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Mathematical analysis. — *Local clustering of the non-zero set of functions in* $W^{1,1}(E)$,
by EMMANUELE DIBENEDETTO, UGO GIANAZZA and VINCENZO VESPRI.

ABSTRACT. — We extend to the $p = 1$ case a measure-theoretic lemma previously proved by DiBenedetto and Vespri for functions $u \in W^{1,p}(K_\rho)$ where K_ρ is an N-dimensional cube of edge ρ. It states that if the set where u is bounded away from zero occupies a sizeable portion of K_ρ, then the set where u is positive clusters about at least one point of K_ρ.

KEY WORDS: $W^{1,1}$ functions; measure theory; positivity set.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 46E35; Secondary 26B35.

1. INTRODUCTION AND STATEMENT OF THE RESULT

For $\rho > 0$, denote by $K_\rho(y) \subset \mathbb{R}^N$ a cube of edge ρ centered at y. If y is the origin of $\mathbb{R}^N$, we write $K_\rho(0) = K_\rho$. For any measurable set $A \subset \mathbb{R}^N$, by $|A|$ we denote its N-dimensional Lebesgue measure. We prove the following Measure-Theoretic Lemma.

LEMMA. *Let* $u \in W^{1,1}(K_\rho)$ *satisfy*

$$\|u\|_{W^{1,1}(K_\rho)} \le \gamma \rho^{N-1} \quad \textit{and} \quad |[u > 1]| \ge \alpha |K_\rho| \tag{1.1}$$

for some $\gamma > 0$ *and* $\alpha \in (0,1)$. *Then for every* $\delta \in (0,1)$ *and* $0 < \lambda < 1$ *there exist* $x_0 \in K_\rho$ *and* $\eta = \eta(\alpha, \delta, \gamma, \lambda, N) \in (0,1)$ *such that*

$$|[u > \lambda] \cap K_{\eta\rho}(x_0)| > (1-\delta)|K_{\eta\rho}(x_0)|. \tag{1.2}$$

Roughly speaking, the Lemma asserts that if the set where u is bounded away from zero occupies a sizeable portion of K_ρ, then there exists at least one point x_0 and a neighborhood $K_{\eta\rho}(x_0)$ such that u remains large in a large portion of $K_{\eta\rho}(x_0)$. Thus the set where u is positive clusters about at least one point of K_ρ.

The Lemma was established in [1] for $u \in W^{1,p}(K_\rho)$ and $p > 1$. Such a limitation on p was essential to the proof. We give a new proof which includes the case $p = 1$ and is simpler.

2. PROOF

It suffices to establish the Lemma for u continuous and $\rho = 1$. For $n \in \mathbb{N}$ partition K_1 into n^N cubes, with pairwise disjoint interiors and each of edge $1/n$. Divide these cubes into two finite subcollections $\mathbf{Q}^+$ and $\mathbf{Q}^-$ by

$$Q_j \in \mathbf{Q}^+ \;\Rightarrow\; |[u > 1] \cap Q_j| > \frac{\alpha}{2}|Q_j|,$$
$$Q_i \in \mathbf{Q}^- \;\Rightarrow\; |[u > 1] \cap Q_i| \le \frac{\alpha}{2}|Q_i|,$$

and denote by $\#(\mathbf{Q}^+)$ the number of cubes in $\mathbf{Q}^+$. By the assumption,

$$\sum_{Q_j \in \mathbf{Q}^+} |[u > 1] \cap Q_j| + \sum_{Q_i \in \mathbf{Q}^-} |[u > 1] \cap Q_i| > \alpha |K_1| = \alpha n^N |Q|$$

where $|Q|$ is the common measure of the Q_ℓ. From the definitions of the classes $\mathbf{Q}^\pm$,

$$\alpha n^N < \sum_{Q_j \in \mathbf{Q}^+} \frac{|[u > 1] \cap Q_j|}{|Q_j|} + \sum_{Q_i \in \mathbf{Q}^-} \frac{|[u > 1] \cap Q_i|}{|Q_i|} < \#(\mathbf{Q}^+) + \frac{\alpha}{2}(n^N - \#(\mathbf{Q}^+)).$$

Therefore

$$\#(\mathbf{Q}^+) > \frac{\alpha}{2-\alpha} n^N. \tag{2.1}$$

Fix $\delta, \lambda \in (0,1)$. The integer n can be chosen depending upon $\alpha, \delta, \lambda, \gamma$ and N, such that

$$|[u > \lambda] \cap Q_j| \ge (1-\delta)|Q_j| \quad \text{for some } Q_j \in \mathbf{Q}^+. \tag{2.2}$$

This would establish the Lemma for $\eta = 1/n$. Let $Q \in \mathbf{Q}^+$ satisfy

$$|[u > \lambda] \cap Q| < (1-\delta)|Q|. \tag{2.3}$$

Then there exists a constant $c = c(\alpha, \delta, \gamma, \eta, N)$ such that

$$\|u\|_{W^{1,1}(Q)} \ge c(\alpha, \delta, \gamma, \lambda, N) \frac{1}{n^{N-1}}. \tag{2.4}$$

From the assumptions,

$$|[u \le \lambda] \cap Q| \ge \delta |Q| \quad \text{and} \quad \left| \left[u > \frac{1+\lambda}{2} \right] \cap Q \right| > \frac{\alpha}{2}|Q|.$$

For fixed $x \in [u \le \lambda] \cap Q$ and $y \in [u > (1+\lambda)/2] \cap Q$,

$$\frac{1-\lambda}{2} \le u(y) - u(x) = \int_0^{|y-x|} Du(x + t\omega) \cdot \omega \, dt \quad \text{where} \quad \omega = \frac{y-x}{|x-y|}.$$

Let $R(x,\omega)$ be the polar representation of ∂Q with pole at x, for the solid angle ω. Integrate the previous relation with respect to y over $[u > (1+\lambda)/2] \cap Q$. Minorize the resulting left hand side, by using the lower bound on the measure of such a set, and majorize the resulting integral on the right hand side by extending the integration over Q. Expressing such integration in polar coordinates with pole at $x \in [u \le \lambda] \cap Q$ gives

$$\begin{aligned}
\frac{\alpha(1-\lambda)}{4}|Q| &\le \int_{|\omega|=1} \int_0^{R(x,\omega)} r^{N-1} \int_0^{|y-x|} |Du(x+t\omega)| \, dt \, dr \, d\omega \\
&\le N^{N/2}|Q| \int_{|\omega|=1} \int_0^{R(x,\omega)} |Du(x+t\omega)| \, dt \, d\omega \\
&= N^{N/2}|Q| \int_Q \frac{|Du(z)|}{|z-x|^{N-1}} \, dz.
\end{aligned}$$

Integrate now with respect to x over $[u \leq \lambda] \cap Q$. Minorize the resulting left hand side by using the lower bound on the measure of such a set, and majorize the resulting right hand side by extending the integration to Q. This gives

$$\frac{\alpha\delta(1-\lambda)}{4N^{N/2}}|Q| \leq \|u\|_{W^{1,1}(Q)} \sup_{z\in Q} \int_Q \frac{1}{|z-x|^{N-1}}\, dx \leq C(N)|Q|^{1/N}\|u\|_{W^{1,1}(Q)}$$

for a constant $C(N)$ depending only upon N.

If (2.2) does not hold for any cube $Q_j \in \mathbf{Q}^+$, then (2.4) is satisfied for all such Q_j. Adding over such cubes and taking into account (2.1) gives

$$\frac{\alpha}{2-\alpha} c(\alpha, \delta, \gamma, N)n \leq \|u\|_{W^{1,1}(K_1)} \leq \gamma.$$

REFERENCES

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