Rend. Lincei Mat. Appl. 17 (2006), 351–366

Optimal control theory. — *On exponential observability estimates for the heat semigroup with explicit rates*, by LUC MILLER, communicated on 12 May 2006.

ABSTRACT. — This note concerns the final time observability inequality from an interior region for the heat semigroup, which is equivalent to the null-controllability of the heat equation by a square integrable source supported in this region. It focuses on exponential estimates in short times of the observability cost, also known as the control cost and the minimum energy function. It proves that this final time observability inequality implies four variants with roughly the same exponential rate everywhere (an integrated inequality with singular weights, an integrated inequality in infinite times, a sharper inequality and a Sobolev inequality) and some control cost estimates with explicit exponential rates concerning null-controllability, null-reachability and approximate controllability. A conjecture and open problems about the optimal rate are stated. This note also contains a brief review of recent or to be published papers related to exponential observability estimates: boundary observability, Schrödinger group, anomalous diffusion, thermoelastic plates, plates with square root damping and other elastic systems with structural damping.

KEY WORDS: Observability; heat; controllability cost; minimum energy function.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 93B07, 35B37, 35K05.

1. INTRODUCTION

The natural setting for the problem to be discussed is on manifolds, but all the statements can be understood, and are already interesting, when the domain M is a smooth bounded open set in $\mathbb{R}^d$ with the flat metric so that the distance is $\operatorname{dist}(x, y)^2 = |y_1 - x_1|^2 + \cdots + |y_d - x_d|^2$ and the Laplacian is $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_d^2$, always considered with the Dirichlet condition on ∂M. We shall refer to this setting as *the Euclidean case*.

Although it can be skipped, for completeness we now describe the general setting. Let (M, g) be a smooth connected compact d-dimensional Riemannian manifold with metric g and smooth boundary ∂M. When $\partial M \neq \emptyset$, M denotes the interior and $\overline{M} = M \cup \partial M$. Let $\operatorname{dist} : \overline{M}^2 \to \mathbb{R}_+$ denote the distance function. Let Δ denote the Dirichlet Laplacian on $L^2(M)$ with domain $H_0^1(M) \cap H^2(M)$.

The *observation region* Ω is a nonempty open subset of M such that $\Omega \neq M$. Unless mentioned otherwise, the range of the time T is $(0, \infty)$ and the range of the initial state u_0 is $L^2(M)$. The corresponding solution of the Cauchy problem for the (forward) heat equation is denoted by $u(T, x) = (e^{T\Delta}u_0)(x)$, briefly $u = e^{T\Delta}u_0$ is the (relative) temperature on $\mathbb{R}_+ \times M$.

In this note, we make some remarks about the following observability inequality from Ω of the final state at time T: for any T,

$$(1) \qquad \forall u_0, \quad \int_M |e^{T\Delta}u_0|^2\,dx \le K \int_0^T \int_\Omega |e^{t\Delta}u_0|^2\,dx\,dt \quad \text{with } K = Ce^{A/T}.$$

Even when K is an unspecified constant, this inequality is interesting from various points of view. If u is always zero on Ω then it implies that u is zero everywhere on M at the final time T, which implies by backward uniqueness that u is always zero everywhere. Thus (1) is a unique continuation estimate. Moreover, by the duality in [DR77], the existence of a constant K such that (1) holds is equivalent to the ability of steering the heat flux from any u_0 to zero in time T by a square integrable source supported in Ω at a *cost* K (hence the optimal K does not increase with T). This property is called *null-controllability* or exact controllability to zero. Its validity in this context was proved a decade ago in [LR95, Èma95].

Indeed (1) specifies how the cost $K = Ce^{A/T}$ depends on T. The first such *exponential cost estimate* is due to Seidman (cf. [Sei84] and the survey [Sei05]). As far as I know, the best results about the validity of this estimate are threefold and use different methods. In the Euclidean case, (1) was proved in [FCZ00] by global Carleman estimates with singular weights as in [Èma95]. Under the geometrical optics condition on Ω (i.e. $L_\Omega < \infty$ with the notation of Theorem 4), (1) was deduced in [Mil04b] by the control transmutation method (for short CTM, cf. Section 2.2) from the observability of the wave group in [BLR92]. In the general setting, (1) is not proved, but a slightly weaker exponential cost estimate was proved in [Mil05b] by the control strategy of [LR95] as implemented in [LZ98]: for all $\beta > 1$, there are positive constants A_β and C_β such that (1) holds with $K = C_\beta e^{A_\beta/T^\beta}$ (Carleman estimates should allow reaching $\beta = 1$ as in (1)).

This note reviews the known bounds on the optimal rate A in (1) (Section 2) and other similar cost estimates (Section 3), and relates (1) to several of the following variants considered in [FCZ00, Zua01] in the Euclidean case (Theorem 1). The method of global Carleman estimates leads more naturally to the following integrated inequality with singular weight:

$$\text{(2)} \qquad \forall u_0, \quad \int_0^T \int_M e^{-\tilde{A}/t} |e^{t\Delta} u_0|^2 \, dx \, dt \le \tilde{C} \int_0^T \int_\Omega |e^{t\Delta} u_0|^2 \, dx \, dt.$$

This is proved in Proposition 6.1 of [FCZ00]. Among open problems, the following variant for infinite time is stated in [Zua01] (equation 4.3):

$$\text{(3)} \qquad \forall u_0, \quad \int_0^\infty \int_M e^{-A/t} |e^{t\Delta} u_0|^2 \, dx \, dt \le C_\infty \int_0^\infty \int_\Omega |e^{t\Delta} u_0|^2 \, dx \, dt.$$

Remark 6.1 of [FCZ00] extracts from the proof of Theorem 6.1 the following inequality for fixed T, which is sharper than (1), at least when $T \ge B$:

$$\text{(4)} \quad \forall u_0, \quad \int_M |e^{-B\sqrt{-\Delta}} u_0|^2 \, dx \le K' \int_0^T \int_\Omega |e^{t\Delta} u_0|^2 \, dx \, dt \quad \text{with } K' = C' e^{A'/T}.$$

Replacing the L^2 norm of the final state in (1) by its norm in a Sobolev space of real order s yields the following inequality, better for positive s:

$$\text{(5)} \quad \forall u_0 \in H^s(M), \quad \|e^{T\Delta} u_0\|_{H^s}^2 \le K_s \int_0^T \int_\Omega |e^{t\Delta} u_0|^2 \, dx \, dt \quad \text{with } K_s = C_s e^{A_s/T}.$$

We prove in this note that (1) for small times implies its four variants (2), (3), (4) and (5), with rates A, $\tilde{A}$, A' and A_s which are roughly the same everywhere. More precisely, in Section 4 we prove

THEOREM 1. *Let $A' > A$ and $B > \sqrt{2A}(1+(A'/A-1)^{-1/2})/2$ ($B > \sqrt{2A}$ if $A' > 2A$). Let $s \in \mathbb{R}$ and $A_s > A$ ($A_s = A$ if $s \leq 0$). If the final time observability inequality* (1) *holds for all $T \leq T_0$, then*

(i) *the integrated inequality* (2) *holds for all $T \leq T_0$ with $\tilde{A} = A$ and $\tilde{C} = CT$,*
(ii) *the infinite time inequality* (3) *holds with $C_\infty = CT_0(1 + e^{A/T_0})$,*
(iii) *the sharp inequality* (4) *holds for all T,*
(iv) *the Sobolev inequality* (5) *holds for all T.*

Conversely, for $A > \tilde{A}$, if the integrated inequality (2) *holds for all $T \leq T_0$, then the final time inequality* (1) *holds for all T.*

Even in the Euclidean case and for fixed T, Theorem 1 simplifies the proof of (4) (Proposition 6.1 in [FCZ00] already uses (1) but also goes back to the global Carleman inequality). The fast cost estimate in (4) seems to be new:

COROLLARY 2. *Under the geometrical optics condition on Ω or in the Euclidean case, there are positive constants B, A' and C' such that*

$$\forall T,\ \forall u_0, \quad \int_M |e^{-B\sqrt{-\Delta}}u_0|^2\,dx \leq C'e^{A'/T}\int_0^T\int_\Omega |e^{t\Delta}u_0|^2\,dx\,dt.$$

Besides null-controllability, the exponential observability estimate (1) implies various reachability results which were known in the Euclidean case (cf. [FCZ00, Phu04] where a time and space dependent potential is emphasized as a preliminary step towards a nonlinear term). The appendix gives simple proofs of these results keeping track of the rate A in (1) explicitly.

The controllability cost (e.g. the optimal K in (1), or sometimes $\sqrt{K}$) is also called the minimum energy function (for normalized initial states). It is connected to the minimum time function (cf. [Câr93, GL99]), also known as the Bellman function of the system. An exponential fast control cost estimate yields a logarithmic modulus of continuity for the minimum time function (cf. Remark 3.6 in [Mil04a]).

The cost of fast controls is also related to the regularity properties of the stochastic diffusion process obtained from the control system by substituting a white noise for the controlled source (more generally, for the input function of the system). The regularity of the generalized solution of Ornstein–Uhlenbeck equations (e.g. the Kolmogorov equation corresponding to this stochastic P.D.E.) and the strong Feller property for the transition semigroup depend on the behaviour of the cost of fast controls (cf. Theorem 8.3.3 in [DP01], Theorem 6.2.2 and Appendix B in [DPZ02]). The introduction of [AL03b] elaborates on this motivation.

2. BOUNDS ON THE OPTIMAL RATE A IN (1)

2.1. *Lower bounds*

It is proved in [Mil04b] that (1) for all small T implies

$$A \geq \sup_{y \in M} \operatorname{dist}(y, \overline{\Omega})^2/2. \tag{6}$$

The proof relies on Varadhan's formula for the heat kernel in small time (cf. [Var67]), which requires very low smoothness assumptions as proved in [Nor97]. This improves on the former lower bound in the Euclidean case stated in Section 4.1 of [Zua01] which was based on a construction made in the proof of Theorem 6.2 in [FCZ00]: $A \geq \sup_{\overline{B}_\rho \subset M \setminus \overline{\Omega}} \rho^2/4$, where the supremum is taken over balls B_ρ of radius ρ.

For finite times T, the lack of observability at a better cost is only due to the finite linear combinations of the eigenmodes corresponding to frequencies lower than a threshold of order $1/T$. To state this result from [Mil04b] more precisely,[1] we introduce the spectral data: $(\omega_k)_{k \in \mathbb{N}^*}$ is a nondecreasing sequence of nonnegative real numbers and $(e_k)_{k \in \mathbb{N}^*}$ is an orthonormal basis of $L^2(M)$ such that e_k is an eigenvector of $-\Delta$ with eigenvalue ω_k^2, i.e.

$$-\Delta e_k = \omega_k^2 e_k \quad \text{and} \quad e_k = 0 \quad \text{on } \partial M. \tag{7}$$

THEOREM 3 ([Mil04b]). *Let $d \in (0, \sup_{y \in M} \operatorname{dist}(y, \overline{\Omega}))$. If* (1) *holds for all small T and for any u_0 in the linear span of $\{e_k\}_{\omega_k \leq d/T}$, then $A \geq d^2/2$.*

2.2. *Upper bounds*

In view of Theorem 1, upper bounds on the optimal rate A in (1) imply upper bounds on the optimal rates in (2)–(5).

THEOREM 4 ([Mil04b]). *Let L_Ω be the length of the longest generalized geodesic*[2] *in $\overline{M}$ which does not intersect Ω. For all $A > (2(36/37)L_\Omega)^2$ there is a positive constant C such that* (1) *holds for all T.*

The same bound is immediately deduced, by Theorem 1.6 in [Mil05c], for the heat semigroup on the product manifold $M \times \tilde{M}$ observed from $\Omega \times \tilde{M}$, where $\tilde{M}$ denotes another smooth complete $\tilde{n}$-dimensional Riemannian manifold (e.g. an infinite

[1] Theorem 3 is not explicitly stated in [Mil04b], but it is roughly explained after Theorem 2.1 there. Moreover, Theorem 3 for the Schrödinger group instead of the heat semigroup is proved by the same method and explicitly stated in [Mil04c].

[2] In this context, the generalized geodesics are continuous trajectories $t \mapsto x(t)$ in $\overline{M}$ which follow geodesic curves at unit speed in M (so that on these intervals $t \mapsto \dot{x}(t)$ is continuous); if they hit ∂M transversely at time t_0, then they reflect as light rays or billiard balls (and $t \mapsto \dot{x}(t)$ is discontinuous at t_0); if they hit ∂M tangentially then either there exists a geodesic in M which continues $t \mapsto (x(t), \dot{x}(t))$ continuously and they branch onto it, or there is no such geodesic curve in M and then they glide at unit speed along the geodesic of ∂M which continues $t \mapsto (x(t), \dot{x}(t))$ continuously until they may branch onto a geodesic in M. The meaning of the geometrical optics condition $L_\Omega < \infty$, due to Bardos–Lebeau–Rauch in [BLR92], is discussed at length in [Mil02].

strip observed from any infinite strip in the interior). To the best of my knowledge, there is no better upper bound of the optimal rate in the literature.

When comparing Theorem 4 to the lower bound in (6), one should bear in mind that L_Ω is always greater than $2\sup_{y\in M}\operatorname{dist}(y,\overline{\Omega})$ (as the length of a generalized geodesic through y which does not intersect Ω is always greater than $2\operatorname{dist}(y,\overline{\Omega})$) and can be infinitely so. But, for some simple geometries,[3] Theorem 4 implies an upper bound on the optimal rate in terms of $\sup_{y\in M}\operatorname{dist}(y,\overline{\Omega})$ as well, e.g.:

COROLLARY 5. *In the Euclidean case, if M is a ball and Ω is a small enough neighborhood of its boundary then for all $A > 16\sup_{y\in M}\operatorname{dist}(y,\overline{\Omega})^2$ there is a $C > 0$ such that* (1) *holds for all T.*

Theorem 4 is deduced from the observability of the wave group (cf. [BLR92]) by the *Control Transmutation Method*, for short CTM. This method applies to control problems the guiding principle of the kernel estimates method of [CGT82]: systems with finite propagation speed yield geometrical information in small times about systems with similar generators but without propagation speed. Here, it consists in constructing a time kernel k, dubbed “the fundamental controlled solution”, which transforms the input function for the wave group in time L into an input function for the heat semigroup in time T; some norms of k must be estimated explicitly in terms of L and T only. Unlike Russell’s harmonic analysis method in [Rus73], it does not use information on the spectrum and it extends to the most general abstract setting (cf. [Mil04a]).[4]

3. A CONJECTURE AND RELATED RESULTS

3.1. *Conjecture and open problems*

Combining the upper and lower bounds for the optimal rate A in (1) in the simple example of Corollary 5 (M is a Euclidean ball and Ω is a small enough neighbourhood of its boundary) yields

$$\alpha := A(\sup_{y\in M}\operatorname{dist}(y,\overline{\Omega}))^{-2} \in [1/2, 16).$$

Since we believe that there is no solution of the heat equation which is more singular than the heat kernel, it is natural to conjecture that the lower bound (6) is also an upper bound: the optimal rate A such that (1) holds for small T is $\sup_{y\in M}\operatorname{dist}(y,\overline{\Omega})^2/2$ for any (M, g) and $\Omega \neq M$ (i.e. $\alpha = 1/2$).

If $K(T)$ denotes the optimal cost in (1) for fixed T, then the function $K : (0,\infty) \to (0,\infty)$ does not increase (as a result of the semigroup property or the duality with null-controllability), but this is not enough to ensure that $\lim_{T\to 0} T\ln K(T)$ exists. The existence of this limit is part of the conjecture but could possibly be

[3] In the Euclidean case, if Ω is a neighbourhood of ∂M then L_Ω is the length of the longest segment in M which does not intersect Ω.

[4] E.g. the abstract presentation of Russell’s method in Section 2 of [FCZ02] assumes that the eigenvalues grow quadratically.

established independently. Until then, the optimal rate can only be defined as[5] $A^* = \limsup_{T\to 0} T \ln K(T)$.

Theorem 1 roughly says that the "optimal rates" A and A' in (1) and (2) are equal.[6] It does not say wether the "optimal rates" A in (1) and (3) are also equal (it roughly says that the "optimal rate" is not greater in (3) than in (1)).

Other related open problems will appear in [Zua06].

3.2. *Boundary observability and window problems*

For steering the temperature to zero with the temperature on $\Gamma \subset \partial M$ as input, the corresponding observability inequality of the final state from Γ is similar to (1):

$$\text{(8)} \qquad \forall u_0, \quad \int_M |e^{T\Delta}u_0|^2\,dx \le K_\nu \int_0^T \int_\Gamma |\partial_\nu u|^2\,dx\,dt \quad \text{with } K_\nu = C_\nu e^{A_\nu/T},$$

where ∂_ν denotes the Neumann derivative at the boundary.

When M is a Euclidean segment and Γ is one endpoint, (8) is an inequality on sums of exponentials called a "window problem" in [SAI00]. A well trodden path in the harmonic analysis of this problem is to construct a Riesz basis of bi-orthogonal functions. This reduces by the Paley–Wiener theorem to the construction of entire functions with zero and growth conditions. Proving exponential cost estimates in this setting is a nonclassical aspect of this problem deeply studied in [SAI00]. We refer to [SAI00, Sei05, Mil04b] for more details and references.

In this context, $L = \sup_{y\in M} \operatorname{dist}(y, \overline{\Gamma})$ is the length of the segment M. The best upper bound obtained so far by this method is (cf. [Mil04b]): for $A_\nu > 2\alpha_* L^2$, (8) holds for all T, where $\alpha_* = 2\,(36/37)^2 < 2$. Any improvement on the value of α_* in this result, and in the analogous result where the Neumann derivative is removed in (8), will improve Theorem 4 to $A > 2\alpha_* L_\Omega^2$. N.b. in the CTM which deduces this theorem from the boundary observability estimate on the segment there is a loss of a factor 4 since $L_\Gamma = 2L$ on the segment.

The CTM has been extended in [Mil04a] to the observability (by unbounded operators) of holomorphic semigroups generated by the generator of a cosine operator function. Since [BLR92] proved the boundary observability for (the real part of) the wave group $\cos(t\sqrt{-\Delta})$, which is the model of all cosine operator evolutions, Theorem 4 still holds when (1) and Ω are replaced by (8) and Γ (this is Theorem 6.1 in [Mil04a]).

By Theorem 1.5 in [Mil05c], these two estimates of the cost in (8) for problems in dimension one and greater extend to the problems that can be deduced from them by a tensor product, e.g. the better one-dimensional result extends to an infinite strip observed from one of the boundary lines.

3.3. *Other evolution systems*

The heat kernel method used to prove the lower bound in Theorem 3 and the control transmutation method (CTM) used to prove the upper bound in Theorem 4 were adapted to

[5] If $A > A^*$ then (1) holds for small T, and conversely, if (1) holds for small T then $A \ge A^*$. Whether (1) holds for small T when $A = A^*$ is an open problem.

[6] Theorem 1 proves: $\tilde{A} > A^*$ implies "(2) holds for small T" implies $\tilde{A} \ge A^*$.

the interior observability of the Schrödinger group in [Mil04c] (n.b. this is the observation-control system to which a transmutation method was first applied in [Phu01]). Thanks to a new necessary and sufficient condition for the observability of unitary groups by unbounded operators, called a “resolvent observability estimate” (this Theorem 5.1 in [Mil05a] is the analogue of the Hautus test for finite-dimensional control systems, cf. [RW94]), the CTM has been extended to this abstract setting. Thus it allows one to deduce from [BLR92] exponential observability estimates from the boundary for the Schrödinger group (Theorem 10.2 in [Mil05a]).

The slightly weaker exponential cost estimates mentioned in the introduction, i.e. $K = C_\beta e^{A_\beta/T^\beta}$ for any $\beta > 1$ and some $A_\beta > 0$ and $C_\beta > 0$, were generalized by the same method to the system of thermoelastic plates without rotatory inertia (in the Euclidean case, with hinged mechanical boundary conditions and Dirichlet thermal boundary condition) observed from Ω by either the mechanical or thermal component (cf. [Mil05d]), and to the plate equation with square root damping observed from Ω (in the Euclidean case, with hinged boundary conditions, cf. [Mil06] where the CTM was also adapted to this system and yields $\beta = 1$ under the geometrical optics condition on Ω). The same method was applied to more general abstract linear elastic systems with structural damping in [Mil06] and yields various ranges for β depending on the strength of the damping. It also applies to anomalous diffusions generated by the fractional Laplacian $-(-\Delta)^p$ for $p > 1/2$, where it yields $\beta > 1/(2p-1)$ (cf. [Mil05b]).

The exponential cost estimates in [Mil06, Mil05d] use earlier *polynomial cost estimates* proved in [Tri03, AL03b, AL03a] in the case $\Omega = M$ which we have excluded at the very beginning of this note because (1) holds with $K = C/T$ when $\Omega = M$. Triggiani, Lasiecka and Avalos proved cost estimates of the form $K = C/T^p$, $p \geq 1$, where p is related to the strength of the damping. These estimates are similar to the optimal cost estimates for finite-dimensional control systems proved in [Sei88] which we now describe. Let A be an $n \times n$ matrix defining a system of linear differential equations in $\mathbb{R}^n$, and let B be the $m \times n$ matrix which prescribes the m observed coordinates in $\mathbb{R}^n$. The observability inequality is

$$\forall x_0 \in \mathbb{R}^n, \quad \|x_0\|^2 \leq K \int_0^T \|Be^{tA}x_0\|^2 dt. \tag{9}$$

Kalman proved that (9) holds if and only if there is an integer $p < n$ such that the $n \times nm$ block matrix $\{B^*, A^*B^*, \ldots, A^{*p}B^*\}$ is of rank n (the star denotes transposed matrices). Seidman proved that, as T tends to zero, the optimal cost in (9) satisfies $K \sim C/T^{1+2p}$ where p is the smallest integer satisfying Kalman's rank condition.

We are still longing for such a complete result regarding infinite-dimensional control systems, at least for distributed systems with infinite propagation speed such as the heat semigroup.

4. Proof of Theorem 1

(i) For $T \leq T' \leq T_0$, multiplying (1) by $e^{-A/T}$, bounding $\int_0^T$ from above by $\int_0^{T'}$, then integrating T over $(0, T')$ yields (2) with $T = T'$, $\tilde{C} = CT'$, $A = \tilde{A}$.

(ii) This point results from the previous one and the following lemma.

LEMMA 6. *If* (1) *and* (2) *hold, then* (3) *holds with* $C_\infty = \tilde{C} + CTe^{A/T}$.

PROOF. Since $e^{-A/t} \leq 1$ and $t \mapsto \|e^{t\Delta}u_0\|_{L^2(M)}$ does not increase, for all $n \in \mathbb{N}^*$ we have

$$\int_{nT}^{(n+1)T}\int_M e^{-A/t}|e^{t\Delta}u_0|^2\,dx\,dt \leq T\int_M |e^{nT\Delta}u_0|^2\,dx$$
$$\leq CTe^{A/T}\int_{(n-1)T}^{nT}\int_\Omega |e^{t\Delta}u_0|^2\,dx\,dt,$$

where (1) with u_0 replaced by $e^{(n-1)T}u_0$ is used in the last step. Summing up over $n \geq 1$ yields

$$\int_T^\infty\int_M e^{-A/t}|e^{t\Delta}u_0|^2\,dx\,dt \leq CTe^{A/T}\int_0^\infty\int_\Omega |e^{t\Delta}u_0|^2\,dx\,dt.$$

Adding this inequality to (2) yields (3) with $C_\infty = \tilde{C} + CTe^{A/T}$. □

(iii) This point results from the first point and the following lemma.

LEMMA 7. *For any* $A' > A$ *and* $B > \sqrt{2A}\,b(A')$, *where* $b(A') = 1$ *for* $A' > 2A$ *and otherwise* $b(A') = (1+(A'/A-1)^{-1/2})/2$ *(n.b.* $\lim_{A'\to A} b(A') = +\infty$*), there is a* $C' > 0$ *such that for all* T,

$$\forall u_0, \quad \int_M |e^{-B\sqrt{-\Delta}}u_0|^2\,dx \leq C'e^{A'/T}\int_0^T\int_M e^{-A/t}|e^{t\Delta}u_0|^2\,dx\,dt.$$

PROOF. Writing $u_0 = \sum_k c_k e_k$ with $\sum_k |c_k|^2 < \infty$ in the eigenbasis (7) yields

$$\int_M |e^{-B\sqrt{-\Delta}}u_0|^2\,dx = \sum_k e^{-2B\omega_k}|c_k|^2 = \sum_k e^{-B\sqrt{2\cdot 2\omega_k^2}}|c_k|^2, \tag{10}$$

$$\int_0^T\int_M e^{-A/t}|e^{t\Delta}u_0|^2\,dx\,dt = \sum_k I_A(T, 2\omega_k^2)|c_k|^2, \tag{11}$$

with

$$I_A(T,\lambda) = \int_0^T e^{-\lambda t - A/t}dt = \sqrt{A/\lambda}\int_0^{T\sqrt{\lambda/A}} e^{-\sqrt{A\lambda}(s+1/s)}\,ds.$$

Henceforth, we keep the same notation ε and C_ε meaning "for all small $\varepsilon > 0$, there is $C_\varepsilon > 0$ independent of λ and T such that..." although their value may change. Setting $a = \min\{1, (A'/A-1)^{1/2}\}$, we have $A' > (1+a^2)A$ and $B > \sqrt{2A}\,b(A')$ with $b(A') = \max\{1, (1+(A'/A-1)^{-1/2})/2\} = (1+1/a)/2 \geq (a+1/a)/2$.

For $T\sqrt{\lambda/A} > a$, we may bound the last integral from below by $\int_{(1-\varepsilon)a}^a \cdots ds$ and use the fact that $f : s \mapsto s + 1/s$ decreases on $(0,1] \supset (0,a]$, hence

$$I_A(T,\lambda) \geq \varepsilon\sqrt{A/\lambda}\,e^{-\sqrt{A\lambda}f((1-\varepsilon)a)} \geq C_\varepsilon e^{-\sqrt{A\lambda}\,(1+\varepsilon)(a+1/a)} \geq C_\varepsilon e^{-(1+\varepsilon)\sqrt{2A}\,b(A')\sqrt{2\lambda}}.$$

For $T\sqrt{\lambda/A} \le a$, i.e. $\lambda \le a^2 A/T^2$, we have

$$I_A(T,\lambda) \ge e^{-\lambda T} \int_{(1-\varepsilon)T}^{T} e^{-A/t}\,dt \ge e^{-a^2 A/T} \varepsilon T e^{-A/((1-\varepsilon)T)} \ge C_\varepsilon e^{-(1+\varepsilon)(a^2+1)A/T}.$$

Multiplying the lower bounds obtained in the two cases yields

$$\forall \lambda > 0,\ \forall T > 0, \qquad e^{-(1+\varepsilon)\sqrt{2A}\,b(A')\sqrt{2\lambda}} \le C_\varepsilon e^{(1+\varepsilon)(1+a^2)A/T} I_A(T,\lambda). \tag{12}$$

Choosing $\varepsilon > 0$ small enough yields $A' \ge (1+\varepsilon)(1+a^2)A$ and $B \ge (1+\varepsilon)\sqrt{2A}\,b(A')$. Hence (12), (10) and (11) complete the proof of the lemma. $\square$

(iv) For negative s, since $L^2(M)$ is continuously embedded in $H^s(M)$, (1) implies (5) with $A_s = A$ by density. Let $s > 0$ from now on. Since $e^{t\Delta}$ is an analytic semigroup, it satisfies the smoothing property: $S_s := \sup_{t>0} \|t^s e^{t\Delta}\|_{\mathcal{L}(L^2;H^s)} < \infty$. Let $K(T)$ and $K_s(T)$ denote the optimal costs K and K_s in (1) and (5). For all $\varepsilon \in (0,1)$ and T, $K_s(T) \le S_s(\varepsilon T)^{-s} K((1-\varepsilon)T)$. Since ε is arbitrarily small, for all $A_s > A$, there is a C'_s such that for any T_0, if $K(T) \le Ce^{A/T}$ for all $T \le T_0$ then $K_s(T) \le C'_s e^{A_s/T}$ for all $T \le T_0$. Therefore, with $C_s = C'_s e^{A_s/T_0}$, if (1) holds for all $T \le T_0$, then (5) holds for all T.

Converse. The last statement of Theorem 1 results from the following lemma.

LEMMA 8. *For all $A > \tilde{A}$, there is a $C > 0$ such that*

$$\forall T,\ \forall u_0, \qquad \int_M |e^{T\Delta}u_0|^2\,dx \le Ce^{A/T} \int_0^T \int_M e^{-\tilde{A}/t} |e^{t\Delta}u_0|^2\,dx\,dt.$$

PROOF. Let $\varepsilon \in (0,1)$. Bounding $\int_0^T$ from below by $\int_{(1-\varepsilon)T}^T$ yields

$$\forall u_0, \qquad \int_0^T \int_M e^{-\tilde{A}/t} |e^{t\Delta}u_0|^2\,dx\,dt \ge (1-\varepsilon)Te^{-\tilde{A}/((1-\varepsilon)T)} \int_M |e^{T\Delta}u_0|^2\,dx,$$

since $t \mapsto e^{-A/t}$ does not decrease and $t \mapsto \|e^{t\Delta}u_0\|_{L^2(M)}$ does not increase. Since ε is arbitrarily small, this completes the proof of the lemma. $\square$

APPENDIX A. REACHABILITY RESULTS RELATED TO THE RATE A IN (1)

A.1. *Reachability set*

The control system corresponding to the observability inequality (1) is

$$\partial_t u - \Delta u = f \quad \text{on } \mathbb{R}_+ \times M, \qquad u(0) = u_0 \in L^2(M),\ f \in L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2_\Omega(M)),$$

where $L^2_\Omega(M)$ denotes the functions in $L^2(M)$ which are zero outside Ω (hence the heat source f is located in the observation-control region Ω). Each input function f defines a unique continuous trajectory $t \mapsto u(t)$ in the state space $L^2(M)$ from the initial state u_0. As f varies, $u(T)$ spans the set of states which are reachable from u_0 in time T, denoted $\mathcal{R}_{T,u_0}(\Omega)$.

As recalled in the introduction, null-controllability in time T holds for this system, i.e. for any u_0, there is an input f steering u_0 to $u(T)=0$. By linearity, the final state $e^{T\Delta}u_0$ is reached from $u(0)=0$ when $-f$ is applied. The simple argument[7] in [Sei79] proves that null-controllability in time T implies $\mathcal{R}_{T,0}(\Omega)=\mathcal{R}_{t,u_0}(\Omega)$ for all u_0 and $t\geq T$. Since null-controllability in time T holds for all T, $\mathcal{R}_{T,u_0}(\Omega)$ does not depend on T and u_0. It is therefore natural to define the reachability set as follows:

DEFINITION 1. *The state $\tilde{u}$ in $L^2(M)$ is in the* reachability set $\mathcal{R}(\Omega)$ *if there is a time T and an input f in $L^2(0,T;L^2_\Omega(M))$ such that: $\tilde{u}=\int_0^T e^{t\Delta}f(T-t)\,dt$.*

The null-reachability cost *in time T of a nonzero $\tilde{u}$ in $\mathcal{R}(\Omega)$ is* $\inf\|f\|^2/\|\tilde{u}\|^2$ *over all such f.*

The fact that exact controllability does not hold, i.e. $L^2(M)\not\subseteq\mathcal{R}(\Omega)$, is often deduced from the hypoellipticity of the heat operator $\partial_t-\Delta$: outside $(0,T)\times\Omega$, the smoothness of the null source implies the smoothness of u, hence all states in $\mathcal{R}(\Omega)$ are smooth outside Ω. We further remark that:

LEMMA 9. (i) $\mathcal{R}(M)=H^1_0(M)$.

(ii) *For any open Ω' such that $\overline{\Omega'}\subset\Omega$ and any $u'\in H^1(\Omega')$, there is a $\tilde{u}$ in $\mathcal{R}(\Omega)$ such that $\tilde{u}-u'$ is smooth on Ω'. (When Ω' is smooth, the same holds with $\Omega'=\Omega$ and $u'\in H^1_0(\Omega)$.)*

(iii) *If $\Omega\neq\Omega'$, then $\mathcal{R}(\Omega)\neq\mathcal{R}(\Omega')$.*

PROOF. (i) We use expansions in the eigenbasis (7): $f(t)=\sum_k f_k(t)e_k$ and $u(T)=\sum_k u_k(T)e_k$. Setting $g_k(t)=e^{-t\omega_k^2}$, $u(T)=\int_0^T e^{t\Delta}f(T-t)\,dt$ is equivalent to $u_k(T)=\int_0^T g_k(t)f_k(T-t)dt$ for all k. The norm of g_k in $L^2(0,T)$ satisfies

$$\text{(A1)}\qquad \frac{\omega_0}{\omega_k}\|g_0\|\leq\|g_k\|=\frac{(1-e^{-2T\omega_k^2})^{1/2}}{\sqrt{2T}\,\omega_k}\leq\frac{1}{\sqrt{2T}\,\omega_k}.$$

The upper bound in (A1) implies

$$\|\nabla u(T)\|^2=\|\sqrt{-\Delta}\,u(T)\|^2=\sum_k|\omega_k u_k(T)|^2\leq\frac{1}{2T}\sum_k\|f_k\|^2=\frac{1}{2T}\|f\|^2<\infty.$$

Hence $\mathcal{R}(M)\subset H^1_0(M)$. Conversely, let $\tilde{u}=\sum_k\tilde{u}_k e_k$ be in $H^1_0(M)$ and set $f_k(t)=g_k(T-t)\|g_k\|^{-2}\tilde{u}_k e_k$. The lower bound in (A1) implies

$$\|f\|^2=\sum_k\|f_k\|^2\leq\frac{1}{\omega_0^2\|g_0\|^2}\sum_k|\omega_k\tilde{u}_k|^2=\frac{\|\nabla\tilde{u}\|}{\omega_0^2\|g_0\|^2}<\infty.$$

Since $u(T)=\tilde{u}$, this proves $\mathcal{R}(M)\supset H^1_0(M)$.

[7] Let $u_0\in L^2(M)$ and $t\geq T$. Since null-controllability in time T holds, there is an input f_0 equal to 0 on $[0,(t-T)]$ which steers u_0 to 0 between 0 and t. If $\tilde{u}\in\mathcal{R}_{T,0}(\Omega)$, then there is an input f equal to 0 on $[0,(t-T)]$ which steers 0 to $\tilde{u}$ between 0 and t, hence $f+f_0$ steers u_0 to $\tilde{u}$ between 0 and t, which proves $\tilde{u}\in\mathcal{R}_{t,u_0}(\Omega)$. Conversely, if f steers u_0 to $\tilde{u}$ between 0 and t, then, since null-controllability in time T holds, there is an f_0 equal to f on $[0,(t-T)]$ steering u_0 to 0 between 0 and t, hence $f-f_0$ steers 0 to $\tilde{u}$ between 0 and t, which proves $\tilde{u}\in\mathcal{R}_{T,0}(\Omega)$.

(ii) Let M' be a smooth open set such that $\overline{\Omega'} \subset M'$ and $\overline{M'} \subset \Omega$, and let $u \in H_0^1(M')$. The previous point implies that there is an input $f \in L^2(0, T; L^2(M'))$ which steers 0 to u on the manifold M'. The extension $\tilde{f}$ of f to M by zero outside M' steers 0 to some final state $\tilde{u} \in \mathcal{R}(M') \subset \mathcal{R}(\Omega)$ on the manifold M. Since $(\partial_t - \Delta)(\tilde{u} - u) = \tilde{f} - f = 0$ on M', $\tilde{u} - u$ is smooth on $M' \supset \Omega$ by hypoellipticity.

(iii) This last point results from the previous point and the already mentioned fact that functions of $\mathcal{R}(\Omega)$ are smooth outside Ω. $\square$

We already mentioned that the final time observability estimate implies null-controllability by the duality argument in [DR77], yielding in turn some information on the reachability set: $\bigcup_{t>0} e^{t\Delta}(L^2(M)) \subset \mathcal{R}(\Omega)$. The sharp observability estimate (4) improves this to $e^{B\sqrt{-\Delta}}(L^2(M)) \subset \mathcal{R}(\Omega)$ by the same argument (cf. (3.22) in [DR77]). Thus, Theorem 4 and the third point in Theorem 1 prove:

COROLLARY 10. *If* (1) *holds for all* $T \leq T_0$, *then* $\bigcup_{B>\sqrt{2A}} e^{B\sqrt{-\Delta}}(L^2(M)) \subset \mathcal{R}(\Omega)$. *Let* L_Ω *be the length of the longest generalized geodesic in* $\overline{M}$ *which does not intersect* Ω. *For all* $B > 2\sqrt{2}(36/37)L_\Omega$, $e^{B\sqrt{-\Delta}}(L^2(M)) \subset \mathcal{R}(\Omega)$.

When M is a segment of length L controlled from one endpoint (cf. Section 3.2), [FR71][8] proves that the reachability set includes $e^{B\sqrt{-\Delta}}(L^2(M))$ for all $B > L$ (this improves Corollary 10, since the analogue of L_Ω is $2L$ here). This result raises the question whether "the optimal" rate B such that $e^{B\sqrt{-\Delta}}(L^2(M)) \subset \mathcal{R}(\Omega)$ can be expressed geometrically (e.g. is it $\sup_{y\in M} \operatorname{dist}(y, \overline{\Omega})$?). More generally, although Lemma 9 proves that $\mathcal{R}(\Omega)$ does depend on Ω, this dependence has not been investigated yet, to my best knowledge.

A.2. *Null-reachability cost*

We already mentioned that any $\tilde{u} = e^{T\Delta}u_0$ is in $\mathcal{R}(\Omega)$ (cf. Definition 1). Indeed, the duality in [DR77] proves more: the best K such that (1) holds is also the best K such that for all u_0, there is an input f in $L^2(0, T; L^2_\Omega(M))$ with $e^{T\Delta}u_0 = \int_0^T e^{t\Delta} f(T-t)\, dt$ and $\|f\|^2 \leq K\|u_0\|^2$. But this is not enough to estimate the null-reachability cost uniformly over $e^{T\Delta}(L^2(M))$.

For any positive frequency threshold μ, the linear span of $\{e_k\}_{\omega_k \leq \mu}$ defined by (7), denoted $\mathcal{S}_{\sqrt{-\Delta}\leq\mu}$, is a finite-dimensional subspace of $e^{T\Delta}(L^2(M))$. In the following proposition, a uniform estimate of the null-reachability cost over $\mathcal{S}_{\sqrt{-\Delta}\leq\mu}$ is deduced from the exponential observability estimate (1). This estimate is exponential with respect to μ with an explicit rate (essentially $2\sqrt{A}$).

LEMMA 11. *If* (1) *holds for any* T, *then there is a* $C' > 0$ *such that, for all* $\mu > 0$ *and* T, *the null-reachability cost of any* $\tilde{u}$ *in* $\mathcal{S}_{\sqrt{-\Delta}\leq\mu}$ *in time* T *(cf. Definition* 1*) is not greater than* $C'e^{A'(\mu,T)}$ *with* $A'(\mu, T) \leq (A + \sqrt{A}\,\mu)(2 + 1/T)$.

[8] In [FR71], we refer to (3.19) rather than Theorem 3.3 where the analogous (3.23) contains a misprint (L should be replaced by π). N.b. (3.20) in [FR71] proves that $e^{B\sqrt{-\Delta}}(L^2(M)) \subset \mathcal{R}(\Omega)$ cannot be proved by the same method for $B < L$. This is an indication that L is "the optimal" rate B for which it holds.

In particular, $A'(\mu,T)\le A^*(2+1/T)\mu$, *where* $A^*=\sqrt{A}(1+\sqrt{A}/\omega_0)$.
More precisely, for all $\varepsilon>0$, $A'(\mu,T)\le\sqrt{A}\max\{\mu,\sqrt{A}\}(\varepsilon+1/\min\{T,\varepsilon\})$.

N.b. the second bound, which is linear in μ for all μ, is easily deduced from the previous one.[9] The proof also shows that $A'(\mu,1)\le 2\sqrt{A}\,\mu$ for $\mu\ge\sqrt{A}$ and $T=1$.

PROOF. Since the cost does not increase with T, it is enough to prove that the cost is not greater than $C'e^{A'(\mu,T)}$ with $A'(\mu,T)\le\sqrt{A}\max\{\mu,\sqrt{A}\}(T+1/T)$ for any T (applying this with $T=\varepsilon$ yields the precise estimate in the lemma). Since $\tilde{u}\in\mathcal{S}_{\sqrt{-\Delta}\le\mu}$, the backward estimate $\|e^{-T\Delta}\tilde{u}\|\le e^{T\mu^2}\|\tilde{u}\|$ holds. The remark beginning Section A.2 and (1) imply that the null-reachability cost of $\tilde{u}$ in time T is not greater than $K\|e^{-T\Delta}\tilde{u}\|/\|\tilde{u}\|\le Ce^{\mu^2T+A/T}$. Thus

$$\forall\tilde{u}\in\mathcal{S}_{\sqrt{-\Delta}\le\mu}\setminus\{0\},\ \forall T,\qquad A'(\mu,T)\le\mu^2T+A/T. \tag{A2}$$

If $\mu\le\sqrt{A}$, then this yields $A'(\mu,T)\le\mu^2T+A/T\le\sqrt{A}\,\mu T+A/T$. If $\mu\ge\sqrt{A}$, then $T\ge T':=T\sqrt{A}/\mu$ and we may use an input function which is zero on $(0,T-T')$ and estimate it on $(T-T',T)$ with (A2) so that $A'(\mu,T)\le\mu^2T'+A/T'=\sqrt{A}\mu(T+1/T)$. In both cases $A'(\mu,T)\le\sqrt{A}\max\{\mu,\sqrt{A}\}(T+1/T)$. □

In the following lemma, a nonlinear initial state observability estimate with explicit rate is also deduced from the final state exponential observability estimate (1). Such "logarithmic observability estimates" have been proved in [Phu04] in the Euclidean case with a bounded potential depending on time and space (with explicit dependence on the norm of the potential but nonexplicit rates).

LEMMA 12. *If* (1) *holds for any* T, *then there is a* $C'>0$ *such that*

$$\forall T,\ \forall u_0\in H_0^1(\Omega)\setminus\{0\},\qquad \int_M|u_0|^2\,dx\le C'e^{A'(\sqrt{2}F(u_0),T)}\int_0^T\int_\Omega|e^{t\Delta}u_0|^2\,dx\,dt,$$

where $F(u_0)=\|\nabla u_0\|/\|u_0\|$ *and* A' *is as in Lemma* 11.[10]

PROOF. As in Lemma 11, the proof reduces to a backward uniqueness estimate

$$\forall T,\ \forall u_0\in H_0^1(\Omega)\setminus\{0\},\qquad \|u_0\|\le e^{TF(u_0)^2}\|e^{T\Delta}u_0\|. \tag{A3}$$

This follows from the well-known log-convexity method (cf. [AN67]). Let $u(t)=e^{t\Delta}u_0$ and $e(t)=\int_M|u(t)|^2\,dx$. Integrating by parts yields $e'(t)=-2\int_M|\nabla u(t)|^2\,dx$ and $e''(t)=4\int_M|\Delta u(t)|^2\,dx$. Hence $(e'(t))^2=(2\int_M u(t)\Delta u(t)\,dx)^2\le e(t)e''(t)$. Introducing the function f defined by $f(t)=\ln e(t)$, this reads $f''(t)\ge 0$. Since f is convex, it satisfies $f(T)\ge f(0)+Tf'(0)$. Since $f'(0)=-2F(u_0)$, exponentiating this inequality yields (A3), which completes the proof of the lemma. □

[9] N.b. $\lambda^*=\omega_0^2$ is the smallest eigenvalue of $-\Delta$ often called the fundamental tone of M.
[10] F is often called the frequency function since $F(e_k)=\omega_k$ with the notations in (7).

A.3. *Approximate controllability cost*

Null-controllability implies approximate controllability, i.e. $\mathcal{R}(\Omega)$ is dense in $L^2(M)$, since $e^{T\Delta}(L^2(M))$ is dense in $L^2(M)$.

DEFINITION 2. *For any $\varepsilon > 0$ and $\tilde{u} \in L^2(M)$ such that $\|\tilde{u}\| = 1$, the* approximate null-reachability cost *is the smallest constant $K_{T,\varepsilon}(\tilde{u})$ such that there is an input function f in $L^2(0,T; L^2_\Omega(M))$ satisfying $\|f\|^2 \leq K_{T,\varepsilon}(\tilde{u})$ and $\|u(T) - \tilde{u}\| \leq \varepsilon$ where $u(T) = \int_0^T e^{t\Delta} f(T-t)\,dt$ is the final state reached with f from the null initial state.*

N.b. replacing the inequalities in this definition by $\|f\|^2 \leq K_{T,\varepsilon}(\tilde{u})\|\tilde{u}\|^2$ and $\|u(T) - \tilde{u}\| \leq \varepsilon\|\tilde{u}\|$, the normalization $\|\tilde{u}\| = 1$ can be dispensed with, thanks to linearity. By a weak compactness argument as ε tends to zero, $K_{T,\varepsilon}(\tilde{u})$ is not bounded for fixed T since exact controllability does not hold. Indeed, the following stronger statement holds.

LEMMA 13. *For any T and $\varepsilon \in (0,1)$, the approximate null-reachability cost $K_{T,\varepsilon}(\tilde{u})$ in Definition 2 is not bounded with respect to $\tilde{u}$.*

PROOF. By integration by parts and a density argument, for all $v_0 \in L^2(M)$,

$$\int_0^T \int_\Omega f e^{t\Delta} v_0\,dx\,dt = \int_M u(T) v_0\,dx = \int_M \tilde{u} v_0\,dx + \int_M (u(T) - \tilde{u}) v_0\,dx.$$

By the Cauchy–Schwarz inequality and the inequalities satisfied by f, this implies

$$\left(\int_M \tilde{u} v_0\,dx - \varepsilon\|v_0\|\right)^2 \leq K_{T,\varepsilon}(\tilde{u}) \int_0^T \int_\Omega |e^{t\Delta} v_0|^2\,dx\,dt.$$

For $v_0 = \tilde{u}$, the left hand side equals $(1-\varepsilon)^2$. Hence, by linearity, for $\varepsilon \in (0,1)$,

$$\forall \tilde{u} \neq 0, \quad \int_M |\tilde{u}|^2\,dx \leq \frac{K_{T,\varepsilon}(\tilde{u}/\|\tilde{u}\|)}{(1-\varepsilon)^2} \int_0^T \int_\Omega |e^{t\Delta}\tilde{u}|\,dx\,dt.$$

If $K_{T,\varepsilon}(\tilde{u})$ did not depend on $\tilde{u}$, this would be an initial time observability inequality equivalent, by the duality in [DR77], to exact controllability. This proves the lemma, since exact controllability does not hold for any T (cf. Lemma 9). □

The next theorem generalizes the estimate of the approximate null-reachability cost proved in Theorem 6.1 of [FCZ00] in the Euclidean case, with $p = 1$, without explicit rate. For fixed $\|(-\Delta)^p\tilde{u}\|^2 = \int_M |(-\Delta)^p\tilde{u}(x)|^2\,dx$, the dependence on ε of this estimate is optimal according to Theorem 6.2 in [FCZ00].[11]

THEOREM 14. *If* (1) *holds for any T, then there is a $C' > 0$ such that, for all $p > 0$ and $\tilde{u}$ in* $\mathrm{D}((-\Delta)^p)$*, for all T and $\varepsilon > 0$, the cost in Definition* 2 *satisfies*

$$K_{T,\varepsilon}(\tilde{u}) \leq C' \exp A'((\|(-\Delta)^p\tilde{u}\|/\varepsilon)^{1/(2p)}, T) \leq C' e^{A^*(2+1/T)(\|(-\Delta)^p\tilde{u}\|/\varepsilon)^{1/2p}},$$

where the function $A'(\mu, T)$ and the rate A^* are as in Lemma 11.

[11] N.b. for $p \in (1/4, 1]$, $\mathrm{D}((-\Delta)^p) = H^{2p}(M) \cap H^1_0(M)$ and $u \mapsto \|(-\Delta)^p u\|$ defines a norm which is equivalent to the norm in $H^{2p}(M)$.

PROOF. Define $g : \mathbb{R}_+ \to \mathbb{R}_+$ by $g(\lambda) = 1 - \lambda$ for $\lambda \leq 1$ and $g(\lambda) = 0$ elsewhere. For any $\mu > 0$, since $g(\lambda^p/\mu^{2p}) = 0$ for $\sqrt{\lambda} \geq \mu$, we have $g((-\Delta)^p/\mu^{2p})\tilde{u} \in \mathcal{S}_{\sqrt{-\Delta}\leq\mu}$. According to Lemma 11, there is an input f in $L^2(0, T; L^2_\Omega(M))$ steering 0 to $u(T) = g((-\Delta)^p/\mu^{2p})\tilde{u}$ at cost $\|f\|^2 \leq C'e^{A'(\mu,T)}\|u(T)\|^2$. Since $0 \leq 1 - g(\lambda) \leq \lambda$, it follows that $\|\tilde{u} - g((-\Delta)^p/\mu^{2p})\tilde{u}\| \leq \|(-\Delta)^p\tilde{u}/\mu^{2p}\|$. By choosing $\mu = (\|(-\Delta)^p\tilde{u}\|/\varepsilon)^{1/(2p)}$, this inequality becomes $\|\tilde{u} - u(T)\| \leq \varepsilon$, which completes the proof of the theorem. $\square$

ACKNOWLEDGMENTS. I am grateful to Enrique Zuazua for raising my attention to this problem in [Zua01, Zua06]. More broadly, I pay a tribute to his always inspiring surveys.

This article and [Mil05b, Mil06, Mil05d] were written while the author was exempt from teaching duties thanks to the CNRS – "accueil en délégation".

REFERENCES

[AL03a] G. AVALOS - I. LASIECKA, *Mechanical and thermal null controllability of thermoelastic plates and singularity of the associated minimal energy function*. Control Cybernet. 32 (2003), 473–490.

[AL03b] G. AVALOS - I. LASIECKA, *Optimal blowup rates for the minimal energy null control of the strongly damped abstract wave equation*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 2 (2003), 601–616.

[AN67] S. AGMON - L. NIRENBERG, *Lower bounds and uniqueness theorems for solutions of differential equations in a Hilbert space*. Comm. Pure Appl. Math. 20 (1967), 207–229.

[BLR92] C. BARDOS - G. LEBEAU - J. RAUCH, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*. SIAM J. Control Optim. 30 (1992), 1024–1065.

[Câr93] O. CÂRJĂ, *The minimal time function in infinite dimensions*. SIAM J. Control Optim. 31 (1993), 1103–1114.

[CGT82] J. CHEEGER - M. GROMOV - M. TAYLOR, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*. J. Differential Geom. 17 (1982), 15–53.

[DP01] G. DA PRATO, *An introduction to infinite dimensional analysis*. Appunti dei Corsi Tenuti da Docenti della Scuola, Scuola Normale Superiore, Pisa, 2001.

[DPZ02] G. DA PRATO - J. ZABCZYK, *Second Order Partial Differential Equations in Hilbert Spaces*. London Math. Soc. Lecture Note Ser. 293, Cambridge Univ. Press, 2002.

[DR77] S. DOLECKI - D. L. RUSSELL, *A general theory of observation and control*. SIAM J. Control Optim. 15 (1977), 185–220.

[Èma95] O. YU. ÈMANUILOV, *Controllability of parabolic equations*. Mat. Sb. 186 (1995), 109–132 (in Russian).

[FCZ00] E. FERNÁNDEZ-CARA - E. ZUAZUA, *The cost of approximate controllability for heat equations: the linear case* Adv. Differential Equations 5 (2000), 465–514.

[FCZ02] E. FERNÁNDEZ-CARA - E. ZUAZUA, *On the null controllability of the one-dimensional heat equation with non-smooth coefficients*. Comput. Appl. Math. 21 (2002), 167–190.

[FR71] H. O. FATTORINI - D. L. RUSSELL, *Exact controllability theorems for linear parabolic equations in one space dimension*. Arch. Ration. Mech. Anal. 43 (1971), 272–292.

[GL99] F. GOZZI - P. LORETI, *Regularity of the minimum time function and minimum energy problems: the linear case*. SIAM J. Control Optim. 37 (1999), 1195–1221.

[LR95] G. LEBEAU - L. ROBBIANO, *Contrôle exact de l'équation de la chaleur*. Comm. Partial Differential Equations 20 (1995), 335–356.

[LZ98] G. LEBEAU - E. ZUAZUA, *Null-controllability of a system of linear thermoelasticity*. Arch. Ration. Mech. Anal. 141 (1998), 297–329.

[Mil02] L. MILLER, *Escape function conditions for the observation, control, and stabilization of the wave equation*. SIAM J. Control Optim. 41 (2002), 1554–1566.

[Mil04a] L. MILLER, *The control transmutation method and the cost of fast controls*. SIAM J. Control Optim., to appear, arXiv:math.OC/0310358 preprint, 2004.

[Mil04b] L. MILLER, *Geometric bounds on the growth rate of null-controllability cost for the heat equation in small time*. J. Differential Equations 204 (2004), 202–226.

[Mil04c] L. MILLER, *How violent are fast controls for Schrödinger and plates vibrations?*. Arch. Ration. Mech. Anal. 172 (2004), 429–456.

[Mil05a] L. MILLER, *Controllability cost of conservative systems: resolvent condition and transmutation*. J. Funct. Anal. 218 (2005), 425–444.

[Mil05b] L. MILLER, *On the controllability of anomalous diffusions generated by the fractional Laplacian*. Math. Control Signals Systems, to appear, //hal.ccsd.cnrs.fr/ccsd-00008809, preprint, 2005.

[Mil05c] L. MILLER, *On the controllability of the heat equation in unbounded domains*. Bull. Sci. Math. 129 (2005), 175–185.

[Mil05d] L. MILLER, *On the cost of fast controls for thermoelastic plates*. //hal.ccsd.cnrs.fr/ccsd-00068757, preprint, 2005.

[Mil06] L. MILLER, *The cost of fast non-structural controls for a linear elastic system with structural damping*. J. Funct. Anal. 236 (2006), 592–608.

[Nor97] J. R. NORRIS, *Heat kernel asymptotics and the distance function in Lipschitz Riemannian manifolds*. Acta Math. 179 (1997), 79–103.

[Phu01] K.-D. PHUNG, *Observability and control of Schrödinger equations*. SIAM J. Control Optim. 40 (2001), 211–230.

[Phu04] K.-D. PHUNG, *Note on the cost of the approximate controllability for the heat equation with potential* J. Math. Anal. Appl. 295 (2004), 527–538.

[Rus73] D. L. RUSSELL, *A unified boundary controllability theory for hyperbolic and parabolic partial differential equations*. Stud. Appl. Math. 52 (1973), 189–211.

[RW94] D. L. RUSSELL - G. WEISS, *A general necessary condition for exact observability* SIAM J. Control Optim. 32 (1994), 1–23.

[SAI00] T. I. SEIDMAN - S. A. AVDONIN - S. A. IVANOV, *The "window problem" for series of complex exponentials*. J. Fourier Anal. Appl. 6 (2000), 233–254.

[Sei79] T. I. SEIDMAN, *Time-invariance of the reachable set for linear control problems*. J. Math. Anal. Appl. 72 (1979), 17–20.

[Sei84] T. I. SEIDMAN, *Two results on exact boundary control of parabolic equations*. Appl. Math. Optim. 11 (1984), 145–152.

[Sei88] T. I. SEIDMAN, *How violent are fast controls?*. Math. Control Signals Systems 1 (1988), 89–95.

[Sei05] T. I. SEIDMAN, *On uniform null controllability and blowup estimates*. In: Control Theory of Partial Differential Equations, Lecture Notes Pure Appl. Math. 242, Chapman & Hall/CRC, 2005, 213–227.

[Tri03] R. TRIGGIANI, *Optimal estimates of norms of fast controls in exact null controllability of two non-classical abstract parabolic systems*. Adv. Differential Equations 8 (2003), 189–229.

[Var67] S. R. S. VARADHAN, *On the behavior of the fundamental solution of the heat equation with variable coefficients*. Comm. Pure Appl. Math. 20 (1967), 431–455.

[Zua01] E. ZUAZUA, *Some results and open problems on the controllability of linear and semilinear heat equations*. In: Carleman Estimates and Applications to Uniqueness and Control Theory (Cortona, 1999), F. Colombini and C. Zuily (eds.), Progr. Nonlinear Differential Equations Appl. 46, Birkhäuser Boston, Boston, MA, 2001, 191–211.

[Zua06] E. ZUAZUA, *Controllability and observability of partial differential equations: Some results and open problems*. In: Evolutionary Equations, Vol. III, C. M. Dafermos and E. Feireisl (eds.), Handb. Differ. Equ., North-Holland, Amsterdam, 2006, //www.uam.es/personal_pdi/ciencias/ezuazua/informweb/survey.pdf, preprint.

Received 30 January 2006,
and in revised form 15 May 2006.

MODAL'X, EA 3454, Bât. G
Université Paris 10 – Nanterre
200 Av. de la République
2001 Nanterre Cedex, France

Centre de Mathématiques Laurent Schwartz
UMR CNRS 7640
École Polytechnique
91128 Palaiseau Cedex, France
E-mail: Luc.Miller@math.polytechnique.fr