*Rend. Lincei Mat. Appl. 17 (2006), 367*–375

**Partial differential equations.** — *Some inequalities of Glaeser–Bronšteĭn type*, by SERGIO SPAGNOLO and GIOVANNI TAGLIALATELA, communicated on 12 May 2006.

ABSTRACT. — The classical Glaeser estimate is a special case of the Bronšteĭn lemma which states the Lipschitz continuity of the roots $\lambda_j(x)$ of a hyperbolic polynomial $P(x, X)$ with coefficients $a_j(x)$ depending on a real parameter. Here we prove a pointwise estimate for higher order derivatives of the $a_j(x)$'s in terms of certain nonnegative functions which are symmetric polynomials of the roots $\lambda_j(x)$ (hence also of the coefficients $a_j(x)$). These inequalities are very helpful in the study of the Cauchy problem for homogeneous weakly hyperbolic equations of higher order.

KEY WORDS: Hyperbolic polynomials; weakly hyperbolic equations; Cauchy problem.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 26C05, 35L30.

## INTRODUCTION

The simplest version of the Glaeser inequality ([G], [Di]) states that

$$|a'(x)| \le C(M)\sqrt{a(x)}, \qquad \forall x \in \mathbb{R}, \tag{1}$$

for every nonnegative function $a \in \mathcal{C}^2(\mathbb{R})$ with $\|a''\|_{L^\infty(\mathbb{R})} \le M$. This inequality holds, in fact, with $C(M) = \sqrt{2M}$, and follows from the Taylor expansion

$$0 \le a(x+h) = a(x) + a'(x)h + a''(\xi)h^2/2 \le a(x) + a'(x)h + Mh^2/2,$$

by noting that the discriminant of the polynomial on the right hand side must be nonnegative.

We can reformulate (1) by saying that the square root of a nonnegative function with second derivative bounded is Lipschitz continuous, or, equivalently, the roots of the polynomial $P(x, X) = X^2 - a(x)$ are Lipschitz continuous functions of the parameter $x \in \mathbb{R}$. Hence, (1) can be viewed as a special case of the following general result on *hyperbolic polynomials* (i.e., monic polynomials having only real roots) depending on a real parameter.

BRONŠTEĬN'S LEMMA ([B1], [M], [T], [W]). *Let*

$$\lambda_1(x) \le \cdots \le \lambda_m(x) \tag{2}$$

*be the roots of a hyperbolic polynomial*

$$P(x, X) = \sum_{j=0}^{m} a_j(x) X^{m-j} = \prod_{j=1}^{m} (X - \lambda_j(x)), \qquad a_0 \equiv 1. \tag{3}$$

*Assume that the coefficients $a_j(x)$ belong to $\mathcal{C}^m(\mathbb{R})$ and satisfy*

$$\|a_j^{(k)}\|_{L^\infty(\mathbb{R})} \le M < \infty, \qquad j, k = 0, \dots, m. \tag{4}$$

*Then each root $\lambda_j(x)$ is a Lipschitz continuous function with*

$$\|\lambda_j'(x)\| \le C(m, M), \quad \text{a.e. in } \mathbb{R}, \quad j = 1, \dots, m. \tag{5}$$

The inequality (1) has been extended in various directions (e.g., [Da], [NS]). In particular Oleĭnik [O1] proved that any symmetric $n \times n$ matrix $A(x) \ge 0$, with $\|A''\| \le M$, satisfies the pointwise estimate

$$|\mathrm{Tr}(A'(x)B)| \le C(n, M)\sqrt{\mathrm{Tr}(B^*A(x)B)}, \quad \forall n \times n \text{ matrix } B, \tag{6}$$

which is essentially equivalent to saying that the square root of $A(x)$ is a Lipschitz continuous matrix function of $x$ (cf. [LV]). The estimate (6) is a key point in the proof of an important result of well-posedness for second order weakly hyperbolic equations in $n$ space variables ([O2]).

When considering the Cauchy problem for a homogeneous weakly hyperbolic equation of higher order in one space variable, we need a pointwise estimate (like (1)) of higher order derivatives of the coefficients $a_j(x)$ of a hyperbolic polynomial of the type (**??**). In this regard we make the obvious remark that, if $a(x) \ge 0$ is a smooth function, then an estimate like $|a''(x)| \le Ca(x)^\delta$ with $\delta > 0$ is in general false.

However, the higher order derivatives of $a_j(x)$ can be majorized by the nonnegative functions

$$\psi_k(x) = \sum_{1 \le j_1 < \dots < j_k \le m} \lambda_{j_1}^2(x) \cdots \lambda_{j_k}^2(x), \quad 1 \le k \le m, \quad \psi_0 \equiv 1. \tag{7}$$

More precisely we prove:

THEOREM 1. *Given a hyperbolic polynomial* (**??**) *with coefficients $a_j \in \mathcal{C}^m(\mathbb{R})$ satisfying* (4), *we have, for all $x \in \mathbb{R}$,*

$$|a_j^{(k)}(x)| \le C(m, M)\sqrt{\psi_{j-k}(x)}, \quad 0 \le k \le j \le m. \tag{8}$$

The proof will be given in §2, while in §1 we shall write (8) in more explicit forms in the case of hyperbolic polynomials of order $m \le 4$.

The estimates (8) provide a useful tool in studying the possible well-posedness for weakly hyperbolic equations of order $m$. For instance, let us consider the model equation, in $\mathbb{R}_t \times \mathbb{R}_x$,

$$\partial_t^2 u - a(x)\partial_x^2 u = 0, \quad a(x) \ge 0, \tag{9}$$

and define the energy

$$E(t) = \frac{1}{2}\int_{\mathbb{R}} (u_t^2 + a(x)u_x^2)\, dx.$$

We compute, by differentiating in time and integrating by parts,

$$E'(t) = -\int_{\mathbb{R}} a'(x) u_x u_t \, dx,$$

so, if $|a''(x)|$ is bounded, (1) yields an a priori estimate $E'(t) \leq CE(t)$ ensuring the well-posedness of the Cauchy problem for (9).

In a similar way Oleĭnik [O2] proved the well-posedness for any second order equation, in $n$ space variables, of the form

$$\partial_t^2 u - \sum_{i,j=1}^{n} a_{ij}(x) \partial_{x_i} \partial_{x_j} u = 0, \qquad \sum a_{ij}(x) \xi_i \xi_j \geq 0.$$

The energy is now

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^n} \Big( u_t^2 + \sum a_{ij}(x) u_{x_i} u_{x_j} \Big) \, dx,$$

hence, in view of an estimate $E'(t) \leq CE(t)$, we resort to the inequality (6) with $A(x) = [a_{ij}(x)]$ and $B \equiv B(x) = [u_{x_i}(x) u_{x_j}(x)]$.

Going back to the case of coefficients depending on a single variable $x \in \mathbb{R}$, we consider the Cauchy problem for a hyperbolic, homogeneous equation of general type, i.e.,

$$\partial_t^m u + a_1(x) \partial_t^{m-1} \partial_x u + \cdots + a_m(x) \partial_x^m u = f(t, x), \tag{10}$$

$$\partial_t^j u(0, x) = \varphi_j(x), \qquad 0 \leq j \leq m-1. \tag{11}$$

In this case we are still able to find an energy $E(t)$ explicitly expressed in terms of the coefficients $a_j(x)$, and we can try to estimate $E'(t)$ in terms of $E(t)$ by resorting to (8). But now the presence of mixed time-space derivatives (note that (9) is the unique equation of type (10) without mixed derivatives) forces us to make some additional assumptions on the equation, besides the hyperbolicity. Indeed, it is well known that the mere hyperbolicity is unable to ensure the $\mathcal{C}^\infty$ well-posedness of (10)–(11); for example we do not have well-posedness for the equation

$$\partial_t^2 u + 2x \partial_t \partial_x u + x^2 \partial_x^2 u = 0.$$

Incidentally, we recall that the case of second order equations

$$\partial_t^2 u - a(x) \partial_x^2 u + b(x) \partial_t \partial_x u = 0, \qquad \Delta(x) = b^2(x) + 4a(x) \geq 0, \tag{12}$$

is well understood thanks to Nishitani [N]. In particular we know that the assumption

$$b^2(x) \leq M \Delta(x) \tag{13}$$

is a sufficient condition for the well-posedness of (12).

On the other hand, in the study of the equation (10) with $m \geq 3$, we run into various difficulties of algebraic nature, which we can partly overcome by the technique of *quasi-symmetrizers* (see [ST]). As a matter of fact, thanks to Theorem 1, we are able to prove:

THEOREM 2. *Consider the equation* (10) *with* $a_j \in \mathcal{C}^\infty(\mathbb{R})$. *Assume that the equation is hyperbolic and its characteristic roots satisfy the uniform estimates*

$$\lambda_i^2(x) + \lambda_j^2(x) \leq M(\lambda_i(x) - \lambda_j(x))^2, \quad 1 \leq i < j \leq m, \quad x \in \mathbb{R}. \tag{14}$$

*Then the Cauchy problem is* $\mathcal{C}^\infty$ *well-posed in a neighborhood of each point* $(0, x_0)$. *Moreover, if the* $a_j(x)$ *are uniformly bounded in* $\mathbb{R}$, *the problem is globally well-posed.*

The proof of Theorem 2 will appear in a forthcoming paper ([ST]).

REMARK 1. In the case $m = 2$, condition (14) reduces to (13). For $m \geq 2$, (14) was introduced by Colombini and Orrù [CO] for equations of type (10) with coefficients $a_j = a_j(t)$ depending only on time. We emphasize that the handling of $x$-dependent coefficients requires a quite different technique from the time dependent case.

## 1. SOME SPECIAL CASES OF THEOREM 1

Recalling the well known Vieta's identities

$$a_j = (-1)^j \sum_{1 \leq h_1 < \cdots < h_j \leq m} \lambda_{h_1} \cdots \lambda_{h_j}, \quad 1 \leq j \leq m, \tag{15}$$

and using Schwarz' inequality, we readily get (8) in the case $k = 0$. On the other hand, the case $k = 1$ is a direct consequence of Bronšteĭn's Lemma. Indeed, we have

$$|a_j'| = \Big|\sum_{l=1}^m \lambda_l' \sum_{\substack{1 \leq k_1 < \cdots < k_{j-1} \leq m \\ k_i \neq l}} \lambda_{k_1} \cdots \lambda_{k_{j-1}}\Big| \leq \Big[\sum_{l=1}^m \lambda_l'^2\Big]^{1/2} \sqrt{\psi_{j-1}}.$$

We also note that (8) is an easy consequence of (15) whenever, for some reason, the roots $\lambda_j(x)$ of (**??**) are smooth functions of class $\mathcal{C}^k$. This lucky circumstance occurs, for instance, when all the roots are simple, or more generally of constant multiplicity, so that they have the same regularity as the coefficients. Another favourable case is when the coefficients of the polynomial are analytic functions of $x \in \mathbb{R}$, since then the roots are also analytic (see [Re]). Other results on the regularity of the roots of hyperbolic polynomials can be found in [AKML] or [M] (see also [Ra] for an overview of all these results).

REMARK 2. Each of the functions $\psi_k$ in (7) can be expressed in terms of the coefficients $a_1, \ldots, a_m$. Indeed, omitting the $x$-dependence for simplicity, we have

$$P(X)P(-X) = (-1)^m \prod_{j=0}^m (X^2 - \lambda_j^2) = \sum_{k=0}^m (-1)^{m+k} \psi_k X^{2(m-k)}.$$

On the other hand, setting $a_0 = 1$, we also have

$$P(X)P(-X) = \sum_{h=0}^{2m} c_h X^{2m-h} \quad \text{with} \quad c_h = \sum_{\substack{i+j=h \\ 0 \leq i,j \leq m}} (-1)^{m-j} a_i a_j.$$

By comparing the two expressions, one concludes

$$\psi_k = \sum_{\substack{i+j=2k \\ 0\le i,j\le m}} (-1)^{k+j} a_i a_j. \tag{16}$$

REMARK 3. By (15) and (16), we can write (8) entirely in terms of the $\lambda_j$'s, or, alternatively, in terms of the $a_j$'s. In particular, for low dimensions we find:

(I) $m = 2$: For a hyperbolic polynomial $P = X^2 + a_1(x)X + a_2(x)$, we have

$$|(\lambda_1\lambda_2)'|^2 \le C(\lambda_1^2 + \lambda_2^2),$$

i.e.,

$$|a_2'(x)|^2 \le C[a_1^2(x) - 2a_2(x)].$$

(II) $m = 3$: For a hyperbolic polynomial $P = X^3 + a_1(x)X^2 + a_2(x)X + a_3(x)$, we have

$$\begin{aligned}|(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)'|^2 &\le C(\lambda_1^2 + \lambda_2^2 + \lambda_3^2),\\ |(\lambda_1\lambda_2\lambda_3)'|^2 &\le C(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2),\\ |(\lambda_1\lambda_2\lambda_3)''|^2 &\le C(\lambda_1^2 + \lambda_2^2 + \lambda_3^2),\end{aligned}$$

i.e.,

$$\begin{aligned}|a_2'(x)|^2 &\le C[a_1^2(x) - 2a_2(x)],\\ |a_3'(x)|^2 &\le C[a_2^2(x) - 2a_1(x)a_3^2(x)],\\ |a_3''(x)|^2 &\le C[a_1^2(x) - 2a_2(x)].\end{aligned}$$

(III) $m = 4$: For a hyperbolic polynomial $P = X^4 + a_1(x)X^3 + a_2(x)X^2 + a_3(x)X + a_4(x)$, we have

$$\begin{aligned}|(\lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_1 + \lambda_4\lambda_3\lambda_1 + \lambda_1\lambda_2\lambda_3)''|^2 &\le C(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2),\\ |(\lambda_1\lambda_2\lambda_3\lambda_4)''|^2 &\le C[(\lambda_1\lambda_2)^2 + (\lambda_1\lambda_3)^2 + (\lambda_1\lambda_4)^2\\ &\quad + (\lambda_2\lambda_3)^2 + (\lambda_2\lambda_4)^2 + (\lambda_3\lambda_4)^2],\\ |(\lambda_1\lambda_2\lambda_3\lambda_3)'''|^2 &\le C(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2),\end{aligned}$$

i.e.,

$$\begin{aligned}|a_3''(x)|^2 &\le C[a_1^2(x) - 2a_2(x)],\\ |a_4''(x)|^2 &\le C[a_2^2(x) - 2a_1(x)a_3^2(x) + 2a_4(x)],\\ |a_4'''(x)|^2 &\le C[a_1^2(x) - 2a_2(x)].\end{aligned}$$

As a consequence, we obtain some nice inequalities:

COROLLARY. *Let $f, g \in \mathcal{C}^3(\mathbb{R})$ be two real-valued functions with*

$$\|f\|_{\mathcal{C}^3(\mathbb{R})} + \|g\|_{\mathcal{C}^3(\mathbb{R})} \le M < \infty.$$

*Assume that, for all $x \in \mathbb{R}$, $g(x) \ge 0$ and*

$$f^2(x) \le g^3(x). \tag{17}$$

*Then, for all $x \in \mathbb{R}$,*

$$|f'(x)| \leq C_1(M)g(x), \tag{18}$$

$$|f''(x)| \leq C_2(M)\sqrt{g(x)}. \tag{19}$$

PROOF. To get (18) and (19), it is sufficient to apply to the polynomial

$$P(x, X) = X^3 - 3g(x)X + 2f(x)$$

the above estimates (II) on $|a_3'(x)|$ and $|a_3''(x)|$ (with $a_1 \equiv 0$). Indeed, under the assumption (17), the discriminant $\Delta = 4(3g)^3 - 27(2f)^2$ is nonnegative for all $x$, thus $P(x, X)$ is hyperbolic. □

REMARK 4. As already noted, the inequality (18) follows directly from the Bronšteĭn Lemma. On the other hand, in order to prove (19) it is not really necessary to appeal to (8) (with $m$=$j$=3, $k$=2). Indeed, in the special case when $a_1 = 0$, (19) can be easily derived from (18) by applying (1) to the nonnegative functions $F = f' + C_1(M)g$ and $g$: putting $C_1 = C_1(M)$, $C = C(M)$, we get

$$\begin{aligned} |f''(x)| &\leq |F'(x)| + C_1|g'(x)| \leq C\sqrt{F(x)} + C_1C\sqrt{g(x)} \\ &\leq C\sqrt{|f'(x)| + C_1 g(x)} + C_1C\sqrt{g(x)} \leq (C\sqrt{2C_1} + C_1C)\sqrt{g(x)}. \end{aligned}$$

## 2. PROOF OF THEOREM 1

Our proof is based on the same idea used by Bronšteĭn in Proposition 3 of [B2] (see also [Nu]). Since (8) is trivial when $k = 0$, we argue by induction on $k$: we assume that (8) holds true for every hyperbolic polynomials of degree $\geq k$, and we prove it, at the level $k + 1$, for a given polynomial $P(x, X)$ (see (**??**)) of degree $m \geq k + 1$.

Writing

$$P_x = \frac{\partial P}{\partial x}, \qquad P_X = \frac{\partial P}{\partial X},$$

we define the auxiliary polynomial (of degree $m - 1$)

$$P_\delta(x, X) := P_X(x, X) + \delta P_x(x, X) = \sum_{j=0}^{m-1} b_j(x) X^{m-1-j}, \tag{20}$$

where $b_j(x) = (m - j)a_j(x) + \delta a_{j+1}'(x)$, and $\delta = \delta(m, M)$ is a small positive constant to be chosen later. For $\delta \leq (m - 1)/M$, we have $b_0(x) = m + \delta a_1'(x) \geq 1$, hence, putting

$$r(x) := \frac{1}{m + \delta a_1'(x)}, \qquad \widetilde{a}_j(x) := r(x)[(m - j)a_j(x) + \delta a_{j+1}'(x)], \tag{21}$$

we can define the *monic* polynomial

$$\widetilde{P}(x, X) := r(x)P_\delta(x, X) = X^{m-1} + \sum_{j=0}^{m-1} \widetilde{a}_j(x) X^{m-1-j}.$$

Now, by Bronšteĭn's Lemma there exist $m$ Lipschitz functions $\lambda_j(x)$, with $|\lambda_j'(x)| \leq C = C(m, M)$ a.e. in $\mathbb{R}$, such that $P(x, X) = \prod_{j=1}^{m}(X - \lambda_j(x))$. Hence we have

$$P_X(x, X) = \sum_{k=1}^{m} P_{(k)}(x, X), \quad \text{where} \quad P_{(k)}(x, X) = \prod_{\substack{j=1,\dots,m \\ j \neq k}} (X - \lambda_j(x)),$$

and

$$|P_x(x, X)| = \Big| - \sum_{k=1}^{m} \lambda_j'(x) P_{(k)}(x, X) \Big| \le C(m, M) \sum_{k=1}^{m} |P_{(k)}(x, X)|.$$

After simplifying the common factors, we find

$$\lim_{X \to \lambda_j(x)} \frac{P_{(k)}(x, X)}{P_X(x, X)} = \delta_{jk}, \qquad \forall x \in \mathbb{R},$$

so that

$$\limsup_{X \to \lambda_j(x)} \left| \frac{P_x(x, X)}{P_X(x, X)} \right| \le C.$$

Hence, recalling (20) and choosing $\delta \le (2C)^{-1}$, we obtain

$$\liminf_{X \to \lambda_j(x)} \frac{P_\delta(x, X)}{P_X(x, X)} \ge 1 - \delta C \ge \frac{1}{2}. \tag{22}$$

By (22), we are in a position to prove that, if $P(x, X)$ is a hyperbolic polynomial, then $\widetilde{P}(x, X)$ is also hyperbolic. For each fixed $x \in \mathbb{R}$, we denote by $\mathring{\lambda}_1 < \cdots < \mathring{\lambda}_\nu$ the *distinct* roots of $P(x, X)$, and by $m_1, \dots, m_\nu$ the corresponding multiplicities. If $m_i \ge 2$, then clearly $\mathring{\lambda}_j$ is also a root of the polynomial $P_\delta(x, X)$, hence of $\widetilde{P}(x, X)$, with multiplicity $m_i - 1$. Thus, noting that

$$\sum_{i=1}^{\nu} (m_i - 1) = m - \nu,$$

we have to find the remaining $\nu - 1$ real roots of $P_\delta(x, \cdot)$. Actually, we prove that $P_\delta(x, \cdot)$ has $\nu - 1$ roots, $\mu_1, \dots, \mu_{\nu-1}$, such that

$$\mathring{\lambda}_1 < \mu_1 < \mathring{\lambda}_2 < \mu_2 < \cdots < \mathring{\lambda}_{\nu-1} < \mu_{\nu-1} < \mathring{\lambda}_\nu.$$

Indeed, we easily see that

$$P_X(x, \mathring{\lambda}_i + \varepsilon) \cdot P_X(x, \mathring{\lambda}_{i+1} - \varepsilon) < 0, \qquad \text{for small } \varepsilon > 0, \tag{23}$$

since $t \mapsto P(x, t)$ is a polynomial function not vanishing in the open interval $\mathring{\lambda}_i < t < \mathring{\lambda}_{i+1}$. Now, (22) says us that $P_\delta(x, t)$ and $P_X(x, t)$ have the same sign for $t$ close to $\mathring{\lambda}_i$, and for $t$ close to $\mathring{\lambda}_{i+1}$, thus (23) holds with $P_\delta$ in place of $P_X$, and consequently the function $t \mapsto P_\delta(x, t)$ must have a zero $\mu_i \in ]\mathring{\lambda}_i, \mathring{\lambda}_{i+1}[$.

In conclusion, the polynomial $\widetilde{P}(x, X)$ has real roots with total multiplicity equal to $m - 1$, i.e., it is hyperbolic.

Next, we put $\widetilde{\psi}_0 \equiv 1$ and we define

$$\widetilde{\psi}_k(x) := \sum_{1 \le j_1 < \cdots < j_k \le m-1} \widetilde{\lambda}_{j_1}^2(x) \cdots \widetilde{\lambda}_{j_k}^2(x), \qquad 1 \le k \le m - 1, \tag{24}$$

where $\widetilde{\lambda}_1(x), \ldots, \widetilde{\lambda}_{m-1}(x)$ are the roots of $\widetilde{P}(x, X)$. Then we have

$$\widetilde{\psi}_k(x) \le \psi_k(x), \qquad 1 \le k \le m-1. \tag{25}$$

Indeed, using the same notation as above, every summand in (24) can be written as

$$\widetilde{\lambda}_{j_1}^2 \cdots \widetilde{\lambda}_{j_k}^2 = \mathring{\lambda}_1^{2\gamma_1} \cdots \mathring{\lambda}_\nu^{2\gamma_\nu} \cdot \mu_1^{2\varepsilon_1} \cdots \mu_{\nu-1}^{2\varepsilon_{\nu-1}}$$

for some set of integers $\gamma_1, \ldots, \gamma_\nu, \varepsilon_1, \ldots, \varepsilon_{\nu-1}$ such that

$$0 \le \gamma_j \le m_j - 1 \quad (1 \le j \le \nu), \qquad \varepsilon_j \in \{0, 1\} \quad (1 \le j \le \nu - 1),$$
$$\gamma_1 + \cdots + \gamma_\nu + \varepsilon_1 + \cdots + \varepsilon_{\nu-1} = k.$$

Moreover, from $\mathring{\lambda}_j < \mu_j < \mathring{\lambda}_{j+1}$, it follows that $\mu_j^2 < \mathring{\lambda}_{j+1}^2$ if $\mu_j \ge 0$, while $\mu_j^2 < \mathring{\lambda}_j^2$ if $\mu_j < 0$. Thus, each term of the sum in (24) is majorized by some term of the type $\lambda_{j_1}^2 \cdots \lambda_{j_k}^2$, and we get (25).

Finally, by differentiating (21) we get the identity

$$\widetilde{a}_j^{(k)}(x) = \sum_{h=0}^{k} \binom{k}{h} r^{(k-h)}(x)[(m-j)a_j^{(h)}(x) + \delta a_{j+1}^{(h+1)}(x)]$$

for $0 \le k \le j < m$, whence we can solve for the highest derivative:

$$a_{j+1}^{(k+1)}(x) = \frac{1}{\delta r(x)}\Big[\widetilde{a}_j^{(k)}(x) - (m-j)\sum_{h=0}^{k}\binom{k}{h} r^{(k-h)}(x) a_j^{(h)}(x) - \delta \sum_{l=0}^{k-1}\binom{k}{l} r^{(k-l)}(x) a_{j+1}^{(l+1)}(x)\Big].$$

Thus, noting that $|r^{(k-h)}(x)| \le C_m M$ since $k - h < m$, and using the inductive hypothesis on the polynomials $\widetilde{P}(x, X)$ and $P(x, X)$ to estimate $|\widetilde{a}_j^{(k)}|$, $|a_j^{(h)}|$, $|a_{j+1}^{(l+1)}|$, we find

$$|a_{j+1}^{(k+1)}(x)| \le C(m+1, M)\Big[\widetilde{\psi}_{j-k}(x) + \sum_{h=0}^{k}\sqrt{\psi_{j-h}(x)} + \sum_{l=0}^{k-1}\sqrt{\psi_{j-l}(x)}\Big].$$

However, from the definition (7), we see immediately that $\psi_k(x) \le C_k \psi_{k'}(x)$ for $k' \le k$, hence by (25) we conclude the proof of Theorem 1. □

## REFERENCES


[AKML] D. ALEKSEEVSKY - A. KRIEGL - P. W. MICHOR - M. LOSIK, *Choosing roots of polynomials smoothly*. Israel J. Math. 105 (1998), 203–233.

[B1] M. D. BRONŠTEĬN, *Smoothness of roots of polynomials depending on parameters*. Sibirsk. Mat. Zh. 20 (1979), 493–501, 690 (in Russian); English transl.: Siberian Math. J. 20 (1980), 347–352.

[B2] M. D. BRONŠTEĬN, *The Cauchy problem for hyperbolic operators with characteristics of variable multiplicity*. Trudy Moskov. Mat. Obshch. 41 (1980), 83–99 (in Russian); English transl.: Trans. Moscow Math. Soc. 1982, no. 1, 87–103.

[CO] F. COLOMBINI - N. ORRÙ, *Well-posedness in $\mathcal{C}^\infty$ for some weakly hyperbolic equations*. J. Math. Kyoto Univ. 39 (1999), 399–420.

[Da] R. DALMASSO, *Un résultat sur les fonctions de classe $\mathcal{C}^{1,\alpha}$ et application au problème de Cauchy*. Ann. Inst. Fourier (Grenoble) 36 (1986), 43–55.

[Di] J. DIEUDONNÉ, *Sur un théorème de Glaeser*. J. Anal. Math. 23 (1970), 85–88.

[G] G. GLAESER, *Racine carrée d'une fonction différentiable*. Ann. Inst. Fourier (Grenoble) 13 (1963), 203–210.

[LV] P.-L. LIONS - C. VILLANI, *Regularité optimale de racines carrés*. C. R. Acad. Sci. Paris 321 (1995), 1537–1541.

[M] T. MANDAI, *Smoothness of roots of hyperbolic polynomials with respect to one-dimensional parameter*. Bull. Fac. Gen. Ed. Gifu Univ. 21 (1985), 115–118.

[N] T. NISHITANI, *The Cauchy problem for weakly hyperbolic equations of second order*. Comm. Partial Differential Equation 5 (1980), 1273–1296.

[NS] T. NISHITANI - S. SPAGNOLO, *An extension of Glaeser inequality and its applications*. Osaka J. Math. 41 (2004), 145–157.

[Nu] W. NUIJ, *A note on hyperbolic polynomials*. Math. Scand. 23 (1968), 69–72.

[O1] O. A. OLEINĬK, *On the smoothness of solutions of degenerate elliptic and parabolic equations*. Dokl. Akad. Nauk SSSR 163 (1965), 577–580 (in Russian); English transl.: Soviet Math. Dokl. 6 (1965), 972–976.

[O2] O. A. OLEINĬK, *On the Cauchy problem for weakly hyperbolic equations*. Comm. Pure Appl. Math. 23 (1970), 569–586.

[Ra] A. RAINER, *Choosing roots of polynomials smoothly and lifting smooth curves over invariants*. Ph.D. thesis, Univ. of Wien, 2004.

[Re] F. RELLICH, *Perturbation Theory of Eigenvalue Problems*. Gordon and Breach, New York, 1969.

[ST] S. SPAGNOLO - G. TAGLIALATELA, *Homogeneous hyperbolic equations with coefficients depending on one space variable*. In preparation.

[T] S. TARAMA, *Note on the Bronshtein theorem concerning hyperbolic polynomials*. Sci. Math. Jpn. 63 (2006), 247–285.

[W] S. WAKABAYASHI, *Remarks on hyperbolic polynomials*. Tsukuba J. Math. 10 (1986), 17–28.

Received 14 April 2006,
and in revised form 13 May 2006.

Sergio Spagnolo
Dipartimento di Matematica
Università di Pisa
Largo Bruno Pontecorvo 5
56127 PISA, Italy
spagnolo@dm.unipi.it

Giovanni Taglialatela
Dipartimento di Scienze Economiche
Università di Bari
via Camillo Rosalba 53
70124 BARI, Italy
taglia@dse.uniba.it