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Partial differential equations. — *Bifurcation points of a degenerate elliptic boundary-value problem*, by GILLES EVÉQUOZ and CHARLES A. STUART, communicated on 12 May 2006.

ABSTRACT. — We consider the nonlinear elliptic eigenvalue problem

$$-\nabla\cdot\{A(x)\nabla u(x)\} = \lambda f(u(x)) \quad \text{for } x\in\Omega,$$
$$u(x) = 0 \quad \text{for } x\in\partial\Omega,$$

where Ω is a bounded open subset of $\mathbb{R}^N$ and $f\in C^1(\mathbb{R})$ with $f(0)=0$ and $f'(0)=1$. The ellipticity is degenerate in the sense that $0\in\Omega$ and $A(x)>0$ for $x\neq 0$ but $\lim_{x\to 0} A(x)/|x|^2 = 1$. We show that there is vertical bifurcation at all points λ in the interval $(N^2/4,\infty)$. Bifurcation also occurs at any eigenvalues of the linearized problem that are below $N^2/4$. Our treatment is based on recent results concerning the bifurcation points of equations with nonlinearities that are Hadamard differentiable, but not Fréchet differentiable.

KEY WORDS: Degenerate elliptic; bifurcation; Hadamard differentiable.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 35J60, 35J70, 35B32.

1. INTRODUCTION

For $N\geq 3$, let Ω be a bounded open subset of $\mathbb{R}^N$ with a Lipschitz boundary and let $0\in\Omega$. We consider the nonlinear degenerate elliptic boundary-value problem

$$-\nabla\cdot\{A(x)\nabla u(x)\} = \lambda f(u(x)) \quad \text{for } x\in\Omega. \tag{1.1}$$
$$u(x) = 0 \quad \text{for } x\in\partial\Omega, \tag{1.2}$$

where

(D1) $A\in C(\overline{\Omega})$ with $A(x)>0$ for all $x\in\overline{\Omega}\setminus\{0\}$ and $\lim_{|x|\to 0} A(x)/|x|^2 = 1$,
(D2) $f\in C^1(\mathbb{R})$ with $f(0)=0$, $f'(0)=1$, $\sup\{|f'(s)| : s\in\mathbb{R}\} = M < \infty$.

Of course, by rescaling λ we can accommodate the more general assumptions

$$\lim_{|x|\to 0} A(x)/|x|^2 = \alpha > 0 \quad \text{and} \quad f'(0)=\beta>0.$$

Furthermore, in Section 5.1, we show how the assumption (D2) can be replaced by the condition

(F) For some $T>0$, $f\in C^1([-T,T])$ is an odd function that is strictly concave on $[0,T]$ with $f(0)=f(T)=0$ and $f'(0)=1$.

The condition (F) does not require f' to be bounded on the whole real line and it enables us to deal with nonlinearities such as $f(s) = s - s^3$.

It follows from (D1) and the boundedness of Ω that

$$\alpha_1|x|^2 \leq A(x) \leq \alpha_2|x|^2 \quad \text{for all } x \in \overline{\Omega} \text{ where } 0 < \alpha_1 \leq 1 \leq \alpha_2 < \infty. \tag{1.3}$$

We are interested in solutions of (1.1), (1.2) that have finite energy

$$E_\lambda(u) = \frac{1}{2}\int_\Omega A(x)|\nabla u(x)|^2\,dx - \lambda\int_\Omega F(u(x))\,dx < \infty \text{ where } F(s) = \int_0^s f(t)\,dt.$$

Since

$$\int_\Omega A(x)|\nabla u(x)|^2\,dx < \infty \Leftrightarrow \int_\Omega |x|^2|\nabla u|^2\,dx < \infty$$

by (1.3), and $|F(s)| \leq Ms^2/2$ by (D2), we seek solutions in the space

$$H = \Big\{u \in L^2(\Omega) : u \text{ admits generalized derivatives } \partial_i u \text{ on } \Omega \setminus \{0\} \text{ and } \int_\Omega |x|^2|\nabla u|^2\,dx < \infty\Big\}.$$

Clearly H, with the scalar product

$$(u, v) = \int_\Omega uv\,dx + \int_\Omega |x|^2\nabla u \cdot \nabla v\,dx, \tag{1.4}$$

is a Hilbert space and (by a slight abuse of notation) $H \subset W^{1,2}(\Omega\backslash\overline{B}_\varepsilon)$ where $\varepsilon > 0$ is small enough so that the closed ball $\overline{B}_\varepsilon = \{x \in \mathbb{R}^N : |x| \leq \varepsilon\} \subset \Omega$. Let

$$H_0 = \{u \in H : \Gamma u = 0\} \tag{1.5}$$

where $\Gamma : W^{1,2}(\Omega\backslash\overline{B_\varepsilon}) \to L^2(\partial\Omega)$ is the usual trace operator (see [2, A 5.7] for example). The continuity of Γ ensures that $(H_0, (\cdot,\cdot))$ is a Hilbert space. We use $|\cdot|_p$ to denote the usual norm on $L^p(\Omega)$. We show in Appendix 1 that if $u \in H_0$, then u admits generalized derivatives on Ω and, in fact, $H_0 \subset W^{1,1}(\Omega)$.

DEFINITION 1.1. *Under the hypotheses* (D1) *and* (D2) *a* solution *of* (1.1), (1.2) *is a pair* $(\lambda, u) \in \mathbb{R} \times H_0$ *such that*

$$\int_\Omega A(x)\nabla u(x) \cdot \nabla\varphi(x)\,dx = \lambda\int_\Omega f(u(x))\varphi(x)\,dx \quad \textit{for all } \varphi \in H_0. \tag{1.6}$$

A point $\Lambda \in \mathbb{R}$ *is a* bifurcation point *for* (1.1), (1.2) *if there is a sequence* $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times [H_0 \setminus \{0\}]$ *of solutions such that* $\lambda_n \to \Lambda$ *and* $|u_n|_2 \to 0$.

We show in Appendix 1 that $C_0^\infty(\Omega)$ is dense in H_0, so our definition is equivalent to requiring that $(\lambda, u) \in \mathbb{R} \times H_0$ be such that

$$\int_\Omega A(x)\nabla u(x) \cdot \nabla\varphi(x)\,dx = \lambda\int_\Omega f(u(x))\varphi(x)\,dx \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

It follows from Proposition 4.1 in Section 4 that our definition of solution of (1.1), (1.2) is equivalent to requiring u to be a stationary point of the energy E_λ in H_0.

Our main results are Theorems 5.1 and 5.2. They show that, under some additional assumptions on the nonlinearity f, but without further assumptions about the domain Ω and the coefficient A, the set of bifurcation points for (1.1), (1.2) contains the interval $[N^2/4, \infty)$. Whether or not there are other bifurcation points depends on additional properties of A.

If the degeneracy of (1.1) is subquadratic, in the sense that $\lim_{|x|\to 0} A(x)/|x|^t = 1$ for some $t \in [0, 2)$, rather than quadratic as in (D1), then the set of bifurcation points for (1.1), (1.2) is a discrete set and, as we show in Theorem 7.3 in Appendix 2, this can be deduced rather easily from standard results concerning compact Fréchet differentiable operators. Other recent work on subquadratic, degenerate elliptic nonlinear boundary value problems can be found in [10], [11] and [21]. A one-dimensional boundary value problem involving quadratic degeneracy is treated in [6], but the nonlinearity is superlinear so there is bifurcation to the left at $1/4$ which is the infimum of the spectrum of the linearized problem. Some nonexistence results are given in [13]. The existence and interesting behaviour of branches of positive solutions for problems in N dimensions with quadratic degeneracy and various types of nonlinearity are studied in [12].

To present our approach to (1.1), (1.2), which is based on our recent work on problems that are differentiable in the sense of Hadamard, but not in the sense of Fréchet, we begin with following result, which will be proved in Appendix 1, providing some basic information for our treatment of the problem.

LEMMA 1.2. (i) *For all $u \in H_0 \setminus \{0\}$, we have*

$$\int_\Omega u^2\,dx < \frac{4}{N^2}\int_\Omega |x|^2|\nabla u|^2\,dx \tag{1.7}$$

and in fact,

$$\sup\left\{\frac{\int_\Omega u^2\,dx}{\int_\Omega |x|^2|\nabla u|^2\,dx} : u \in H_0 \setminus \{0\}\right\} = \frac{4}{N^2}.$$

Hence

$$\langle u, v\rangle = \int_\Omega |x|^2 \nabla u \cdot \nabla v\,dx \quad \text{for } u, v \in H_0 \tag{1.8}$$

defines a scalar product on H_0 with a norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$ which is equivalent to $(\cdot,\cdot)^{1/2}$.

(ii) *For a function A that satisfies* (D1)*, we can define another scalar product on H_0 by*

$$\langle u, v\rangle_A = \int_\Omega A(x)\nabla u \cdot \nabla v\,dx \quad \text{for } u, v \in H_0. \tag{1.9}$$

Then $\|\cdot\|_A = \langle\cdot,\cdot\rangle_A^{1/2}$ is a norm that is also equivalent to $(\cdot,\cdot)^{1/2}$ on H_0 and

$$|u|_2 \le \frac{2}{N}\|u\| \le \frac{2}{N\sqrt{\alpha_1}}\|u\|_A \quad \text{for all } u \in H_0.$$

Suppose that the conditions (D1) *and* (D2) *are satisfied.*

(iii) *If* $\{(\lambda_n, u_n)\}$ *is a sequence of solutions of* (1.1), (1.2) *and* $\lambda_n \to \Lambda$, *then*

$$|u_n|_2 \to 0 \quad \Leftrightarrow \quad \|u_n\|_A \to 0 \text{ and } |u_n|_2 \to 0 \;\Rightarrow\; E(u_n) \to 0.$$

(iv) *If* (λ, u) *is solution of* (1.1), (1.2) *and* $u \neq 0$, *then*

$$|\lambda| \geq \frac{N^2 \alpha_1}{4M}.$$

From now on we use $(H_A, \langle \cdot, \cdot \rangle_A)$ to denote the Hilbert space H_0 equipped with the scalar product $\langle \cdot, \cdot \rangle_A$. Using part (i) of the lemma and the fact that $f(u) \in L^2(\Omega)$ for all $u \in H_0$ by (D2), we can define $K(u), G(u) \in H_0$ by the relations

$$\text{(1.10)} \quad \langle K(u), v\rangle_A = \int_\Omega uv\,dx \quad \text{and} \quad \langle G(u), v\rangle_A = \int_\Omega f(u)v\,dx \quad \text{for all } u, v \in H_0.$$

Note that $(\lambda, u) \in \mathbb{R} \times H_0$ satisfies (1.6) if and only if

$$\text{(1.11)} \qquad u = \lambda G(u).$$

It follows from the lemma that $K \in B(H_A, H_A)$ and we shall show that $G : H_A \to H_A$ is Hadamard differentiable at $u = 0$ with $G'(0) = K$.

In the next section we recall some general results, which we have established recently, about bifurcation for equations like (1.11) involving a function G that is differentiable in the sense of Hadamard, but not necessarily in the sense of Fréchet. Our conclusions concerning the bifurcation points of (1.1), (1.2) are deduced from these abstract results in Section 5. Sections 3 and 4 are devoted to proving the requisite properties of the operators K and G. The fact that every point in the interval $[N^2/4, \infty)$ is a bifurcation point for the problem (1.1), (1.2) is related to the facts that $4/N^2$ is the supremum of the essential spectrum of the self-adjoint operator $K \in B(H_A, H_A)$ and G is Hadamard differentiable, but not Fréchet differentiable, at $u = 0$ with $G'(0) = 0$.

Our abstract results apply directly to (1.1), (1.2) under the hypotheses (D1) and (D2). When (D2) is replaced by (F), we show that f can be extended beyond $[-T, T]$ in such a way that the extension satisfies (D2) and all solutions (λ, u) of (1.1), (1.2) for this extension have $|u|_\infty \leq T$.

In [16], we have made a more detailed study of the radially symmetric version of (1.1), (1.2) under the hypotheses (D1) and (F). We find that at every bifurcation point λ, and hence at every $\lambda \geq N^2/4$, there is a sequence $\{(\lambda_n, u_n)\}$ of nontrivial radially symmetric solutions of (1.1), (1.2) such that $\lambda_n \to \lambda$ and $|u_n|_p \to 0$ as $n \to \infty$ for all $p \in [1, \infty)$. However, it is not the case that $|u_n|_\infty \to 0$ as $n \to \infty$. Indeed, $|u_n|_\infty = T$ for all n and the solutions u_n concentrate to a spike at the origin in the sense that u_n converges uniformly to zero on all compact subsets of $\overline{\Omega}$ that do not contain the origin.

The Bernoulli–Euler model for the buckling of a heavy tapered rod under its own weight leads to a one-dimensional problem with the same structure as the radially symmetric case (1.1), (1.2) with $f(s) = \sin s$ which satisfies (F) with $T = \pi$. It has been treated in [23]–[26] where the quadratic degeneracy required in (D1) corresponds to what is called the critical case for the tapering of the column. In this critical case, we again encounter a nonlinearity which is Hadamard, but not Fréchet, differentiable.

The existence of an interval of bifurcation points for a nonlinear eigenvalue problem was first established by Benci and Fortunato [4] for the problem

$$-\Delta u + q(x)|u(x)|^{\sigma} u(x) = \lambda u(x) \quad \text{with } u \in W^{1,2}(\mathbb{R}^N) \tag{1.12}$$

where $\sigma > 0$ and $q \in C(\mathbb{R}^N)$ with $q(x) \geq C|x|^t$ for all $x \in \mathbb{R}^N$ for some constants $C > 0$ and $t > N\sigma/2$. In [5], they gave a more abstract formulation of their approach which is based on Lyusternik–Schnirelman theory and Clark's use of the genus. Further progress in understanding the bifurcation theory of (1.12) was made in [7] and [19]. In [15], we show how our approach using Hadamard differentiability can be used to deal with (1.12) as well as problems of the form

$$-\Delta u(x) + p(x)u(x) = \lambda q^{-1}(x) f(q(x)u(x)) \quad \text{for } u \in W^{1,2}(\mathbb{R}^N) \tag{1.13}$$

and

$$-\Delta u(x) + p(x)u(x) - q^{-1}(x) f(q(x)u(x)) = \lambda u(x) \quad \text{for } u \in W^{1,2}(\mathbb{R}^N) \tag{1.14}$$

where f satisfies (D2), $p \in L^{\infty}(\mathbb{R}^N)$ and q is a positive measurable function such that

$$\int_{|x|\geq R} q(x)^{-2}\, dx < \infty \quad \text{for some } R > 0.$$

2. Hadamard differentiability and bifurcation

In this section we recall the relevant parts of a general study of bifurcation in the context of Hadamard differentiable functions.

Throughout this section, $(H, \langle \cdot, \cdot \rangle, \| \cdot \|)$ denotes any real Hilbert space.

DEFINITION 2.1. *A function $F : H \to H$ is* Hadamard differentiable *at $u \in H$ if there exists $T \in B(H, H)$ such that*

$$\lim_{n\to\infty} \frac{F(u + t_n v_n) - F(u)}{t_n} = Tv \quad \text{for all } v \in H$$

for all $\{t_n\} \subset \mathbb{R} \setminus \{0\}$ with $t_n \to 0$ and all $\{v_n\} \subset H$ with $v_n \to v$.

Replacing strong convergence by weak convergence, we arrive at the final notion of differentiability that we shall use. These definitions are examined in detail in our paper [14].

DEFINITION 2.2. *A function $F : H \to H$ is* w-Hadamard differentiable *at $u \in H$ if there exists $T \in B(H, H)$ such that*

$$\lim_{n\to\infty} \left\langle \frac{F(u + t_n v_n) - F(u)}{t_n}, \varphi \right\rangle = \langle Tv, \varphi \rangle \quad \text{for all } v \in H \text{ and all } \varphi \in H$$

for all $\{t_n\} \subset \mathbb{R} \setminus \{0\}$ with $t_n \to 0$ and all $\{v_n\} \subset H$ with $v_n \rightharpoonup v$ weakly in H.

Now consider an equation of the form

$$G(u) = \mu u \quad \text{for } \mu \in \mathbb{R} \text{ and } u \in H, \tag{2.1}$$

where $G : H \to H$ with $G(0) = 0$. We shall use the following terminology.

A real number μ is called a *bifurcation point* for (2.1) if there exists a sequence $\{(\mu_n, u_n)\} \subset \mathbb{R} \times H$ of solutions of (2.1) such that $u_n \neq 0$, $\|u_n\| \to 0$ and $\mu_n \to \mu$. The set of all bifurcation points for (2.1) is denoted by B_G. We say that there is *vertical bifurcation* at μ if the sequence $\{(\mu_n, u_n)\}$ can be chosen with the additional property that $\mu_n = \mu$ for all $n \in \mathbb{N}$. We say that there is *bifurcation to the right* (*left*) at μ if the sequence $\{(\mu_n, u_n)\}$ can be chosen with the additional property that $\mu_n > (<) \mu$ for all $n \in \mathbb{N}$.

For a bounded linear operator $T : H \to H$ we denote its spectrum by $\sigma(T)$ and its essential spectrum by

$$\sigma_e(T) = \{\lambda \in \mathbb{R} : T - \lambda I : H \to H \text{ is not a Fredholm operator}\}.$$

The first result, which is part of Corollary 4.3 in [15], gives necessary conditions for bifurcation at μ.

PROPOSITION 2.3. *Let $G : H \to H$ be a function such that $G(0) = 0$ and G is w-Hadamard differentiable at $u = 0$ with $G'(0) = G'(0)^*$. If $\mu \in B_G \cap (\Lambda^e, \infty)$ where $\Lambda^e = \sup \sigma_e(G'(0))$ and*

$$\limsup_{\|u\| \to 0} \frac{\langle G(u) - G'(0)u, u \rangle}{\|u\|^2} < d(\mu, \sigma(G'(0))),$$

then $\mu \in \sigma(G'(0))$.

In formulating some sufficient conditions for μ to be a bifurcation point we suppose that G is a gradient map with the following properties.

(H1) There exists an even potential $\psi \in C^1(H, \mathbb{R})$ with $\psi(0) = 0$ such that

$$\lim_{\|u\| \to \infty} \psi(u)/\|u\|^2 = 0$$

and

$$\psi'(u)u < 2\psi(u) \quad \text{for all } u \in H \setminus \{0\}.$$

We use G to denote the gradient of ψ defined by

$$\langle G(u), v \rangle = \psi'(u)v \quad \text{for all } u, v \in H$$

and make the following additional assumptions:

(H2) $G : H \to H$ is compact.

(H3) $G : H \to H$ is either Hadamard or w-Hadamard differentiable at $u = 0$ with a derivative $G'(0) \in B(H, H)$ that is self-adjoint.

We set

$$\Lambda_e = \inf \sigma_e(G'(0)) \quad \text{and} \quad \Lambda^e = \sup \sigma_e(G'(0)).$$

PROPOSITION 2.4. *Suppose that the conditions* (H1)–(H3) *are satisfied.*

(A) *If* $\Lambda^e > 0$, *then* $[0, \Lambda^e] \subset B_G$, *and there is vertical bifurcation at every* $\mu \in (0, \Lambda^e)$.
(B) *Also* $(\Lambda^e_+, \infty) \cap \sigma(G'(0)) \subset B_G$ *where* $\Lambda^e_+ = \max\{0, \Lambda^e\}$ *and there is bifurcation to the left at every* $\mu \in (\Lambda^e_+, \infty) \cap \sigma(G'(0))$, *If in addition* G *is w-Hadamard differentiable at* $u = 0$, *then* $(\Lambda^e_+, \infty) \cap \sigma(G'(0)) = (\Lambda^e_+, \infty) \cap B_G$.

This is part of Corollary 5.2 in [15].

3. PROPERTIES OF THE OPERATOR K

We expose the main features of the spectrum of the linear operator $K \in B(H_A, H_A)$ defined by (1.10).

PROPOSITION 3.1. *Under the hypothesis* (D1), $K \in B(H_A, H_A)$ *is a positive self-adjoint operator with*

(i) $\sup \sigma(K) = \|K\| = \sup\{\langle Ku, u\rangle_A / \|u\|_A^2 : u \in H_A \setminus \{0\}\} \leq 4/N^2\alpha_1$ *and* $\inf \sigma(K) = 0$,
(ii) $\sup \sigma_e(K) = 4/N^2$,
(iii) $\mu \in (4/N^2, \infty) \cap \sigma(K)$ *if and only if there exists* $u \in H_A \setminus \{0\}$ *such that*

$$\int_\Omega A\nabla u \cdot \nabla v\, dx = \frac{1}{\mu} \int_\Omega uv\, dx \quad \textit{for all } v \in H_A.$$

PROOF. (i) For any $u, v \in H_A$,

$$|\langle Ku, v\rangle_A| = \left| \int_\Omega uv\, dx \right| \leq |u|_2 |v|_2 \leq \frac{4}{N^2\alpha_1} \|u\|_A \|v\|_A$$

so $\|K\| \leq 4/N^2\alpha_1$. Clearly K is self-adjoint and positive. Hence $\sigma(K) \subset [0, \|K\|]$ and $\sup \sigma(K) = \|K\| = \sup\{\langle Ku, u\rangle / \|u\|_A^2 : u \in H_A \setminus \{0\}\}$. To see that $0 \in \sigma(K)$, consider any ball $B = B(x_0, r)$ such that $\overline{B} \subset \Omega \setminus \{0\}$ and any nonzero function $z \in H^1(B)$. For $n \in \mathbb{N}$, set

$$z_n(x_0 + x) = \begin{cases} n^{N/2} z(x_0 + nx) & \text{for } |x| \leq r/n, \\ 0 & \text{otherwise,} \end{cases}$$

where x_0 is the centre of B and let

$$m = \min_{x \in B} A(x).$$

Then $z_n \in H_A$ with $\langle K z_n, z_n\rangle_A = |z_n|_2^2 = |z|_2^2$ and

$$\|z_n\|_A^2 = \int_\Omega A(x) |\nabla z_n|^2\, dx \geq m \int_{|x_0 - x| \leq r/n} |\nabla z_n(x)|^2\, dx = mn^2 \int_{|x_0 - y| \leq r} |\nabla z(y)|^2\, dy$$

where $m > 0$, showing that $\langle K z_n, z_n\rangle_A / \|z_n\|_A^2 \to 0$.

(ii) We begin by recalling (see Theorem 7.24 of [27], for example) that $\lambda \in \sigma_e(K)$ if and only if there is a singular sequence for $K - \lambda$, that is, a sequence $\{v_n\}$ having the following properties:

$$v_n \rightharpoonup 0 \quad \text{in } H_A, \qquad \liminf \|v_n\|_A > 0 \quad \text{and} \quad \|(K-\lambda)v_n\|_A \to 0.$$

We split the proof into two parts. First we construct a singular sequence for $K - 4/N^2$, then we show that for $\lambda > 4/N^2$ there are no singular sequences for $K - \lambda$.

Part 1. By part (i) of Lemma 1.2, there exists a sequence $\{u_n\} \subset H_A$ such that $\operatorname{supp} u_n \subset \overline{B}_\varepsilon \subset \Omega$ with

$$|u_n|_2 = 1 \quad \text{and} \quad \int_\Omega |x|^2 |\nabla u_n|^2\, dx \le \frac{N^2}{4} + \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

Now define v_n by

$$v_n(x) = \begin{cases} n^{N/2} u_n(nx) & \text{for } |x| \le \varepsilon/n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $v_n \in H_A$ with $\operatorname{supp} v_n \subset \overline{B}_{\varepsilon/n} \subset \Omega$ and

$$\begin{aligned} |v_n|_2^2 &= \int_{|x|\le\varepsilon/n} n^N u_n(nx)^2\, dx = \int_{|y|\le\varepsilon} u_n(y)^2\, dy = 1, \\ \|v_n\|^2 &= \int_\Omega |x|^2 |\nabla v_n(x)|^2\, dx = \int_{|x|\le\varepsilon/n} |x|^2 n^{2+N} |\nabla u_n(nx)|^2\, dx \\ &= \int_{|y|\le\varepsilon} |y|^2 |\nabla u_n(y)|^2 dy \le \frac{N^2}{4} + \frac{1}{n}. \end{aligned}$$

Since $1 = |v_n|_2^2 \le \frac{4}{N^2}\|v_n\|^2$ by (1.7), it follows that

$$\frac{N^2}{4} \le \|v_n\|^2 \le \frac{N^2}{4} + \frac{1}{n}.$$

For any $u \in H_A$,

$$\begin{aligned} \langle v_n, u\rangle &= \int_{|x|\le\varepsilon/n} |x|^2 \nabla v_n \cdot \nabla u\, dx \\ &\le \left\{\int_{|x|\le\varepsilon/n} |x|^2 |\nabla v_n|^2\, dx\right\}^{1/2} \left\{\int_{|x|\le\varepsilon/n} |x|^2 |\nabla u|^2\, dx\right\}^{1/2} \\ &= \|v_n\| \left\{\int_{|x|\le\varepsilon/n} |x|^2 |\nabla u|^2\, dx\right\}^{1/2} \\ &\le \sqrt{\frac{N^2}{4} + \frac{1}{n}} \left\{\int_{|x|\le\varepsilon/n} |x|^2 |\nabla u|^2\, dx\right\}^{1/2} \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

showing that $v_n \rightharpoonup 0$ weakly in H_A. Since $\|v_n\|_A \to N/2$, this means that $\{v_n\}$ has no subsequence converging strongly in H_A. Furthermore, for all $u \in H_A$,

$$\begin{aligned}\left|\left\langle\left(K-\frac{4}{N^2}\right)v_n, u\right\rangle_A\right| &= \left|\int_\Omega\left(v_n u - \frac{4}{N^2}A(x)\nabla v_n\cdot\nabla u\right)dx\right| \\ &\leq \left|\int_\Omega\left(v_n u - \frac{4}{N^2}|x|^2\nabla v_n\cdot\nabla u\right)dx\right| + \frac{4L_n}{N^2}\|v_n\|\,\|u\|\end{aligned}$$

where $L_n = \sup_{0<|x|\leq\varepsilon/n}|A(x)/|x|^2 - 1| \to 0$ as $n\to\infty$. But

$$\int_\Omega\left(v_n u - \frac{4}{N^2}|x|^2\nabla v_n\cdot\nabla u\right)dx = \left\langle\left(\widetilde{K}-\frac{4}{N^2}\right)v_n, u\right\rangle$$

where $\widetilde{K}: H_A \to H_A$ is defined by

$$\langle\widetilde{K}w, u\rangle = \int_\Omega wu\,dx \qquad \text{for all } u, w \in H_A$$

and so

$$\left|\int_\Omega\left(v_n u - \frac{4}{N^2}|x|^2\nabla v_n\cdot\nabla u\right)dx\right| \leq \left\|\left(\widetilde{K}-\frac{4}{N^2}\right)v_n\right\|\|u\|.$$

It follows that

$$\begin{aligned}\left|\left\langle\left(K-\frac{4}{N^2}\right)v_n, u\right\rangle_A\right| &\leq \left\{\left\|\left(\widetilde{K}-\frac{4}{N^2}\right)v_n\right\| + \frac{4L_n}{N^2}\|v_n\|\right\}\|u\| \\ &\leq \left\{\left\|\left(\widetilde{K}-\frac{4}{N^2}\right)v_n\right\| + \frac{4L_n}{N^2}\|v_n\|\right\}\frac{\|u\|_A}{\sqrt{\alpha_1}}\end{aligned}$$

and hence that

$$\left\|\left(K-\frac{4}{N^2}\right)v_n\right\|_A \leq \frac{1}{\sqrt{\alpha_1}}\left\{\left\|\left(\widetilde{K}-\frac{4}{N^2}\right)v_n\right\| + \frac{4L_n}{N^2}\|v_n\|\right\}.$$

But

$$|\langle\widetilde{K}w, u\rangle| = \left|\int_\Omega wu\,dx\right| \leq |w|_2|u|_2 \leq \frac{4}{N^2}\|w\|\,\|u\|$$

showing that $\|\widetilde{K}w\| \leq (4/N^2)\|w\|$ for all $w \in H_A$. Thus

$$\begin{aligned}\left\|\left(\widetilde{K}-\frac{4}{N^2}\right)v_n\right\|^2 &= \|\widetilde{K}v_n\|^2 - \frac{8}{N^2}\langle\widetilde{K}v_n, v_n\rangle + \frac{16}{N^4}\|v_n\|^2 \\ &\leq \frac{16}{N^4}\|v_n\|^2 - \frac{8}{N^2}\int_\Omega v_n^2\,dx + \frac{16}{N^4}\|v_n\|^2 \\ &\leq \frac{32}{N^4}\left(\frac{N^2}{4}+\frac{1}{n}\right) - \frac{8}{N^2} = \frac{32}{N^4 n}\end{aligned}$$

and

$$\left\|\left(K-\frac{4}{N^2}\right)v_n\right\|_A \leq \frac{1}{\sqrt{\alpha_1}}\left\{\sqrt{\frac{32}{N^4 n}} + \frac{4L_n}{N^2}\sqrt{\frac{N^2}{4}+\frac{1}{n}}\right\}.$$

Thus $\|(K-4/N^2)v_n\|_A \to 0$ and we have shown that $\{v_n\}$ is a singular sequence for $K-4/N^2$. This implies that $4/N^2 \in \sigma_e(K)$.

Part 2. Fix $\lambda > 4/N^2$. Consider a sequence $\{v_n\} \subset H_A$ such that

$$v_n \rightharpoonup 0 \quad \text{in } H_A \quad \text{and} \quad \|(K-\lambda)v_n\|_A \to 0.$$

It is enough to prove that these two properties imply that $\|v_n\|_A \to 0$.

First we observe that $v_n \rightharpoonup 0$ in $W^{1,2}(\Omega\backslash\overline{B}_\varepsilon)$ for all $\varepsilon > 0$ such that $\overline{B}_\varepsilon \subset \Omega$ where $B_\varepsilon = \{x \in \mathbb{R}^N : |x| < \varepsilon\}$. This implies that

$$\int_{\Omega\backslash\overline{B}_\varepsilon} v_n^2\,dx \to 0 \quad \text{for all } \varepsilon > 0 \text{ such that } \overline{B}_\varepsilon \subset \Omega. \tag{3.1}$$

Since $\lambda > 4/N^2$, we can choose $\delta > 0$ such that $\lambda(1-\delta) > 4/N^2$. Then we can choose $\varepsilon > 0$ such that $\overline{B}_\varepsilon \subset \Omega$ and $A(x) \geq (1-\delta)|x|^2$ for all $x \in \overline{B}_\varepsilon$. Let $\varphi \in C^\infty(\mathbb{R}^N)$ be such that

$$\varphi(x) = 1 \text{ for } |x| \leq \varepsilon/2, \quad 0 \leq \varphi(x) \leq 1 \text{ for } \varepsilon/2 \leq |x| < \varepsilon, \quad \varphi(x) = 0 \text{ for } |x| \geq \varepsilon.$$

Then $\varphi v_n \in H_A$ and

$$\begin{aligned}
&\int_\Omega (\lambda A(x)|\nabla(\varphi v_n)|^2 - (\varphi v_n)^2)\,dx \\
&\quad \geq \int_\Omega \left(\lambda A(x)|\nabla(\varphi v_n)|^2 - \frac{4}{N^2}|x|^2|\nabla(\varphi v_n)|^2\right)dx \qquad \text{by (1.7)} \\
&\quad = \int_{B_\varepsilon} \left(\lambda A(x)|\nabla(\varphi v_n)|^2 - \frac{4}{N^2}|x|^2|\nabla(\varphi v_n)|^2\right)dx \\
&\quad \geq \int_{B_\varepsilon} \left(\lambda(1-\delta)|x|^2|\nabla(\varphi v_n)|^2 - \frac{4}{N^2}|x|^2|\nabla(\varphi v_n)|^2\right)dx \\
&\quad = \left\{\lambda(1-\delta) - \frac{4}{N^2}\right\}\int_{B_\varepsilon} |x|^2|\nabla(\varphi v_n)|^2\,dx = \left\{\lambda(1-\delta) - \frac{4}{N^2}\right\}\|\varphi v_n\|^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_\Omega (\lambda A(x)|\nabla(\varphi v_n)|^2 - (\varphi v_n)^2)\,dx \\
&\qquad = \int_\Omega (\lambda A(x)\{\varphi\nabla v_n + v_n\nabla\varphi\}\cdot\nabla(\varphi v_n) - \varphi^2 v_n^2)\,dx \\
&\qquad = \int_\Omega (\lambda A(x)\nabla v_n\cdot\nabla(\varphi v_n) - v_n\varphi v_n)\,dx + \int_\Omega \varphi(1-\varphi)v_n^2\,dx \\
&\qquad\quad + \int_\Omega \lambda A(x)\{(\varphi-1)\nabla v_n + v_n\nabla\varphi\}\cdot\nabla(\varphi v_n)\,dx \\
&\qquad = \langle(\lambda-K)v_n, \varphi v_n\rangle_A + \int_\Omega \varphi(1-\varphi)v_n^2\,dx \\
&\qquad\quad + \int_\Omega \lambda A(x)\{(\varphi-1)\nabla v_n + v_n\nabla\varphi\}\cdot\nabla(\varphi v_n)\,dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega \lambda A(x)\{(\varphi-1)\nabla v_n + v_n\nabla\varphi\}\cdot\nabla(\varphi v_n)\,dx\\
&\quad= \int_\Omega \lambda A(x)\{(\varphi-1)\varphi|\nabla v_n|^2 + (\varphi-1)v_n\nabla v_n\cdot\nabla\varphi + v_n\nabla\varphi\cdot\nabla(\varphi v_n)\}\,dx\\
&\quad\le \int_\Omega \lambda A(x)\{(\varphi-1)v_n\nabla v_n\cdot\nabla\varphi + v_n\nabla\varphi\cdot\nabla(\varphi v_n)\}\,dx\\
&\quad= \int_{\Omega\setminus\overline{B}_{\varepsilon/2}} \lambda A(x)\{(\varphi-1)v_n\nabla v_n\cdot\nabla\varphi + v_n\nabla\varphi\cdot\nabla(\varphi v_n)\}\,dx\\
&\quad= \int_{\Omega\setminus\overline{B}_{\varepsilon/2}} \lambda A(x)\{(2\varphi-1)v_n\nabla v_n\cdot\nabla\varphi + v_n^2|\nabla\varphi|^2\}\,dx.
\end{aligned}$$

Hence

$$\begin{aligned}
\left\{\lambda(1-\delta)-\frac{4}{N^2}\right\}\|\varphi v_n\|^2 &\le \langle(\lambda-K)v_n,\varphi v_n\rangle_A + \int_\Omega \varphi(1-\varphi)v_n^2\,dx\\
&\quad+\int_\Omega \lambda A(x)\{(\varphi-1)\nabla v_n + v_n\nabla\varphi\}\cdot\nabla(\varphi v_n)\,dx\\
&\le |\langle(\lambda-K)v_n,\varphi v_n\rangle_A| + \int_{\Omega\setminus\overline{B}_{\varepsilon/2}} v_n^2\,dx\\
&\quad+\int_{\Omega\setminus\overline{B}_{\varepsilon/2}} \lambda A(x)\{(2\varphi-1)v_n\nabla v_n\cdot\nabla\varphi + v_n^2|\nabla\varphi|^2\}\,dx.
\end{aligned}$$

Setting $\rho=\{\lambda(1-\delta)-4/N^2\}$, we see that $\rho>0$ and

$$\begin{aligned}
|\langle(\lambda-K)v_n,\varphi v_n\rangle_A| &\le \|(K-\lambda)v_n\|_A\|\varphi v_n\|_A \le \sqrt{\alpha_2}\|(K-\lambda)v_n\|_A\|\varphi v_n\|\\
&\le \frac{\alpha_2}{2\rho}\|(K-\lambda)v_n\|_A^2 + \frac{\rho}{2}\|\varphi v_n\|^2.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\frac{\rho}{2}\|\varphi v_n\|^2 &\le \frac{\alpha_2}{2\rho}\|(K-\lambda)v_n\|_A^2 + \int_{\Omega\setminus\overline{B}_{\varepsilon/2}} v_n^2\,dx\\
&\quad+\int_{\Omega\setminus\overline{B}_{\varepsilon/2}} \lambda A(x)\{(2\varphi-1)v_n\nabla v_n\cdot\nabla\varphi + v_n^2|\nabla\varphi|^2\}\,dx.
\end{aligned}$$

But $\int_{\Omega\setminus\overline{B}_{\varepsilon/2}} v_n^2\,dx \to 0$ by (3.1), and

$$\int_{\Omega\setminus\overline{B}_{\varepsilon/2}} |\nabla v_n|^2\,dx \le \int_{\Omega\setminus\overline{B}_{\varepsilon/2}} \frac{|x|^2}{(\varepsilon/2)^2}|\nabla v_n|^2\,dx \le \frac{4}{\varepsilon^2}\int_\Omega |x|^2|\nabla v_n|^2\,dx \le \frac{4}{\varepsilon^2\alpha_1}\|v_n\|_A^2,$$

showing that

$$\int_{\Omega\setminus\overline{B}_{\varepsilon/2}} v_n^2\,dx + \int_{\Omega\setminus\overline{B}_{\varepsilon/2}} \lambda A(x)\{(2\varphi-1)v_n\nabla v_n\cdot\nabla\varphi + v_n^2|\nabla\varphi|^2\}\,dx \to 0.$$

This proves that $\|\varphi v_n\| \to 0$ and consequently $\int_{B_{\varepsilon/2}} v_n^2\,dx \to 0$ because

$$\int_{B_{\varepsilon/2}} v_n^2\,dx \le \int_\Omega (\varphi v_n)^2\,dx \le \frac{4}{N^2}\|\varphi v_n\|^2.$$

Recalling (3.1), we now deduce that $\langle K v_n, v_n\rangle_A = \int_\Omega v_n^2\,dx \to 0$. But

$$\begin{aligned}\lambda\|v_n\|_A^2 &= \langle K v_n, v_n\rangle_A - \langle (K-\lambda)v_n, v_n\rangle_A \\ &\le |\langle K v_n, v_n\rangle_A| + \|(K-\lambda)v_n\|_A\|v_n\|_A\end{aligned}$$

where $\{\|v_n\|_A\}$ is bounded and $\|(K-\lambda)v_n\|_A \to 0$ by hypothesis. This proves that $\|v_n\|_A \to 0$ and so $\lambda \notin \sigma_e(K)$.

(iii) This follows from part (ii) and the definition (1.10) of K. □

4. PROPERTIES OF THE OPERATOR G

We now turn to the nonlinear operator $G : H_A \to H_A$ defined by (1.10) and the corresponding potential $\psi : H_A \to \mathbb{R}$ defined by

$$\psi(u) = \int_\Omega F(u(x))\,dx \quad \text{where} \quad F(s) = \int_0^s f(t)\,dt. \tag{4.1}$$

PROPOSITION 4.1. *Under the hypotheses* (D1) *and* (D2)*, we have:*

(i) *$G : H_A \to H_A$ is Lipschitz continuous.*
(ii) *$G : H_A \to H_A$ is both Hadamard and w-Hadamard differentiable at $u = 0$ with $G'(0) = K$.*
(iii) *$\psi : H_A \to \mathbb{R}$ is continuously Fréchet differentiable and $\psi'(u)v = \langle G(u), v\rangle_A$ for all $u, v \in H_A$.*
(iv) *If in addition,*

(D3) $\sup\{|f(s)| : s \in \mathbb{R}\} = m < \infty$,

then $G : H_A \to H_A$ is compact and $G : H_A \to H_A$ is not Fréchet differentiable at $u = 0$.

PROOF. Suppose throughout that $u, v, w \in H_A$.

(i) We have

$$\begin{aligned}|\langle G(u) - G(v), w\rangle_A| &\le \int_\Omega |f(u) - f(v)|\,|w|\,dx \le \int_\Omega M|u-v|\,|w|\,dx \\ &\le M|u-v|_2|w|_2 \le \frac{4M}{N^2\alpha_1}\|u-v\|_A\|w\|_A\end{aligned}$$

and hence

$$\|G(u) - G(v)\|_A \le \frac{4M}{N^2\alpha_1}\|u-v\|_A.$$

(ii) In view of (i), to establish Hadamard differentiability at $u = 0$, it is enough to prove that G is Gateaux differentiable at $u = 0$. For $t \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned}
\left|\left\langle \frac{G(tv) - G(0)}{t} - K(v), w \right\rangle_A\right| &= \left|\int_\Omega \left\{\frac{f(tv)}{t} - v\right\} w\, dx\right| \\
&= \left|\int_\Omega \int_0^1 \left\{\frac{1}{t}\frac{d}{ds} f(stv) - v\right\} w\, ds\, dx\right| = \left|\int_\Omega \int_0^1 \{f'(stv) - 1\} vw\, ds\, dx\right| \\
&\le \int_\Omega \int_0^1 |v|\,|w|\,|f'(stv) - 1|\, ds\, dx \\
&\le \left\{\int_\Omega \int_0^1 |v|^2 |f'(stv) - 1|^2\, ds\, dx\right\}^{1/2} \left\{\int_\Omega \int_0^1 |w|^2\, ds\, dx\right\}^{1/2} \\
&= |w|_2 \left\{\int_\Omega \int_0^1 |v|^2 |f'(stv) - 1|^2\, ds\, dx\right\}^{1/2} \\
&\le \frac{2}{N\sqrt{\alpha_1}} \|w\|_A \left\{\int_\Omega \int_0^1 |v|^2 |f'(stv) - 1|^2\, ds\, dx\right\}^{1/2}
\end{aligned}$$

so that

$$\left\| \frac{G(tv) - G(0)}{t} - K(v) \right\|_A \le \frac{2}{N\sqrt{\alpha_1}} \left\{\int_\Omega \int_0^1 |v|^2 |f'(stv) - 1|^2\, ds\, dx\right\}^{1/2}.$$

But $v \in L^2(\Omega)$ and $|f'(stv(x)) - 1| \le M + 1$ for all $|t| \le 1$, $s \in [0, 1]$ and almost all $x \in \Omega$. It follows from dominated convergence that G is Gateaux differentiable at $u = 0$.

For the w-Hadamard differentiability, we now consider sequences $\{v_n\} \subset H_A$ and $\{t_n\} \subset \mathbb{R} \setminus \{0\}$ such that $v_n \rightharpoonup v$ weakly in H_A and $t_n \to 0$. Then

$$\langle K(v_n) - K(v), w\rangle_A = \langle v_n - v, Kw\rangle_A \to 0$$

and

$$\left|\left\langle \frac{G(t_n v_n) - G(0)}{t_n} - K(v_n), w \right\rangle_A\right| \le \int_\Omega \int_0^1 |v_n|\,|w|\,|f'(st_n v_n) - 1|\, ds\, dx.$$

Since $\{v_n\}$ is bounded in H_A, there is a constant $C > 0$ such that

$$|v_n|_2 \le \frac{2}{N\sqrt{\alpha_1}} \|v_n\|_A \le C \quad \text{for all } n \in \mathbb{N}.$$

Then, for any $\varepsilon > 0$ such that $\overline{B}_\varepsilon \subset \Omega$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\int_{B_\varepsilon} \int_0^1 |v_n|\,|w|\,|f'(st_n v_n) - 1|\, ds\, dx &\le (M + 1)|v_n|_2 \left\{\int_{B_\varepsilon} w^2\, dx\right\}^{1/2} \\
&\le (M + 1)C \left\{\int_{B_\varepsilon} w^2\, dx\right\}^{1/2}.
\end{aligned}$$

We claim that

$$\int_{\Omega\setminus\overline{B}_\varepsilon}\int_0^1 |v_n|\,|w|\,|f'(st_nv_n)-1|\,ds\,dx \to 0 \quad \text{as } n\to\infty.$$

Indeed, if this were false, there would exist $\varepsilon\in(0,1)$, $\delta>0$ and a subsequence $\{n_k\}$ such that

$$\int_{\Omega\setminus\overline{B}_\varepsilon}\int_0^1 |v_{n_k}|\,|w|\,|f'(st_{n_k}v_{n_k})-1|\,ds\,dx \geq \delta \quad \text{for all } n_k. \tag{4.2}$$

But the fact that $v_n \rightharpoonup v$ weakly in H_A implies that $v_n \rightharpoonup v$ weakly in $H^1(\Omega\setminus\overline{B}_\varepsilon)$ and hence strongly in $L^2(\Omega\setminus\overline{B}_\varepsilon)$. Therefore, passing to a further subsequence, we may suppose (see Theorem IV.6 in [8], for example) that there exists $z\in L^2(\Omega\setminus\overline{B}_\varepsilon)$ such that

$$|v_{n_k}|\leq z \quad \text{a.e. on } \Omega\setminus\overline{B}_\varepsilon \quad \text{and} \quad v_{n_k}\to z \quad \text{a.e. on } \Omega\setminus\overline{B}_\varepsilon.$$

Hence

$$\int_0^1 |v_{n_k}|\,|w|\,|f'(st_{n_k}v_{n_k})-1|\,ds \leq (M+1)|v_{n_k}|\,|w| \leq (M+1)|z|\,|w| \quad \text{a.e. on } \Omega\setminus\overline{B}_\varepsilon$$

where $|z|\,|w|\in L^1(\Omega\setminus\overline{B}_\varepsilon)$. Recalling that $f'(0)=1$, dominated convergence now shows that

$$\int_{\Omega\setminus\overline{B}_\varepsilon}\int_0^1 |v_{n_k}|\,|w|\,|f'(st_{n_k}v_{n_k})-1|\,ds\,dx \to 0,$$

contradicting (4.2). Thus we find that, for any $\varepsilon>0$ such that $\overline{B}_\varepsilon\subset\Omega$,

$$\begin{aligned}\limsup_{n\to\infty}\int_\Omega\int_0^1 |v_n|\,|w|\,|f'(st_nv_n)-1|\,ds\,dx \\ &\leq \limsup_{n\to\infty}\int_{B_\varepsilon}\int_0^1 |v_n|\,|w|\,|f'(st_nv_n)-1|\,ds\,dx \\ &\leq (M+1)C\left\{\int_{B_\varepsilon} w^2\,dx\right\}^{1/2}.\end{aligned}$$

But $w\in L^1(\Omega)$ so $\{\int_{B_\varepsilon} w^2\,dx\}^{1/2}\to 0$ as $\varepsilon\to 0+$ and we have proved that

$$\int_\Omega\int_0^1 |v_n|\,|w|\,|f'(st_nv_n)-1|\,ds\,dx \to 0.$$

Finally,

$$\begin{aligned}\left|\left\langle \frac{G(0)-G(t_nv_n)}{t_n} - K(v), w\right\rangle_A\right| \\ &\leq \left|\left\langle \frac{G(0)-G(t_nv_n)}{t_n} - K(v_n), w\right\rangle_A\right| + |\langle K(v_n)-K(v), w\rangle_A| \\ &\leq \int_\Omega\int_0^1 |v_n|\,|w|\,|f'(st_nv_n)-1|\,ds\,dx + |\langle K(v_n)-K(v), w\rangle_A|\end{aligned}$$

and it follows that G is w-Hadamard differentiable at $u=0$.

(iii) By part (i), it is enough to prove that ψ is Gateaux differentiable at u with $\psi'(u)v = \langle G(u), v\rangle_A$ for all $u, v \in H_A$. For any $t \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned}\left|\frac{\psi(u+tv)-\psi(u)}{t} - \langle G(u), v\rangle_A\right| &= \left|\int_\Omega \left(\frac{1}{t}\int_0^1 \frac{d}{ds}F(u+stv)ds - f(u)v\right)dx\right| \\ &= \left|\int_\Omega \int_0^1 \{f(u+stv) - f(u)\}v\,ds\,dx\right| \\ &\le \int_\Omega \int_0^1 M|stv|\,|v|\,ds\,dx \le \frac{M|t|}{2}|v|_2^2\end{aligned}$$

and the result follows.

(iv) Consider a sequence $\{v_n\} \subset H_A$ such that $v_n \rightharpoonup v$ weakly in H_A. For any $\varepsilon > 0$ such that $\overline{B}_\varepsilon \subset \Omega$ and any $w \in H_A$, we have

$$\begin{aligned}&|\langle G(v_n) - G(v), w\rangle_A| \\ &\qquad \le \int_\Omega |f(v_n) - f(v)|\,|w|\,dx \\ &\qquad \le 2m\int_{B_\varepsilon} |w|\,dx + \int_{\Omega\setminus B_\varepsilon} |f(v_n) - f(v)|\,|w|\,dx \\ &\qquad \le 2m\left[\int_{B_\varepsilon} dx\right]^{1/2}|w|_2 + \left[\int_{\Omega\setminus B_\varepsilon} |f(v_n)-f(v)|^2\,dx\right]^{1/2}|w|_2 \\ &\qquad \le \left\{2m\left[\int_{B_\varepsilon} dx\right]^{1/2} + \left[\int_{\Omega\setminus B_\varepsilon} |f(v_n)-f(v)|^2\,dx\right]^{1/2}\right\}\frac{2}{N\sqrt{\alpha_1}}\|w\|_A\end{aligned}$$

and hence

$$\begin{aligned}\|G(v_n) - G(v)\|_A &\le \left\{2m\left[\int_{B_\varepsilon} dx\right]^{1/2} + \left[\int_{\Omega\setminus B_\varepsilon} |f(v_n)-f(v)|^2\,dx\right]^{1/2}\right\}\frac{2}{N\sqrt{\alpha_1}} \\ &\le \left\{2m\left[\int_{B_\varepsilon} dx\right]^{1/2} + M\left[\int_{\Omega\setminus B_\varepsilon} |v_n - v|^2\,dx\right]^{1/2}\right\}\frac{2}{N\sqrt{\alpha_1}}.\end{aligned}$$

Since $v_n \to v$ strongly in $L^2(\Omega\setminus\overline{B}_\varepsilon)$, it follows that

$$\limsup_{n\to\infty} \|G(v_n) - G(v)\|_A \le \frac{4m}{N\sqrt{\alpha_1}}\left[\int_{B_\varepsilon} dx\right]^{1/2}$$

for any $\varepsilon > 0$ such that $\overline{B}_\varepsilon \subset \Omega$. Thus $\|G(v_n) - G(v)\|_A \to 0$ as $n \to \infty$, establishing the compactness of $G : H_A \to H_A$. Since $\sup \sigma_e(K) > 0$, $K = G'(0) : H_A \to H$ is not a compact linear operator and so G cannot be Fréchet differentiable at $u = 0$. □

5. Bifurcation for (1.1), (1.2)

As we have shown in Section 1, $(\lambda, u) \in \mathbb{R} \times H_0$ is a solution of (1.1), (1.2) if and only if (λ, u) satisfies (1.11). Furthermore, if (λ, u) is a solution and $u \neq 0$, then $|\lambda| \ge$

$N^2\alpha_1/4M$ and so λ is a bifurcation point for (1.1), (1.2) if and only if $\mu = 1/\lambda \in B_G$ in the terminology of Section 2 with $G : H_A \to H_A$ defined by (1.10).

Similarly $\mu \in \sigma(K) \cap (4/N^2, \infty)$ if and only if the linear boundary value problem

$$\begin{aligned} -\nabla \cdot \{A(x)\nabla u(x)\} &= \lambda u(x) \quad \text{for } x \in \Omega, \\ u &= 0 \quad \text{for } x \in \partial\Omega, \end{aligned}$$

has a nontrivial solution $u \in H_A$ for $\lambda = 1/\mu$. Let

$$\Sigma = \{1/\mu : \mu \in \sigma(K) \cap (4/N^2, \infty)\}$$

be the set of all such eigenvalues of this linearization of (1.1), (1.2).

Under the assumption (D1), the set Σ may be empty. For example, if $A(x) \geq |x|^2$ for all $x \in \Omega$, it follows from Proposition 3.1(i) that

$$\begin{aligned} \|K\| &= \sup\left\{ \frac{\langle Ku, u\rangle_A}{\|u\|_A^2} : u \in H_A \setminus \{0\} \right\} = \sup\left\{ \frac{\int_\Omega u^2\,dx}{\int_\Omega A|\nabla u|^2\,dx} : u \in H_A \setminus \{0\} \right\} \\ &\leq \sup\left\{ \frac{\int_\Omega u^2\,dx}{\int_\Omega |x|^2|\nabla u|^2\,dx} : u \in H_A \setminus \{0\} \right\} = \frac{4}{N^2} \end{aligned}$$

and so $\sigma(K) \cap (4/N^2, \infty) = \emptyset$. Hence $\Sigma = \emptyset$ if $A(x) \geq |x|^2$ for all $x \in \Omega$.

On the other hand, there are coefficients A satisfying (D1) for which Σ contains many points. Let Λ_1 denote the first eigenvalue of the Laplacian with Dirichlet boundary condition on Ω. Then

$$\Lambda_1 = \inf\left\{ \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega u^2\,dx} : u \in W_0^{1,2}(\Omega) \setminus \{0\} \right\}$$

and so, since $W_0^{1,2}(\Omega) \subset H_A$,

$$\begin{aligned} \|K\| &= \sup\left\{ \frac{\int_\Omega u^2\,dx}{\int_\Omega A|\nabla u|^2\,dx} : u \in H_A \setminus \{0\} \right\} \\ &\geq \sup\left\{ \frac{\int_\Omega u^2\,dx}{\int_\Omega A|\nabla u|^2\,dx} : u \in W_0^{1,2}(\Omega) \setminus \{0\} \right\} \\ &\geq \frac{1}{\max_\Omega A} \sup\left\{ \frac{\int_\Omega u^2\,dx}{\int_\Omega |\nabla u|^2\,dx} : u \in W_0^{1,2}(\Omega) \setminus \{0\} \right\} = \frac{1}{\Lambda_1 \max_\Omega A}. \end{aligned}$$

If A satisfies (D1) and

$$\max_\Omega A < \frac{N^2}{4\Lambda_1}, \tag{5.1}$$

we see that $\|K\| > 4/N^2 = \sup \sigma_e(K)$, and consequently $\|K\| \in \sigma(K) \cap (4/N^2, \infty)$ by the self-adjointness of K in H_A. This shows that $1/\|K\| \in \Sigma$ when (5.1) is satisfied.

THEOREM 5.1. *Suppose that the conditions* (D1) *and* (D2) *are satisfied and let B denote the set of all bifurcation points for the problem* (1.1), (1.2).

(i) *If* $0 \leq f(s)/s \leq 1$ *for all* $s \neq 0$, *then* $B \subset \Sigma \cup [N^2/4, \infty)$.
(ii) *If f is odd with*

$$\sup_{s\in\mathbb{R}} |f(s)| < \infty \quad \text{and} \quad sf(s) < 2\int_0^s f(t)\,dt \quad \text{for all } s > 0,$$

then $\Sigma \cup [N^2/4, \infty) \subset B$. *More precisely, there is bifurcation to the right at every* $\lambda \in \Sigma$, *vertical bifurcation at every* $\lambda \in (N^2/4, \infty)$ *and* $B \cap (0, \infty) = \Sigma \cup [N^2/4, \infty)$.

PROOF. Under our hypotheses, $G : H_A \to H_A$ is both Hadamard differentiable and w-Hadamard differentiable at $u = 0$ with $G'(0) = K = K^*$. We also have $\sigma(K) \subset [0, 4/N^2\alpha_1]$ and $\sup \sigma_e(K) = 4/N^2$.

(i) By the assumptions about f,

$$0 \leq \int_\Omega f(u)u\,dx \leq \int_\Omega u^2\,dx \quad \text{for all } u \in H_A \subset L^2(\Omega).$$

If $(\lambda, u) \in \mathbb{R} \times H_A$ is a solution of (1.1), (1.2), then by (1.6),

$$\int_\Omega A(x)|\nabla u(x)|^2\,dx = \lambda \int_\Omega f(u)u\,dx$$

and so $\lambda \geq 0$. Using Lemma 1.2(iv), we then deduce that $B \subset [N^2/4\alpha_1, \infty) \subset (0, \infty)$. Thus if $\lambda \in B$, then $\mu = 1/\lambda \in B_G \cap (0, \infty)$.

Suppose that $\lambda \in B \cap [N^2/4\alpha_1, N^2/4)$. Then $\mu = 1/\lambda > 4/N^2 = \sup \sigma_e(K)$ and $\mu \in B_G$. Furthermore,

$$\frac{\langle G(u) - G'(0)u, u\rangle_A}{\|u\|_A^2} = \frac{\int_\Omega \{f(u) - u\}u\,dx}{\|u\|_A^2} \leq 0,$$

and it follows from Proposition 2.3 that $\mu \notin \sigma(G'(0)) = \sigma(K)$ and consequently, $\lambda \in \Sigma$.

(ii) Here we use Proposition 2.4 with $\psi : H_A \to \mathbb{R}$ defined by (4.1). By Proposition 4.1, we know that $\psi \in C^1(H_A, \mathbb{R})$ with

$$\psi'(u)v = \langle G(u), v\rangle_A = \int_\Omega f(u)v\,dx \quad \text{for all } u, v \in H_A.$$

Also, for all $u, v \in H_A$,

$$|\langle G(u), v\rangle_A| \leq m|\Omega|^{1/2}|v|_2 \leq \frac{2m|\Omega|^{1/2}}{N\sqrt{\alpha_1}}\|v\|_A$$

where $m = \sup_{s\in\mathbb{R}} |f(s)|$ and $|\Omega|$ is the N-dimensional volume of Ω. Hence $\|G(u)\|_A \leq 2m|\Omega|^{1/2}/N\sqrt{\alpha_1}$ for all $u \in H_A$, and

$$\psi(u) = \left|\int_0^1 \langle G(tu), u\rangle_A\,dt\right| \leq \frac{2m|\Omega|^{1/2}}{N\sqrt{\alpha_1}}\|u\|_A.$$

Thus we see that

$$\lim_{\|u\|_A \to \infty} \psi(u)/\|u\|_A^2 = 0.$$

Furthermore, for $u \in H_A \setminus \{0\}$,

$$2\psi(u) - \psi'(u)u = \int_\Omega \left\{ 2\int_0^{u(x)} f(t)\,dt - f(u(x))u(x) \right\} dx > 0.$$

Referring to Proposition 4.1, we have now shown that the hypotheses (H1) to (H3) of Proposition 2.4 are satisfied and $\Lambda^e = \sup \sigma_e(K) = 4/N^2$ by Proposition 3.1. It follows that $\Sigma \cup [N^2/4, \infty) \subset B$.

Finally, we observe that, since $\psi'(u)u < 2\psi(u)$ for all $u \in H \setminus \{0\}$, we have

$$\frac{d}{dt}\frac{\psi(tu)}{t^2} < 0 \quad \text{for all } u \in H\backslash\{0\} \text{ and } t > 0$$

and hence

$$\frac{\langle G'(0)u, u\rangle}{2} = \lim_{t\to 0} \frac{\langle G(tu), u\rangle}{2t} = \lim_{t \to 0} \frac{\psi(tu)}{t^2} > \psi(u).$$

Hence

$$\langle G'(0)u, u\rangle_A > 2\psi(u) > \psi'(u)u = \langle G(u), u\rangle_A \quad \text{for all } u \in H\backslash\{0\}$$

and so

$$\limsup_{\|u\|_A \to 0} \frac{\langle G(u) - G'(0)u, u\rangle_A}{\|u\|_A^2} \leq 0.$$

Using Proposition 2.3, it follows that $(4/N^2, \infty) \cap B_G \subset \sigma(K)$ and hence that $B \cap (0, N^2/4) \subset \Sigma$. □

EXAMPLES. Consider an odd function $f \in C^1(\mathbb{R})$ that is positive, strictly concave and bounded on $[0, \infty)$ with $f'(0) = 1$. Then $\lim_{s\to\infty} f'(s) = 0$ and f satisfies all the hypotheses of Theorem 5.1. For such functions, $B = \Sigma \cup [N^2/4, \infty)$. The functions $f(s) = \tanh s$ and $f(s) = \arctan s$ have these properties.

On the other hand, for any $\alpha > 2$, one finds that the function $f(s) = s(1+s^2)^{-\alpha}$ also satisfies all of the hypotheses of Theorem 5.1, but it is not concave on $[0, \infty)$. In fact, f'' changes sign exactly once in $[0, \infty)$.

5.1. *More general nonlinearities*

As we have shown in Theorem 5.1, the abstract results in Section 2 can be applied directly to some problems of the type (1.1), (1.2) but the nonlinear term f is required to be bounded on the whole real line. However, it is possible to deduce from these results similar conclusions for equations with unbounded nonlinearities. In fact, it is sufficient to assume that f satisfies the following condition.

(F) For some $T > 0$, $f \in C^1([-T, T])$ is an odd function that is strictly concave on $[0, T]$ with $f(0) = f(T) = 0$ and $f'(0) = 1$.

Given such a function, we set

$$F(s) = \int_0^s f(t)\,dt \qquad \text{for } s \in [-T, T]$$

and then extend F to $\mathbb{R}$ as an even function having the following properties:

$$F \in C^2(\mathbb{R}), \qquad F'(s) < 0 \quad \text{for all } s > T,$$
$$\lim_{s\to\infty} F(s) = \lim_{s\to\infty} F'(s) = \lim_{s\to\infty} F''(s) = 0.$$

It follows that $F(s) > 0$ for $s \neq 0$ and the function $f = F'$ satisfies the condition (D2) and also the conditions of Theorem 5.1(ii). However, if A satisfies (D1) and $(\lambda, u) \in (0, \infty) \times H_0$ is a solution of (1.1), (1.2) for $f = F'$, it turns out that $|u(x)| \leq T$ a.e. on Ω and so we obtain the following result.

THEOREM 5.2. *Suppose that the conditions* (D1) *and* (F) *are satisfied. Then all points in the set* $\Sigma \cup [N^2/4, \infty)$ *are bifurcation points for* (1.1), (1.2) *in the sense that, for any* $\lambda \in \Sigma \cup [N^2/4, \infty)$, *there exists a sequence* $\{(\lambda_n, u_n)\} \subset (0, \infty) \times [H_0 \setminus \{0\}]$ *having the following properties: for all* $n \in \mathbb{N}$, $|u_n(x)| \leq T$ *a.e. on* Ω *and* (1.6) *holds with* $u = u_n$, $\lambda_n \to \lambda$ *and* $|u_n|_2 \to 0$ *as* $n \to \infty$.

REMARK. Since

$$|u|_1 \leq |\Omega|^{1/2}|u|_2 \quad \text{and} \quad |u|_p \leq |u|_1^{1/p}|u|_\infty^{1-1/p} \qquad \text{for all } p \in [1, \infty),$$

it follows that $|u_n|_p \leq |\Omega|^{1/2p}|u_n|_2^{1/2}T^{1-1/p}$ and so $|u_n|_p \to 0$ as $n \to \infty$ for all $p \in [1, \infty)$. However, as we have shown in [16] for the radially symmetric case of (1.1), (1.2), we do not have $|u_n|_\infty \to 0$ as $n \to \infty$. In fact, in that case, $|u_n|_\infty = T$ for all n and the bifurcating solutions concentrate at $x = 0$ in the sense that the sequence $\{u_n\}$ tends uniformly to zero on all compact subsets of $\overline{\Omega}$ that do not contain $x = 0$.

PROOF. We consider an extension $f = F'$ of the type described above. By Theorem 5.1(ii), there is bifurcation at every point in $\Sigma \cup [N^2/4, \infty)$ for (1.1), (1.2) with this function $f : \mathbb{R} \to \mathbb{R}$. It is therefore enough to show that if $(\lambda, u) \in (0, \infty) \times H_0$ satisfies (1.6) for $f = F'$, then $|u(x)| \leq T$ a.e. on Ω. Given such a pair (λ, u), let $v = (u - T)^+$ and $\omega = \{x \in \Omega : u(x) > T\}$. Since $u \in W^{1,1}(\Omega)$ by Lemma 6.1(ii) below, it follows from Lemma 7.6 of [18] that $v \in W^{1,1}(\Omega)$ with $\nabla v = \nabla u$ if $u > T$ and $\nabla v = 0$ if $u \leq T$. Hence $|\nabla v| \leq |\nabla u|$ a.e. on Ω and it follows that $v \in H$. Finally, using Proposition 5.3 of [20], we easily deduce that $v \in H_0$. Putting $\varphi = v$ in (1.6), we obtain

$$\int_\Omega A(x)\nabla u(x).\nabla v(x)\,dx = \lambda \int_\Omega f(u(x))v(x)\,dx$$

where the left hand side equals $\int_\omega A(x)|\nabla u(x)|^2\,dx \geq 0$, while the right hand side equals $\lambda \int_\omega F'(u(x))v(x)\,dx \leq 0$ since $F'(s) < 0$ for all $s > T$. But, if ω has positive measure, then

$$\lambda \int_\omega F'(u(x))v(x)\,dx < 0,$$

and we have a contradiction. Hence ω must have measure zero and $u(x) \le T$ a.e. on Ω. Using the oddness of F', we can replace u by $-u$ and obtain the same conclusion, showing that $|u(x)| \le T$ a.e. on Ω as required. □

EXAMPLES. For any $\sigma > 0$, the function $f(s) = s - |s|^\sigma s$ satisfies the condition (F) with $T = 1$. The function $f(s) = \sin s$ satisfies the condition (F) with $T = \pi$.

6. APPENDIX 1: PROPERTIES OF H_0

First we prove the lemma stated in the introduction.

PROOF OF LEMMA 1.2. (i) Let $u \in H_0 \setminus \{0\}$ and set $v(x) = ru(x)$ where $r = |x|$. Then

$$\partial_i v = \frac{x_i}{r} u + r\partial_i u \quad \text{and} \quad |\nabla v|^2 = u^2 + |x|^2 |\nabla u|^2 + x \cdot \nabla(u^2).$$

For any $\varepsilon > 0$ such that $\overline{B}_\varepsilon \subset \Omega$, we have

$$\int_{\Omega\setminus\overline{B}_\varepsilon} x \cdot \nabla(u^2)\, dx = -\varepsilon \int_{|x|=\varepsilon} u^2\, dy - N \int_{\Omega\setminus\overline{B}_\varepsilon} u^2\, dx \le -N \int_{\Omega\setminus\overline{B}_\varepsilon} u^2\, dx$$

since $\Gamma u = 0$, and so

$$\int_{\Omega\setminus\overline{B}_\varepsilon} |\nabla v|^2\, dx \le \int_{\Omega\setminus\overline{B}_\varepsilon} (|x|^2 |\nabla u|^2 - (N-1)u^2)\, dx$$

from which it follows that

$$\int_\Omega |\nabla v|^2\, dx + (N-1) \int_\Omega u^2\, dx \le \int_\Omega |x|^2 |\nabla u|^2\, dx < \infty. \tag{6.1}$$

By Lemma 6.1(i) below, u and hence v admits a generalized derivative on Ω. Also $|v| \le C|u|$ where $C = \max_{x\in\Omega} |x|$ and $u \in L^2(\Omega)$. Thus (6.1) implies that $v \in W_0^{1,2}(\Omega) \setminus \{0\}$ and Hardy's inequality (see [9] or [1]) then yields

$$\int_\Omega |\nabla v|^2\, dx > \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{v^2}{|x|^2}\, dx = \left(\frac{N-2}{2}\right)^2 \int_\Omega u^2\, dx.$$

Combined with (6.1), this yields

$$\frac{N^2}{4} \int_\Omega u^2\, dx < \int_\Omega |x|^2 |\nabla u|^2\, dx \quad \text{and} \quad \sup\left\{ \frac{\int_\Omega u^2\, dx}{\int_\Omega |x|^2 |\nabla u|^2\, dx} : u \in H_0 \setminus \{0\} \right\} \le \frac{4}{N^2}.$$

Setting $u_\alpha(x) = |x|^\alpha - \varepsilon$ for $x \in \overline{B}_\varepsilon$ and $u_\alpha(x) = 0$ for $x \in \Omega\setminus\overline{B}_\varepsilon$, we find that $u_\alpha \in H_0$ for all $\alpha > -N/2$ and a little calculation shows that

$$\lim_{\alpha\to -N/2+} \frac{\int_\Omega u_\alpha^2\, dx}{\int_\Omega |x|^2 |\nabla u_\alpha|^2\, dx} = \frac{4}{N^2},$$

completing the proof of part (i).

(ii) By part (i), $\int_\Omega |x|^2|\nabla u|^2\,dx$ is a norm on H_0 equivalent to $\langle\cdot,\cdot\rangle^{1/2}$, and using (1.3) we obtain

$$\alpha_1\int_\Omega |x|^2|\nabla u|^2\,dx \le \int_\Omega A(x)|\nabla u|^2\,dx \le \alpha_2\int_\Omega |x|^2|\nabla u|^2\,dx.$$

(iii) & (iv) Since (λ_n, u_n) satisfies (1.6) we have

$$\|u_n\|_A^2 = \int_\Omega A(x)|\nabla u_n(x)|^2\,dx = \lambda_n\int_\Omega f(u_n(x))u_n(x)\,dx \le |\lambda_n|M|u_n|_2^2$$

and

$$|u_n|_2^2 \le \frac{4}{N^2}\int_\Omega |x|^2|\nabla u_n|^2\,dx \le \frac{4}{N^2\alpha_1}\|u_n\|_A^2. \qquad \square$$

Now we provide some additional properties of the space H_0 defined by (1.5). Recall that in our definition, $\nabla u = (\partial_1 u,\ldots,\partial_N u)$ where $\partial_i u$ denotes the generalized derivative of u on the open set $\Omega\setminus\{0\}$. We use the norm on H_0 defined by (1.8).

LEMMA 6.1. (i) *If $u\in H_0$, then $\nabla u\in L^1(\Omega)$ and $\partial_i u$ is also the generalized derivative of u on Ω for $i=1,\ldots,N$.*
(ii) *H_0 is continuously embedded in $W^{1,1}(\Omega)$.*
(iii) *$C_0^\infty(\Omega)$ is dense in H_0.*

PROOF. Let $u\in H_0$. As usual, for $B_\varepsilon = \{x\in\mathbb{R}^N : |x|<\varepsilon\}$ with $\varepsilon>0$, but small enough so that $\overline{B}_\varepsilon\subset\Omega$, we have

$$\int_{\Omega\setminus B_\varepsilon}|\nabla u|\,dx \le \left\{\int_{\Omega\setminus B_\varepsilon}|x|^2|\nabla u|^2\,dx\right\}^{1/2}\left\{\int_{\Omega\setminus B_\varepsilon}|x|^{-2}\,dx\right\}^{1/2} \le C\|u\|$$

where $C = \{\int_\Omega |x|^{-2}\,dx\}^{1/2}<\infty$ since $N\ge 3$. Hence $\nabla u\in L^1(\Omega)$ and $|\nabla u|_1\le C\|u\|$.

Now consider any $\varphi\in C_0^\infty(\Omega)$. Then, since $u,\partial_i u\in L^1(\Omega)$,

$$\int_\Omega (u\partial_i\varphi+\varphi\partial_i u)\,dx = \lim_{\varepsilon\to 0}\int_{\Omega\setminus B_\varepsilon}\partial_i(u\varphi)\,dx = -\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon}\frac{x_i}{|x|}u(x)\varphi(x)\,dx.$$

But $H_0\subset L^2(\Omega)$ and so there must exist a sequence $\{\varepsilon_n\}\subset(0,\infty)$ such that $\varepsilon_n\to 0$ and

$$\int_{|y|=\varepsilon_n} u^2 dy \le \frac{1}{\varepsilon_n}.$$

But then

$$\int_{\partial B_{\varepsilon_n}}|u|\,dy \le \left\{\int_{|y|=\varepsilon_n}u^2\,dy\right\}^{1/2}\left\{\int_{|y|=\varepsilon_n}dy\right\}^{1/2} \le \varepsilon_n^{-1/2}\{\omega_N\varepsilon_n^{N-1}\}^{1/2} = \omega_N^{1/2}\varepsilon_n^{(N-2)/2}$$

and

$$\left|\int_{\partial B_{\varepsilon_n}}\frac{x_i}{|x|}u(x)\varphi(x)\,dx\right| \le |\varphi|_\infty\int_{\partial B_{\varepsilon_n}}|u|\,dy \le |\varphi|_\infty\omega_N^{1/2}\varepsilon_n^{(N-2)/2}.$$

Hence

$$\lim_{n\to\infty}\int_{\partial B_{\varepsilon_n}}\frac{x_i}{|x|}u(x)\varphi(x)\,dx=0$$

and it follows that

$$\int_\Omega (u\partial_i\varphi+\varphi\partial_i u)\,dx=0.$$

Since $u, \partial_i u \in L^1(\Omega)$, this proves that u admits a generalized derivative on Ω and indeed that $u \in W^{1,1}(\Omega)$ with

$$|u|_1+|\nabla u|_1 \le |\Omega|^{1/2}|u|_2+C\|u\| \le \{2|\Omega|^{1/2}/N+C\}\|u\|.$$

This proves parts (i) and (ii).

Let $\xi \in C^\infty(\mathbb{R}^N)$ with $\xi(x)=1$ for $|x|\ge 2$ and $\xi(x)=0$ for $|x|\le 1$. Consider $n \ge 2/\varepsilon$ where $\varepsilon > 0$ is such that $\overline{B}_\varepsilon \subset \Omega$ and set $\xi_n(x)=\xi(nx)$. Note that $|\nabla\xi_n(x)| \le n|\nabla\xi|_\infty$. Then for any $u \in H_0$, $\xi_n u \in W_0^{1,2}(\Omega)$ and

$$\|u-\xi_n u\|^2=\int_\Omega |x|^2|\nabla[(1-\xi_n)u]|^2\,dx \le 2\int_\Omega |x|^2\{|\nabla\xi_n|^2u^2+(1-\xi_n)^2|\nabla u|^2\}\,dx.$$

But

$$\begin{aligned}\int_\Omega |x|^2|\nabla\xi_n|^2u^2\,dx &= \int_{1/n<|x|<2/n}|x|^2|\nabla\xi_n|^2u^2\,dx \le \int_{1/n<|x|<2/n}\left(\frac{2}{n}\right)^2|\nabla\xi_n|^2u^2\,dx\\ &\le 4|\nabla\xi|_\infty^2\int_{1/n<|x|<2/n}u^2\,dx\end{aligned}$$

where $\lim_{n\to\infty}\int_{1/n<|x|<2/n}u^2\,dx=0$ since $u\in H_0\subset L^2(\Omega)$, and

$$\int_\Omega |x|^2(1-\xi_n)^2|\nabla u|^2\,dx=\int_{|x|\le 2/n}|x|^2(1-\xi_n)^2|\nabla u|^2\,dx \le \int_{|x|\le 2/n}|x|^2|\nabla u|^2\,dx$$

where $\lim_{n\to\infty}\int_{|x|\le 2/n}|x|^2|\nabla u|^2\,dx=0$ since $\int_\Omega |x|^2|\nabla u|^2\,dx<\infty$. Thus $\|u-\xi_n u\| \to 0$, showing that $W_0^{1,2}(\Omega)$ is dense in H_0.

But $C_0^\infty(\Omega)$ is dense in $W_0^{1,2}(\Omega)$ with its Dirichlet norm

$$\|u\|_{W_0^{1,2}(\Omega)}=\left\{\int_\Omega |\nabla u|^2\,dx\right\}^{1/2}$$

and, since Ω is bounded, there exists a constant D_A such that

$$\int_\Omega |x|^2|\nabla u|^2\,dx \le D_A\int_\Omega |\nabla u|^2\,dx \quad \text{for all } u\in W_0^{1,2}(\Omega).$$

This proves that $C_0^\infty(\Omega)$ is dense in H_0 for the norm $\|\cdot\|$. □

7. APPENDIX 2: WEAKER DEGENERACY

To place our results concerning (1.1), (1.2) in better perspective, we make some remarks about what happens when the assumption (D1) is replaced by

$(D1)_t$ $A \in C(\overline{\Omega})$ with $A(x) > 0$ for all $x \in \overline{\Omega} \setminus \{0\}$ and $\lim_{|x|\to 0} A(x)/|x|^t = 1$ for some $t \in [0, 2]$.

We can still define a Hilbert space $(H_A, \langle\cdot,\cdot\rangle_A)$ by

$$H_A = \left\{ u \in L^2(\Omega) : \int_\Omega A(x)|\nabla u(x)|^2\,dx < \infty \text{ and } \Gamma u = 0 \right\}, \tag{7.1}$$

$$\langle u, v\rangle_A = \int_\Omega A(x)\nabla u(x)\cdot\nabla v(x)\,dx, \tag{7.2}$$

but for $t < 2$, this space has much better properties than in the case $t = 2$ and the boundary-value problem behaves like the uniformly elliptic case $t = 0$ for all $t \in [0, 2)$. In particular, the set of bifurcation points is an increasing sequence $\{\lambda_i : i \in \mathbb{N}\} \subset (0, \infty)$ with $\lim_{i\to\infty} \lambda_i = \infty$.

DEFINITION 7.1. *Under the hypotheses* $(D1)_t$ *and* (D2) *a* solution *of* (1.1), (1.2) *is a pair* $(\lambda, u) \in \mathbb{R} \times H_A$ *such that*

$$\int_\Omega A(x)\nabla u(x)\cdot\nabla\varphi(x)\,dx = \lambda \int_\Omega f(u(x))\varphi(x)\,dx \quad \text{for all } \varphi \in H_A. \tag{7.3}$$

A point $\Lambda \in \mathbb{R}$ *is a* bifurcation point *for* (1.1), (1.2) *if there is a sequence* $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times [H_A \setminus \{0\}]$ *of solutions such that* $\lambda_n \to \Lambda$ *and* $|u_n|_2 \to 0$.

Here are some basic properties of the space H_A.

LEMMA 7.2. *Let the function A satisfy* $(D1)_t$ *for some* $t \in [0, 2]$. *Then*

(i) $(H_A, \langle\cdot,\cdot\rangle_A)$ *is continuously embedded in the space* $(H_0, (\cdot,\cdot))$ *defined by* (1.5), *and hence also in* $L^2(\Omega)$. *Let* $J : H_A \to L^2(\Omega)$ *denote this embedding and* C_A *its norm so that*

$$|u|_2 \le C_A\|u\|_A \quad \text{for all } u \in H_A.$$

(ii) $(H_A, \langle\cdot,\cdot\rangle_A)$ *is continuously embedded in* $W^{1,p}(\Omega)$ *for* $1 \le p < 2N/(N + t)$.

(iii) $(H_A, \langle\cdot,\cdot\rangle_A)$ *is compactly embedded in* $L^q(\Omega)$ *for* $1 \le q < t^* = 2N/(N + t - 2)$.

REMARK. Note that $t^* > 2$ if and only if $t < 2$. For $t = 2$, we have seen that $(H_A, \langle\cdot,\cdot\rangle)$ is continuously embedded in $L^2(\Omega)$, but the embedding is not compact.

PROOF. (i) By $(D1)_t$ and the boundedness of Ω, there exist constants $\beta \ge \alpha > 0$ such that

$$\alpha|x|^t \le A(x) \le \beta|x|^t \quad \text{for all } x \in \Omega. \tag{7.4}$$

The conclusion now follows easily.

(ii) Let $u \in H_A$. By part (i) and Lemma 6.1, u admits generalized derivatives on Ω and $u \in W^{1,1}(\Omega)$. Furthermore, for $1 \le p < 2N/(N+t)$,

$$\int_\Omega |\nabla u|^p \, dx \le \left\{ \int_\Omega |x|^t |\nabla u|^2 \, dx \right\}^{p/2} \left\{ \int_\Omega |x|^{-tp/(2-p)} \, dx \right\}^{(2-p)/2}$$
$$= C_p \left\{ \int_\Omega |x|^t |\nabla u|^2 \, dx \right\}^{p/2}$$

where $C_p < \infty$ since $tp/(2-p) < N$. The conclusion now follows from (7.4).

(iii) This follows from (ii) and the Sobolev embedding of $W^{1,p}(\Omega)$ in $L^q(\Omega)$ (see part 3) of 8.7 in [2], for example). □

Using this lemma, for any $t \in [0, 2]$, we can define a bounded linear operator $L : L^2(\Omega) \to H_A$ by

$$\langle Lu, v \rangle_A = \int_\Omega uv \, dx \quad \text{for all } u \in L^2(\Omega) \text{ and } v \in H_A$$

and we set $K = LJ$. It follows that $K \in B(H_A, H_A)$ and $K : H_A \to H_A$ is compact if $t < 2$. Hence $\sigma_e(K) = \{0\}$ if $t < 2$, whereas for $t = 2$, we have shown that $\sup \sigma_e(K) = 4/N^2$ and consequently K is not compact.

For a function f that satisfies (D1), $f(u) \in L^2(\Omega)$ for all $u \in L^2(\Omega)$ and we define an operator $F : L^2(\Omega) \to L^2(\Omega)$ by

$$F(u) = f(u) \quad \text{for all } u \in L^2(\Omega).$$

Clearly $F : L^2(\Omega) \to L^2(\Omega)$ is Lipschitz continuous and as is well known (see Theorem 2.7 in [3], for example), it is also Gateaux differentiable at every $u \in L^2(\Omega)$. It follows that $F : L^2(\Omega) \to L^2(\Omega)$ is also Hadamard differentiable at every $u \in L^2(\Omega)$. Setting $G = LFJ$, we infer that $G : H_A \to H_A$ is continuous and Gateaux differentiable with

$$\langle G(u), v \rangle_A = \int_\Omega f(u) v \, dx \qquad \text{for all } u, v \in H_A, \text{ and}$$
$$\langle G'(u)w, v \rangle_A = \int_\Omega f'(u) w v \, dx \quad \text{for all } w \in H_A,$$

for all $t \in [0, 2]$. For $t \in [0, 2)$ and $q \in (2, t^*)$, there is a constant D_q such that

$$|\langle [G'(u) - G'(z)]w, v \rangle_A| \le \int_\Omega |f'(u) - f'(z)| \, |w| \, |v| \, dx \le |f'(u) - f'(z)|_p |w|_q |v|_q$$
$$\le |f'(u) - f'(z)|_p D_q^2 \|w\|_A \|v\|_A$$

where $p = q/(q-2)$. It follows that

$$\|G'(u) - G'(z)\|_{B(H_A, H_A)} \le D_q^2 |f'(u) - f'(z)|_p \quad \text{for all } u, z \in H_A.$$

Since $|f'(s)| \le M$ for all $s \in \mathbb{R}$, the mapping $u \mapsto f'(u)$ is continuous from $L^2(\Omega)$ into $L^p(\Omega)$ (see Theorem 2.2 in [3], for example) and so a fortiori from H_A into $L^p(\Omega)$. This proves that $G' : H_A \to B(H_A, H_A)$ is continuous and it follows that $G : H_A \to H_A$ is continuously differentiable in the sense of Fréchet for $t \in [0, 2)$.

By Proposition 4.1, $\psi : H_0 \to \mathbb{R}$ is continuously differentiable and

$$\psi'(u)v = \int_\Omega f(u)v\,dx \quad \text{for all } u, v \in H_0.$$

But for $0 \leq t \leq 2$, H_A is continuously embedded in H_0 and so $\psi \in C^1(H_A, \mathbb{R})$ with

$$\psi'(u)v = \langle G(u), v\rangle_A \quad \text{for all } u, v \in H_A.$$

Thus, in fact, $\psi \in C^2(H_A, \mathbb{R})$ when $t \in [0, 2)$.

From the compactness of the injection $J : H_A \to L^2(\Omega)$ when $t < 2$, it follows that $G = LFJ : H_A \to H_A$ is also compact when $t \in [0, 2)$.

Recalling from Section 2 that B_G denotes the set of all bifurcation points for the equation (2.1), standard results from abstract bifurcation theory (for example, Theorems 4.1 and 7.1 in [22]) show that for $t \in [0, 2)$, $B_G = \sigma(G'(0))$ where $G'(0) = K : H_A \to H_A$ is a compact, self-adjoint operator with $\langle Ku, u\rangle_A > 0$ for all $u \in H_A \setminus \{0\}$. Thus for $t \in [0, 2)$, $\sigma(G'(0)) = \{\mu_i : i \in \mathbb{N}\} \cup \{0\}$ where μ_i is an eigenvalue of finite multiplicity of K, $0 < \mu_{i+1} < \mu_i$ for all i and $\lim_{i\to\infty} \mu_i = 0$. Setting $\lambda_i = 1/\mu_i$, we obtain the following result.

THEOREM 7.3. *Suppose that the conditions* (D1)$_t$ *and* (D2) *are satisfied for some* $t \in [0, 2)$ *and let* B *denote the set of all bifurcation points for the problem* (1.1), (1.2). *Then* $B = \{\lambda_i : i \in \mathbb{N}\}$ *where* $0 < \lambda_i < \lambda_{i+1}$ *with* $\lim_{i\to\infty} \lambda_i = \infty$.

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