Relativistic theories. — *Positivity of the Bondi mass in Bondi's radiating spacetimes*, by
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ABSTRACT. — We introduce two conditions related to the *news functions* of Bondi's radiating vacuum spacetimes. We provide a complete proof of the positivity of the Bondi mass by using Schoen–Yau's method under one condition and by using Witten's method under the other condition.

1. INTRODUCTION

Gravitational waves are predicted by Einstein's general relativity. They are time dependent solutions of the Einstein field equations which radiate or transport energy. Although they have not been detected yet, the existence of gravitational waves has been proved indirectly from observations of the pulsar PSR 1913+16. This rapidly rotating binary system should emit gravitational radiation, hence lose energy and rotate faster. The observed relative change in period agrees remarkably with the theoretical value.

A fundamental conjecture is that gravitational waves cannot carry away more energy than they have initially in an isolated gravitational system. It is usually referred as the positive mass conjecture at null infinity. In Bondi's radiating vacuum spacetime, this conjecture is equivalent to the positivity of the Bondi mass. In the pioneering work of Bondi, van der Burg, Metzner and Sachs on the gravitational waves in vacuum spacetimes, the Bondi mass associated to each null cone is defined and their main result asserts that this Bondi mass is always nonincreasing with respect to the retarded time [2, 15, 18]. Therefore, the Bondi mass can be interpreted as the total mass of the isolated physical system measured after the loss due to the gravitational radiation up to that time. The proof of the positivity of the Bondi mass was outlined by Schoen–Yau modifying their arguments in the proof of the positivity of the ADM mass [17]. It was also outlined by physicists applying Witten's spinor method (see, e.g., [11, 9, 1, 13, 14, 10]). The main goal of this paper is to find a complete proof of the positivity. Indeed, we find certain conditions related to the *news functions* of the system. The Bondi mass is nonnegative under these conditions.

It is an open problem whether vacuum Einstein field equations always develop logarithmic singularities at null infinity. In [7], the authors studied polyhomogeneous Bondi expansions. The u-evolution equations actually indicate that logarithmic singularities at null infinity can be removed in the axisymmetric case (Appendix D of [7]) if the free function $\gamma_2(u, x^\alpha)$ is chosen to be zero and $\gamma_{3,1}(u_0, x^\alpha)$ is chosen to be zero for some u_0. It

is quite possible that Bondi's radiating vacuum spacetime does not develop any logarithmic singularity at null infinity after suitable "gauge fixing". Therefore we do not consider polyhomogeneous Bondi expansions [3] in the present paper.

The paper is organized as follows. In Section 2, we state some well-known formulas and results of Bondi, van der Burg, Metzner and Sachs. We employ two fundamental assumptions: *Condition A* and *Condition B*. We also derive a generalized Bondi mass loss formula under these two conditions. In Section 3, we study some basic geometry of asymptotically null spacelike hypersurfaces. We compute the asymptotic behaviors of the induced metric and the second fundamental form of an asymptotically null spacelike hypersurface given by a certain graph. In Section 4, we use Schoen–Yau's method to prove that, under Conditions A and B, if there is a retarded time u_0 such that $\mathcal{M}(u_0, \theta, \psi)$ defined in Section 2 is constant, then the Bondi mass is nonnegative in the region $\{u \leq u_0\}$, and the Bondi mass is zero at $u \in (-\infty, u_0]$ if and only if the spacetime is flat in a neighborhood of a spacelike hypersurface $\{u = u_0 + \sqrt{1 + r^2} - r\}$. In Section 5, we use Witten's method to prove that, under Conditions A and B, if there is a retarded time u_0 such that $c(u_0, \theta, \psi) = d(u_0, \theta, \psi) = 0$, then the Bondi mass is nonnegative in the region $\{u \leq u_0\}$, and the Bondi mass is zero at $u \in (-\infty, u_0]$ if and only if the spacetime is flat in a neighborhood of a spacelike hypersurface $\{u = u_0 + \sqrt{1 + r^2} - r\}$. In Section 6, we modify the definition of the Bondi energy-momentum and prove its positivity without Condition B.

2. BONDI'S RADIATING SPACETIMES

We assume that $(\mathbb{L}^{3,1}, \tilde{g})$ is a vacuum spacetime (possibly with black holes) and $\tilde{g}$ is the following Bondi's radiating metric:

$$
\begin{aligned}
(2.1) \quad \tilde{g} = \Bigg(& \frac{V}{r} e^{2\beta} + r^2 e^{2\gamma} U^2 \cosh 2\delta + r^2 e^{-2\gamma} W^2 \cosh 2\delta \\
& + 2r^2 U W \sinh 2\delta \Bigg) du^2 - 2e^{2\beta} du\,dr \\
& - 2r^2 (e^{2\gamma} U \cosh 2\delta + W \sinh 2\delta) du\,d\theta \\
& - 2r^2 (e^{-2\gamma} W \cosh 2\delta + U \sinh 2\delta) \sin\theta\, du\,d\psi \\
& + r^2 (e^{2\gamma} \cosh 2\delta\, d\theta^2 + e^{-2\gamma} \cosh 2\delta \sin^2\theta\, d\psi^2 + 2\sinh 2\delta \sin\theta\, d\theta\, d\psi)
\end{aligned}
$$

in coordinates (u, r, θ, ψ) (u is retarded time) where

$$-\infty < u < \infty, \quad r > 0, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \psi \leq 2\pi.$$

Write

$$x^0 = u, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \psi.$$

We suppose that β, γ, δ, U, V and W are smooth functions of u, r, θ, ψ. We write $f_{,v} = \partial f / \partial x^v$ for $v = 0, 1, 2, 3$ throughout the paper. The metric (2.1) was studied by Bondi, van der Burg, Metzner and Sachs in the theory of gravitational waves in general relativity [2,

15, 18]. They proved that the following asymptotic behavior holds for r sufficiently large if the spacetime satisfies the outgoing radiation condition [18]:

$$\gamma = \frac{c(u,\theta,\psi)}{r} + \frac{C(u,\theta,\psi) - \frac{1}{6}c^3 - \frac{3}{2}cd^2}{r^3} + O\left(\frac{1}{r^4}\right),$$

$$\delta = \frac{d(u,\theta,\psi)}{r} + \frac{H(u,\theta,\psi) + \frac{1}{2}c^2 d - \frac{1}{6}d^3}{r^3} + O\left(\frac{1}{r^4}\right),$$

$$\beta = -\frac{c^2 + d^2}{4r^2} + O\left(\frac{1}{r^4}\right),$$

$$U = -\frac{l(u,\theta,\psi)}{r^2} + \frac{p(u,\theta,\psi)}{r^3} + O\left(\frac{1}{r^4}\right),$$

$$W = -\frac{\bar{l}(u,\theta,\psi)}{r^2} + \frac{\bar{p}(u,\theta,\psi)}{r^3} + O\left(\frac{1}{r^4}\right),$$

$$V = -r + 2M(u,\theta,\psi) + \frac{\bar{M}(u,\theta,\psi)}{r} + O\left(\frac{1}{r^2}\right),$$

where

$$l = c_{,2} + 2c\cot\theta + d_{,3}\csc\theta,$$
$$\bar{l} = d_{,2} + 2d\cot\theta - c_{,3}\csc\theta,$$
$$p = 2N + 3(cc_{,2} + dd_{,2}) + 4(c^2 + d^2)\cot\theta - 2(c_{,3}d - cd_{,3})\csc\theta,$$
$$\bar{p} = 2P + 2(c_{,2}d - cd_{,2}) + 3(cc_{,3} + dd_{,3})\csc\theta,$$
$$\bar{M} = N_{,2} + \cot\theta + P_{,3}\csc\theta - \frac{c^2 + d^2}{2} - [(c_{,2})^2 + (d_{,2})^2] - 4(cc_{,2} + dd_{,2})\cot\theta$$
$$\quad - 4(c^2 + d^2)\cot^2\theta - [(c_{,3})^2 + (d_{,3})^2]\csc^2\theta$$
$$\quad + 4(c_{,3}d - cd_{,3})\csc\theta\cot\theta + 2(c_{,3}d_{,2} - c_{,2}d_{,3})\csc\theta.$$

Here M is referred to as the *mass aspect* and $c_{,0}$, $d_{,0}$ are called the *news functions*. The u-derivatives of certain functions are

$$C_{,0} = \frac{c^2 c_{,0}}{2} + cdd_{,0} - \frac{c_{,0}d^2}{2} + \frac{cM}{2} + \frac{d\lambda}{4} - \frac{N_{,2} - N\cot\theta - P_{,3}\csc\theta}{4},$$

$$H_{,0} = -\frac{c^2 d_{,0}}{2} + cc_{,0}d + \frac{d_{,0}d^2}{2} + \frac{dM}{2} - \frac{c\lambda}{4} - \frac{P_{,2} - P\cot\theta + N_{,3}\csc\theta}{4},$$

$$M_{,0} = -[(c_{,0})^2 + (d_{,0})^2] + \frac{1}{2}(l_{,2} + l\cot\theta + \bar{l}_{,3}\csc\theta)_{,0},$$

$$3N_{,0} = -M_{,2} - \frac{\lambda_{,3}\csc\theta}{2} - (c_{,0}c_{,2} + d_{,0}d_{,2}) - 3(cc_{,02} + dd_{,02}) - 4(cc_{,0} + dd_{,0})\cot\theta$$
$$\quad + (c_{,0}d_{,3} - c_{,3}d_{,0} + 3c_{,03}d - 3cd_{,03})\csc\theta,$$

$$3P_{,0} = -M_{,3}\csc\theta + \frac{\lambda_{,2}}{2} + (c_{,2}d_{,0} - c_{,0}d_{,2}) + 3(cd_{,02} - c_{,02}d) + 4(cd_{,0} - c_{,0}d)\cot\theta$$
$$\quad - (c_{,0}c_{,3} + d_{,0}d_{,3} + 3cc_{,03} + 3dd_{,03})\csc\theta$$

where

$$\lambda = \bar{l}_{,2} + \bar{l} \cot \theta - l_{,3} \csc \theta.$$

Define

(2.2) $\mathcal{M}(u, \theta, \psi) = M(u, \theta, \psi) - \frac{1}{2}(l_{,2} + l \cot \theta + \bar{l}_{,3} \csc \theta)$

$= M(u, \theta, \psi) - \frac{1}{2}[-2c(u, \theta, \psi) + c_{,22}(u, \theta, \psi)$

$- \csc^2 \theta c_{,33}(u, \theta, \psi) + 2 \csc \theta d_{,23}(u, \theta, \psi)$

$+ 3 \cot \theta c_{,2}(u, \theta, \psi) + 2 \cot \theta \csc \theta d_{,3}(u, \theta, \psi)].$

Its u-derivative is

(2.3) $$\mathcal{M}_{,0} = -[(c_{,0})^2 + (d_{,0})^2].$$

There are some physical conditions [2, 15, 18] ensuring the regularity of (2.1). In this paper, however, we assume

CONDITION A.   Each of the six functions β, γ, δ, U, V, W together with its derivatives up to the second order has equal values at $\psi = 0$ and 2π.

CONDITION B.   For all u,

$$\int_0^{2\pi} c(u, 0, \psi) \, d\psi = 0, \qquad \int_0^{2\pi} c(u, \pi, \psi) \, d\psi = 0.$$

Let $\mathbb{N}_{u_0}$ be a null hypersurface which is given by $u = u_0$. The Bondi energy-momentum of $\mathbb{N}_{u_0}$ is defined by (see [2, 5])

(2.4) $$m_\nu(u_0) = \frac{1}{4\pi} \int_{S^2} M(u_0, \theta, \psi) n^\nu \, dS$$

where $\nu = 0, 1, 2, 3$, $n^0 = 1$, and n^i is the restriction of the natural coordinate x^i to the unit sphere, i.e.,

$$n^0 = 1, \quad n^1 = \sin \theta \cos \psi, \quad n^2 = \sin \theta \sin \psi, \quad n^3 = \cos \theta.$$

Under Conditions A and B, we have (see [2, 15, 22])

(2.5) $$\frac{d}{du} m_\nu = -\frac{1}{4\pi} \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] n^\nu \, dS$$

for $\nu = 0, 1, 2, 3$. When $\nu = 0$, this is the famous Bondi mass loss formula.

The following proposition can be viewed as a generalized Bondi mass loss formula. It does not seem to appear in the literature.

PROPOSITION 2.1.   *Let* $(\mathbb{L}^{3,1}, \tilde{g})$ *be a vacuum Bondi's radiating spacetime with metric* $\tilde{g}$ *given by* (2.1). *Suppose that Conditions A and Condition B hold. Then*

(2.6) $$\frac{d}{du} \left(m_0 - \sqrt{\sum_{1 \leq i \leq 3} m_i^2} \right) \leq 0.$$

PROOF. Define $|m| = \sqrt{m_1^2 + m_2^2 + m_3^2}$. We assume $|m| \neq 0$, as otherwise the proof reduces to the Bondi mass loss formula. We have

$$\frac{d}{du}(m_0 - |m|) = \frac{dm_0}{du} - \frac{1}{|m|} \sum_{1 \leq i \leq 3} \frac{dm_i}{du} m_i$$

$$= -\frac{1}{4\pi} \left\{ \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] \, dS \right.$$

$$\left. - \frac{1}{|m|} \sum_{1 \leq i \leq 3} m_i \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] n^i \, dS \right\}.$$

Thus $\frac{d}{du}(m_0 - |m|) \leq 0$ is equivalent to

$$\sum_{1 \leq i \leq 3} m_i \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] n^i \, dS \leq |m| \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] \, dS.$$

Using $(n^1)^2 + (n^2)^2 + (n^3)^2 = 1$, we obtain

$$(2.7) \quad \sum_{1 \leq i \leq 3} \left\{ \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] n^i \, dS \right\}^2$$

$$\leq \sum_{1 \leq i \leq 3} \left\{ \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] \, dS \right\} \left\{ \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] (n^i)^2 \, dS \right\}$$

$$= \left\{ \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] \, dS \right\}^2.$$

Then by the Cauchy–Schwarz inequality,

$$\sum_{1 \leq i \leq 3} m_i \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] n^i \, dS \leq |m| \sqrt{\sum_{1 \leq i \leq 3} \left\{ \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] n^i \, dS \right\}^2}$$

$$\leq |m| \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] \, dS.$$

Therefore (2.6) holds. $\square$

3. ASYMPTOTICALLY NULL SPACELIKE HYPERSURFACES

The hypersurface $u = \sqrt{1 + r^2} - r$ in the Minkowski spacetime is a hyperbola equipped with the standard hyperbolic 3-metric $\breve{g}$. Let $\{\breve{e}_i\}$ be the frame

$$\breve{e}_1 = \sqrt{1 + r^2} \frac{\partial}{\partial r}, \quad \breve{e}_2 = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad \breve{e}_3 = \frac{1}{r \sin \theta} \frac{\partial}{\partial \psi}.$$

Let $\{\breve{e}^i\}$ be the coframe. Write $\breve{\nabla}_i = \breve{\nabla}_{\breve{e}_i}$, etc., where $\breve{\nabla}$ is the Levi-Civita connection of $\breve{g}$. The connection 1-forms $\{\breve{\omega}_{ij}\}$ are given by $d\breve{e}^i = -\breve{\omega}_{ij} \wedge \breve{e}^j$ or $\breve{\nabla}\breve{e}_i = -\breve{\omega}_{ij} \otimes \breve{e}_j$. They

are

$$\breve{\omega}_{12} = -\frac{\sqrt{1+r^2}}{r}\breve{e}^2, \quad \breve{\omega}_{13} = -\frac{\sqrt{1+r^2}}{r}\breve{e}^3, \quad \breve{\omega}_{23} = -\frac{\cot\theta}{r}\breve{e}^3.$$

Let $\mathbb{X}$ be a spacelike hypersurface in a vacuum Bondi's radiating spacetime $(\mathbb{L}^{3,1}, \tilde{g})$ with metric (2.1), which is given by the inclusion

$$i : \mathbb{X}^3 \to \mathbb{L}^{3,1}, \quad (y^1, y^2, y^3) \mapsto (x^0, x^1, x^2, x^3),$$

for r sufficiently large, where

$$x^0 = u(y^1, y^2, y^3), \quad x^1 = y^1 = r, \quad x^2 = y^2 = \theta, \quad x^3 = y^3 = \psi.$$

Let $g = i^*\tilde{g}$ be the induced metric of $\mathbb{X}$ and h be the second fundamental form of $\mathbb{X}$. Let $\tilde{\nabla}$ be the Levi-Civita connection of $\mathbb{L}^{3,1}$. For any tangent vectors $Y_i, Y_j \in T\mathbb{X}$, i_*Y_i, i_*Y_j are tangent vectors along $\mathbb{X}$, and

$$g(Y_i, Y_j) = \tilde{g}(i_*Y_i, i_*Y_j).$$

Let e_n be the downward unit normal of $\mathbb{X}$. The second fundamental form is defined as

$$h(Y_i, Y_j) = \tilde{g}(\tilde{\nabla}_{i_*Y_i} i_*Y_j, e_n).$$

Now it is a straightforward computation that

$$i_* \frac{\partial}{\partial y^i} = \frac{\partial}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y^i} = \frac{\partial}{\partial x^0} \frac{\partial x^0}{\partial y^i} + \frac{\partial}{\partial x^i}.$$

Set $e_i = i_*\breve{e}_i$. Then

$$(3.1) \qquad e_1 = \sqrt{1+r^2}u_{,1}\partial_0 + \breve{e}_1, \quad e_2 = \frac{u_{,2}}{r}\partial_0 + \breve{e}_2, \quad e_3 = \frac{u_{,3}}{r\sin\theta}\partial_0 + \breve{e}_3.$$

DEFINITION 3.1. *A spacelike hypersurface* $(\mathbb{X}, g, h)$ *in an asymptotically flat spacetime is asymptotically null of order* $\tau > 0$ *if, for* r *sufficiently large,* $g(\breve{e}_i, \breve{e}_j) = \delta_{ij} + a_{ij}$, $h(\breve{e}_i, \breve{e}_j) = \delta_{ij} + b_{ij}$, *where* a_{ij}, b_{ij} *satisfy*

$$(3.2) \qquad \{a_{ij}, \breve{\nabla}_k a_{ij}, \breve{\nabla}_l \breve{\nabla}_k a_{ij}, b_{ij}, \breve{\nabla}_k b_{ij}\} = O(1/r^\tau).$$

Let $(\mathbb{X}, g, h)$ be an asymptotically null spacelike hypersurface with the induced metric g and the second fundamental form h in a vacuum Bondi's radiating spacetime $(\mathbb{L}^{3,1}, \tilde{g})$, which is given by

$$(3.3) \qquad u = \sqrt{1+r^2} - r + \frac{(c^2+d^2)_{u=0}}{12r^3} + \frac{a_3(\theta, \psi)}{r^4} + a_4$$

where $a_4(r, \theta, \psi)$ is a smooth function such that in the Euclidean coordinate systems $\{\breve{z}^i\}$, $|\breve{z}| = r$,

$$a_4 = o(1/r^4), \quad \partial_k a_4 = o(1/r^5), \quad \partial_k \partial_l a_4 = o(1/r^6)$$

as $r \to \infty$. We will compute asymptotic behaviors of the induced metric and the second fundamental form of $\mathbb{X}$. The induced metric can be obtained by substituting du into (2.1).

Let X_n be the downward normal vector

$$X_n = -\frac{\partial}{\partial x^0} - \varrho^i \frac{\partial}{\partial x^i}.$$

Let e_i be given by (3.1). Since X_n is orthogonal to e_i, we obtain

$$\tilde{g}(e_i, X_n) = 0.$$

This implies that ϱ^i satisfies the following equations:

$$(u_{,i}\tilde{g}_{00} + \tilde{g}_{0i}) + \varrho^j (u_{,i}\tilde{g}_{0j} + \tilde{g}_{ij}) = 0$$

for $i = 1, 2, 3$. Therefore ϱ^i can be found by solving linear algebraic equations and the unit normal vector is

$$e_n = \frac{X_n}{\sqrt{-\tilde{g}(X_n, X_n)}}.$$

The second fundamental form is then given by

$$h(\breve{e}_i, \breve{e}_j) = \tilde{g}(\tilde{\nabla}_{e_i} e_j, e_n)$$

for $1 \leq i, j \leq 3$. Now we define $a \approx b$ if and only if $a = b + o(1/r^3)$. For r sufficiently large, we expand c, d and M at $u = 0$ in Taylor series:

$$(3.4) \qquad c(u, \theta, \psi) \approx c(0, \theta, \psi) + c_{,0}(0, \theta, \psi)u + \frac{c_{,00}(0, \theta, \psi)}{2}u^2,$$

$$(3.5) \qquad d(u, \theta, \psi) \approx d(0, \theta, \psi) + d_{,0}(0, \theta, \psi)u + \frac{d_{,00}(0, \theta, \psi)}{2}u^2,$$

$$(3.6) \qquad M(u, \theta, \psi) \approx M(0, \theta, \psi) + M_{,0}(0, \theta, \psi)u + \frac{M_{,00}(0, \theta, \psi)}{2}u^2.$$

With the help of Mathematica 5.0, we obtain the asymptotic behaviors of the metric g:

$$g(\breve{e}_1, \breve{e}_1) \approx 1 + \frac{16a_3 + M - cc_{,0} - dd_{,0}}{2r^3},$$

$$g(\breve{e}_1, \breve{e}_2) \approx -\frac{l}{2r^2} + \frac{12N - 3l_{,0} + 4(cc_{,2} + dd_{,2})}{12r^3},$$

$$g(\breve{e}_1, \breve{e}_3) \approx -\frac{\bar{l}}{2r^2} + \frac{12P - 3\bar{l}_{,0} + 4\csc\theta(cc_{,3} + dd_{,3})}{12r^3},$$

$$g(\breve{e}_2, \breve{e}_2) \approx 1 + \frac{2c}{r} + \frac{2(c^2 + d^2) + c_{,0}}{r^2} + \frac{c^3 + cd^2 + 2C + 2(cc_{,0} + dd_{,0}) + c_{,00}/4}{r^3},$$

$$g(\breve{e}_2, \breve{e}_3) \approx \frac{2d}{r} + \frac{d_{,0}}{r^2} + \frac{c^2d + d^3 + 2H + d_{,00}/4}{r^3},$$

$$g(\breve{e}_3, \breve{e}_3) \approx 1 - \frac{2c}{r} + \frac{2(c^2 + d^2) - c_{,0}}{r^2} + \frac{-c^3 - cd^2 - 2C + 2(cc_{,0} + dd_{,0}) - c_{,00}/4}{r^3},$$

$$h(\breve{e}_1, \breve{e}_1) \approx 1 + \frac{c^2 + d^2}{r^2} + \frac{16a_3 - M}{r^3},$$

$$h(\check{e}_1, \check{e}_2) \approx \frac{l}{2r^2} + \frac{1}{2r^3}\left[\frac{l_{,0}}{2} - 2(c^2 + d^2)\cot\theta - 4N\right.$$

$$\left. + (-cd_{,3} + c_{,3}d)\csc\theta - \frac{13}{3}(cc_{,2} + dd_{,2})\right],$$

$$h(\check{e}_1, \check{e}_3) \approx \frac{\bar{l}}{2r^2} + \frac{1}{2r^3}\left[\frac{\bar{l}_{,0}}{2} + cd_{,2} - c_{,2}d - 4P - \frac{13}{3}(cc_{,3} + dd_{,3})\csc\theta\right],$$

$$h(\check{e}_2, \check{e}_2) \approx 1 + \frac{c}{r} + \frac{c_{,0}}{r^2} + \frac{1}{4r^3}\left[3M - 16a_3 - 4C - 2l_{,2}\right.$$

$$\left. - 2c(c^2 + d^2) + 5(cc_{,0} + dd_{,0}) + \frac{3}{2}c_{,00}\right],$$

$$h(\check{e}_2, \check{e}_3) \approx \frac{d}{r} + \frac{d_{,0}}{r^2} + \frac{1}{4r^3}\left[-2d(c^2 + d^2) + 2d\cot^2\theta\right.$$

$$\left. + 2d\csc^2\theta - 4c_{,3}\cot\theta\csc\theta - d_{,33}\csc^2\theta - d_{,2}\cot\theta - d_{,22} - 4H + \frac{3}{2}d_{,00}\right],$$

$$h(\check{e}_3, \check{e}_3) \approx 1 - \frac{c}{r} - \frac{c_{,0}}{r^2} + \frac{1}{4r^3}\left[3M - 16a_3 + 4C\right.$$

$$\left. + 2c(c^2 + d^2) + 5(cc_{,0} + dd_{,0}) - \frac{3}{2}c_{,00} - 2l\cot\theta - 2\bar{l}_{,3}\csc\theta\right].$$

Here all functions on the right hand sides are evaluated at $u = 0$ and all derivatives with respect to x^2 and x^3 are taken after substituting $u = 0$. Therefore $(\mathbb{X}, g, h)$ is asymptotically null of order 1.

4. POSITIVITY: SCHOEN–YAU'S METHOD

In this section, we will complete the argument in [17]. Denote by $(\mathbb{X}, g, h)$ the asymptotically null spacelike hypersurface which is given by (3.3) for r sufficiently large. In [17], Schoen–Yau solved the following Jang's equation on $\mathbb{X}$:

$$(4.1) \qquad \left(g^{ij} - \frac{f^i f^j}{1 + |\nabla f|^2}\right)\left(\frac{f_{,ij}}{\sqrt{1 + |\nabla f|^2}} - h_{ij}\right) = 0$$

under the suitable boundary condition

$$(4.2) \qquad\qquad\qquad\qquad f \to f_0$$

as $r \to \infty$ such that the metric

$$(4.3) \qquad\qquad\qquad\qquad \bar{g} = g + \nabla f \otimes \nabla f$$

is asymptotically flat. Denote by $J(f)$ the left hand side of Jang's equation (4.1). Note that in the standard hyperbolic 3-space, (4.1) has a solution $f = \sqrt{1 + r^2}$. Therefore it is reasonable to set

$$f_0 = \sqrt{1 + r^2} + o(r).$$

Let f be a function on $\mathbb{X}$ which has asymptotic expansion

(4.4) $$f = \sqrt{1 + r^2} + p(\theta, \psi) \ln r + q(r, \theta, \psi),$$

for r sufficiently large, where $p(\theta, \psi)$ is a smooth function on S^2 and q is a smooth function on $\mathbb{R}^3$ which satisfies the following asymptotic conditions: In the Euclidean coordinate systems $\{\breve{z}^i\}$, $|\breve{z}| = r$,

$$q = o(1), \quad \partial_k q = o(1/r), \quad \partial_k \partial_l q = o(1/r^2), \quad \partial_k \partial_l \partial_j q = o(1/r^3)$$

as $r \to \infty$.

Let the standard metric of S^2 be $d\theta^2 + \sin^2 \theta d\psi^2$. The Laplacian operator for this metric is

$$\triangle_{S^2} = \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \csc^2 \theta \frac{\partial^2}{\partial \psi^2}.$$

The *spherical harmonics* w_j are the eigenfunctions of $\triangle_{S^2}$, i.e., $\triangle_{S^2} w_j = j(j-1)w_j$ for $j = 1, 2, \ldots$.

PROPOSITION 4.1. *If Jang's equation* (4.1) *has a solution f which has the asymptotic expansion* (4.4) *for r sufficiently large, then $p(\theta, \psi)$ and $\mathcal{M}(0, \theta, \psi)$ must be constant.*

PROOF. A lengthy computation with the help of Mathematica 5.0 shows that

$$J(f) \approx \frac{\ln r}{r^3} \triangle_{S^2} p + \frac{p - 2\mathcal{M}(0, \theta, \psi)}{r^3}$$

for r sufficiently large. That $J(f) = 0$ implies

$$\triangle_{S^2} p = 0, \quad p - 2\mathcal{M} = 0.$$

As there is no nonconstant harmonic function on S^2, the proposition follows. $\square$

The existence for (4.1) under the boundary condition (4.2) with

(4.5) $$f_0(r) = \sqrt{1 + r^2} + p \ln r$$

for a certain constant p can be established as follows: We extend f_0 to the whole $\mathbb{X}$ and denote as f_0 also. Denote by B_R the ball of radius R in $\mathbb{R}^3$. If $\mathbb{X}$ has no apparent horizon, the existence theorem for the Dirichlet problem [20] indicates that there exists a (smooth) solution $\bar{f}_R$ of (4.1) on B_R such that

$$\bar{f}_R|_{\partial B_R} = 0$$

for sufficiently large R. By the translation invariance of (4.1) in the vertical direction, we find that

$$f_R = \bar{f}_R + f_0(R)$$

is a solution of (4.1) which is $f_0(R)$ on ∂B_R. Now the estimates in [16] show that

$$f_R \to f$$

on any compact subset of $\mathbb{X}$, where f is a (smooth) solution of (4.1). Write $f = f_0 + f_1$ where $\lim_{r \to \infty} f_1 = 0$. Substitute it into Jang's equation (4.1) and obtain an equation for f_1. Then using a similar argument to the proof of Proposition 3 in [16], we can show that for any $\varepsilon \in (0, 1)$, there is a constant $C(\varepsilon)$ depending only on ε and the geometry of $\mathbb{X}$ such that

$$|f_1(\check{z})| + |\check{z}| \, |\partial f_1(\check{z})| + |\check{z}|^2 |\partial \partial f_1(\check{z})| + |\check{z}|^3 |\partial \partial \partial f_1(\check{z})| \leq C(\varepsilon)|\check{z}|^\varepsilon.$$

Therefore f has asymptotic behaviors (4.4), (4.5) for r sufficiently large.

By adding one-point compactification, the existence for (4.1) can be extended to $\mathbb{X}$ with apparent horizons. See [16] for details.

The following lemma was proved in [22].

LEMMA 4.1. *Let $(\mathbb{L}^{3,1}, \tilde{g})$ be a vacuum Bondi's radiating spacetime with metric $\tilde{g}$ given by (2.1). Suppose that Conditions A and B hold. Then*

$$\int_{S^2} (l_{,2} + l \cot \theta + \bar{l}_{,3} \csc \theta) n^\nu \, dS = 0$$

for $\nu = 0, 1, 2, 3$.

THEOREM 4.1. *Let $(\mathbb{L}^{3,1}, \tilde{g})$ be a vacuum Bondi's radiating spacetime with metric $\tilde{g}$ given by (2.1). Suppose that Conditions A and B hold. If there exists a constant u_0 such that $\mathcal{M}(u_0, \theta, \psi)$ is constant, then*

$$m_0(u) \geq \sqrt{\sum_{1 \leq i \leq 3} m_i^2(u)}$$

for all $u \leq u_0$. If equality holds for some $u \in (-\infty, u_0]$, then $\mathbb{L}^{3,1}$ is flat in the region foliated by all spacelike hypersurfaces which are given by

$$u = u_0 + \sqrt{1 + r^2} - r + o(1/r^4)$$

for r sufficiently large. In particular, if equality holds for all $u \leq u_0$, then $\mathbb{L}^{3,1}$ is flat in the region $\{u \leq u_0\}$.

PROOF. Suppose $\mathcal{M}(u_0, \theta, \psi) = p/2$. By the translation invariance of Jang's equation, we can assume that $u_0 = 0$. The assumption of the theorem ensures that there exists a smooth solution f of Jang's equation (4.1) under the boundary condition (4.2) with f_0 given by (4.5). It is obvious that the metric $\bar{g}$ given by (4.3) is asymptotic flat. Now we show its ADM total energy is p. Denote by g_0 the flat metric of $\mathbb{R}^3$ in polar coordinates. Let $\{e_i^0\}$ be the frame of g_0,

$$e_1^0 = \frac{\partial}{\partial r}, \quad e_2^0 = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad e_3^0 = \frac{1}{r \sin \theta} \frac{\partial}{\partial \psi}.$$

Let $\{e_0^i\}$ be the coframe of g_0. Define $\alpha_{ij} = \bar{g}(e_i^0, e_j^0) - \delta_{ij}$. Now we use the ADM energy expression in polar coordinates

$$E(\bar{g}) = \frac{1}{16\pi} \lim_{r \to \infty} \int_{S_r} [(\nabla^0)^j \alpha_{1j} - (\nabla^0)_1 \operatorname{tr}_{g_0}(\alpha)] e_0^2 \wedge e_0^3$$

where ∇^0 is the Levi-Civita connection of g_0. Since

$$(\nabla^0)^j \alpha_{1j} - (\nabla^0)_1 \operatorname{tr}_{g_0}(\alpha) = \frac{\ln r}{r^2} \triangle_{S^2} p + \frac{4p}{r^2} + o\left(\frac{1}{r^2}\right),$$

we obtain

$$E(\bar{g}) = p.$$

Since it satisfies vacuum Einstein field equations, the Bondi's radiating metric satisfies the dominant energy condition automatically. Therefore the scalar curvature $\bar{R}$ of $\bar{g}$ satisfies

$$\bar{R} \geq 2|Y|_{\bar{g}}^2 - 2\operatorname{div}_{\bar{g}} Y$$

for a certain vector field Y in $\bar{\mathbb{X}}$. Therefore a standard positive mass argument [16, 12] shows that

$$E(\bar{g}) = p \geq 0.$$

And $p = 0$ if and only if the metric $\bar{g}$ is flat, which implies that $(\mathbb{X}, g, h)$ can be embedded into the Minkowski spacetime as a spacelike hypersurface with the metric g induced from the Minkowski metric and the second fundamental form h.

Integrating $\mathcal{M}(0, \theta, \psi) = p/2$ over unit S^2 and using Lemma 4.1, we obtain the Bondi energy-momentum of slice $u = 0$,

$$m_0(0) = p/2, \quad m_1(0) = m_2(0) = m_3(0) = 0.$$

Thus the theorem follows from Proposition 2.1. $\qquad \square$

5. POSITIVITY: WITTEN'S METHOD

In this section, we will use Witten's [19] method and the positive mass theorem near null infinity proved by the third author [21, 23] to study the positivity of the Bondi mass. Let $(\mathbb{X}, g, h)$ be an asymptotically null spacelike hypersurface. Define

$$E_\nu(\mathbb{X}) = \frac{1}{16\pi} \lim_{r \to \infty} \int_{S_r} \mathcal{E} n^\nu r \, \check{e}^2 \wedge \check{e}^3, \quad P_\nu(\mathbb{X}) = \frac{1}{8\pi} \lim_{r \to \infty} \int_{S_r} \mathcal{P} n^\nu r \, \check{e}^2 \wedge \check{e}^3,$$

where

$$\mathcal{E} = \check{\nabla}^j a_{1j} - \check{\nabla}_1 \operatorname{tr}_{\check{g}}(a) - [a_{11} - \delta_{11} \operatorname{tr}_{\check{g}}(a)], \quad \mathcal{P} = b_{11} - \delta_{11} \operatorname{tr}_{\check{g}}(b).$$

Theorem 4.1 in [21] indicates if $(\mathbb{X}, g, h)$ is an asymptotically null spacelike hypersurface of order $\tau > 3/2$ in a vacuum Bondi's radiating spacetime (2.1), then

(5.1)
$$E_0(\mathbb{X}) - P_0(\mathbb{X}) \geq \sqrt{\sum_{1 \leq i \leq 3} [E_i(\mathbb{X}) - P_i(\mathbb{X})]^2}$$

and equality implies the spacetime is flat over $\mathbb{X}$. (Theorem 4.1 in [21] was proved for $\tau = 3$. However, the argument goes through if $c|_{u=0} = d|_{u=0} = 0$ for the above $(\mathbb{X}, g, h)$ in the Bondi's radiating spacetimes. See also Theorem 3.1 and Remark 3.1 in [23]. The sharp order $\tau > 3/2$ together with certain integrable conditions was also given in [8, 6] to ensure the argument to work.) In general, the hyperbolic mass of an asymptotically null

spacelike hypersurface is different from the Bondi mass of the null cone. For instance, if $c|_{u=0}$ or $d|_{u=0}$ is nonzero, $E_0(\mathbb{X}) - P_0(\mathbb{X})$ may not be finite.

LEMMA 5.1. *Let $(\mathbb{L}^{3,1}, \tilde{g})$ be a vacuum Bondi's radiating spacetime with metric $\tilde{g}$ given by (2.1). Let $(\mathbb{X}, g, h)$ be a spacelike hypersurface u which is given by (3.3) for r sufficiently large. Define $L(\phi, \psi) = l(0, \phi, \psi)$, $\bar{L}(\phi, \psi) = \bar{l}(0, \phi, \psi)$. Then*

$$\mathcal{E} \approx \frac{12}{r^2}(c^2 + d^2)_{u=0} + \frac{1}{r^3}(M + 16a_3 + 15cc_0 + 15dd_0)_{u=0}$$
$$- \frac{1}{2r^3}(L_{,2} + L\cot\theta + \bar{L}_{,3}\csc\theta),$$
$$\mathcal{P} \approx -\frac{1}{2r^3}(3M - 16a_3 + 5cc_0 + 5dd_0)_{u=0} + \frac{1}{2r^3}(L_{,2} + L\cot\theta + \bar{L}_{,3}\csc\theta).$$

PROOF. Note that

$$a_{22} + a_{33} \approx \frac{4}{r^2}(c^2 + d^2)_{u=0} + \frac{4}{r^3}(cc_{,0} + dd_{,0})_{u=0},$$
$$b_{22} + b_{33} \approx \frac{1}{2r^3}(3M - 16a_3 + 5cc_{,0} + 5dd_{,0})_{u=0} - \frac{1}{2r^3}(L_{,2} + L\cot\theta + \bar{L}_{,3}\csc\theta).$$

Using the formula

$$\check{\nabla}_k a_{ij} = \check{e}_k(a_{ij}) - a_{jl}\check{\omega}_{li}(\check{e}_k) - a_{il}\check{\omega}_{lj}(\check{e}_k),$$

we obtain

$$\mathcal{E} = \check{e}_j(a_{1j}) - a_{jl}\check{\omega}_{l1}(\check{e}_j) - a_{1l}\check{\omega}_{lj}(\check{e}_j) - \check{\nabla}_1 \text{tr}_{\check{g}}(a) + a_{22} + a_{33}$$
$$\approx -\frac{1}{2r^3}(L_{,2} + L\cot\theta + \bar{L}_{,3}\csc\theta) + \frac{1}{r^3}(M + 16a_3 - cc_{,0} - dd_{,0})_{u=0}$$
$$+ \frac{8\sqrt{1 + r^2}}{r^3}(c^2 + d^2)_{u=0} + \frac{12\sqrt{1 + r^2}}{r^4}(cc_{,0} + dd_{,0})_{u=0}$$
$$+ \frac{4}{r^3}(c^2 + d^2)_{u=0} + \frac{4}{r^4}(cc_{,0} + dd_{,0})_{u=0}$$
$$\approx -\frac{1}{2r^3}(L_{,2} + L\cot\theta + \bar{L}_{,3}\csc\theta) + \frac{1}{r^3}(M + 16a_3 + 15cc_{,0} + 15dd_{,0})_{u=0}$$
$$+ \frac{12}{r^2}(c^2 + d^2)_{u=0} + O\left(\frac{1}{r^4}\right),$$
$$\mathcal{P} = -b_{22} - b_{33}$$
$$\approx -\frac{1}{2r^3}(3M - 16a_3 + 5cc_{,0} + 5dd_{,0})_{u=0} + \frac{1}{2r^3}(L_{,2} + L\cot\theta + \bar{L}_{,3}\csc\theta). \qquad \square$$

THEOREM 5.1. *Let $(\mathbb{L}^{3,1}, \tilde{g})$ be a vacuum Bondi's radiating spacetime with metric $\tilde{g}$ given by (2.1). Suppose that Conditions A and B hold and $c|_{u=u_0} = d|_{u=u_0} = 0$ for some u_0. Then*

$$m_0(u) \geq \sqrt{\sum_{1 \leq i \leq 3} m_i^2(u)}$$

for all $u \leq u_0$. If equality holds for some $u \in (-\infty, u_0]$, then $\mathbb{L}^{3,1}$ is flat in the region foliated by all spacelike hypersurfaces which are given by

$$u = u_0 + \sqrt{1 + r^2} - r + o(1/r^4)$$

for r sufficiently large. In particular, if equality holds for all $u \leq u_0$, then $\mathbb{L}^{3,1}$ is flat in the region $\{u \leq u_0\}$.

PROOF. By translation, we can assume that $u_0 = 0$. Choose an asymptotically null spacelike hypersurface $\mathbb{X}$ which is given by (3.3) with $a_3 = 0$ for r sufficiently large. By Lemma 5.1, we obtain

$$\mathcal{E} - \mathcal{P} \approx -\frac{1}{r^3}(L_{,2} + L\cot\theta + \bar{L}_{,3}\csc\theta) + \frac{5M(0, \theta, \psi)}{2r^3}.$$

Then Lemma 4.1 implies that

$$E_\nu(\mathbb{X}) - P_\nu(\mathbb{X}) = \frac{5}{8}m_\nu(0).$$

Therefore the first part of the theorem follows from (5.1) and Proposition 2.1. For the second part, if equality holds for some $u \in (-\infty, u_0]$, then it holds for $u = u_0$ by Proposition 2.1. Thus $\mathbb{L}^{3,1}$ is flat over $\mathbb{X}$ and the assertion follows. □

6. MODIFIED BONDI ENERGY-MOMENTUM

We can modify the definition of the Bondi energy-momentum to remove Condition B. Define the modified Bondi energy-momentum as

$$(6.1) \qquad \mathbf{m}_\nu(u_0) = \frac{1}{4\pi}\int_{S^2}\mathcal{M}(u_0, \theta, \psi)n^\nu \, dS$$

for $\nu = 0, 1, 2, 3$. Then we can prove that

$$(6.2) \qquad \frac{d}{du}\mathbf{m}_\nu = -\frac{1}{4\pi}\int_{S^2}((c_{,0})^2 + (d_{,0})^2)n^\nu \, dS$$

for $\nu = 0, 1, 2, 3$, and

$$(6.3) \qquad \frac{d}{du}\left(\mathbf{m}_0 - \sqrt{\sum_{1 \leq i \leq 3}\mathbf{m}_i^2}\right) \leq 0$$

under Condition A only.

Now choose the spacelike asymptotically null hypersurface $\mathbb{X}$ given by (3.3) with $a_3 = -M(0, \theta, \psi)/16$. If $c|_{u=0} = 0$, $d|_{u=0} = 0$, then

$$\mathcal{E} - \mathcal{P} \approx \frac{2\mathcal{M}(0, \theta, \psi)}{r^3}.$$

Therefore the following theorem is a direct consequence.

THEOREM 6.1. *Let* $(\mathbb{L}^{3,1}, \tilde{g})$ *be a vacuum Bondi's radiating spacetime with metric* $\tilde{g}$ *given by* (2.1). *Suppose that Condition A holds. If either* (i) $\mathcal{M}(u_0, \theta, \psi)$ *is constant, or* (ii) $c|_{u=u_0} = d|_{u=u_0} = 0$ *for some* u_0, *then*

$$\mathbf{m}_0(u) \geq \sqrt{\sum_{1 \leq i \leq 3} \mathbf{m}_i^2(u)}$$

for all $u \leq u_0$. *If equality holds for some* $u \in (-\infty, u_0]$, *then* $\mathbb{L}^{3,1}$ *is flat in the region foliated by all spacelike hypersurfaces which are given by*

$$u = u_0 + \sqrt{1 + r^2} - r - \frac{M(u_0, \theta, \psi)}{16r^4} + o\left(\frac{1}{r^4}\right)$$

for r sufficiently large. In particular, if the equality holds for all $u \leq u_0$, *then* $\mathbb{L}^{3,1}$ *is flat in the region* $\{u \leq u_0 - M(u_0, \theta, \psi)/16r^4\}$.

REFERENCES

[1] A. ASHTEKAR - G. HOROWITZ, *Energy-momentum of isolated systems cannot be null*. Phys. Lett. A 89 (1982), 181–184.

[2] H. BONDI - M. VAN DER BURG - A. METZNER, *Gravitational waves in general relativity VII. Waves from axi-symmetric isolated systems*. Proc. Roy. Soc. London Ser. A 269 (1962), 21–52.

[3] P. CHRUŚCIEL - J. JEZIERSKI - J. KIJOWSKI, *Hamiltonian Field Theory in the Radiating Regime*. Lecture Notes in Phys. Monogr. 70, Springer, Berlin, 2002.

[4] P. CHRUŚCIEL - J. JEZIERSKI - S. LESKI, *The Trautman–Bondi mass of initial data sets*. Adv. Theor. Math. Phys. 8 (2004), 83–139,

[5] P. CHRUŚCIEL - J. JEZIERSKI - M. MACCALLUM, *Uniqueness of the Trautman–Bondi mass*. Phys. Rev. D 58 (1998), 084001.

[6] P. CHRUŚCIEL - M. HERZLICH, *The mass of asymptotically hyperbolic Riemannian manifolds*. Pacific J. Math. 212 (2003), 231–264.

[7] P. CHRUŚCIEL - M. MACCALLUM - D. SINGLETON, *Gravitational waves in general relativity XIV. Bondi expansions and the "polyhomogeneity" of Scri*. Philos. Trans. Roy. Soc. Ser. A 350 (1995), 113–141.

[8] P. CHRUŚCIEL - G. NAGY, *The mass of spacelike hypersurfaces in asymptotically anti-de Sitter space-times*. Adv. Theor. Math. Phys. 5 (2001), 697–754.

[9] G. HOROWITZ - M. PERRY, *Gravitational energy cannot become negative*. Phys. Rev. Lett. 48 (1982), 371–374.

[10] G. HOROWITZ - P. TOD, *A relation between local and total energy in general relativity*. Comm. Math. Phys. 85 (1982), 429–447.

[11] W. ISRAEL - J. NESTER, *Positivity of the Bondi gravitational mass*. Phys. Lett. A 85 (1981), 259–260.

[12] C.-C. LIU - S. T. YAU, *Positivity of quasi-local mass II*. J. Amer. Math. Soc. 19 (2006), 181–204.

[13] M. LUDVIGSEN - J. VICKERS, *A simple proof of the positivity of the Bondi mass*. J. Phys. A: Math. Gen. 15 (1982), L67–L70.

[14] O. REULA - K. TOD, *Positivity of the Bondi energy*. J. Math. Phys. 25 (1984), 1004–1008.

[15] R. SACHS, *Gravitational waves in general relativity VIII. Waves in asymptotically flat space-time*. Proc. Roy. Soc. London Ser. A 270 (1962), 103–126.

[16] R. SCHOEN - S. T. YAU, *Proof of the positive mass theorem II*. Comm. Math. Phys. 79 (1981), 231–260.

[17] R. SCHOEN - S. T. YAU, *Proof that the Bondi mass is positive*, Phys. Rev. Lett. 48 (1982), 369–371.

[18] M. VAN DER BURG, *Gravitational waves in general relativity IX. Conserved quantities*. Proc. Roy. Soc. London Ser. A 294 (1966), 112–122.

[19] E. WITTEN, *A new proof of the positive energy theorem*. Comm. Math. Phys. 80 (1981), 381–402.

[20] S. T. YAU, *Geometry of three manifolds and existence of black hole due to boundary effect*. Adv. Theor. Math. Phys. 5 (2001), 755–767.

[21] X. ZHANG, *A definition of total energy-momenta and the positive mass theorem on asymptotically hyperbolic 3-manifolds I*. Comm. Math. Phys. 249 (2004), 529–548.

[22] X. ZHANG, *On the relation between ADM and Bondi energy-momenta*. gr-qc/0511036, Adv. Theor. Math. Phys., to appear.

[23] X. ZHANG, *The positive mass theorem near null infinity*. Invited talk at ICCM 2004, Hong Kong.

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