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Partial differential equations. — *Asymptotic profiles for the Kirchhoff equation,* by TOKIO MATSUYAMA, communicated on 23 June 2006 by S. Spagnolo.

Dedicated to Professor Mitsuhiro Nakao on his 60th birthday

ABSTRACT. — The first aim of this paper is to find asymptotic profiles for the Kirchhoff equation. More precisely, it will be shown that there exists a small amplitude solution which is not asymptotically free. The second aim is to prove the existence of scattering states for small amplitude solutions with data belonging to $H^{\sigma(p),p} \times H^{\sigma(p)-1,p}$, where $\sigma(p) = n(2/p - 1) + 1$, $p \in [1, 2(n-1)/(n+1))$ and $n \geq 4$.

KEY WORDS: Kirchhoff equation; asymptotic profiles; scattering states.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 35L05, 35L10.

1. INTRODUCTION

We consider the following Cauchy problem for the Kirchhoff equation:

$$\text{(K)} \begin{cases} \partial_t^2 u - \left(1 + \int_{\mathbb{R}^n} |\nabla u|^2 \, dx\right) \Delta u = 0, & (x,t) \in \mathbb{R}_x^n \times \mathbb{R}_t, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), & x \in \mathbb{R}_x^n, \end{cases}$$

where $\partial_t = \partial/\partial t$, $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ and Δ is the Laplacian in $\mathbb{R}^n$ defined by $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$. The aim of this paper is to find asymptotic profiles of solutions for the Kirchhoff equation. In [16] Yamazaki introduced a set Y_k, $k > 1$ (see §2.1), of initial data to obtain a global-in-time existence theorem with small data of low regularity, and then the scattering operator was constructed in the case when $k > 2$. This generalized [8] (see also [9] for $n = 1$). *A motivation for this paper arises from the problem whether or not the scattering states exist in the case when* $1 < k \leq 2$. Roughly speaking, it depends on the decay rate of data: if the data behave like $\{u_0(x), u_1(x)\} = \{\delta\langle x\rangle^{-\ell}, \delta|D|\langle x\rangle^{-\ell}\}$ with $0 < \delta \ll 1$ and $\ell > (n+1)/2$, then they belong to Y_k for some $k > 2$ (see Theorem 2), hence, the solutions are asymptotically free. On the other hand, for each $k \in (1,2]$, if we choose the data $u_0(x) = \delta|D|^{-1}\langle x\rangle^{-(n+k-1)/2}$ and $u_1(x) = 0$ with $0 < \delta \ll 1$, then the corresponding solution of (K) is not asymptotically free. To derive these asymptotics, we need a delicate analysis of an oscillatory integral associated with the Kirchhoff equation, which was introduced by Greenberg and Hu [9] in the one-dimensional case (see also [4, 5, 16]). We will develop an asymptotic expansion of this oscillatory integral.

Let us introduce some notation. For $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, let $\dot{H}^{s,p}(\mathbb{R}^n)$ and $H^{s,p}(\mathbb{R}^n)$ be the Riesz and Bessel potential spaces which are the completions of $C_0^\infty(\mathbb{R}^n)$

with the seminorm or norm

$$\|u\|_{\dot{H}^{s,p}(\mathbb{R}^n)} = \|\mathcal{F}^{-1}[|\xi|^s \hat{u}(\xi)]\|_{L^p(\mathbb{R}^n)} \equiv \| |D|^s u\|_{L^p(\mathbb{R}^n)},$$
$$\|u\|_{H^{s,p}(\mathbb{R}^n)} = \|\mathcal{F}^{-1}[\langle\xi\rangle^s \hat{u}(\xi)]\|_{L^p(\mathbb{R}^n)} \equiv \|\langle D\rangle^s u\|_{L^p(\mathbb{R}^n)},$$

respectively. Here $\hat{}$ denotes the Fourier transform, $\mathcal{F}^{-1}$ is its inverse, and $\langle\xi\rangle = \sqrt{1+|\xi|^2}$. Throughout this paper, we write

$$\dot{H}^{s,p} = \dot{H}^{s,p}(\mathbb{R}^n), \quad H^{s,p} = H^{s,p}(\mathbb{R}^n), \quad \dot{H}^s = \dot{H}^{s,2}(\mathbb{R}^n), \quad H^s = H^{s,2}(\mathbb{R}^n).$$

We also put, for $s \geq 1$,

$$X^s(\mathbb{R}) = C(\mathbb{R}; H^s) \cap C^1(\mathbb{R}; H^{s-1}) \cap C^2(\mathbb{R}; H^{s-2}),$$
$$\dot{X}^s(\mathbb{R}) = C(\mathbb{R}; \dot{H}^s) \cap C^1(\mathbb{R}; \dot{H}^{s-1}) \cap C^2(\mathbb{R}; \dot{H}^{s-2}).$$

Finally, we denote by $\mathcal{S}$ the Schwartz space on $\mathbb{R}^n$.

2. Statement of results

2.1. *Global existence theorem*

In order to find the asymptotic profiles for the solutions to the Kirchhoff equation, we refer to a general theorem of Yamazaki (see [16]). For this purpose, let us introduce the set

$$Y_k := \{\{\phi, \psi\} \in \dot{H}^{3/2} \times H^{1/2};\ |\{\phi, \psi\}|_{Y_k} < \infty\}, \quad k > 1,$$

where

$$|\{\phi, \psi\}|_{Y_k} := \sup_{\tau\in\mathbb{R}} (1+|\tau|)^k \left| \int_{\mathbb{R}^n} e^{i\tau|\xi|} |\xi|^3 |\hat{\phi}(\xi)|^2 \, d\xi \right| + \sup_{\tau\in\mathbb{R}} (1+|\tau|)^k \left| \int_{\mathbb{R}^n} e^{i\tau|\xi|} |\xi|\, |\hat{\psi}(\xi)|^2 \, d\xi \right| + \sup_{\tau\in\mathbb{R}} (1+|\tau|)^k \left| \int_{\mathbb{R}^n} e^{i\tau|\xi|} |\xi|^2 \Re(\hat{\phi}(\xi)\overline{\hat{\psi}(\xi)}) \, d\xi \right|.$$

Then we have the following:

THEOREM A ([16]). *Let $n \geq 1$ and $s_0 \geq 3/2$. If $u_0 \in \dot{H}^{s_0} \cap H^1$, $u_1 \in H^{s_0-1}$, and*

$$\delta_1 := \|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2 + |\{u_0, u_1\}|_{Y_k} \ll 1 \quad \text{for some } k > 1, \tag{2.1}$$

then the problem (K) *has a unique solution $u(x,t) \in \dot{X}^{s_0}(\mathbb{R})$ having the following property*: *there exists a constant $c_{\pm\infty} \equiv c_{\pm\infty}(u_0, u_1) > 0$ such that*

$$1 + \|\nabla u(\cdot, t)\|_{L^2}^2 = c_{\pm\infty}^2 + O(|t|^{-k+1}) \quad \text{as } t \to \pm\infty.$$

Furthermore, if (2.1) *holds with* $k > 2$*, then* $c_{+\infty} = c_{-\infty} =: c_\infty$ *and each solution* $u(x,t) \in \dot{X}^{s_0}(\mathbb{R})$ *is asymptotically free in* $\dot{H}^\sigma \times \dot{H}^{\sigma-1}$ *for all* $\sigma \in [1, s_0]$ *as* $t \to \pm\infty$*, i.e., there exists a solution* $v_\pm = v_\pm(x,t) \in \dot{X}^\sigma(\mathbb{R})$ *of the equation*

$$(\partial_t^2 - c_\infty^2 \Delta) v_\pm = 0 \quad \textit{on } \mathbb{R}^n \times \mathbb{R}$$

such that

$$\|u(\cdot,t) - v_\pm(\cdot,t)\|_{\dot{H}^\sigma} + \|\partial_t u(\cdot,t) - \partial_t v_\pm(\cdot,t)\|_{\dot{H}^{\sigma-1}} \to 0 \quad (t \to \pm\infty).$$

The inclusions among the classes Y_k are as follows:

$$Y_k \subset Y_\ell \quad \text{if } k > \ell > 1, \quad \text{and} \quad \mathcal{S} \subset Y_k \quad \text{for all } k \in (1, n+1].$$

The latter inclusion can be shown by using the asymptotic expansion of an oscillatory integral $I(\tilde{\vartheta}(t), 0)$ which will be defined in §4.

The definition of Y_k is somewhat complicated, hence we give some explicit examples of spaces contained in Y_k.

EXAMPLE 2.1. (i) Let $n \geq 4$. For all $p \in (1, 2(n-1)/(n+1))$ we put

$$s(p) = \max\left\{\frac{n}{2}\left(\frac{2}{p} - 1\right) + 1, 3\right\}, \quad r(p) = \frac{n+1}{2}\left(\frac{2}{p} - 1\right) + 1,$$
$$k(p) = \frac{n-1}{2}\left(\frac{2}{p} - 1\right).$$

Then it was proved in [17] that

$$(H^{s(p)} \cap H^{r(p),p}) \times (H^{s(p)-1} \cap H^{r(p)-1,p}) \subset Y_{k(p)}.$$

(ii) Let $n \geq 1$ and $k \in (1, n+1]$. Then it was proved in [5] and [16] (see also [4, 6, 9, 12, 18]) that the space

$$X_k \equiv \{(\phi, \psi) \in H^2 \times H^1;\ \|\langle x\rangle^k \phi\|_{H^2} + \|\langle x\rangle^k \psi\|_{H^1} < \infty\}$$

is contained in Y_k.

(iii) It will be proved in §7 that

$$\{\langle x\rangle^{-n}, |D|\langle x\rangle^{-n}\} \in Y_{n+2}, \quad \{\langle x\rangle^{-\ell}, |D|\langle x\rangle^{-\ell}\} \in Y_{n+3} \quad \forall \ell > n.$$

2.2. *Main results*

Keeping in mind Theorem A, we can find the asymptotic profiles for the solutions to (K). Namely, each solution u can be decomposed into a *free wave*, a *nonfree wave* and a remainder term. Let us present the definitions of free and nonfree waves.

DEFINITION. (i) *We say that* $v_\pm = v_\pm(x,t) = \{v_+(x,t), v_-(x,t)\}$ *is a* free wave *if it satisfies the equation*

$$(\partial_t^2 - c_{\pm\infty}^2 \Delta) v_\pm = 0 \quad \textit{on } \mathbb{R}^n \times \mathbb{R}.$$

(ii) *Let* $\sigma \geq 1$. *We say that* $v = v(x,t)$ *is* asymptotically free *in* $\dot{H}^\sigma \times \dot{H}^{\sigma-1}$ *if it is asymptotic to some free wave* $v_\pm$ *in* $\dot{H}^\sigma \times \dot{H}^{\sigma-1}$, *i.e.,*

$$\|v(\cdot,t) - v_\pm(\cdot,t)\|_{\dot{H}^\sigma} + \|\partial_t v(\cdot,t) - \partial_t v_\pm(\cdot,t)\|_{\dot{H}^{\sigma-1}} \to 0 \quad (t \to \pm\infty).$$

(iii) *Let* $\sigma \geq 1$. *We say that* $w = w(x,t)$ *is a* nonfree wave *in* $\dot{H}^\sigma \times \dot{H}^{\sigma-1}$ *if it is not asymptotically free.*

Theorem A states that each solution u of (K) with initial data satisfying (2.1) with $k > 2$ is asymptotically free. On the other hand, the first theorem of the present paper states that the bound $k > 2$ is sharp. More precisely, we shall prove the following:

THEOREM 1. *Assume that*

$$\text{either} \quad n \geq 2 \text{ and } 1 < k \leq 2, \quad \text{or} \quad n = 1 \text{ and } 1 < k < 2.$$

Then there exists a solution $u(x,t) \in \bigcap_{s\geq 1} \dot{X}^s(\mathbb{R})$ *of* (K) *with data satisfying* (2.1), *which is a nonfree wave in* $\dot{H}^\sigma \times \dot{H}^{\sigma-1}$ *for all* $\sigma \geq 1$.

In order to improve Example 2.1(i), we introduce the space $H^{\sigma(p),p} \times H^{\sigma(p)-1,p}$, where

$$\sigma(p) = n\left(\frac{2}{p} - 1\right) + 1, \qquad p \in \left[1, \frac{2(n-1)}{n+1}\right), \quad n \geq 4.$$

Note that for $p \in (1, 2(n-1)/(n+1))$ we have

$$H^{\sigma(p),p} \times H^{\sigma(p)-1,p} \subset H^m \times H^{m-1} \quad \text{with } m = \frac{\sigma(p)+1}{2},$$

by the Sobolev imbedding theorem, while for $p = 1$ the above inclusion holds only for $m < (\sigma(p)+1)/2 = (n+2)/2$. Also note that if $2n/(n+4) < p < 2(n-1)/(n+1)$, then $(\sigma(p)+1)/2 < 3$, thus, for such p, the space $H^{\sigma(p),p} \times H^{\sigma(p)-1,p}$ becomes wider than that of Example 2.1(i).

With the above notation, we have the following:

PROPOSITION 2.2. *Let* $n \geq 4$. *Then, for all* $p \in [1, 2(n-1)/(n+1))$, *we have the inclusion*

$$H^{\sigma(p),p} \times H^{\sigma(p)-1,p} \subset Y_{k(p)},$$

where

$$\sigma(p) = n\left(\frac{2}{p} - 1\right) + 1, \qquad k(p) = \frac{n-1}{2}\left(\frac{2}{p} - 1\right).$$

As a consequence of Proposition 2.2 and Theorem A, we conclude that, for all $s \geq 1$, the problem (K) has a unique solution $u(x,t) \in \dot{X}^s(\mathbb{R})$ provided that $\{u_0, u_1\}$ belongs to $(H^s \cap H^{\sigma(p),p}) \times (H^{s-1} \cap H^{\sigma(p)-1,p})$ and $\|u_0\|_{H^{\sigma(p),p}} + \|u_1\|_{H^{\sigma(p)-1,p}} \ll 1$.

REMARK. In the special case $n = 4$ and $p = 1$, hence $\sigma(p) = 5$, we conclude that for each $\{u_0, u_1\} \in H^{5,1} \times H^{4,1}$, the solution $u(x,t)$ belongs to $X^{3-\varepsilon}(\mathbb{R})$ for all $\varepsilon > 0$. Indeed the Sobolev imbedding theorem ensures the inclusion $H^{5,1} \times H^{4,1} \subset H^{3-\varepsilon} \times H^{2-\varepsilon}$.

For all $n \geq 4$ and $p \leq 2(n-1)/(n+1)$, one has $k(p) \geq 1$. In view of the existence of scattering states, we are interested in the case when $k(p) > 2$. In this regard, we note that

$$k(p) \leq 2 \quad \text{iff} \quad p \geq \frac{2(n-1)}{n+3}.$$

Hence for $n = 4$ and $1 \leq p \leq 6/5$, $n = 5$ and $1 \leq p \leq 4/3$, or more generally, $n \geq 6$ and $2(n-1)/(n+3) \leq p \leq 2(n-1)/(n+1)$, Theorem A does not ensure the existence of scattering states for all solutions of (K) with initial data as in Proposition 2.2.

Our next main result states that in any case there exist some scattering states. More precisely, we shall prove

THEOREM 2. *Let $n \geq 4$. Then, for all $p \in [1, 2(n-1)/(n+1))$ there is some number $\ell = \ell(p) > 2$ such that*

$$\{(H^s \cap H^{r,p}) \times (H^{s-1} \cap H^{r-1,p})\} \cap Y_\ell \neq \{0\}$$

for all $s, r \geq 1$. In particular, if we take $s = 1$ and $r = \sigma(p)$ as in Proposition 2.2 (e.g., $p = 1$ and $r = n+1$), we obtain the existence of nontrivial scattering states for some data in the class $H^{\sigma(p),p} \times H^{\sigma(p)-1,p}$ ($\subset H^{3/2} \times H^{1/2}$) even in the case $k(p) \leq 2$.

3. LINEAR THEORY

The Kirchhoff equation inherits various properties from linear equations, hence the structure of solutions for the Kirchhoff equation can be determined by the linear theory. In this section we introduce the linear theory established through the asymptotic integrations of ordinary differential equations (see Ascoli [3] and Wintner [15]). For more details, see [10, 11]. Consider the following strictly hyperbolic Cauchy problem:

$$\text{(L)} \begin{cases} (\partial_t^2 - c(t)^2 \Delta) u(x,t) = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), & x \in \mathbb{R}^n. \end{cases}$$

We make the following assumption on $c(t)$:

ASSUMPTION B. *The function $c(t)$ is of class $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ and*

$$\inf_{t \in \mathbb{R}} c(t) > 0, \quad c'(t) \in L^1(\mathbb{R}),$$
$$c(t) = c_{\pm\infty} + o(1) \quad \textit{for some constant } c_{\pm\infty} > 0 \textit{ as } t \to \pm\infty.$$

In the following we shall use the notation:

$$\vartheta(t) = \int_0^t c(\tau)\, d\tau, \quad \gamma_\pm(t) = \left| \int_t^{\pm\infty} |c'(\tau)|\, d\tau \right|.$$

First we develop an asymptotic integration of ordinary differential equations. We let

$$W(\xi, t) = \begin{pmatrix} v_0(\xi,t) & v_1(\xi,t) \\ v_0'(\xi,t) & v_1'(\xi,t) \end{pmatrix}$$

be the fundamental matrix of the following ODE:

$$v'' + c(t)^2|\xi|^2 v = 0. \tag{3.1}$$

This means that $v_0(\xi, t)$ is the solution of (3.1) with $v_0(\xi, 0) = 1$, $v_0'(\xi, 0) = 0$, while $v_1(\xi, t)$ is the solution of (3.1) with $v_1(\xi, 0) = 0$, $v_1'(\xi, 0) = 1$. Hence the solution $v(\xi, t)$ of (3.1) can be written

$$\begin{pmatrix} v(\xi, t) \\ v'(\xi, t) \end{pmatrix} = W(\xi, t) \begin{pmatrix} v(\xi, 0) \\ v'(\xi, 0) \end{pmatrix}.$$

Now we define

$$Y(\xi, t) = \begin{pmatrix} \cos(\vartheta(t)|\xi|) & \dfrac{\sin(\vartheta(t)|\xi|)}{c(0)|\xi|} \\ -c(t)|\xi| \sin(\vartheta(t)|\xi|) & \dfrac{c(t) \cos(\vartheta(t)|\xi|)}{c(0)} \end{pmatrix}$$

to be the fundamental matrix of the perturbed ODE

$$v'' - \frac{c'(t)}{c(t)} v' + c(t)^2 |\xi|^2 v = 0.$$

Hence,

$$Y(\xi, 0) = I, \qquad \det Y(\xi, t) = \frac{c(t)}{c(0)},$$

$$Y(\xi, t)^{-1} = \begin{pmatrix} \cos(\vartheta(t)|\xi|) & -\dfrac{\sin(\vartheta(t)|\xi|)}{c(t)|\xi|} \\ c(0)|\xi| \sin(\vartheta(t)|\xi|) & \dfrac{c(0) \cos(\vartheta(t)|\xi|)}{c(t)} \end{pmatrix}.$$

Then by using hyperbolic energy estimates, we have the following:

LEMMA 3.1 (see Ascoli [3] and Wintner [15]). *The limit*

$$\lim_{t \to \pm\infty} \{Y(\xi, t)^{-1} W(\xi, t)\} \equiv L_\pm(\xi)$$

exists, and is C^∞ in ξ. Moreover, putting

$$L_\pm(\xi) = \begin{pmatrix} \alpha_0^\pm(\xi) & \alpha_1^\pm(\xi) \\ \beta_0^\pm(\xi) & \beta_1^\pm(\xi) \end{pmatrix},$$

we have

$$|\alpha_\ell^\pm(\xi)| \lesssim |\xi|^{-\ell}, \quad |\beta_\ell^\pm(\xi)| \lesssim |\xi|^{1-\ell}, \quad \ell = 0, 1.$$

Furthermore, putting

$$R_\pm(\xi, t) \equiv Y(\xi, t)^{-1} W(\xi, t) - L_\pm(\xi) = \begin{pmatrix} \varepsilon_{1,\pm}^{(0)}(\xi, t) & \varepsilon_{1,\pm}^{(1)}(\xi, t) \\ \varepsilon_{2,\pm}^{(0)}(\xi, t) & \varepsilon_{2,\pm}^{(1)}(\xi, t) \end{pmatrix}, \tag{3.2}$$

we have

$$|\varepsilon_{j,\pm}^{(\ell)}(\xi, t)| \lesssim \gamma(t) |\xi|^{(j-1)-\ell}, \quad \ell = 0, 1, \ j = 1, 2.$$

Now writing

$$\cos(\vartheta(t)|\xi|) = \cos(c_{\pm\infty}|\xi|t) + \varphi_c(|\xi|, t),$$
$$\sin(\vartheta(t)|\xi|) = \sin(c_{\pm\infty}|\xi|t) + \varphi_s(|\xi|, t)$$

with

$$\varphi_c(|\xi|, t) = 2\sin\left(\frac{c_{\pm\infty}|\xi|t + \vartheta(t)|\xi|}{2}\right)\sin\left(\frac{c_{\pm\infty}|\xi|t - \vartheta(t)|\xi|}{2}\right),$$
$$\varphi_s(|\xi|, t) = -2\cos\left(\frac{c_{\pm\infty}|\xi|t + \vartheta(t)|\xi|}{2}\right)\sin\left(\frac{c_{\pm\infty}|\xi|t - \vartheta(t)|\xi|}{2}\right),$$

we split the matrix $Y(\xi, t)$ as

$$Y(\xi, t) = Y_{\pm\infty}(\xi, t) + \tilde{Y}(\xi, t),$$

where

$$Y_{\pm\infty}(\xi, t) = \begin{pmatrix} \cos(c_{\pm\infty}|\xi|t) & \dfrac{\sin(c_{\pm\infty}|\xi|t)}{c(0)|\xi|} \\ -c_{\pm\infty}|\xi|\sin(c_{\pm\infty}|\xi|t) & \dfrac{c_{\pm\infty}}{c(0)}\cos(c_{\pm\infty}|\xi|t) \end{pmatrix}$$

and

$$\tilde{Y}(\xi, t) = \begin{pmatrix} \varphi_c(|\xi|, t) & \dfrac{\varphi_s(|\xi|, t)}{c(0)|\xi|} \\ \varphi_c'(|\xi|, t) & \dfrac{\varphi_s'(|\xi|, t)}{c(0)|\xi|} \end{pmatrix}.$$

Thus by (3.2) we can write

$$W(\xi, t) = Y_{\pm\infty}(\xi, t)L_\pm(\xi) + \tilde{Y}(\xi, t)L_\pm(\xi) + R_{0,\pm}(\xi, t)$$

with

$$R_{0,\pm}(\xi, t) = Y(\xi, t)R_\pm(\xi, t).$$

Summarizing the above argument, we conclude that the solution $\boldsymbol{v}(\xi, t) = \binom{v(\xi,t)}{v'(\xi,t)}$ of (3.1) with data $\boldsymbol{v}_0(\xi) = \binom{v(\xi,0)}{v'(\xi,0)}$ splits as

$$\boldsymbol{v}(\xi, t) = Y_{\pm\infty}(\xi, t)L_\pm(\xi)\boldsymbol{v}_0(\xi) + \tilde{Y}(\xi, t)L_\pm(\xi)\boldsymbol{v}_0(\xi) + R_{0,\pm}(\xi, t)\boldsymbol{v}_0(\xi).$$

Since the solution $u(x, t)$ of our problem (L) is represented by

$$u(x, t) = \mathcal{F}^{-1}[v_0(\xi, t)\hat{u}_0(\xi) + v_1(\xi, t)\hat{u}_1(\xi)](x),$$
$$\partial_t u(x, t) = \mathcal{F}^{-1}[v_0'(\xi, t)\hat{u}_0(\xi) + v_1'(\xi, t)\hat{u}_1(\xi)](x),$$

we arrive at the following:

PROPOSITION 3.2. *Let $n \geq 1$ and $s_0 \geq 1$. Suppose that $c(t)$ satisfies Assumption B. Then each solution $u(x, t) \in \dot{X}^1(\mathbb{R}) \cap \dot{X}^{s_0}(\mathbb{R})$ of* (L) *admits the following decomposition:*

$$u(x, t) = v_\pm(x, t) + w_\pm(x, t) + r_\pm(x, t) \quad \textit{for } t \gtrless 0,$$

where $v_\pm(x,t) \in \dot{X}^1(\mathbb{R}) \cap \dot{X}^{s_0}(\mathbb{R})$ *is a free wave,* $w_\pm(x,t) \in \dot{X}^1(\mathbb{R}) \cap \dot{X}^{s_0}(\mathbb{R})$ *satisfies*

$$(\partial_t^2 - c_{\pm\infty}^2 \Delta) w_\pm(x,t) = \frac{c(t)^2 - c_{\pm\infty}^2}{c(t)} \partial_t f_{1,\pm}(x,t) + c'(t) f_{1,\pm}(x,t)$$

with

$$\begin{aligned} f_{1,\pm}(x,t) &= \mathcal{F}^{-1}[|\xi|\{-a_\pm(\xi)\sin(\vartheta(t)|\xi|) + b_\pm(\xi)\cos(\vartheta(t)|\xi|)\}](x), \\ a_\pm(\xi) &= \alpha_0^\pm(\xi)\hat{u}_0(\xi) + \alpha_1^\pm(\xi)\hat{u}_1(\xi), \\ b_\pm(\xi) &= \frac{1}{c(0)|\xi|}(\beta_0^\pm(\xi)\hat{u}_0(\xi) + \beta_1^\pm(\xi)\hat{u}_1(\xi)), \end{aligned}$$

while the remainder term $r_\pm(x,t) \in \dot{X}^1(\mathbb{R}) \cap \dot{X}^{s_0}(\mathbb{R})$ *satisfies*

$$\|r_\pm(\cdot,t)\|_{\dot{H}^\sigma} \lesssim \gamma_\pm(t)(\|u_0\|_{\dot{H}^\sigma} + \|u_1\|_{\dot{H}^{\sigma-1}}) \quad \textit{for all } \sigma \in [1, s_0].$$

More precisely, we have

$$\begin{aligned} v_\pm(x,t) &= \mathcal{F}^{-1}[a_\pm(\xi)\cos(c_{\pm\infty}|\xi|t) + b_\pm(\xi)\sin(c_{\pm\infty}|\xi|t)](x), \\ w_\pm(x,t) &= \mathcal{F}^{-1}[a_\pm(\xi)\varphi_c(|\xi|,t) + b_\pm(\xi)\varphi_s(|\xi|,t)](x), \end{aligned}$$

while the remainder term $r_\pm(x,t)$ *has the following form*:

$$r_\pm(x,t) = \mathcal{F}^{-1}[e_c(\xi,t)\cos(\vartheta(t)|\xi|) + e_s(\xi,t)\sin(\vartheta(t)|\xi|)](x),$$

where

$$\begin{aligned} e_c(\xi,t) &= \varepsilon_{1,\pm}^{(0)}(\xi,t)\hat{u}_0(\xi) + \varepsilon_{1,\pm}^{(1)}(\xi,t)\hat{u}_1(\xi), \\ e_s(\xi,t) &= \frac{1}{c(0)|\xi|}\{\varepsilon_{2,\pm}^{(0)}(\xi,t)\hat{u}_0(\xi) + \varepsilon_{2,\pm}^{(1)}(\xi,t)\hat{u}_1(\xi)\}. \end{aligned}$$

The crucial tool in our proof of Theorem 1 is the following:

LEMMA 3.3. *Let* $n \geq 1$. *Suppose that* $c(t)$ *satisfies Assumption B. Assume further that* $\psi_\pm(t) = \int_0^t (c(\tau) - c_{\pm\infty})\, d\tau$ *satisfies*

$$\lim_{t \to \pm\infty} |\psi_\pm(t)| = \infty. \tag{3.3}$$

Then, for the data

$$u_0(x) = |D|^{-1}\langle x\rangle^{-\ell}, \quad u_1(x) = 0$$

with $\ell \in (n/2, (n+1)/2]$, *the solution* $u(x,t) \in \bigcap_{s\geq 1} \dot{X}^s(\mathbb{R})$ *of* (L) *is a nonfree wave in* $\dot{H}^s \times \dot{H}^{s-1}$ *for all* $s \geq 1$.

PROOF. See [10]. A more general result concerning the initial-boundary value problem on bounded domains is due to Arosio [2].

4. Some lemmas

In this section we prepare some lemmas which will be used in the next section to prove Theorem 1. Given $\Lambda > 1$, $K > 0$ and $k > 1$, we say that $c(t)$ belongs to $\mathcal{K}(\Lambda, K, k)$ if it belongs to $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ and satisfies

$$1 \le c(t) \le \Lambda,$$
$$|c'(t)| \le K(1+|t|)^{-k} \quad \text{a.e. in } \mathbb{R}.$$

Fixing data satisfying the assumptions of Theorem A, we consider the solution $u(x,t)$ of the linear problem (L) in §3, and define

$$\tilde{c}(t) = \sqrt{1 + \|\nabla u(\cdot,t)\|_{L^2}^2}.$$

This defines a mapping $\Theta : c \mapsto \tilde{c}$. By using the Schauder–Tychonoff fixed point theorem as in [4, 5] we can show that Θ has a fixed point in $\mathcal{K}(\Lambda, K, k)$ for suitable Λ, K and k, i.e., $\tilde{c}(t) = c(t)$, and hence, (K) has a unique solution $u(x,t)$ as in Theorem A. Thus, by using the method of [5], we obtain

Lemma 4.1. *Let $u(x,t) \in \dot{X}^{s_0}(\mathbb{R})$ be the solution of* (K) *given by Theorem A. Then, for some constant M independent of u, we have*:

$$1 \le \tilde{c}(t) \le 1 + \|\nabla u_0\|_{L^2} + \frac{M\delta_1}{k-1},$$
$$|\tilde{c}'(t)| \le M\delta_1(1+|t|)^{-k}, \tag{4.1}$$

where $\delta_1 \equiv \delta_1(u_0, u_1)$ is the size of the initial data (*see* (2.1)).

In the proof of Theorem 1 we also use three more lemmas. In order to state them we introduce several functions as follows:

$$\tilde{\vartheta}(t) = \int_0^t \tilde{c}(s)\,ds, \qquad v_\pm(\cdot,t) = \frac{e^{\pm i\tilde{\vartheta}(t)|D|}}{\sqrt{\tilde{c}(t)}}\{\partial_t u(\cdot,t) \mp i\tilde{c}(t)|D|u(\cdot,t)\},$$

where the operator $e^{i\tau|D|}$ is defined by

$$(e^{i\tau|D|}f)(x) = \mathcal{F}^{-1}[e^{i\tau|\xi|}\hat{f}(\xi)](x), \qquad \tau \in \mathbb{R}.$$

Define

$$I(r,t) = (|D|e^{2ir|D|}v_-(\cdot,t), v_+(\cdot,t))_{L^2},$$
$$J(r,t) = (|D|e^{2ir|D|}v_+(\cdot,t), v_+(\cdot,t))_{L^2} + (|D|e^{2ir|D|}v_-(\cdot,t), v_-(\cdot,t))_{L^2}$$

for $r, t \in \mathbb{R}$, where $(f,g)_{L^2}$ denotes the $L^2(\mathbb{R}^n)$ inner product. We recall the following equations.

LEMMA 4.2 ([5, 17]). *Let $u(x,t) \in \dot{X}^{s_0}(\mathbb{R})$ be the solution of* (K) *given by Theorem A. Then, with $\Im$ denoting imaginary part,*

$$2\tilde{c}(t)\tilde{c}'(t) = \Im I(\tilde{\vartheta}(t), t), \tag{4.2}$$

and further, $I(r,t)$ satisfies the integral equation

$$\begin{aligned} I(r,t) &= I(r,0) - \frac{1}{2}\int_0^t \frac{\tilde{c}'(s)}{\tilde{c}(s)} J(r - \tilde{\vartheta}(s), 0)\, ds \\ &+ \frac{1}{2}\int_0^t \frac{\tilde{c}'(s)}{\tilde{c}(s)} \int_0^s \frac{\tilde{c}'(\sigma)}{\tilde{c}(\sigma)} \Big(I(r - \tilde{\vartheta}(s) + \tilde{\vartheta}(\sigma), \sigma) + \overline{I(-r + \tilde{\vartheta}(s) + \tilde{\vartheta}(\sigma), \sigma)} \Big)\, d\sigma\, ds. \end{aligned} \tag{4.3}$$

The next lemma can be found in [17] (see also [5]).

LEMMA 4.3. *Let $u(x,t) \in \dot{X}^{s_0}(\mathbb{R})$ be the solution of* (K) *given by Theorem A. Then, for some constant M independent of u,*

$$(1 + |r|)^k (|I(r,t)| + |J(r,t)|) \le M\delta_1 \tag{4.4}$$

for all $r, t \in \mathbb{R}$, where δ_1 is defined by (2.1).

We conclude this section by stating the following lemma which will be used in order to develop an asymptotic expansion of $I(\tilde{\vartheta}(t), 0)$.

LEMMA 4.4 (Aronszajn and Smith [1], cf. [13, Ch. V, §3]). *Let $\ell > 0$. Then the Fourier transform of $\langle x\rangle^{-\ell}$ is of the form $\mathcal{F}[\langle x\rangle^{-\ell}](\xi) = G_\ell(\xi)$, where*

$$G_\ell(\xi) = \frac{1}{2^{(n+\ell-2)/2}\pi^{n/2}\Gamma(\ell/2)} K_{(n-\ell)/2}(|\xi|)|\xi|^{(\ell-n)/2},$$

and $K_\nu(z)$ is the modified Bessel function of the third kind with order ν. In particular, $G_\ell(\xi)$ has the following asymptotics:

$$G_\ell(\xi) \sim \begin{cases} \dfrac{\Gamma((n-\ell)/2)}{2^\ell \pi^{n/2}\Gamma(\ell/2)}|\xi|^{\ell-n} & \text{for } \ell \in (0,n), \\ \dfrac{1}{2^{n-1}\pi^{n/2}\Gamma(n/2)} \log\dfrac{1}{|\xi|} & \text{for } \ell = n, \qquad (|\xi| \to 0), \\ \dfrac{\Gamma((\ell-n)/2)}{2^\ell \pi^{n/2}\Gamma(\ell/2)} & \text{for } \ell > n, \end{cases} \tag{4.5}$$

$$G_\ell(\xi) \sim \frac{1}{2^{(n+\ell-2)/2}\pi^{(n-1)/2}\Gamma(\ell/2)}|\xi|^{(\ell-n-1)/2}e^{-|\xi|} \quad \text{for } \ell > 0 \quad (|\xi| \to \infty). \tag{4.6}$$

Moreover,

$$G_\ell(\xi) = C_{n,\ell}|\xi|^{\ell-n} + \sigma_\ell(\xi) \quad \text{for } \ell \in (0,n), \quad \text{with} \tag{4.7}$$
$$\sigma_\ell^{(j)}(\xi) \sim C_{n,\ell,j}|\xi|^{\ell-n-1-j}, \quad j = 0,1,2, \text{ as } |\xi| \to 0,$$

$$G_\ell^{(j)}(\xi) \sim \tilde{C}_{n,\ell,j}|\xi|^{(\ell-n-1)/2}e^{-|\xi|} \quad \text{for } \ell > 0 \text{ and } j = 1,2,\ldots, \text{ as } |\xi| \to \infty, \tag{4.8}$$

with certain constants $C_{n,\ell}$, $C_{n,\ell,j}$ and $\tilde{C}_{n,\ell,j}$.

PROOF. Put $z = |\xi|$ and $\nu = (n-\ell)/2$. Then the asymptotics (4.5)–(4.6) follow from

$$K_\nu(z) \sim 2^{\nu-1}\Gamma(\nu)z^{-\nu} \quad \text{as } z \to 0 \text{ for } \nu > 0,$$
$$K_0(z) \sim \log\frac{1}{z} \quad \text{as } z \to 0,$$
$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \quad \text{as } z \to \infty \text{ for all } \nu \in \mathbb{R}.$$

On the other hand, we have

$$\left(\frac{1}{z}\frac{d}{dz}\right)^m [z^{-\nu}K_\nu(z)] = (-1)^m z^{-\nu-m}K_{\nu+m}(z), \qquad m = 1, 2, \dots$$

(see Watson [14], cf. [1, Ch. 1]). Hence

$$\frac{d}{dz}[z^{-\nu}K_\nu(z)] = -z^{-\nu}K_{\nu+1}(z),$$
$$\left(\frac{d}{dz}\right)^2 [z^{-\nu}K_\nu(z)] = z^{-\nu}K_{\nu+2}(z) - z^{-\nu-1}K_{\nu+1}(z).$$

Combining these formulas with (4.5)–(4.6), we get (4.7)–(4.8) for $\nu > 0$. □

5. PROOF OF THEOREM 1

For any $k \in (1,2)$, or even $k = 2$ if $n \geq 2$, we write

$$k = 2\ell - n + 1, \quad \text{whence} \quad \ell = \frac{k+n-1}{2} \in \left(\frac{n}{2}, \frac{n+1}{2}\right],$$

and we take the same initial data of Lemma 3.3, i.e.,

$$u_0(x) = \delta|D|^{-1}\langle x\rangle^{-\ell}, \qquad u_1(x) = 0 \quad (0 < \delta \ll 1).$$

By Lemma 4.4, we have

$$\hat{u}_0(\xi) = \delta|\xi|^{-1}G_\ell(\xi), \qquad \hat{u}_1(\xi) = 0 \quad (0 < \delta \ll 1), \tag{5.1}$$

where $G_\ell(\xi)$ is the kernel of Bessel potentials. These data satisfy the assumptions of Theorem A. Indeed, we have the following:

PROPOSITION 5.1. *For the data* $\{u_0, u_1\}$ *satisfying* (5.1), *we have*

$$\{u_0, u_1\} \in (\dot{H}^s \times H^{s-1}) \cap Y_k \quad \textit{for all } s \geq 1 \textit{ with } k = 2\ell - n + 1 > 1.$$

Hence, by Theorem A, for these data the problem (K) *has a unique solution* $u(x,t) \in \dot{X}^s(\mathbb{R})$ *for all* $s \geq 1$ *provided that* $0 < \delta \ll 1$. *Moreover, the function*

$$\tilde{c}(t) = \sqrt{1 + \|\nabla u(\cdot,t)\|_{L^2}^2}$$

satisfies (4.1) *in Lemma* 4.1.

Proposition 5.1 follows from Proposition 5.2 below, and hence we postpone its proof.

Now, by resorting to the linear theory and in particular to Lemma 3.3, we prove that the solution given by Proposition 5.1 is not asymptotically free.

Indeed, $u(x, t)$ is also a solution of the linear equation (L) with the coefficient given by

$$\tilde{c}(t) = \sqrt{1 + \|\nabla u(\cdot, t)\|_{L^2}^2},$$

and by Lemma 4.1 we know that $\tilde{c}(t)$ satisfies Assumption B. Thus in order to apply Lemma 3.3 it remains to prove that the function $\psi_\pm(t) := \int_0^t (\tilde{c}(\tau) - c_{\pm\infty})\, d\tau$ satisfies (3.3). Recalling equation (4.3) in Lemma 4.2, we have

$$I(\tilde{\vartheta}(t), t) = I(\tilde{\vartheta}(t), 0) + A_1(t) + A_2(t), \tag{5.2}$$

where we set

$$A_1(t) = -\frac{1}{2}\int_0^t \frac{\tilde{c}'(s)}{\tilde{c}(s)} J(\tilde{\vartheta}(t) - \tilde{\vartheta}(s), 0)\, ds,$$

$$A_2(t) = \frac{1}{2}\int_0^t \frac{\tilde{c}'(s)}{\tilde{c}(s)} \int_0^s \frac{\tilde{c}'(\sigma)}{\tilde{c}(\sigma)} \times \left(I(\tilde{\vartheta}(t) - \tilde{\vartheta}(s) + \tilde{\vartheta}(\sigma), \sigma) + \overline{I(-\tilde{\vartheta}(t) + \tilde{\vartheta}(s) + \tilde{\vartheta}(\sigma), \sigma)} \right) d\sigma\, ds.$$

The crucial point is the following asymptotic expansion:

$$\Im I(\tilde{\vartheta}(t), 0) = \mp c_{n,k}\delta^2 \tilde{c}(0)|t|^{-k} + o(|t|^{-k}) \quad \text{as } t \to \pm\infty, \tag{5.3}$$

where $k = 2\ell - n + 1$ (so that $1 < k \le 2$) and $c_{n,k}$ is some constant depending on n and k. We postpone the proof of (5.3) to Proposition 5.2 below. Herafter c_k (resp. $c_{n,k}$) stands for various constants depending on k (and n). Notice that δ_1 of (2.1) in Lemma 4.3 is the size of data. Hence, thanks to (5.1), there exists a constant $c_{n,k}$ such that $\delta_1 = c_{n,k}\delta^2$. Then, combining (4.1) with (4.4), we can conclude that

$$\begin{aligned} |A_1(t) + A_2(t)| &\le C(M\delta_1 + M^2\delta_1^2)M\delta_1(1 + |t|)^{-k} \\ &= c_{n,k}(M\delta^2 + M^2\delta^4)M\delta^2(1 + |t|)^{-k}. \end{aligned} \tag{5.4}$$

Now, we go back to equation (4.2) of Lemma 4.2. Then, using (5.2)–(5.4), and integrating equation (4.2), we see that, for large $|t|$,

$$\begin{aligned} \tilde{c}(t)^2 - c_{\pm\infty}^2 &= -\int_t^{\pm\infty} \Im I(\tilde{\vartheta}(\tau), 0)\, d\tau - \int_t^{\pm\infty} \Im(A_1(\tau) + A_2(\tau))\, d\tau \\ &\ge \gamma(k, n, \delta)|t|^{-k+1}, \end{aligned} \tag{5.5}$$

where we set

$$\gamma(k, n, \delta) \equiv c_{n,k}\delta^2\tilde{c}(0) - c'_{n,k}(M\delta^2 + M^2\delta^4)M\delta^2.$$

Here we note that $\gamma(k, n, \delta) > 0$, since $\tilde{c}(0) = \sqrt{1 + c_{n,k}\delta^2}$ and δ is small. Thus, dividing (5.5) by $\tilde{c}(t) + c_{\pm\infty}$ $(\le C_0)$, we get, for large $|t|$,

$$\tilde{c}(t) - c_{\pm\infty} \ge \gamma(k, n, \delta)C_0^{-1}|t|^{-k+1},$$

which implies that $|\psi_\pm(t)| \to +\infty$ as $t \to \pm\infty$ on account of our assumption $1 < k \le 2$. This proves (3.3) and concludes the proof of Theorem 1.

PROOF OF (5.3). Let us prove (5.3) and Proposition 5.1. Note that we have put $u_1(x) = 0$ in (5.1). It follows from the definition of $I(r, t)$ and $v_\pm(\cdot, t)$, and from Plancherel's theorem, that $I(\tilde{\vartheta}(t), 0)$ for data satisfying (5.1) can be rewritten as

$$I(\tilde{\vartheta}(t), 0) = -\tilde{c}(0) \int_{\mathbb{R}^n} e^{2i\tilde{\vartheta}(t)|\xi|} |\xi|^3 |\hat{u}_0(\xi)|^2 \, d\xi.$$

Then we prove the following:

PROPOSITION 5.2. *Let* $\{u_0(x), u_1(x)\}$ *be as in* (5.1). *Then* $I(\tilde{\vartheta}(t), 0)$ *has the asymptotic expansion* (5.3). *Moreover*

$$\Re I(\tilde{\vartheta}(t), 0) = \pm c_{n,k} \delta^2 \tilde{c}(0) |t|^{-k} + o(|t|^{-k}) \quad (t \to \pm\infty). \tag{5.6}$$

PROOF. Passing to polar coordinates $\xi = \rho\omega$, $\rho = |\xi|$, $\omega \in S^{n-1}$, we derive from Lemma 4.4 and (5.1) that

$$|\xi|^3 |\hat{u}_0(\xi)|^2 |\xi|^{n-1} = c_{n,k} \delta^2 \rho^{k-1} + r_k(\rho), \quad \text{with} \tag{5.7}$$
$$r_k^{(j)}(\rho) \sim c_{n,k,j} \rho^{k-j-1}, \quad j = 0, 1, 2, \text{ as } \rho \to 0,$$

$$|\xi|^3 |\hat{u}_0(\xi)|^2 |\xi|^{n-1} = c_{n,k} \delta^2 \rho^{(n+k+1)/2} e^{-2\rho} + \tilde{r}_k(\rho), \quad \text{with} \tag{5.8}$$
$$\tilde{r}_k^{(j)}(\rho) \sim c_{n,k,j} \rho^{(n+k+1)/2} e^{-2\rho}, \quad j = 0, 1, 2, \ldots, \text{ as } \rho \to \infty.$$

Now we introduce two C^∞ cut-off functions $\chi_1(\rho)$ and $\chi_3(\rho)$ on $[0, \infty)$ such that $0 \le \chi_1(\rho), \chi_3(\rho) \le 1$ for $\rho \ge 0$, $\chi_1(\rho) = 1$ for $0 \le \rho \le 1/2$ and $= 0$ for $\rho \ge 1$, and $\chi_3(\rho) = 1$ for $\rho \ge L$ and $= 0$ for $\rho \le L - 1$ with sufficiently large L. We set $\chi_2(\rho) = 1 - \chi_1(\rho) - \chi_3(\rho)$. Then we can write, by using (5.7)–(5.8),

$$I(\tilde{\vartheta}(t), 0) = -\tilde{c}(0) c_{n,k} \delta^2 (J_1(t) + J_2(t) + J_3(t)) + R(t), \tag{5.9}$$

where

$$J_1(t) = \int_0^\infty e^{2i\tilde{\vartheta}(t)\rho} \chi_1(\rho) \rho^{k-1} \, d\rho,$$
$$J_2(t) = \int_0^\infty e^{2i\tilde{\vartheta}(t)\rho} \chi_2(\rho) \rho^{n+2} |G_{(n+k-1)/2}(\rho)|^2 \, d\rho,$$
$$J_3(t) = \int_0^\infty e^{2i\tilde{\vartheta}(t)\rho} \chi_3(\rho) \rho^{(n+k+1)/2} e^{-2\rho} \, d\rho,$$
$$R(t) = \int_0^\infty e^{2i\tilde{\vartheta}(t)\rho} \chi_1(\rho) r_k(\rho) \, d\rho + \int_0^\infty e^{2i\tilde{\vartheta}(t)\rho} \chi_3(\rho) \tilde{r}_k(\rho) \, d\rho.$$

Let us develop asymptotic expansions of $J_m(t)$ $(m = 1, 2, 3)$ and $R(t)$. Having in mind that $1 < k \le 2$ and integrating by parts, we see that

$$\begin{aligned} J_1(t) = {} & -(2i\tilde{\vartheta}(t))^{-1}(k-1) \int_0^\infty e^{2i\tilde{\vartheta}(t)\rho} \rho^{k-2} \chi_1(\rho) \, d\rho \\ & - (2i\tilde{\vartheta}(t))^{-1} \int_0^\infty e^{2i\tilde{\vartheta}(t)\rho} \rho^{k-1} \chi_1'(\rho) \, d\rho. \end{aligned} \tag{5.10}$$

Let us rewrite the first member of the right hand side as

(5.11)
$$-(2i\tilde{\vartheta}(t))^{-1}\int_0^\infty e^{2i\tilde{\vartheta}(t)\rho}\rho^{k-2}\,d\rho+(2i\tilde{\vartheta}(t))^{-1}\int_0^\infty e^{2i\tilde{\vartheta}(t)\rho}\rho^{k-2}(1-\chi_1(\rho))\,d\rho.$$

We need the following lemma.

LEMMA 5.3. *If $0<\alpha<1$, then, for some complex constant c_α,*

$$\int_0^\infty e^{-ix\rho}\rho^{-1+\alpha}\,d\rho=c_\alpha x^{-\alpha},\qquad \forall x>0. \tag{5.12}$$

Moreover, if $\chi_1(\rho)$ is the cut-off function defined above, we have

$$\int_0^\infty e^{-ix\rho}\chi_1(\rho)\,d\rho=-i\,\frac{1}{x}+o(x^{-1})\qquad (x\to+\infty).$$

PROOF. To get the first equality it is sufficient to effect the change of variable $x\rho=\rho'$. To prove the second one, integrate by parts and observe that $\chi_1'(\rho)$ has compact support in $(0,\infty)$. □

Asymptotic expansion of $\Im J_1(t)$. Taking the imaginary part of the first term in (5.11) and using (5.12), we see that for $1<k<2$,

$$\Im(\text{1st term of (5.11)})=c_k(2\tilde{\vartheta}(t))^{-1}|2\vartheta(t)|^{-(k-1)}=\pm c_k|2\tilde{\vartheta}(t)|^{-k}. \tag{5.13}$$

Taking into account that $v(\rho):=\rho^{k-2}(1-\chi_1(\rho))$ is of class C^∞ and $v^{(j)}(0)=v^{(j)}(\infty)=0$, $j=0,1,\dots$, we have, by N-fold integration by parts for every $N=1,2,\dots$,

$$\text{2nd term of (5.11)}=o(\tilde{\vartheta}(t)^{-N})\quad \text{as } t\to\infty. \tag{5.14}$$

Hence we conclude from (5.11), (5.13) and (5.14) that

$$\Im\left(-(2i\tilde{\vartheta}(t))^{-1}\int_0^\infty e^{2i\tilde{\vartheta}(t)\rho}\rho^{k-2}\chi_1(\rho)\,d\rho\right)=\pm c_k|2\tilde{\vartheta}(t)|^{-k}+o(|\tilde{\vartheta}(t)|^{-k}) \tag{5.15}$$

as $t\to\pm\infty$. In a similar way, we have, by N-fold integration by parts,

$$-(2i\tilde{\vartheta}(t))^{-1}\int_0^\infty e^{2i\tilde{\vartheta}(t)\rho}\rho^{k-1}\chi_1'(\rho))\,d\rho=o(\tilde{\vartheta}(t)^{-N}) \tag{5.16}$$

for every $N\geq 2$ as $t\to\infty$. Summarizing (5.10), (5.15) and (5.16), we arrive at

$$\Im J_1(t)=\pm c_k|2\tilde{\vartheta}(t)|^{-k}+o(|\tilde{\vartheta}(t)|^{-k})\quad \text{as } t\to\pm\infty. \tag{5.17}$$

In the case when $n\geq 2$ and $k=2$, we integrate (5.10) by parts once more to obtain (5.17).

Asymptotic expansion of $\Re J_1(t)$. We go back to (5.11). Then, proceeding in the same way as for the imaginary part, we arrive at

$$\Re J_1(t)=-c_k|\tilde{\vartheta}(t)|^{-k}+o(|\tilde{\vartheta}(t)|^{-k})\quad \text{as } t\to\pm\infty. \tag{5.18}$$

Asymptotic expansions of $J_2(t)$, $J_3(t)$ and $R(t)$. Set $v(\rho) := \chi_2(\rho)\rho^n|G_{(n+k-1)/2}(\rho)|^2$. Since $v(\rho)$ is of class C_0^∞ with support in $\{\rho; 1/2 \le \rho \le L\}$, we have, by N-fold integration by parts,

(5.19) $$J_2(t) = o(\tilde{\vartheta}(t)^{-N}) \quad \text{for every } N \ge 2 \text{ as } t \to \pm\infty.$$

It remains to obtain asymptotic expansions of $J_3(t)$ and $R(t)$. Notice that $v(\rho) := \chi_3(\rho)\rho^{(n+k-3)/2}e^{-2\rho}$ is of class C^∞ and satisfies $v^{(j)}(0) = v^{(j)}(\infty) = 0$, $j = 0, 1, \dots$. Then we obtain, by N-fold integration by parts,

(5.20) $$J_3(t) = o(\tilde{\vartheta}(t)^{-N}) \quad \text{for every } N \ge 2 \text{ as } t \to \pm\infty.$$

In a similar way, we conclude that

(5.21) $$R(t) = o(|\tilde{\vartheta}(t)|^{-k}) \quad \text{as } t \to \pm\infty.$$

Finally, noting $|\tilde{\vartheta}(t)| \cong |t|$, and summarizing (5.17)–(5.21), we arrive at

(5.22) $$\Im \sum_{m=1}^{3} J_m(t) = \pm c_k|t|^{-k} + o(|t|^{-k}) \quad \text{as } t \to \pm\infty,$$

(5.23) $$R(t) = o(|t|^{-k}) \qquad \text{as } t \to \pm\infty.$$

Completion of proof of Proposition 5.2. We combine equation (5.9) with (5.22)–(5.23) to deduce (5.3). In a similar way, by (5.18) and (5.19)–(5.21) we get (5.6). The proof of Proposition 5.2 is now finished. □

Thanks to Proposition 5.2, we are now in a position to prove Proposition 5.1:

PROOF OF PROPOSITION 5.1. By Proposition 5.2 we have

$$|I(\tilde{\vartheta}(t), 0)| \lesssim |t|^{-k} \quad \text{for all } t \in \mathbb{R}.$$

Therefore, the data $\{u_0, u_1\}$ satisfying (5.1) belongs to Y_k. On the other hand, we can conclude from Lemma 4.4 that $\{u_0, u_1\} \in \dot{H}^s \times H^{s-1}$ for all $s \ge 1$. Thus these data satisfy the assumptions of Theorem A. Hence (K) has a unique solution $u(x,t) \in \dot{X}^s(\mathbb{R})$ provided the size δ of $u_0(x)$ is sufficiently small. Furthermore, for this solution $u(x,t)$, the function $\tilde{c}(t)$ satisfies (4.1) in Lemma 4.1. The proof of Proposition 5.1, hence also of Theorem 1, is complete. □

CONCLUDING REMARK. We go back to the choice (5.1), i.e., $u_1(x) = 0$. We may take $u_1(x) = \delta\langle x\rangle^{-(n+k-1)/2}$. For such a choice we need to obtain an asymptotic expansion of $\Re J_1(t)$. Actually, we can derive $\Re J_1(t) \sim |t|^{-k}$ $(t \to \pm\infty)$ by minor modifications of the former computations. In this way, we get Proposition 5.2.

6. PROOF OF PROPOSITION 2.2

Our argument follows [17]. Recall the definition of $I(r,t)$ in §4:

$$I(r,t) = (|D|e^{2ir|D|}v_-(\cdot,t), v_+(\cdot,t))_{L^2}, \qquad r, t \in \mathbb{R}.$$

Let $\{u_0, u_1\} \in H^{\sigma(p),p} \times H^{\sigma(p)-1,p}$, where $\sigma(p) = n(2/p-1)+1$. Notice that $k(p) > 1$ if and only if $1 \le p < 2(n-1)/(n+1)$, where

$$k(p) = \frac{n-1}{2}\left(\frac{2}{p} - 1\right) = \frac{n-1}{2}\left(\frac{1}{p} - \frac{1}{q}\right) \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Then the classical Strichartz estimates imply that, for $n \ge 4$,

$$\begin{aligned}
\sup_{r\in\mathbb{R}} (1+|r|)^{k(p)}|I(r,0)| &= \sup_{r\in\mathbb{R}} (1+|2r|)^{k(p)}|(|D|e^{2ir|D|}v_-(\cdot,0), v_+(\cdot,0))_{L^2}| \\
&\le \sup_{r\in\mathbb{R}} (1+|2r|)^{k(p)}\|e^{2ir|D|}v_-(\cdot,0)\|_{L^q}\| \, |D|v_+(\cdot,0)\|_{L^p} \\
&\lesssim \|v_-(\cdot,0)\|_{H^{n(1/p-1/q),p}}\| \, |D|v_+(\cdot,0)\|_{L^p} \\
&\lesssim (\|u_0\|_{H^{n(1/p-1/q)+1,p}} + \|u_1\|_{H^{n(1/p-1/q),p}})(\|u_0\|_{H^{2,p}} + \|u_1\|_{H^{1,p}}) \\
&\lesssim (\|u_0\|_{H^{n(1/p-1/q)+1,p}} + \|u_1\|_{H^{n(1/p-1/q),p}})^2 = (\|u_0\|_{H^{\sigma(p),p}} + \|u_1\|_{H^{\sigma(p)-1,p}})^2
\end{aligned}$$

since $H^{\sigma(p),p} = H^{n(1/p-1/q)+1,p} \subset H^{2,p}$. Hence we get

$$\sup_{r\in\mathbb{R}} (1+|r|)^{k(p)}|I(r,0)| \le C_1(\|u_0\|_{H^{\sigma(p),p}} + \|u_1\|_{H^{\sigma(p)-1,p}})^2.$$

On the other hand, since $H^{\sigma(p)-1,p} \subset H^{n/(n-1)} \subset H^1$ by the Sobolev imbedding theorem, we can write $I(r,0)$ as an oscillatory integral by using Plancherel's theorem,

$$\begin{aligned}
(6.1)\quad I(r,0) &= \int_{\mathbb{R}^n} e^{ir|\xi|}|\xi|^3|\hat{u}_0(\xi)|^2\,d\xi + \int_{\mathbb{R}^n} e^{ir|\xi|}|\xi|\,|\hat{u}_1(\xi)|^2\,d\xi \\
&\quad + \int_{\mathbb{R}^n} e^{ir|\xi|}|\xi|^2\Re(\hat{u}_0(\xi)\overline{\hat{u}_1(\xi)})\,d\xi \equiv I_1(r,0) + I_2(r,0) + I_3(r,0).
\end{aligned}$$

A similar computation to that made above shows that $I_1(r,0)$, $I_2(r,0)$ and $I_3(r,0)$ satisfy the same estimate as that proved above for $I(r,0)$, thus we conclude that $\{u_0, u_1\} \in Y_{k(p)}$. The proof of Proposition 2.2 is now complete. □

7. Proof of Theorem 2

We go back to (6.1). We divide the proof into two cases: $p > 1$ and $p = 1$.

CASE I: $p \in (1, 2(n-1)/(n+1))$. In this case, for any $\ell_0 \in (n/p, n)$, take the data so that

$$(7.1)\qquad \hat{u}_0(\xi) = \delta|\xi|^{\ell_0-n}e^{-|\xi|/2}, \quad \hat{u}_1(\xi) = \delta|\xi|^{\ell_0-n+1}e^{-|\xi|/2}, \quad 0 < \delta \ll 1.$$

Then

$$(7.2)\qquad u_0(x) \in H^s \cap H^{r,p}, \quad u_1(x) \in H^{s-1} \cap H^{r-1,p}$$

for all $s, r \geq 0$. Hence it follows from Proposition 2.2 that (K) has a solution $u(x,t) \in X^s(\mathbb{R})$ (for all $s \geq 3/2$) with data satisfying (7.1). We can derive, passing to the polar coordinates $\xi = \rho\omega$, $\rho = |\xi|$, $\omega \in S^{n-1}$,

$$\begin{aligned}|\xi|^3|\hat{u}_0(\xi)|^2|\xi|^{n-1} = |\xi|\,|\hat{u}_1(\xi)|^2|\xi|^{n-1} &= \Re(|\xi|^2\hat{u}_0(\xi)\overline{\hat{u}_1(\xi)})|\xi|^{n-1}\\ &= \delta^2\rho^{2\ell_0-n+2}e^{-\rho},\end{aligned}$$

hence,

$$I(r,0) = \delta^2|S^{n-1}|(-\tilde{c}(0)+\tilde{c}(0)^{-1}+2i)\int_0^\infty e^{2ir\rho}\rho^{2\ell_0-n+2}e^{-\rho}\,d\rho.$$

The next lemma is convenient for deriving asymptotic expansions of oscillatory integrals.

LEMMA 7.1 (Erdélyi [7]). *Let $w(\rho)$ be of class C^N on $[\alpha,\beta]$ $(-\infty \leq \alpha < \beta \leq \infty)$. Then*

$$\int_\alpha^\beta e^{ir\rho}w(\rho)\,d\rho = \sum_{m=0}^{N-1} i^{m-1}w^{(m)}(\rho)r^{-m-1}e^{ir\rho}\Big|_{\rho=\alpha}^{\rho=\beta} + o(r^{-N}) \quad \text{as } r \to \pm\infty$$

provided that $w^{(N)}(\rho)$ is integrable over (α,β). For the remainder term $R_N(r)$ we obtain

$$R_N(r) = (-ir)^{-N}\int_\alpha^\beta e^{ir\rho}w^{(N)}(\rho)\,d\rho.$$

Since $v(\rho) := \rho^{2\ell_0-n+2}e^{-\rho}$ is at least of class $C^{[2\ell_0-n+2]}$ and satisfies $v^{(j)}(0) = v^{(j)}(\infty) = 0$ for $j = 0, 1, \dots, [2\ell_0-n+1]$, it follows from Lemma 7.1 that

$$I(r,0) = o(|r|^{-[2\ell_0-n+2]}) \quad \text{as } |r| \to \infty.$$

Therefore, it follows from the definition of Y_k that the data (7.1) belong to $Y_{[2\ell_0-n+2]}$. Notice that

$$2\ell_0 - n + 2 > \frac{2n}{p} - n + 2 > 4 + \frac{2}{n-1}, \quad \text{i.e.,} \quad 4 \leq [2\ell_0-n+2] \leq n+1$$

because $\ell_0 \in (n/p, n)$ and $1 < p < 2(n-1)/n + 1$.

In the limiting case $\ell_0 = n$, take the data so that

$$\hat{u}_0(\xi) = \delta\log\frac{1}{|\xi|}e^{-|\xi|/2}, \quad \hat{u}_1(\xi) = \delta|\xi|\log\frac{1}{|\xi|}e^{-|\xi|/2} \tag{7.3}$$

for $0 < \delta \ll 1$. These data also satisfy (7.2), and we can derive

$$\begin{aligned}|\xi|^3|\hat{u}_0(\xi)|^2|\xi|^{n-1} = |\xi|\,|\hat{u}_1(\xi)|^2|\xi|^{n-1} &= \Re(|\xi|^2\hat{u}_0(\xi)\overline{\hat{u}_1(\xi)})|\xi|^{n-1}\\ &= \delta^2\rho^{n+2}(\log\rho)^2e^{-\rho},\end{aligned}$$

hence, $v(\rho) = \delta^2\rho^{n+2}(\log\rho)^2e^{-\rho}$ is at least of class C^{n+1} and satisfies $v^{(j)}(0) = v^{(j)}(\infty) = 0$ for $j = 1, \dots, n+1$. Thus we conclude from Lemma 7.1 that

$$I(r,0) = o(|r|^{-(n+2)}) \quad \text{as } |r| \to \infty.$$

Therefore the data satisfying (7.3) belong to Y_{n+2}.

CASE II: $p = 1$. In this case, take the data so that

$$\hat{u}_0(\xi) = \delta e^{-|\xi|/2}, \quad \hat{u}_1(\xi) = \delta|\xi|e^{-|\xi|/2}, \quad 0 < \delta \ll 1. \tag{7.4}$$

Clearly, these data satisfy (7.2) and

$$\begin{aligned} |\xi|^3|\hat{u}_0(\xi)|^2|\xi|^{n-1} = |\xi|\,|\hat{u}_1(\xi)|^2|\xi|^{n-1} &= \Re(|\xi|^2\hat{u}_0(\xi)\overline{\hat{u}_1(\xi)})|\xi|^{n-1} \\ &= \delta^2\rho^{n+2}e^{-\rho}. \end{aligned}$$

Since $v(\rho) := \delta^2\rho^{n+2}e^{-\rho}$ satisfies $v^{(n+2)}(0) = c_n(n+2)!$ and $v^{(n+2)}(\infty) = v^{(j)}(0) = v^{(j)}(\infty) = 0$, $j = 1, \dots, n+1$, it follows from Lemma 7.1 that

$$I(r, 0) = (n+2)!c_n|r|^{-(n+3)} + o(|r|^{-(n+3)}) \quad \text{as } |r| \to \infty,$$

hence, the data (7.4) belong to Y_{n+3}. The proof of Theorem 2 is now complete. □

FINAL REMARK. (i) In view of Lemma 4.4, the data satisfying (7.1), (7.3) and (7.4) have a similar behaviour to $\{\delta\langle x\rangle^{-\ell}, \delta|D|\langle x\rangle^{-\ell}\}$ with $\ell \in (n/p, n)$ and $p \in (1, 2(n-1)/(n+1))$, $\ell = n$, $\ell > n$, respectively.

(ii) By the argument of the proof of Theorem 2 we can prove that the Schwartz space $\mathcal{S}$ is contained in Y_k for all $k \in (1, n+1]$.

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