Mathematical physics. — *A mathematical model of frying processes*, by ALBERTO MANCINI, MARIO PRIMICERIO and YIQING YANG, communicated on 21 April 2006.

ABSTRACT. — We present a mathematical model for the process of frying a rather thick sample of an indeformable porous material saturated with water (e.g. a potato slice).

The model is based on thermodynamical arguments and results in an initial-boundary value problem for a system of equations satisfied by the temperature and vapour content, with a free boundary separating the region saturated with water and the vapour region.

We provide some results of numerical simulations.

1. INTRODUCTION

In the last decade, several papers have been devoted to the mathematical model of frying, starting from [2], [3] that abandoned the purely phenomenological approach of previous attempts and analyzed and discussed the different phenomena of mass and energy transfer that are involved in the process.

Referring to one-dimensional geometry with planar symmetry, a slab $x \in (-L, L)$ of porous material saturated with water and having a given temperature is considered to be put in contact, on the faces $x = \pm L$, with an oil bath kept at temperature T_∞ above the boiling point of water. This is assumed to happen starting from time $t = 0$ and frying is described as a coupling between heat transfer (conduction and convection) and vapour migration in an undeformable porous medium (see [1] and [8]).

As a matter of fact, this idealized situation exhibits most of the basic ingredients that seem to be relevant e.g. in the process of frying relatively thick slices of potato. Moreover it is suitable for rather easy numerical simulation and hence for possible experimental validation. Of course, once this preliminary check is obtained, modification induced in the organic material by the thermal history (see [8], [9] and e.g. [7] for an introduction to the influence of the process on the result) as well as more realistic geometry will be taken into account.

But, even in the idealized situation described above, the problem is far from being trivial. In recent papers [5], [6] a complete analysis of the phenomenon, based on a correct application of the basic balance laws, has been performed. There the mathematical model is formulated in terms of a non-standard free boundary problem for a system of parabolic equations. Indeed, the region $x \in (-L, L)$, $t > 0$ (of course, using symmetry the analysis is confined to $(0, L) \times \mathbb{R}^+$) is the union of four subregions separated by a free boundary: the water-saturated part, the region of coexistence of vapour and water, the pure vapour region and the crust.

Here, we present a simplified model in which the region of coexistence is assumed to have a negligible thickness, while incorporating in the model the correct Rankine–Hugoniot type conditions. Moreover, in the model that will be presented and discussed, crust formation is neglected although a simple modification of the boundary condition on $x = L$ is suggested in order to take this effect into account.

Thus, only one free boundary and two regions are considered: region 1, where the porous medium is completely saturated by water, and region 2 in which the pores are filled by water vapour.

A crucial point is to assume that the porous medium is non-deformable, and thus frying processes of thin layers (e.g. tortilla chips, see [9]) will need a substantially different model.

After evaluating the time scales, it turns out to be possible to partially decouple the problem for the unknown temperature from the problem for the unknown pressure, suggesting how to investigate the well-posedness of the problem.

Finally, numerical simulations are shown for some specific cases.

2. THE GOVERNING EQUATIONS AND THE BOUNDARY CONDITIONS

According to the discussion of the previous section, we write the balance equations in the two regions.

In region 1 (water-saturated porous medium), the pressure is constant, since $\partial p/\partial x = 0$ at $x = 0$ and the medium is assumed to be non-deformable (and water compressibility is neglected). Thus, the only equation we have to consider is the energy balance

$$(\rho c)_1 \frac{\partial T}{\partial t} - k_1 \frac{\partial^2 T}{\partial x^2} = 0,$$

where $(\rho c)_1$ and k_1 are respectively heat capacity and conductivity in region 1, which are essentially given by

$$(2.1) \qquad\qquad (\rho c)_1 = (1 - \varepsilon)\rho_s c_s + \varepsilon \rho_w c_w,$$

$$(2.2) \qquad\qquad k_1 = (1 - \varepsilon)k_s + \varepsilon k_w.$$

In (2.1) and (2.2), we denote by ε the porosity, and ρ, c and k represent density, specific heat and conductivity respectively, while the subscripts s and w refer to solid matter and water, respectively.

In region 2 (solid + vapour) we have to consider mass balance in addition to energy conservation. Indeed, vapour migration is induced by the pressure gradient, so that Darcy's law implies

$$\varepsilon \frac{\partial \rho_v}{\partial t} - \frac{K}{\mu} \frac{\partial}{\partial x}\left(\rho_v \frac{\partial p}{\partial x}\right) = 0,$$

where the subscript v refers to vapour and K, μ denote intrinsic permeability of the porous medium and dynamic viscosity of vapour.

Passing to energy balance, we should consider that, in principle, heat capacity and conductivity depend on ρ_v, but it is reasonable to assume that

$$(\rho c)_2 = (1 - \varepsilon)\rho_s c_s + \varepsilon \rho_v c_v \approx (1 - \varepsilon)\rho_s c_s,$$
$$k_2 = (1 - \varepsilon)k_s + \varepsilon k_v(\rho_v) \approx (1 - \varepsilon)k_s.$$

The most important difference with energy balance in region 1 is that now a relevant role is played by convection. Indeed, we have

$$(\rho c)_2 \frac{\partial T}{\partial t} - k_2 \frac{\partial^2 T}{\partial x^2} - c_v \frac{K}{\mu} \rho_v \frac{\partial p}{\partial x} \frac{\partial T}{\partial x} = 0,$$

where T_0 is the boiling point temperature at atmospheric pressure.

Let us turn to discussing the conditions on the external boundary and at the initial time. For the latter, we have e.g.

$$T(x, 0) = \bar{T} < T_0, \quad 0 < x < L,$$

where we may, in general, allow $\bar{T}$ to depend on x. Moreover, for symmetry reasons,

$$\frac{\partial T}{\partial x}(0, t) = 0, \quad t > 0.$$

On the boundary $x = L$, we write

$$-k_1 \frac{\partial T}{\partial x}(L, t) = \gamma_1(T(L, t) - T_\infty), \quad 0 < t < t^*,$$

and

$$-k_2 \frac{\partial T}{\partial x}(L, t) = \gamma_2(T(L, t) - T_\infty), \quad t > t^*,$$

where T_∞ is the (prescribed) temperature of the oil bath, γ_i are thermal exchange coefficients (in principle, γ_2 depends on the thermal history of the surface and on the discharge of vapour) and

$$t^* = \sup_t \{T(L, t) < T_0\}.$$

Moreover,

$$p(L, t) = p_0, \quad t > t^*,$$

where p_0 is the atmospheric pressure.

Finally, we have to write the conditions on the free boundary $x = s(t)$, $t > t^*$, separating the two regions. To simplify notation we write $T^\pm \equiv T(s(t)\pm 0, t)$ and similarly for the other quantities. Thus, we have

(2.3) $$T^- = T_0.$$

Pressure is given by the Clapeyron law

(2.4) $$p^+ = p_0 \exp\left\{\frac{\lambda}{R}\left(\frac{1}{T_0} - \frac{1}{T^+}\right)\right\},$$

where λ is the latent heat of vapourization and R is the gas constant. Note that on the r.h.s. we had to write the temperature of vapour, since temperature will be discontinuous across the free boundary, as a consequence of the fact that in our scheme the thickness of the region occupied by saturated vapour is neglected.

Imposing global mass and energy balance and differentiating as in the procedure used to obtain Rankine–Hugoniot jump conditions, we obtain

$$\varepsilon(\rho_w - \rho_v^+)s'(t) = \frac{K}{\mu}\rho_v^+ p_x^+, \quad t > t^*,$$

which will be approximated by

(2.5) $$\varepsilon\rho_w s'(t) = \frac{K}{\mu}\rho_v^+ p_x^+, \quad t > t^*,$$

and

(2.6) $$-\varepsilon\lambda\rho_w s'(t) + k_1 T_x^- - k_2 T_x^+ - (\rho c)_2(T^+ - T_0)s'(t)$$
$$- c_v(T^+ - T_0)\frac{K}{\mu}\rho_v^+ p_x^+ = 0, \quad t > t^*,$$

We will finally assume that, in region 2, ρ_v, p and T are related by a known state equation. For simplicity, we take

$$p = RT\rho_v.$$

Of course, the "initial" condition for $s(t)$ is

$$s(t^*) = L.$$

3. RESCALING

We set

$$x = \xi L, \quad T = uT_0, \quad p = vp_0,$$

and we rescale time by a constant σ that will be chosen later,

$$t = \sigma\tau.$$

Then, we define

$$\rho_0 = p_o/(RT_0), \quad \tau^* = t^*/\sigma,$$

so that $\tau^* = \sup_\tau\{u(1, \tau) < 1\}$. We also define the rescaled free boundary between the water-saturated region and the vapour region,

$$z(\tau) = \begin{cases} 1, & \tau \le \tau^*, \\ s(\sigma\tau)/L, & \tau > \tau^*. \end{cases}$$

Then, after simple calculation, we get

$$\frac{\partial u}{\partial \tau} = \frac{\sigma}{t_1}\frac{\partial^2 u}{\partial \xi^2}, \quad 0 < \xi < z(\tau), \ \tau > 0,$$

(3.1) $$\frac{\partial u}{\partial \tau} = \frac{\sigma}{t_2}\frac{\partial^2 u}{\partial \xi^2} + \varepsilon\frac{\rho_0 c_v}{(\rho c)_2}\frac{\sigma}{t_3}\frac{v}{u}\frac{\partial v}{\partial \xi}\frac{\partial u}{\partial \xi}, \quad z(\tau) < \xi < 1, \ \tau > \tau^*,$$

(3.2) $$\frac{\partial v}{\partial \tau} - \frac{v}{u}\frac{\partial u}{\partial \tau} = \frac{\sigma}{t_3}u\frac{\partial}{\partial \xi}\left[\frac{v}{u}\frac{\partial v}{\partial \xi}\right], \quad z(\tau) < \xi < 1, \ \tau > \tau^*,$$

where

$$t_1 = \frac{L^2(\rho c)_1}{k_1}, \quad t_2 = \frac{L^2(\rho c)_2}{k_2}, \quad t_3 = \frac{L^2 \varepsilon}{p_0} \frac{\mu}{K}.$$

Initial and fixed boundary conditions read

$$u(\xi, 0) = \bar{u}(\xi) = \bar{T}(\xi)/T_0, \quad 0 < \xi < 1,$$

$$\frac{\partial u}{\partial \xi}(0, \tau) = 0, \quad \tau > 0,$$

$$-\frac{\partial u}{\partial \xi}(1, \tau) = \Gamma_1[u(1, \tau) - u_\infty], \quad 0 < \tau < \tau^*,$$

$$-\frac{\partial u}{\partial \xi}(1, \tau) = \Gamma_2[u(1, \tau) - u_\infty], \quad \tau > \tau^*,$$

$$v(1, \tau) = 1, \quad \tau > \tau^*.$$

where $\Gamma_1 = \gamma_1 L/k_1$, $\Gamma_2 = \gamma_2 L/k_2$, $u_\infty = T_\infty/T_0$.

Next, we consider the interface conditions. From (2.5), we have

$$(3.3) \qquad \frac{dz}{d\tau} = \frac{\sigma}{t_3} \frac{\rho_0}{\rho_w} \frac{v^+}{u^+} \left(\frac{\partial v}{\partial \xi}\right)^+, \quad \tau > \tau^*,$$

From (2.6), we obtain

$$(3.4) \qquad \frac{dz}{d\tau} = \frac{(\rho c)_1 T_0}{\varepsilon \lambda \rho_w} \frac{\sigma}{t_1} u_\xi^- - \frac{(\rho c)_2 T_0}{\varepsilon \lambda \rho_w} \frac{\sigma}{t_2} u_\xi^+$$

$$- \frac{(\rho c)_2 T_0}{\varepsilon \lambda \rho_w}(u^+ - 1)\frac{dz}{d\tau} - \frac{\rho_0 c_v T_0}{\lambda \rho_w} \frac{\sigma}{t_3} v^+ v_\xi^+ \frac{u^+ - 1}{u^+}, \quad \tau > \tau^*.$$

Finally, (2.3) and (2.4) become respectively

$$u^- = 1,$$

$$v(z(\tau), \tau) = \exp\left\{ \frac{\lambda}{RT_0}\left(1 - \frac{1}{u^+}\right) \right\}, \quad \tau > \tau^*.$$

At this point, it is natural to choose

$$\sigma = \frac{\varepsilon \lambda \rho_w L^2}{k_1 T_0},$$

so that, setting

$$\theta = \frac{(\rho c)_2 T_0}{\varepsilon \lambda \rho_w},$$

condition (3.4) becomes

$$(3.5) \qquad \frac{dz}{d\tau} = u_\xi^- - \frac{k_2}{k_1} u_\xi^+ - \theta(u^+ - 1)\frac{dz}{d\tau} - \varepsilon \frac{\rho_0 c_v}{(\rho c)_1} \frac{t_1}{t_3} v^+ v_\xi^+ \frac{u^+ - 1}{u^+}, \quad \tau > \tau^*.$$

4. A REASONABLE SIMPLIFIED MODEL

It is immediately checked that, while σ and the characteristic diffusion times t_1 and t_2 are of the same order of magnitude (using the data of [2] we have indeed $\sigma \simeq 325$, $t_1 \simeq 846$, $t_2 \simeq 870$), $t_3 = 0.38$ and thus σ/t_3 is of the order 10^3.

Therefore, it is reasonable to substitute (3.2) with the much simpler relationship

$$\frac{\partial}{\partial \xi}\left(\frac{v}{u}\frac{\partial v}{\partial \xi}\right) = 0, \quad z(\tau) < \xi < 1, \ \tau > \tau^*.$$

Note that (3.1), (3.3) and (3.5) contain the ratios σ/t_3, t_2/t_3, t_1/t_3, but there are also multiplicative factors of the order $\rho_0/\rho_w \sim 10^{-3}$.

Now, we use again symbols x and t instead of ξ and τ and denoting by capital letters $A, B, C, \ldots$ the constants $O(1)$ appearing in the equations written in Section 3, we can state the following classical formulation of our problem:

Find a constant t^*, a decreasing function $z(t)$, $z(t^*) = 1$, $z(t) > 0$ and two functions $u(x, t)$, $v(x, t)$ possessing all the regularity we will need and such that:

(4.1)
$$\begin{cases} u_t = Au_{xx} & \text{in } D_0 = (0, 1) \times (0, t^*), \\ u(x, 0) = \bar{u}, & x \in (0, 1), \\ u_x(0, t) = 0, & t \in (0, t^*), \\ -u_x(1, t) = \Gamma_1[u(1, t) - u_\infty], & t \in (0, t^*), \end{cases}$$

where $\bar{u} \in (0, 1)$ and $A = \sigma/t_1$,

(4.2) $$0 < u(x, t) < 1 \quad \text{in } D_0,$$

(4.3) $$u(1, t^*) = 1;$$

(4.4)
$$\begin{cases} u_t = Au_{xx} & \text{in } D_1 = \{(x, t) : 0 < x < z(t), \ t > t^*\}, \\ u(x, t^*+) = u(x, t^*-), & x \in (0, 1), \\ u_x(0, t) = 0, & t > t^*, \\ u(z(t), t) = 1, & t > t^*, \end{cases}$$
$$0 < u(x, t) < 1 \quad \text{in } D_1;$$

$$\begin{cases} u_t = Bu_{xx} + Cu_x vv_x/u & \text{in } D_2 = \{(x, t) : z(t) < x < 1, \ t > t^*\}, \\ -u_x(1, t) = \Gamma_2[u(1, t) - u_\infty], & t > t^*, \end{cases}$$
$$B = \sigma/t_2, \quad C = \varepsilon\rho_0 c_v \sigma/[(\rho c)_2 t_3],$$

(4.5) $$\left(\frac{vv_x}{u}\right)_x = 0 \quad \text{in } D_2,$$

$$v(z(t), t) = \exp\{G(1 - 1/u^+)\}, \quad t > t^*,$$
$$v(1, t) = 1, \quad t > t^*,$$

and

$$z'(t) = Hv^+ v_x^+/u^+, \quad t > t^*,$$
$$z'(t) = u_x^- - Mu_x^+ - \theta(u^+ - 1)z'(t) - Nv^+ v_x^+\left(\frac{u^+ - 1}{u^+}\right), \quad t > t^*,$$
$$1 < u(x, t) < u_\infty \quad \text{in } D_2,$$
$$v(x, t) > 1 \quad \text{in } D_2,$$

where $G = \lambda/[RT_0]$, $H = \sigma\rho_0/[t_3\rho_w]$, $M = k_2/k_1$ and $N = \varepsilon\rho_0 c_v t_1/[(\rho c)_1 t_3]$. Since (4.1) is a standard heat conduction problem we can solve it. Then (4.2) is a consequence of the maximum principle, and the existence of t^* follows immediately from (4.3) if $u_\infty > 1$.

Let us write, from (4.5),

$$\frac{v v_x}{u} = -f(t) \quad \text{in } D_2$$

for an unknown positive function $f(t)$, and consider the free boundary problem for $u(x, t)$ consisting of (4.4) and

$$u_t = B u_{xx} - C f(t) u_x \quad \text{in } D_2,$$
$$-u_x(1, t) = \Gamma_2(u(1, t) - u_\infty), \quad t > t^*,$$
$$z'(t) = -H f(t), \quad t > t^*,$$
$$z'(t) = u_x^- - M u_x^+ - \theta(u^+ - 1) z'(t) + N f(t)(u^+ - 1), \quad t > t^*,$$

with the condition on $f(t)$,

$$(v^2)_x = -2 f(t) u(x, t) \quad \text{in } D_2,$$

i.e.

(4.6)
$$f(t) = -\frac{1 - \exp\{2G(1 - 1/u^+)\}}{2 \int_{z(t)}^1 u(x, t) \, dx}.$$

Thus, for any given $f(t)$, we have transformed the original problem into a free boundary problem (in two "phases") for a single unknown function $u(x, t)$, whose solution in turn gives $f(t)$. The analysis of the well-posedness of this non-standard problem will be performed in a forthcoming paper.

5. NUMERICAL SIMULATION

We show here some numerical simulations where the simplified adimensional model has been used. With reference to the situation sketched in Fig. 1 (where the conditions on the
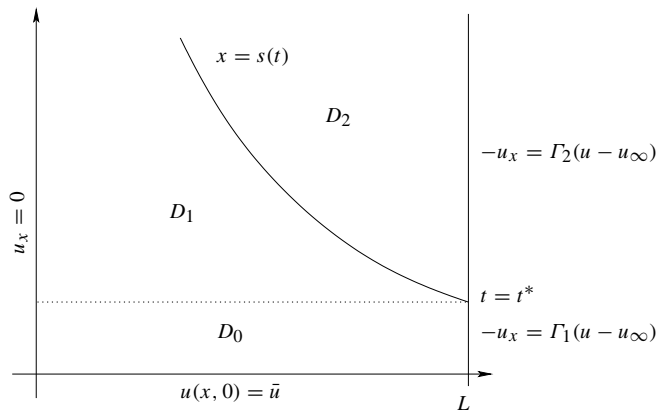

FIG. 1. Sketch of the problem.

fixed boundaries are shown) we have solved

$$(5.1) \qquad u_t = A u_{xx} \qquad \text{in } D_0 \cup D_1,$$

$$(5.2) \qquad u_t = B u_{xx} - C f(t) u_x \quad \text{in } D_2,$$

where $f(t)$ is given by (4.6) and $z(t)$ $(z(t^*) = 1)$ is the free boundary on which the following conditions are prescribed:

$$(5.3) \qquad u(z(t)^-, t) = 1,$$

$$(5.4) \qquad z'(t) = -H f(t),$$

$$(5.5) \qquad z'(t)[1 + \theta(u^+ - 1)] = u_x^- - M u_x^+ + N f(t)(u^+ - 1).$$

The following values have been taken for the physical quantities, according to [10], [11] and to the values used in the quoted literature (although some of them seem to be of rather difficult determination):

ρ_s	1.6	g/cm^3
ρ_v	0.001	g/cm^3
ρ_w	1.0	g/cm^3
c_s	0.74	cal/(g K)
c_v	0.48	cal/(g K)
c_w	1.0	cal/(g K)
k_s	0.00136	cal/(sec cm K)
k_w	0.00127	cal/(sec cm K)
L	1.0	cm
ε	0.3	
K	$1.0E-9$	darcy
μ	0.0013	poise
R	0.11	cal/(g $*$ K)
γ_1	0.002	cal/(sec $*$ cm^2 $*$ K)
γ_2	0.003	cal/(sec $*$ cm^2 $*$ K)
T_0	373.16	K
λ	540.2	cal/g
p_0	1013250.0	g/(cm $*$ sec^2)

Consequently, the constants appearing in (5.1)–(5.5) and (4.6) are the following:

$$A = 0.38, \quad B = 0.37,$$
$$C = 0.08,$$
$$H = 0.49, \quad \theta = 1.9,$$
$$M = 0.71, \quad N = 0.16,$$
$$G = 13.16.$$

The constants appearing in the initial and boundary conditions are

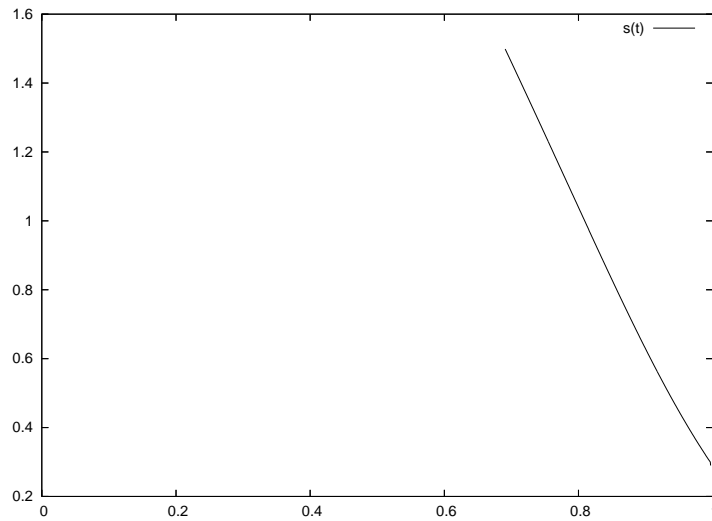$$u_0 = 0.745, \quad u_\infty = 1.428, \quad \Gamma_1 = 1.50, \quad \Gamma_2 = 3.15.$$

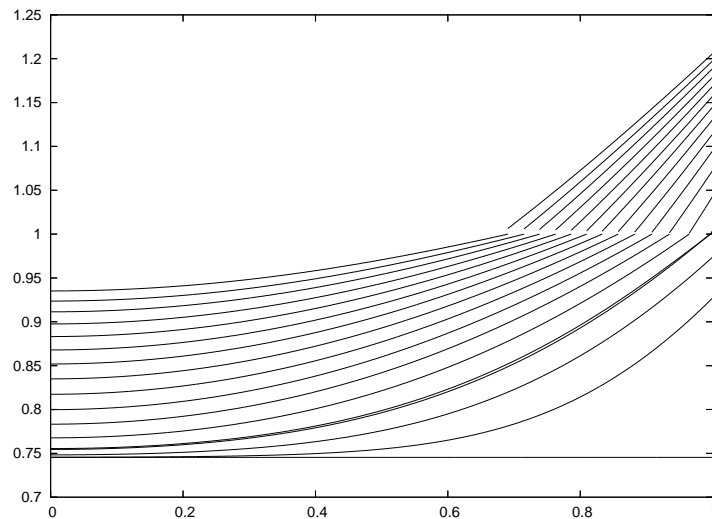FIG. 2. Evolution of the free boundary.

FIG. 3. Evolution of temperature profile.

In Figure 2 we show the evolution of the rescaled free boundary as a function of the rescaled time.

Figure 3 gives the rescaled temperature profile in the two zones, at different times.

In Figures 4 and 5 we display the same quantities, but we simulate the crust formation as a relaxation in the heat exchange coefficient Γ_2. In particular we assume that

$$\Gamma_2(0) = \Gamma_2, \qquad \dot{\Gamma}_2(t) = \gamma\,(u(1,t) - 1)_+\,.$$
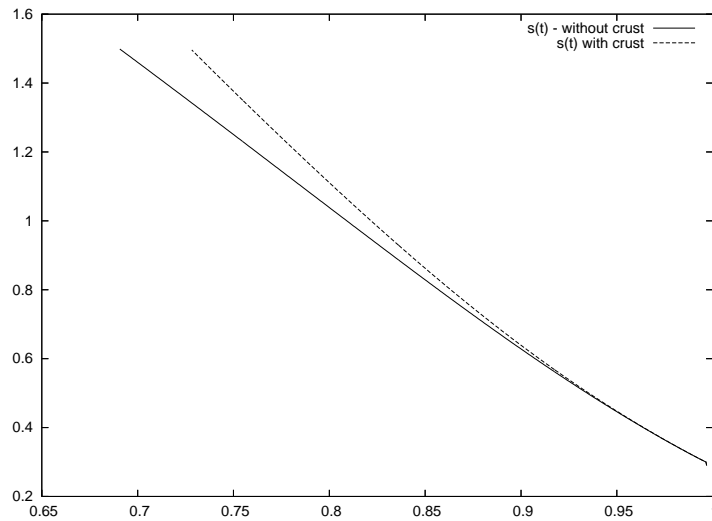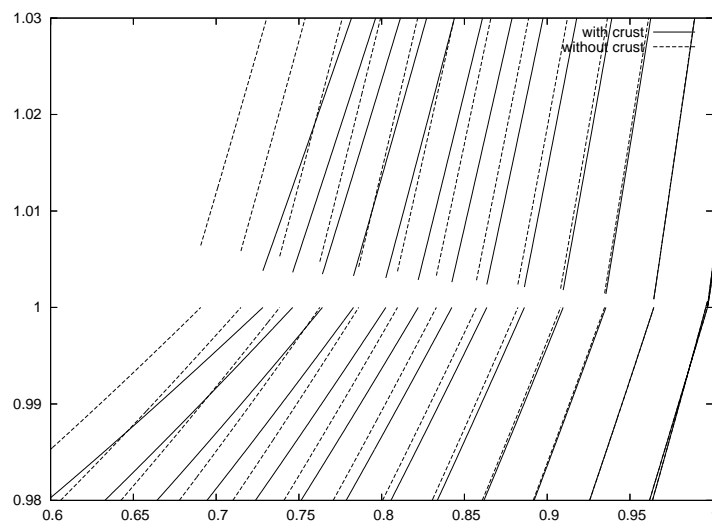
FIG. 4. Free boundary with and without "crust formation".

FIG. 5. Evolution of temperature with and without "crust formation".

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