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Partial differential equations. — *Concentration at manifolds of arbitrary dimension for a singularly perturbed Neumann problem*, by FETHI MAHMOUDI and ANDREA MALCHIODI, communicated on 10 March 2006.

ABSTRACT. — We consider the equation $-\varepsilon^2 \Delta u + u = u^p$ in $\Omega \subseteq \mathbb{R}^N$, where Ω is open, smooth and bounded, and we prove concentration of solutions along k-dimensional minimal submanifolds of $\partial\Omega$, for $N \geq 3$ and for $k \in \{1, \ldots, N-2\}$. We impose Neumann boundary conditions, assuming $1 < p < (N-k+2)/(N-k-2)$ and $\varepsilon \to 0^+$. This result settles in full generality a phenomenon previously considered only in the particular case $N = 3$ and $k = 1$.

KEY WORDS: Singularly perturbed elliptic problems; differential geometry; local inversion; Fourier analysis.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 35B25, 35B34, 35J20, 35J60, 53A07.

1. INTRODUCTION

We study concentration phenomena for the problem

$$(P_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $p > 1$, and where ν denotes the unit normal to $\partial\Omega$. Problem (P_ε) arises in different contexts, as a nonlinear Schrödinger equation or from modeling reaction-diffusion systems (see for example [1], [8], [20]). A typical phenomenon one observes is the existence of solutions which are sharply concentrated near some subsets of their domain.

When dealing with reaction-diffusion systems, this phenomenon is related to the so-called Turing's instability [25], according to which reaction-diffusion systems whose reactants have very different diffusivities might generate stable non-trivial patterns. Well-known examples of solutions (P_ε) are *spike-layers*, namely solutions which concentrate at one or multiple points of $\overline{\Omega}$ (see [5], [6], [9], [10], [12], [21], [22]). The profile of these solutions, which exist only for $p < (N+2)/(N-2)$, after a scaling in ε converges to a function w_0 which solves

$$(1) \qquad -\Delta u + u = u^p \quad \text{in } \mathbb{R}^N \quad (\text{or in } \mathbb{R}^N_+ = \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_N > 0\}),$$

and which tends to zero at infinity. The limit domain depends on whether the spikes sit in the interior or at the boundary of Ω.

In recent years, some new types of solutions have been constructed: for example in [16], [17] it has been shown that given any smooth bounded domain $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, and any $p > 1$, there exists a sequence $\varepsilon_j \to 0$ such that (P_{ε_j}) possesses solutions concentrating at $\partial\Omega$ along this sequence. Their profile is still a solution of (1), but in this case it does not decay to zero at infinity, and depends on one variable only. Indeed the profile, viewed as a function of one variable only, is a solution of the ODE $-u'' + u = u^p$ on the real half-line $\{x_1 > 0\}$, and satisfies the condition $u'(0) = 0$.

Later in [15] it has been proved that, if $N = 3$ and if h is a closed, simple non-degenerate geodesic on $\partial\Omega$, then there exists again a sequence $(\varepsilon_j)_j$ converging to zero such that (P_{ε_j}) admits solutions u_{ε_j} concentrating along h as j tends to infinity. In this case the profile of u_{ε_j} is a decaying solution of (1) in $\mathbb{R}^2_+$, extended to a cylindrical solution in higher dimensions.

The goal of this note is to describe the recent progress in [13], where the last result has been extended to the case of general dimension N and general codimension of the limit set. Here and below, for brevity reasons, we shall be sketchy and sometimes not completely rigorous, referring to [13] for details. The main result we want to illustrate is the following one.

THEOREM 1.1. *Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 3$, be a smooth and bounded domain, and let $K \subseteq \partial\Omega$ be a compact embedded non-degenerate minimal submanifold of dimension $k \in \{1, \dots, N-2\}$. Then, if $p \in (1, \frac{N-k+2}{N-k-2})$, there exists a sequence $\varepsilon_j \to 0$ such that (P_{ε_j}) admits positive solutions u_{ε_j} concentrating along K as $j \to \infty$. The profile of u_{ε_j}, scaled in any plane orthogonal to K, is the unique radial solution of*

$$\begin{cases} -\Delta u + u = u^p & \text{in } \mathbb{R}^{N-k}_+, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathbb{R}^{N-k}_+, \\ u > 0, \ u \in H^1(\mathbb{R}^{N-k}_+). \end{cases} \tag{2}$$

REMARKS 1.2. (a) In contrast to the previous works concerning the case $N = 3$, or concentration on the whole $\partial\Omega$, an upper bound on p is needed. This condition is indeed natural, since (2) is well known to be solvable if and only if $p < (N-k+2)/(N-k-2)$.

(b) Observe that we have concentration along a sequence $\varepsilon_j \to 0$ and not for any small ε. This is caused by a resonance phenomenon, namely the existence of ε's for which the linearized operator at the solution u_ε is not invertible. Similar phenomena also appear in different contexts: see for example [2], [7], [14] and [19].

In the next sections we sketch the main ideas of the proof of the above result.

2. GEOMETRIC PRELIMINARIES

In this section we recall some basic facts in differential geometry; we refer for example to [3] and [24] for the details.

We endow $\partial\Omega$ with the metric $\overline{g}$ induced from $\mathbb{R}^N$, and we let K be a k-dimensional submanifold of $(\partial\Omega, \overline{g})$ $(1 \leq k \leq N-2)$. We denote by ∇ the connection induced by the metric $\overline{g}$ and by Δ_K the Laplace–Beltrami operator on K.

The normal bundle NK of K consists of the tangent vectors to $\partial\Omega$ at points of K which are perpendicular to K. The normal connection ∇^N on a normal field V is defined as the projection of ∇V onto NK. Let $C^\infty(K, NK)$ be the space of smooth normal vector fields on K. The normal Laplacian $\Delta_K^N : C^\infty(K, NK) \to C^\infty(K, NK)$ is defined by duality as $\int_K \langle \nabla^N V, \nabla^N W\rangle_N \, dV_{\overline{g}} = -\int_K \langle \Delta_K^N V, W\rangle_N \, dV_{\overline{g}}$ for any normal vector fields V and W, where $\langle \cdot, \cdot \rangle_N$ denotes the restriction of $\overline{g}$ to NK.

For $\Phi \in C^\infty(K, NK)$, we can define the one-parameter family of submanifolds $t \mapsto K_{t,\Phi}$ by

$$K_{t,\Phi} := \{\exp_x^{\partial\Omega}(t\Phi(x)) : x \in K\}, \tag{3}$$

where $\exp_x^{\partial\Omega}$ is the exponential map at $x \in K$ in $\partial\Omega$. The first variation formula of the volume is the equation

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Vol}(K_{t,\Phi}) = \int_K \langle \Phi, \mathbf{h}\rangle_N \, dV_K, \tag{4}$$

where $\mathbf{h}$ is the *mean curvature* (vector) of K in $\partial\Omega$ and dV_K the volume element of K.

The submanifold K is said to be *minimal* if it is a critical point for the volume functional, that is, if

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Vol}(K_{t,\Phi}) = 0 \quad \text{for any } \Phi \in C^\infty(K, NK) \tag{5}$$

or, equivalently by (4), if the mean curvature $\mathbf{h}$ is identically zero on K.

The *Jacobi operator* $\mathfrak{J}$ appears in the expression of the second variation of the volume functional for a minimal submanifold K:

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Vol}(K_{t,\Phi}) = -\int_K \langle \mathfrak{J}\Phi, \Phi\rangle_N \, dV_K, \qquad \Phi \in C^\infty(K, NK), \tag{6}$$

and is given by

$$\mathfrak{J}\Phi := -\Delta_K^N \Phi + \mathfrak{D}\Phi, \tag{7}$$

where $\mathfrak{D} : NK \to NK$ is a bounded linear operator (of order zero) which depends on the geometries of K and $\partial\Omega$. A submanifold K is said to be *non-degenerate* if the Jacobi operator $\mathfrak{J}$ is invertible, or equivalently if the equation $\mathfrak{J}\Phi = 0$ has only the trivial solution among the sections of NK.

Let $(\rho_i)_{i\geq 0}$ and $(\omega_j)_{j\geq 0}$ denote the eigenvalues of $-\Delta_K$ (respectively of $-\Delta_K^N$) chosen to be non-decreasing in i (respectively in j) and counted with multiplicity. Weyl's asymptotic formula tells that

$$\rho_i \sim C_k \left(\frac{i}{\mathrm{Vol}(K)}\right)^{2/k} \quad \text{as } i \to \infty \quad \text{and} \quad \omega_j \sim C_{N,k} \left(\frac{j}{\mathrm{Vol}(K)}\right)^{2/k} \quad \text{as } j \to \infty, \tag{8}$$

where C_k (respectively $C_{N,k}$) depends on k (respectively on N and k) only. Observe that, since $\mathfrak{J}$ differs from $-\Delta_K^N$ only by a bounded quantity, the eigenvalues $(\mu_l)_l$ of $\mathfrak{J}$ satisfy the same asymptotic formula of the ω_j's.

Our next goal is to define a metric $\hat{g}$ on NK. A tangent vector $V \in T_v NK$, where $v \in NK$, can be identified with the velocity of a curve $v(t)$ in NK which is equal to v

at time $t = 0$. The same holds true for another tangent vector $W \in T_v NK$, to which we associate a curve $w(t)$. Then the metric $\hat{g}$ on NK can be defined on the couple (V, W) in the following way (see [3, p. 79]):

$$\hat{g}(V, W) = \overline{g}(\pi_* V, \pi_* W) + \left\langle \left.\frac{D^N v}{dt}\right|_{t=0}, \left.\frac{D^N w}{dt}\right|_{t=0} \right\rangle_N .$$

In this formula π denotes the natural projection from NK onto K, and $D^N v/dt$ is the (normal) covariant derivative of the vector field $v(t)$ along the curve $\pi(v(t))$.

3. A MODEL LINEAR PROBLEM

In this section we introduce a model problem for the linearization of (P_ε) at approximate solutions. To do this we need first to study a suitable eigenvalue problem in a half-space of $\mathbb{R}^{n+1}$, where $n = N - k - 1$. We denote points of $\mathbb{R}^{n+1}$ by $(n+1)$-tuples $(\zeta_1, \dots, \zeta_n, \zeta_{n+1}) = (\zeta', \zeta_{n+1})$, and we let

$$\mathbb{R}^{n+1}_+ = \{(\zeta_1, \dots, \zeta_n, \zeta_{n+1}) \in \mathbb{R}^{n+1} : \zeta_{n+1} > 0\}.$$

For $p \in (1, \frac{n+3}{n-1})$ ($\frac{n+3}{n-1}$ is the critical Sobolev exponent in $\mathbb{R}^{n+1}$) we consider problem (2) which, with our definition of n, is

$$\begin{cases} -\Delta u + u = u^p & \text{in } \mathbb{R}^{n+1}_+, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial \mathbb{R}^{n+1}_+, \\ u > 0, \ u \in H^1(\mathbb{R}^{n+1}_+). & \end{cases}$$

It is well known that this problem has a radial solution $w_0(r)$, $r^2 = \sum_{i=1}^{n+1} \zeta_i^2$, with the properties

$$(9) \qquad \begin{cases} w_0'(r) < 0 \quad \text{for every } r > 0, \\ \lim\limits_{r\to\infty} e^r r^{n/2} w_0(r) = \alpha_{n,p} > 0, \qquad \lim\limits_{r\to\infty} \dfrac{w_0'(r)}{w_0(r)} = -1, \end{cases}$$

where $\alpha_{n,p}$ is a positive constant depending only on n and p. Solutions of (2) can be found as critical points of the functional $\bar{J}$ defined by

$$(10) \qquad \bar{J}(u) = \frac{1}{2} \int_{\mathbb{R}^{n+1}_+} (|\nabla u|^2 + u^2) - \frac{1}{p+1} \int_{\mathbb{R}^{n+1}_+} |u|^{p+1}, \qquad u \in H^1(\mathbb{R}^{n+1}_+),$$

and it turns out that w_0 is a mountain-pass critical point of $\bar{J}$. Since (2) is invariant under translation in the directions $\zeta_1, \dots, \zeta_n$, w_0 is a degenerate solution of (2), since $\partial_{\zeta_1} w_0, \dots, \partial_{\zeta_n} w_0$ belong to the kernel of $\bar{J}''(w_0)$. Actually, also the converse is true, in the sense specified by the following proposition (see [23]).

PROPOSITION 3.1. *The kernel of* $\bar{J}''(w_0)$ *is generated by the functions* $\partial w_0/\partial\zeta_1, \dots, \partial w_0/\partial\zeta_n$. *More precisely,*

$$\bar{J}''(w_0)[w_0, w_0] = -(p-1)\|w_0\|^2_{H^1(\mathbb{R}^{n+1}_+)},$$

and

$$\bar{J}''(w_0)[v, v] \geq C^{-1}\|v\|^2_{H^1(\mathbb{R}^{n+1}_+)}, \qquad \forall v \in H^1(\mathbb{R}^{n+1}_+),\ v \perp w_0, \partial_{\zeta_1} w_0, \dots, \partial_{\zeta_n} w_0,$$

for some positive constant C. *In particular, we have* $\eta < 0$, $\sigma = 0$ *and* $\tau > 0$, *where* η, σ *and* τ *are respectively the first, second and third eigenvalue of* $\bar{J}''(w_0)$. *Furthermore, the eigenvalue* η *is simple while* σ *has multiplicity* n.

We notice that if a function u satisfies the eigenvalue equation $\bar{J}''(w_0)u = \lambda u$ in $H^1(\mathbb{R}^{n+1}_+)$, then

$$\begin{cases} -\Delta u + u - p w_0^{p-1} u = \lambda(-\Delta u + u) & \text{in } \mathbb{R}^{n+1}_+, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{11}$$

Our next goal is to consider a variant of problem (11). Precisely, for $\varepsilon > 0$ and for $\gamma \in (0, 1)$ small we define

$$B_{\varepsilon,\gamma} = \{x \in \mathbb{R}^{n+1}_+ : |x| < \varepsilon^{-\gamma}\}, \tag{12}$$

and let

$$H^1_\varepsilon = \{u \in H^1(B_{\varepsilon,\gamma}) : u(x) = 0 \text{ for } |x| = \varepsilon^{-\gamma}\}.$$

After these definitions we consider the following problem, for $u \in H^1_\varepsilon$:

$$\begin{cases} -\Delta u + (1+\alpha)u - p w_0^{p-1} u = \lambda(-\Delta u + (1+\alpha)u) & \text{in } B_{\varepsilon,\gamma}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \{\zeta_{n+1} = 0\}, \end{cases} \tag{13}$$

where $\alpha \geq 0$. Notice that when ε tends to zero, $B_{\varepsilon,\gamma}$ *approaches* $\mathbb{R}^{n+1}_+$, so when $\alpha = 0$ the two problems (11) and (13) almost coincide. It is convenient to view (13) as an abstract eigenvalue equation in $H_{\alpha,\varepsilon}$, which is nothing but the space H^1_ε endowed with the equivalent norm

$$\|u\|^2_{\alpha,\varepsilon} = \int_{B_{\varepsilon,\gamma}} [|\nabla u|^2 + (1+\alpha)u^2], \qquad u \in H^1_\varepsilon,$$

(and the corresponding scalar product $(\cdot,\cdot)_{\alpha,\varepsilon}$), and define $T_{\alpha,\varepsilon}$ by duality as

$$(T_{\alpha,\varepsilon}u, v)_{\alpha,\varepsilon} = \int_{B_{\varepsilon,\gamma}} [(\nabla u \cdot \nabla v) + (1+\alpha)uv - p w_0^{p-1} uv], \qquad u, v \in H_{\alpha,\varepsilon}.$$

In fact, by integration by parts it can be shown that $T_{\alpha,\varepsilon}u = \lambda u$ in $H_{\alpha,\varepsilon}$ if and only if u satisfies (13). We are interested in the first two eigenvalues of $T_{\alpha,\varepsilon}$ (or equivalently of (11)), depending on the parameter α, and in the symmetries of the corresponding eigenfunctions. One can prove the following result.

PROPOSITION 3.2. *There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ the following properties hold true. Let $\eta_{\alpha,\varepsilon}$, $\sigma_{\alpha,\varepsilon}$ and $\tau_{\alpha,\varepsilon}$ denote the first three eigenvalues of $T_{\alpha,\varepsilon}$. Then $\eta_{\alpha,\varepsilon}$, $\sigma_{\alpha,\varepsilon}$ and $\tau_{\alpha,\varepsilon}$ are non-decreasing in α. For every α, $\eta_{\alpha,\varepsilon}$ is simple, $\partial\eta_{\alpha,\varepsilon}/\partial\alpha > 0$ and $\eta_{\alpha,\varepsilon} \to 1$ as $\alpha \to +\infty$. For α belonging to any fixed bounded subset of $\mathbb{R}_+$, we also have $\partial\eta_{\alpha,\varepsilon}/\partial\alpha > \tilde{\eta}$ for some positive $\tilde{\eta}$. For α sufficiently small, $\sigma_{\alpha,\varepsilon}$ has multiplicity n and $\partial\sigma_{\alpha,\varepsilon}/\partial\alpha > 0$. Furthermore $\eta_{0,\varepsilon}$ (resp. $\sigma_{0,\varepsilon}$) converges to η (resp. σ) exponentially fast as $\varepsilon \to 0^+$. The eigenfunction $u_{\alpha,\varepsilon}$ corresponding to $\eta_{\alpha,\varepsilon}$ is radial in ζ and radially decreasing, while the eigenspace corresponding to $\sigma_{\alpha,\varepsilon}$ is spanned by functions $v_{\alpha,\varepsilon,i}$ of the form $v_{\alpha,\varepsilon,i}(\zeta) = \hat{v}_{\alpha,\varepsilon}(|\zeta|)\zeta_i/|\zeta|$, $i = 1, \ldots, n$, for some radial function $\hat{v}_{\alpha,\varepsilon}(|\zeta|)$. The eigenvector $u_{\alpha,\varepsilon}$ (resp. $v_{\alpha,\varepsilon,i}$), normalized by $\|u_{\alpha,\varepsilon}\|_{\alpha,\varepsilon} = 1$ (resp. $\|v_{\alpha,\varepsilon,i}\|_{\alpha,\varepsilon} = 1$), corresponding to $\eta_{\alpha,\varepsilon}$ (resp. $\sigma_{\alpha,\varepsilon}$ for α small) depends smoothly on α. Moreover, for some fixed $C > 0$,*

$$|\nabla^{(l)} u_{\alpha,\varepsilon}(\zeta)| + |\nabla^{(l)} v_{\alpha,\varepsilon,i}(\zeta)| \le C_l e^{-|\zeta|/C_l} \qquad \text{for } i = 0, \ldots, n \tag{14}$$

provided α stays in a fixed bounded subset of $\mathbb{R}$. Furthermore, $\tau_{\alpha,\varepsilon} \ge \tau$ for all α and ε.

We now turn to the construction of the model operator. For $\gamma \in (0, 1)$ and ε small, we consider the set

$$S_\varepsilon = \{(v, \zeta_{n+1}) \in NK \times \mathbb{R}_+ : (|v|^2 + \zeta_{n+1}^2)^{1/2} \le \varepsilon^{1-\gamma}\}, \qquad \mathbb{R}_+ = \{\zeta_{n+1} : \zeta_{n+1} > 0\}.$$

We introduce a metric on S_ε inherited from $\hat{g}$ simply by

$$\tilde{g} = \hat{g} \otimes d\zeta_{n+1}^2.$$

Choosing *Fermi coordinates* $(\overline{y}, \zeta)$, $\overline{y} = (\overline{y}_a)_{a=1,\ldots,k}$, at some point $q \in K$ (see [13, Section 2]), we have

$$\Delta_{\tilde{g}} u = \partial^2_{aa} u + \partial^2_{ii} u + \partial^2_{\zeta_{n+1}\zeta_{n+1}} u; \tag{15}$$

here and throughout, we are using the summation convention for repeated indices.

If $v = v(|\zeta|)$ is a radial function in ζ, one can prove that, in the above coordinates, for any function ϕ defined on K and for any normal section ψ, at the point q we have

$$\Delta_{\tilde{g}} \left(\phi(\overline{y}) v(|\zeta|)\right) = (\Delta_K \phi(\overline{y})) v(|\zeta|) + \phi(\overline{y}) \Delta_\zeta v(|\zeta|); \tag{16}$$

$$\Delta_{\tilde{g}} \left(\psi^h \frac{\zeta_h}{|\zeta|} v(|\zeta|) \right) = (\Delta_K^N \psi)^h (\overline{y}) \frac{\zeta_h}{|\zeta|} v(|\zeta|) + \psi^h(\overline{y}) \Delta_\zeta \left(v(|\zeta|) \frac{\zeta_h}{|\zeta|} \right). \tag{17}$$

Here $\psi = \psi^h E_h$, where $(E_h)_{h=1,\ldots,n}$ is an orthonormal frame for NK in $\partial\Omega$ associated to the Fermi coordinates.

Now we introduce the function space H_{S_ε} defined as the family of functions in $H^1(S_\varepsilon)$ which vanish on $\{|v|^2 + \zeta_{n+1}^2 = \varepsilon^{2-2\gamma}\}$, endowed with the scalar product

$$(u, v)_{H_{S_\varepsilon}} = \int_{S_\varepsilon} (\nabla_{\tilde{g}} u \cdot \nabla_{\tilde{g}} v + uv)\, dV_{\tilde{g}}, \tag{18}$$

and the operator $T_{S_\varepsilon} : H_{S_\varepsilon} \to H_{S_\varepsilon}$ defined by duality as

$$(T_{S_\varepsilon} u, v)_{H_{S_\varepsilon}} = \int_{S_\varepsilon} (\nabla_{\tilde{g}} u \cdot \nabla_{\tilde{g}} v + uv - p w_0^{p-1}(|\zeta|/\varepsilon) uv)\, dV_{\tilde{g}} \tag{19}$$

for arbitrary $u, v \in H_{S_\varepsilon}$.

We will use the operator T_{S_ε} as our model for the linearization of (P_ε), and therefore it is important to understand the structure of its spectrum and in particular of the small eigenvalues. For any fixed $\overline{y} \in K$, we can decompose any function $u(\overline{y}, \zeta)$, $u \in H_{S_\varepsilon}$, into spherical harmonics with respect to the variable ζ. From formulas (16), (17) it can be shown that T_{S_ε} preserves the subspaces spanned by functions of the form $\phi(\overline{y}) v(|\zeta|)$ or of the form $\psi^h(\zeta_h/|\zeta|) v(|\zeta|)$, and hence T_{S_ε} has some eigenfunctions belonging to these spaces. It turns out that indeed these kinds of eigenfunctions correspond to the smallest eigenvalues of T_{S_ε}. More precisely, in the next proposition we characterize the eigenvalues smaller than $\tau/4$ and the corresponding eigenfunctions (see the notation of Proposition 3.2).

PROPOSITION 3.3. *Let $\varepsilon_0, \varepsilon$ be as in Proposition 3.2. Let $\lambda < \tau/4$ be an eigenvalue of T_{S_ε}. Then either $\lambda = \eta_{j,\varepsilon}$ for some j, or $\lambda = \sigma_{l,\varepsilon}$ for some l. Here we have set $\eta_{j,\varepsilon} = \eta_{\varepsilon^2 \rho_j, \varepsilon}$ and $\sigma_{l,\varepsilon} = \sigma_{\varepsilon^2 \omega_l, \varepsilon}$, where $(\rho_j)_j$, $(\omega_l)_l$ are as in (8). The corresponding eigenfunctions u are of the form*

$$u(y, \zeta) = \sum_{\{j : \eta_{j,\varepsilon} = \lambda\}} \alpha_j \phi_j(\overline{y}) u_{j,\varepsilon}(\zeta/\varepsilon) + \sum_{\{l : \sigma_{l,\varepsilon} = \lambda\}} \beta_l \varphi_l^i(\overline{y}) v_{l,\varepsilon,i}(\zeta/\varepsilon), \tag{20}$$

where (y, ζ) denote the above coordinates on S_ε; $(\alpha_j)_j$, $(\beta_l)_l$ are arbitrary constants; and $u_{j,\varepsilon} = u_{\varepsilon^2 \rho_j, \varepsilon}$, $v_{l,\varepsilon,i} = v_{\varepsilon^2 \omega_l, \varepsilon, i}$. Vice versa, every function of the form (20) is an eigenfunction of T_{S_ε} with eigenvalue λ. The functions $(\phi_j)_j$ and $(\varphi_l)_l$ are respectively the eigenfunctions of $-\Delta_K$ and $-\Delta_K^N$ corresponding to $(\rho_j)_j$ and $(\omega_l)_l$. In particular the eigenvalues of T_{S_ε} which are smaller than $\tau/4$ coincide with those numbers $(\eta_{j,\varepsilon})_j$ or $(\sigma_{l,\varepsilon})_l$ which are smaller than $\tau/4$.

4. PROOF OF THEOREM 1.1

Since the solutions we are looking for have a given asymptotic profile, it is convenient to prove the theorem using local inversion arguments. We divide the proof into three steps: first of all we construct approximate solutions $u_{I,\varepsilon}$ to (P_ε). Then we perform a careful analysis in order to understand the structure of the small eigenvalues of the linearization of (P_ε) at $u_{I,\varepsilon}$. Finally, we establish invertibility of the linearized operator for suitable values of ε and we prove Theorem 1.1 via a contraction mapping argument.

Step 1: finding approximate solutions. Given any positive integer I, we construct functions $u_{I,\varepsilon}$ which solve (P_ε) up to an error of order ε^I. First of all, we identify S_ε with a neighborhood of K in Ω using the map

$$(v, \zeta_{n+1}) \mapsto \exp_{\overline{y}}^{\partial\Omega}(v) + \nu(\exp_{\overline{y}}^{\partial\Omega}(v)),$$

where $\overline{y}$ denotes the base point of v, $\exp^{\partial\Omega}$ the exponential map from K into the boundary of Ω, and ν the inner unit normal to $\partial\Omega$.

Given a smooth cutoff function $\chi(t)$ which is identically 1 near $t = 0$ and identically zero for $t \geq 1$, locally near $q \in K$ we define $u_{I,\varepsilon}$ (using the above Fermi coordinates) as

$$\chi\left(\frac{|\zeta|}{\varepsilon^{1-\gamma}}\right)\left(w_0\left(\frac{\zeta'}{\varepsilon}+\Phi(\overline{y}), \frac{\zeta_{n+1}}{\varepsilon}\right)+\varepsilon w_1\left(\overline{y}, \frac{\zeta'}{\varepsilon}+\Phi(\overline{y}), \frac{\zeta_{n+1}}{\varepsilon}\right)\right.$$
$$\left.+\cdots+\varepsilon^I w_I\left(\overline{y}, \frac{\zeta'}{\varepsilon}+\Phi(\overline{y}), \frac{\zeta_{n+1}}{\varepsilon}\right)\right), \tag{21}$$

where $\Phi(\overline{y}) = \Phi_0(\overline{y}) + \cdots + \varepsilon^{I-2}\Phi_{I-2}(\overline{y})$ is a smooth normal section, and where the functions $(w_i)_i$ are determined by an iteration procedure.

One can indeed expand the expression $-\Delta u_{I,\varepsilon} + u_{I,\varepsilon} - u_{I,\varepsilon}^p$ formally in powers of ε and set each term of the expansion identically equal to zero. If we do this we find that for any integer $\tilde{I} \leq I$ the function $w_{\tilde{I}}$ satisfies the equation

$$\begin{cases} \mathcal{L}_\Phi w_{\tilde{I}} = F_{\tilde{I}}(\overline{y}, \zeta, w_0, w_1, \dots, w_{\tilde{I}-1}, \Phi_0, \dots, \Phi_{\tilde{I}-2}) & \text{in } \mathbb{R}^{n+1}_+, \\ \dfrac{\partial w_{\tilde{I}}}{\partial \zeta_{n+1}} = 0 & \text{on } \{\zeta_{n+1} = 0\}, \end{cases} \tag{22}$$

where $\mathcal{L}_\Phi$ is defined by

$$\mathcal{L}_\Phi u = -\Delta u + u - p w_0^{p-1}(\zeta' + \Phi(\overline{y}), \zeta_{n+1})u,$$

and where $F_{\tilde{I}}$ is some smooth function of its arguments (which we are assuming to be determined by induction on the index $\tilde{I}$). By Fredholm's alternative, (22) is solvable if and only if the right-hand side of the equation is orthogonal to the kernel of $\mathcal{L}_\Phi$, which, by Proposition 3.1, is spanned by $\partial_{\zeta_i} w_0(\cdot + \Phi, \cdot_{n+1})$, $i = 1, \dots, n$. Imposing orthogonality we find that $\Phi_{\tilde{I}-2}$ satisfies the equation

$$\mathfrak{J}\Phi_{\tilde{I}-2} = G_{\tilde{I}-2}(\overline{y}, \zeta, w_0, w_1, \dots, w_{\tilde{I}-1}, \Phi_0, \dots, \Phi_{\tilde{I}-3})$$

for some expression $G_{\tilde{I}-2}$, where $\mathfrak{J}$ is the Jacobi operator of K. The latter equation is solvable by the non-degeneracy assumption on K, and hence also (22) is solvable in $w_{\tilde{I}}$.

Using this procedure, and letting $J_\varepsilon : H^1(\Omega) \to \mathbb{R}$ be the Euler–Lagrange functional associated to (P_ε), namely

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega (\varepsilon^2|\nabla u|^2 + u^2) - \frac{1}{p+1}\int_\Omega |u|^{p+1},$$

one can prove the following result.

PROPOSITION 4.1. *For any $I \in \mathbb{N}$ there exists a function $u_{I,\varepsilon} : \Omega_\varepsilon \to \mathbb{R}$ with the following properties:*

$$\|J'_\varepsilon(u_{I,\varepsilon})\|_{H^1(\Omega)} \leq C_I \varepsilon^{I+1+(N-k)/2}; \quad u_{I,\varepsilon} \geq 0 \quad \text{in } \Omega; \quad \frac{\partial u_{I,\varepsilon}}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{23}$$

where C_I depends only on Ω, K, p and I.

The functional J_ε is indeed well defined on $H^1(\Omega)$ only if $p \leq (N+2)/(N-2)$. The cases $p \in (\frac{N+2}{N-2}, \frac{N-k+2}{N-k-2})$ can be handled using a truncation procedure, as described in [17, Section 5].

Step 2: study of the linearization at $u_{I,\varepsilon}$. Having obtained approximate solutions to (P_ε) via Proposition 4.1, we can now study the spectrum of $J''_\varepsilon(u_{I,\varepsilon})$ and in particular its invertibility. Using the Rayleigh quotient and some elementary estimates one can prove the following result about the eigenvalues of T_{S_ε} and those of $J''_\varepsilon(u_{I,\varepsilon})$.

LEMMA 4.2. *There exists a fixed constant C, depending on Ω, K and p, such that the eigenvalues of $J''_\varepsilon(u_{I,\varepsilon})$ and T_{S_ε} satisfy*

$$\left|\lambda_j(J''_\varepsilon(u_{I,\varepsilon})) - \lambda_j(T_{S_\varepsilon})\right| \leq C\varepsilon^{1-\gamma}, \qquad \textit{provided } \lambda_j(J''_\varepsilon(u_{I,\varepsilon})) \leq \tau/2.$$

Here we are indexing the eigenvalues in non-decreasing order, counted with multiplicity.

By the Weyl asymptotic formulas in (8) and by Proposition 3.3, one can check that T_{S_ε} possesses eigenvalues which approach zero at rate $\min\{\varepsilon^k, \varepsilon^2\}$ as ε tends to zero. Therefore the estimate in Lemma 4.2 cannot yield invertibility for $J''_\varepsilon(u_{I,\varepsilon})$, and a more sophisticated argument is needed. Nevertheless, still from (8) and by Proposition 3.3, we can obtain an asymptotic estimate on the Morse index of $J''_\varepsilon(u_{I,\varepsilon})$ as ε tends to zero, which is of order ε^{-k}.

To understand better the spectral structure of $J''_\varepsilon(u_{I,\varepsilon})$ we expand formally in powers of ε some of its eigenfunctions, perturbing the eigenspaces of T_{S_ε} generated by the functions $\varphi^i_l(\overline{y})v_{l,\varepsilon,i}(\zeta/\varepsilon)$ for *small* values of the index l. Through such an expansion we can produce a sequence $(\Psi_l)_l$ of approximate eigenfunctions with the following property.

LEMMA 4.3. *There exist a polynomial $P(\zeta)$, a positive constant $\tilde{C}$ and a sequence $(C_l)_l$ of positive constants, depending on Ω, K, p and I, such that*

$$|-\Delta\Psi_l + \Psi_l - pu^{p-1}_{I,\varepsilon}\Psi_l - \varepsilon^2\tilde{C}\mu_l(-\Delta\Psi_l + \Psi_l)| \leq C_l\varepsilon^3 P(\zeta/\varepsilon)e^{-|\zeta/\varepsilon|}.$$

As a consequence, Ψ_l satisfies the eigenvalue equation $J''_\varepsilon(u_{I,\varepsilon})\Psi_l = \varepsilon^2\tilde{C}\mu_l\Psi_l$ (in the space $H^1(\Omega)$) up to an error of order ε^3.

Since the numbers $(\mu_l)_l$ are the eigenvalues of the Jacobi operator, we see from the above result that the invertibility of $\mathfrak{J}$ keeps some of the eigenvalues of $J''_\varepsilon(u_{I,\varepsilon})$ away from zero at an order ε^2, and this plays a crucial role in proving its invertibility. Letting $(\phi_j)_j$, $(\psi_l)_l$ denote the eigenfunctions of $-\Delta_K$ and $\mathfrak{J}$ corresponding to $(\rho_j)_j$ and $(\mu_l)_l$, we define the following subspaces:

$$H_1 = \operatorname{span}\{\phi_i(\overline{y})u_{i,\varepsilon}(\zeta/\varepsilon) : i = 0, \ldots, \infty\}; \tag{24}$$

$$\begin{aligned} \hat{H}_2 &= \operatorname{span}\{\Psi_l : l = 0, \ldots, \varepsilon^{-\delta}\}; \\ \tilde{H}_2 &= \operatorname{span}\{\psi^m_l(\overline{y})\hat{v}_{l,\varepsilon,m}(\zeta/\varepsilon) : l = \varepsilon^{-\delta}+1, \ldots, \overline{C}\varepsilon^{-k}\}; \end{aligned} \tag{25}$$

$$H_2 = \hat{H}_2 \oplus \tilde{H}_2; \qquad H_3 = (H_1 \oplus H_2)^\perp, \tag{26}$$

where $\overline{C}$ and δ are some small constants. It turns out that $H^1(\Omega)$ decomposes uniquely as a direct sum of H_1, H_2, H_3, that $J''_\varepsilon(u_{I,\varepsilon})$ is positive-definite on H_3 (with a uniform positive lower bound), and that $J''_\varepsilon(u_{I,\varepsilon})$ is nearly diagonal with respect to this decomposition (see Sections 5 and 6 in [13] for precise statements).

Step 3: invertibility of the linearized operator. The main difficulty in proving Theorem 1.1 is the presence of the resonance phenomenon described above. In order to overcome this problem we look at small eigenvalues of $J''_\varepsilon(u_{I,\varepsilon})$ as functions of ε and then, using a classical theorem due to Kato (see [11, p. 45]), we estimate the derivatives of these eigenvalues with respect to ε. We prove that if $\lambda = o(\varepsilon^2)$ is an eigenvalue of the linearized operator, then $\partial\lambda/\partial\varepsilon$ is close to a number depending on ε, N, p and K only. As a consequence, the spectral gaps near zero will *shift without squeezing*, as ε varies, and we obtain invertibility for suitable values of the parameter. This method also provides estimates on the norm of the inverse operator, which blows up with rate $\max\{\varepsilon^{-k}, \varepsilon^{-2}\}$ as ε tends to zero. The result then follows by using the contraction mapping theorem near the approximate solution $u_{I,\varepsilon}$.

Using the analysis of $J''_\varepsilon(u_{I,\varepsilon})$ one can prove that, for an eigenfunction corresponding to a resonant eigenvalue, the H_2 and H_3 components tend to zero as ε tends to zero. More precisely, one has the following result.

PROPOSITION 4.4. *For ε sufficiently small, let λ be an eigenvalue of $J''_\varepsilon(u_{I,\varepsilon})$ such that $|\lambda| \leq \varepsilon^\varsigma$ for some $\varsigma > 2$, and let $u \in H^1(\Omega)$ be an eigenfunction of $J''_\varepsilon(u_{I,\varepsilon})$ corresponding to λ with $\|u\|_{H^1(\Omega)} = 1$. In the above notation, let $u = u_1 + u_2 + u_3$, with $u_i \in H_i$, $i = 1, 2, 3$. If $u_1 = \sum_{j=0}^{\infty} \alpha_j \phi_j(\overline{y}) u_{j,\varepsilon}(|\zeta/\varepsilon|)$, then*

$$\Big\| u - \sum_{\{j\,:\,|\eta_{j,\varepsilon}| \leq \varepsilon^{(1-\gamma)/2}\}} \alpha_j \phi_j u_{j,\varepsilon} \Big\|_{H^1(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Using Kato's theorem, one can prove that the resonant eigenvalues of $J''_\varepsilon(u_{I,\varepsilon})$ are differentiable with respect to ε and if λ is such an eigenvalue, then

$$\frac{\partial\lambda}{\partial\varepsilon} = \{\text{eigenvalues of } Q_\lambda\}, \tag{27}$$

where $Q_\lambda : \mathcal{H}_\lambda \times \mathcal{H}_\lambda \to \mathbb{R}$ is the quadratic form given by

$$Q_\lambda(u, v) = (1-\lambda)\frac{2}{\varepsilon}\int_\Omega \nabla u \cdot \nabla v - p(p-1)\int_\Omega uv u_{I,\varepsilon}^{p-2}\left(\frac{\partial u_{I,\varepsilon}}{\partial\varepsilon}\right). \tag{28}$$

Here $\mathcal{H}_\lambda \subseteq H^1(\Omega)$ stands for the eigenspace of $J''_\varepsilon(u_{I,\varepsilon})$ corresponding to λ. Notice that, since λ might have multiplicity greater than 1, when we vary ε this eigenvalue can split into a multiplet, which is allowed by formula (27). Taking λ small, we can apply (27), and evaluate the quadratic form in (28) on the couples of eigenfunctions in $\mathcal{H}_\lambda$, which are characterized by Proposition 4.4. Reasoning as in [15, Proposition 5.1], one can prove the following result.

PROPOSITION 4.5. *Let λ be as in Proposition 4.4. Then for ε small one has*

$$\frac{\partial\lambda}{\partial\varepsilon} = \frac{1}{\varepsilon}(\overline{F} + o_\varepsilon(1)),$$

where $\overline{F}$ is a positive constant depending only on N, k and p.

Using the last proposition and choosing suitable values of the parameter ε, one can obtain the invertibility of $J''_\varepsilon(u_{I,\varepsilon})$ along a sequence $\varepsilon_j \to 0$, with a quantitative estimate on the inverse operator.

PROPOSITION 4.6. *For a suitable sequence $\varepsilon_j \to 0$, the operator $J''_\varepsilon(u_{I,\varepsilon}) : H^1(\Omega) \to H^1(\Omega)$ is invertible and the inverse operator satisfies*

$$\|(J''_{\varepsilon_j}(u_{I,\varepsilon_j}))^{-1}\|_{H^1(\Omega)} \leq \frac{C}{\min\{\varepsilon_j^k, \varepsilon_j^2\}}$$

for all $j \in \mathbb{N}$ and for some fixed positive constant C.

Having the invertibility of the linearized operator, the existence of a critical point of J_ε follows easily by applying the contraction mapping theorem near $u_{I,\varepsilon}$ for I sufficiently large, depending on k, N and p. In fact, since the norm of the inverse operator blows up as ε tends to zero, it is necessary to choose approximate solutions which solve (P_ε) with a good accuracy (see Proposition 4.1). When the exponent p is supercritical, problem (P_ε) is not variational any more, but in this case it is sufficient to use some truncations on the Euler functional and to apply standard elliptic regularity estimates.

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