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Ordinary differential equations. — *Existence of periodic solutions for some second order quasilinear Hamiltonian systems*, by MARIO GIRARDI and MICHELE MATZEU, communicated on 12 May 2006.

ABSTRACT. — A class of second order nonautonomous quasilinear Hamiltonian systems (S) is considered. We show that, for any $T < T_0$, where T_0 depends on the growth coefficients of the Hamiltonian function H, there exists a T-periodic and $T/2$-antiperiodic solution of the system (S) below, provided two symmetry conditions hold for H.

KEY WORDS: Quasilinear Hamiltonian systems; periodic solutions; mountain pass techniques; iteration methods.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 34C25, 49R99, 58E99, 47H15.

INTRODUCTION

In the present paper we are interested in the existence of periodic solutions to the following second order quasilinear Hamiltonian system

$$(S) \qquad -\ddot{u}(t) = b(t)\nabla H(u(t), \dot{u}(t)), \qquad t \in \mathbb{R},$$

where ∇H denotes the gradient of H with respect to its first variable in $\mathbb{R}^N$, and b is a periodic function.

In [2] the authors, in collaboration with D. De Figueiredo, introduced a method in order to solve a quasilinear elliptic equation by a variational approach using mountain pass techniques and some estimates for mountain pass solutions of semilinear problems suitably approximating the quasilinear problem.

A similar approach allows one to solve another quasilinear problem with more general assumptions (see [3]), a semilinear integro-differential equation with nonsymmetric kernel (see [5]), finally a fully nonlinear elliptic equation (see [4]).

The aim of the present paper is to use some basic ideas of [2]–[5] in order to find periodic solutions of (S).

As a matter of fact, in [2]–[5] it is very important that the Poincaré inequality holds in the variational approach in order to show that the approximating solutions actually converge to a solution of the initial problem. In the present case, we consider a variational approach in which, thanks to some symmetry assumptions on H, one can use the Wirtinger inequality, which still allows us to prove the convergence of the approximating solutions.

The T-periodicity of the solution found is ensured for any $T < T_0$, where T_0 depends on the growth coefficients of H with respect to its two variables and on the maximum of $b(\cdot)$, which is further supposed to be $T/2$-periodic. Indeed one finds that this solution is even $T/2$-antiperiodic.

Obviously, as a particular case, one can also consider autonomous Hamiltonian systems, if one chooses $b(t) \equiv 1$.

1. The result

Let $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a function continuously differentiable in the first variable (let ∇H denote the corresponding gradient) and continuous in the second variable. Let b be a continuous periodic function on $\mathbb{R}$.

The problem is to find a nonzero periodic solution to the following second order quasilinear Hamiltonian system:

$$(S) \qquad -\ddot{u}(t) = b(t)\nabla H(u(t), \dot{u}(t)), \qquad t \in \mathbb{R}.$$

We prove the following

THEOREM 1. *Let b and H be as above and suppose that:*

$(H0)$ *∇H is Lipschitz continuous in a sufficiently large neighbourhood of the origin,*
$(H1)$ *$H(x, y) = H(-x, y)$ for all $x, y \in \mathbb{R}^N$,*
$(H2)$ *$H(x, y) = H(x, -y)$ for all $x, y \in \mathbb{R}^N$,*
$(H3)$ *$\lim_{x\to 0} H(x, y)/|x|^2 = 0$ uniformly with respect to $y \in \mathbb{R}^N$,*
$(H4)$ *there exist $a_1 > 0$, $p > 1$, and $r \in (0, 1)$ such that*

$$|\nabla H(x, y)| \le a_1(1 + |x|^p)(1 + |y|^r) \qquad \forall x, y \in \mathbb{R}^N,$$

$(H5)$ *there exist $\vartheta > 2$ such that*

$$0 < \vartheta H(x, y) \le x\nabla H(x, y) \qquad \forall x \in \mathbb{R}^N \setminus \{0\},\ \forall y \in \mathbb{R}^N,$$

$(H6)$ *there exist $a_2, a_3 > 0$ such that*

$$H(x, y) \ge a_2|y|^\vartheta - a_3 \qquad \forall x, y \in \mathbb{R}^N.$$

Then there exists a positive number T_0, explicitly depending on $p, r, \vartheta, a_1, a_2, a_3, N$ and $B = \max\{|b(t)| : t \in [0, T]\}$, such that, for all $T \in (0, T_0)$, if b is a $T/2$-periodic function, then there exists a nonzero classical T-periodic and $T/2$-antiperiodic solution of (S).

REMARK 1. From $(H4)$ and $(H5)$ it follows that $\vartheta \le p + 1$.

REMARK 2. Let us point out that, if $b(t) \equiv 1$, then Theorem 1 applies and yields the existence of a T-periodic and $T/2$-antiperiodic nonzero solution, for any $T < T_0$, of the *autonomous* Hamiltonian system

$$-\ddot{u}(t) = \nabla H(u(t), \dot{u}(t)), \qquad t \in \mathbb{R},$$

if $(H0), \dots, (H6)$ hold.

REMARK 3. Note that, as a consequence of the T-periodicity and $T/2$-antiperiodicity of the solution $u(\cdot)$, one gets

$$\int_0^T u(t)\,dt = 0.$$

As a standard example of a function H satisfying $(H0), \ldots, (H6)$, one can consider $\nabla H(x, y) = h(x, y)$ with

$$h(x, y) = \begin{cases} \tilde{a}_1 x|x|^{p-1}(1 + |y|^{\beta}), & |x| \le \delta_1,\ |y| \le \delta_2, \\ x|x|^{p-1}|y|^r, & |x| \ge \tilde{\delta}_1,\ |y| \ge \tilde{\delta}_2, \end{cases}$$

where $p, \beta > 1$, $r \in (0, 1)$, and $\tilde{a}_i$, δ_j, $\tilde{\delta}_j$ $(j = 1, 2)$ are fixed positive constants with $\tilde{\delta}_j > \delta_j$.

2. A VARIATIONAL APPROACH TO THE PROBLEM

Let us consider the space

$$V = \{v \in H^1([0, T]; \mathbb{R}^N) : v(0) = v(T)\}$$

and its L^2 orthogonal decomposition

$$V = V_0 \oplus V_1 \oplus V_2$$

where

$$V_0 = \mathbb{R}^N = \left\{ \frac{1}{T} \int_0^T v(t)\,dt : v \in V \right\},$$

$$V_1 = \left\{ v \in V : v(t) = \sum_{\substack{k=2h+1 \\ h \in \mathbb{Z}}} a_k e^{i2k\pi t/T},\ a_{-k} = \overline{a}_k \right\},$$

$$V_2 = \left\{ v \in V : v(t) = \sum_{\substack{k=2h \\ h \in \mathbb{Z}\setminus\{0\}}} a_k e^{i2k\pi t/T},\ a_{-k} = \overline{a}_k \right\}.$$

Fix w in V_1 and consider the functional

$$I_w(v) = \frac{1}{2} \int_0^T |\dot{v}(t)|^2\,dt - \int_0^T b(t) H(v(t), \dot{w}(t))\,dt \qquad \forall v \in V.$$

PROPOSITION 1. *Let $b(\cdot)$ be $T/2$-periodic. Then any critical point u_w of I_w on V_1 is a weak solution of the following second order Hamiltonian system with T-periodic boundary conditions:*

$$(S_w) \qquad \begin{cases} -\ddot{u}_w(t) = b(t) \nabla H(u_w(t), \dot{u}_w(t)), \\ u_w(0) = u_w(T), \quad \dot{u}_w(0) = \dot{u}_w(T). \end{cases}$$

Moreover, u_w is $T/2$-antiperiodic.

PROOF. Let u_w be critical on V_1, that is,

$$\langle I'_w(u_w), v_1\rangle = 0 \quad \forall v_1 \in V_1.$$

We have to prove that

$$\langle I'_w(u_w), v_0 + v_1 + v_2\rangle = 0 \quad \forall v = v_0 + v_1 + v_2 \in V = V_0 \oplus V_1 \oplus V_2$$

(as the critical points of I_w on the whole space V are the weak solutions of (S_w), and, if $u_w \in V_1$, then u_w is $T/2$-antiperiodic).

It is enough to prove that

$$\langle I'_w(u_w), v_0\rangle = \langle I'_w(u_w), v_2\rangle = 0 \quad \forall v_0 \in V_0,\ \forall v_2 \in V_2.$$

On the other hand,

$$I'_w(u_w(t)) = -u''_w(t) - b(t)\nabla H(u_w(t), \dot w(t))$$

and one has, by $(H1)$, $(H2)$ and the $T/2$-periodicity of $b(\cdot)$,

$$\begin{aligned} (1)\qquad b(t+T/2)\nabla H(u_w(t+T/2), \dot w(t+T/2)) &= b(t+T/2)\nabla H(-u_w(t), -\dot w(t)) \\ &= -b(t)\nabla H(u_w(t), -\dot w(t)) = -b(t)\nabla H(u_w(t)), \dot w(t)). \end{aligned}$$

Therefore the function $b(t)\nabla H(u_w(t), \dot w(t))$ is $T/2$-antiperiodic, so it is orthogonal to V_2 as well as to V_0, as it has zero mean.

Since $-u''_w$ obviously belongs to V_1, the proof is complete. □

In the following we will put

$$\|v\| = \left(\int_0^T |\dot v(t)|^2\, dt\right)^{1/2} \quad \forall v \in V_1,$$

which is a norm equivalent to the H^1-norm of V, as the Wirtinger inequality holds in the space V_1.

Therefore the functional I_w, for any $w \in V_1$, has the form

$$I_w(v) = \frac{1}{2}\|v\|^2 - \int_0^T b(t)H(v(t), \dot w(t))\, dt \quad \forall v \in V_1.$$

3. PROOF OF THEOREM 1

First of all, fix $R > 0$ and put

$$C_R = \{v \in V_1 \cap C^2([0,T]) : \|v\|_{C^2([0,T])} \le R\}.$$

PROPOSITION 2. *For any $w \in C_R$, there exists a mountain pass critical point of $u_w \in C_R$ for I_w on V_1 (as defined in Step 3 below).*

We prove Proposition 2 in several steps.

STEP 1. *Let $w \in C_R$. Then there exist $\rho_R, \alpha_R > 0$ depending on R, but not on w, such that*

$$I_w(v) \geq \alpha_R \quad \forall v \in V_1 : \|v\| = \rho_R.$$

PROOF. From $(H1)$ it follows that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$H(v(t), \dot{w}(t)) < \frac{1}{2}\varepsilon|v(t)|^2 \quad \forall t \in [0, T] : |v(t)| \leq \delta,$$

hence

$$\int_0^T H(v(t), \dot{w}(t))\, dt < (\varepsilon/2)\int_0^T |\dot{v}(t)|^2\, dt + K(1+R)^r \int_0^T |v(t)|^{p+1}\, dt$$
$$\leq K'(\varepsilon/2 + K(1+R)^r \|v\|^{p-1})\|v\|^2$$

with a constant K' depending on the Wirtinger and Sobolev inequalities. Choosing

$$\|v\| = \left(\frac{\varepsilon}{2K(1+R)^r}\right)^{1/p-1} = \rho_R$$

one gets

$$\int_0^T H(v, \dot{w}) \leq K'\varepsilon\|v\|^2.$$

Recalling that $B = \max_{t\in[0,T]} |b(t)|$, if one chooses $\varepsilon < 1/2BK'$ and $\alpha_R = (1/2 - K'\varepsilon)\rho_R^2$, the assertion follows. □

STEP 2. *Let $w \in C_R$ and fix $\tilde{v}$ in V_1 with $\|\tilde{v}\| = 1$. Then there exists $\tilde{s} > 0$, independent of w and R, such that*

$$I_w(s\tilde{v}) \leq 0 \quad \forall s \geq \tilde{s}; \tag{1}$$

then $\overline{v} = \tilde{s}\tilde{v}$ satisfies

$$\|\overline{v}\| > \rho_R, \quad I_w(\overline{v}) \leq 0.$$

PROOF. It follows from $(H6)$ that

$$I_w(s\tilde{v}) \leq \frac{1}{2}s^2 - a_2|s|^\vartheta \int_0^T |\overline{v}|^\vartheta\, dt + a_3 T.$$

By the Sobolev embedding theorem, as $\vartheta \leq p+1$ (see Remark 1), one gets

$$I_w(s\tilde{v}) \leq \frac{1}{2}s^2 - a_2|s|^\vartheta (S_\vartheta)^\vartheta + a_3 T$$

where S_ϑ is the embedding constant of V_1 in $L^\vartheta([0, T])$. Since $\vartheta > 2$, one gets some $\tilde{s}$ such that (1) holds.

STEP 3. *Let $w \in C_R$. Then there exists a mountain pass critical point u_w for I_w on V_1, that is,*

$$I_w(u_w) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) \tag{2}$$

where

$$\Gamma = \{\gamma \in C^0([0,1];\mathbb{R}^N) : \gamma(0) = 0,\ \gamma(1) = \overline{v}\}$$

and

$$I_w(u_w) \geq \alpha_R > 0 \quad (\Rightarrow u_w \not\equiv 0).$$

PROOF. This is a consequence of a theorem by Ambrosetti and Rabinowitz (see [1]), as $I_w(0) = 0$, Steps 1 and 2 hold, and the Palais–Smale condition is trivially satisfied, due to the continuous embedding of V_1 into $L^\infty(0, T; \mathbb{R}^N)$ and the fact that $p > 2$. □

PROPOSITION 3. *Let $w \in C_R$ and let u_w be a mountain pass solution given by Proposition 2. Then there exists a positive constant $c_1(R)$ depending on R, but not on w, such that*

$$\|u_w\| \geq c_1(R).$$

PROOF. Actually the estimate holds for any critical point u_w of I_w on V_1 with $u_w \not\equiv 0$ and one does not use the mountain pass nature of u_w. Indeed, if one puts $v = u_w$ in the relation

$$\int_0^T \dot{u}_w \dot{v} = \int_0^T b(t)\nabla H(u_w, \dot{w})v \quad \forall v \in V_1,$$

one gets

$$\|u_w\|^2 = \int_0^T b(t)\nabla H(u_w, \dot{w})u_w\, dt. \tag{3}$$

From $(H3)$, $(H4)$, $(H5)$, it follows that, for any $\varepsilon > 0$, there exists a positive constant $c_{\varepsilon,R}$, depending on ε and R, but not on w, such that

$$|\nabla H(u_w, \dot{w})| \leq \varepsilon|u_w| + c_{\varepsilon,R}|u_w|^p.$$

Together with this inequality, (3) yields

$$\|u_w\|^2 \leq B\left(\varepsilon \int_0^T |u_w|^2 + c_{\varepsilon,R}\int_0^T |u_w|^{p+1}\right),$$

hence, by the Wirtinger inequality and the continuous Sobolev embedding,

$$(1 - \varepsilon(T/2\pi)^2)\|u_w\|^2 \leq \tilde{c}_{\varepsilon,R}\|u_w\|^{p+1},$$

which implies the assertion if we choose $\varepsilon < (2\pi/T)^2$, as $p + 1 > 2$. □

PROPOSITION 4. *4 Let $w \in C_R$ and let u_w be a mountain pass solution given by Proposition 2. Then there exists a constant $c_2 > 0$, independent of w and R, such that*

$$\|u_w\| \leq c_2.$$

PROOF. From the inf-maximum characterization of u_w, by choosing γ in Γ as the line segment joining 0 and $\overline{v}$, one gets

$$I_w(u_w) \le \sup_{s \ge 0} I_w(s\overline{v}),$$

hence, by $(H6)$,

$$I_w(u_w) \le B \sup_{s\ge 0}\left\{ \frac{s^2}{2}\int_0^T |\dot{\overline{v}}|^2 - a_2|s|^\vartheta \int_0^T |\overline{v}|^\vartheta + a_3 T\right\}.$$

Since $\vartheta > 2$, such an upper bound is a maximum and it does not depend on R and w, hence

$$I_w(u_w) \le \text{ const } \quad \forall R > 0,\ \forall w \in C_R.$$

At this point, using the criticality of u_w for I_w, $(H5)$ and (3), one gets

$$\frac{1}{2}\|u_w\|^2 \le \text{const} + \frac{1}{\vartheta}\int_0^T b(t)\nabla H(u_w, \dot w)u_w = \text{const} + \frac{1}{\vartheta}\|u_w\|^2,$$

from which the estimate follows, as $\vartheta > 2$. $\square$

Actually, for any $w \in C_R$, any mountain pass solution of (S_w) is not only weak, but even of class C^2, so a classical solution, since it solves a problem of the type

$$\begin{cases} -\ddot u_w(t) = \varphi(t)\ (= b(t)\nabla H(u_w(t), \dot w(t))), & t \in [0, T],\\ u_w(0) = u_w(T), \quad \dot u_w(0) = \dot u_w(T), \end{cases} \tag{4}$$

where φ belongs to $C^0([0, T])$.

PROPOSITION 5. *There exists a constant $\mu > 0$ such that the mountain pass solutions u_w of* (4) *satisfy, for any $R > 0$, and any $w \in C_R$,*

$$\|u_w\|_{C^2([0,T])} \le \mu(1 + R^r).$$

PROOF. This is a consequence of $(H4)$, Proposition 4 and the fact that u_w solves (4) in the classical sense. $\square$

PROPOSITION 6. *There exists a constant $\overline{R} > 0$ such that*

$$w \in C_{\overline{R}} \ \Rightarrow\ u_w \in C_{\overline{R}}$$

for any mountain pass solution u_w of (4).

PROOF. This easily follows from Proposition 5 and the fact that $r < 1$. $\square$

At this point it is very natural to introduce an iterative scheme in the following way. Let $\overline{R}$ be given by Proposition 6 and let u_0 be arbitrarily fixed in $C_{\overline{R}}$. Let us define u_n as a mountain pass solution of the following problem, for any $n \in \mathbb{N}$:

$$(S_n)\ \begin{cases} -\ddot u_n(t) = b(t)\nabla H(u_n(t), \dot u_{n-1}(t)), & t \in [0, T],\\ u_n(0) = u_n(T), \quad \dot u_n(0) = \dot u_n(T). \end{cases}$$

Obviously, by Proposition 6, one has $u_n \in C_{\overline{R}}$ for any $n \in \mathbb{N}$.

Now we are in a position to give

Proof of Theorem 1. Let u_n be a mountain pass solution of (S_n) for any $n \in \mathbb{N}$. First of all we prove that there exists some positive number T_0 such that, if $T < T_0$, then the whole sequence $\{u_n\}$ strongly converges in V_1.

Indeed, using (S_n) and (S_{n+1}), one gets

$$\int_0^T \dot{u}_{n+1}(t)(\dot{u}_{n+1}(t) - \dot{u}_n(t))\,dt = \int_0^T b(t)\nabla H(u_{n+1}(t), \dot{u}_n(t))(u_{n+1}(t) - u_n(t))\,dt,$$
$$\int_0^T \dot{u}_n(t)(\dot{u}_{n+1}(t) - \dot{u}_n(t))\,dt = \int_0^T b(t)\nabla H(u_n(t), \dot{u}_{n-1}(t))(u_{n+1}(t) - u_n(t))\,dt,$$

which yields

$$\begin{aligned} (5) \qquad & \|u_{n+1} - u_n\|^2 \\ & \leq B\int_0^T |\nabla H(u_{n+1}(t), \dot{u}_n(t)) - \nabla H(u_n(t), \dot{u}_{n-1}(t))|\,|u_{n+1}(t) - u_n(t)|\,dt \\ & \leq B\int_0^T |\nabla H(u_{n+1}(t), \dot{u}_n(t)) - \nabla H(u_n(t), \dot{u}_n(t))|\,|u_{n+1}(t) - u_n(t)|\,dt \\ & \quad + B\int_0^T |\nabla H(u_n(t), \dot{u}_n(t)) - \nabla H(u_n(t), \dot{u}_{n-1}(t))|\,|u_{n+1}(t) - u_n(t)|\,dt. \end{aligned}$$

Denoting by $c'_{\overline{R}}$, $c''_{\overline{R}}$ the best Lipschitz constants of ∇H with respect to its two variables in the set $B_{\overline{R}} \times B_{\overline{R}}$ where

$$B_{\overline{R}} = \{x \in \mathbb{R}^N : |x| \leq \overline{R}\},$$

one gets from (5) the relation

$$\|u_{n+1} - u_n\|^2 \leq B\left(c'_{\overline{R}}\int_0^T |u_{n+1} - u_n|^2\,dt + c''_{\overline{R}}\|u_n - u_{n-1}\|\left(\int_0^T |u_{n+1} - u_n|^2\,dt\right)^{1/2}\right).$$

Using the Wirtinger inequality (as u_n and u_{n+1} are T-periodic with zero mean), one obtains

$$\|u_{n+1} - u_n\|^2 \leq B\left(c'_{\overline{R}}\left(\frac{T}{2\pi}\right)^2\|u_{n+1} - u_n\|^2 + c''_{\overline{R}}\left(\frac{T}{2\pi}\right)\|u_n - u_{n-1}\|\,\|u_{n+1} - u_n\|\right),$$

from which

$$(6) \qquad \|u_{n+1} - u_n\| \leq \frac{Bc''_{\overline{R}}(T/2\pi)}{1 - Bc'_{\overline{R}}(T/2\pi)^2}\|u_n - u_{n-1}\| = \gamma\|u_n - u_{n-1}\|,$$

where γ is positive if

$$T < \frac{2\pi}{\sqrt{Bc'_{\overline{R}}}}.$$

Actually, by putting

$$T_0 = \min\left(\frac{2\pi}{\sqrt{Bc'_{\overline{R}}}}, \frac{\sqrt{B^2(c''_{\overline{R}})^2 - 4Bc'_{\overline{R}}} - Bc''_{\overline{R}}}{2Bc'_{\overline{R}}}\right) \tag{7}$$

and if $T < T_0$, the constant γ in (6) is less than 1. Therefore if $T < T_0$, given by (7), then (6) implies that $\{u_n\}$ is a Cauchy sequence in V_1, so it strongly converges to some u in V_1.

At this point, from the Ascoli–Arzelà's theorem and the fact that $\{u_n\}$ is contained in $C_{\overline{R}}$, it follows that the whole sequence $\{u_n\}$ converges in $C^2([0, T])$. Then it is easily verified that it converges to a classical solution u of (S). The fact that u is not identically zero is an immediate consequence of Proposition 3, if we put $R = \overline{R}$. □

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