$$

because $\int_0^{2\pi} g\bar u = 0$. Since $\bar u(t) = \dot g(t)/\dot g(0)$ and $g(0) = 0$, we have

$$\int_0^t g\bar u = \frac{1}{2\dot g(0)}g^2(t), \qquad \int_0^{2\pi}\left(\int_0^t g\bar u\right)\bar v(t)g(t)\,dt = \frac{1}{2\dot g(0)}\int_0^{2\pi} g^3\bar v,$$

so it remains to show that

$$\int_0^{2\pi} g^3\bar v = \frac{\rho\dot g(0)}{2A(g)}. \tag{29}$$

Since g solves (16), multiplying by $\bar v$ and integrating yields

$$\int_0^{2\pi}[\bar v(t)\ddot g(t) + 3A(g)\langle g^2\rangle g(t)\bar v(t) + A(g)g^3(t)\bar v(t)]\,dt = 0.$$

Since $\bar v$ solves (HOM), multiplying by g and integrating gives

$$\int_0^{2\pi}[g(t)\ddot{\bar v}(t) + 3A(g)\langle g^2\rangle \bar v(t)g(t) + 3A(g)g^3(t)\bar v(t)]\,dt = 0.$$

Subtracting the last two equalities we get

$$\int_0^{2\pi}[\bar v(t)\ddot g(t) - g(t)\ddot{\bar v}(t)]\,dt = 2A(g)\int_0^{2\pi} g^3\bar v.$$

Integrating by parts the left hand side, since $g(0) = g(2\pi) = 0$, $\bar u(0) = 1$ and (26), we obtain

$$\int_0^{2\pi}[\bar v(t)\ddot g(t) - g(t)\ddot{\bar v}(t)]\,dt = \dot g(0)[v(2\pi) - v(0)] = \rho\dot g(0).$$

So $2A(g)\int_0^{2\pi} g^3\bar v = \rho\dot g(0)$. □

2.2. *Explicit computations*

We now give the explicit construction of g by means of the Jacobi elliptic sine defined as follows. Let $\mathrm{am}(\cdot, m) : \mathbb{R} \to \mathbb{R}$ be the inverse function of the Jacobi elliptic integral of the first kind

$$\varphi \mapsto F(\varphi, m) := \int_0^{\varphi}\frac{d\vartheta}{\sqrt{1 - m\sin^2\vartheta}}.$$

The Jacobi elliptic sine is defined by

$$\mathrm{sn}(t,m) := \sin(\mathrm{am}(t,m)).$$

It is $4K(m)$-periodic, where $K(m)$ is the complete elliptic integral of the first kind

$$K(m) := F\left(\frac{\pi}{2}, m\right) = \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - m\sin^2\vartheta}},$$

and admits an analytic extension with a pole at $iK(1-m)$ for $m \in (0,1)$ and at $iK(1/(1-m))/\sqrt{1-m}$ for $m < 0$. Moreover, since $\partial_t \,\mathrm{am}(t,m) = \sqrt{1 - m\,\mathrm{sn}^2(t,m)}$, the elliptic sine satisfies

$$(\dot{\mathrm{sn}})^2 = (1 - \mathrm{sn}^2)(1 - m\,\mathrm{sn}^2). \tag{30}$$

LEMMA 6. *There exist $V, \Omega > 0$ and $m \in (-1,0)$ such that $g(t) := V\,\mathrm{sn}(\Omega t, m)$ is an odd, analytic, 2π-periodic solution of* (16) *with pole at $iK(1/(1-m))/(\Omega\sqrt{1-m})$.*

PROOF. Differentiating (30) we have $\ddot{\mathrm{sn}} + (1+m)\,\mathrm{sn} - 2m\,\mathrm{sn}^3 = 0$. Therefore $g_{(V,\Omega,m)}(t) := V\,\mathrm{sn}(\Omega t, m)$ is an odd, $(4K(m)/\Omega)$-periodic solution of

$$\ddot{g} + \Omega^2(1+m)g - 2m\frac{\Omega^2}{V^2}g^3 = 0. \tag{31}$$

The function $g_{(V,\Omega,m)}$ will be a solution of (16) if (V, Ω, m) satisfy

$$\begin{cases} \Omega^2(1+m) = 3A(g_{(V,\Omega,m)})\langle g^2_{(V,\Omega,m)}\rangle, \\ -2m\Omega^2 = V^2 A(g_{(V,\Omega,m)}), \\ 2K(m) = \Omega\pi. \end{cases} \tag{32}$$

Dividing the first equation of (32) by the second yields

$$-\frac{1+m}{6m} = \langle \mathrm{sn}^2(\cdot, m)\rangle. \tag{33}$$

The right hand side can be expressed as

$$\langle \mathrm{sn}^2(\cdot, m)\rangle = \frac{K(m) - E(m)}{mK(m)}, \tag{34}$$

where $E(m)$ is the complete elliptic integral of the second kind,

$$E(m) := \int_0^{\pi/2} \sqrt{1 - m\sin^2\vartheta}\, d\vartheta = \int_0^{K(m)} (1 - m\,\mathrm{sn}^2(\xi, m))\, d\xi$$

(in the last passage we make the change of variable $\vartheta = \mathrm{am}(\xi, m)$).

Now, we show that system (32) has a unique solution. By (33) and (34),

$$(7+m)K(m) - 6E(m) = 0.$$

By the definitions of $E(m)$ and $K(m)$ we have

$$\psi(m) := (7+m)K(m) - 6E(m) = \int_0^{\pi/2} \frac{1+m(1+6\sin^2\vartheta)}{(1-m\sin^2\vartheta)^{1/2}}\, d\vartheta.$$

We have $\psi(0) = \pi/2 > 0$ and $\psi(-1) = -\int_0^{\pi/2} 6\sin^2\vartheta\ (1+\sin^2\vartheta)^{-1/2}\, d\vartheta < 0$. Since ψ is continuous there exists $\bar{m} \in (-1, 0)$ such that $\psi(\bar{m}) = 0$. Next the third equation in (32) fixes $\bar{\Omega}$ and finally we find $\bar{V}$. Hence $g(t) = \bar{V}\,\mathrm{sn}(\bar{\Omega}t, \bar{m})$ solves (16).

Analyticity and poles follow from [1, 16.2, 16.10.2, pp. 570, 573].

Finally, $\bar{m}$ is unique because $\psi'(m) > 0$ for $m \in (-1, 0)$ as can be verified by differentiating the formula for ψ. One can also compute that $\bar{m} \in (-0.30, -0.28)$. □

REMARK. We can explicitly compute the sign of dT/dE and ρ of (28) in the following way. The functions $g_{(V,\Omega,m)}$ are solutions of the Hamiltonian system (27)

$$\begin{cases} \Omega^2(1+m) = \alpha, \\ -2m\Omega^2 = V^2\beta, \end{cases} \tag{35}$$

where $\alpha := 3A(g)\langle g^2\rangle$, $\beta := A(g)$ and g is the solution constructed in Lemma 6.

We solve (35) with respect to m to find the one-parameter family (y_m) of odd periodic solutions $y_m(t) := V(m)\,\mathrm{sn}(\Omega(m)t, m)$, close to g, with energy and period

$$E(m) = \frac{1}{2}V^2(m)\Omega^2(m) = -\frac{1}{\beta}m\Omega^4(m), \qquad T(m) = \frac{4K(m)}{\Omega(m)}.$$

We have

$$\frac{dT(m)}{dm} = \frac{4K'(m)\Omega(m) - 4K(m)\Omega'(m)}{\Omega^2(m)} > 0,$$

because $K'(m) > 0$ and from (35), $\Omega'(m) = -\Omega(m)(2(1+m))^{-1} < 0$. Then

$$\frac{dE(m)}{dm} = -\frac{1}{\beta}\Omega^4(m) - \frac{1}{\beta}m4\Omega^3(m)\Omega'(m) < 0,$$

so

$$\frac{dT}{dE} = \frac{dT(m)}{dm}\left(\frac{dE(m)}{dm}\right)^{-1} < 0$$

as stated by general arguments in the proof of Lemma 3.

We can also write an explicit formula for ρ,

$$\rho = \frac{m}{m-1}\left[2\pi + (1+m)\int_0^{2\pi} \frac{\mathrm{sn}^2(\Omega t, m)}{\mathrm{dn}^2(\Omega t, m)}\, dt\right].$$

From this formula it follows that $\rho > 0$ because $-1 < m < 0$.

3. Case $f(x,u) = a_2 u^2 + a_3(x) u^3 + O(u^4)$

We have to prove the existence of *nondegenerate* critical points of the functional $\Phi_n(v) := \Phi_0(\mathcal{H}_n v)$ where Φ_0 is defined in (12).

LEMMA 7 (see [6]). *Φ_n has the following development: for $v(t,x) = \eta(t+x) - \eta(t-x) \in V$,*

$$\Phi_n(\beta n v) = 4\pi\beta^2 n^4 \left[\Psi(\eta) + \frac{\beta^2}{4\pi} \left(\frac{R_2(\eta)}{n^2} + R_3(\eta) \right) \right],$$

where

$$\Psi(\eta) := \frac{s^*}{2} \int_{\mathbb{T}} \dot\eta^2 + \frac{\beta^2}{4\pi} \left[\alpha \left(\int_{\mathbb{T}} \eta^2 \right)^2 + \gamma \int_{\mathbb{T}} \eta^4 \right],$$
$$R_2(\eta) := -\frac{a_2^2}{2} \left[\int_\Omega v^2 \Box^{-1} v^2 - \frac{\pi^2}{6} \left(\int_{\mathbb{T}} \eta^2 \right)^2 \right],$$
$$R_3(\eta) := \frac{1}{4} \int_\Omega (a_3(x) - \langle a_3 \rangle)(\mathcal{H}_n v)^4,$$

$\alpha := (9\langle a_3\rangle - \pi^2 a_2^2)/12$, $\gamma := \pi\langle a_3\rangle/2$, *and*

$$\beta = \begin{cases} (2|\alpha|)^{-1/2} & \text{if } \alpha \neq 0, \\ (\pi/\gamma)^{1/2} & \text{if } \alpha = 0. \end{cases}$$

PROOF. By Lemma 4.8 in [6] with $m(s_1,s_2) = (\eta(s_1) - \eta(s_2))^2$, for $v(t,x) = \eta(t+x) - \eta(t-x)$ the operator Φ_n admits the development

$$\Phi_n(v) = 2\pi s^* n^2 \int_{\mathbb{T}} \dot\eta^2(t)\,dt - \frac{\pi^2 a_2^2}{12} \left(\int_{\mathbb{T}} \eta^2(t)\,dt \right)^2$$
$$- \frac{a_2^2}{2n^2} \left(\int_\Omega v^2 \Box^{-1} v^2 - \frac{\pi^2}{6} \left(\int_{\mathbb{T}} \eta^2(t)\,dt \right)^2 \right)$$
$$+ \frac{1}{4}\langle a_3\rangle \int_\Omega v^4 + \frac{1}{4} \int_\Omega (a_3(x) - \langle a_3\rangle)(\mathcal{H}_n v)^4.$$

Since

$$\int_\Omega v^4 = 2\pi \int_{\mathbb{T}} \eta^4 + 3 \left(\int_{\mathbb{T}} \eta^2 \right)^2,$$

we write

$$\Phi_n(v) = 2\pi s^* n^2 \int_{\mathbb{T}} \dot\eta^2 - \frac{\pi^2 a_2^2}{12} \left(\int_{\mathbb{T}} \eta^2 \right)^2 + \frac{1}{4}\langle a_3\rangle \left[2\pi \int_{\mathbb{T}} \eta^4 + 3 \left(\int_{\mathbb{T}} \eta^2 \right)^2 \right]$$
$$+ \frac{R_2(\eta)}{n^2} + R_3(\eta),$$

where R_2, R_3 defined above are both homogeneous of degree 4. So

$$\Phi_n(v) = 2\pi s^* n^2 \int_{\mathbb{T}} \dot\eta^2 + \alpha \left(\int_{\mathbb{T}} \eta^2 \right)^2 + \gamma \int_{\mathbb{T}} \eta^4 + \frac{R_2(\eta)}{n^2} + R_3(\eta),$$

where α, γ are defined above. The rescaling $\eta \mapsto \eta\beta n$ concludes the proof. □

In order to find for n large a nondegenerate critical point of Φ_n, by the decomposition of Lemma 7 it is sufficient to find critical points of Ψ on $E = \{\eta \in H^1(\mathbb{T}) \mid \eta \text{ odd}\}$ (as in Lemma 6.2 of [7], also the term $R_3(\eta)$ tends to 0 with its derivatives).

If $\langle a_3\rangle \in (-\infty, 0) \cup (\pi^2 a_2^2/9, +\infty)$, then $\alpha \neq 0$ and we must choose $s^* = -\text{sign}(\alpha)$, so that the functional becomes

$$\Psi(\eta) = \text{sign}(\alpha)\left(-\frac{1}{2}\int_{\mathbb{T}} \dot{\eta}^2 + \frac{1}{8\pi}\left[\left(\int_{\mathbb{T}} \eta^2\right)^2 + \frac{\gamma}{\alpha}\int_{\mathbb{T}} \eta^4\right]\right).$$

Since in this case $\gamma/\alpha > 0$, the functional Ψ clearly has a mountain-pass critical point which solves

$$\ddot{\eta} + \langle \eta^2\rangle \eta + \lambda \eta^3 = 0, \qquad \lambda = \frac{\gamma}{2\pi\alpha} > 0. \tag{36}$$

The proof of the nondegeneracy of the solution of (36) is very simple by using the analytic arguments of the previous section (since $\lambda > 0$ a positivity argument is sufficient).

If $\langle a_3\rangle = 0$, then the equation becomes $\ddot{\eta} + \langle \eta^2\rangle\eta = 0$, so we find again what was proved in [7] for $a_3(x) \equiv 0$.

If $\langle a_3\rangle = \pi^2 a_2^2/9$, then $\alpha = 0$. We must choose $s^* = -1$, so that we obtain

$$\Psi(\eta) = -\frac{1}{2}\int_{\mathbb{T}} \dot{\eta}^2 + \frac{1}{4}\int_{\mathbb{T}} \eta^4, \qquad \ddot{\eta} + \eta^3 = 0.$$

This equation has periodic solutions which are nondegenerate because of non-isochronicity (see Proposition 2 in [8]).

Finally, if $\langle a_3\rangle \in (0, \pi^2 a_2^2/9)$, then $\alpha < 0$ and there are solutions for both $s^* = \pm 1$. The functional

$$\begin{aligned}\Psi(\eta) &= \frac{s^*}{2}\int_{\mathbb{T}} \dot{\eta}^2 + \frac{1}{8\pi}\left[-\left(\int_{\mathbb{T}} \eta^2\right)^2 + \frac{\gamma}{|\alpha|}\int_{\mathbb{T}} \eta^4\right] \\ &= \frac{s^*}{2}\int_{\mathbb{T}} \dot{\eta}^2 + \frac{1}{4}\int_{\mathbb{T}} \eta^4[\lambda - Q(\eta)],\end{aligned}$$

where

$$\lambda := \frac{\gamma}{2\pi|\alpha|} > 0, \qquad Q(\eta) := \frac{(\int_{\mathbb{T}} \eta^2)^2}{2\pi \int_{\mathbb{T}} \eta^4},$$

has mountain-pass critical points for any $\lambda > 0$ because (as in Lemma 3.14 of [6])

$$\inf_{\eta \in E\setminus\{0\}} Q(\eta) = 0, \qquad \sup_{\eta \in E\setminus\{0\}} Q(\eta) = 1$$

(for $\lambda \geq 1$ if $s^* = -1$, and for $0 < \lambda < 1$ for both $s^* = \pm 1$).

Such critical points satisfy the Euler–Lagrange equation

$$-s^*\ddot{\eta} - \langle \eta^2\rangle\eta + \lambda\eta^3 = 0 \tag{37}$$

but their nondegeneracy is not obvious. For this, it is convenient to express these solutions in terms of the Jacobi elliptic sine.

PROPOSITION 4. (i) *Let $s^* = -1$. Then for every $\lambda \in (0, +\infty)$ there exists an odd, analytic, 2π-periodic solution $g(t)$ of* (37) *which is nondegenerate in E. It is given by $g(t) = V\,\mathrm{sn}(\Omega t, m)$ for suitable constants $V, \Omega > 0$ and $m \in (-\infty, -1)$.*

(ii) *Let $s^* = 1$. Then for every $\lambda \in (0, 1)$ there exists an odd, analytic, 2π-periodic solution $g(t)$ of* (37) *which is nondegenerate in E. It is given by $g(t) = V\,\mathrm{sn}(\Omega t, m)$ for suitable constants $V, \Omega > 0$ and $m \in (0, 1)$.*

We prove Proposition 4 in several steps. First we construct the solution g as in Lemma 6.

LEMMA 8. (i) *Let $s^* = -1$. Then for every $\lambda \in (0, +\infty)$ there exist $V, \Omega > 0$ and $m \in (-\infty, -1)$ such that $g(t) = V\,\mathrm{sn}(\Omega t, m)$ is an odd, analytic, 2π-periodic solution of* (37) *with a pole at $iK(1/(1-m))/(\Omega\sqrt{1-m})$.*

(ii) *Let $s^* = 1$. Then for every $\lambda \in (0, 1)$ there exist $V, \Omega > 0$ and $m \in (0, 1)$ such that $g(t) = V\,\mathrm{sn}(\Omega t, m)$ is an odd, analytic, 2π-periodic solution of* (37) *with a pole at $iK(1-m)/\Omega$.*

PROOF. We know that $g_{(V,\Omega,m)}(t) := V\,\mathrm{sn}(\Omega t, m)$ is an odd, $(4K(m)/\Omega)$-periodic solution of (31) (see Lemma 6). So it is a solution of (37) if (V, Ω, m) satisfy

$$\begin{cases} \Omega^2(1+m) = s^* V^2 \langle \mathrm{sn}^2(\cdot, m)\rangle, \\ 2m\Omega^2 = s^* V^2 \lambda, \\ 2K(m) = \Omega\pi. \end{cases} \tag{38}$$

Conditions (38) give the connection between λ and m:

$$\lambda = \frac{2m}{1+m}\langle \mathrm{sn}^2(\cdot, m)\rangle. \tag{39}$$

Moreover system (38) imposes

$$\begin{cases} m \in (-\infty, -1) & \text{if } s^* = -1, \\ m \in (0, 1) & \text{if } s^* = 1. \end{cases}$$

We know that $m \mapsto \langle \mathrm{sn}^2(\cdot, m)\rangle$ is continuous, strictly increasing on $(-\infty, 1)$, and tends to 0 as $m \to -\infty$ and to 1 as $m \to 1$ (see Lemma 12 below). So the right hand side of (39) covers $(0, +\infty)$ for $m \in (-\infty, 0)$, and it covers $(0, 1)$ for $m \in (0, 1)$. For this reason for every $\lambda > 0$ there exists a unique $\bar{m} < -1$ satisfying (39), and for every $\lambda \in (0, 1)$ there exists a unique $\bar{m} \in (0, 1)$ satisfying (39).

The value $\bar{m}$ and system (38) uniquely determine the values $\bar{V}, \bar{\Omega}$.

Analyticity and poles follow from [1, 16.2, 16.10.2, pp. 570, 573]. □

Now we have to prove the nondegeneracy of g. The linearized equation of (37) at g is

$$\ddot{h} + s^*(\langle g^2\rangle - 3\lambda g^2)h = -2s^*\langle gh\rangle g.$$

Let L be the Green operator, i.e. for $f \in E$, let $H := L(f)$ be the unique solution belonging to E of the nonhomogeneous linear system

$$\ddot{H} + s^*(\langle g^2\rangle - 3\lambda g^2)H = f.$$

We can write the linearized equation as $h = -2s^*\langle gh\rangle L(g)$. Multiplying by g and integrating we get

$$\langle gh\rangle[1 + 2s^*\langle gL(g)\rangle] = 0.$$

If $A_0 := 1 + 2s^*\langle gL(g)\rangle \neq 0$, then $\langle gh\rangle = 0$, so $h = 0$ and the nondegeneracy is proved.

It remains to show that $A_0 \neq 0$. As before, the key is to express $L(g)$ in a suitable way. We first look for a fundamental set of solutions of the homogeneous equation

$$\ddot{h} + s^*(\langle g^2\rangle - 3\lambda g^2)h = 0. \tag{40}$$

LEMMA 9. *There exist two linearly independent solutions of* (40), $\bar{u}$ *even,* 2π*-periodic and* $\bar{v}$ *odd, not periodic, such that* $\bar{u}(0) = 1$, $\dot{\bar{u}}(0) = 0$, $\bar{v}(0) = 0$, $\dot{\bar{v}}(0) = 1$, *and*

$$\bar{v}(t + 2\pi) - \bar{v}(t) = \rho\bar{u}(t) \quad \forall t \tag{41}$$

for some $\rho \neq 0$. *Moreover*

$$\bar{u}(t) = \dot{g}(t)/\dot{g}(0) = \dot{\mathrm{sn}}(\bar{\Omega}t, \bar{m}),$$

$$\bar{v}(t) = \frac{1}{\bar{\Omega}(1-\bar{m})}\,\mathrm{sn}(\bar{\Omega}t) + \frac{\bar{m}}{\bar{m}-1}\dot{\mathrm{sn}}(\bar{\Omega}t)\left[t + \frac{1+\bar{m}}{\bar{\Omega}}\int_0^{\bar{\Omega}t}\frac{\mathrm{sn}^2(\xi,\bar{m})}{\mathrm{dn}^2(\xi,\bar{m})}\,d\xi\right].$$

PROOF. g solves (37) so $\dot{g}$ solves (40); normalizing we find $\bar{u}$.

By (31), the function $y(t) = V\,\mathrm{sn}(\Omega t, m)$ solves

$$\ddot{y} + s^*\langle g^2\rangle y - s^*\lambda y^3 = 0 \tag{42}$$

if (V, Ω, m) satisfy

$$\begin{cases} \Omega^2(1+m) = s^*\langle g^2\rangle, \\ 2m\Omega^2 = s^*V^2\lambda. \end{cases}$$

We solve this system with respect to m. We obtain a one-parameter family (y_m) of odd periodic solutions of (42), $y_m(t) = V(m)\,\mathrm{sn}(\Omega(m)t, m)$. So $l(t) := (\partial_m y_m)_{|m=\bar{m}}$ solves (40). We normalize $\bar{v}(t) := l(t)/\dot{l}(0)$ and we compute the coefficients by differentiating the system with respect to m. From the definitions of the Jacobi elliptic functions we find that

$$\partial_m\,\mathrm{sn}(x, m) = -\dot{\mathrm{sn}}(x, m)\frac{1}{2}\int_0^x\frac{\mathrm{sn}^2(\xi, m)}{\mathrm{dn}^2(\xi, m)}\,d\xi;$$

thanks to this formula we obtain the expression of $\bar{v}$.

Since $2\pi\bar{\Omega} = 4K(\bar{m})$ is the period of the Jacobi functions sn and dn, from the formulae for $\bar{u}$, $\bar{v}$ we obtain (41) with

$$\rho = \frac{\bar{m}}{\bar{m}-1}2\pi\left(1 + (1+\bar{m})\left\langle\frac{\mathrm{sn}^2}{\mathrm{dn}^2}\right\rangle\right).$$

If $s^* = 1$, then $\bar{m} \in (0, 1)$ and we can see directly that $\rho < 0$. If $s^* = -1$, then $\bar{m} < -1$. From the equality $\langle\mathrm{sn}^2/\mathrm{dn}^2\rangle = (1-m)^{-1}(1 - \langle\mathrm{sn}^2\rangle)$ (see [3, Lemma 3, (L.2)]), it follows that $\rho > 0$. □

Note that the integral representation of the Green operator L holds again in the present case. The formula and the proof are just as for Lemma 4.

LEMMA 10. *We can write* $A_0 := 1 + 2s^*\langle gL(g)\rangle$ *as a function of* λ, $\bar{m}$,

$$A_0 = \frac{\lambda(1-\bar{m})^2 q - (1-\lambda)^2(1+\bar{m})^2 + \bar{m}q^2}{\lambda(1-\bar{m})^2 q}$$

where $q = q(\lambda, \bar{m}) := 2 - \lambda(1+\bar{m})^2(2\bar{m})^{-1}$. *Moreover,* $q > 0$.

PROOF. First, we calculate $\langle gL(g)\rangle$ by means of the integral formula of Lemma 4. The first two equalities in the proof of Lemma 5 still hold, while similar calculations give $\int_0^{2\pi} g^3\bar{v} = -s^*\dot{g}(0)\rho/2\lambda$ instead of (29). So

$$\langle gL(g)\rangle = -s^*\frac{\rho}{4\pi\lambda} + \frac{1}{2\pi\rho}\Big(\int_0^{2\pi} g\bar{v}\Big)^2 \tag{43}$$

and the sign of A_0 is not obvious. We calculate $\int_0^{2\pi} g\bar{v}$ recalling that $g(t) = \bar{V}\,\mathrm{sn}(\bar{\Omega}t, \bar{m})$, using the formula for $\bar{v}$ of Lemma 9 and integrating by parts:

$$\int_0^{2\pi} \mathrm{sn}(\bar{\Omega}t)\dot{\mathrm{sn}}(\bar{\Omega}t)\mu(t)\,dt = -\frac{1}{2\bar{\Omega}}\int_0^{2\pi} \mathrm{sn}^2(\bar{\Omega}t)\dot{\mu}(t)\,dt,$$

where $\mu(t) := t + (1+\bar{m})\bar{\Omega}^{-1}\int_0^{\bar{\Omega}t} \mathrm{sn}^2(\xi)/\,\mathrm{dn}^2(\xi)\,d\xi$. From [3, (L.2), (L.3) in Lemma 3], we obtain the formula

$$\Big\langle \frac{\mathrm{sn}^4}{\mathrm{dn}^2}\Big\rangle = \frac{1 + (m-2)\langle \mathrm{sn}^2\rangle}{m(1-m)}$$

and consequently

$$\int_0^{2\pi} g\bar{v} = \frac{\pi\bar{V}}{\bar{\Omega}(1-\bar{m})^2}(1 + \bar{m} - 2\bar{m}\langle \mathrm{sn}^2\rangle).$$

By the second equality of (38) and (43),

$$A_0 = 1 + \frac{2}{\lambda}\Big[-\frac{\rho}{4\pi} + \frac{\pi\bar{m}}{\rho(1-\bar{m})^4}(1 + \bar{m} - 2\bar{m}\langle \mathrm{sn}^2\rangle)^2\Big]$$

for both $s^* = \pm 1$. From the proof of Lemma 9 we have $\rho = -2\pi\bar{m}q/(1-\bar{m})^2$, where q is defined above; inserting this expression of ρ in the last equality we obtain the formula for A_0.

Finally, for $\bar{m} < -1$ we have immediately $q > 0$, while for $\bar{m} \in (0, 1)$ we get $q = 2 - (1+\bar{m})\langle \mathrm{sn}^2\rangle$ by (39). Since $\langle \mathrm{sn}^2\rangle < 1$, it follows that $q > 0$. □

LEMMA 11. $A_0 \neq 0$. *More precisely,* $\mathrm{sign}(A_0) = -s^*$.

PROOF. From Lemma 10, $A_0 > 0$ iff $\lambda(1-\bar{m})^2 q - (1-\lambda)^2(1+\bar{m})^2 + \bar{m}q^2 > 0$. This expression is equal to $-(1-\bar{m})^2 p$, where

$$p = p(\lambda, \bar{m}) = \frac{(1+\bar{m})^2}{4\bar{m}}\lambda^2 - 2\lambda + 1,$$

so $A_0 > 0$ iff $p < 0$. The polynomial $p(\lambda)$ has degree 2 and its determinant is $\Delta = -(1-\bar{m})^2/\bar{m}$. So, if $s^* = 1$, then $\bar{m} \in (0,1)$, $\Delta < 0$ and $p > 0$, so that $A_0 < 0$.

It remains to consider the case $s^* = -1$. For $\lambda > 0$, we have $p(\lambda) < 0$ iff $\lambda > x^*$, where x^* is the positive root of p, $x^* := 2R(1+R)^{-2}$, $R := |\bar{m}|^{1/2}$. By (39), $\lambda > x^*$ iff

$$\langle \mathrm{sn}^2(\cdot, \bar{m})\rangle > \frac{R-1}{(R+1)R}.$$

By formula (34) and by definition of complete elliptic integrals K and E we can write this inequality as

$$\int_0^{\pi/2}\left(\frac{R-1}{(R+1)R} - \sin^2\vartheta\right)\frac{d\vartheta}{\sqrt{1+R^2\sin^2\vartheta}} < 0. \tag{44}$$

We put $\sigma := (R-1)/((R+1)R)$ and note that $\sigma < 1/2$ for every $R > 0$.

We have $\sigma - \sin^2\vartheta > 0$ iff $\vartheta \in (0, \vartheta^*)$, where $\vartheta^* := \arcsin(\sqrt{\sigma})$, i.e. $\sin^2\vartheta^* = \sigma$. Moreover $1 < 1 + R^2\sin^2\vartheta < 1 + R^2$ for every $\vartheta \in (0, \pi/2)$. So

$$\int_0^{\pi/2}\frac{\sigma - \sin^2\vartheta}{\sqrt{1+R^2\sin^2\vartheta}}\,d\vartheta \;<\; \int_0^{\vartheta^*}(\sigma - \sin^2\vartheta)\,d\vartheta + \int_{\vartheta^*}^{\pi/2}\frac{\sigma-\sin^2\vartheta}{\sqrt{1+R^2}}\,d\vartheta. \tag{45}$$

Thanks to the formula

$$\int_a^b \sin^2\vartheta\,d\vartheta = \frac{b-a}{2} - \frac{\sin(2b)-\sin(2a)}{4}$$

the right hand side of (45) is equal to

$$\frac{\sin(2\vartheta^*)}{4}\left[(2\sigma-1)\left(\frac{2\vartheta^*}{\sin(2\vartheta^*)} + \frac{1}{\sqrt{1+R^2}}\frac{\pi-2\vartheta^*}{\sin(2\vartheta^*)}\right) + \left(1 - \frac{1}{\sqrt{1+R^2}}\right)\right].$$

Since $2\sigma - 1 < 0$ and $\alpha > \sin\alpha$ for every $\alpha > 0$, this quantity is less than

$$\frac{\sin(2\vartheta^*)}{4}\left[(2\sigma-1)\left(1 + \frac{1}{\sqrt{1+R^2}}\right) + \left(1 - \frac{1}{\sqrt{1+R^2}}\right)\right].$$

By definition of σ, the last quantity is negative for every $R > 0$, so (44) is true. Consequently, $\lambda > x^*$, $p < 0$ and $A_0 > 0$. □

APPENDIX

We show the properties of the function $m \mapsto \langle \mathrm{sn}^2(\cdot, m)\rangle$ used in the proof of Lemma 8.

LEMMA 12. *The function $\varphi : (-\infty, 1) \to \mathbb{R}$, $m \mapsto \langle \mathrm{sn}^2(\cdot, m)\rangle$, is continuous, differentiable, and strictly increasing. It tends to zero as $m \to -\infty$ and to* 1 *as $m \to 1$.*

PROOF. By (34) and by definition of complete elliptic integrals K and E,

$$\varphi(m) = \frac{K(m) - E(m)}{mK(m)} = \int_0^{\pi/2} \frac{\sin^2\vartheta\, d\vartheta}{\sqrt{1 - m\sin^2\vartheta}} \left(\int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - m\sin^2\vartheta}} \right)^{-1},$$

so the continuity of φ is evident.

Using the equality $\sin^2 + \cos^2 = 1$ and the change of variable $\vartheta \mapsto \pi/2 - \vartheta$ in the integrals which define K and E, we obtain, for every $m < 1$,

$$K(m) = \frac{1}{\sqrt{1-m}} K\left(\frac{m}{m-1}\right), \qquad E(m) = \sqrt{1-m}\, E\left(\frac{m}{m-1}\right).$$

We put $\mu := m/(m-1)$, so

$$\varphi(m) = 1 - \frac{1}{\mu} + \frac{E(\mu)}{\mu K(\mu)}.$$

Since μ tends to 1 as $m \to -\infty$ and $E(1) = 1$ and $\lim_{\mu\to 1} K(\mu) = +\infty$, the last formula gives $\lim_{m\to-\infty} \varphi(m) = 0$. Since $E(m)/K(m)$ tends to 0 as $m \to 1$, (34) implies that $\lim_{m\to 1} \varphi(m) = 1$.

Differentiating the integrals which define K and E with respect to m we obtain

$$E'(m) = \frac{E(m) - K(m)}{2m}, \qquad K'(m) = \frac{1}{2m}\left(\int_0^{\pi/2} \frac{d\vartheta}{(1 - m\sin^2\vartheta)^{3/2}} - K(m) \right)$$

so

$$\varphi'(m) = \frac{1}{2m^2K^2(m)}\left[E(m) \int_0^{\pi/2} \frac{d\vartheta}{(1 - m\sin^2\vartheta)^{3/2}} - K^2(m) \right].$$

The term in square brackets is positive by the strict Hölder inequality for $(1-m\sin^2\vartheta)^{-3/4}$ and $(1 - m\sin^2\vartheta)^{1/4}$. □

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