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Mathematical analysis. — *Dissipative stochastic equations in Hilbert space with time dependent coefficients*, by GIUSEPPE DA PRATO and MICHAEL RÖCKNER, communicated on 23 June 2006.

ABSTRACT. — We prove existence and, under an additional assumption, uniqueness of an evolution system of measures $(\nu_t)_{t\in\mathbb{R}}$ for a stochastic differential equation with time dependent dissipative coefficients. We prove that if $P_{s,t}$ denotes the corresponding transition evolution operator, then $P_{s,t}\varphi$ behaves asymptotically as $t \to +\infty$ like a limit curve (which is independent of s) for any continuous and bounded "observable" φ.

KEY WORDS: Dissipative stochastic equations; evolution systems of measures; mixing.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 47D07, 37L40, 35B40, 47H06.

1. INTRODUCTION

We are given a separable Hilbert space H (with norm $|\cdot|$ and inner product $\langle\cdot,\cdot\rangle$); we denote by $L(H)$ the space of all bounded linear operators in H. We are also given a cylindrical Wiener process defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge 0}, \mathbb{P})$ in H.

We are concerned with the following stochastic differential equation:

$$dX = (AX + F(t,X))dt + \sqrt{C}\, dW(t), \qquad X(s) = x \in H, \tag{1.1}$$

where $A\colon D(A) \subset H \to H$ is the infinitesimal generator of a C_0 semigroup e^{tA} in H, $C \in L(H)$ is symmetric nonnegative and $F\colon D(F) \subset \mathbb{R}\times H \to H$ is such that $F(t,\cdot)$ is dissipative for all $t \in \mathbb{R}$.

When s is negative, in order to give a meaning to equation (1.1), we shall extend $W(t)$ and the filtration $(\mathscr{F}_t)_{t\ge 0}$ for all $t < 0$. To do so we take another cylindrical process $W_1(t)$ independent of $W(t)$ and set

$$\overline{W}(t) = \begin{cases} W(t) & \text{if } t \ge 0, \\ W_1(-t) & \text{if } t \le 0. \end{cases}$$

Moreover, we denote by $\overline{\mathscr{F}}_t$ the σ-algebra generated by $\overline{W}(s)$, $s \le t$, $t \in \mathbb{R}$, $k \in \mathbb{N}$.

Concerning A, C, F we shall assume

HYPOTHESIS 1.1. (i) *There is $\omega > 0$ such that $\langle Ax, x\rangle \le -\omega|x|^2$ for all $x \in D(A)$.*
(ii) *$C \in L(H)$ is symmetric, nonnegative and such that*

$$\int_0^{+\infty} \mathrm{Tr}[e^{tA}Ce^{tA^*}]\,dt < +\infty.$$

(iii) *$F : \mathbb{R} \times H \to H$ is continuous and there exist $M > 0$ and $K > 0$ such that*

$$|F(t, 0)| \leq M, \quad |F(t, x) - F(t, y)| \leq K|x - y| \quad \text{for all } x, y \in H,\ t \in \mathbb{R}.$$

Moreover,

$$\langle F(t, x) - F(t, y), x - y \rangle \leq 0 \quad \text{for all } x, y \in H,\ t \in \mathbb{R}.$$

A *mild solution* $X(t, s, x)$ of (1.1) is an adapted stochastic process $X \in C([s, T]; L^2(\Omega, \mathscr{F}, \mathbb{P}))$ such that

$$X(t, s, x) = e^{(t-s)A}x + \int_s^t e^{(t-r)A} F(r, X(r, s, x))\, dr + W_A(t, s), \quad t \geq s, \tag{1.2}$$

where $W_A(t, s)$ is the *stochastic convolution*

$$W_A(t, s) = \int_s^t e^{(t-r)A}\sqrt{C}\, d\overline{W}(r), \quad t \geq s. \tag{1.3}$$

It is well known that, in view of Hypothesis 1.1(ii), $W_A(t, s)$ is a Gaussian random variable in H with mean 0 and covariance operator $Q_{t,s}$ given by

$$Q_{t.s}x = \int_s^t e^{rA}Ce^{rA^*}x\, dr, \quad t \geq s,\ x \in H, \tag{1.4}$$

and that there exists a unique mild solution of (1.1) (see e.g. [5]). We define the transition evolution operator

$$P_{s,t}\varphi(x) = \mathbb{E}[\varphi(X(t, s, x))], \quad t \geq s,\ \varphi \in C_b(H),$$

where $C_b(H)$ is the Banach space of all continuous and bounded mappings $\varphi : H \to \mathbb{R}$ endowed with the sup norm

$$\|\varphi\|_0 = \sup_{x \in H} |\varphi(x)|.$$

REMARK 1.2. Since $(t, s, x) \mapsto X(t, s, x)$ is mean square continuous, hence continuous in distribution, it follows by definition that $P_{s,t}$ is *Feller*, that is, $P_{s,t}\varphi \in C_b(H)$ for all $\varphi \in C_b(H)$ and any $s < t$.

The aim of this paper is to prove the existence and, under a suitable condition, uniqueness of an *evolution system of measures* $(\nu_t)_{t \in \mathbb{R}}$ indexed by $\mathbb{R}$ (see [2]). This means that each ν_t is a probability measure on H and

$$\int_H P_{s,t}\varphi(x)\, \nu_s(dx) = \int_H \varphi(x)\, \nu_t(dx) \quad \text{for all } \varphi \in C_b(H),\ s < t. \tag{1.5}$$

This concept is the natural generalization of the notion of an invariant measure to nonautonomous systems. We notice that an evolution system of measures indexed by $\mathbb{R}$ is a measure solution of the corresponding (dual) Kolmogorov equation on all the real line. So, it is a generalization of a measure solution of the Kolmogorov equations on half-lines (see [1]).

Using the system $(\nu_t)_{t\in\mathbb{R}}$ we are able to study the asymptotic behaviour of $P_{s,t}\varphi(x)$. We prove that

$$\lim_{s\to-\infty} P_{s,t}\varphi(x) = \int_H \varphi(y)\,\nu_t(dy), \qquad \forall t \in \mathbb{R},\ x \in H \tag{1.6}$$

and

$$\lim_{t\to+\infty}\left[P_{s,t}\varphi(x) - \int_H \varphi(y)\,\nu_t(dy)\right] = 0, \qquad \forall s \in \mathbb{R},\ x \in H. \tag{1.7}$$

The second result implies that $P_{s,t}\varphi(x)$ approaches as $t \to +\infty$ a curve, parametrized by t, which is independent of s and x. This is the natural generalization of the strongly mixing property for an autonomous dissipative system (see e.g. [4, §3.4]).

In a paper in preparation we shall study the case when the coefficient $F(t,x)$ is singular, generalizing the results in [3].

2. EXISTENCE AND UNIQUENESS OF AN EVOLUTION FAMILY OF MEASURES INDEXED BY $\mathbb{R}$

It is convenient to write equation (1.1) as a family of deterministic equations indexed by $\omega \in \Omega$. Setting $Y(t) = X(t,s,x) - W_A(t,s)$, we see that $Y(t)$ is a mild solution of the deterministic evolution equation

$$Y'(t) = AY(t) + F(t, Y(t) + W_A(t,s)), \quad Y(s) = x. \tag{2.1}$$

In the following we shall treat $Y(t)$ as a classical solution for simplicity. This argument can be made rigorous by approximating W_A with $n(n-A)^{-1}W_A$ and x with $n(n-A)^{-1}x$ and then letting $n \to \infty$.

LEMMA 2.1. *For any $m \in \mathbb{N}$ there is $C_m > 0$ such that*

$$\mathbb{E}(|X(t,s,x)|^{2m}) \le C_m(1 + e^{-m\omega(t-s)}|x|^{2m}). \tag{2.2}$$

PROOF. Multiplying (2.1) by $|Y(t)|^{2m-2}Y(t)$ and taking into account Hypothesis 1.1 yields, for a suitable constant C_m^1,

$$\begin{aligned}
\frac{1}{2m}\frac{d}{dt}|Y(t)|^{2m} &\le -\omega|Y(t)|^{2m} + \langle F(t, W_A(t,s)), Y(t)\rangle |Y(t)|^{2m-2}\\
&\quad + \langle F(t, Y(t) + W_A(t,s)) - F(t, W_A(t,s)), Y(t)\rangle |Y(t)|^{2m-2}\\
&\le -\omega|Y(t)|^{2m} + \langle F(t, W_A(t,s)), Y(t)\rangle |Y(t)|^{2m-2}\\
&\le -\frac{\omega}{2}|Y(t)|^{2m} + C_m^1|F(W_A(t,s))|^{2m}.
\end{aligned}$$

By a standard comparison result it follows that

$$|Y(t)|^{2m} \le e^{-m\omega(t-s)}|x|^{2m} + 2mC_m^1\int_s^t e^{-m\omega(t-\sigma)}|F(\sigma, W_A(t,\sigma))|^{2m}\,d\sigma,$$

and finally we find that, for some constant C_m^2,

$$|X(t,s,x)|^{2m} \le C_m^2 e^{-m\omega(t-s)}|x|^{2m} + C_m^2\left(\int_s^t e^{-m\omega(t-\sigma)}|F(\sigma, W_A(t,\sigma))|^{2m}\,d\sigma + |W_A(t,s)|^{2m}\right). \tag{2.3}$$

Now the conclusion follows by taking expectation, recalling that in view of Hypothesis 1.1,

$$|F(t,x)| \le |F(t,0)| + |F(t,x) - F(t,0)| \le M + K|x|, \qquad t \in \mathbb{R},\ x \in H,$$

and using the fact that

$$\sup_{t\in\mathbb{R},\, t\ge s} \mathbb{E}|W_A(t,s)|^{2m} < +\infty. \qquad \square$$

The following lemma gives a generalization to the time dependent case of a result proved in [4].

LEMMA 2.2. *Assume that Hypothesis* 1.1 *holds. Then for any* $t \in \mathbb{R}$, *there exists* $\eta_t \in L^2(\Omega, \mathscr{F}, \mathbb{P})$ *(independent of* x*) such that*

$$\lim_{s\to-\infty} X(t,s,x) = \eta_t \quad \text{in } L^2(\Omega, \mathscr{F}, \mathbb{P}). \tag{2.4}$$

Moreover,

$$\mathbb{E}|X(t,s,x) - \eta_t|^2 \le 2e^{-2\omega(t-s)}(|x|^2 + C_2). \tag{2.5}$$

PROOF. Let $h > 0$ and set $Z(t) = X(t,s,x) - X(t,s-h,x),\ t \ge s$. Then $Z(t)$ is a mild solution of the following problem:

$$\begin{cases} Z'(t) = AZ(t) + F(t, X(t,s,x)) - F(t, X(t,s-h,x)), \\ Z(s) = x - X(s,s-h,x). \end{cases} \tag{2.6}$$

Multiplying (2.6) by $Z(t)$ and taking into account Hypothesis 1.1 yields

$$\frac{1}{2}\frac{d}{dt}|Z(t)|^2 \le -\omega|Z(t)|^2.$$

Therefore

$$|X(t,s,x) - X(t,s-h,x)|^2 = |Z(t)|^2 \le e^{-2\omega(t-s)}|x - X(s,s-h,x)|^2.$$

Now, by Lemma 2.1 it follows that

$$\mathbb{E}|X(t,s,x) - X(t,s-h,x)|^2 \le 2e^{-2\omega(t-s)}(|x|^2 + C_2(1 + e^{-2\omega h}|x|^2)). \tag{2.7}$$

Consequently, by the completeness of $L^2(\Omega, \mathscr{F}, \mathbb{P})$, for any $t \in \mathbb{R}$ and any $x \in H$, there exists the limit

$$\lim_{s\to-\infty} X(t,s,x) := \eta_t(x) \quad \text{in } L^2(\Omega, \mathscr{F}, \mathbb{P}).$$

Moreover, letting $h \to \infty$ yields (2.5) (if we know that $\eta_t(x)$ is independent of x).

It remains to show that $\eta_t(x)$ is independent of x.

Let $x, y \in H$ and set $V(t) = X(t,s,x) - X(t,s,y)$. Then $V(t)$ is the solution of the following problem:

$$\begin{cases} V'(t) = AV(t) + F(t, X(t,s,x)) - F(t, X(t,s,y)), \\ V(s) = x - y. \end{cases} \tag{2.8}$$

Multiplying (2.8) by $V(t)$ and taking into account Hypothesis 1.1 yields

$$\frac{1}{2}\frac{d}{dt}|V(t)|^2 \le -\omega |V(t)|^2, \quad t \in \mathbb{R},$$

so that

$$|X(t,s,x) - X(t,s,y)|^2 = |V(t)|^2 \le e^{-2\omega(t-s)}|x-y|^2, \quad t, s \in \mathbb{R}.$$

Letting $s \to -\infty$ we see that $\eta_t(x) = \eta_t(y)$, as required. □

In the following we shall denote by ν_t the law of η_t, $t \in \mathbb{R}$.

PROPOSITION 2.3. *$(\nu_t)_{t\in\mathbb{R}}$ is an evolution system of measures indexed by $\mathbb{R}$,*

$$\int_H P_{s,t}\varphi(x)\,\nu_s(dx) = \int_H \varphi(x)\,\nu_t(dx), \quad s \le t,\ \varphi \in C_b(H). \tag{2.9}$$

Moreover, for all $\varphi \in C_b(H)$ we have

$$\lim_{s\to-\infty} P_{s,t}\varphi(x) = \int_H \varphi(y)\,\nu_t(dy), \quad x \in H. \tag{2.10}$$

PROOF. Let us first prove (2.10). Let $\varphi \in C_b(H)$. Letting $s \to -\infty$ in the identity

$$P_{s,t}\varphi(x) = \mathbb{E}[\varphi(X(t,s,x))]$$

and recalling (2.4) yields

$$\lim_{s\to-\infty} P_{s,t}\varphi(x) = \mathbb{E}[\varphi(\eta_t)] = \int_H \varphi(y)\,\nu_t(dy),$$

and (2.10) is proved. Let us prove (2.9). Let $s < t < \tau$. Letting $s \to -\infty$ in the identity

$$P_{s,t}P_{t,\tau}\varphi(x) = P_{s,\tau}\varphi(x),$$

recalling Remark 1.2 and taking into account (2.10) yields

$$\int_H P_{t,\tau}\varphi(y)\,\nu_t(dy) = \int_H \varphi(y)\,\nu_\tau(dy). \quad □$$

The following result gives information on the asymptotic behaviour of $P_{s,t}\varphi(x)$ when $t \to +\infty$.

PROPOSITION 2.4. *Let $\varphi \in C_b^1(H)$. Then for any $s \in \mathbb{R}$ and $x \in H$, we have*

$$\lim_{t\to+\infty}\left[P_{s,t}\varphi(x) - \int_H \varphi(x)\,\nu_t(dx)\right] = 0. \tag{2.11}$$

PROOF. Fix $s \in \mathbb{R}$ and $x \in H$ and choose $s_1 < s$. Set, for $t > s$,

$$X(t) = X(t, s, x), \qquad Y(t) = X(t, s_1, x), \qquad Z(t) = X(t) - Y(t).$$

Then we have

$$\frac{d}{dt}Z(t) = AZ(t) + F(t, X(t)) - F(t, Z(t)), \qquad Z(s) = x - X(s, s_1, x).$$

Multiplying scalarly both sides of this identity by $Z(t)$ and taking into account the dissipativity of $F(t,\cdot)$ yields

$$\frac{d}{dt}|Z(t)|^2 \le 2\omega|Z(t)|^2,$$

so that

$$|X(t, s, x) - X(t, s_1, x)|^2 = |Z(t)|^2 \le e^{-2\omega(t-s)}|x - X(s, s_1, x)|^2.$$

Letting $s_1 \to -\infty$ yields

$$|X(t, s, x) - \eta_t|^2 = |Z(t)|^2 \le e^{-2\omega(t-s)}|x - \eta_s|^2.$$

Consequently,

$$\begin{aligned}\left|P_{s,t}\varphi(x) - \int_H \varphi(x)\,\nu_t(dx)\right|^2 &= |\mathbb{E}[\varphi(X(t, s, x))] - \mathbb{E}[\varphi(\eta_t)]|^2\\ &\le \|\varphi\|^2_{C_b^1(H)}\mathbb{E}(|X(t, s, x) - \eta_t|^2) \le \|\varphi\|^2_{C_b^1(H)}e^{-2\omega(t-s)}\mathbb{E}(|x - \eta_s|^2),\end{aligned}$$

which yields the conclusion. □

We end the paper with a uniqueness result.

PROPOSITION 2.5. *Assume that $(\zeta_t)_{t\in\mathbb{R}}$ is an evolution system of measures indexed by $\mathbb{R}$ and that there exists $C > 0$ such that*

$$\sup_{t\in\mathbb{R}}\int_H |x|^2\,\zeta_t(dx) \le C.$$

Then $\zeta_t = \nu_t$ for all $t \in \mathbb{R}$.

PROOF. Let $\varphi \in C_b^1(H)$. By the assumption we have, for $s < t$,

$$\int_H P_{s,t}\varphi(x)\,\zeta_s(dx) = \int_H \varphi(x)\,\zeta_t(dx).$$

We claim that

$$\lim_{s\to-\infty}\int_H P_{s,t}\varphi(x)\,\zeta_s(dx) = \int_H \varphi(x)\,\nu_t(dx). \tag{2.12}$$

By the claim it follows that $\zeta_t = \nu_t$ by the arbitrariness of φ. To prove the claim write

$$\int_H P_{s,t}\varphi(x)\,\zeta_s(dx) = \int_H\left(P_{s,t}\varphi(x) - \int_H \varphi(y)\,\nu_t(dy)\right)\zeta_s(dx) + \int_H \varphi(y)\,\nu_t(dy). \tag{2.13}$$

But, since

$$P_{s,t}\varphi(x) - \int_H \varphi(y)\,\nu_t(dy) = \mathbb{E}(\varphi(X(t,s,x) - \varphi(\eta_t)),$$

we have, taking into account (2.5),

$$\begin{aligned}\left|P_{s,t}\varphi(x) - \int_H \varphi(y)\,\nu_t(dy)\right|^2 &\le \|\varphi\|^2_{C_b^1(H)}\mathbb{E}(|X(t,s,x) - \eta_t)|^2)\\ &\le 2e^{-2\omega(t-s)}(|x|^2 + C_2)\|\varphi\|^2_{C_b^1(H)}.\end{aligned}$$

So,

$$\left|\int_H \left(P_{s,t}\varphi(x) - \int_H \varphi(y)\,\nu_t(dy)\right)\zeta_s(dx)\right|^2 \le 2\|\varphi\|^2_{C_b^1(H)} e^{-2\omega(t-s)}\left(C_2 + \int_H |x|^2\,\zeta_s(dx)\right),$$

and the conclusion follows. □

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